Formulas notebook CP

Angel, Cristian y Jesus

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1 Properties Binomial Coefficients

$$\sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}$$

$$\sum_{i=0}^{n} \binom{n}{i} i = n2^{n-1}$$

$$\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$$

$$(x+y)^n = \sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k}$$

2 Star and bars

Also know as composition, count the number of solution of the equation

$$x_1 + x_2 + \cdots + x_k = n \;\; \& \;\; x_i \geq 0$$

$$\binom{n+k-1}{n}.$$

3 Catalan number

There are many counting problems in combinatorics whose solution is given by the Catalan numbers. The book Enumerative Combinatorics: Volume 2 by combinatorialist Richard P. Stanley contains a set of exercises which describe 66 different interpretations of the Catalan numbers. Following are some examples, with illustrations of the cases C3 = 5 and C4 = 14.

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!} = \prod_{k=2}^{n}\frac{n+k}{k} \qquad \text{for n} \geq 0.$$

There are many counting problems in combinatorics whose solution is given by the Catalan numbers. Like:

- C_n is the number of Dyck words of length 2n. A Dyck word is a string consisting of n X's and n Y's such that no initial segment of the string has more Y's than X's.
- Re-interpreting the symbol X as an open parenthesis and Y as a close parenthesis, Cn counts the number of expressions containing n pairs of parentheses which are correctly matched:
- It follows that C_n is the number of full binary trees with n + 1 leaves

Some Catalan numbers: 1, 1, 2, 5, 14, 42, 132, 429, 1.430

4 Super Catan number (Schröder–Hipparchus number)

- The nth number in the sequence counts the number of different ways of subdividing of a polygon with n + 1 sides into smaller polygons by adding diagonals of the original polygon.
- The nth number counts the number of different plane trees with n leaves and with all internal vertices having two or more children.
- The nth number counts the number of different ways of inserting parentheses into a sequence of n + 1 symbols, with each pair of parentheses surrounding two or more symbols or parenthesized groups, and without any parentheses surrounding the entire sequence.
- The nth number counts the number of faces of all dimensions of an associahedron Kn + 1 of dimension n 1, including the associahedron itself as a face, but not including the empty set. For instance, the two-dimensional associahedron K4 is a pentagon; it has five vertices, five faces, and one whole associahedron, for a total of 11 faces.

$$S_n = \frac{1}{n}\left((6n-9)S_{n-1} - (n-3)S_{n-2}\right), with: S_1 = S_2 = 1$$

Some super catalan numbers: 1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049

5 Inclusion-Exclusion (generalization)

If s_i = sum of carnality of sets A with a mask of bits with i elements (properties). Number of ways of satisfy at least i properties. And if e_r = Number of ways of satisfy exactly r properties, then:

$$e_r = \sum_{i=r}^{k} (-1)^{i-r} \binom{i}{i-r} s_i$$

6 Luca's Theorem

For non-negative integers m and n and a prime p, the following congruence relation holds:

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \pmod{p},$$

where

$$m = m_k p^k + m_{k-1} p^{k-1} + \cdots + m_1 p + m_0$$

and

$$n = n_k p^k + n_{k-1} p^{k-1} + \cdots + n_1 p + n_0$$

are the base "p" expansions of "m" and "n" respectively. This uses the convention that

$$\tbinom{m}{n} = 0$$

if

$$n < m$$

7 Lagrange Interpolation

Assume that A is a polynomial of degree n, and assume that we know the value of A when evaluated at $\{x_1, ..., X_{n+1}\}$. Let us say that $A(x_i) = y_i$. Then A is completely determined as the following polynomial

$$A(X) = \sum_{i=1}^{n+1} (y_i \prod_{j=1, i!=j}^{n} \frac{X - x_j}{x_i - x_j})$$