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NOTE

THE n-QUEENS PROBLEM

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We present some new solutions to the problem of arranging n queens on an $n \times n$ chessboard with no two taking each other. Recent related work of other authors is also discussed.

An old problem in combinatorics, known as the n-queens problem, is the determination of all possible arrangements of n queens on an $n \times n$ chessboard with no two taking each other. In this note we focus on the case $n = p$ (see Remark (C)). Let $K = \{0, 1, \ldots, p-1\}$. We can then uniquely assign to each position on the board a pair (i, j) of coordinates in the usual manner, with $i, j \in K$. Then a solution can be thought of as a permutation f from K to K satisfying (1) and (2) below for all x, y in K, $x \neq y$.

(1) $f(x) - x \neq f(y) - y$,

(2) $f(x) + x \neq f(y) + y$.

Such a permutation f will be called an *ordinary solution*. Instead of conditions (1) and (2) one might also consider permutations f satisfying (a) and (b) below for all x, y in K with $x \neq y$.

(a) $f(x) - x \neq f(y) - y \pmod p$,

(b) $f(x) + x \neq f(y) + y \pmod p$.

A permutation f satisfying (a) and (b) above is called a *modular solution*. Clearly, any modular solution is also an ordinary solution. The only known solutions, modular or ordinary (see [1]), appear to be the linear functions $f(x) \equiv ux + v \pmod p$ with $u, v \in K$ and $u \neq 0, 1, -1$. Note that, in the above examples, f is actually a modular solution. Our purpose here is to present a new class of non-linear, modular solutions for all $p \geqslant 13$. Using the elements of K we construct the field $F = \mathrm{GF}(p)$. The non-zero squares in F form a subgroup S of the multiplicate group of non-zero elements F^* of F. We assign to each element of S a plus (+) symbol, and each element of F^* not in S is assigned a minus symbol. Let us write down the elements of F^* in the usual order. Then any t consecutive elements of F^* yield a *parity pattern of length t*, that is, an ordered t-tuple of pluses and minuses. (For example, for $p = 5$, the four elements of F^* yield the parity pattern $+ - - +$ of length 4.)

Lemma 1. *Let $p > 11$. Then there are at least two distinct (consecutive) triples having the same parity pattern of length 3.*

Proof. There are exactly eight possible parity patterns of length 3. Thus there is at least one repetition of patterns once $|F^*| > 10$. As an immediate consequence we have:

Theorem 2. *Suppose $p > 11$. Let u and v denote the middle elements of two distinct consecutive triples having the same parity pattern of length 3. Define a permutation g from F^* to F^* as follows:*

$g(0) = 0$,

$g(x) \equiv ux \pmod{p}$ *if* $x \in S$,

$g(x) \equiv vx \pmod{p}$ *if* $0 \neq x \notin S$.

Set $f(x) \equiv g(x) + w \pmod{p}$, with $w \in F$. Then $f(x)$ is a non-linear modular solution.

Remarks.

(A). The lemma just fails for $p = 11$. In fact, for $p \leqslant 11$, it appears that the only solutions, modular or ordinary, are linear.

(B). It would be of interest to see if there exist any ordinary solutions which are not modular solutions.

(C). Modular solutions are of particular interest in the theory of finite planes, orthogonal Latin squares and related areas. For example, if f is a modular solution, then f yields a transversal $T = \{(x, f(x))\}$ to the subnet N of the Desarguesian plane of order p, where N corresponds to the slope set $\{(0), (\infty), (1), (-1)\}$. The only known planes of order p are Desarguesian: for $p \leqslant 11$ it is known that planes of order p must be Desarguesian.

(D). It is easy to see that the theorem above can be generalized by using, instead of S, arbitrary subgroups of F^*.

(E). Each function f in the theorem is a function from F to F and we may form the set T of all "difference quotients", that is, all elements in F of the form $(f(x) - f(y))(x - y)^{-1}$ with $x \neq y$ and with division in F. Since f is injective and satisfies the two modular conditions (a) and (b), we have $|T| \leqslant p - 3$. On the other hand, in his book [2] Redei has shown that, for any function f from F to F, $|T| \geqslant \frac{1}{2}(p + 3)$ unless f is linear. Thus, for small p at least, this bound of Redei is close to being "best possible".

(F) As mentioned earlier, we have focussed on the case $n = p$. However, analogous definitions can be made for any n, and one can speak of modular solutions over $K = Z_n$, the ring of integers mod n. It is well-known that for a modular solution to exist, n must be odd (see [3]). Let f be any solution (ordinary or modular). Since f is a permutation,

$$\sum_{x \in K} f(x) = \sum_{x \in K} x \ .$$

Thus

$$\sum_x (f(x) + x) = \sum_x f(x) + \sum_x x = 2 \sum x = \text{even integer} .$$

Thus, the number of elements x for which $x + f(x)$ is odd must be even, as is pointed out in [1].

References

[1] B. Hansche and W. Vucenic, On the n-queens problem, Notices Am. Math. Soc. (October 1973) A-568 (abstract).

[2] L. Redei, Lückenhafte Polynome über endlichen Körpern (Birkhäuser, Basel, 1970).

[3] Solution to E 2302, Am. Math. Monthly 79 (May 1972) 522.