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A survey of known results and research areas for n-queens

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Abstract

In this paper we survey known results for the n-queens problem of placing n nonattacking queens on an $n \times n$ chessboard and consider extensions of the problem, e.g. other board topologies and dimensions. For all solution constructions, we either give the construction, an outline of it, or a reference. In our analysis of the modular board, we give a simple result for finding the intersections of diagonals. We then investigate a number of open research areas for the problem, stating several existing and new conjectures. Along with the known results for n-queens that we discuss, we also give a history of the problem. In particular, we note that the first proof that n nonattacking queens can always be placed on an $n \times n$ board for $n > 3$ is by E. Pauls, rather than by W. Ahrens who is typically cited. We have attempted in this paper to discuss all the mathematical literature in all languages on the n-queens problem. However, we look only briefly at computational approaches.

Keywords: n-queens problem; Modular n-queens problem; Queens graph; Chessboard graph; Chessboard problems

1. Introduction and background

The n-queens problem is to place n nonattacking queens on an $n \times n$ board. This is a generalization of the problem of putting eight nonattacking queens on a chessboard, which was first posed in 1848 by M. Bezzel, a German chess player, in the *Berliner Schachzeitung* [20]. The earliest paper on the general n-queens problem we have been able to find is F.J.E. Lionnet's 1869 work [132]. We discuss Lionnet's arithmetic representation of the n-queens problem shortly, in Section 4 of this paper. We have not found any solutions to the original 8-queens problem by Bezzel himself, but two nonattacking board configurations were published as partial answer to his problem in the next edition of that serial. Starting in 1850, the 8-queens problem was then studied by C.F. Gauss, for example in his letters to Schumacher in [74]. According to Ahrens in [1], and consistent with our research, the first to solve the problem by finding all 92 solutions was Nauck in [141], in 1850; Gauss later claimed this was the total number of solutions, saying that it is possible in principle to use brute force computation to show this, and that 92 is indeed the total number of solutions was shown by Pauls in [148].

Gauss is often cited as the originator of this problem or the first to solve it, but this is almost certainly a case of "broken telephone", which P. J. Campbell notes in a paper [30] about this historical error. Indeed, as we have noted,

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Bezzel seems to have come up with the problem before Gauss, and Gauss was not the first to solve the problem, having found only 72 of the 92 total solutions given by Nauck, and proved complete by Pauls. (Sprague in [173] and É. Lucas in the section "Quatrième récréation" of [134] both give excellent summaries of the work that has been done by many mathematicians on the original 8-queens problem; Sprague also gives the total number of solutions for $n = 4, \ldots, 11$, respectively 2, 10, 4, 40, 92, 352, 724, 2680, and the number of fundamental solutions, those solutions invariant under the symmetry group of the square, respectively 1, 2, 1, 6, 12, 46, 92, 341, for these board sizes.)

We discuss proofs of n-queens in Section 4, but for background we note that the first proof that n nonattacking queens can be placed on the $n \times n$ chessboard, seems to have been given by Pauls in 1874 in [147] and [148]. However, the literature typically cites Ahrens [1] as the first to prove the n-queens problem. In fact Ahrens himself in [1] and also [2] notes that his method is due to Pauls, so this may be another case of broken telephone. Aside from Ahrens' papers, the only other paper that we have found which notes that Ahrens' proof is due to Pauls is [87] by Gu.

In this paper we consider the results that are known about the original n-queens problem (including a few results we developed in our study of n-queens), known results for generalizations of the problem, and open research areas for the problem and extensions of it. Under the known results, we discuss all the important results that we are aware of, either giving proofs or references to proofs. For all solution constructions, we give either the construction or give an outline of it, but for independent solutions that are close to completely the same, we just give references.

2. Applications

The n-queens problem is often studied as a "mathematical recreation", but Erbas, Tanik and Aliyazicioglu in [60] note several applications: parallel memory storage schemes, VLSI testing, traffic control and deadlock prevention. Erbas and Tanik [58] and Erbas, Tanik and Nair [62] introduce a memory storage scheme for conflict free access for parallel memory systems using n-queens solutions. Kunde and Gürtzig [124] use modular n-queens solutions to study reconfigurable meshes with buses (RMB). The modular n-queens problem is equivalent to finding "valid" periodic skewing schemes for parallel memories [168]. For deadlock prevention, Tanik in [179] shows that given a solution to the n-queens problem, one can find a set of deadlock free paths. As well, Erbas, Sarkeshik and Tanik in [55] note that the n-queens problem has applications in neural networks and constraint satisfaction problems. Another example of an application of the n-queens problem is in image processing, discussed by Yang, Wang, Liu and Chang in [170]. Li, Guangxi and Xiao [131] use n-queens solutions in low-density parity-check codes. Wang, Yang, Liu and Chiang in [193,194] consider applications of n-queens solutions to motion estimation. Dean and Parisi, in [46], develop a statistical mechanical model of "glassy" phase transitions and note that modular n-queens solutions are the zero-energy points in their Hamiltonian. Yamamoto, Kitamura and Yoshikura [198] give a computer analysis of the statistical secondary structure of nucleic acids, and remark that this computation is related to searching for n-queens solutions. Erdem and Lifschitz note that n-queens problems are examples of *finite normal, absolutely tight* logic problems and of interest to the study of theories of negation [63]. Colbourn and Sagols, in [159], define two weaker variants of the n-queens problem and use these to construct anti-Pasch Steiner Triple Systems in $O(n^2)$ time rather than $O(n^3)$; triple systems, themselves, have many applications [43]. In [182], Taylor also observes that a weaker variant of the n-queens problem is equivalent to the problem of finding the "Florentine rows" on n symbols. Heden, in [95–97], uses partial n-queens solutions to construct maximal partial spreads of many sizes in $PG(3, q)$, the three-dimensional projective space over the finite field $\mathbb{F}_q$. Finally, in [104], Hsiang, Shieh and Chen note that complete mapping problems, a variant of the n-queens problem, are ideal to test propositional solvers. Thus the n-queens problem is also of interest in physics and computer science and has some industrial applications. We give a brief overview of information about the computational approach to n-queens in Section 7, for computing all or some solutions, which can used to generate conjectures for confirmation.

Next we give the definitions of a number of terms and introduce notation which we will use in this paper.

3. Definitions and notation

As we have already noted, the *standard n-queens problem* is to place n nonattacking queens on the $n \times n$ chessboard, which we analyze in Section 4, and we speak of the *standard $n \times n$ chessboard*. The *modular n-queens problem* is to place n nonattacking queens on the $n \times n$ modular board, i.e. toroidal board, which is the chessboard with opposite sides identified to make a torus, and we discuss this in Section 5. We speak of the *modular $n \times n$ chessboard*. Generalizing the n-queens problem to the $n \times n \times n$ cube, we speak of the *standard n-cube* which can be represented by a *standard Latin queen square*, a Latin square with each entry signifying the level the queen is on. We can also speak of the *modular n-cube* which represents an n-cube with opposite faces identified, which can be represented by a *modular Latin queen square*. We discuss Latin queen squares in Section 6. Unless we specify otherwise, by a *solution* for some board we mean a set of n nonattacking queens for the standard and modular $n \times n$ boards and a set of n^2 nonattacking queens for the n-cube. Sometimes we speak of *partial solutions*, which we define as a set of nonattacking queens; typically we are interested in maximum partial solutions, when full solutions are impossible. We call a set of solutions *fundamental* if they are all non-equivalent under the group of symmetries of the board and they generate all solutions under this action. A queen that also has the movement of the knight will be called a *knight-queen*.

We will number the rows of the chessboard from 1 at the top to n at the bottom, and the columns from 1 at the left to n at the right, and refer to squares by 2-tuples of the row number in the first entry and the column number in the second entry, so (i, j) is the square on the ith row and the jth column sometimes it will be convenient to number the rows and columns from 0 to $n - 1$, and this will be clear from context. When we do arithmetic modulo n, it will be clear from the context whether the row and column numbers take the values from 0 to $n - 1$ or 1 to n. By the lth *sum diagonal* we mean the diagonal of which each square (i, j) in it has $i + j = l$ (or for the modular board $i + j \equiv l (\text{mod } n)$), and by the lth *difference diagonal* we mean the diagonal for which each square (i, j) in it has $i - j = l$ (or for the modular board $i - j \equiv l (\text{mod } n)$). We can also represent the $n \times n$ chessboard by a simple graph of n^2 vertices with vertex adjacency determined by queen attacks. In this paper unless we specify otherwise, for a graph G, by $\beta(G)$ we mean the independence number of G and by $\chi(G)$ we mean the chromatic number of G. Also, when the type of queens graph (e.g. standard or modular) is clear from context, β_n and χ_n will represent the independence and chromatic numbers of the queens graph of order n. The first author we are aware of to present the n-queens problem in graph-theoretical terms is Berge in 1970, in Chapter 13 of [18].

Kreuzer and Robbiano [123, Tutorial 71] give an interpretation of the n-queens problem using commutative algebra, where the moves of a queen are ideals.

Since a full solution on an $n \times n$ board contains one queen in each row and column, we can write a full solution ϕ as a permutation, such as $(\phi(1), \phi(2), \ldots, \phi(n))$ rather than $\{(\phi(1), 1), (\phi(2), 2), \ldots (\phi(n), n)\}$. For example, for $n = 4$ the solution $\{(2, 1), (4, 2), (1, 3), (3, 4)\}$ can be written as the shorter $(2, 4, 1, 3)$ or even (2413) (we do not use the latter notation for $n \geq 10$). This permutation can be thought of as a map from the columns to the rows. We note that a permutation ϕ of $\{0, 1, \ldots, n - 1\}$ represents a solution for the $n \times n$ standard board if and only if $\phi(x) + x$ and $\phi(x) - x$ are injective functions, and that a permutation ϕ of $\{0, 1, \ldots, n - 1\}$ represents a solution for the $n \times n$ modular board if and only if $\phi(x) + x (\text{mod } n)$ and $\phi(x) - x (\text{mod } n)$ are permutations.

A *regular solution* (also called *linear solution*) is a solution that is generated by a starting square (m, n) and a fixed movement (a, b) that places a queen at $(m + a \bmod n, n + b \bmod n)$, $(m + 2a \bmod n, n + 2b \bmod n)$, etc. A solution that is not regular we call an *irregular solution* (*nonlinear solution*). We define a *symmetric solution* to be a solution that is invariant under 180 degrees rotation, and a *doubly symmetric solution* to be a solution that is invariant under 90 degrees rotation. We call a family of solutions *superimposable solutions* if they are disjoint, that is can be placed on the chessboard without overlap. A *doubly periodic* n-queens solutions is a solution such that when y is added modulo n to each row number of the solution, or x is added to each column number in the solution, or both, for any integer x and y, it remains a solution; as we remark later, a doubly periodic solution is precisely a solution for the modular board. We express the number of solutions for the standard $n \times n$ board (counting solutions in the same equivalence class as distinct) as $Q(n)$, and the number of solutions for the modular board as $M(n)$.

A $k \times n$ *Latin rectangle* is a $k \times n$ array with precisely n distinct entries such that each entry appears precisely once in each row and at most once in each column. An $n \times n$ *Latin square* is a Latin rectangle where $k = n$. An $n \times n$ *pandiagonal Latin square* is a Latin square such that each modular diagonal has precisely n distinct entries. We define an $n \times n$ *magic square* to be an $n \times n$ array of integers such that the sums of the entries in each row, column, and the center sum and difference diagonals are equal to a single integer that is called the *magic constant* of the magic square. A square in which the entries in the rows and columns but not necessarily the center sum and difference diagonal is equal to a magic constant is called a *semi-magic square*. A *panmagic square* is a magic square such that the sum of all modular diagonals (i.e. diagonals that carry over) are also equal to the magic constant of that square. For an $n \times n$ magic/semi-magic/panmagic square, if the n^2 entries are each of the integers from 1 to n^2 (e.g. 1 is an entry, 2 is an entry, etc., and n^2 is an entry), it is called a *normal magic/semi-magic/panmagic square*.

For q a prime power, we denote the finite field with q elements by $\mathbb{F}_q$. We denote the integers by $\mathbb{Z}$.

4. Standard n-queens

The problem of placing nonattacking n-queens on an $n \times n$ chessboard is not trivial (as, for example, placing n rooks would be). There is one solution for $n = 1$, and no solutions for $n = 2$ or $n = 3$. Repeating from Section 1, for $n = 4, \ldots, 11$ there are respectively 2, 10, 4, 40, 92, 352, 724, 2680 solutions. Ahrens in [1] notes that the reflection about the middle row of the board of a solution generates a distinct solution while this is not necessarily true for the rotation of a solution, although it always gives a solution. For example, for $n = 4$ where (2413) is a solution, the reflection of it (3142) is a distinct solution but the 90 degrees rotation of it (2413) is not a distinct solution. Note that for a solution ϕ written as a permutation, ϕ^{-1} is also a solution, although it is not in general distinct. Cull and Pandey discuss this in [44].

Remark 1. *The equivalence classes generated by a set of fundamental solutions of the $n \times n$ chessboard are disjoint and partition the set of all solutions for the $n \times n$ chessboard.*

The number of solutions for particular values of n is available in Sloane's [172], where Sequence A000170 is the total number of solutions and Sequence A002562 is the number of fundamental solutions. Sequences A065256, A032522, A033148 and A065258 are also related. As we noted in Section 1, there are 92 solutions for the 8×8 board as found by Nauck, and they break into 12 fundamental solutions. There is a nice discussion of a two-dimensional space-group that contains seven of these twelve, in [102] by D.H. Hollander. The earliest work we have found that gives the 352 solutions for $n = 9$ is the 1883 paper [146, Note 2] by Th. Parmentier, which gives $Q(n)$ for $n \leq 9$. He distinguishes the 46 fundamental solutions for $n = 9$, gives 92 fundamental solutions for $n = 10$, and conjectures that there are around 350 fundamental solutions and a total of 2800 solutions for $n = 11$ (recall from Section 1 that there are in total 2680 solutions for $n = 11$). Recently, finding the number of solutions for certain n was independently repeated by Panayotopoulos in the form of finding *stable permutations* in [145]. Recently also, Kise, Katagiri, Honda and Yuba, in [113], were able to determine that the number of solutions for $n = 24$ is in fact 227,514,171,973,736. To do this, they optimized their code for memory referencing, control structures and developed a parallelization scheme. Moreover, it has recently been determined by the OASIS research group (from INRIA Sophia Antipolis, the I3S CNRS laboratory, and the University of Nice Sophia Antipolis) using grid computing that the number of solutions for $n = 25$ is 2,207,893,435,808,352 (cf. Sequence A000170 of Sloane's [172]). As we note in Section 7, there is no closed form expression for the total number of solutions for the standard or modular board of arbitrary size. Additionally, the only asymptotic lower bound of which we are aware derives from modular board solutions, which we discuss in Section 5, although we give a conjecture on lower bounds in Section 8. Since queens are more powerful than rooks, $n!$ is clearly an upper bound. Rivin, Vardi and Zimmermann in [157] prove the following result:

Theorem 2 (*Rivin, Vardi and Zimmermann*). *For all m, n for which $m \geq 3$ and $\gcd(n, 6) = 1$, it holds that $Q(mn) > (Q(m))^n M(n)$. In particular, if $\gcd(N, 30) = 5$ then $Q(N) > 4^{N/5}$.*

Clark and Shisha in [42] show that for a "standard infinite" $\aleph_0 \times \aleph_0$ chessboard, that is the top-right quadrant in the grid plane in which a solution is considered to be a placement of queens such that each row and column has a queen, the set of solutions has the cardinality of the continuum.

Of course queens are precisely the combinations of bishops and rooks. A basic discussion on the placement of nonattacking rooks and bishops on the $n \times n$ standard board is given in by J. S. Madachy in [135]. We observe that for the $n \times n$ standard board, there are $2n - 1$ sum diagonals and $2n - 1$ difference diagonals (whereas for the modular board there are n of each). We can put a bishop in each of the sum diagonals except one, because the two end sum diagonals are on the same difference diagonal. Thus we can always place at most $2n - 2$ bishops on a board. This can be achieved by placing n bishops in the first column and $n - 2$ bishops in the second to second last squares of the last column.

The conditions for a permutation to represent solutions are equivalent to the arithmetical question of whether the first n positive integers can be arranged in an n-tuple $(i_1, \ldots, i_n)$ such that the difference between any two entries is not equal to the absolute value of the difference in their indices. This type of arithmetic representation was used by Gauss in [74] in analyzing the original 8-queens problem; T. Ginsburg in [77] gives an equivalent approach to Gauss' for the 8-queens problem. For the general n-queens problem, Lionnet in [132] seems to be the first to give the arithmetic representation just mentioned, and indeed he is the first author we have found who talks about the n-queens problem in general. Lionnet also asks how many solutions exist for different values of n; later, in 1894,

residue class	solution
$n = 6k, 6k+4$	$A_1 = \{(2i, i) \mid 1 \le i \le \frac{n}{2}\}$,
	$A_2 = \{(2i-1, \frac{n}{2}+i) \mid 1 \le i \le \frac{n}{2}\}$,
$n = 6k+1, 6k+5$	$B_1 = \{(n, 1)\}$,
	$B_2 = \{(2i, i+1) \mid 1 \le i \le \frac{n-1}{2}\}$,
	$B_3 = \{(2i-1, \frac{n+1}{2}+i) \mid 1 \le i \le \frac{n-1}{2}\}$,
$n = 6k+2$	$C_1 = \{(4, 1)\}$,
	$C_2 = \{(n, \frac{n}{2}-1)\}$,
	$C_3 = \{(2, \frac{n}{2})\}$,
	$C_4 = \{(n-1, \frac{n}{2}+1)\}$,
	$C_5 = \{(1, \frac{n}{2}+2)\}$,
	$C_6 = \{(n-3, n)\}$,
	$C_7 = \{(n-2i, i+1) \mid 1 \le i \le \frac{n}{2}-3\}$,
	$C_8 = \{(n-2i-3, \frac{n}{2}+i+2) \mid 1 \le i \le \frac{n}{2}-3\}$,
$n = 6k+3$	put solution for $(n-1) \times (n-1)$ board in bottom left corner, add queen to top right.

Fig. 1. Pauls' n-queens solution.

Lucas in [133] independently poses the same question. In 1922, Poulet in [154] similarly notes that the n-queens problem is equivalent to finding an n-tuple $(a_1, \dots, a_n)$ where for any a_i, a_j, $a_i - i \ne a_j - j$ unless $i = j$ and $a_i + i \ne a_j + j$ unless $i = j$; in this paper he gives 8 solutions for $n = 13$. L. M. Blumenthal in [21] also applies the arithmetic representation of a solution, giving a solution for the $n \times n$ board when $\gcd(n, 6) = 1$. In 1900, C. Planck in [151] notes that the arithmetic representation can be considered apart from the chessboard. Planck uses a magic square to give n-queens solutions, and we give his solution later. The arithmetic representation of n-queens is related quite naturally to finding solutions for the reflecting board, which we consider in Section 8.

We momentarily consider representing solutions as systems of distinct representatives (SDR's) of families of sets. Let us have permutations θ, σ of $\{1, \dots, n\}$, and permutations ϕ, ψ of $\{1, \dots, 2n-1\}$. For the standard board, if we can make n 4-tuples, $q_i = (r_{\theta(i)}, c_{\sigma(i)}, s_{\phi(i)}, d_{\psi(i)})$, with r a row, c a column, s a sum diagonal and d a difference diagonal, that do not coincide with each other in any of their respective entries we have solved n-queens. For $i \ne j$, q_i cannot share any entries with q_j. This suggests finding SDR's for families of rows, columns and sum and difference diagonals. Clearly finding a common SDR for all four families is equivalent to finding a solution to the n-queens problem. Yet for three families even, there is a fairly obvious reduction from three-dimensional matching to a simultaneous SDR for three families of sets, so it is an NP-hard problem [72]. Thus we see the difficulty of the n-queens problem reflected in the complexity of a related graph-theory problem.

The computational complexity of completing partial n-queens solutions is studied by Cadoli and Schaerf [27].

Another similar approach to the problem is given by Knuth [118]. He formulates the problem as a generalized exact cover problem by constructing a matrix from the chess board. Define the $n^2 \times (6n-6)$ matrix M_n that has one primary column for each of the n ranks of the board, one primary column for each of the n files, and one secondary column for each of the $4n-6$ nontrivial diagonals of the board. The matrix has n^2 rows: one for each possible queen placement. Each row has a 1 in the columns corresponding to that square's rank, file, and diagonals and a 0 in all the other columns. The n-queens problem is equivalent to choosing a subset of the rows of this matrix such that every primary column has a 1 in precisely one of the chosen rows and every secondary column has a 1 in at most one of the chosen rows; it might be interesting to consider non-integer relaxations of this problem.

In the remainder of this section, we give all the constructions in the literature for solutions to the standard n-queens problem. There are many different proofs from different authors, and many independent proofs of the same solution construction. We first consider the solutions that involve, or are very similar to, dividing the interval from 1 to n into integer subintervals, and then giving a (usually linear) placement function for each interval. Then we look at solutions that involve constructing an array of some sort (e.g. Latin square, magic square), and taking certain squares from it as a solution.

We will show that for all $n > 3$ there is at least one solution to the n-queens problem on an $n \times n$ standard chessboard. The first proof for all n is given by Pauls in [147] and [148]. The second paper also includes an exhaustive algorithm to find *all* solutions for a given n. He gives his solutions as permutations. We give Pauls' solutions in Fig. 1.

For Pauls, all solutions are permutations so there are no row or column attacks. Within each interval (e.g. between the queens in A_1) there are no attacks on sum or difference diagonals, since each set places queens by knight's moves

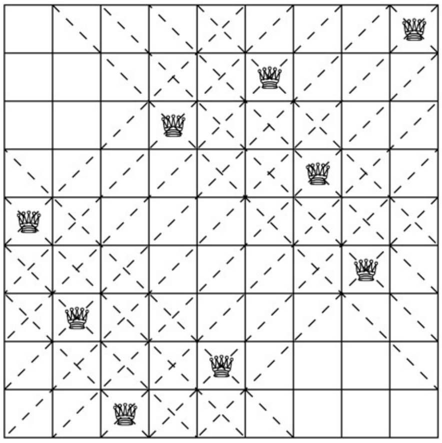

Fig. 2. Pauls' nonattacking placement for the 9×9 board.

residue class	solution
$n \equiv 2, 4 \pmod 6$	$\{(\frac{n}{2} + 2(i-1) \bmod n, i) \mid 1 \le i \le \frac{n}{2}\}$,
	$\{(\frac{n}{2} + 2i + 1 \bmod n, i + \frac{n}{2}) \mid 1 \le i \le \frac{n}{2}\}$,
$n \equiv 3 \pmod 6$	$\{(1,1)\}$,
	$\{((\frac{n-1}{2} + 2(i-1) \bmod (n-1)) + 1, i+1) \mid 1 \le i \le \frac{n-1}{2}\}$,
	$\{((\frac{n+1}{2} + 2i \bmod (n-1)) + 1, i + \frac{n+1}{2}) \mid 1 \le i \le \frac{n-1}{2}\}$,
$n \equiv 1, 5 \pmod 6$	$\{(r + 2i \bmod n, i) \mid 1 \le i \le n\}$ for any r,
$n \equiv 0 \pmod 6$	$\{(2i, i) \mid 1 \le i \le \frac{n}{2}\}$,
	$\{(2i-1, i + \frac{n}{2}) \mid 1 \le i \le \frac{n}{2}\}$.

Fig. 3. Franel's n-queens solution.

going from the top of the board to the bottom and then stopping; for example, in A_1 clearly there are no sum diagonal attacks, and there are no difference diagonal attacks because the attacks on difference diagonals have a slope of -1, whereas the knight's move placement is moving two rows ahead each time so any two queens are on a line with slope -2 only. For each residue class modulo 6, we have checked that having a sum diagonal or difference diagonal attack between two intervals (e.g. for $n = 6k$, between A_1 and A_5) produces a contradiction; for $n = 6k, 6k + 4$, there are two cases (of a sum diagonal attack between the intervals and a difference diagonal attack), for $n = 6k + 1, 6k + 5$ there are six cases, and for $n = 6k + 2$ there are 56 cases. For $n = 6k + 3$, it is immediate that there are no row, column or diagonal attacks between intervals, and there is a sum diagonal attack if and only if the solution for the $n = 6k + 2$ board places a queen on the main sum diagonal, i.e. the diagonal with each sum of coordinates $n + 1$. Checking $C_1, \ldots, C_8$ finds that none of these sums of coordinates are $n + 1$. Hence we have:

Theorem 3 (*Pauls*). *For all $n > 3$, n nonattacking queens can be placed on the $n \times n$ standard chessboard.*

In other words, the above theorem shows that for all $n > 3$, $Q(n) > 0$. A solution made by Pauls method is shown in Fig. 2, in which we show only the diagonal attacks of the queens, which clearly do not attack each other on rows or columns.

In Section 72 of [160], A. Sainte-Laguë discusses the problem of n-queens, and gives Pauls solution above, citing Ahrens, and does not give an independent solution himself.

Following Pauls' proof, the next earliest solution seems to have been in 1894 by Franel in [67]. He makes different permutations of $\{1, \ldots, n\}$ for congruency classes modulo 6, and we give these in Fig. 3.

Another early author on the n-queens problem is Tarry in [181] from 1897, in which he talks about a method of finding solutions to the n-queens problem for arbitrary n, and says particularly that it can be used to solve the cases $n = 21$ to $n = 71$. However, this is in a conference proceedings and we have only been able to find the abstract.

In 1911, H. Behmann in [12] shows for the $n \times n$ board such that $\gcd(n, 6) = 1$ that four superimposable solutions exist that each include one of the four corners of the board: starting at the top-left corner, place a queen successively one row down and two columns to the right (left), and starting at the top-right, place a queen successively one row

residue class	solution
$n = 4 + 6j$	$(2, 4, \ldots, n, 1, 3, \ldots, n-1)$,
$n = 5 + 6j$	$(2, 4, \ldots, n-1, 1, 3, \ldots, n)$,
$n = 6 + 6j$	$(2, 4, \ldots, n, 1, 3, \ldots, n-1)$,
$n = 7 + 6j$	$(2, 4, \ldots, n-1, 1, 3, \ldots, n)$,
$n = 8 + 12j$	$(2, 4, \ldots, n, 3, 1, 7, 5, \ldots, n-1, n-3)$,
$n = 9 + 12j$	$(n, 4, 6, \ldots, n-1, 3, 1, 7, 5, \ldots, n-2, n-4, 2)$,
$n = 14 + 12j$	$(n-1, 2, 4, \ldots, n, 3, 1, 7, 5, \ldots, n-3, n-5)$,
$n = 15 + 12j$	$(n, n-2, 2, 4, \ldots, n-1, 3, 1 \ldots, n-4, n-6)$.

Fig. 4. Scheid's n-queens solutions.

residue class	solution
$n = 6k, 6k+4$	$\{(j, 2j) \mid 1 \le j \le \frac{n}{2}\}$,
	$\{(\frac{n}{2} + j, 2j - 1) \mid 1 \le j \le \frac{n}{2}\}$,
$n = 6k+2, 6k+4$	$\{(j, 1 + [2(j-1) + \frac{n}{2} - 1 \pmod{n}]) \mid 1 \le j \le \frac{n}{2}\}$,
	$\{(n+1-j, n-[2(j-1)+\frac{n}{2}-1 \pmod{n})) \mid 1 \le j \le \frac{n}{2}\}$,
$n = 2k+1$	solution for $n-1$ even in cells $(1,1)$ to $(n-1, n-1)$
	and add queen at (n, n).

Fig. 5. Hoffman, Loessi and Moore's n-queens solutions.

down and two columns to the right (left), and the vertical reflection of this yields two solutions that include respectively the bottom-left and bottom-right corners, and then these four solutions are superimposable.

In 1912/1913, Laparewicz in [125] gives explicit solutions for $n = 6k + 1$ and $n = 6k + 5$, of the knight's move placement $(1, 3, 5, \ldots, 2, 4, 6, \ldots)$, and for even n makes the observation that certainly this is no longer a solution, as the first and last queens are at opposite corners (e.g. at top-left and bottom-right). He notes the result that if n is prime, then cyclic permutations of a solution remain a solution. Laparewicz also gives the fundamental solutions for $n = 4$ to $n = 8$. As well, he remarks that around 1900 there was a game sold in toy shops that was equivalent to the n-queens problem for $n = 6$. He also analyzes the n-queens problem as finding the determinant of a matrix constructed in a certain way, which we comment on in Section 7.

In 1925, G. Sforza in [166] gives a placement for the $n \times n$ board for n prime: for h such that $1 < h < n - 1$, take $a_2 \equiv a_1 + h \pmod{n}$, $a_3 \equiv a_2 + h \pmod{n}$, $\ldots$, $a_n \equiv a_{n-1} + h \pmod{n}$, and then $(a_1, a_2, \ldots, a_n)$ is a solution. He shows that this will produce $n(n - 3)$ solutions, but that these are not necessarily all the solutions.

In 1960, F. Scheid, in [161], who also gives results for the number of nonattacking other chess pieces that can be placed on the $n \times n$ standard board, solves the n-queens problem by giving the eight congruence classes of solutions in Fig. 4, which form a complete residue system modulo 12.

In 1969, Hoffman, Loessi and Moore in [101] give the solution shown in Fig. 5. Falkowski and Schmitz's solution in [64] is given in Fig. 6. Their solution is reduced into four cases in [156] by M. Reichling, and into five cases by J. B. Wu in [196].

Yaglom and Yaglom [197, Section 3] give the solution in Fig. 7.

In [76], a book on mathematics and chess, E. Ya. Gik looks at the n-queens problem, and gives some solutions for small n. Gik [75, Chapter 8] gives the same solution (up to numbering rows and columns) as Yaglom and Yaglom in Fig. 7, but he also cites Okunev in [144] for the proof. He also considers the modular board, the cube, which we discuss in Section 6, and the projective plane, which we discuss in Section 8. In these cases he does not give general results.

W.S. Qiu's solution from [155] is given in Fig. 8, where $n \ge 4$. In his paper Qiu also gives necessary and sufficient conditions for a permutation to be a symmetric or doubly symmetric solution. Additionally he uses these to prove that for $n \ge 6$ there exists symmetric solutions, and he finds sufficient conditions on n for doubly symmetric solutions and gives lower bounds for the number of these.

Erbas, Tanik and Aliyazicioglu in [60] use congruence equations to generate solutions. (In their earlier technical report [61], they use their linear congruence method to repeat the solutions made by Falkowski and Schmitz.) We give their solutions in Fig. 9; their use of the integer set $S(n)$ makes their method give increasing numbers of solutions as n gets bigger. Note that they take the residues modulo n of 0 to $n - 1$, and they denote the first column as $y = 0$, first row as $x = 0$, etc.

residue class	solution
$n = 6k, 6k-2$	$\{(2i, i) \mid 1 \le i \le \frac{1}{2}n\}$, $\{(2i-1, \frac{1}{2}n+i) \mid 1 \le i \le \frac{1}{2}n\}$,
$n = 6k-1, 6k+1$	$\{(2i, i) \mid 1 \le i \le \frac{1}{2}(n-1)\}$, $\{(2i-1, \frac{1}{2}(n-1)+i) \mid 1 \le i \le \frac{1}{2}(n-1)\}$, $\{(n, n)\}$,
$n = 12k-4$	$\{(2i, i+1) \mid 1 \le i \le \frac{1}{2}n\}$, $\{(4i+3, \frac{1}{2}n+2(i+1)) \mid 0 \le i \le \frac{1}{4}(n-4)\}$, $\{(4i+1, \frac{1}{2}n+2(i+1)+1) \mid 0 \le i \le \frac{1}{4}(n-8)\}$, $\{(n-3, 1)\}$,
$n = 12k+2$	$\{(2i, i+1) \mid 1 \le i \le \frac{1}{2}n\}$, $\{(4i+3, \frac{1}{2}n+2(i+1)) \mid 0 \le i \le \frac{1}{4}(n-6)\}$, $\{(4i+1, \frac{1}{2}n+2(i+1)+1) \mid 0 \le i \le \frac{1}{4}(n-6)\}$, $\{(n-1, 1)\}$,
$n = 12k+3$	$\{(2i+1, i+1) \mid 1 \le i \le \frac{1}{2}(n-1)\}$, $\{(4i+4, \frac{1}{2}(n-1)+2(i+1)) \mid 0 \le i \le \frac{1}{4}(n-7)\}$, $\{(4i+2, \frac{1}{2}(n-1)+2(i+1)+1) \mid 0 \le i \le \frac{1}{4}(n-7)\}$, $\{(n-1, 1)\}$,
$n = 24k-15$	$\{(2i+1, \frac{1}{4}(3n+1)-i) \mid 0 \le i \le \frac{1}{2}(n-1)\}$, $\{(4i, \frac{1}{4}(n-1)-2(i-1)) \mid 1 \le i \le \frac{1}{8}(n-1)\}$, $\{(4i, n-2(i-\frac{1}{8}(n+7))) \mid \frac{n+7}{8} \le i \le \frac{1}{4}(n-1)\}$, $\{(4i+2, \frac{1}{4}(n-5)-2i) \mid 0 \le i \le \frac{1}{8}(n-9)\}$, $\{(4i+2, n-1-2(i-\frac{1}{8}(n-1))) \mid \frac{n-1}{8} \le i \le \frac{1}{4}(n-5)\}$,
$n = 24k-3$	$\{(2i+1, \frac{1}{4}(3n+1)-i) \mid 0 \le i \le \frac{1}{2}(n-1)\}$, $\{(4i, \frac{1}{4}(n+3)-1-2(i-1)) \mid 1 \le i \le \frac{1}{8}(n+3)\}$, $\{(4i, n-1-2(i-\frac{1}{8}(n+11))) \mid \frac{n+11}{8} \le i \le \frac{1}{4}(n-1)\}$, $\{(4i+2, \frac{1}{4}(n-5)-2i) \mid 0 \le i \le \frac{1}{8}(n-13)\}$, $\{(4i+2, n-2(i-\frac{1}{8}(n-5))) \mid \frac{n-5}{8} \le i \le \frac{1}{4}(n-5)\}$.

Fig. 6. Falkowski and Schmitz's n-queens solutions.

residue class	solution
$n = 6k, 6k+4$	$\{(n-2i+1, i) \mid 1 \le i \le \frac{n}{2}\}$, $\{(2n-2i+2, i) \mid \frac{n}{2}+1 \le i \le n\}$,
$n = 6k+2$	$\{(n-2i+2, i) \mid 2 \le i \le \frac{n}{2}-2\}$, $(2n-2i+1, i) \mid \frac{1}{2}+3 \le i \le n-1\}$, $\{(4,1), (n, \frac{n}{2}-1), (2, \frac{n}{2}), (n-1, \frac{n}{2}+1), (1, \frac{n}{2}+2), (n-3, n)\}$,
$n = 2k+1$	solution for $n-1$ in cells $(n, 1)$ to $(2, n-1)$ and queen in $(1, n)$.

Fig. 7. Yaglom and Yaglom's n-queens solutions.

D.S. Clark in [40] constructs a solution for n-queens using d-circulant matrices, which are matrices where each row is the row above it cyclically shifted d places to the right. Erbas and Tanik in [59] produce a superset of Clark's construction, which we describe: For $n \equiv 0, 4 (\text{mod } 6)$, a $n/2 \times n$ 2-circulant is made from $(1, \ldots, n)$ and another from $(2, 3, 4, \ldots, n, 1)$. The vertical concatenation of these two is a Latin square. Erbas and Tanik place a queen on each of the cells that contain a 2 and prove that this is a nonattacking placement. For $n \equiv 2, 4 (\text{mod } 6)$ they define a set $D(n) = \{d \in \mathbb{Z} \mid d = 6i+3, 0 \le i \le \lfloor \frac{\alpha-1}{3} \rfloor\}$. They then make a 2-circulant from $(1, 2, 3, \ldots, n)$ and for $d \in D(n)$, make a 2-circulant from $(n-d+1, n-d+2, \ldots, n, 1, 2, 3, \ldots, n-d)$. They then define the set $S(n, d) = \{s \in \mathbb{Z} \mid d+1 \le s \le n-(2d-2)\}$, and prove that for the $n \times n$ matrix made by combining each of these Latin rectangles, placing a queen on the squares containing $s \in S(n, d)$ yields a nonattacking arrangement of n queens. Finally, for n odd, they add a queen to the bottom-right corner of an $(n-1) \times (n-1)$ solution.

For the $n \times n$ board if the numbers $1, \ldots, n^2$ are written across the rows, carrying over from the end of the first row to the start of the second row etc., the entries in the squares which the n queens of any n-queens solution are on sum to the magic constant $\frac{n(n^2+1)}{2}$ of the square, as M. Petković observes in Chapter I of [149]. In fact magic squares can also be used to construct n-queens solutions and vice versa. The first instance of this we have found is [151] by Planck. Planck states without proof that panmagic squares (he calls them Nasik squares), with all entries distinct can be used to find n-queens solutions, with the necessary and sufficient condition that $\gcd(n, 6) = 1$. Planck observes

residue class	solution
$n = 1, 5, 7, 11 \pmod{12}$	$y = \lambda(x - l) + l \pmod{n}, 1 \le x \le n, \lambda = 2, 3,$ and $0 \le l \le n-1$ fixed
$n = 4, 6, 10, 12 \pmod{12}$	$y = \lambda x \pmod{n+1}, 1 \le x \le n,$ delete $(n+1, n+1), \lambda = 2, 3,$
$n = 3 \pmod{12}$	$y = -3x + n - 2, 1 \le x \le \frac{n}{3} - 1,$ $y = 3, x = \frac{n}{3},$ $y = -3x + 2n + 2, \frac{n}{3} + 1 \le x \le \frac{2n}{3},$ $y = n - 2, x = \frac{2}{3}n + 1,$ $y = -3x + 3n + 6, \frac{2}{3}n + 2 \le x \le n,$
$n = 9 \pmod{12}$	$y = -2x + \frac{n-1}{2}, 1 \le x \le \frac{n-5}{4},$ $y = \frac{n-1}{2}, x = \frac{n-1}{4},$ $y = -2x + \frac{3n+3}{2}, \frac{n+3}{4} \le x \le \frac{3n+1}{4},$ $y = \frac{n+3}{2}, x = \frac{3n+5}{4},$ $y = -2x + \frac{5n+7}{2}, \frac{3n+9}{4} \le x \le n,$
$n = 4, 8 \pmod{12}$	$y = 2x + \frac{n}{2} - 2, 1 \le x \le \frac{n}{4} + 1,$ $y = 2x - \frac{n}{2} - 2, \frac{n}{4} + 2 \le x \le \frac{n}{2},$ $y = 2x - \frac{n}{2} + 1, \frac{n}{2} + 1 \le x \le \frac{3n}{4} - 1,$ $y = 2x - \frac{3n}{2} + 1, \frac{3n}{4} \le x \le n,$
$n = 2, 10$	$y = 2x + \frac{n}{2} - 2, 1 \le x \le \frac{n+2}{2},$ $y = 2x - \frac{n}{2} - 2, \frac{n+6}{4} \le x \le \frac{n}{2},$ $y = 2x - \frac{n}{2} + 1, \frac{n}{2} + 1 \le x \le \frac{3n-2}{4},$ $y = 2x - \frac{3n}{2} + 1, \frac{3n+2}{4} \le x \le n.$

Fig. 8. Qiu's n-queens solutions.

residue class	solution
$n = 6\alpha \pm 1$	$ax + y \equiv c \pmod{n}, x = 0, \ldots, n-1, c \in \{0, \ldots, n-1\}, a \in S(n) = \{s \mid 2 \le s \le n-2, \gcd(s(s-1)(s+1), n) = 1\},$
$n = 6\alpha \pm 2$	$(n-2)x + y \equiv c \pmod{n}, x = 0, \ldots, \frac{n}{2} - 1, c \in S(n) = \{s \mid 3 \le s \le n-5\},$ $(n-2)x + y \equiv c \pmod{n}, x = \frac{n}{2}, \ldots, n-1, c \in S(n),$
$n = 6\alpha + 3$	$(n-3)x + y \equiv c \pmod{n}, x = 0, \ldots, \frac{n}{3} - 1, c \in S(n) = \{s \mid 4 \le s \le n-10\},$ $(n-3)x + y \equiv c + 4 \pmod{n}, x = \frac{n}{3}, \ldots, \frac{2}{3}n - 1, c \in S(n),$ $(n-3)x + y \equiv c + 8 \pmod{n}, x = \frac{2}{3}n, \ldots, n-1, c \in S(n),$
$n = 6\alpha$	$(n-2)x + y \equiv 1 \pmod{n}, x = 0, \ldots, \frac{n}{2} - 1,$ $(n-2)x + y \equiv 0 \pmod{n}, x = \frac{n}{2}, \ldots, n-1,$
$n = 6\alpha - 2$	$(n-2)x + y \equiv 1 \pmod{n}, x = 0, \ldots, \frac{n}{2} - 1,$ $(n-2)x + y \equiv 0 \pmod{n}, x = \frac{n}{2}, \ldots, n-1$
$n = 12\alpha - 3$	$(n-3)x + y \equiv c \pmod{n}, x = 0, \ldots, \frac{n}{3} - 1, c \in S(n) = \{(\frac{n+3}{2}, \frac{n+5}{2})\},$ $(n-3)x + y = c - 1 \pmod{n}, x = \frac{n}{3}, \ldots, \frac{2}{3}n - 1, c \in S(n),$ $(n-3)x + y \equiv c - 2 \pmod{n}, x = \frac{2n}{3}, \ldots, n-1, c \in S(n).$

Fig. 9. Erbas, Tanik and Aliyazicioglu's n-queens solutions.

that while this method yields all solutions for the square of order 5, it does not give all the solutions for squares of larger order. He produces a solution for the 11×11 board by placing queens on the cells labeled 1–11 on the magic square in Fig. 10. Similarly the cells that contain 12–22, and so on, also work.

Stern in Section C of [175] and Section B of [176] uses pandiagonal Latin squares to construct solutions to the n-queens problem, and magic squares. He shows that in an $n \times n$ pandiagonal Latin square, for each particular number in it, placing a queen on all squares with this number yields a solution to the n-queens problem (clearly this is also a solution to the modular n-queens problem, although he does not say this). He mentions that for $n = 5$ this gives all solutions, but that this is not necessarily true for all n. He remarks that the solutions obtained in this way are doubly periodic.

40	6	93	59	25	112	78	55	21	108	74
32	119	85	51	17	104	70	36	2	89	66
13	100	77	43	9	96	62	28	115	81	47
5	92	58	24	111	88	54	20	107	73	39
118	84	50	16	103	69	35	1	99	65	31
110	76	42	8	95	61	27	114	80	46	12
91	57	23	121	87	53	19	107	72	38	4
83	49	15	102	68	34	11	98	64	30	117
75	41	7	94	60	26	113	79	45	22	109
56	33	120	86	52	18	105	71	37	3	90
48	14	101	67	44	10	97	63	29	116	82

Fig. 10. 11×11 panmagic square for Planck's n-queens solution.

In [48], Demirörs, Rafraf and Tanik give a procedure for moving back and forth between $n \times n$ normal semi-magic squares and n-queens solutions when $\gcd(n, 6) = 1$. The proofs for the methods they use are given in [49] by Demirörs and Tanik. Their method is further discussed by Pickover in [150]. We now give Demirörs, Rafraf and Tanik's two constructions.

Theorem 4 (*Demirörs, Rafraf and Tanik*). *For* $\gcd(n, 6) = 1$*, given an* $n \times n$ *magic square made either by the de la Loubére or Bachet or Méziriac methods* [7,122]*, a solution for the* n*-queens problem can be found in the following way:*

1. For each entry a_{ij} *in the* $n \times n$ *magic square, replace it with* $b_{ij} \equiv a_{ij} \pmod n$

2. Replace all 0*'s in the modified magic square with* n*'s, so all entries are from* 1 *to* n *rather than* 0 *to* $n - 1$*.*

3. Every row, column and modular diagonal of the final magic square is a permutation ϕ *of* $\{1, \ldots, n\}$ *that places* n *nonattacking queens on the standard board. Furthermore, the* n *permutations from the rows and the* n *permutations from the columns place* n *superimposable sets of* n *nonattacking queens each on the board.*

Theorem 5 (*Demirörs, Rafraf and Tanik*). *Given a permutation describing a solution for the* n*-queens problem, an* $n \times n$ *normal semi-magic square can be constructed:*

1. Given the permutation $\phi = (a_1, \ldots, a_n)$ *of* $\{1, \ldots, n\}$*, form an* $n \times n$ *matrix* $P = (p_{ij})$ *from* ϕ*, where* ϕ *is the first row and each other row is the row above cyclically shifted one unit to the right.*

2. Have $\tau = (\lceil n/2 \rceil, \lceil n/2 \rceil + 1, \lceil n/2 \rceil + 2, \ldots, \lceil n/2 \rceil + n - 1) - \phi$*, subtracting componentwise. Then make a matrix* $R = (r_{ij})$ *from* τ*, where* τ *is the first row and each other row is the row above cyclically shifted one unit to the left.*

3. Make a new $n \times n$ *matrix* $M = (m_{ij})$ *such that for all* $1 \le i, j \le n$*,* $m_{ij} = nr_{ij} + p_{ij}$*.* M *is a normal semi-magic square.*

In [53], Erbas, Rafraf and Tanik show how to get n-queens solutions from magic squares constructed by the uniform step method.

K. Zhao in [199] gives the following solution for the $p \times p$ chessboard, p prime: Let p be a prime number greater than 3, $(V, *)$ a p-quasigroup defined on set $V = \{1, 2, 3, \ldots, p\}$ under operation $(*)$ defined by $r * s = 2s - r \pmod p$. Let $K(p)$ be the multiplication table of $(V, *)$ (recall that the multiplication table of any quasigroup is a Latin square). If $K(p)$ is superimposed onto a $p \times p$ chessboard and queens are placed on the squares that contain the integer k where $k \in V$, then the resultant board configuration corresponds to an n-queens solution.

Let us define the queens separation number $s_Q(m, n)$ to be the minimum number of pawns that can be placed on an $n \times n$ chessboard such that m nonattacking queens can be placed on the board, where queens do not attack through the pawns. In other words, the queens separation number is the minimum number of squares that can be removed from the $n \times n$ board such that m nonattacking queens can be placed. Chatham, Fricke and Skaggs [34] prove that $s_Q(n + 1, n) = 1$, that is, that $n + 1$ queens can be separated by 1 pawn. In [33] it is proved that for a given k, for all $N > \max\{87 + k, 25k\}$, $s_Q(N + k, N) = k$.

It follows from a very nice result proved in 1918 by G. Pólya in [153] that if we have an $n \times n$ array F that is a solution for the $n \times n$ modular board (cf. Section 5, and Theorem 2 from Section 1) and an $m \times m$ array G that is a solution for a standard board, by placing a copy of F in place of each queen in G we obtain a solution for the $mn \times mn$ standard board, hence:

Theorem 6 (*Pólya*). *For given $m, n > 3$ such that $\gcd(n, 6) = 1$, $f_1, \dots, f_Q(m)$ are all solutions for the $m \times m$ standard board, and g a solution for the $n \times n$ modular board, then for each map $\pi : \{0, \dots, n-1\} \to \{1, \dots, Q(m)\}$ the function $h(an + b) = f_{\pi(b)}(a)n + g(b)$ gives a distinct $mn \times mn$ solution for the standard board.*

Pólya showed as a corollary of this that doubly periodic (modular) solutions exist if and only if $\gcd(n, 6) = 1$. In fact, as Watkins observes in [195], an $n \times n$ doubly periodic solution is precisely one where if the grid plane is tessellated with it, an $n \times n$ board placed anywhere on the grid plane will be a solution.

The $n \times n$ board can be extended to the entire plane as an infinite chessboard, and indeed on the infinite board we can put in place of each queen itself a solution for the modular board, giving a board whose "queens" are $n \times n$ solutions themselves, and do this forever, such that we achieve a maximal number of nonattacking queens on the infinite board; Clark and Shisha give a nice "proof without words" of this fact in [41], and cf. [163, Chapter 1, pp. 4–7]. This was first observed by Pólya in [153], and we discuss Pólya's result more in Section 5.

We know that there is always at least one solution for the $n \times n$ board for $n > 3$, but we can also try to find n solutions such that no two solutions have any squares in common. Clearly these solutions could then be superimposed on the $n \times n$ board. In [45], Cvetković discusses this problem, and notes that for $n = 8$ there are only 6 superimposable solutions. This fact was first shown by Bennett in [17]; the following proof is a simplified version from the same journal by Gosset in [86]. Consider the symmetric octagon containing 20 squares that is formed by the four middle squares from each side, and the squares diagonally connecting each of these sets of four squares. For each of the 12 fundamental solutions, at least three of their squares are contained in this octagon. Hence it is impossible to have more than six superimposed solutions, as this would involve having at least 21 queens on 20 squares.

Kraitchik in [122] notes that any two-dimensional lattice $(a, b) + r(u, v) (\bmod n)$ with $\gcd(uv, n) = 1$ can be obtained by replacing (u, v) with (ru, rv) for any r such that $\gcd(r, n) = 1$, and that for r such that $ru \equiv 1 (\bmod n)$, any two elements of the original lattice will be on the same diagonal if and only if $rv \equiv \pm 1 (\bmod n)$: thus to have a regular solution it is necessary and sufficient that the solution be generated by a motion $(1, v)$ with $\gcd(v(v - 1)(v + 1), n) = 1$, from which it follows that there are no regular solutions for n divisible by 2 or 3. Kraitchik observes that if any regular solutions occur, they will occur in the n sets obtained from each other by changing the origin of the lattice, and that none of these solutions intersect, so they can be superimposed on the same board without overlapping. He then shows that a regular solution is symmetric if in the lattice $(a, b) + r(1, v)$ it holds that $a + r \equiv \frac{n+1}{2} (\bmod n)$ and $b + rv \equiv \frac{n+1}{2} (\bmod n)$, and observes that regular solutions are cyclic, such that their rows and columns can be permuted cyclically without destroying their character as solutions, although the individual solutions may be changed. He notes that it is necessary to have a doubly symmetric solution that $n = 4k$ or $n = 4k+1$, and that the number of doubly symmetric solutions will always be divisible by 2^k.

Sumitaka in [178] considers conditions under which certain arrangements on the $n \times n$ board become solutions to the n-queens problem, and also the conditions under which solutions have certain symmetries.

Larson in [126] shows that a doubly symmetric solution for the $p \times p$ board for p a prime yields positive integers u and v such that $u^2 + v^2 = p$. He further shows that there is a regular doubly symmetric solution for the $p \times p$ board whenever p is a prime of the form $p = 4k + 1$. He remarks about Pólya's result that there is a doubly symmetric solution for the modular board only if $n \equiv 1 (\bmod 4)$ and $\gcd(n, 6) = 1$ (observe that all primes congruent to 1 modulo 4 are of the form $6k \pm 1$). This gives an alternate proof of Fermat's two-square theorem that all primes of the form $4k + 1$ can be expressed as the sum of two squares. He conversely shows that this theorem implies the existence of a regular doubly symmetric solution for the $p \times p$ board with p prime. M. Gardner in [69] gives a clear, elementary description of Larson's result, and gives a sketch of the proof.

As we noted earlier, the n-queens problem can naturally be thought of as the graph-theoretical problem of showing that the independence number of the queens graph with n^2 vertices is always n for $n > 3$, or moreover, presenting such a set. In a survey [68, Section 3.1] on graph-theoretic properties of chessboards, Fricke and a number of other authors discuss some other parameters of the queens graph Q_n. The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G. The independent domination number $i(G)$ is the minimum cardinality of an independent dominating set. The upper domination number, $\Gamma(G)$, is the maximum cardinality of a minimal dominating set of vertices. The irredundance number, $ir(G)$, is the minimum cardinality of a maximal irredundant set of vertices. The upper irredundance number, $IR(G)$, is the maximum cardinality of an irredundant set. They note that in general

$$ir(G) \leq 2\gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G).$$

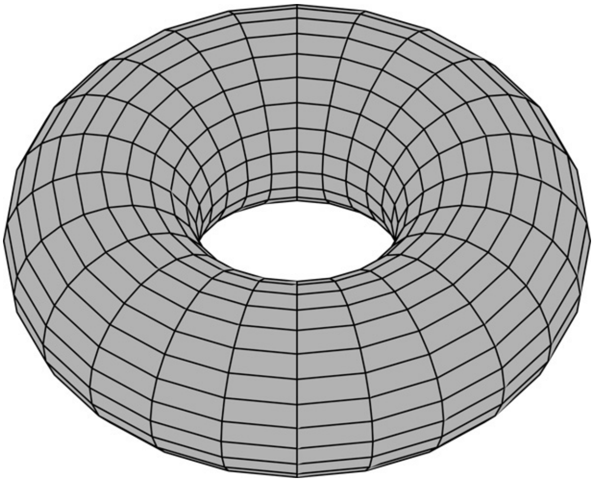

Fig. 11. The modular chessboard.

They give many other results about these parameters for the queens graph and the other chess graphs, and note the result that $\Gamma(Q_n) > \beta(Q_n)$ is possible. Their other results do not involve $\beta(Q_n)$ and thus we do not discuss them further. M. Katzman in [109] has looked at the domination number by enumerating monomials in graded modules. This work is related to Hilbert functions in commutative algebra.

We also find an analysis of chessboard domination and independence with graph theory in [128], by Laskar and Wallis. They relate this area to known results in design theory and the study of projective planes. However, most of their attention is on studying designs related to domination by rooks, not queens. Further, Iyer and Menon in [108] show that for the queens graph, with n^2 vertices such that two vertices are adjacent if a queen can attack one from the other, if n is not divisible by 2 or 3, the chromatic number χ_n of the graph is n. This is a weaker case of Latin queen squares, which we consider in Section 5.

In [13], Beineke, Broere and Henning begin a characterization of induced subgraphs of the queens graph. They show that certain classes can be identified by forbidden induced subgraphs and others can be explicitly given. This work is continued by Ambrus and Barát in [6].

R.K. Guy in Section C18 of [89] gives a brief survey on the n-queens problem, along with other chessboard problems such as finding the minimum number of queens needed to attack every square on the board and the reflecting queens problem where the board has a reflecting strip at the top, which we describe in Section 8. Hedetniemi, Hedetniemi and Reynolds in [98] give a comprehensive graph-theoretical survey of general combinatorial problems on the chessboard (e.g. independence, domination, irredundance, independent domination parameters of the queens graph), and they briefly discuss the n-queens problem along with other problems.

5. Modular *n*-queens

The modular chessboard can be thought of as a toroidal board, with left and right edges identified and top and bottom identified. Clearly for f to be a solution to the $n \times n$ modular board it is a necessary condition for f to be a solution to the $n \times n$ standard board, but this is not sufficient. We depict the modular board in Fig. 11, as a ring torus with a board imposed on it. (In variant chess literature, the modular chessboard is also called an "anchor-ring" chessboard.) We could also only connect the left and right sides, which would make a cylinder, but as B. Eickenscheidt notes in [52], queens on the cylinder attack precisely the same set of squares as on the torus, so any results for the modular board immediately apply to the cylindrical board; this is also observed independently by A. Tolpygo in [185]. The attacks of a queen on the modular board is shown in Fig. 12. We consider other topologies of chessboard surfaces in Sections 6 and 8.

The modular $n \times n$ board seems to us much nicer to analyze than the standard board, because, for example, all the diagonals have the same number of entries (as each other, and the rows and columns), and there are n of each type of diagonal, n rows, and n columns. An important property of them is that if there is one solution for the $n \times n$ modular board, there exist n superimposable solutions for the board.

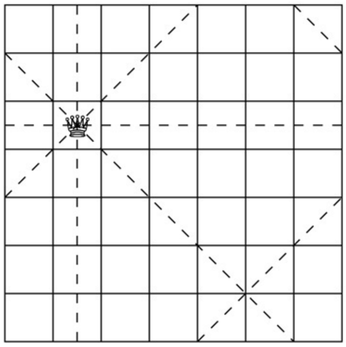

Fig. 12. A queen's attacks on the modular board.

We start with a very elementary result that seems to be rarely stated explicitly. For a sum diagonal $\alpha \equiv i + j \pmod{n}$ and a difference diagonal $\beta \equiv k - l$, the equations $i = k$ and $j = l$ hold if and only if there is an intersection between these diagonals. Using linear algebra to solve this linear system we get

$$\left[\begin{array}{cccccc|c} i & j & k & l & \alpha & \beta & \\ \hline 1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \end{array}\right] \tag{1}$$

For n odd, we can row reduce (1) to the following:

$$\left[\begin{array}{cccccc|c} i & j & k & l & \alpha & \beta & \\ \hline 1 & 0 & 0 & 0 & (n-1)/2 & -(n+1)/2 & 0 \\ 0 & 1 & 0 & 0 & -(n+1)/2 & (n+1)/2 & 0 \\ 0 & 0 & 1 & 0 & (n-1)/2 & -(n+1)/2 & 0 \\ 0 & 0 & 0 & 1 & -(n+1)/2 & (n+1)/2 & 0 \end{array}\right] \tag{2}$$

Thus we find the following, which follows from (2).

Remark 7. *Let n be odd and use 0 to $n-1$ as residues modulo n. Then the system of linear equations $\alpha \equiv i + j \pmod{n}$ and $\beta \equiv k - l \pmod{n}$, has a single solution $i \equiv [-(n-1)\alpha]/2 + [(n+1)\beta]/2 \pmod{n}$ and $j \equiv [(n+1)\alpha] - [(n+1)\beta]/2 \pmod{n}$.*

For n even, we can similarly row reduce (1) to show that for α, β with the same parity, $2i \equiv \alpha + \beta \pmod{n}$, and so we have the following:

Remark 8. *Let n be even and use 0 to $n-1$ as residues modulo n. Then the system of linear equations $\alpha \equiv i + j \pmod{n}$ and $\beta \equiv k - l \pmod{n}$, has exactly two solutions if and only if $\alpha \equiv \beta \pmod{2}$. They are $i \equiv (\alpha + \beta)/2 \pmod{n}$ or $i \equiv (\alpha + \beta + n)/2 \pmod{n}$, and $j \equiv (\alpha - \beta)/2 \pmod{n}$ or $j \equiv (\alpha - \beta + n)/2 \pmod{n}$, respectively. Otherwise there are no solutions.*

In the rest of this section, first we give linear solutions for the $n \times n$ modular board and then we give nonlinear solutions, as well as other results.

Pólya's paper [153] is the first to consider placing queens on the modular chessboard. He proves that there is a solution for the $n \times n$ modular board if and only if $\gcd(n, 6) = 1$. Pólya proves this by tiling the top-left quadrant of the grid plane with $n \times n$ chessboards, which yields a proof of the modular n-queens problem.

Theorem 9 (*Pólya*). *The $n \times n$ modular board has a solution if and only if* $\gcd(n, 6) = 1$.

Proof. For the forward implication, we assume that there is a solution permutation σ of $\{0, \ldots, n-1\}$ for the $n \times n$ modular board. Since $\sigma + i$ also a permutation it holds that $\sum_{i=0}^{n-1}(\sigma(i) + i) \equiv \sum_{i=0}^{n-1} i \pmod{n}$, therefore

$\sum_{i=0}^{n-1} \sigma(i) \equiv 0 (\text{mod } n)$. But by the summation formula, $\sum_{i=0}^{n-1} \sigma(i) = \frac{n(n-1)}{2}$ (since σ is a permutation), so $\frac{n(n-1)}{2} \equiv 0 (\text{mod } n)$ and n must be odd.

Similarly $\sum_{i=0}^{n-1} (\sigma(i) + i)^2 \equiv \sum_{i=0}^{n-1} i^2 (\text{mod } n)$, hence $\sum_{i=0}^{n-1} (\sigma(i)^2 + 2i\sigma(i)) \equiv 0 (\text{mod } n)$. But since also $\sum_{i=0}^{n-1} (\sigma(i) - i)^2 \equiv \sum_{i=0}^{n-1} i^2 (\text{mod } n)$, it follows that $\sum_{i=0}^{n-1} 2i\sigma(i) \equiv 0 (\text{mod } n)$, and substituting this back in we obtain $\sum_{i=0}^{n-1} (\sigma(i))^2 \equiv 0 (\text{mod } n)$. But by the summation formula for squares, $\sum_{i=0}^{n-1} i^2 = \frac{n(n-1)(2n-1)}{6}$, therefore $\frac{n(n-1)(2n-1)}{6} \equiv 0 (\text{mod } n)$ and we can conclude that n is not divisible by 3.

Now we prove backwards implication, and assume that $\gcd(n, 6) = 1$. Consider $\sigma(i) \equiv 2i (\text{mod } n)$ a permutation. Using this to place queens gives n queens with all row and column numbers distinct. If $\sigma(i) + i \equiv \sigma(j) + j (\text{mod } n)$, this implies that $3i \equiv 3j (\text{mod } n)$, and since $n \neq 3k$, 3 is not a divisor of 0, so $i \equiv j (\text{mod } n)$, so no two distinct queens share a sum diagonal. Similarly, no two distinct queens are on the same difference diagonal. □

In other words, the above theorem shows that $M(n) > 0$ if and only if $\gcd(n, 6) = 1$.

In [116], Kløve gives an independent proof of the result that the modular $n \times n$ board has a solution if and only if $\gcd(n, 6) = 1$. His proof is essentially the same as Theorem 9, and this is the case for all direct proofs that we have seen.

T. Koshy [119, Section 5.4] proves that for $p > 3$, then $f(i) \equiv \frac{(p+1)i}{2} (\text{mod } p)$ is a solution for the $p \times p$ modular board.

In 1922, W. H. Bussey in [26] finds a solution for certain cases of the (modular) n-queens problem, for when $n = p$ with p a prime number. Bussey considers the board as a finite Euclidean plane geometry with p points to the line, and notes that to have a queen attack another, the queens must share a vertical or horizontal line, or line with slope 1 or $-1 (\text{mod } p)$. If the p queens are placed on the points of a line whose slope is any one of $2, 3, 4, \ldots, p-2$ there can be no queen attacks; equivalently $\sigma(i) \equiv mi + b (\text{mod } p)$ gives a solution for any $m \neq -1, 0, 1$ and any b. He observes that there are $p^2 - 3p$ such lines, and that while his method gives all 10 solutions for $p = 5$, it only gives 28 out of the 40 solutions for $p = 7$, and for higher values of p it yields only a subset of all the solutions.

Blumenthal's 1928 paper [21] gives a solution for the case $\gcd(n, 6) = 1$ of the standard board. He does not mention it, but his solution is also a solution for the modular board. His solution for odd n is the permutation $\sigma = (1, 3, 5, \ldots, 2, 4, \ldots n-1)$. He shows that the sequences $\sigma(i) + 1$ and $\sigma(i) - i$ are distinct when n is not divisible by 3. Erbas and Tanik, in [57], are two of the people to independently derive some of these results again. Their expositions relate the linear modular n-queens problem to certain polygons.

In [32], A. K. Chandra gives an independent proof that the modular $n \times n$ board has a solution if and only if $\gcd(n, 6) = 1$ as a corollary of his theorem about the largest number $S(n)$ of independent permutations on the set $D_n = \{0, \ldots, n-1\}$. He further relates this to the problem of Moser, of the maximum number, $f(n, d)$, of nodes of a d-dimension hypercube of side n such that no n of these nodes are collinear; for Moser's problem he proves that if $1 \leq d \leq S(n)$ then $f(n, d) = n^d - n^{d-1}$ and that given any d there is an $n > 1$ for which $f(n, d) = n^d - n^{d-1}$.

For a given permutation P on D_n, $a \in D_n$ and $b \in \{0, 1, -1\}$, Chandra defines the function $P' : D_n \to D_n$ by $P'(x) = P((a + bx)(\text{mod } n))$. He calls a set $\{P_1, \ldots, P_d\}$ of permutations to be *independent* if, for each i we have a modification, P'_i of P_i, not all constant, then the function $P'_1 + \ldots + P'_d$ is also a permutation. He proves the lower bound $S(n) \geq \lfloor \log_2(m) \rfloor, n > 1$, m the smallest prime factor of n, and the upper bound $S(n) \leq \frac{m}{2}, n > 1$, m the smallest prime factor of n. He also proves that for any $n > 1$, the number of solutions for the $n \times n$ modular board, counting solutions in the same equivalence class as distinct, is equal to the number of permutations P on D_n such that $\{I, P\}$ is independent where I the identity permutation. Chandra also shows that the smallest irregular solution to the $n \times n$ modular board is for $n = 13$. As well, Chandra proves a result of Golomb in [81] as a corollary of his work, that n knight-queens can be placed on the modular $n \times n$ board if and only if $n \geq 11$ and $\gcd(n, 6) = 1$, and he proves the more general result for the K-queen, which is a knight-queen that can move any number of knight's moves in one direction; for this he proves that n of the K-queens can be placed on the $n \times n$ board if and only if $\gcd(n, 210) = 1$.

A. Stoffel in [177] defines a totally diagonal Latin square (TDLS), which is a Latin square with distinct entries in each sum and difference diagonal. Clearly we can place n queens for each distinct entry i in the TDLS of order n, and this yields n superimposable solutions for the $n \times n$ modular board, which indeed Stoffel observes. Stoffel proves that TDLS's of order n exist for all $n = 6k \pm 1$, hence proves that $\gcd(n, 6) = 1$ is a sufficient condition for there to exist n superimposable solutions on the $n \times n$ board. He also proves that there are q mutually orthogonal TDLS's of order n if n a prime power, and q the number of integers b such that $2 \leq b \leq n-2$ with $\gcd(b(b-1)(b+1), n) = 1$.

Stoffel further defines a completely diagonal Latin square (CDLS) which is a TDLS such that for all i, j and $k \neq \alpha n$, $(i, j) \neq (i + k, j - k)$ and $(i, j) \neq (i + k, j + k)$, and he proves that the existence of a CDLS of order n_1 and the existence of a CDLS of n_2 implies the existence of a CDLS of order $n_1 n_2$.

A. Hedayat in [93] defines a Knut Vik design equivalently to a TDLS, and he proves that Knut Vik designs exist for all n not divisible by 2 or 3, i.e. that $\gcd(n, 6) = 1$ is a sufficient condition for n superimposable solutions to be placed on the $n \times n$ board. Hedayat also proves some results about orthogonal Knut Vik designs, in particular the theorem that if n is prime there are $n - 3$ pairwise orthogonal Knut Vik designs, and that if n is not prime, not divisible by 2 or 3, then there is at least a pair of orthogonal Knut Vik designs.

Atkin, Hay and Larson, in [8], study the same question under the name of pandiagonal Latin squares. They give a complete enumeration and algebraic description of all such squares up to $n = 13$ and give constructions for larger primes.

In [138], Monsky poses the question of for what n there exist solutions to the modular $n \times n$ board, apparently not having seen the earlier results on this. Goldstein in [80] gives a solution to this with another statement and proof of Theorem 9. Monsky also asks if superimposable solutions exist for any other n. We will return to this question later. Shapiro in [167] considers Latin squares with opposite sides identified, and proves a result about linear Latin squares, Latin squares generated in a special way. He obtains as corollaries of his theorem, Theorem 9 in this article and Chandra's result $f(n, d) = n^d - n^{d-1}$.

In [107], Hwang and Lih give a theorem concerning Latin squares. They define the (n, k) property for Latin squares, where for each choice of k from 0 to $n - 1$, every row, column and modular diagonal on the Latin square has k entries less than k, which for full diagonals is a Knut Vik design. They prove that it is necessary and sufficient to have $2|n \Rightarrow 2|k$ and $3|n \Rightarrow 3|k$, and that it is also necessary and sufficient for the Latin square to have the (n, k) superqueen property, that n queens can be placed on the $n \times n$ modular board such that each queen attacks precisely $k - 1$ other queens. For $k = 1$ this proves Pólya's result for the modular board. They prove the following more general theorem:

Theorem 10 (*Hwang and Lih*). *There exists a Latin square of order n with the (n, k) property simultaneously for:*

$$\begin{aligned}
&k = 1, 2, 3, \ldots, \quad \textit{if } 2 \nmid n \textit{ and } 3 \nmid n,\\
&k = 2, 4, 6, \ldots, n \quad \textit{if } 2|n \textit{ and } 3 \nmid n,\\
&k = 3, 6, 9, \ldots, n \quad \textit{if } 2 \nmid n \textit{ and } 3|n,\\
&k = 6, 12, 18, \ldots, n \quad \textit{if } 2|n \textit{ and } 3|n.
\end{aligned}$$

Burger, Mynhardt and Cockayne in [24] and [25] show that the independence number of the modular $n \times n$ chessboard graph is n if and only if $\gcd(n, 6) = 1$, another equivalent version of Theorem 9. They also give results about partial solutions and on the number of fundamental regular solutions, which we discuss in a moment.

Rivin, Vardi and Zimmermann in [157] give the following result that yields solutions for the $p \times p$ modular board for p prime:

Theorem 11 (*Rivin, Vardi and Zimmermann*). *Each map $\sigma : \{1, \ldots, (p - 1)/4\} \to \{\pm 1\}$ yields a distinct solution for the $p \times p$ modular board. Moreover in general, for q the smallest divisor of $p - 1$ that is even and greater than 2, each map $\sigma : \{1, \ldots, (p - 1)/q\} \to \{\pm 1\}$ leads to a distinct solution for the $p \times p$ modular board.*

In discussing the modification of the chessboard to the $n \times n$ cylindrical board in Chapter 7 of [11], Beasley proves that it is possible to arrange n nonattacking queens on the $n \times n$ cylindrical board if and only if the $\gcd(n, 6) = 1$. Recall that the cylindrical and toroidal boards are equivalent.

In [25], Burger, Mynhardt and Cockayne prove some results about linear solutions for the modular board. Let

$$R_n = \{k | 0 \leq k \leq n - 1, \gcd((k - 1)k(k + 1), n) = 1\}.$$

They prove the following result:

Theorem 12 (*Burger, Mynhardt and Cockayne*). *Let $n = \prod_{i=1}^{t} p_i^{m_i}$, where $p_i \geq 5$ is prime for each i. Then the number ϵ_n of non-isometric regular solutions to the n-queens problem is given by:*

$$\epsilon_n = \begin{cases} \frac{1}{4}(|R_n| + 2^t) = \frac{1}{4}\left(2^t + n\prod_{i=1}^{t}\left(1 - \frac{3}{p^i}\right)\right) & \text{if } p_i \equiv 1 (\text{mod } 4) \text{ for each } i, \\ \frac{1}{4}|R_n| = \frac{n}{4}\prod_{i=1}^{t}\left(1 - \frac{3}{p^i}\right) & \text{otherwise.} \end{cases}$$

Let us now consider constructions of nonlinear solutions for the modular board. Bruen and Dixon in [23] give a theorem that gives nonlinear solutions in the case of the $p \times p$ modular board for p prime, $p \geq 13$. The nonzero squares S in $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$, form a multiplicative subgroup of $\mathbb{F}_p \setminus \{0\}$. We assign to each element of S a plus (+) symbol, and each element of $\mathbb{F}_p \setminus S$ is assigned a minus symbol. Writing the elements of $\mathbb{F}_p$ in the usual order, any t consecutive elements of $\mathbb{F}_p$ yields an ordered t-tuple of pluses and minuses. They prove the following:

Theorem 13 (*Bruen and Dixon*). *Suppose $p > 11$, and let u and v denote the middle elements of two distinct consecutive triples having the same parity pattern of length* 3. *Define a permutation g of $\mathbb{F}_p$ by:*

$$g(x) \equiv ux (\text{mod } p) \quad \text{if } x \in S,$$
$$g(x) \equiv vx (\text{mod } p) \quad \text{if } x \notin S.$$

Set $f(x) \equiv g(x) + w (\text{mod } p)$, with $w \in \mathbb{F}_p$. Then $f(x)$ is a nonlinear modular n-queens solution.

Proof. Since there are exactly eight possible parity patterns of length N, when $p > 11$ there must be at least one repetition. Thus there are at least two distinct (consecutive) triples having the same parity pattern. That the queens are nonattacking can then be checked. □

For $p = 13$, $S = \{1, 3, 4, 9, 10, 12\}$, for which 2, 3, 4 and 8, 9, 10 constitute two consecutive triples with the same parity pattern $- + +$, giving $u = 3$ and $v = 9$. Then $g(x) = (0, 3, 5, 9, 12, 6, 2, 11, 7, 1, 4, 8, 10)$ is a nonlinear modular solution. Bruen and Dixon also note the following for p prime:

> For example, if f is a modular solution, then f yields a transversal $T = \{(x, f(x))\}$ to the subnet N of the Desarguesian plane of order p, where N corresponds to the slope set $\{(0), (\infty), (1), (-1)\}$. The only known planes of order p are Desarguesian: for $p \leq 11$ it is known that planes of order p must be Desarguesian.

Kløve in [116] gives the following nonlinear solution for the $n \times n$ modular board, which gives nonlinear solutions for all n which divide at least one prime to the second power:

Theorem 14 (*Kløve*). *Let $n = \prod_{i=1}^{s} p_i^{\alpha_i}$, $P_n = \prod_{i=1}^{s} p_i$. Let* $\gcd(n, 6) = 1$. *If*

$$\gcd((a_1 - 1)a_1(a_1 + 1), n) = 1, \quad \text{and} \quad a_i \equiv 0 (\text{mod } P_n) \quad \text{for } i \geq 2,$$

then $f(x) \equiv \sum_{i=0}^{r} a_i x^i (\text{mod } n)$ is a solution to the modular n-queens problem.

The following gives the number of distinct solutions generated by this:

Theorem 15 (*Kløve*). *Let us have n, P_n as defined in Theorem 14, with $n^* = nP_n^{-1}$. Then N is the number of these solutions:*

$$N = n\prod_{p|n}(p-3)\prod_{k\geq 1}\frac{n^*}{\gcd(n^*, k!)}.$$

For the modular $p \times p$ board for p prime, it is natural to ask whether there is a modular solution g given by powers of a primitive root modulo p. For any such p and g, $g \not\equiv 1 (\text{mod } p)$ hence $g - 1 \equiv g^\lambda$ for some λ. Thus $(g-1)g^\lambda \equiv 1 (\text{mod } p)$ and so $g^{\lambda+1} - g^\lambda \equiv 1 (\text{mod } p)$, giving $g^{\lambda+1} - (\lambda - 1) \equiv g^\lambda - \lambda (\text{mod } p)$, i.e. $g^x + x (\text{mod } p)$ is not injective, a contradiction. Therefore no modular n-queens solution is given by powers of a primitive root modulo a prime.

Let $p \equiv 1 (\text{mod } 4)$ be a prime. Define W_β by $W_\beta = 0$ if $\beta = 0$ and $W_\beta = p^{\beta-1}(p-1)/4$ if $\beta > 0$. Also define $\langle r, s\rangle = \{(r, s), (s, -r), (-r, -s), (-s, r)\}$. Kløve in [117] gives the following construction of doubly symmetric solutions for the modular $p^\alpha \times p^\alpha$ board, for p a prime:

Theorem 16 (*Kløve*). *Let $p \equiv 1 \pmod 4$ be a prime and let g be a primitive root modulo p. For $\beta = 1, 2, \ldots, \alpha$, let*
(1) *δ_β be an odd integer such that $1 \le \delta_\beta \le \beta$,*
(2) *λ_β be an odd integer,*
(3) *V_β, U_β be integers such that U_β is odd and $V_\beta U_\beta = W_{\delta_\beta}$.*
Then

$$\{(0,0)\} \cup \bigcup_{\beta=1}^{\alpha} \bigcup_{j=1}^{V_\beta} \bigcup_{i=1}^{U_\beta} \langle p^{\alpha-\beta} g^{2iV_\beta+i+\lambda_\beta W_{\delta_\beta}}, p^{\alpha-\beta} g^{2iV_\beta+i} \rangle$$

is a doubly symmetric solution of the modular $p^\alpha \times p^\alpha$ board.

We have shown that there are solutions if and only if $\gcd(n, 6) = 1$, and thus it is natural to ask for the maximum placements of queens for other n. The first authors we have found to give nontrivial results for this are Kazarin, Kopylov and Timofeev in [110], who give results about the independence number β_n of the $n \times n$ modular queens graph (they actually speak of the largest m such that n disjoint independent sets of m vertices exist in the modular queens graph, which is equivalent to speaking of the independence number of this graph). They show (cf. Theorem 9) that $\beta_n = n$ if and only if $n \equiv \pm 1 \pmod 6$, and for partial solutions that if $n \equiv \pm 2 \pmod{12}$ then $\beta_n = n - 1$, and that if $n \equiv 0 \pmod 3$ then $\beta_n \le n - 2$. The next author to consider partial solutions is Kløve in [117]. Letting $\mu(n)$ be the maximum number of queens placeable on an $n \times n$ modular board, he proves the following:

Theorem 17 (*Kløve*). *For a modular board,*

$$\begin{aligned}
&\mu(n) = n \quad \textit{if } \gcd(n,6) = 1,\\
&\mu(n) = n - 2 \quad \textit{if } \gcd(n,6) = 3,\\
&n - 3 \le \mu(n) \le n - 1 \quad \textit{if } \gcd(n,6) = 2,\\
&n - 5 \le \mu(n) \le n - 1 \quad \textit{if } \gcd(n,6) = 6.
\end{aligned}$$

M. Chen in [35], O. Heden in [94], Chen, Sun and Zhu in [36–38], Burger, Mynhardt and Cockayne in [24], and Schlude and Specker in [162] all improve on this result. In [139,140], Monsky solves the problem, giving

Theorem 18 (*Monsky*). *For all n, there exists a partial solution of $n - 2$ queens for the $n \times n$ modular board. There exists a partial solution of $n - 1$ queens for the modular board if and only if n is not divisible by either 3 or 4.*

Heden uses partial solutions to produce maximal partial spreads in $PG(3, q)$ of many different sizes in [95–97].

Pólya in [153] proves that $n \equiv 1 \pmod 4$ and $\gcd(n, 6) = 1$ are necessary conditions for doubly symmetric solution to exist for the modular board. In [117], Kløve derives other necessary conditions for doubly symmetric solutions. He produces some general constructions (in particular the construction we give as Theorem 16) and also uses his conditions to make computer search feasible.

For a solution to the $n \times n$ modular board, for the n queens $(r_i, c_j) 1 \le i \le n$, Hansche and Vucenic in [90] conjecture that there is always an even number of i such that $r_i + c_i$ is odd; this is proved by Bruen and Dixon in [23]: for f a solution, since f is a permutation, for $K = \{0, \ldots, n-1\}$, $\sum_{x \in K} f(x) = \sum_{x \in K} x$, and so $\sum_x (f(x) + x) = \sum_x f(x) + \sum_x x = 2 \sum_x = 2k$, and so the number of elements x for which $x + f(x)$ is odd must be even.

Alvis and Kinyon [5] use nonlinear modular n-queens solutions to construct panstochastic matrices, which are matrices with entries from a field such that the sum of the entries in each row, column, modular sum diagonal and modular difference diagonal is 1.

We give in [15] a construction for orthogonal pandiagonal Latin squares and panmagic squares from modular n-queens solutions f such that $f(x) + x \pmod n$ is a permutation.

We can also study the total number of solutions and number of fundamental solutions for each value of n; in Sloane's [172], the total number of solutions for particular n with isometric solutions counted as distinct is in Sequence A007705, and the total number of solutions with isometric solutions counted only once, i.e. the number of equivalence classes, is in Sequence A053994. See also Sequence A051906. The standard board gives $Q(n)$ as an upper bound. Lucas in [134] gives the lower bound $M(p) \ge p(p-3)$ if p is prime. In [157], Rivin, Vardi and Zimmermann give the following nontrivial lower bound for the number of modular solutions $M(n)$:

Theorem 19 (*Rivin, Vardi and Zimmerman*). *If p is a prime such that $(p-1)/2$ is not prime, then $M(p) > 2^{(p-1)/(2d)}$, with d the smallest nontrivial divisor of $(p-1)/2$. If $p \equiv 1 (\mathrm{mod}\ 4)$ then $M(p) > 2^{(p-1)/4}$, but in general we only have $M(p) > 2^{\sqrt{(p-1)/2}}$.*

If n is divisible by a prime p such that $p \equiv 1 (\mathrm{mod}\ 4)$ then $M(n) > 2^{n/5}$.

They observe that the inequality $M(n) > 2^{n/5}$ holds for almost all n, since the set of all integers not divisible by a prime p such that $p \equiv 1 (\mathrm{mod}\ 4)$ has density zero.

A complete mapping of $\{0, 1, \dots, n-1\}$ is a permutation f of $\{0, 1, \dots, n-1\}$ such that $f(x) + x (\mathrm{mod}\ n)$ is also a permutation. Clearly every modular n-queens solution is a complete mapping. Indeed, it is easy to see that f is a modular n-queens solution if and only if both f and $-f$ are complete mappings. Let $C(n)$ denote the number of complete mappings of $\{0, 1, \dots, n-1\}$. It is clear that for all n, $M(n) \le C(n)$. Thus we can use upper bounds on the number of complete mappings to give upper bounds on the number of modular n-queens solutions.

I.N. Kovalenko [121] proves that $C(n) \le n! \mathrm{e}^{-c(n-1)}$ with $c \ge \frac{\ln 2}{2} \approx 0.35$.

Vardin [188, Section 6.3] conjectures that there exist constants c_1, c_2 with $0 < c_1 < c_2 < 1$, such $c_1^n n! \le C(n) \le c_2^n n!$ for all odd $n \ge 3$.

McKay, McLeod and Wanless [137] prove the upper bound of Vardi's conjecture. They show that $C(n) \le c_2^n n!$ for $c_2 = 3/4$, and that for all sufficiently large n, $c_2 = 0.614$.

6. Latin queen squares, and other generalizations

(We also discuss some of the extensions we mention here in Section 8 on open research areas.) A natural generalization of the n-queens problems is to an $n \times n \times n$ chessboard, i.e. the n-cube. Can we place n^2 nonattacking queens on it, with queens now being able to attack in any of the 26 directions represented by $\{0, \pm 1\}^3$? Since each vertical line can have at most one nonattacking queen in it, we can represent this problem by putting numbers from 1 to n on an $n \times n$ chessboard, with the number on a square representing the level of its queen in the cube, which we call a standard Latin queen square, and we can also speak of the modular Latin queen square as a cube with opposite faces identified. For a nonattacking placement of n^2 queens in the standard n-cube represented by a Latin square $A = (a_{i,j})$ with each entry representing the queen's level, that the following equations all hold for all k such that $1 \le k \le n-1$ is both necessary and sufficient:

$$\begin{aligned}
&a_{i+k,j} - a_{i,j} \ne 0, a_{i+k,j} - a_{i,j} \ne k, a_{i+k,j} - a_{i,j} \ne -k,\\
&a_{i,j+k} \ne a_{i,j} \ne 0, a_{i,j+k} \ne a_{i,j} \ne k, a_{i,j+k} \ne a_{i,j} \ne -k,\\
&a_{i+k,j+k} - a_{i,j} \ne 0, a_{i+k,j+k} - a_{i,j} \ne k, a_{i+k,j+k} - a_{i,j} \ne -k
\end{aligned} \tag{3}$$

The differences not being equal to 0 enforces no attack on each level, and not being equal to k no attacks between levels (i.e., i and j cannot be placed $|i-j|$ apart). This generalization was first formally proposed in 1978 by McCarty in [136] (he calls them queen squares), although Ahrens does discuss three-dimensional chess in [1], and Günther shows that his representation of solutions by the irreducible terms in determinants of the matrix in Fig. 15 can be generalized to the n-cube.

The first construction for McCarty's queen squares seems to have been in 1979/1980 by Klarner in [115], in which he gives the following theorem which gives solutions for all n whose largest prime factor exceeds 7. A solution using his method for $n = 11$ is given in Fig. 13.

Theorem 20 (*Klarner*). *If $\gcd(n, 210) = 1$ then n^2 nonattacking queens can be placed on the standard $n \times n \times n$ board.*

Proof. Take an $n \times n$ array $A = (a_{i,j})$ such that $a_{i,j} = ai + bj (\mathrm{mod}\ n)$ for some a, b. The conditions in Eq. (3) are satisfied whenever each $e_0 + e_1 a + e_2 b$, with $e_0, e_1, e_2 \in \{-1, 0, 1\}$ shares no prime factors with n, since $k < n$. It can be seen that this is impossible when 2, 3, 5 or 7 are a factor of n. But if $\gcd(n, 210) = 1$ then it can be $a = 3, b = 5$, which yields a solution for any such n. This completes the proof. □

We note that the construction in the above theorem is in fact a solution to the modular case.

In 1980, G.H.J. Van Rees proves the same result (along with other results for Latin queen squares and n-queens, which we have already discussed) in [187], and proves that modular Latin queen squares do not exist of order n for

8	2	7	1	6	11	5	10	4	9	3
11	5	10	4	9	3	8	2	7	1	6
3	8	2	7	1	6	11	5	10	4	9
6	11	5	10	4	9	3	8	2	7	1
9	3	8	2	7	1	6	11	5	10	4
1	6	11	5	10	4	9	3	8	2	7
4	9	3	8	2	7	1	6	11	5	10
7	1	6	11	5	10	4	9	3	8	2
10	4	9	3	8	2	7	1	6	11	5
2	7	1	6	11	5	10	4	9	3	8
5	10	4	9	3	8	2	7	1	6	11

Fig. 13. 11×11 standard Latin queen square made by Klarner's solution.

$\gcd(n, 6) > 1$. In 1981, in [100,73], Herzberg and Garner give the weaker result that for all prime n greater than or equal to 11, n^2 nonattacking queens can be placed in the standard n-cube. However, their construction can easily be extended to give Latin queen squares of all orders n such that $\gcd(n, 210) = 1$. Alavi, Lick and Liu [3] prove an equivalent result for "strongly diagonal permutation cubes", which are n-cubes for which all the modular lines parallel to lines joining pairs of vertices are permutations of $\{0, 1, \ldots, n-1\}$.

Van Rees [187] gives a construction to prove the following theorem:

Theorem 21 (*Van Rees*). *If there exists an $m \times m$ modular Latin queen square and an $n \times n$ modular Latin queen square, then there exists a $mn \times mn$ modular Latin queen square.*

Nudelman's 1995 paper [143] generalizes the problem to higher dimensions in a different way. On the two-dimensional board a queen's attack directions can be thought of as either those in a $\{0, \pm 1\}^2$ direction *or* as attacks in the subspaces orthogonal to these vectors. Nudelman generalizes queen attacks in higher dimensions in the latter sense; he allows queens to attack on any hyperplane, orthogonal to a vector in $\{0, \pm 1\}^d$. We discuss a further possible generalization of this in Section 8. He proves that for a d-dimensional chessboard with side length n, if $n \leq 2^d - 1$ then we can place only 1 queen in the entire system, and proves that $\gcd(n, (2^d - 1)!) = 1$ is a sufficient condition for being able to place n of these queens on the d-dimensional chessboard. Nudelman conjectures that this is also a necessary condition. He also shows that if $\gcd(n, (2^d - 1)!) \neq 1$ then the solution must be nonlinear and if $\gcd(n, (2d - 1)!) \neq 1$ then n queens are not possible.

Gómez in [84] and Gómez, Montellano and Strausz [85] give results on the composition of higher-dimensional modular n-queens solutions. They prove a special case of Nudelman's conjecture that $\gcd(n, (2^d - 1)!) = 1$ is necessary for the existence of solutions for modular n-queens solutions for the d-dimensional board, for polynomial solutions.

The n-queens problem in higher dimensions is also studied by Barr and Rao [9].

A weaker version of an $n \times n$ Latin queen square is an $n \times n$ board on which n solutions are superimposed. As we have noted, several authors have independently proved the result that $\gcd(n, 6) = 1$ is a sufficient condition to have n superimposable solutions. The first proof of this we have found is by Iyer and Menon in [108]. However, they do not prove that this is also a necessary condition for n superimposable solutions. Gardner in [70] states without proof that $\gcd(n, 6) = 1$ is a sufficient condition to have n superimposable solutions, and also does not give necessary conditions. Gik [76, Chapter 4] asks whether it is possible to superimpose n solutions for the $n \times n$ board such that every square on the board has a queen in it, and claims that this is only possible when $\gcd(n, 6) = 1$, although he does not give a proof of this. In fact though, Vasquez in [189] proves that $\gcd(n, 6) = 1$ is not a necessary condition for the queens graph of n^2 vertices to be n-colourable, that is, this is not a necessary condition for superimposable solutions. He shows computationally that for $n = 12$, $n = 14$ and $n = 16$, n solutions can be superimposed on the $n \times n$ board, which are counter-examples; Stertenbrink independently arrived at these results. Goldberg [79] shows that $\chi_n = n$ if $\gcd(n, 6) = 1$, that if $n \equiv 0, 4 \pmod 6$ then $\chi_n = n + 1$, if $n \equiv 3 \pmod 6$ then $n < \chi_n \leq n + 2$, and if $n \equiv 2 \pmod 6$ then $n < \chi_n \leq n + 3$. Vasquez and Habet in [192] devise an algorithm to find superimposable solutions for the $n \times n$ board and use it to show that there are n superimposable solutions for $n = 15, 16, 18, 20, 21, 22, 24, 28, 32$. Vasquez constructs an algorithm in [190] that can be used to show that there exist infinitely many integers n a multiple of two or three such that n superimposable sets can be placed on the $n \times n$ board. An example found by Stertenbrink (cf. Chvátal in [39]) of twelve superimposed solutions on the 12×12 standard board is given in Fig. 14. We discuss superimposing solutions further in Section 8.

0	5	9	6	3	8	4	1	10	11	7	2
7	11	4	2	1	6	10	3	0	8	9	5
8	1	10	9	5	2	0	7	11	6	3	4
10	0	3	8	7	11	9	5	4	1	2	6
5	6	11	4	2	1	3	0	8	9	10	7
11	7	0	1	10	4	8	6	3	2	5	9
2	8	6	3	9	5	7	11	1	10	4	0
3	4	5	0	11	10	6	9	2	7	8	1
9	2	1	10	4	7	5	8	6	3	0	11
4	10	7	11	0	3	1	2	9	5	6	8
6	3	2	5	8	9	11	4	7	0	1	10
1	9	8	7	6	0	2	10	5	4	11	3

Fig. 14. Stertenbrink's 12 superimposed solutions on the 12×12 standard board.

Clearly the n-queens problem is equivalent to the $k = 1$ instance of the problem of placing kn queens on the $n \times n$ chessboard such that in each row there are k queens, in each column k queens, and in each diagonal at most k queens. Le, Li and Wang prove the following theorem for $k > 1$ in [129,130]:

Theorem 22 (*Le, Li and Wang*). *If $n \geq 4$ and $n \geq k \geq 1$, then for the $n \times n$ board, we can place kn queens on the board, such that there are k queens in each row, k queens in each column,and at most k queens in each diagonal.*

Kim in [112] asks what the maximum number of queens is that can be placed on the standard $n \times n$ chessboard such that each queen is attacking exactly k others, for $1 \leq k \leq 4$; recall that this question is also asked by Gik. For $k = 0$, this is the standard n-queens problem. In [10], Barwell solves this for $k = 1$, proving that the answer is 10 queens for the standard 8×8 chessboard. Hayes in [92] proves that for $k \geq 5$ there are no solutions for a board of any order. He defines $Q_k(n)$ to be the number of solutions for particular k for the $n \times n$ board. He notes that $Q_1(n) \leq \lfloor 4n/3 \rfloor$. For $k = 2$, he proves that for all $n \geq 3$, $Q_2(n) = 2n - 2$. For $k = 3$, he proves that for all $n \geq 2$, $Q_3(n) \leq 2\lfloor (6n-2)/5 \rfloor$, and that for $k = 4$, $Q_4(4) = 8$, $Q_4(5) = 11$, and that for all $n \geq 6$, $Q_4(n) = 3n - 3$.

In [180], Tarry poses the problem of determining formulas, for each value of m, to place m nonattacking queens on the $m \times n$ standard board. In [120], Kotěšovec gives the following polynomials that give the number of ways of placing k nonattacking queens on the $k \times n$ board. $2 \times n : (n-1)(n-2)$, $3 \times n : (n-3)(n^2 - 6n + 12)$, $4 \times n : n^4 - 18n^3 + 139n^2 - 534n + 840$, $n \geq 7$, $5 \times n : n^5 - 30n^4 + 407n^3 - 3098n^2 + 13104n - 24332$, $n \geq 11$, $6 \times n : n^6 - 45n^5 + 943n^4 - 11\,755n^3 + 91\,480n^2 - 418\,390n + 870\,920$, $n \geq 17$, $7 \times n : n^7 - 63n^6 + 18\,790 - 34\,411n^4 + 417\,178n^3 - 3336\,014n^2 + 16\,209\,916n - 36\,693\,996$, $n \geq 23$.

Kotěšovec also gives, in [120], the polynomials that give the number of ways of placing k nonattacking queens on the $n \times n$ board, i.e. the number of partial solutions for particular k. For $k = 2$, $\frac{n(n-1)(n-2)(3n-1)}{6}$. For $k = 3$, if n is even then $\frac{n(n-2)^2(2n^3-12n^2+23n-10)}{6}$ and if n is odd then $\frac{(n-1)(n-3)(2n^4-12n^3+25n^2-14n+1)}{12}$. For $k = 4$ with $n \geq 2$, $\frac{n^8}{24} - \frac{5n^7}{6} + \frac{65n^6}{9} - \frac{1051n^5}{30} + \frac{817n^4}{8} + Q$; if $n = 6l$, then $Q = -\frac{4769n^3}{27} + \frac{1963n^2}{12} - \frac{1769n}{30}$, if $n = 6l+1$, then $Q = -\frac{9565n^3}{54} + \frac{1013n^2}{6} - \frac{6727n}{90} + \frac{257}{27}$, if $n = 6l+2$, then $Q = -\frac{4769n^3}{27} + \frac{1963n^2}{12} - \frac{5467n}{90} + \frac{28}{27}$, if $n = 6l+3$, then $Q = -\frac{9565n^3}{54} + \frac{1013n^2}{6} - \frac{2189n}{30} + 7$, if $n = 6l+4$, then $Q = -\frac{4769n^3}{27} + \frac{1963n^2}{12} - \frac{5467n}{90} + \frac{68}{27}$, and if $n = 6l+5$, then $Q = -\frac{9565n^3}{54} + \frac{1013n^2}{6} - \frac{6727n}{90} + \frac{217}{27}$. If $B_k(n)$ is the number of ways of placing k nonattacking queens on the $n \times n$ board, it is known that $\sum_{n=0}^{\infty} B_k(n)x^n$ is a rational power series [174, Chapter 4, Exercise 15].

The extension of the modular board to the $m \times n$ modular chessboard is proposed by Cairns in [29], who gives the result that for $m < n$, the maximum number of queens that we can place on the $m \times n$ modular board is $\gcd(m, n)$, except for $m = 3$, $n = 6$, where only 2 queens can be placed.

Gik [76] (mainly Chapter 4) suggests a number of variations of the n-queens problem, for some problems he is the originator but not the solver. He poses the problem of placing as many queens as possible on the chessboard such that each queen attacks at most one other queen, and observes that for the 8×8 board 10 is the maximal number. He generalizes this to the problem of finding the maximal number of queens that can be put on the $n \times n$ board such that each queen attacks at most p others, with attacking being considered to stop once they hit a queen. He suggests the problem of finding the minimum number of queens such that every row, column and diagonal of the standard board has a queen in it; since there are $2n - 1$ of each kind of diagonal for the standard board, this problem is nontrivial. He also suggests several variations on the chessboard, in particular into a projective plane: add four points to the board, one a point at which all verticals intersect, one a point at which all horizontals intersect, one a point at which all sum diagonals intersect, and one a point at which all negative diagonals intersect. However, he does not give results about the n-queens problem on this projective plane chessboard.

We extend the n-queens problem to the Möbius strip in [14]. There we have shown that solutions exist for all $m \times n$ Möbius boards such that $m \leq n$ and $n > 3$, and another set of (m, n), which altogether have an asymptotic density of $25/48$. However in fact there are infinitely many (m, n) for which the $m \times n$ Möbius board has no solution.

The weaker case of queens that can move along rows and columns but can only move on either sum diagonals or difference diagonals has been studied by several authors. In [52], Eickenscheidt proves that n queens that can only move on difference diagonals can be placed on the cylindrical (modular) board if and only if n is odd. Bennett and Potts in [16] find an isomorphism between the restricted queens problem and zero-sum arrays. This variant of the n-queens problem that considers attacks only on one diagonal has arisen in some applications [104,159], and Carter and Weakley consider in [31] the n-queens problem with diagonal constraints.

Dietrich and Harborth in [50] study the triangular triangle board, the board in the shape of a triangle with triangular cells. On this they define the chess pieces, in particular the rook which attacks in straight lines from side of the triangle to side of the triangle, forming rhombuses, the bishop 1 which attacks from vertex to side, side to vertex etc. in straight lines, forming diamonds, and the bishop 2: the triangular triangle board can be 2-coloured with cells sharing an edge of different colour, and the bishop 2 moves as bishop 1, but attacks only cells of the same colour. Dietrich and Harborth define the queen 1 (2) attack as the union of all squares attacked by the rook and bishop 1 (2), and they show that the independence number of the queen 1 graph is bounded above by the independence number of the queen 2 graph, which is bounded above by the independence number of the rook graph, which they prove is equal for all n to $\lfloor \frac{2n+1}{3} \rfloor$. Furthermore, for the queen 1 graph, for $n = 3, 4, 6, 7, 13, 16, 19$, $\beta_n = \lfloor \frac{2n+1}{3} \rfloor - 1$, and for all other $n \leq 36$, $\beta_n = \lfloor \frac{2n+1}{3} \rfloor$, but they have not found a pattern for this sequence.

In [127], Laskar, McRae and Wallis consider chessboard-like graphs defined upon the triangulated association scheme design, and demonstrate [127, Theorem 22] for such graphs that the independence number for queens is $\lfloor \frac{n}{2} \rfloor$ for $n > 4$.

Harborth, Kultan, Nyaradyova and Spendelova in [91] consider the triangular hexagon board, in which the cells are hexagons and the board is a triangle. In this bishops attack in straight lines through the vertices of their cells, rooks attack along straight lines through the centers of the edges of their cells, and queens have both attacks. The only general upper bound they are able to give on the independence number of the queens graph is by the rooks bound, which is $\lfloor \frac{2n+1}{3} \rfloor$ for all n. For $n = 3, 4, 6, 7, 13, 16, 19, 25, 31$, they find that $\beta_n = \lfloor \frac{2n+1}{3} \rfloor - 1$, and for the other $n \leq 31$, $\beta_n = \lfloor \frac{2n+1}{3} \rfloor$.

Nivasch and Lev [142] give a generalization of the n-queens problem to the triangular board with side length n and $\frac{n^2+n}{2}$ cells. A queen moves along the lines parallel to the sides of the board. It is clear that there can be no more than n queens, since each of the n rows can contain at most one queen. Nivasch and Lev prove that the maximum number q of nonattacking queens that can be placed on the triangular board of side length n has the upper bound $q \leq \lfloor \frac{2n+1}{3} \rfloor$. They further prove that this upper bound can always be satisfied, that is, that we can always place $\lfloor \frac{2n+1}{3} \rfloor$ nonattacking queens on the triangular board of side length n. Vaderlind, Guy and Larson [186] also look at this problem.

B. Polster [152, Section 4.1] looks at the queens problem on a “doily”.

In [184], Theron and Burger define the hexagonal hive board with hexagonal cells, with queens defined to attack along lines through each of the six sides of its cell. They prove [184, Theorem 3] that n nonattacking queens can be placed for all n.

Klarner in [114] modifies the n-queens problem to have a “reflecting strip” at the top (i.e. in row 0), that queens bounce off. For example, if a queen is on the difference diagonal from (1, 2), it can attack (1, 4), (2, 5), etc. He notes that this increases the number of fundamental solutions, because each fundamental solution produces less equivalent solutions under mapping by the symmetry group of the square, as there is only one axis of symmetry (only horizontal reflection gives an equivalent solution). This is related to a problem posed by Shen and Shen in [169], who ask, is it possible for all $n \geq 3$ to make n pairs (a_1, b_1) with no shared entries of the numbers from 1 to $2n$ with $b_i > a_i$ and such that for all $1 \leq i \leq n$, $a_i + b_i$, $b_i - a_i$, $2n$ all distinct? M. Slater in [171] sharpens the question to whether the integers 1 through n can be paired with the integers $n + 1$ through $2n$ so that no two of the $2n$ sums and differences $b_i \pm i$ are equal. Slater notes that this is a refinement of the problem of putting n nonattacking queens on the $n \times n$ board. He conjectures that this is possible for all other n. Klarner also notes this variant and calculates that there is only one solution each for $n = 4$,$n = 5$, and no solutions for $n = 6$, and conjectures that for all $n > 6$ there is at least one solution. In [164], J. D. Sebastian shows that there are solutions to the pairing i.e. reflecting queens problem for $n = 9, \dots, 27$. For $n = 9$ a solution is (11, 16, 14, 17, 10, 13, 15, 12, 18). G.B. Huff in [105] answers the Shen problem in the affirmative; however, his solution is not a solution for reflecting queens. J. L. Selfridge gives the abstract [165] for another solution of the Shen problem, but he does not seem to have published his results. We can modify the reflecting queen problem to have a different offset for where the reflection occurs (i.e. no square in between the sum diagonal and reflected difference diagonal), and mirrors on different sides. The consistency of the colour of cells in a diagonal attack depends on the position of the mirror.

Zhao in [199] defines the (maximum) queens problem on a partial chessboard, which is a chessboard with cells removed or blocked such that queen attacks do not continue through them. With m cells removed, the basic problem asks for an arrangement of more than n queens on the board such that no queen attacks another, and the maximum problem asks, what is the maximum number of queens one can place on an $n \times n$ board with as many cells removed as you please? For example, four nonattacking queens can be put in the corners of the 3×3 board by removing the other cells. Zhao proves that the maximum number of squares that need to be blocked in order to place 6 queens on a 5×5 board is 3. Zhao further applies results about point-lattice coalgebra to the partial chessboard; with this machinery she proves that on the 6×6 board with two squares blocked out, there is a maximum of four nonattacking queens.

We can also consider the more powerful knight-queen piece: this seems to have been first considered mathematically by Sainte-Laguë in [160], although according to the entry on the queen in H. Golombeck [83], in Russian chess of the 18th century the queen also sometimes had the movement of the knight. In [81], S.W. Golomb proves as a corollary of his work on perfect Lee metric codes that for p prime such that $p > 7$, p nonattacking queens with knight's moves can be placed on the $p \times p$ modular board. He also proves that if $n \geq 10$ is a prime or 1 less than a prime, n nonattacking queens with knight's moves can be placed on the $n \times n$ standard board. For $n = 10$, Petković in [149] shows that there is one fundamental solution (3, 6, 9, 1, 4, 7, 10, 2, 5, 8), and three more solutions (7, 3, 10, 6, 2, 9, 5, 1, 8, 4), (4, 8, 1, 5, 9, 2, 6, 10, 3, 7), (8, 5, 2, 10, 7, 4, 1, 9, 6, 3). Sequence A051223 in Sloane [172] is the total number of ways of placing n nonattacking knight-queens (it calls them superqueens) on the $n \times n$ standard board, and Sequence A051224 is the number of fundamental solutions. Golomb's work also relates the queen problem to prime ideals in quadratic number fields. Gik [75, Chapter 8] shows that n nonattacking knight-queens cannot be placed on the $n \times n$ standard board for $n = 8$ or $n = 9$, but gives a solution for $n = 10$.

A figure on an $n \times n$ chessboard is said to have *constant width* w if every row, column or diagonal that intersects it, intersects it in exactly w squares. A figure is of type (n, k, w) when it is contained in an $n \times n$ chessboard, has constant width w, and is comprised of exactly kw squares. This problem is studied by Hernández and Robert [99]. A figure of type $(n, n, 1)$ is precisely an n-queens solution.

7. Computation

The n-queens problem is well suited for solution by backtracking algorithms. As Bernhardsson notes in [19] though, it is not necessary to find solutions with algorithms, because there are explicit ways to construct solutions. However, the n-queens problem is still an interesting test case for algorithms, and since there does not exist a closed form solution yielding all solutions for arbitrary n, computation is necessary to enumerate the solutions in general.

In [158], Rouse Ball discusses S. Günther's approach of 1874 in [88] to the n-queens problem using determinants. Günther represents the n-queens problem as finding the determinant of the matrix in Fig. 15. This matrix is made such that each difference diagonal has all the same letters, and each sum diagonal has all the same number subscripts. He also observes that his method of using determinants to find all the solutions to the n-queens problem could be generalized to the n^3 problem of placing n^2 queens in the n-cube in which none of the lines from the points hit another queen.

By elementary linear algebra we know that each term in the determinant of the matrix from Fig. 15, will have one entry from each row and column. Thus if it can be shown that each determinant will always have a term with no repeating letters or subscripts, we will have shown that there is always a solution for the standard n-queens problem. In fact since we know that the answer to the n-queens problem is affirmative for $n \geq 4$ we are able to state that such terms must exist in the determinant. Is this fact useful in linear algebra? Glaisher in [78] develops Günther's suggested approach noting ways to calculate the determinants more efficiently, also describing a procedure to find a solution for

$$\begin{bmatrix} a_1 & c_2 & e_3 & g_4 & k_5 & \cdots & \cdots \\ b_2 & a_3 & c_4 & e_5 & g_6 & \cdots & \cdots \\ d_3 & b_4 & a_5 & c_6 & e_7 & \cdots & \cdots \\ f_4 & d_5 & b_6 & a_7 & c_8 & \cdots & \cdots \\ h_5 & f_6 & d_7 & b_8 & a_9 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \beta_{2n-2} & a_{2n-1} \end{bmatrix}$$

Fig. 15. Günther's n-queens matrix.

the $(n+1) \times (n+1)$ board given a solution for the $n \times n$ board, with $n > 3$. In fact, Sprague in [173] further develops the procedure for finding solutions using determinants.

Foulds and Johnston in [66] show that since the n-queens problem in graph-theory is equivalent to finding an independent set of n vertices in the queens graph of n^2 vertices, the problem is equivalent to finding the maximal clique with the most vertices in the complement of the queens graph, and note that there are many well-known results for finding maximal cliques. They also discuss finding solutions to the n-queens problem from the operations research technique of 0–1 integer programming, and describe an algorithm using this approach.

A very good and thorough study of the algorithms for finding solutions to the n-queens problem is in [56] by Erbas and Tanik. They give the primary division of algorithms into those that generate all solutions, those that generate only necessarily the fundamental solutions and those that generate only a subset of all solutions. For algorithms that find all solutions, they study brute-force trial and error algorithms, backtracking algorithms and permutation generation algorithms (with Rohl's algorithm). Under algorithms generating fundamental solutions, they consider generate and test algorithms, partial elimination of symmetries (with Naur's algorithm), and an algorithm based on the group properties of the solutions (with Topor's algorithm). Finally, for algorithms for forming partial solutions, they note nondeterministic algorithms, backtracking algorithms, probabilistic algorithms (with a Las Vegas algorithm), a divide and conquer algorithm, a magic square based algorithm, an algorithm using regular polygons, and another algorithm which they call the constructions algorithm that is based on the approach to constructing solutions used by Hoffman, Loessi and Moore in [101]. There is also a discussion of algorithmic approaches to n-queens by Erbas, Sarkeshik and Tanik in [54,55], and Allison, Yee and McGaughey in [4].

None of the solutions we have considered so far for n-queens have given all the solutions for each n, and indeed it is demonstrated by Hsiang, Hsu and Shieh in [103] which we give as the following theorem, that finding all the solutions for the n-queens problem and the modular n-queens problem are both beyond the $\#P$-class, which proves the conjecture of H. E. Dudeney given in his book of math recreations [51] that no closed form solution exists.

Theorem 23 (*Hsiang, Hsu and Shieh*). *Finding all the solutions for the n-queens problem and the modular n-queens problem are both beyond the $\#P$-class.*

This is obviously related to finding the number of solutions for each n. Hsiang, Hsu and Shieh's result shows that there is no closed form solution to this.

In [106], Hukushima uses a statistical mechanical simulation to estimate the number of solutions and shows consistency with the conjecture by Rivin, Vardin and Zimmermann that $Q(n) \sim n^{\beta n}$ for some $\beta > 0$. Recently a quantum algorithm for the n-queens problem has been developed [47]. Superposition offers the potential to enumerate all solutions very quickly.

Computer searches have been very helpful in our work on n-queens, as, in particular, visually seeing a complete set of solutions for the problem or variations of it (e.g. modular board, reflecting queens) can give some insight on fruitful approaches to take to describe general solutions.

8. Open areas

We consider several conjectures given in the literature on n-queens. In [187], Van Rees conjectures that $\gcd(n, 210) = 1$ is a necessary condition for the existence of solutions to the modular Latin queen squares problem. Van Rees also conjectures that $\gcd(n, 210) = 1$ is a necessary condition for the existence of a solution for standard Latin queen squares. Clearly the second conjecture implies the first, and Van Rees feels the second is more difficult. The only support for either of these conjectures is the lack of counter-examples.

In [82], Golomb and Taylor consider Costas arrays, which are permutations g of $\{0, \ldots, n-1\}$ such that $g(j+k) - g(j) = g(l+k) - g(k)$ if and only if $k = 0$ or $j = l$, and ask whether there exist Costas arrays which are also n-queens solutions. It remains unknown whether there exist Costas arrays which are n-queens solutions.

We say that an $n \times n$ Latin square is *centrosymmetric* if $L(i, j) = L(n+1-i, n+1-j)$. Taylor [183, Conjecture 3] conjectures that if $n \times n$ Latin square exists in which each of $n-2$ marks a pattern of n centrosymmetric nonattacking queens, then $n+1$ must be a prime number.

Petković, in [149], which is a book about mathematical problems on the chessboard, gives several interesting problems that are related to the specific n-queens problem; many of these Petković only mentions the 8×8 case of, and we easily generalize these to the $n \times n$ board. In Chapter I, Petković shows that the probability that two randomly placed queens on the 8×8 board will be nonattacking is $1 - \frac{1456}{4032}$, and it would be interesting to consider this question for $2 \leq k \leq n$ queens and on the $n \times n$ board. In Chapter IV he asks whether it is possible to place n pieces on the $n \times n$ board such that the pairwise distances of all of them are different, with distance of queens measured between centers of squares and squares of side length 1; for the 7×7 board for example there is only one solution, which however is not an n-queens solution. It is interesting to study for what (if any) n it is possible for an n queens solution to have distinct pairwise distances. In Chapter VI he discusses the modification of the $n \times n$ board to a projective chessboard, which we discussed from Gik earlier in this paper.

Counting the number of solutions is not possible in closed form as we have already noted, but finding asymptotic lower bounds for the number of solutions for both the standard and modular boards may be tractable. Rivin, Vardi and Zimmermann study this in [157], and we state their conjectures the end of this section. Benoit Cloitre (cf. of Sloane's [172, Sequence A000170]) conjectures that $\lim_{n\to\infty} \frac{\ln(\frac{n!}{Q(n)})}{n}$ is equal to a constant $0.90\ldots$, and further conjectures that based on computational evidence that there is a constant c near 2.54 such that $Q(n)$ is asymptotic to $\frac{n!}{c^n}$.

Recall from Section 6 that Kim in [112] asks what the maximum number of queens $Q_k(n)$ is that can be placed on the standard $n \times n$ chessboard such that each queen is attacking exactly k others, for $1 \leq k \leq 4$, and this has been solved conclusively by Hayes in [92] for $k = 2$, $k = 4$ and $k \geq 5$. However for $k = 1$ it has only been shown by Hayes that $Q_1(n) \leq \lfloor 4n/3 \rfloor$, and for $k = 3$ that $Q_3(n) \leq 2\lfloor (6n-2)/5 \rfloor$.

We have not seen a general solution of the related problem of placing n queens on the board to maximize the number of nonattacked cells. Sequences A019317, A001366 by M. Velucchi in Sloane [172] give the number of solutions for particular n. Gardner [71] further discusses this problem.

Recall from Section 6 the problem of having n superimposable solutions, which is equivalent to the queens graph having chromatic number n. In 1976, Gik [75, Chapter 9] considers colouring the $n \times n$ chessboard, that is, considers the minimum number of independent sets of queens needed to partition the set of all squares of the chessboard. He shows that $\chi_8 = 9$, and conjectures that in general $\chi_n = n$ or $\chi_n = n+1$. Although several authors have shown that $\gcd(n, 6) = 1$ is a sufficient condition for $\chi_n = n$, necessary conditions for this are unknown. In [138], Monsky asks, is $\gcd(n, 6) = 1$ also a necessary condition for there to exist n superimposable solutions for the $n \times n$ board? As we have seen from Vasquez in Section 6, the answer is no. Chvátal has asked in [39] whether for all sufficiently large n the $n \times n$ queens graph is n-colourable. Recall from Section 6 that Vasquez [191] has shown that $\chi_n = n$ for infinitely many $n = 2p, 3p$ for p a prime.

Chandra's solution which we discussed in Section 5 to Moser's problem of putting non-collinear points in a d-dimensional hypercube of side n can be viewed as a higher-dimensional analog of queens which attack along lines. Nudelman generalizes the n-queens problem for boards of arbitrary dimension, d, but has queens attack along particular hyperplanes. We would like to unify these two movements into a single generalization with queens with an l-dimensional attack and ask to place n^{d-l} queens. Then, with $d = 2$ and $l = 1$ the standard problem emerges. With $d = 3$ and $l = 1$ Latin queen squares emerge. Nudelman's instance would emerge when $l = d-1$. However we cannot allow any direction of attack. One idea would be to permit attacks along any l-dimensional subspace spanned by l independent vectors from $\{0, \pm 1\}^d$. Another might be to permit attacks in any l-dimensional subspace that is orthogonal to a $(d-l)$-dimensional subspace spanned by $d-l$ independent vectors from $\{0, \pm 1\}^d$. For $l = d-1$ all the spaces of the second type are also of the first type, but the two notions are not the same in general. For instance the subspace orthogonal to $\mathrm{Span}\{(1, 1, 1), (1, -1, 0)\}$ does not have a basis vector from $\{0, \pm 1\}^3$. It would be interesting to determine the conditions under which these ideas do coincide.

In [111], S. U. Khan poses the problem of the number of queens that can be placed on the d-dimensional chessboard of side length n for incomplete solutions, which can be thought of as an analog to Monsky's result that for all n, $n-2$ queens can be placed on the modular board.

It would be interesting to do more geometric research on the n-queens problem. As we have noted, Erbas and Tanik have looked at the connections between it and certain polygons in [57].

Further investigating magic squares and Latin squares with n-queens solutions and vice versa is recommended, especially for constructing panmagic squares. Also, we think it would be fruitful to consider this in higher dimensions.

Klarner in [114] gives the reflecting queens problem, and although its motivating partition problem has been solved, [105], the n-queens variant of this remains open.

In Problem 300 of [51], Dudeney asks whether it is possible for the 8×8 board to place 8 nonattacking queens such that no three of them are of the form (a, b), $(a-1, b+2)$, $(a-2, b+4)$. This is satisfied for only one of the fundamental solutions for the 8×8 board, and this problem can clearly be generalized for the $n \times n$ board, for either 3-progressions or different lengths.

Kløve notes in [117] from earlier work by Pólya that $n \equiv 1 (\mathrm{mod}\ 4)$ and $\gcd(n, 6) = 1$ are necessary conditions for there to exist a symmetric solution for the modular $n \times n$ board, but that it is an open question whether these are also sufficient conditions for there to be solutions.

We can also consider Möbius chessboards, with the ith row continuing to the $n - i (\mathrm{mod}\ n)$th row, and the Klein Bottle chessboard, which is the same as the Möbius board except that its top and bottom edges are connected orientably. These topologies can also be extended to the $m \times n$ board. As well, we can study chessboards with more or less than four sides, such as hexagonal hive boards studied by Gik [76, Chapter 11]; however, this is a fundamental change of the problem, since all the other variations we have mentioned so far do not change the basic movement of the queen. In fact, as Watkins in [195] notes, to have a chess piece be able to move in four directions, a surface must have Euler characteristic 0, and the torus and Klein Bottles are the only surfaces with Euler characteristic 0; as we have observed, the queen has the same movement on the cylinder as on the torus, and the standard chessboard and Möbius board are boundried analogs to the toroidal and Klein Bottle chessboard. Watkins also notes the possibility of having an annular chessboard, a chessboard on the outer ring of a rectangle, i.e. an inner rectangle is removed from a larger rectangle, and a chessboard is made from the remaining surface. As well, he poses having a chessboard in the cuboid, the three-dimensional torus, in which each of the three pairs of opposite faces are connected (identified), and having chessboards in and on the surface of twisted square tori.

Recall from Section 5 Chandra's result that n nonattacking K-queens can be placed on the $n \times n$ modular board if and only if $\gcd(n, 210) = 1$. There are only regular solutions for $n < 29$, and Chandra asks if there exist non-regular solutions.

We discussed in Section 6 Zhao's partial chessboard from [199], which has cells removed. Zhao asks further for a certain n, what the maximum number of nonattacking queens that can be placed on the n-cube with as many cells removed as we please.

It is interesting to ask, if given a queen already put on some square, when can $n-1$ other queens always be placed to give a solution, and if this is not possible, for what n it is possible or if there is a weaker condition that can be met for all n.

One can pose many other interesting questions about nonattacking queens on a chessboard. A very nice problem is to construct an $n \times n$ magic square (with magic diagonals) with the integers from 1 to n^2, and to ask when n nonattacking queens can be placed only on the cells which contain prime numbers. This is possible, for example, for $n = 8$.

Additionally, Bode and Harborth in [22] look at some other fairly natural variants of the chessboard have been looked at. In this paper they determine or give bounds for the independence number β_n (i.e. the maximum number of vertices in the chessboard graph which are adjacent to no other vertices, for adjacency determined by queen attacks) for the graphs of the chess-pieces in the Euclidean tessellations of triangles and hexagons; of course as we have already seen, for a chessboard with square entries, the independence numbers are is $\beta_1 = 1, \beta_2 = 1, \beta_3 = 2, \beta_n = n$ for $n > 3$. For the hexagonal board, they find the independence numbers to be: $\beta_1 = 1, \beta_2 = 1, \beta_3 = 1, \beta_4 = 3, \beta_5 = 3, \beta_6 = 4, \beta_7 = 7, \beta_8 = 7, \beta_9 = 7, \beta_{10} = 9, \beta_{11} = 9, \beta_{12} = 11, \beta_{13} = 12, \beta_{14} = 13, \beta_{15} = 15$. The strongest upper bound they give for the queens graph of the hexagonal chessboard is the upper bound due to the rooks graph, that is $\beta_n \le 2\lceil n/2 \rceil - 1$. For the triangular graph they define two different queens attacks: the first has the queen attack all the straight lines of cells with which it shares a side or a vertex, while the second has it only attack lines of cells with which it shares a side. For the first stronger type of queen they give the following independence numbers: $\beta_1 = 1, \beta_2 = 2, \beta_3 = 2, \beta_4 = 3, \beta_5 = 4, \beta_6 = 6, \beta_7 = 7, \beta_8 = 8, \beta_9 = 8, \beta_{10} = 10, \beta_{11} = 10, \beta_{12} = 12, \beta_{13} = 12, \beta_{14} = 14, \beta_{15} = 15$, and for the second weaker type of queen they give: $\beta_1 = 1, \beta_2 = 2, \beta_3 = 2, \beta_4 = 4, \beta_5 = 4, \beta_6 = 6, \beta_7 = 7, \beta_8 = 8, \beta_9 = 8, \beta_{10} = 10, \beta_{11} = 10, \beta_{12} = 12, \beta_{13} = 12, \beta_{14} = 14, \beta_{15} = 15$. They note certainly that the independence number of the weaker queens graph is always equal to or greater than the independence number of the stronger queens graph, which is always less than or equal to the independence number of rooks on the triangular board, which is always equal to n; however, they say that it remains open whether $\beta_n = n$ can be attained infinitely often.

Finch [65, Section 5.12] asks what the "hard square entropy constant" is for queens on the chessboard.

Finally, Cairns in [28] introduces the pillow chessboard, which is equivalent to applying the chessboard to the surface of the sphere. The left and right sides are identified with each other, in the sense of the modular chessboard, with the left and right sides of the top (and also the bottom) identified with each other, in the sense for example that on the 8×8 board, when a queen moves vertically up on $(1, 2)$ it continues vertically going down at $(1, 7)$. As Cairns observes, unlike the torus and like the projective plane, such a surface has corners. Cairns studies the n-queens problem on the pillow board, restricting it to boards of the form $n \times 2m$ in order to retain the usual black–white pattern of squares. He notes that the pillow board is an orbifold, that is a manifold that looks locally like the quotient of Euclidean space by a finite group action, and indeed the pillow is homeomorphic to the sphere S^2, with it globally the quotient of the torus by an action of the group with two elements, C_2. Cairns proves that the $n \times 2m$ pillow board is equivalent to the $m \times 2n$ pillow board; e.g. that the 4×10 board is equivalent to the 5×8 board, and also the $2m \times 2m$ board is equivalent to the $m \times 4m$ board, and thus he notes that we can place at most m nonattacking queens on the $2m \times 2m$ pillow board. Indeed, Cairns proves the theorem that for all m, the $2m \times 2m$ pillow chessboard can have m nonattacking queens. Cairns recommends for future study extending his results for the $n \times 2m$ board to the general $n \times m$ board.

We give now several conjectures about n-queens and the extensions of it that we have discussed.

Conjecture 24 (*Van Rees*). *For all n, n^2 nonattacking queens can be placed on the modular n-cube if and only if* $\gcd(n, 210) = 1$.

This can be generalized to the following:

Conjecture 25. *For all n and d, n^{d-1} nonattacking queens can be placed on the modular d-dimensional board if and only if* $\gcd(n, 2^d!) = 1$.

One way of approaching this could be to prove that this is true for linear solutions and then use Shapiro's method to show that a nonlinear solution for a particular n and d implies a linear solution [167]. This approach will likely not work for non-modular boards.

Conjecture 26 (*Rivin, Vardin and Zimmermann*). *For $M(n)$ the total number of solutions to for the modular board and $Q(n)$ the total number of solutions for the standard board (counting solutions equivalent under the symmetry group of the square as distinct solutions)* $\lim_{\substack{n \to \infty \\ \gcd(n,6)=1}} \frac{\log M(n)}{n \log n} > 0$, $\lim_{n\to\infty} \frac{\log Q(n)}{n \log n} > 0$.

Conjecture 27 (*Rivin, Vardin and Zimmermann*). *For $M(n)$ and respectively the total number of distinct solutions, counting solutions equivalent under action of the symmetry group of the square as distinct, the generating function $\sum_{n=1}^{\infty}(M(n)/n!)x^n$ has a closed form.*

Conjecture 28 (*Klarner*). *For all $n > 6$, there exists a solution to the reflecting queens problem.*

Conjecture 29 (*Nudelman*). *For $\mu(n, d)$ the maximum number of queens on a d-dimensional modular board such that no two attack each other, for all positive integers n and d, $\mu(n, d) = n$ if and only if* $\gcd(n, (2^{d-1})!) = 1$.

Conjecture 30. *For all $n \geq 10$, and for $n = 1, 2, 7$, we can place $\lceil n/2 \rceil$ nonattacking queens on the $n \times n$ Klein Bottle board, such that the left and right sides are twisted connected and the top and bottom sides are directly connected, as a cylinder.*

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