



# A parsimonious arbitrage-free implied volatility parameterization with application to the valuation of volatility derivatives

---

*Jim Gatheral*  
*Global Derivatives & Risk Management 2004*  
*Madrid*  
*May 26, 2004*

The opinions expressed in this presentation are those of the author alone, and do not necessarily reflect the views of of Merrill Lynch, its subsidiaries or affiliates.

## Outline of this talk

- Roger Lee's moment formula
- A stochastic volatility inspired (SVI) parameterization of the implied volatility surface
- No-arbitrage conditions
- SVI fits to market data
- SVI fits to theoretical models
- Carr-Lee valuation of volatility derivatives under the zero correlation assumption
- Valuation of volatility derivatives in the general case

# Roger Lee's moment formula

Define  $k := \log(K / F)$ .

Let  $q^* := \sup \{q : \mathbb{E} S_T^{-q} < \infty\}$  and  $\mathbf{b}^* := \limsup_{k \rightarrow -\infty} \frac{\mathbf{s}_{\text{BS}}^2(k, T)T}{k}$

Then  $\mathbf{b}^* \in [0, 2]$  and  $q^* = \frac{1}{2} \left( \frac{1}{\sqrt{\mathbf{b}^*}} - \frac{\sqrt{\mathbf{b}^*}}{2} \right)^2$

Also let  $p^* := \sup \{p : \mathbb{E} S_T^{1+p} < \infty\}$  and  $\mathbf{a}^* := \limsup_{k \rightarrow +\infty} \frac{\mathbf{s}_{\text{BS}}^2(k, T)T}{k}$

Then  $\mathbf{a}^* \in [0, 2]$  and  $p^* = \frac{1}{2} \left( \frac{1}{\sqrt{\mathbf{a}^*}} - \frac{\sqrt{\mathbf{a}^*}}{2} \right)^2$

- The derivation assumes only the existence of a Martingale measure: it makes no assumptions on the distribution of  $S_T$ .

## Implications of the moment formula

- Implied variance is always linear in  $k$  as  $|k| \rightarrow \infty$
  - For many models,  $p^*$  and  $q^*$  may be computed explicitly in terms of the model parameters.
- 
- So, if we want a parameterization of the implied variance surface, it needs to be linear in the wings!
  - ... and it needs to be curved in the middle - many conventional parameterizations of the volatility surface are quadratic for example.

## The SVI (“stochastic volatility inspired”) parameterization

- For each timeslice,

$$\text{var}(k; a, b, \mathbf{S}, \mathbf{r}, m) = a + b \left\{ \mathbf{r}(k - m) + \sqrt{(k - m)^2 + \mathbf{S}^2} \right\}$$

Left and right asymptotes are:

$$\text{var}_L(k; a, b, \mathbf{S}, \mathbf{r}, m) = a - b(1 - \mathbf{r})(k - m)$$

$$\text{var}_R(k; a, b, \mathbf{S}, \mathbf{r}, m) = a + b(1 + \mathbf{r})(k - m)$$

- Variance is always positive
- Variance increases linearly with  $|k|$  for  $k$  very positive or very negative
  - Intuition is that the more out-of-the-money an option is, the more volatility convexity it has.

## What the SVI parameters mean

$a$  gives the overall level of variance

$b$  gives the angle between the left and right asymptotes

$s$  determines how smooth the vertex is

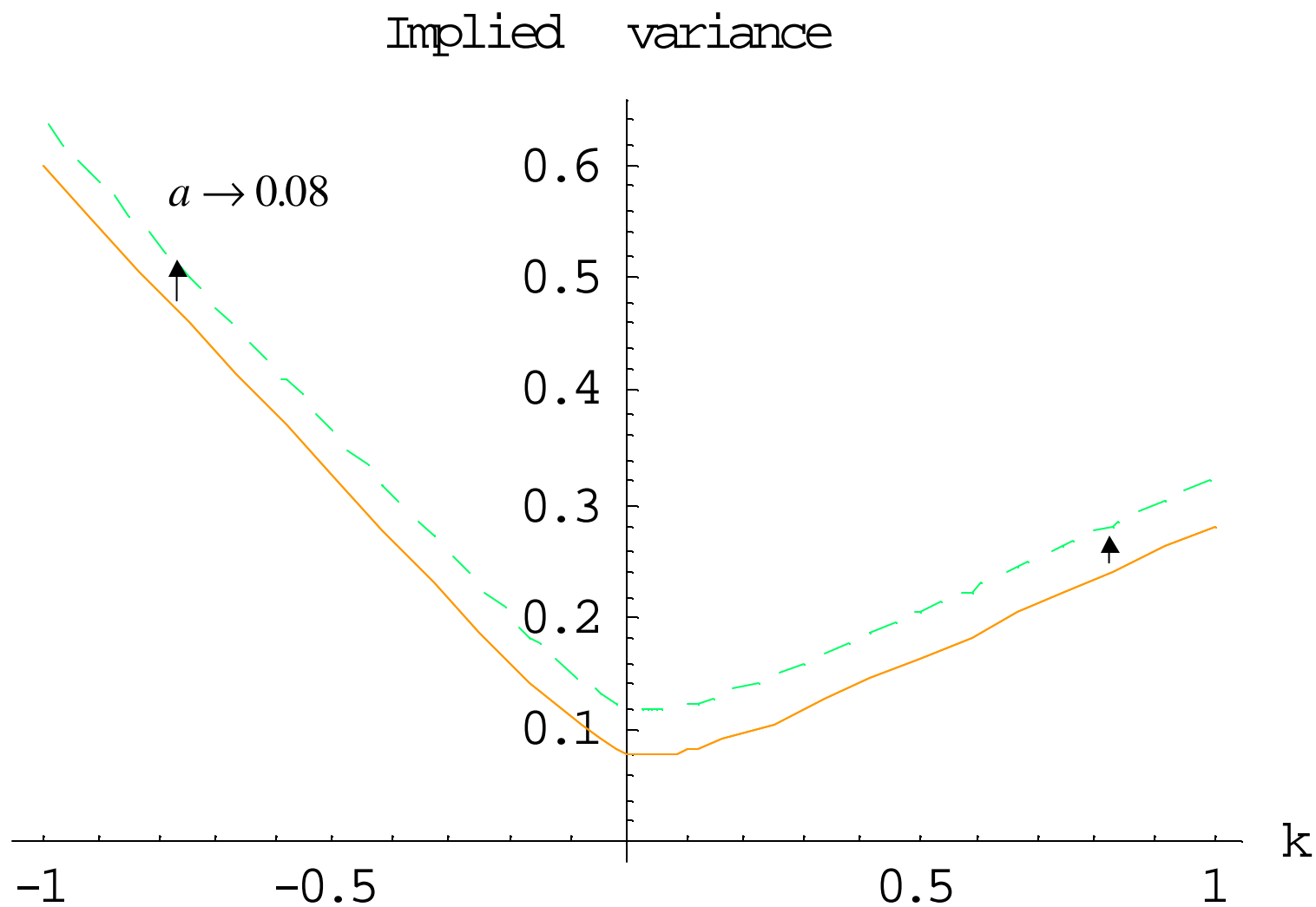
$r$  determines the orientation of the graph

changing  $m$  translates the graph

The next few slides show this graphically with base case

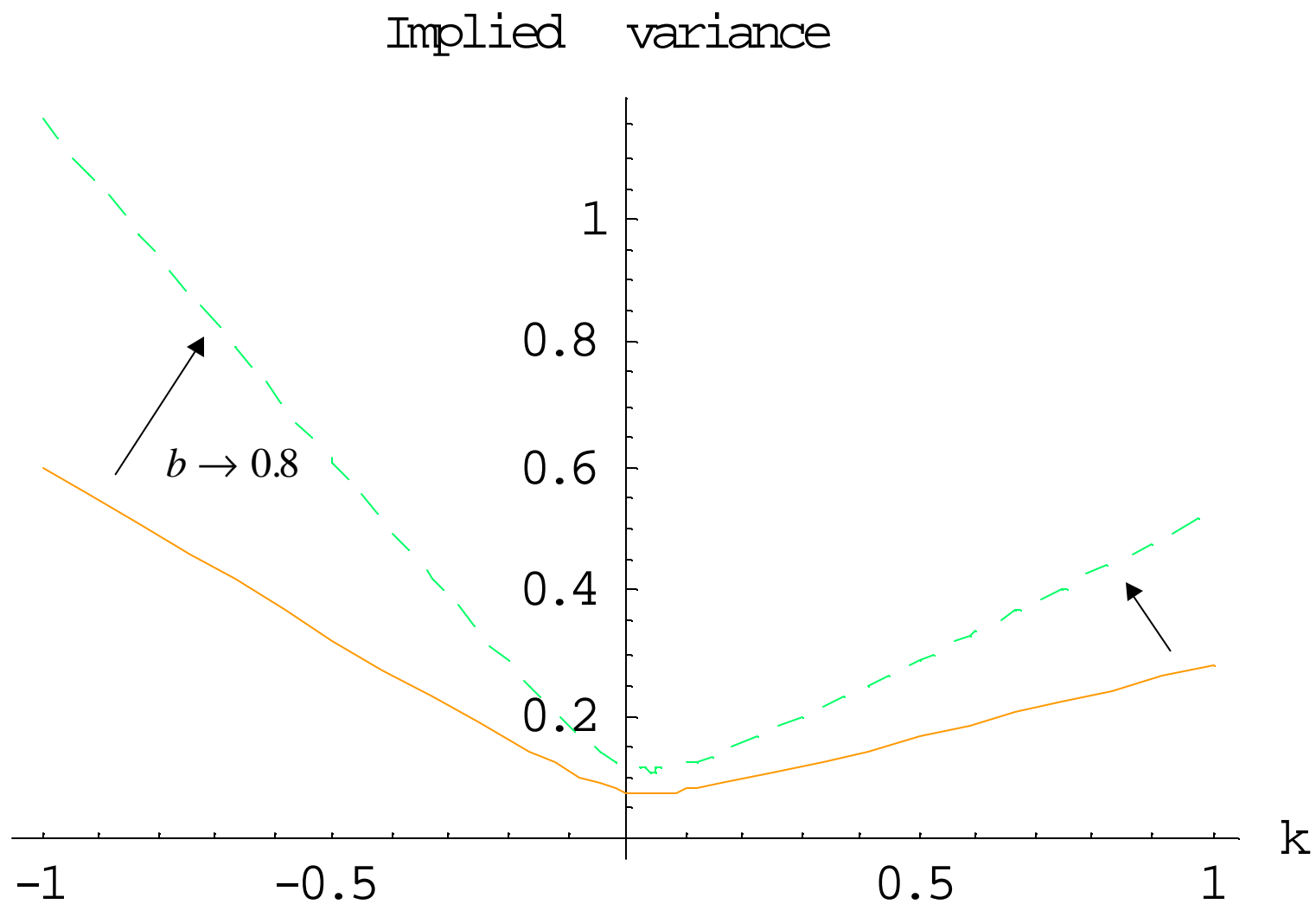
$a = 0.04, b = 0.4, s = 0.1, r = -0.4, m = 0$

## Changing a

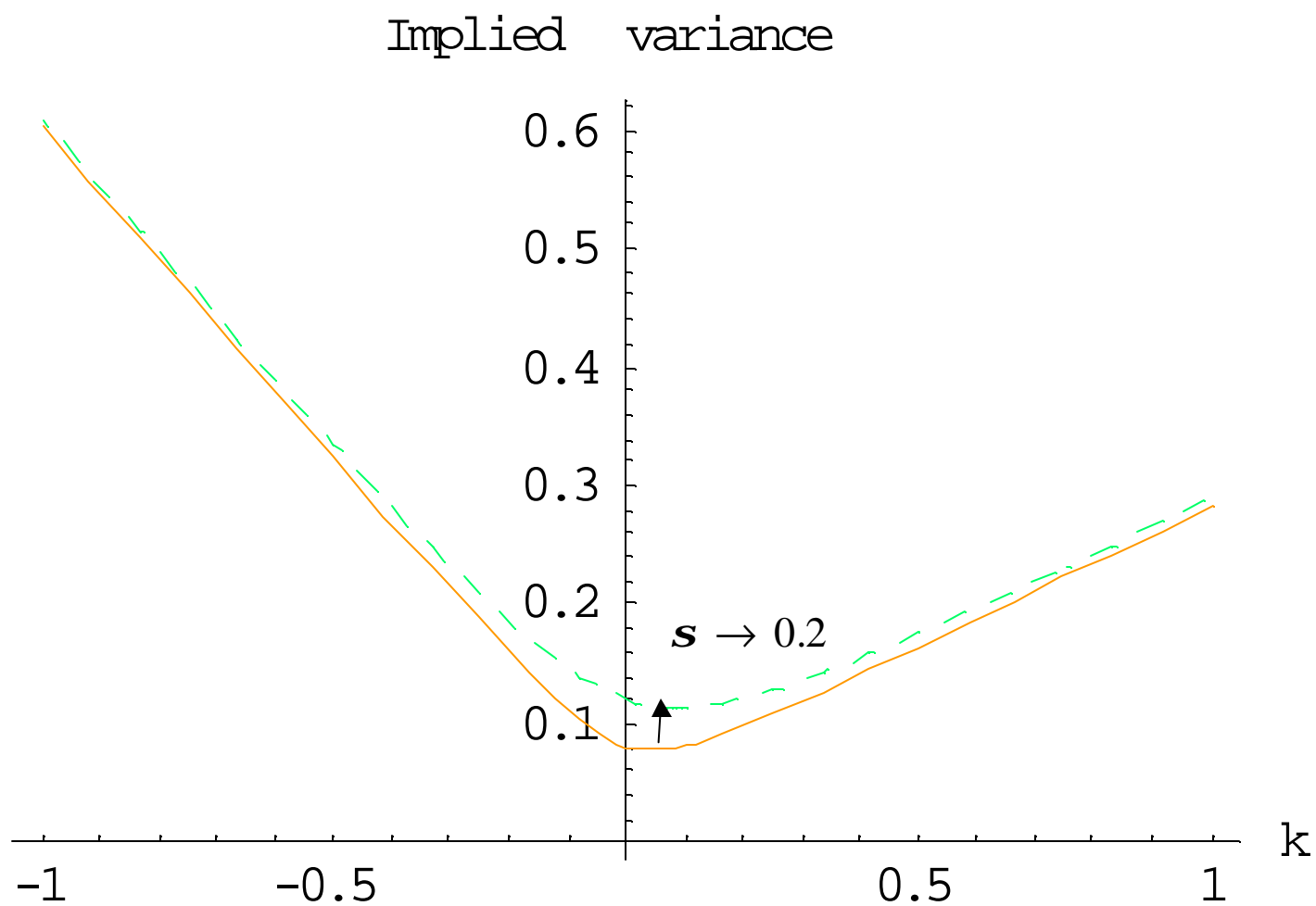




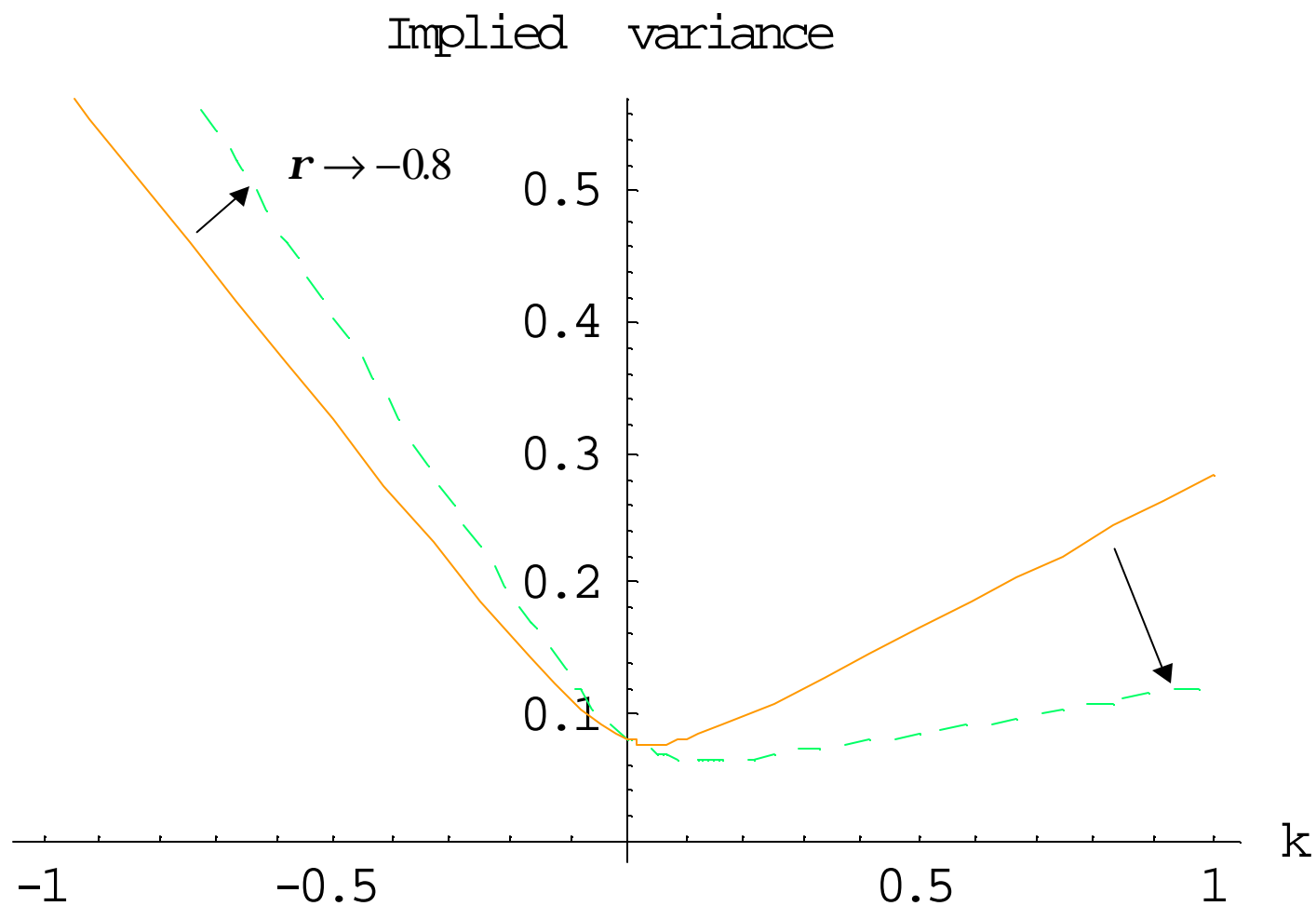
## Changing b



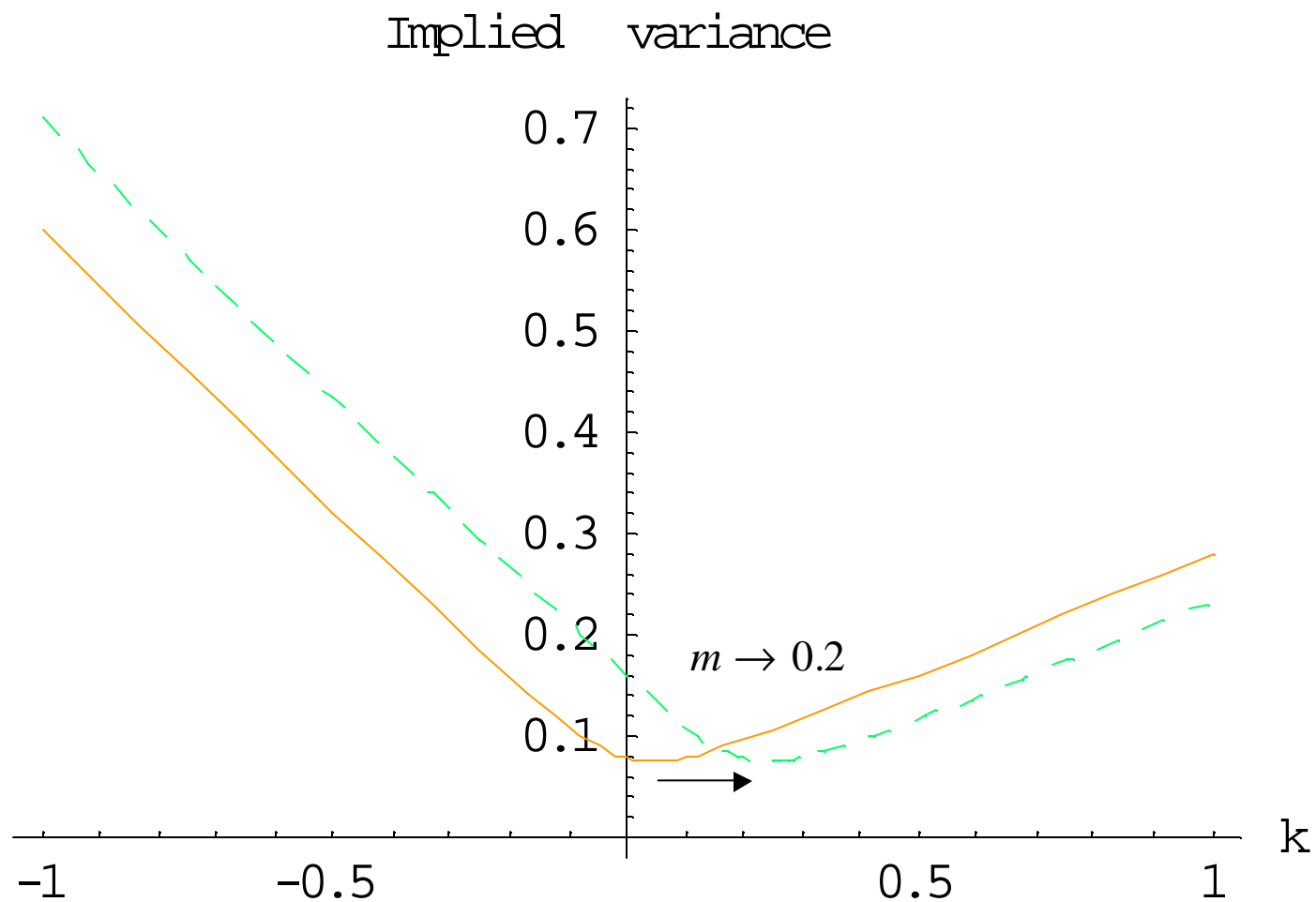
# Changing $s$



# Changing $r$



# Changing $m$



## Preventing arbitrage

- In practice, if SVI is fitted to actual option price data, negative vertical spreads never arise. In terms of the SVI (implied variance fit) parameters, the condition is

$$b(1 + |\mathbf{r}|) \leq \frac{4}{T}$$

- It turns out that if there are no negative vertical spreads, negative butterflies are also excluded.
- In contrast, it is not obvious how to prevent negative time spreads. We now show what conditions a parameterization needs to satisfy to exclude arbitrage between expirations (calendar spread arbitrage).

## Necessary and sufficient condition for no calendar spread arbitrage

- First we note that for any Martingale  $X_t$  and  $t_2 \geq t_1$  it is easy to show that

$$\mathbb{E}(X_{t_2} - L)^+ \geq \mathbb{E}(X_{t_1} - L)^+$$

- Now consider the non-discounted values  $C_1$  and  $C_2$  of two options with strikes  $K_1$  and  $K_2$  and expirations  $t_1$  and  $t_2$  with  $t_2 > t_1$ . Suppose the two options have the same moneyness so that

$$\frac{K_1}{F_1} = \frac{K_2}{F_2} \equiv k$$

- Consider the martingale  $X_t \equiv S_t / F_t$ . Then, in the absence of arbitrage, we must have

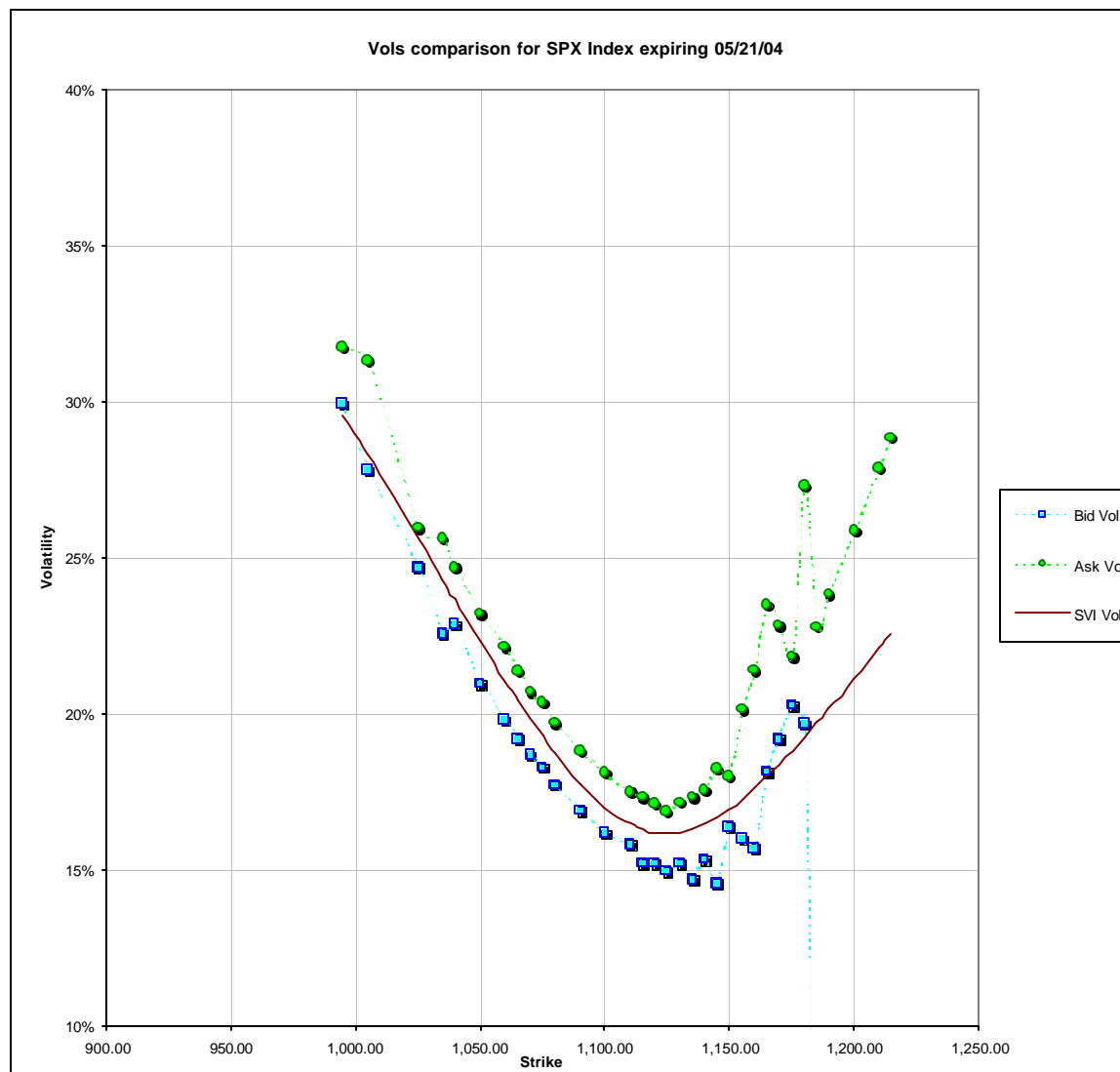
$$\frac{C_2}{K_2} = \frac{1}{k} \mathbb{E}(X_{t_2} - k)^+ \geq \frac{1}{k} \mathbb{E}(X_{t_1} - k)^+ = \frac{C_1}{K_1}$$

- So, keeping the moneyness constant, option prices are non-decreasing in time to expiration.

## Derivation of no arbitrage condition continued

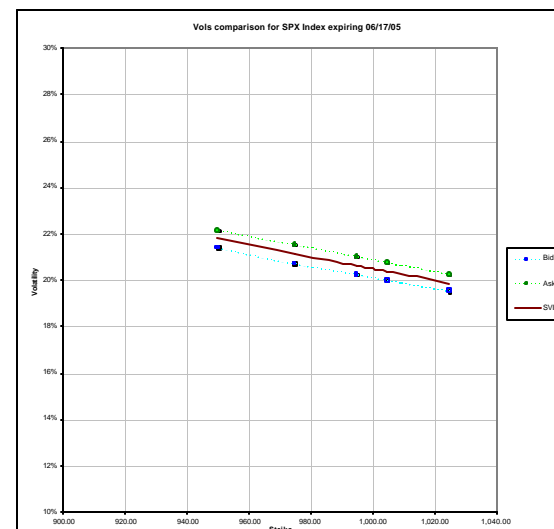
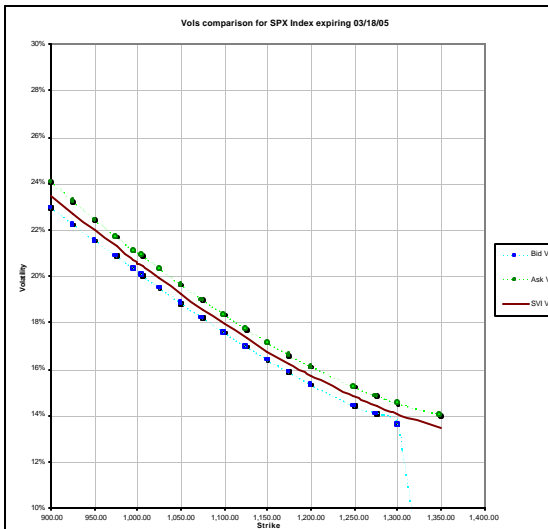
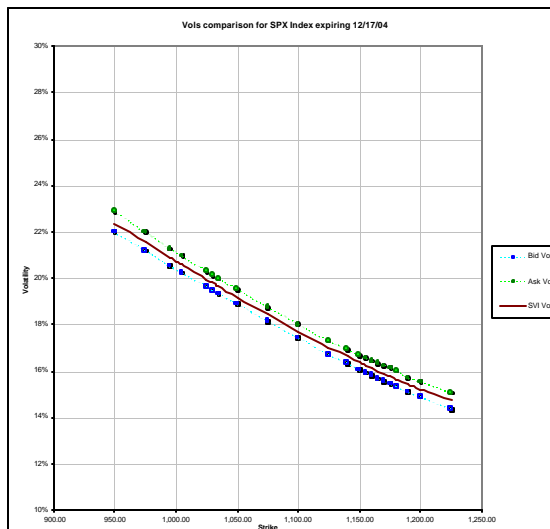
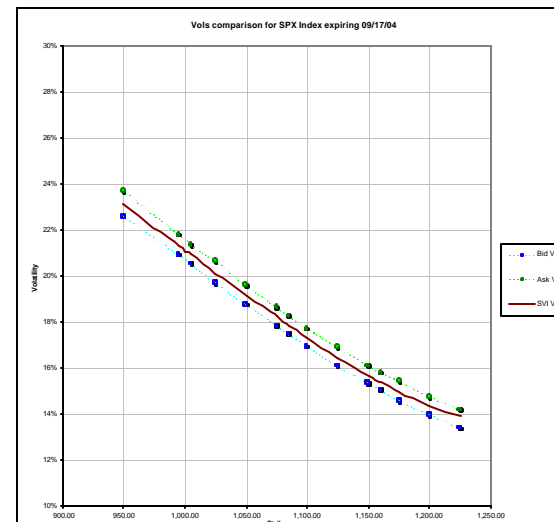
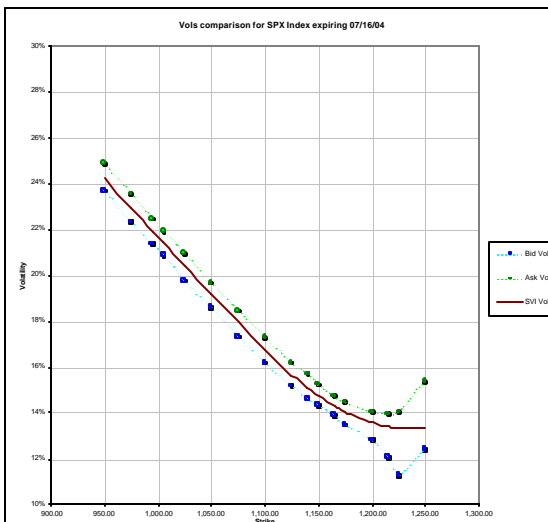
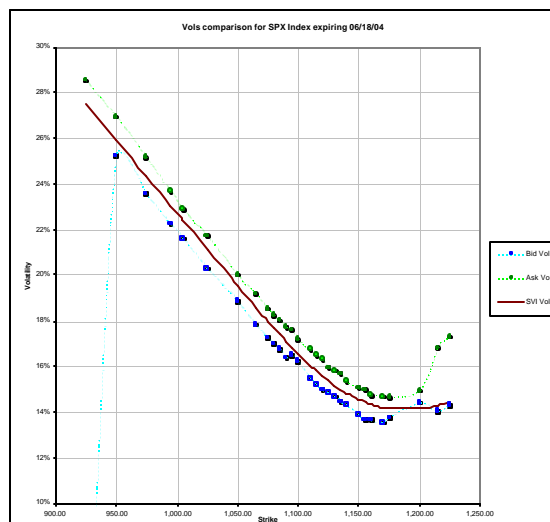
- Let  $w_t \equiv S_{BS}(k, t)^2 t$
- The Black-Scholes formula for the non-discounted value of an option may be expressed in the form  $C_{BS}(k, w_t)$  with  $C_{BS}$  strictly increasing in the total implied variance  $w_t$ .
- It follows that for fixed  $k$ , we must have the total implied variance  $w_t$  non-decreasing with respect to time to expiration.

# SVI fit to SPX with one week to expiration

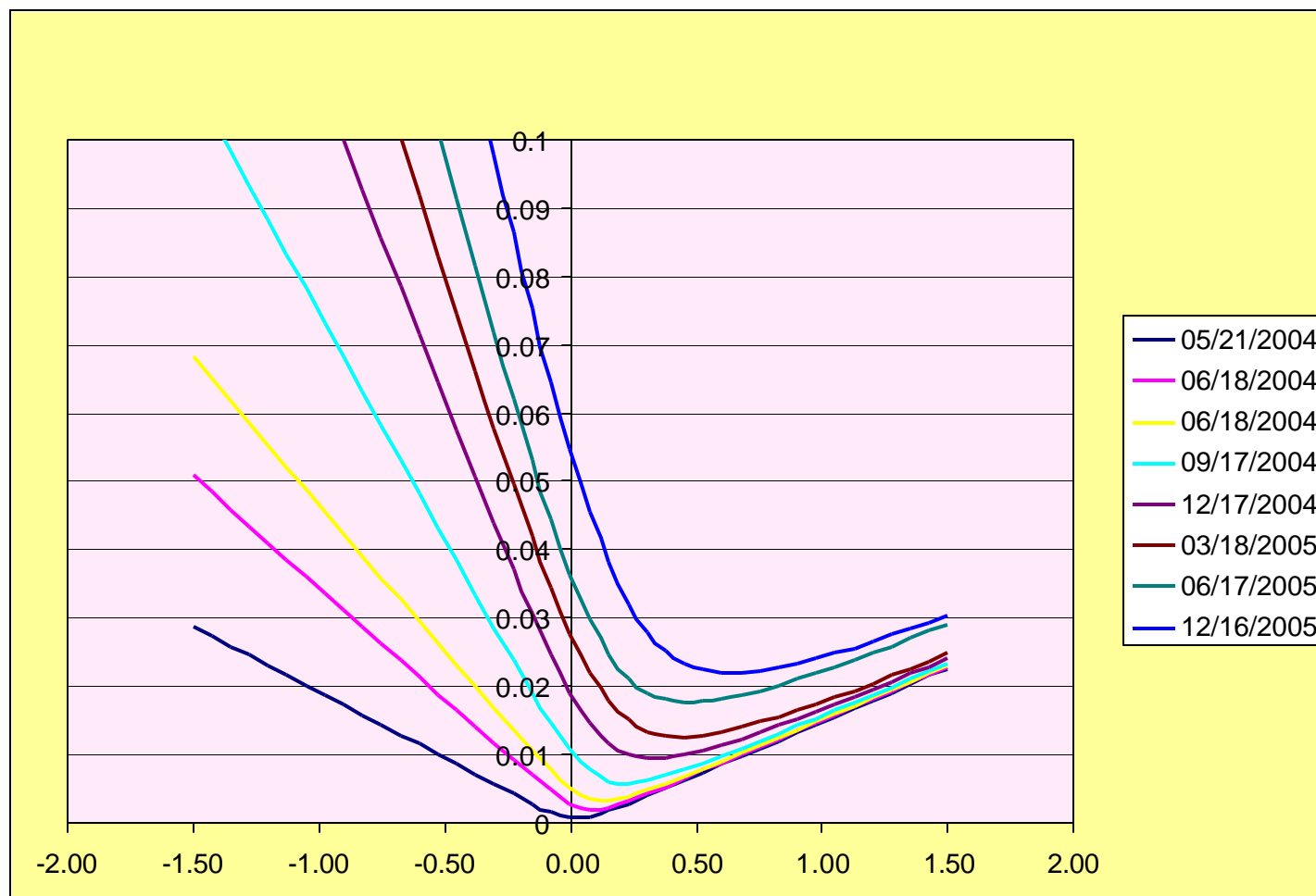




# Other expirations



## The total variance plot

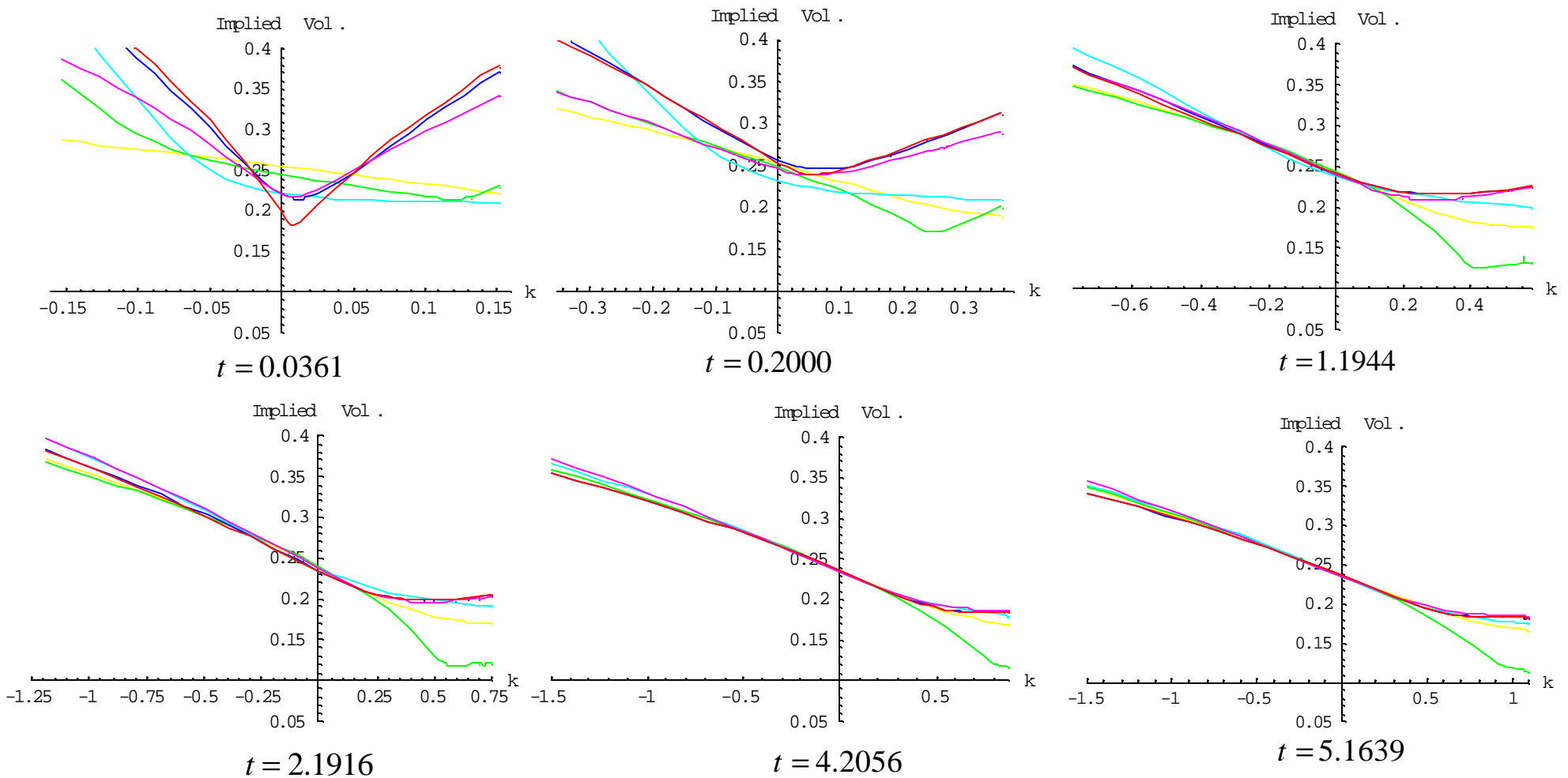


- Note no crossed lines so no arbitrage!

## Review of the work of Schoutens et al.

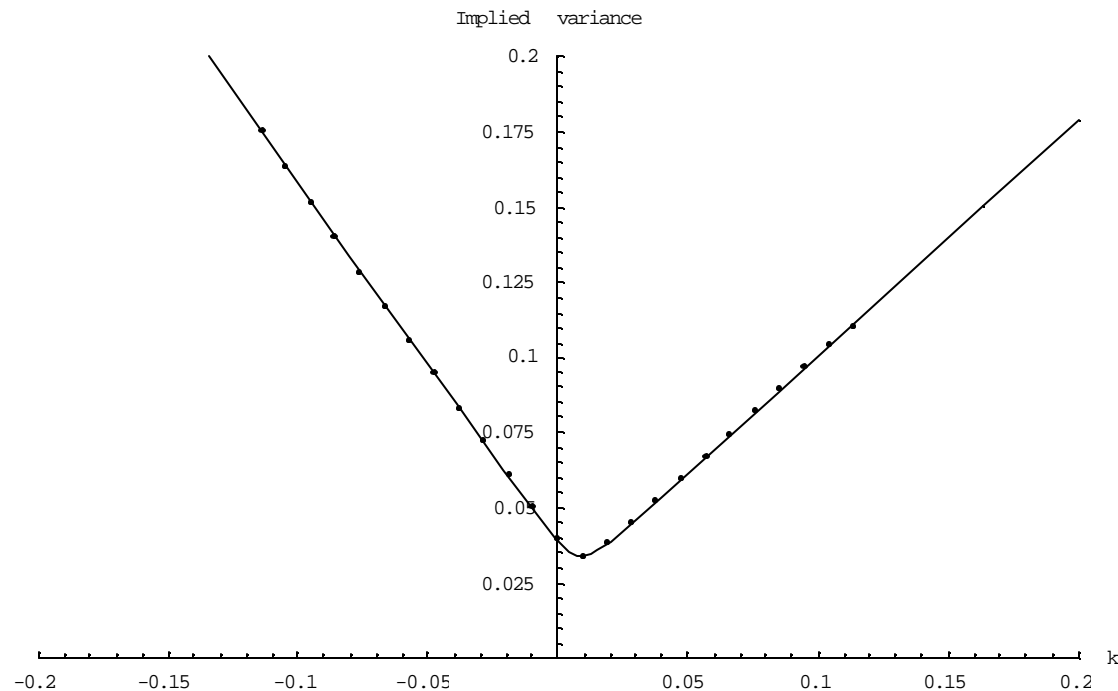
- In their recent Wilmott Magazine paper, Schoutens, Simons and Tistaert carefully fitted 7 different models to the STOXX50 index option market observed on 7-Oct-2003.
- The models fitted were
  - HEST, HESJ, BNS, VG-CIR, VG-OUT, NIG-CIR, NIG-OUT
- These models represent a range of quite different dynamics. For example, HEST is a pure stochastic volatility model and VG-CIR is a pure jump model with subordinated time.

# Plots of implied vol. vs log-strike $k$ for 6 of the Schoutens et al. models



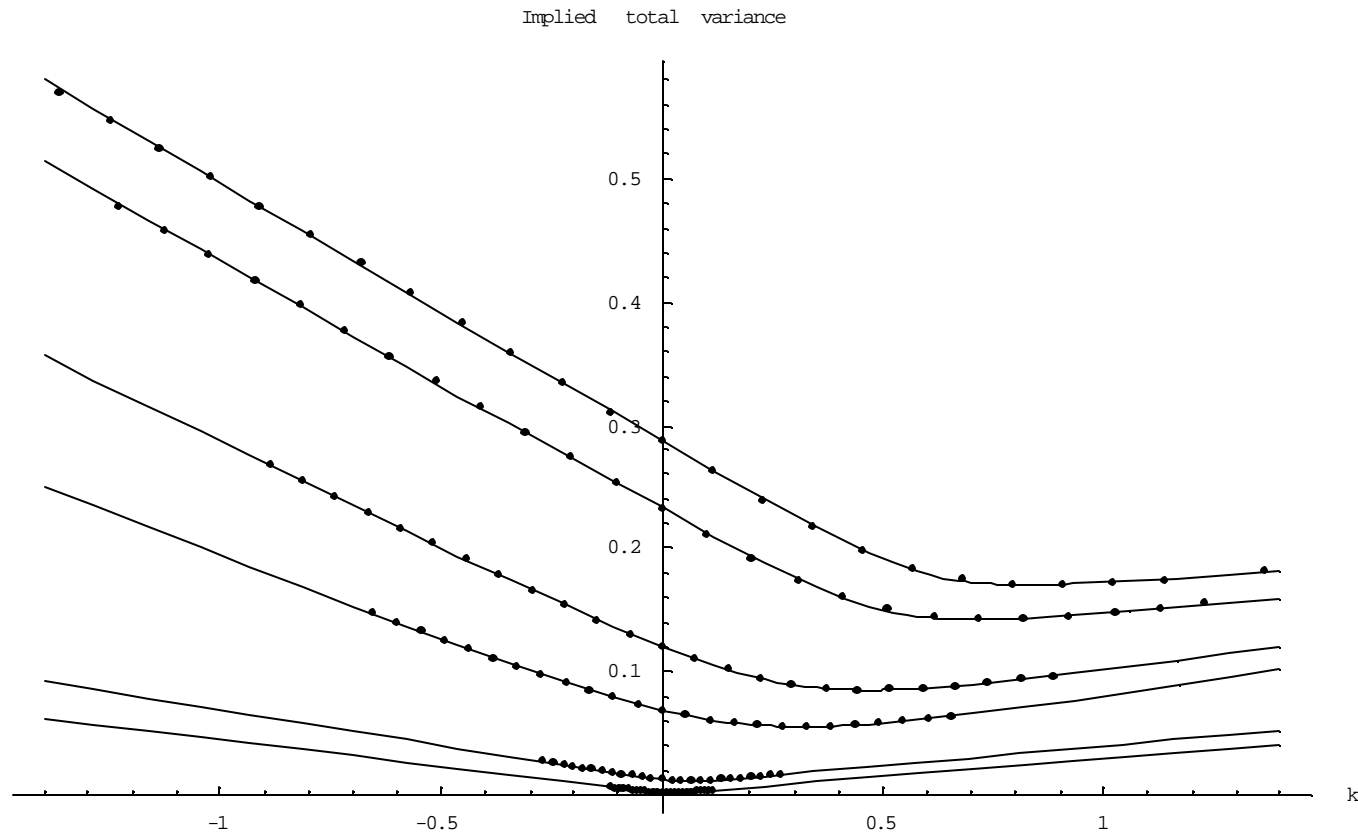
## SVI fit to VG-OUG model with Schoutens et al. parameters

$c \text{ ® } 6.1610, g \text{ ® } 9.6443, m \text{ ® } 16.026, l \text{ ® } 1.679, a \text{ ® } 0.3484, b \text{ ® } 0.7664, y0 \text{ ® } 1$



- The above graph shows the SVI fit to the first timeslice ( $t=0.0361$  years). This slice is invariably the hardest to fit in practice; SVI fits almost perfectly.

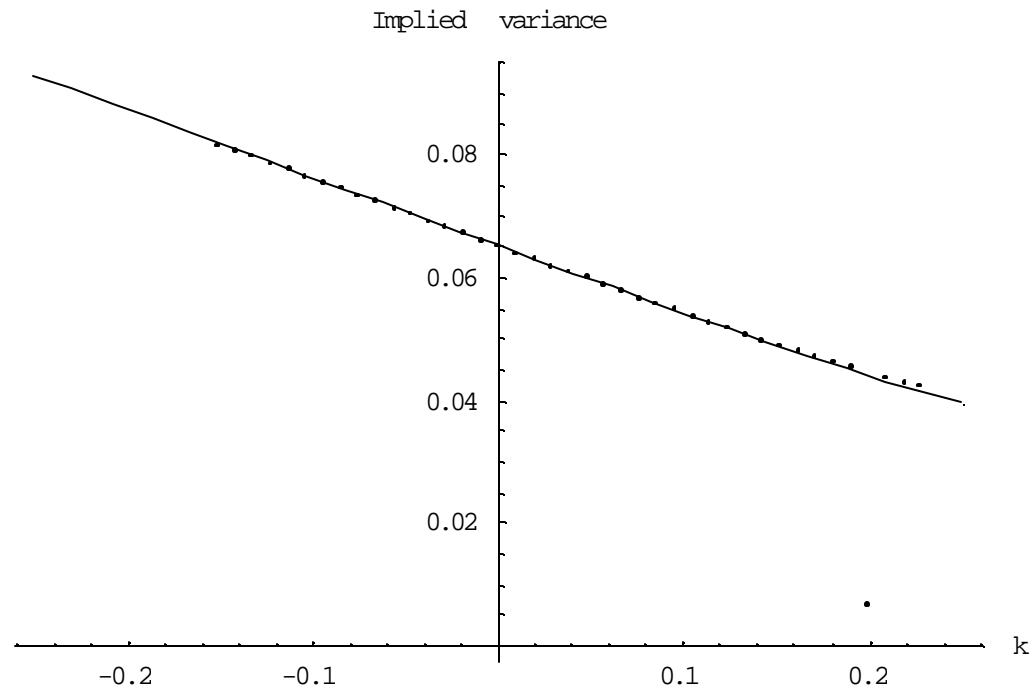
## Total variance plot for VG-OUG with Schoutens et al. parameters



- Note that the surface has to be arbitrage-free because it is generated from explicit arbitrage-free dynamics.
- Note also that SVI fits every slice extremely well!

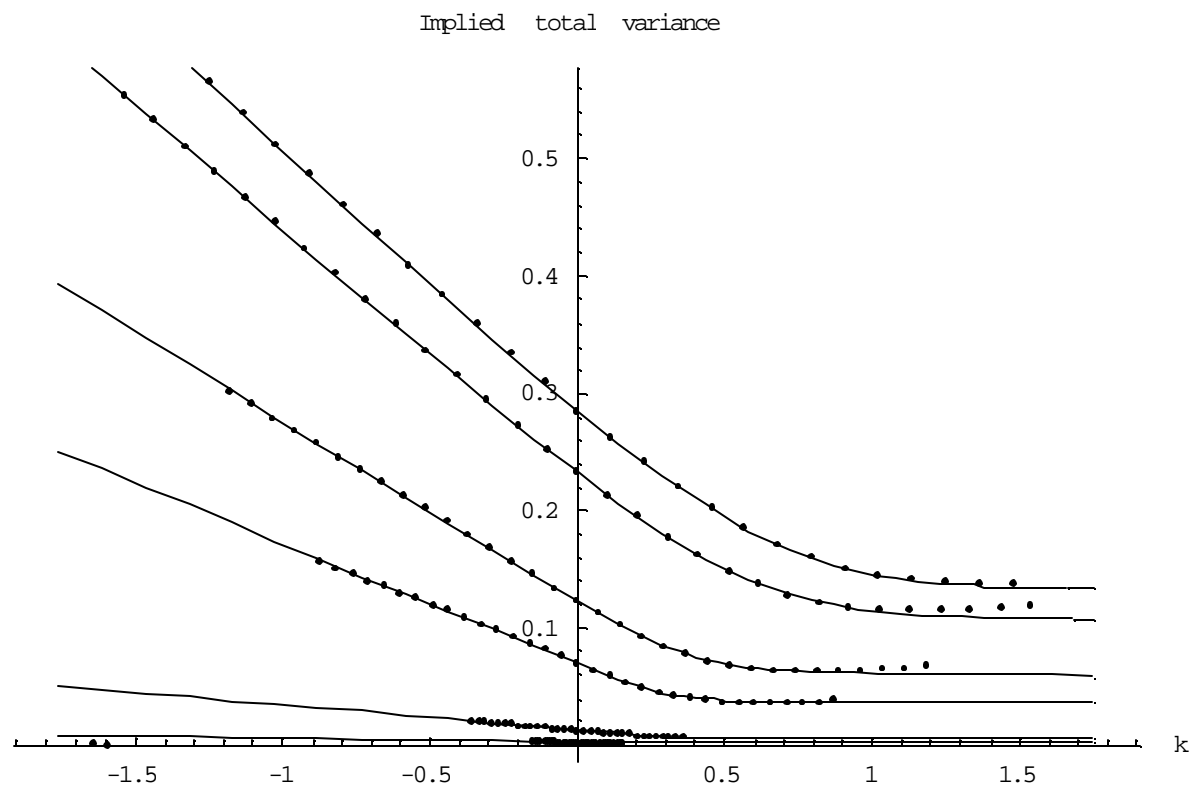
## SVI fit to the Heston model with Schoutens et al. parameters

$\lambda \rightarrow 0.6067$ ,  $\rho \rightarrow -0.7571$ ,  $\eta \rightarrow 0.2928$ ,  $\bar{v} \rightarrow 0.0707$ ,  $v \rightarrow .0654$



- The above graph shows the SVI fit to the first timeslice ( $t=0.0361$  years). Again the fit is very good but fitting this slice on its own is not asking very much...

## Total variance plot for Heston with Schoutens et al. parameters



- SVI fits somewhat less well this time.



## The Heston Model

- The model is given by

$$dx = -\frac{\nu}{2} dt + \sqrt{\nu} dZ_1$$

$$d\nu = -\mathbf{l}(\nu - \bar{\nu}) dt + \mathbf{h} \sqrt{\nu} dZ_2$$

$$\langle dZ_1, dZ_2 \rangle = \mathbf{r} dt$$

with  $x = \log(F / S_0)$

## The Heston model as $T \rightarrow \infty$

- From (e.g.) Lewis, we can derive the the level, slope and curvature of at-the-money forward (ATMF) implied volatility from the parameters of the model. In the case of the Heston model as  $T \rightarrow \infty$ , these are

$$v_T(0) \sim \frac{4l'\bar{v}'}{h^2(1-r)^2} \left\{ \sqrt{4l'^2 + h^2(1-r)^2} - 2l' \right\}$$

$$v'_T(0) \sim \frac{4r}{h(1-r)^2 T} \left\{ \sqrt{4l'^2 + h^2(1-r)^2} - 2l' \right\}$$

$$v''_T(0) \sim \frac{2}{l'\bar{v}'T^2} \left\{ \sqrt{4l'^2 + h^2(1-r)^2} - 2l' \right\}$$

with  $l' = l - rh/2$ ;  $l'\bar{v}' = l\bar{v}$ .

- From Roger Lee, we can compute the slope of the  $|k| \rightarrow \infty$  asymptotes of the implied variance by finding the poles of the Heston characteristic function. The resulting slopes of the asymptotes again as  $T \rightarrow \infty$  are given by

$$v'_T(+\infty)T = \mathbf{a}^*(T) \rightarrow \frac{2}{h(1-r)} \left\{ \sqrt{4l'^2 + h^2(1-r)^2} - 2l' \right\}$$

$$v'_T(-\infty)T = \mathbf{b}^*(T) \rightarrow \frac{2}{h(1+r)} \left\{ \sqrt{4l'^2 + h^2(1-r)^2} - 2l' \right\}$$

## Consistency with SVI

- It turns out that these formulae are totally consistent with SVI if we have the following correspondence between SVI parameters and Heston parameters (as  $T \rightarrow \infty$ ):

$$a = \frac{w}{2}(1 - r^2)$$

$$b = \frac{w}{2}q$$

$$r = r$$

$$m = -\frac{r}{q}$$

$$s = \frac{\sqrt{1 - r^2}}{q}$$

with

$$w = \frac{4l'\vec{v}}{h^2(1 - r^2)} \left\{ \sqrt{4l'^2 + h^2(1 - r)^2} - 2l' \right\}$$

$$q = \frac{h}{l'\vec{v}T}$$

## Review of Carr-Lee valuation of volatility derivatives

- Assumes zero correlation between quadratic variation and returns.
- Carr and Lee show how under this assumption, the prices of volatility derivatives are completely determined by the prices of European options (which we assume to be known). Formally

$$\mathbb{E} f \left( \langle x \rangle_t \right) = \int dk w_f(k) c(k, T)$$

with

$$w_f(k) = 4 \frac{e^{k/2}}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(l) l \cosh \left[ k \sqrt{1/4 + 2l} \right] dl$$

whenever  $f(y)$  has a Laplace transform  $F(l)$  for  $a \in \mathbb{R}$  on the right of all singularities of  $F$

- Friz and Gatheral show that more often than not, the weights  $w_f$  are ill-defined and the problem of inverting to find the weights is ill-posed.
- However, they also show how to compute the value of volatility derivatives from the values of European options to any desired accuracy.

## Extension to non-zero correlation

- Suppose that (*e.g.*) Heston dynamics really drive the market.
- We note that the values of volatility derivatives are independent of the correlation between stock returns and volatility moves.
- To compute the fair value of a volatility derivative, all we would need to do is to set  $\mathbf{r}$  to zero and apply the inversion method described in Friz and Gatheral.
  
- This suggests the following general recipe:
- Start with some parameterization of the volatility surface.
- Noting that when  $\mathbf{r}$  is zero, the implied variance smile is symmetric around  $k=0$ , rotate the curve keeping its overall level the same.
  - Although this step is intuitively clear, its meaning is far from precise and its implementation is far from obvious
- Apply the Friz algorithm to compute the value of the volatility derivative.

## The Friz inversion algorithm

- We recall from Hull and White that in a zero-correlation world, we may write

$$c(k) = \int dy g(y) c_{BS}(k, y)$$

where  $y := \mathbf{S}_{BS}^2 T$  is the total variance. In words, we can compute an option price by averaging over Black-Scholes option prices conditioned on the BS total implied variance.

- We discretize the above to obtain

$$c_i = \sum_{j=1}^m A_{ij} g_j \quad (i = 1, \dots, n)$$

with  $m \gg n$ .

- We may then compute the probability vector  $g$  using the Moore-Penrose pseudo-inverse as

$$\mathbf{g} = \mathbf{A}^T (\mathbf{A} \mathbf{A}^T)^{-1} \mathbf{c} =: \mathbf{M} \mathbf{c}$$

- This is effectively a method for deducing the law of implied variance from option prices. Then, the expectation of any function of variance may be computed.

## Example: SVI

- Although SVI has  $r$  explicitly as a parameter, it's not enough to set  $r = 0$  and apply the Friz method.
  - This can be seen easily by computing the value of a variance swap before and after setting  $r = 0$ .
- Other of the SVI parameters may and do in fact depend on  $r$ . To see how they should transform, we refer back to the mapping between SVI and Heston parameters in the  $T \rightarrow \infty$  limit. Specifically,

$$r \rightarrow 0$$

$$m \rightarrow 0$$

$$s \rightarrow \frac{s}{\sqrt{1-r^2}}$$

Instead of specifying exactly how  $a$  and  $b$  transform (which is pretty complicated) we note that they transform as

$$a \rightarrow y a$$

$$b \rightarrow y b(1-r^2)$$

We find  $y$  by imposing that the expected variance be conserved under the transformation.

## A formula for computing expected total variance

- It may be shown that for diffusion processes

$$\mathbb{E}\left[\langle x \rangle_T\right] = \int_0^\infty dz N'(z) \mathbf{s}_{BS}^2(z)$$

with

$$z \equiv d_2(k) = \frac{-k}{\mathbf{s}_{BS}(k, T)\sqrt{T}} - \frac{\mathbf{s}_{BS}(k, T)\sqrt{T}}{2}$$



## Recipe for valuation of volatility derivatives

- Fit SVI parameterization to the implied volatility surface
- Transform the surface so it is symmetric around the forward price as follows:

$$a \rightarrow y a$$

$$b \rightarrow y b (1 - r^2)$$

$$r \rightarrow 0$$

$$m \rightarrow 0$$

$$s \rightarrow \frac{s}{\sqrt{1 - r^2}}$$

- Adjust  $y$  using the total variance formula to keep the expected total variance invariant.
- Apply the Friz method or Carr-Lee directly to compute the value of the volatility derivative directly in this new zero-correlation world.

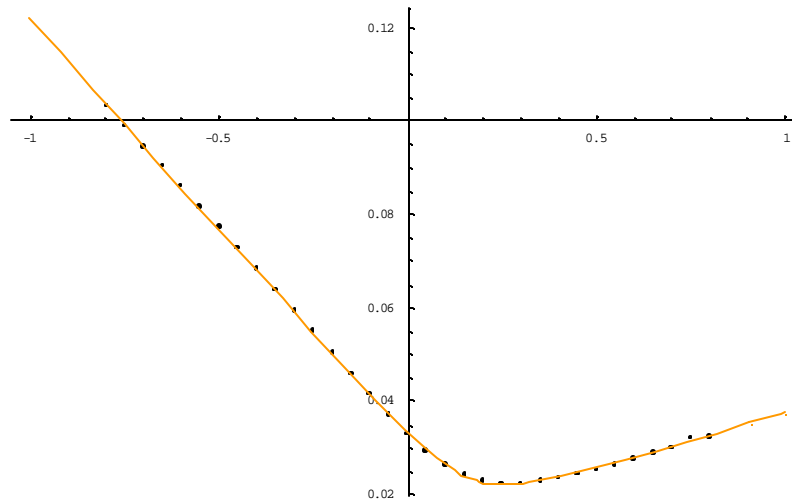
## Example: One year Heston with BCC parameters

- We compute one year European option prices in the Heston model using parameters from Bakshi, Cao and Chen. Specifically

$$\nu = \bar{\nu} = 0.04; \mathbf{h} = 0.39; \mathbf{l} = 1.15; \mathbf{r} = -0.64$$

- We fit SVI and obtain the following SVI parameters:

$$a = .0159479; b = 0.0577371; \mathbf{r} = -0.568899; \mathbf{s} = 0.127445; m = 0.165476$$



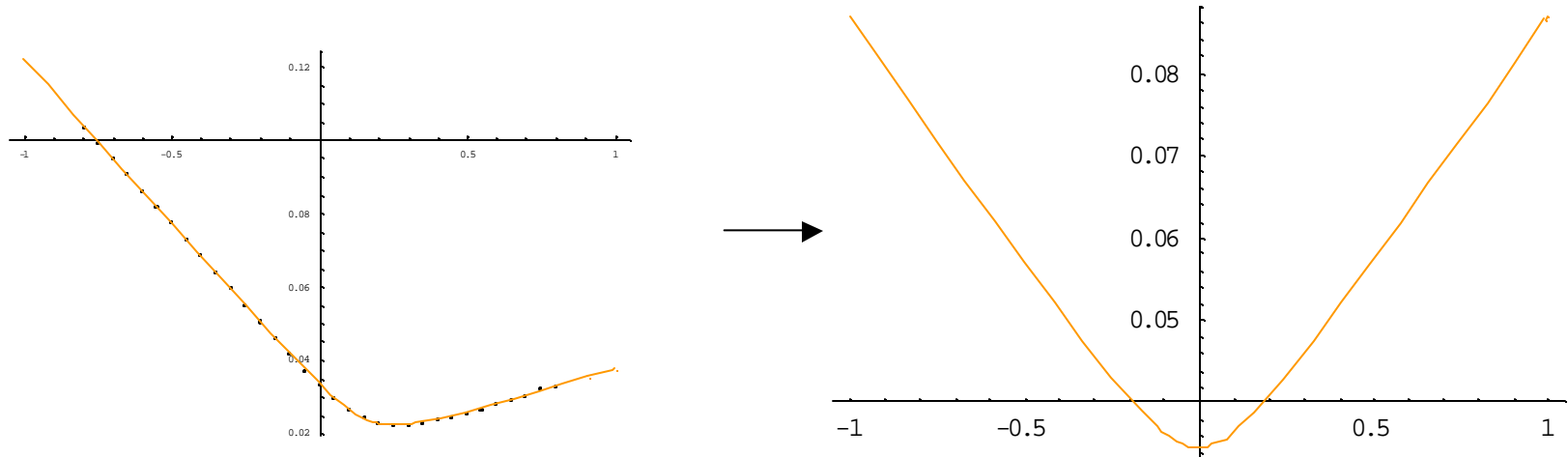
- Applying the total variance formula gives  $\mathbb{E}[\langle x \rangle_T] = 0.0400846$  which is not too far off the exact value of 0.04.

## Example: continued

- We rotate the SVI parameters according to our recipe to obtain the new SVI parameters

$$a = 0.024973; b = 0.0611502; \mathbf{r} = 0; \mathbf{s} = 0.154966; m = 0;$$

- Graph is as expected and total variance is 0.0400846 by construction

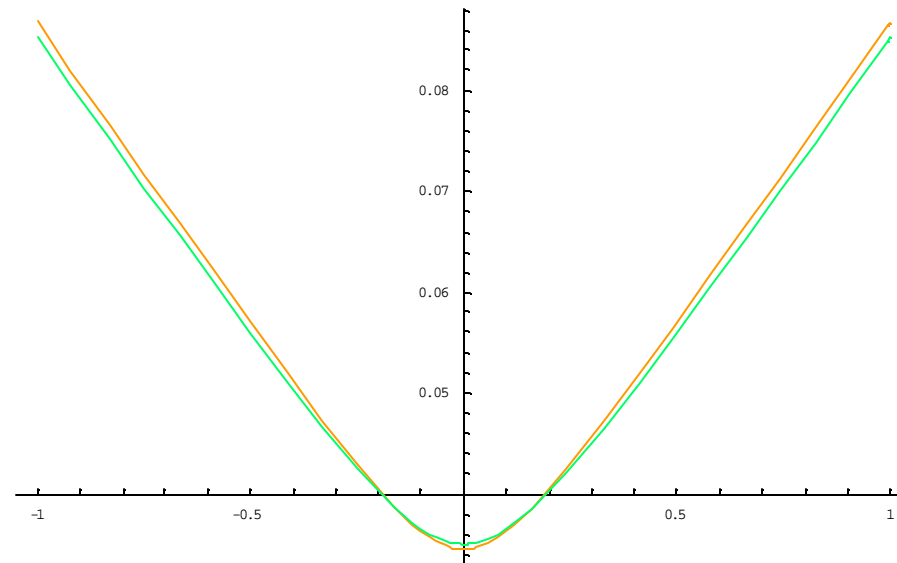


- As a check, regenerate Heston option prices with  $\mathbf{r} = 0$  and refit SVI. We obtain

$$a = 0.0233521; b = 0.06087352; \mathbf{r} = 0; \mathbf{s} = 0.1914214; m = 0;$$

- To recap, we guessed a SVI rotation transformation from the  $T \rightarrow \infty$  limit. This transformation seems to work pretty well even for  $T=1$ !

# True zero-correlation Heston and rotated SVI smiles



- The closeness of agreement of the two curves is all we need to show that the values of all volatility derivatives are correctly obtained through the rotation recipe (at least in this case).
- We now compute the value of the volatility swap to verify this.

## A formula for computing expected volatility

- It may be shown that for diffusion processes under the zero-correlation assumption

$$\mathbb{E} \sqrt{\langle x \rangle_t} = \int dk w_f(k) c(k, T)$$

with

$$w_f(k) = 2\mathbf{p} \mathbf{d}(k) + \sqrt{\frac{\mathbf{p}}{2}} e^{k/2} I_1\left(\frac{k}{2}\right)$$

where  $I_1(\cdot)$  is a modified Bessel function of the first kind.

- To motivate this formula, recall that

$$c(0, t) \simeq \frac{\mathbf{s}_{BS}(0) \sqrt{T}}{\sqrt{2\mathbf{p}}}$$

## Heston formula for expected volatility

- By a well-known formula

$$\sqrt{y} = \frac{1}{2\sqrt{p}} \int_0^\infty \frac{1 - e^{-ly}}{l^{3/2}} dl$$

- Then, taking expectations

$$\mathbb{E} \left[ \sqrt{\langle x \rangle_T} \right] = \frac{1}{2\sqrt{p}} \int_0^\infty \frac{1 - \mathbb{E} \left[ e^{-l \langle x \rangle_T} \right]}{l^{3/2}} dl$$

- We know the Laplace transform  $\mathbb{E} \left[ e^{-l \langle x \rangle_T} \right]$  from the CIR bond formula.
- So we can also compute expected volatility explicitly in terms of the Heston parameters.

## Results of volatility swap computation

- Applying the CIR bond formula and computing Heston one year expected volatility exactly gives

$$\mathbb{E}\left[\sqrt{\langle x \rangle_T}\right] = 0.187429$$

- Applying the SVI rotation recipe and computing expected volatility using the Carr-Lee weights gives

$$\mathbb{E}\left[\sqrt{\langle x \rangle_T}\right] = 0.185871$$

- Error is only 0.16 vol. points!!

## Summary

- Motivated by the asymptotic behavior of the implied volatility smile at extreme strikes, we introduced a parameterization (SVI).
- We demonstrated that this parameterization not only fits SPX option prices extremely well but that it fits option prices generated by many theoretical models. These models include stochastic volatility and pure jump models.
- By identifying the form of the implied volatility smile as SVI for the Heston model in the limit of infinite time to expiration, we were able to map the SVI parameters to parameters of the Heston model (in this limit).
- Noting that volatility derivatives (under a stochastic volatility assumption) cannot depend on the correlation between underlying returns and volatility changes, we developed a recipe transforming a given set of SVI parameters into their zero-correlation world equivalents.
- We then applied Carr-Lee to the option prices thus generated to value volatility derivatives.
- We demonstrated that in the particular case of Heston one year European options with BCC parameters, agreement was very close.



## References

- Bakshi, G., Cao C. and Chen Z. (1997). Empirical performance of alternative option pricing models. *Journal of Finance*, 52, 2003-2049.
- Carr, Peter and Lee, Roger (2003), Robust Replication of Volatility Derivatives, *Courant Institute, NYU and Stanford University*.
- Friz, Peter and Gatheral, J. (2004), Valuation of Volatility Derivatives as an Inverse Problem, *Merrill Lynch and Courant Institute, NYU*.
- Gatheral, J. (2003). Case studies in financial modeling lecture notes.  
[http://www.math.nyu.edu/fellows\\_fin\\_math/gatheral/case\\_studies.html](http://www.math.nyu.edu/fellows_fin_math/gatheral/case_studies.html)
- Heston, Steven (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies* 6, 327-343.
- Hull, John and White Alan (1987), The pricing of options with stochastic volatilities, *The Journal of Finance* 19, 281-300.
- Lee, Roger (2002). The Moment Formula for Implied Volatility at Extreme Strikes. Forthcoming in *Mathematical Finance*.
- Lewis, Alan R. (2000), *Option Valuation under Stochastic Volatility : with Mathematica Code*, Finance Press.
- Schoutens W., Simons E. and Tistaert J. (2004). A Perfect Calibration! Now What?, *Wilmott Magazine*, March, pages 66-78.