Gamma Convergence

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Gamma Convergence is a notion of convergence that fits naturally into the study of the asymptotic behavior of functionals. In particular, in this paper we are interested in studying integral functionals of this type

$$F(u) = \int_\Omega f(x, \nabla u)dx, \quad F(u) = \int_\Omega f(u, \nabla u)dx$$

the integrand function f, can depend on the spatial variable x and on the gradient of a function u, or on the function u and its gradient ∇u. Ω is a generic open set of $\mathbb{R}^N$
Functionals of this type are thought to be set in a certain space of functions X and therefore a functional is a function of functions, that associates a real number to each element of the space of functions X.
Suppose to do not have two single functionals but dealing with a sequence of functionals that depend on a parameter ϵ. The integrand function is supposed to be periodic in the first variable and the parameter ϵ looks like this form

$$F_\epsilon(u) = \int_\Omega f\left(\frac{x}{\epsilon}, \nabla u\right)dx, \quad F_\epsilon(u) = \int_\Omega f\left(\frac{u}{\epsilon}, \nabla u\right)dx$$

In general it is possible, given a phenomenon (for example a physical phenomenon), to model its energy through functionals of this type and therefore the interest in the study of functionals is linked to the study of a physical phenomenon and to the study of the properties of this physical phenomenon through the study of its energy. The presence of a small parameter ϵ can be determined by the fact that, for example, microscopic interactions are important to describe the energy of this phenomenon. So ϵ represents the microscopic scale and we describe the energy of the physical system through a functional that takes into account the microscopic structure. We are actually interested on the macroscopic behavior of the phenomenon and therefore what we want to give is a macroscopic description of it. Since generally between the macroscopic scale and the microscopic scale there is a substantial difference of orders of magnitude, we suppose that a good description of the macroscopic phenomenon exists for ϵ very small (i.e. making ϵ tend to zero) . This is why we are interested in studying a convergence of functions i.e. the asymptotic behavior for $\epsilon \to 0$ of functionals.
For example, suppose we have a point particle that moves along a straight line and suppose that t is the time and a $u(t)$ represents the position of my particle at the instant t. Suppose that the particle has unit mass and therefore the kinetic energy of the particle is $\frac{1}{2}\dot{u}(t)$ and then suppose that the particle is also subject to an external force (for example a gravity

force) and therefore a potential energy $gu(t)$. If we study the phenomenon for an interval of time $I = (0, T)$, the action of the phenomenon is the following

$$F_\epsilon(u) = \int_I \left(\frac{1}{2}\dot{u}(t) - gu(t)\right) dt$$

and its equilibrium configurations are, for the principle of least action, the minima of the action. Therefore, it is important in the description of the phenomenon to study the minima of this functional.
If, instead of having a functional, we have a sequence of functionals which describe the energy of my phenomenon, this notion of convergence must carry the information on the minima, which are the equilibrium configurations. Gamma Convergence is precisely a notion of convergence suitable for the study of minimization problems.

0.0.1 Direct methods of the calculus of variations

An example can be done considering a function $f : \mathbb{R} \to \mathbb{R}$ where we want to study the stationary points of it because, the equilibrium configurations, are the stationary points of the function and in particular the minimum points are the configuration of equilibrium stable.
Supposing that the function is regular, to find the stationary points, the following equation must be solved

$$f'(x) = 0 \tag{1}$$

with the condition that the derivative of the function exist must be done.
An analogous thing, can also be done when we want to study the problem of the stationary points of the equilibrium configurations of a system, whose energy is described by a functional. Naturally, it is necessary to specify in which space of functions X we are studying the problem and then it is necessary to give a notion of derivation for functionals.
If the functional is sufficiently regular to lend itself to this operation, the study of the stationary points of the functional leads it back to the study of determined equations such as (1) in which the unknown, this time, are functions and therefore they will be suitable differential equations, the so-called Euler equations. We pass from the study of the problem of the stationary points of a functional to the study of the existence of differential equations (the Euler equation).
This is the most classic approach which however has disadvantages because it requires a regularity of the functional. If the function is not smooth, the problem of stationary points (or minimum points) remains the same but we cannot deal with it in this way, so we have to see if it is possible to deal with it in another way.
Since precisely the definition of extremal point exists regardless of the regularity of a function, we can appeal for example to Weierstrass' Theorem. The 1D version of the Weierstrass' Theorem for function of $f : \mathbb{R} \to \mathbb{R}$ is the following

Theorem 0.1 (Weierstrass' Theorem)**.** *Let $[a,b] \subseteq \mathbb{R}$ be a non-empty closed bounded interval and let a function $f : [a,b] \to \mathbb{R}$ a continuous function. Then f will have at least a minimum and maximum absolute points in $[a,b]$*

in this theorem no derivability of the function is required but regularity properties (continuity) of it are required.

Definition 0.1. *A function is continuous if $x_n \to x$ then $f(x) = f(\lim x_n) = \lim f(x_n)$*

Since we are interested only in the study of the minimum problem, the continuity of the function is too restrictive condition. It is important to define the lower-Semi continuity that is a weak property.
The fact that $[a,b]$ should be a closed bounded interval, in a general space of functions does not mean anything. In a closed bounded interval happens that if we have a sequence in that interval, unless passing to a subsequence, can always assume that this sequence converges. When we are in $\mathbb{R}$, a numerical sequence can diverge, converge or oscillate (e.g. -1^n).
The property of compactness says that from every bounded sequence it is possible to extract a convergent subsequence.
Subsequence means that we take a subfamily of the family of the sequence (i.e. -1^n) made of infinity index chose as we want. For example, we can take the subfamily with all even indices and so we have the sequence $1, 1, 1...1$ that converge to 1. We can take the subfamily with all odd indices and so we have the sequence $-1, -1, -1... - 1$ that converge to -1. We can also chose a subsequence defining a law on the index that we will take (for example we will take one yes and five no). A subsequence is a sequence that we can build imposing a law on the index starting from a given sequence.
Not all subsequence will converge, but there will be at least one that is convergent.
We want to contextualize this result in the realm of functionals and at a certain point we have to say what is the space X of the functions we want to consider, we have to define a notion of convergence on this space X and starting from this notion of convergence we have to see when the these compactness properties are valid.

Definition 0.2 (Lower-Semi continuity)**.** *Let $F : X \to \mathbb{R}$ where X is a metric space. F is l.s.c. on X if $\forall x \in X$ and $\forall \{x_n\} \subseteq X$ s.t. $x_n \to x$, we have that $F(x) \leq \liminf_{n \to +\infty} F(x_n)$*

F in a functional and associate to a function of a certain space X a real number. A space X is metric when is defined a notion of convergence on it.
In the definition of continuous function there is the equality instead here there is an inequality and instead of limit there is a $\liminf$.
$\liminf$ is the least limit of convergent subsequences. We take all the subsequences, of a given sequence, that are convergent, then we take all the limits of the convergent subsequence and at the end we take the smallest one and this is the $\liminf$.

Definition 0.3 (Coercivity)**.** *Let $F : X \to \mathbb{R}$ where X is a normed space. F is coercive on X*
if $\exists \alpha > 0, \alpha \in \mathbb{R}$ s.t. $F(x) \geq \alpha ||x||_X$ when $||x||_X \to +\infty$

In a vectorial space V of $\mathbb{R}^N$ is it to possible to define a normal norm that is the euclidean norm, that measure the length of a vector. A normed space is a vectorial space in which is defined the notion of length of a vector. In our case the space X is a a space in which the vectors are functions and it is defined the length of a function (norm of a function).
If the norm of x diverge, then $F(x)$ is bigger then a fixed positive quantity α times the norm of x and so means that $F(x)$ will not remain limited but will diverge.
The coercivity property is connect to the property of compactness. On a normed space is defined a notion of convergence.

Theorem 0.2 (existence of minimum)**.** *Let $F : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, X is a reflexive normed space. Supposing that F is coercive and is l.s.c. respect to the weak convergence. Then the problem $\min\limits_{x \in X} F(x)$ has at least one solution.*

Considering X a normed space, in which there are some vector $x \in X$. It is possible to define the dual space of X, defined with X^*, that for definition is a normed vectorial space of continuous linear functionals on X.
A function is linear if preserve the property of vectorial space (sum of vectors and product of a scalar):

$$f(\alpha v_1 + \beta v_2) = \alpha f(v_1) + \beta f(v_2)$$

The dual of X^* is called bi-dual and is defined as X^{**}.
In general is always valid that $X \subseteq X^{**}$ but when X is reflessive is valid also the opposite inclusion, and so means that $X = X^{**}$.

Considering $X = \mathbb{R}$ and in this space is defined, from all the norms, a canonical norm that is the euclidean norm, which is the absolute value $|x|$ and respect to this is defined the notion of convergence of numeric sequences: $x_n \to x \Leftrightarrow |x_n - x| \to 0$.
In general, if X is a normed space (i.e. space of functions) on which is defined an opportune norm (that is not the absolute value of numbers but will be another norm), $\{x_n\} \subseteq X$, the notion of convergence connected with the norm is $||x_n - x||_X \to 0$. This is what is called strong convergence.

We will say that $x_n \rightharpoonup x$ weakly, if $\forall f \in X^*$ (where f is a linear functional of the dual) we have that $f(x_n) \to f(x)$. The function of the dual space are called test functions.
If a sequence converges in the sense of the norm (strongly) then it will converges also tested with any linear continuous functional (weakly). In the case of $\mathbb{R}^N$, the strong and weak convergence are the same thing. This is valid when the space is finite (N is a finite number), while is a property that is not valid in the space of infinite dimensions. The spaces of functions are spaces of infinite dimensions and so these two notion of convergence are different.

If we are in an infinite dimensional space, there is no opportunity to have property of compactness respect to the strong convergence, while in case of reflexivity the propriety of

compactness works well respect to the weak convergence.
Weak convergence means that we have a lot of convergent sequences and the compactness says that from every limited sequence we can extract a convergent subsequence. If we have a lot of sequences that converges, it will be easier to find one subsequence of my sequence.

The property of continuity are easier to achive when we have less test to do (less sequences to check).
The property of compactness and the property of continuity (and analogus semi continuity) are topological in competitions because in the case of the regularity properties of the function we would like to have as few convergent sequences as possible, so we run as few tests as possible and it is easier for the test to work (if we have to check fewer things it is easier for these things to go well) on the other hand though the properties of compactness to work instead require many convergent sequences. In reality the middle ground will be dictated by compactness.

Proof. Inside the theorem of existence of minimum, we supposed that the function can have also infinite value (because we chose $\overline{\mathbb{R}}$). There will be two cases:
1) $F(x) \equiv +\infty, \forall x \in X$ a constant functional with value $+\infty$. This is the trivial case because all point are points of minimum (and so at least one exist)
2) $\exists$ at least a point $x \in X$ s.t. $F(x) < +\infty \Rightarrow \inf_{x \in X} F(x)$ is finite (because inf is the largest value below which the function does not go).
In this second case, we can build a minimizing sequence $\{x_n\}$ that for definition is of this type:
$F(x_n) \leq \inf F + \frac{1}{n}$, where $\frac{1}{n}$ is a small quantity.
In general, whatever point x we are going to consider, the value of the functional (or function) at that point, is every time verified that $F(x_n) \geq \inf F$.
Since that $\inf F$ is finite, we can say that $F(x_n) \leq \inf F + \frac{1}{n} \leq C$ where C is a constant.
Since that F is coercive, that means that $\lim_{||x_n||_X \to +\infty} \frac{F(x_n)}{||x_n||_X} \geq \alpha > 0$, if the sequence $\{x_n\}$ was not limited (so the norm diverge), the functional $F(x_n)$ to satisfy the coercive property, should diverge also. The coercivity implies that $\{x_n\}$ is limited (the minimizing sequence are limitated).
If X is a reflexive normed space, then from any limited sequence it is possible to extract a weakly convergent subsequence with the limit in the space X we are considering.
From the reflexivity of $X \Rightarrow \exists\, \bar{x} \in X$ and a subsequence $\{x_{n_k}\}$ s.t. $\{x_{n_k}\} \rightharpoonup \bar{x}$ weakly.
At the end, from the weak l.s.c. we obtain $F(\bar{x}) \leq \liminf_{k \to +\infty} F(x_{n_k})$. For a generic point $\bar{x}$ of the space, it is always true that $F(\bar{x}) \geq \inf F$. In particular since that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ it is valid that $F(x_{n_k}) \leq \inf F + \frac{1}{n_k}$ where $\frac{1}{n_k}$ goes to 0, and so we can write that $\inf F \leq F(x_{n_k}) \leq \inf F$ but this means that $F(\bar{x}) = \inf F$.
This means that if there is a point in which we reach the $\inf F$, that point is a point of

minimum ($\bar{x}$ is a minimum point). $\square$

Definition 0.4 (strict convexity). *Let $F : X \to \overline{\mathbb{R}}$ where X is a vector space, F is strict convex if $\forall x, y \in X$ s.t. $F(x), F(y) < +\infty$, $x \neq y$ and $\forall \alpha \in (0, 1)$ we have*

$$F(\alpha x + (1-\alpha)y) < \alpha F(x) + (1-\alpha)F(y)$$

Remark. *Supposing that $\overline{x}_1, \overline{x}_2$, $\overline{x}_1 \neq \overline{x}_2$ are two minimum points for F on X. We call $\overline{x}$ the convex combination of $\overline{x}_1$ and $\overline{x}_2$ with coefficient i.e. $\alpha = 1/2$*

$$\overline{x} = \frac{1}{2}\overline{x}_1 + \frac{1}{2}\overline{x}_2$$

but from the strict convexity

$$F(\overline{x}) = F\left(\frac{1}{2}\overline{x}_1 + \frac{1}{2}\overline{x}_2\right) < \frac{1}{2}F(\overline{x}_1) + \frac{1}{2}F(\overline{x}_2) = \frac{1}{2}\min F + \frac{1}{2}\min F = \min F$$

but this is a contradiction because $F(\overline{x})$ should be $\geq$ that its minimum.
If the functional is strictly convex, there cannot be two distinct minimum points.

We are intresting to have functional of this kind

$$F(u) = \int_\Omega f(x, \nabla u)dx$$

where $\Omega \subseteq \mathbb{R}^N$ is an open, bounded and regular frontier (edge) set.
u is a function $u : \Omega \to \mathbb{R}^M$. The space of the functions X should be a Sobolev space. Every time we will require that either the size of the starting space N, or the size of the ending space M, should be one (so, either at the start they are vector functions of one variable or they are scalar function of multi variables).
The space $C^k(\Omega; \mathbb{R}^M)$ is a space of functions in which they have regular derivative, instead the Sobolev space like $W^{1,p}(\Omega; \mathbb{R}^M), W_0^{1,p}(\Omega; \mathbb{R}^M)$ with $1 < p < \infty$, the functions are derivable in an integral sense.
When the arrival space is vectorial, we have to indicate the dimension of the space, unless we can indicate in this way $W^{1,p}(\Omega)$.
A space of summable functions is defined as

$$L^p(\Omega) = \left\{ f : \int_\Omega |f|^p dx < +\infty \right\}$$

Since that $\Omega \subseteq \mathbb{R}^N$ that meas an integration in N variables.

$$W^{1,p}(\Omega) = \left\{ f \in L^p(\Omega), \nabla f \in L^p(\Omega) \right\}$$

$$W_0^{1,p}(\Omega) = \left\{ f \in W^{1,p}(\Omega) : f = 0 \,\text{at}\, \partial\Omega \right\}$$

The Sobolev Spaces are vector spaces and so if we do the sum or more in general the linear combination of functions, we have the same properties. Sobolev spaces are also normed space and so we can introduce a norm.
Supposing to have a sequence of functions $\{u_n\} \subseteq W^{1,p}(\Omega), u \subseteq W_0^{1,p}(\Omega)$.
We know that $u_n \to u$ strongly $\Leftrightarrow ||u_n - u|| \to 0$.
The norm of a generic function $f \in W_0^{1,p}(\Omega)$ is defined in this way

$$||f|| = \Big(\int_\Omega |f|^p dx \Big)^{1/p} + \Big(\int_\Omega |\nabla f|^p dx \Big)^{1/p}$$

In all normed space, is possible to define a weak convergence. To understand what $u_n \rightharpoonup u$ weakly, we need to use the test functions that are element of the dual of $W_0^{1,p}(\Omega)$. The dual of $W_0^{1,p}(\Omega)$, that for abbreviation we call it as X^* and it is identified with another Sobolev space but the power is not p but q, $X^* = W^{1,q}(\Omega)$.
The relation between q and p follow the following equation

$$\frac{1}{p} + \frac{1}{q} = 1$$

It is very easy to notice that if $p = 2$ that $q = 2$.
X^* is the space of linear functionals and $u_n \rightharpoonup u$ weakly, means that a sequence function u_n will converge to a function u if u_n will converge tested through the linear functionals of the dual (tested means just multiply the sequence function u_n by the test function) in this way:
If $\forall g \in W^{1,q}$, where g are the test functions, we have

$$\int_\Omega g(x)u_n(x)dx + \int_\Omega \frac{\partial g(x)}{\partial x_j}\frac{\partial u_n(x)}{\partial x_j}dx \to \int_\Omega g(x)u(x)dx + \int_\Omega \frac{\partial g(x)}{\partial x_j}\frac{\partial u(x)}{\partial x_j}dx$$

If we have a sequence that strongly converge, this will of course converge also in the case of the test function (weakly) but the vice-versa is not true.
An example can be seen showing the Riemann-Lebesgue's Lemma.

Lemma 0.3 (Riemann-Lebesgue's Lemma)**.** *Let the sequence of functions $u_n(x) = sin(nx)$. As can be seen $u_n(x) \in W^{1,p}$ and this will not converge in the sense of the norm (strong convergence) because oscillate when $n \to +\infty$. If we test $u_n(x)$, it will converge to the function 0: (in weak sense)*

$$\int_\mathbb{R} g(x)sin(nx)dx \to \int_\mathbb{R} g(x)0dx = 0 \qquad \forall g(x) \in W^{1,q}$$

This result can be generalized to any periodic function. In reality, 0 represent the average of the function

$$\frac{1}{2\pi}\int_0^{2\pi} sin(x)dx = 0$$

Now, we want to study the following problem of the minimum of the functional

$$\min_{u \in W_0^{1,p}(\Omega)} \int_\Omega f(x, \nabla u) dx$$

where f should be a Carathéodory function. It is expressed as $f(x, \xi)$, where the function should be mesurable respect to the variable x and continua from the variable ξ. In general (but not always true) all the functions are mesurable.
The important factors for a problem of minimization are the coercivity and the lower semi continuity.

Lemma 0.4 (Fatou's Lemma). *Let a sequence of functions $\{f_n\}$ that are mesurabes and $f_n(x) \geq 0$. So*

$$\int_\Omega \left(\liminf_{n \to +\infty} f_n(x) \right) dx \leq \liminf_{n \to +\infty} \int_\Omega f_n(x) dx$$

lim not always exist, because exist only if the sequence diverge or converge, instead lim inf always exist because is the smallest limit of the convergent or divergent subsequences. The consequence of this lemma is that we are able to obtain the l.s.c. respect to the convergence given by the norm (strong convergence).
Let a sequence of function of Sobolev u_n that converges to the Sobolev norm $u_n \to u$ in $S - W_0^{1,p}(\Omega)$, where S means strong convergence, this implies that $||\nabla u_n - \nabla u||_{L^p} \to 0$ (since that the functional that we are considering depends only on the gradient of u, we focus the attention only on the part of the gradient). This implies that $\nabla u_n(x) \to \nabla u(x)$ almost everywhere in Ω (we have a punctual convergence).
Supposing that $f \geq 0$,since that the function f in the variable ξ is continue, for definition we have $f(x, \nabla u_n(x)) \to f(x, \nabla u(x))$.
Applying the Fatou' lemma

$$F(u) = \int_\Omega \left(\liminf_{n \to +\infty} f(x, \nabla u_n(x)) \right) dx = \int_\Omega f\left(x, \nabla u(x)\right) \leq \liminf_{n \to +\infty} \int_\Omega f(x, \nabla u_n(x)) dx = \liminf_{n \to +\infty} F(u_n)$$

In the convergence of the norm there is no hope of having the compactness. The fact that we have the l.s.c. respect to the strong convergence is not enough.
The goal is to find some conditions in such a way we have the l.s.c. respect to the weak convergence.
We need to guarantied that the function is controlled from up and down (because we have the modulus):

$$\exists \lambda > 0 s.t. |f(x, \xi)| \leq \lambda(1 + |\xi|^p) \tag{2}$$

The function $f(x, \xi)$ should be measurable respect to the variable x, continua and convex from the variable ξ.

Theorem 0.5 (Tonelli's Theorem). *Let $f : \Omega \times \mathbb{R}^{M \times N} \to \mathbb{R}$ (with M or N equal to 1) be a Carathéodory function that satisfay (2). So the following statements are equivalents:*
1) $f(x, \cdot)$ is convex
2) F is convex
3) F is l.s.c. respect to the weak convergence.

The convexity of the integrated function implies the convexity of the functional is easy to proof because the integral is linear. We know that if f is of charateodory and there are some condition of controll (i.e. (2)), we have the l.s.c. respect to the strong convergence (from Fatou's lemma). If we add also the convexity (the function f is also convex), this implies also that we have the l.s.c. from the weak convergence.
The fact that 2) imply 3) is general. The fact that 3) implies 2) is fundamental to have that or M or N that are one. This means that the condition of convexity, according to this theorem, it is a necessary and sufficient condition for the l.s.c. weak of the functional.

Theorem 0.6. *Let $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ a Carathéodory function, convex on the second variable ξ and s.t. $\exists$ a $0 < \lambda \leq \Lambda < +\infty$ s.t.*

$$\lambda(|\xi|^p - 1) \leq f(x, \xi) \leq \Lambda(1 + |\xi|^p)$$

So, the problem of minimum

$$\min_{u \in W_0^{1,p}(\Omega)} \int_\Omega f(x, \nabla u)dx = \min_{u \in W_0^{1,p}(\Omega)} F(u)$$

admits solution (at lest a minimum)

Definition 0.5 (Poincare's inequality)**.** *If $u \in W_0^{1,p}(\Omega)$ then*

$$\int_\Omega |u|^p dx \leq C \int_\Omega |\nabla u|^p dx$$

This basically said me that, if we use $\int_\Omega |\nabla u|^p dx$, we have a norm equivalent at that one in which we use function u plus its gradient, because the function u can be controlled by its gradient, using this inequality.
So in $W_0^{1,p}$, we can take the following norm

$$||u||_{W_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u|^p dx \right)^{1/p}$$

Now we want to proof the theorem. If the function f not only is convex in ξ but is strictly convex, the minimum is unique.
We know that is l.s.c. respect to weak convergence but now we have to verify the coercivity

Proof. The space in which we are working is a reflexive space because the dual of $W_0^{1,p}$ is $W^{1,q}$ and doing again the dual we have again $W_0^{1,p}$. This is valid if $1 < p < +\infty$.
The functional F is l.s.c. weakly (from Tonelli's theorem).
To guarantied the coercivity we have to satisfy that

$$\lim_{||u||_X \to +\infty} \frac{F(u)}{||u||_{X = W_0^{1,p}}} \geq \alpha > 0$$

but using the Poincare's inequality, we have to verify that

$$\lim_{||u||_X \to +\infty} \frac{F(u)}{\left(\int_\Omega |\nabla u|^p dx\right)^{1/p}} \geq \alpha > 0$$

To shows that this is true we can use the condition of the control of the function f from the bottom, that follows this inequality $\lambda(|\xi|^p - 1) \leq f(x, \xi)$. Doing the integral of both members of the inequality

$$F(u) = \int_\Omega f(x, \nabla u)dx \geq \lambda \left(\int_\Omega |\nabla u|^p dx - |\Omega| \right)$$

and then dividing by the norm of u, yields

$$\frac{\int_\Omega f(x, \nabla u)dx}{\left(\int_\Omega |\nabla u|^p dx\right)^{1/p}} \geq \lambda \frac{\left(\int_\Omega |\nabla u|^p dx - |\Omega|\right)}{\left(\int_\Omega |\nabla u|^p dx\right)^{1/p}} = \lambda \left(\int_\Omega |\nabla u|^p dx \right)^{1-\frac{1}{p}} + O(\lambda)$$

When we do the limit of the norm that tend to infinity, we obtain that $\lambda \left(\int_\Omega |\nabla u|^p dx\right)^{1-\frac{1}{p}} \to +\infty$, and so f is coercive.
Since that it is guarantied that we have l.s.c. respect to the weak convergence, the reflexivity of the space that guarantied that the limited sequences converge in the space and the coercivity that guarantied that the minimizing sequences are limited, the proof is done. □

The theorem that admits solution was done posing the condition that $X = W_0^{1,p}$. That theorem is also true when instead to have a 0 condition at the boundary of the domain Ω, we have the following space $X = W_0^{1,p} + g = W_g^{1,p}$ where g is a function of $W^{1,p}$ ($g \in W^{1,p}$). This means that instead to have 0 at the boundary, we have the function g.

Theorem 0.7 (Immersion Sobolev theorem). *If $\{u_n\} \subseteq W^{1,p}(\Omega)(or\, W_0^{1,p}(\Omega))$ and $u_n \rightharpoonup u \in W^{1,p}(\Omega)(or\, W_0^{1,p}(\Omega))$ weakly, so this implies that $u_n \to u$ strongly in $L^p(\Omega)$*

This is just a small part of the Immersion Sobolev theorem. Basically on the gradients remains the weak convergence but on the function u we have a strong convergence in L^p.

Definition 0.6 (Hölder's inequality). *Let $f \in L^p$ and $g \in L^q$ with p, q coniugate exponents from the equation $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$fg \in L^p \ , \quad \int_\Omega |fg| \leq ||f||_{L^p} ||g||_{L^q}$$

We want to avoid that $p = 1$ because $W^{1,1}$ is not a reflexive space.
We know that if we have a limited sequence in $W^{1,p}$ with $p > 1$, we have a subsequence theta converge weakly in $W^{1,p}$. When $p = 1$ this result of compactness is partial because from a limited sequence in $W^{1,1}$ we can only obtain the information that the sequence will converge in L^1. We cannot sustains that the limit is in $W^{1,1}$. The compactness theorem says only that the sequence will strongly converge in L^1. We do not have the weak convergence of the sequence in $W^{1,1}$ and so the Immersion Sobolev theorem cannot be applied.

0.1 Example

Supposing to have an homogeneous isolated (no forces acting on it) electric conductor Ω with electric potential u, so u is a scalar function $u : \Omega \to \mathbb{R}$.
The potential energy of the conductor is governed by the following functional

$$F(u) = \int_\Omega |\nabla u|^2 dx$$

The integrand function is $f(x, \xi)$, but in reality the function f will not depends on x because the conductor is homogeneous and so in each point the electrical conduction is the same, but depends only on the gradient of u. So $f(x, \xi) = |\xi|^2$ that is a parabola that is strictly convex. Since that $f(x, \xi) = |\xi|^2$, it will check the controlling conditions from the top and bottom. The space is $W_g^{1,2}$ and since that the exponent $p = 2$, this space can be also called with H_g^1. On the space, the functional is coercive, convex, the space is reflexive and so the problem of the minimum is solvable and the minimum is unique because the function is strictly convex (there just one configuration of stable equilibrium that is also of equilibrium stable of the potential).

Supposing now that the conductor is not isolated but there is a source of electric field from the outside. At that energy we have to add another part

$$G(u) = \underbrace{\int_\Omega |\nabla u|^2 dx}_{F(u)} - \underbrace{\int_\Omega gudx}_{H(u)}$$

where g is the external electric source (perturbation) with $g \in L^2$. $H(u)$ as a continuous perturbation respect at the weak convergence. The presence of g will not give a trivial solution if we are in the space $H_0^1(\Omega)$. We know from the previous discussion that $F(u)$ is a l.s.c. respect to weak convergence. $H(u)$ is continuous from the Immersion's theorem, because if u_n converges weak to u in $H_0^1(\Omega)$, then u_n will converge strong to u in L^2.
we can say that

$$\int_\Omega gu_n dx \to \int_\Omega gudx$$

The new functional $G(u)$ is functional l.s.c. It is equivalent to say that

$$\Big| \int_\Omega gu_n dx - \int_\Omega gudx \Big| \to 0$$

Since that the integral is a linear operator we can write it as

$$\Big| \int_\Omega g(u_n - u)dx \Big| \leq \int_\Omega |g||u_n - u|$$

Now we can use Hölder's inequality

$$\Big| \int_\Omega g(u_n - u)dx \Big| \leq \int_\Omega |g||u_n - u| \leq ||g||_{L^2} ||u_n - u||_{L^2}$$

But since that $u_n \to u\ S-L^2$, that means $||u_n - u||_{L^2} \to 0$ and so the previous quantity goes to 0 and so $H(u)$ is continuous respect to the strong convergence of L^2 and so also respect to the weak convergence of $H^1(\Omega)$.
Now we have to guarantied the corcivity of $G(u)$ and we want so see if this is valid

$$G(u) = \int_\Omega |\nabla u|^2 dx - \int_\Omega g\, u\, dx \geq \alpha\,(||\nabla u||_{L^2} - 1)$$

Using the Hölder's inequality

$$\int_\Omega g\, u\, dx \leq ||g||_{L^2}||u||_{L^2}$$

And knowing that

$$(|a| - |b|)^2 \geq 0 \Rightarrow a^2 + b^2 - 2|ab| \geq 0 \Rightarrow ab \leq |ab| \leq \frac{1}{2}\left(a^2 + b^2\right)$$

The absolute value of the product of two numbers, can be always increased it, up to a constant that in this case is $\frac{1}{2}$, as the sum of the square of them.
Since that $||g||_{L^2}, ||u||_{L^2}$ are two numbers, we can apply the same thing. We can multiply and divide by δ (a small constant quantity) to have

$$\int_\Omega g\, u\, dx \leq \frac{1}{\delta}\,(||g||_{L^2})\,(\delta||u||_{L^2}) \leq C\left(\frac{1}{\delta^2}||g||^2_{L^2} + \delta^2||u||^2_{L^2}\right)$$

Now applying the Poincare's inequality for u

$$\int_\Omega g\, u\, dx \leq \frac{1}{\delta}\,(||g||_{L^2})\,(\delta||u||_{L^2}) \leq C\left(\frac{1}{\delta^2}||g||^2_{L^2} + \delta^2||u||^2_{L^2}\right) \leq C\left(\frac{1}{\delta^2}||g||^2_{L^2} + \delta^2||\nabla u||^2_{L^2}\right)$$

and so yields

$$G(u) = \int_\Omega |\nabla u|^2 dx - \int_\Omega g\, u\, dx \geq ||\nabla u||^2_{L^2} - C(\delta)||g||^2_{L^2} - C\delta^2||\nabla u||^2_{L^2} = (1 - C\delta^2)(||\nabla u||^2_{L^2} - C(\delta, g)$$

where $C(\delta) = C\frac{1}{\delta^2}$ is a constant number that depends on δ and $C(\delta, g)$ is another constant that depends on δ and g. Now to have the coercivity, is necessary to have that $\alpha = (1 - C\delta^2) > 0$ and so chose $\delta < \sqrt{1/C}$.

Remark. *A sufficient but absolutely non necessary for the uniqueness of the solution, is that the integrand function should be strictly convex in the variable ξ. This is not necessary because an example can be done choosing the function $f(\xi) = |\xi|$ in which is not convex but has a minimum.*

Theorem 0.8. *Let M=1 (dimension of the arrival space) and let* $1 < p < +\infty$, $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ *a Carathéodory function s.t. exist two constants* λ, Λ, $0 < \lambda \leq \Lambda < +\infty$, *with the property that*
$\lambda(|\xi|^p - 1) \leq f(x, \xi) \leq \Lambda(|\xi|^p + 1), \forall \xi \in \mathbb{R}^N$ *and for almost* $x \in \Omega$. *Assuming that* f *is convex in the second variable, then the problem*

$$\min_{u \in W_0^{1,p}(\Omega)} \int_\Omega f(x, \nabla u) dx$$

have at least one solution.

0.1.1 Example with no corcivity of functional

Let $\Omega = I = (0, 1)$ (in this case N=M=1 that means scalar both in the arriving and in the departing), consider the following functional

$$F(u) = \int_I x(u')^2 dx$$

This functional is well defined if $u \in H_0^1(\Omega) + \phi$, with $\phi = 1 - x$ and we want to study the following problem of minimum

$$\min_{H_0^1(\Omega)+\phi} F(u)$$

We are taking the functions u of $H^1(\Omega)$ in which their value at the boundary is dictated from the ϕ, which means $u(0) = 1, u(1) = 0$.
As can be notice the functional is $F(u) \geq 0$. To find the inf of $F(u)$, we need to take a sequence u_n and for all u_n we will take, is always true that

$$F(u_n) \geq \inf_{u \in H_0^1(\Omega)+\phi} F(u)$$

But since that the functional $F(u) \geq 0$ we can also say that

$$F(u_n) \geq \inf_{u \in H_0^1(\Omega)+\phi} F(u) \geq 0$$

We choose the following sequence of functions

$$u_n = \begin{cases} 1 & 0 \leq x \leq \frac{1}{n} \\ -\frac{log(x)}{log(n)} & \frac{1}{n} \leq x \leq 1 \end{cases}$$

for each n, u_n is a function of the space because satisfy the boundary conditions. It is integrable with the power that we want and so is inside L^2. If we do the derivative (in the sence of Sobolve space) of this function is also integrable with all the powers that we want and so we can conclude that $u_n(x) \in H_0^1(\Omega) + \phi$

$$F(u_n) = \int_0^1 x(u_n')^2 dx$$

but the derivative of the u_n is zero for $0 \leq x \leq \frac{1}{n}$ and so we can also write that

$$F(u_n) = \int_{1/n}^{1} x \left(\frac{1}{xlog(n)} \right)^2 dx = \frac{1}{xlog^2(n)} \int_{1/n}^{1} \frac{1}{x} dx = \frac{1}{xlog^2(n)} \Big[log(x) \Big]_{1/n}^{1} = \frac{1}{log(n)} \to 0$$

The functional F, evaluated on this sequence of functions that belong to the space, tends to 0 and this implies that

$$\inf_{H_0^1(\Omega)+\phi} F = 0$$

If the minimum exist, it will be coincident with the inf. But the minimum cannot be 0 because $F(u) = 0$ can happen if we impose that $u' = 0$ almost everywhere. This cannot be satisfied because the function is not constant due to the fact that we imposed the boundary conditions in which the function is different from each other.
$\nexists u \in H_0^1(\Omega) + \phi$ s.t. $F(u) = 0$ and so the problem of minimum does not have solution.
The fact that there is no solution is related to the fact that the functional $F(u)$ is not coercive.
To have that the functional is coercive, we must guarantide that

$$F(u_n) \geq \lambda(||u_n||_{H^1} - 1)$$

Since that u_n satisfy the value at the boundary, the norm of u_n can be controlled in this way using the triangle inequality

$$||u_n||_{H^1} \leq ||u_n - \phi||_{H_0^1} + ||\phi||_{H^1}$$

where $||\phi||_{H^1}$ is a constant and C is an arbitrary constant. Applying Poincare, we can say that

$$||u_n||_{H^1} \leq ||u_n - \phi||_{H_0^1} + ||\phi||_{H^1} \leq C||\nabla u_n - \nabla\phi||_{L^2} + ||\phi||_{H^1}$$

and applying again the triangle inequality

$$||u_n||_{H^1} \leq ||u_n - \phi||_{H_0^1} + ||\phi||_{H^1} \leq C||\nabla u_n - \nabla\phi||_{L^2} + ||\phi||_{H^1} \leq C||\nabla u_n||_{L^2} + C||\phi||_{H^1}$$

the norm of u_n in H^1, except for a constant C, can be controlled with the norm of its gradient.
Now let's calculate the derivative of u_n

$$\int_0^1 |u_n'|^2 dx = \int_{1/n}^{1} \left| \frac{1}{xlog(n)} \right|^2 dx = \frac{1}{= log^2(n)} \int_{1/n}^{1} \frac{1}{x^2} dx = -\frac{1}{log^2(n)} + \frac{n}{log^2(n)}$$

and for $n \to +\infty$, that derivative will diverge.
And so we cannot verify the coercivity because $F(u_n) \to 0$ and $\lambda(||u_n||_{H^1} - 1) \to +\infty$.

0.1.2 Example with no convexity of the functional

Losing the convexity means that we are losing the l.s.c. property Let's take the following functional in the real interval $\Omega = I = (0,1)$

$$F(u) = \int_0^1 \left(1-(u')^2\right)^2 dx + \int_0^1 u^2 dx$$

the space is $X = W_0^{1,4}(I)$ (there is 4 because inside the functional we have u^4).
The problem of non convex is relates to the energy part $\int_0^1 ((1-(u')^2)^2)$ and so this part is not l.s.c., instead $\int_0^1 u^2$ is continuous and will not give me problems. Globally the functional $F(u)$ is not l.s.c.
Globally we can say that $F(u) \geq 0$ and the inf is 0.
Let's choose the following function

$$u(x) = \begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ 1-x & \frac{1}{2} \leq x \leq 1 \end{cases}$$

The minimizing sequence we choose in this way

$$u_n(x) = \frac{1}{n} u(n\,x)$$

where the characteristic is that $(u_n') = 1 \,\text{or}\, -1$ and so

$$F(u_n) = \underbrace{\int_0^1 \left(1-(u_n')^2\right)^2 dx}_{=0} + \int_0^1 u_n^2 dx \to 0$$

and so this implies that $\inf\limits_{W_0^{1,4}(I)} F = 0$ but this value cannot be reached (it is not the minimum) because $u_n' \neq 0$ and so the problem of minimum cannot be solved.

In the next example we want to show that even if we do not have the convexity (we do not have l.s.c.) sometimes the minimum exist. The fact that the hypothesis are not satisfy, does not mean that we do not have the minimum but says that we cannot guarantied the existence of the minimum.
Let take the following functional

$$F(u) = \int_0^1 \left(1-(u')^2\right)^2 dx$$

this problem have infinite minimum, the functional is coercive but since the the integrand function is not convex this implies that the functional is not l.s.c.
There are some cases in which $p = 1$ and so to be in $W^{1,1}$. If we have the correct condition of growing of the integrand function, this is not sufficient to guarantied the compactness of the minimizing sequences because the space is not good, because $W^{1,1}$ is not a reflexive space

and there is not valid the result of compactness.
From limited sequence in $W^{1,1}$, we can only say that there will exist a limit function of L^1 in which the sequence will converge strong in L^1 (we are losing the informations of the derivatives). To recover these limit functions(to recover the compactness)that are no more in $W^{1,1}$ we have to chose a bigger space. The thing that is done is to extend the functional on this bigger space.
There is a method called theory of relaxation. The idea is to pass to a new functional.

Let's assume to have a functinal F defines on a metrix space X with values in $\mathbb{R}$, $F : X \to \overline{\mathbb{R}}$. We define the relaxed of F (indicated with $sc^- F$ or $\overline{F}$) the biggest functional l.s.c. which minors F. $sc^- F$ is l.s.c., $sc^- F \leq F$. Of course if F is l.s.c., it will be coincide with its relaxed.
The relaxed function can be characterized through the sequences in this way, let fixed $u \in X$

$$sc^- F(u) = \inf\{\liminf_{n\to+\infty} F(u_n), u_n \to u\, in\, X\}$$

we are fixing a point u of X, in which on this metric space we have a notion of convergence, we will take all the sequences u_n that will converge to u, we do $F(u_n)$ and for each u_n we do the liminf and we take the inf of these and this for definition is the relaxed of F in the point u.
Another equivalent way to characterize the relaxed of a functional, is the following:
we need to guarantide these two conditions:
1)$\forall u \in X$ and $\forall u_n \to u$ we have that

$$sc^- F(u) \leq \liminf_{n\to+\infty} F(u_n)$$

2) $\forall u \in X$, $\exists \overline{u}_n \to u$ s.t.

$$sc^- F(u) \geq \limsup_{n\to+\infty} F(\overline{u}_n)$$

The limsup is the sup of the all limits of all subsequences and is always true that $\limsup \geq \liminf$. In reality the second condition can be written in this way

$$sc^- F(u) = \lim_{n\to+\infty} F(\overline{u}_n)$$

we call $\overline{u}_n$ optimal sequence.
Some properties of the relaxed: if G is continuous $\Rightarrow sc^-(F+G) = sc^- F + G$. This means that adding a continuum perturbation to the functional F, will not change the problem of the relaxation.
we want that, supposing that exist, $minsc^- F = infF$. This will happens under some hypothesis (is not always true).

The problem works in this way: we have the problem of min of F that cannot be solved with direct methods because F does not satisfy the request of existence of minimum and so we

relax the problem passing to the relaxed of F. Now for definition the relaxed is l.s.c. but we have also to guarantied that will be coercive in such a way the problem of minimum have a solution and so if all these requests are satisfied we can say that $minsc^{-}F = infF$. The idea now is to look who are the minimums of the relaxed functional and we go to see if in these minimums there is some functions that in reality will cause F to reach the inf and so is a minimum for F. The min of F are recovered looking for the minimums of the relaxed functions.
There is a property that says that if F has a minimum, then that minimum is always in its relaxed.

If we are in the situation in which $X = W_0^{1,p}(\Omega)$ and $F(u) = \int_\Omega f(x, \nabla u)dx$ in which f is a Carathéodory function. If in $W_0^{1,p}(\Omega)$ we consider the strong convergence and $f \geq 0$, then the Fatou's lemma will guarantied me the l.s.c. of the functional $F(u)$ respect to the strong convergence and so we do not need to relax. The problem of the strong convergence is that we do not have the compactness and so we want to use the weak convergence and to have the l.s.c. we need that the functional should be convex (that for the strong convergence is not needed).
The problem is that with the weak convergence, the characterization of the relaxed that we wrote (using the sequences) is not appropriate.
This because in the definition we supposed that X is a metric space, but the weak convergence it is not induced by a metric topology, the characterization of the relax respect to the sequences it is not a real characterization. To recover this problem and to use the sequence is done imposing some conditions of growing.

Theorem 0.9. *Let $f : \Omega \subseteq \mathbb{R}^N \to \mathbb{R}$ a Carathéodory function s.t. exist two constants λ, Λ, $0 < \lambda \leq \Lambda < +\infty$, with the property that*
$\lambda(|\xi|^p - 1) \leq f(x, \xi) \leq \Lambda(|\xi|^p + 1), \forall \xi \in \mathbb{R}^N$ and for almost every $x \in \Omega$. Then

$$sc_{L^p}^{-}F(u) = sc_{W^{1,p}}^{-}F(u) \quad \forall u \in W_0^{1,p}$$

where $sc_{L^p}^{-}F(u)$ is the relaxed of $F(u)$ with respect to the strong convergence of L^p and $sc_{W^{1,p}}^{-}F(u)$ is the relaxed of $F(u)$ with respect to the weak convergence of $W^{1,p}$. The first is well characterized with the sequences, because the strong topology of L^p will give a metric, instead the other term no but in this case yes. So when we talk about relaxed in a cortex of this kind, we can use both without any difference.

Proof. $sc_{L^p}^{-}F(u) \leq sc_{W^{1,p}}^{-}F(u)$ is true because for definition $sc_{L^p}^{-}F(u) = \inf\{\liminf_{n\to+\infty} F(u_n), u_n \to u(strongly)\, in\, L^p\}$, instead $sc_{W^{1,p}}^{-}F(u) = \inf\{\liminf_{n\to+\infty} F(u_n), u_n \rightharpoonup u(weakly)\, in\, W^{1,p}\}$ but the Immersion theorem says me that if we have a sequence that converges weakly in $W^{1,p}$, then it will converge strongly in L^p. So, all the sequences that we need, there are in $sc_{W^{1,p}}^{-}F(u)$, surely will stay also in $sc_{L^p}^{-}F(u)$ (then inside $sc_{L^p}^{-}F(u)$ we have also all the sequence that converge strongly only in L^p that in general are not present in $sc_{W^{1,p}}^{-}F(u)$). So, when we do

the inf of $sc^-_{L^p}F(u)$, we are doing the inf on a bigger set and so is a lower inf respect to the inf of $sc^-_{W^{1,p}}F(u)$.
Now, to show the condition that $sc^-_{L^p}F(u) \geq sc^-_{W^{1,p}}F(u)$ we have to use the growing conditions.
Of course we must impose that $sc^-_{L^p}F(u) \leq +\infty$ because if is $+\infty$ is always the biggest. From the characterization of the relaxed respect to the topology L^p we know that $\exists\, \overline{u}_n \to u in L^p$ s.t. $F(\overline{u}_n) \to sc^-F(u)$ that is finite. Since that $sc^-F(u) < +\infty \Rightarrow \exists C s.t. C \geq F(\overline{u}_n)$. But we know that F should satisfy the growing conditions, an so we can say that:

$$C \geq F(\overline{u}_n) \geq \lambda\left(\int_\Omega |\nabla\overline{u}_n|^p - |\Omega|\right) \Rightarrow ||\overline{u}_n||_{W^{1,p}} \leq C$$

and from the compactness theorem, if we have a sequences that is equilimited in $W^{1,p} \Rightarrow \overline{u}_n \rightharpoonup u(weakly)\, in\, W^{1,p}_0$
So endly

$$sc^-_{W^{1,p}}F(u) = \inf\liminf F(u_n) \leq \liminf F(\overline{u}_n) = \lim F(\overline{u}_n) = sc^-_{L^p}F(u)$$

□

Theorem 0.10. *Let $X = W^{1,p}_0(\Omega)$, f with the same conditions of before, and sc^-F is the relaxed of F in $W^{1,p}_0(\Omega)$. Then the problem*

$$\min_{u\in X} sc^-F(u)$$

have solution. We have some connection between the minimum problem of the relaxed functional and the inf of the starting problem that a priory cannot have solutions. These are:
1)$\min sc^-F = \inf F$
2)if u_0 is a minimum point for sc^-F, exist a minimizing sequence of F that converges to u_0
3)every minimizing sequence of F, converges at a minimum for sc^-F.

Proof. Here we want to proof the 1) of the theorem.
we want to proof that the relaxed of F satisfy the same growing conditions of F. we have soon the coercivity and since that the relaxed for definition is l.s.c. , and of course the space is reflexive, the existence of the minimum is guarantied.
Integrating the growing conditions satisfied from the integrand function f we obtain

$$\lambda\left(\int_\Omega |\nabla u|^p - |\Omega|\right) \leq F(u) \leq \Lambda\left(\int_\Omega |\nabla u|^p + |\Omega|\right)$$

we know that $sc^-F(u)$ is always the biggest functional l.s.c. that is under F, and so we write

$$sc^-F(u) \leq F(u)$$

but since that $F(u) \leq \Lambda\left(\int_\Omega |\nabla u|^p + |\Omega|\right)$ the inequality from the top is automatic.
From bottom instead we can take the optimal sequence

$$sc^-F(u) = \lim_{n\to+\infty} F(\overline{u}_n)$$

and since that F satisfy the conditions of growing, we can say that

$$sc^{-}F(u) = \lim_{n\to+\infty} F(\overline{u}_n) \geq \liminf_{n\to+\infty} \lambda(||\nabla\overline{u}_n||^p_{L^p} + |\Omega|)$$

In general the norm in every space, is l.s.c. respect to the weak convergence. Applying this property we can write that

$$sc^{-}F(u) = \lim_{n\to+\infty} F(\overline{u}_n) \geq \liminf_{n\to+\infty} \lambda(||\nabla\overline{u}_n||^p_{L^p} + |\Omega|) \geq \lambda(||\nabla\overline{u}||^p_{L^p} + |\Omega|)$$

At the end, we can write

$$\lambda\left(\int_\Omega |\nabla u|^p - |\Omega|\right) \leq sc^{-}F(u) \leq \Lambda\left(\int_\Omega |\nabla u|^p + |\Omega|\right)$$

□

Proof. Here we want to proof the 2) of the theorem.
The relaxed of $F(u)$ is coercive, the space $W^{1,p}_0(\Omega)$ is reflexive, the relaxed for definition is l.s.c. and so the problem of minimum has a solution.

Supposing that u_0 is a minimum point for the relaxed $sc^{-}F$. Let's take the optimal sequence $\overline{u_n} \to u_0$. It is possible to write that

$$\min_X sc^{-}F = \min_X sc^{-}F(u_0)$$

Since that $\overline{u_n}$ is the optimal sequence we can write that

$$\min_X sc^{-}F = \min_X sc^{-}F(u_0) = \lim_{n\to+\infty} F(\overline{u}_n)$$

Since that $\overline{u}_n$ is an element of X, $F(\overline{u}_n)$ will be always bigger or equal that inf of F

$$\min_X sc^{-}F = \min_X sc^{-}F(u_0) = \lim_{n\to+\infty} F(\overline{u}_n) \geq \inf_X F$$

Since that the relaxed is below F (less or equal of F), then inf F and inf $sc^{-}F$ must satisfy the same inequality

$$\inf_X F \geq \min_X sc^{-}F = \min_X sc^{-}F(u_0) = \lim_{n\to+\infty} F(\overline{u}_n) \geq \inf_X F$$

But this implies two things:

$$\inf_X F = \min_X sc^{-}F$$

$$\lim_{n\to+\infty} F(\overline{u}_n) = \inf_X F$$

and so $\overline{u}_n$ is a minimizing sequence because for definition the minimizing sequence is a sequence in which its limit will give the inf of F. □

Proof. Here we proof the 3) of the theorem.
Let u_n a minimizing sequence for F. Since that F satisfy the conditions of growing and since that the inf of F is finite, because satisfy also the condition og growing from the top, this minimizing sequence is limited in $W_0^{1,p}(\Omega)$ and so, at least to switch to a sub-sequence, it will converge to something (weakly in $W^{1,p}(\Omega)$ and strongly to L^p). So, from growing conditions it is possible to say that $u_n \to u$ (W and S).
We know that for definition of relaxed

$$sc^- F(u) \leq \liminf_{n \to +\infty} F(u_n)$$

but since that u_n is a minimizing sequence for F and so the $\liminf$ becomes a lim and the lim is coincident with the inf

$$sc^- F(u) \leq \liminf_{n \to +\infty} F(u_n) = \inf F$$

In the previous proof we proved that $\inf_X F = \min_X sc^- F$ and so

$$sc^- F(u) \leq \liminf_{n \to +\infty} F(u_n) = \inf_X F = \min_X sc^- F$$

But for definition of minimum, the relaxed calculate in a generic point u is bigger or equal than its minimum

$$\min_X sc^- F \leq sc^- F(u) \leq \liminf_{n \to +\infty} F(u_n) = \inf_X F = \min_X sc^- F$$

and so It is possible to say that

$$sc^- F(u) = \min_X sc^- F$$

and so u is a point of minimum for $sc^- F$. $\square$

Supposing that u_0 is a minimum of F. So, we can say that $F(u_0) = \min F$. Of course we know that $\min F = \inf F$ and we proved that $\inf F = \min sc^- F$. The minimum of a functional is the lowest value than the values that the functional has in any function of space. So we can say that

$$F(u_0) = \min F = min sc^- F \leq sc^- F(u_0)$$

But we know also that $sc^- F$ is below F for definition of relaxed, and so

$$sc^- F(u_0) \leq F(u_0) = \min F = \min sc^- F \leq sc^- F(u_0)$$

So, this implies that

$$sc^- F(u_0) = \min sc^- F$$

This says that u_0 not only is a minimum for F but also for its relaxed. The minimums of F are inside the relaxed of F.

All this discussion leaves open a not indifferent problem which is the following:
We start from a functional, in the abstract we define what the relaxed of this functional is and then we can do certain things on the relaxed if certain hypotheses are satisfied. We started from an integral functional with a certain integrand. This unfortunately did not satisfy all the hypotheses to apply the direct methods and therefore we relaxed it obtaining $sc^{-}F$, which we can characterize from the point of view of the behavior along the sequences, but of which we know nothing else.
we have links with the initial problem, but we would like to be able to understand how the minima of $sc^{-}F$ are made in order to understand if they are also minima of the initial problem. How do we figure out what these minima are like from an object that is totally abstract?
The first problem that arises is: but if we start from an integral functional, can we also represent the relaxed in integral form? and if so, who is the integrand function that represents it?
So together with the problem of relaxation there is what is called the problem of integral representation of the relaxed

Theorem 0.11. *In the hypothesis of before, we have that*

$$sc^{-}F(u) = \int_{\Omega} (co\, f)(x, \nabla u)dx \quad \forall u \in W_0^{1,p}(\Omega)$$

where $co\, f$ is the convexification of f that represent the biggest convex function that is below f. If the starting finction f is alrady convex, then $co\, f$ will be coincident with f.

Now coming back to the previous problem

$$F(u) = \int_0^1 \left(1 - (u')^2\right)^2 dx + \int_0^1 u^2 dx$$

Since that is not l.s.c. we need to relax it. But since that the second part is continuos, from the property of the relaxed, the relaxed of a function plus a continuos perturbation, we have so not have to relax the continuos part

$$sc^{-}F(u) = \int_0^1 cof(u')dx + \int_0^1 u^2 dx$$

the function is flat and equal to 0 in the range of $-1 < x < 1$. In the relaxed problem the min is $u = 0$.

0.2 Gamma Convergence

Suppose we have a sequence of functionals $F_\epsilon(u)$ which are indexed by a parameter ϵ which is a small parameter. For example, if the functional somehow represents the microscopic energy of a certain physical system, then we want to see what its macroscopic behavior is and therefore we want to make sense of the limit as ϵ tends to zero of this sequence of functional; an appropriate sense that it carries information on the minimum problem with its. If the functional represents a succession of energies, the minimum points represent the equilibrium configurations (in particular the stable equilibrium of my system). A priori we don't know if this minimum problem will have a solution, we are going to consider $\inf F_\epsilon(u)$, and we want to see if we can make sense of giving an appropriate definition of convergence of this sequence of functionals to an appropriate functional which is precisely that it carries with it the information on the equilibrium configurations (actually on the configurations of stable equilibrium) $F_\epsilon(u) \xrightarrow{"\epsilon \to 0"} \overline{F}$; for example that it has some properties, including for example the fact that when we pass to the limit of $\inf F_\epsilon(u)$, this sequence of inf converges to the minimum of $\overline{F}$ and that the minimum of $\overline{F}$ exists (that this problem is solvable) and that there is convergence of the inf at least of an appropriate functional.
A notion of convergence which is called variational convergence, therefore which actually carries with it the information on minima on associated minima problems, is the notion of Gamma Convergence which is a notion which was introduced by Ennio De Giorgi around the 70s.

Definition 0.7. *Suppose we have a sequence of functionals $\{F_\epsilon\}$ defined on a metric space X (therefore a space where there is a metric convergence). The sequence $\{F_\epsilon\}$ gamma converges to a certain functional $\overline{F}$ ($F_\epsilon \xrightarrow{\Gamma} \overline{F}$), always defined on X, if $\overline{F}$ satisfies this condition:*

$$\forall u \in X \;\; \overline{F}(u) = \inf\{\liminf_{\epsilon \to 0} F_\epsilon(u_\epsilon), \;\; \{u_\epsilon\} \in X s.t. u_\epsilon \to u\}$$

The inf of the liminf it is done by varying all possible sequences $\{u_\epsilon\}$. This is the same definition of the relaxed of a functional but here we have a sequences of functional. Also here, this definition can be characterized (is equivalent) to the fact that the functional should satisfy two inequality:

$$\forall u \in X, \; u_\epsilon \to u \;\; \Rightarrow \overline{F}(u) \leq \liminf_{\epsilon \to 0} F_\epsilon(u_\epsilon)$$

$$\forall u \in X, \; \exists\, \overline{u}_\epsilon \to u \;\; s.t. \; \overline{F}(u) = \lim_{\epsilon \to 0} F_\epsilon(\overline{u}_\epsilon)$$

where $\overline{u}_\epsilon$ is the optimal sequence.
The main properties of Gamma Convergence are:
1)The Gamma limit, if it exists, is unique;

2) If $F_\epsilon \xrightarrow{\Gamma} \overline{F} \iff$ from every subsequence of $\{F_\epsilon\}$ is possible to extract another subsequence that always Gamma converges to $\overline{F}$(Urysohn's property);
3)The gamma limit is l.s.c. respect to the notion of convergence that was used to define the Gamma limit;
4)The Gamma limit of homogeneous functionals of degree α it is also positively homogeneous of degree α;
A functional is positively homogeneous of degree α means that:

$$F(\lambda u) = \lambda^\alpha F(u)$$

e.g. $f(x) = x^2$ is a positivle homogeneous function of grade 2, because $f(\lambda x) = (\lambda x)^2 = \lambda^2 x^2 = \lambda^2 f(x)$.
5)The gamma limit of quadratic form is a quadratic form.
A quadratic form means that

$$F(u+v) + F(u-v) = 2\left[F(u) + F(v)\right]$$

6) If we have a sequence of functional F_ϵ that is constant and is equal to a functional $\overline{F}$ ($F_\epsilon \equiv \overline{F}$), the gamma limit of a constant functional is not like the limit in numbers/functions that is itself, but is the relaxed $\overline{F} = sc^{-}\overline{F}$.
This property said me that the Gamma convergence does not have connections with the punctual convergence. An example can be done considering the metric space $X = \mathbb{R}$ (with the Euclidean metric) and so we are using a sequences of functions. Considering the following sequences of functions

$$f_n(x) = nxe^{-n^2x^2}$$

This is a odd function defined on all $\mathbb{R}$, and so we can draw just the right side and then we do the symmetric respect to the origin. The derivative is

$$f_n'(x) = -ne^{-n^2x^2}\left[1 - 2n^2x^2\right]$$

The maximum (speculated the minimum) of the function is in $x = \frac{1}{\sqrt{2}n}$ The value of the function at the minimum point is

$$f_n\left(-\frac{1}{\sqrt{2}n}\right) = -\frac{1}{\sqrt{2}}e^{-1/2}$$

when $n \to +\infty$ the minimum point moves toward the origin and pointwise $f_n(x) \to f(x) \equiv 0 \;\; \forall x \in \mathbb{R}$.
The gamma limit, instead, do not converge to 0 but $f_n(x) \xrightarrow{\Gamma} \tilde{f}(x) \not\equiv 0$. This because the gamma limit must carry the minima information with it. Since that Gamma limit for definition is the inf of liminfs, if we take a point $x \neq 0$ and we take any sequence of points $\{x_n\}$ that converges to this value x and since that the sequence has the maximum/minimum that tens to 0, the limit of $f_n(x_n) \to 0$, for any sequence $\{x_n\}$. For $x = 0$, the point of minimum it is moving toward 0 and should maintain the information of the value of the

minimum and so if we take $x_n = \frac{1}{\sqrt{2n}}$, $f_n(x_n) = -\frac{1}{\sqrt{2}}e^{-1/2}$. The Gamma limit should be lower that this value, but since that this value is reached, this correspond to the value of the Gamma limit and so $\{x_n\}$ is the optimal sequence.
The Gamma limit does not have any relationship with the pointwise limit.
The Gamma limit is not a continuous function but is a l.s.c. function.
7) If G is a continuous functional $\Rightarrow \Gamma\text{-lim } (F_\epsilon + G) = \Gamma\text{-lim } F_\epsilon + G$

Theorem 0.12. *Let $\{F_\epsilon\}$ a functional sequence on X metric space, then $\exists\, F : X \to \overline{\mathbb{R}}$ and a subsequence $\{F_{\epsilon_n}\}$ of $\{F_\epsilon\}$ s.t. $F_{\epsilon_n} \xrightarrow{\Gamma} \overline{F}$*

Unless passing to subsequences the Gamma limit always exists.
We are interested in doing everything with respect to the weak convergence in $W^{1,p}$ because it is there that we will have the compactness of the minimizing sequences and we also need the l.s.c. to have the existence of minima. The weak convergence, in general, does not come from a metric, but if some growth conditions are satisfied (in particular the growth conditions from the bottom), as it has been demonstrated for the relaxed, it is similarly true for the Gamma convergence and therefore we can say that working with weak convergence of $W^{1,p}$ is the same thing as working with strong convergence of L_p. The strong convergence comes from a norm (and so it is associated a metric of the norm).
Given the definition along the sequences, for this case, is correct.
We are working with sequence of functionals of this type

$$F_\epsilon(u) = \int_\Omega f\left(\frac{x}{\epsilon}, \nabla u\right) dx, \quad F_\epsilon(u) = \int_\Omega f\left(\frac{u}{\epsilon}, \nabla u\right) dx$$

where f should be continuous in the second variable and in the first variable the function must be measurable and also $1-$periodic.
The growing conditions: exist two constants λ, Λ, $0 < \lambda \leq \Lambda < +\infty$, with the property that $\lambda(|\xi|^p - 1) \leq f(x, \xi) \leq \Lambda(|\xi|^p + 1), \forall \xi \in \mathbb{R}^N$ and for almost every $x \in \Omega$.
If these hypothesis are satisfied, the following theorem is valid

Theorem 0.13. *In the previous hypothesis, if $F_\epsilon \xrightarrow{\Gamma} \overline{F}$, we have*
1) the problem of minimum for $\overline{F}$ has sulution;
2) $\inf F_\epsilon \to \min \overline{F}$*;*
3) every minimizing sequence of F_ϵ, at least of subsequences, converges to a minimum point of $\overline{F}$;
4) every minimum point of $\overline{F}$ is reached from a minimizing sequence.

Let's introduce the following set $\mathcal{A}(\Omega)$, that is the set of all open subsets that are in Ω. Considering the following functional

$$F_\epsilon(u) = \int_\Omega f\left(\frac{x}{\epsilon}, \nabla u\right) dx$$

This representation can be read as a functional F defined on u (a function defined on the functions u of a Sobolev space) fixing the set Ω. The following can be read in another way, fixing the u function and read it as a function defined on the set Ω that we are integrating. So this becomes a function of two variables

$$F_\epsilon(\Omega, u) = \int_\Omega f\left(\frac{x}{\epsilon}, \nabla u\right) dx$$

In this case we are looking which property has the integral, fixing the u function, respect to the set Ω.
Fixing Ω, this can be repeated for all subsets of Ω

$$F_\epsilon(A, u) = \int_A f\left(\frac{x}{\epsilon}, \nabla u\right) dx \quad u \in W^{1,p}(\Omega), \;\; A \in \mathcal{A}(\Omega)$$

where $F(\cdot, u)$ is the restriction of the opens of a measure (of regular Borel). The measure is a function of set that has a the property that is defined only on a class of sets that is called "measurable sets" with the property that is the sigma additivity. A function of set is called measure if

$$\mu(\bigcup A_h) = \sum_h \mu(A_h)$$

this is valid if A_h are disjoint (they do not intersect).
If we want that that the functional Gamma limit is also a functional that can be represented in integral form, also it should have a property of that is a measure.
The compactness theorem if valid, in general form, in this sense. If we have a sequence of functionals of the couple (set,function), so we see it as an object defined on a couple, we can extract a subsequence Gamma convergent to a certain functional (the gamma limit) and this gamma limit functional does not depend on the subsequence that we have chosen so that for each couple (A, u) this happens.
The compactness theorem that we announced earlier, taking fixed Ω, actually holds, in a more general context, if we consider the sequence of functional as functionals of the couple (set, function).
So, at least of subsequence, we can pass to the gamma limit even when looking at the functional in this more general form.
Locality Property: $\forall A \in \mathcal{A}(\Omega), \;\; u, v \in W^{1,p}(\Omega)$ that are coincident on A, then $F_\epsilon(A, u) = F_\epsilon(A, v)$.
So, fixing the open A and taking two functions that outside that open are able to do what they like, but inside that open are the same function, since that when we integrate over A, what happen outside it is not looked, the two functional calculated on u and on v are coincident. Locally on A, they are the same functional because they are integral functional.
Another property that is satisfy from Gamma limit is the l.s.c.
Since that the functional has some growing, because the integrand function has some growing, also the relaxed functional will respect the same growing of the sequences of starting functionals

$$\lambda\left(\int_A |\nabla u|^p - |A|\right) \leq \overline{F}(A,u) \leq \Lambda\left(|A| + \int_A |\nabla u|^p\right)$$

For integrals of the type $f\left(\frac{x}{\epsilon}, \nabla u\right)$ also the property of translation invariance should be satisfy:

$$F(A, u+c) = F(A,u) \;\; \forall u \in W^{1,p}(\Omega), \;\; \forall A \in \mathcal{A}(\Omega), \;\; \forall c \in \mathbb{R}$$

since that the integrad function do not depend from u this is valid.

Theorem 0.14 (abstract integral rappresentation). *Let $F : \mathcal{A}(\Omega) \times W^{1,p}(\Omega) \to \mathbb{R}$, a functional s.t. satisy the following hypothesis:*
1) $F(\cdot, u)$ it is a measure $\quad \forall u \in W^{1,p}(\Omega)$;
2) F is local (assigned an open and assign two functions that are coincident in that open, the functional on both functions assume the same value);
3)F is l.s.c.;
4) F is invariant for translation in u;
5) F must satisfy the growing conditions.

implies that $\exists\, \varphi$ a charateodory function that is convex in ξ s.t.

$$F(A,u) = \int_A \varphi(x, \nabla u) dx$$

and also that φ satisfy the growing conditions.

Now we want to see what happens to the Gamma limit of functional that are re-scaled of periodic integrand.
When we are in a periodic case we talk about Homogenization and so the Gamma limit is called homogenized of the functional.
If we have a periodic function and I re-scale it by ϵ, If we want to see what happen to the limit of $\epsilon \to 0$, that function will not converge anywhere in pointwise way, because it is too mush oscillating. But If we consider the weak convergence, in integral sense, we consider any test function $\varphi(x)$ that is in the the dual of $f\left(\frac{x}{\epsilon}\right)$, and we do now the limit, the limit exist and is a constant and correspond the the mean value (does not depend anymore from the oscillating variable):

$$\int_{\mathbb{R}^N} \varphi(x) f\left(\frac{x}{\epsilon}\right) dx \to \int_{\mathbb{R}^N} \varphi(x) f_{mean} dx$$

where

$$f_{mean} = \frac{1}{T}\int_0^T f(x)dx$$

We call this procedure homogenization because at the $\lim_{\epsilon\to 0}$ the functional that describe the macroscopic system is homogeneous on the oscillating variable (does not depend anymore on that variable).
Consider the following functional

$$F_\epsilon(u) = \int_\Omega f\left(\frac{x}{\epsilon}, \nabla u\right) dx$$

we want to see what happen when $\epsilon \to 0$.
If we pass to a subsequence F_{ϵ_n}, we know that this will Gamma converge to a certain functional gamma limit $\overline{F}$ (from the compactness theorem).
Some properties of the functional Gamma limit $\overline{F}$ are already satisfy: it satisfy the growing conditions, is l.s.c., is translation invariant (because are translation invariant the starting functionals). Also the other two properties of the are satisfied (it is possible to prove it), and so from the theorem we can say that

$$\overline{F}(u) = \int_\Omega \varphi(x, \nabla u) dx$$

Thinking the functional has the couple (set, function), we can consider the Gamma limit functional of a ball on center x_0 and radius R ($B(x_0, R)$) and let's consider the optimal sequence that converge to the linear function $x\xi$:

$$\overline{F}(B(x_0, R), x\xi) = \lim_{n\to+\infty} \int_{B(x_0,R)} f\left(\frac{x}{\epsilon_n}, \nabla\overline{u}_n\right)$$

where $\overline{u}_n$ is the optimal sequence of the Gamma convergence that brings me to the linear function $x\xi$.
we can represent it using the abstract integral representation theorem, where $\nabla(x\xi) = \xi$:

$$\overline{F}(B(x_0, R), x\xi) = \int_{B(x_0,R)} \varphi(x, \xi) dx$$

Now we want to go to another ball but centered in y_0, ($B(y_0, R)$). If $x - x_0$ is inside the ball of centre R, we can say that $x - x_0 = y - y_0$ and so we can a change of variable $x = y + (x_0 - y_0)$

$$\overline{F}(B(x_0, R), x\xi) = \lim_{n\to+\infty} \int_{B(x_0,R)} f\left(\frac{x}{\epsilon_n}, \nabla\overline{u}_n\right) = \lim_{n\to+\infty} \int_{B(y_0,R)} f\left(\frac{y}{\epsilon_n} + \frac{(x_0 - y_0)}{\epsilon_n}, \nabla\overline{u}_n\right) dy$$

If we take the integer part (the biggest integer number that is below the assigned number) $\left[\frac{(x_0-y_0)}{\epsilon_n}\right]$ this is always less equal than the object that we take the integer part end in turn, this number is less equal then the integer part+1:

$$\left[\frac{(x_0 - y_0)}{\epsilon_n}\right] \leq \frac{(x_0 - y_0)}{\epsilon_n} < \left[\frac{(x_0 - y_0)}{\epsilon_n}\right] + 1$$

we do not know if $\frac{(x_0-y_0)}{\epsilon_n}$ is an integer number but we know that is between two integer numbers.
Now multiplying by ϵ_n

$$\epsilon_n \left[\frac{(x_0 - y_0)}{\epsilon_n} \right] \leq (x_0 - y_0) < \left(\left[\frac{(x_0 - y_0)}{\epsilon_n} \right] + 1 \right) \epsilon_n$$

we can say that $(x_0 - y_0)$ is in between to integer numbers multiplied by ϵ_n. When $\epsilon_n \to 0$, we can approximate $(x_0 - y_0)$ with a sequence of integers multiplied by ϵ_n.

$$\lim_{n \to +\infty} \int_{B(y_0,R)} f\left(\frac{y}{\epsilon_n} + \frac{(x_0 - y_0)}{\epsilon_n}, \nabla \overline{u}_n \right) dy \sim \lim_{n \to +\infty} \int_{B(y_0,R)} f\left(\frac{y}{\epsilon_n} + \frac{\cancel{\epsilon_n} M_n}{\cancel{\epsilon_n}}, \nabla \overline{u}_n \right) dy$$

where M_n are some integers. Since that the function f is 1-periodic, M_n is a multiple of the period and so we can delated it from the dependence

$$\lim_{n \to +\infty} \int_{B(y_0,R)} f\left(\frac{y}{\epsilon_n} + M_n, \nabla \overline{u}_n \right) dy = \lim_{n \to +\infty} \int_{B(y_0,R)} f\left(\frac{y}{\epsilon_n}, \nabla \overline{u}_n \right) dy \geq \int_{B(y_0,R)} \varphi(y, \xi) dy$$

and so we obtained that

$$\int_{B(x_0,R)} \varphi(x, \xi) dx = \int_{B(y_0,R)} \varphi(y, \xi) dy$$

Dividing everything by the measure of the ball times a constant (measure of the unitary ball in the dimension that we are considering) $\frac{1}{cr^n}$ and we send r to 0 (Lebesgue's derivation theorem)

$$\frac{1}{cr^n} \int_{B(x_0,R)} \varphi(x, \xi) dx = \frac{1}{cr^n} \int_{B(y_0,R)} \varphi(y, \xi) dy$$

and we obtain that, almost everywhere

$$\varphi(x_0, \xi) = \varphi(y_0, \xi)$$

and this implies that

$$\varphi(x, \xi) = \varphi(\xi)$$