Mathematical tools to study the movement of rigid bodies

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Abstract

In this paper the following topics will be covered: reference systems, rotation matrix and coordinate transformation, rotation velocity tensor, Poisson's theorem, composition of finite rotations; Euler angles.

1 Reference frame

Consider a reference frame Σ and a base composed by three unitary vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$. The order of the unit vectors is important when defining a reference frame. The coordinate system (Ω, $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$) is defined as *right-handed*, while $(\Omega, \vec{e}_1, \vec{e}_3, \vec{e}_2)$ is stemmed *left-handed*; clearly, they are different from each other. Generally, right-handed convention is considered, as shown in Figure 1.

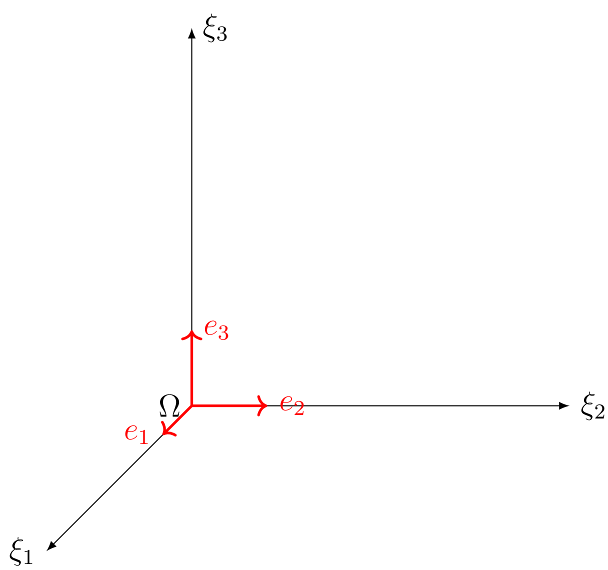


Figure 1: Schematic of right-handed reference frame Σ

Let's consider an example:

let's envisage to have two vectors $\vec{u}$ and $\vec{v}$ belonging to a plane called π and we want to define the positive normal unit vector of this plane. We consider the following vectors order $(\vec{u}, \vec{v})$. We let the vector $\vec{u}$ rotate by a certain angle so as to overlap the vector $\vec{v}$, while maintaining this angle the smallest possible.
Let's consider two observers, A and B. The former is located above the plane and monitors an anticlockwise rotation; the latter is located below the plane and observes a clockwise rotation. Observer A, then, identifies a positive half-space and a positive normal unit vector $\vec{n}^{+}$; observer B pinpoints a negative half-space and a negative normal unit vector $\vec{n}^{-}$, as shown in Figure 2.
Now let's change the order of the vectors (see Figure 3) and, following the same reasoning, it can be checked out that the positive half-plane will be below plane π and the negative half-plane above plane π. It can be concluded that the order of the vectors changes the normal vector direction.

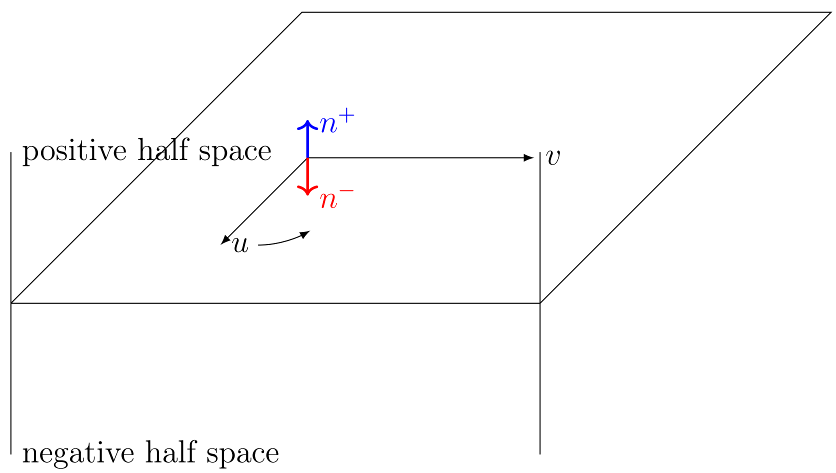


Figure 2: Positive and negative normal unit vectors for the triplet $(\vec{u}, \vec{v}, \vec{w})$

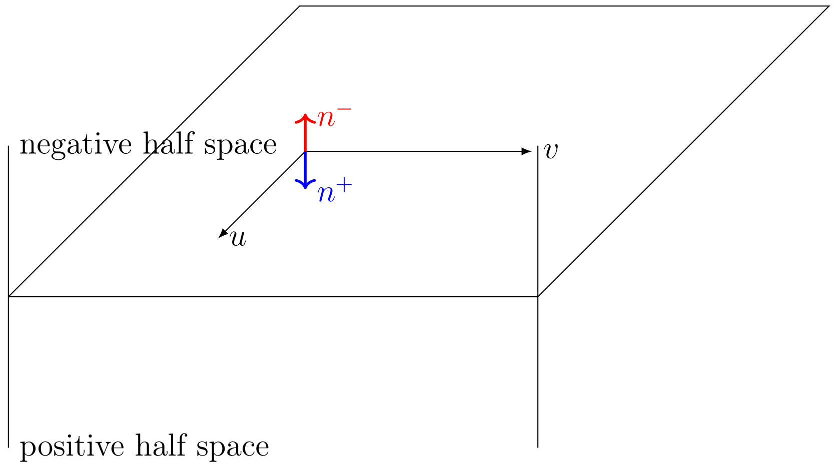


Figure 3: Positive and negative unit vectors for the vector order $(\vec{v}, \vec{u})$

Let's consider three vectors ($\vec{u}$, $\vec{v}$, $\vec{w}$). Since vector $\vec{u}$ and $\vec{v}$ belongs to the same plane, it is possible to determine the normal positive vector $\vec{n}^+$. To have a right-handed triplet, vector $\vec{w}$ has to belong to the positive half-space, as shown in Figure 4. In other words, this condition has to be fulfilled:

$$\vec{w} \cdot \vec{n}^+ > 0, \mid \beta \mid < \pi/2 \tag{1}$$

If the order of the vectors is changed, e.g. ($\vec{v}$, $\vec{u}$, $\vec{w}$),),it is possible to observe that the normal positive vector heads downwards (see Figure 5):

$$\vec{w} \cdot \vec{n}^+ < 0, \mid \beta \mid > \pi/2 \tag{2}$$

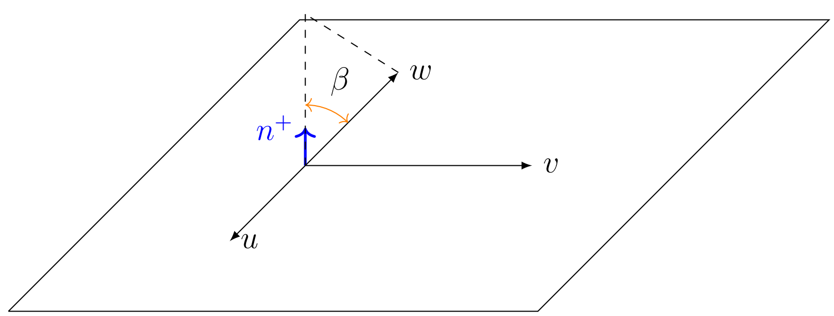


Figure 4: Positive and negative normal unit vectors for the triplet $(\vec{u}, \vec{v}, \vec{w})$

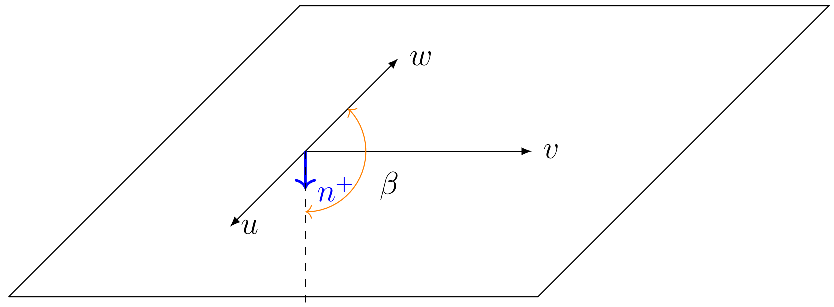


Figure 5: Positive and negative normal unit vectors for the triplet $(\vec{v}, \vec{u}, \vec{w})$

1.1 Vector product

Let's consider two vectors $\vec{u}$ and $\vec{v}$. What we need to do is to rotate the first vector, $\vec{u}$, so to overlap the second vector, $\vec{v}$, so to trace the smallest angle between them. In particular $\vec{w}$ must be contained in the positive half-space and it is defined as:

$$| \vec{w} |=| \vec{u} || \vec{v} | \sin \alpha$$

Furthermore,$\vec{w}$ is orthogonal to the plane π, defined by $\vec{u}$ and $\vec{v}$, and has the same direction of $\vec{n}^+$. It is important to remember that the vector product does not follow the commutative rule

$$\vec{u} \wedge \vec{v} \neq \vec{v} \wedge \vec{u}$$

Making use of the right-hand rule, the direction of $\vec{w}$ has to be changed Therefore, as the vector product depends on the particular rule followed, it is a pseudovector.

2 Orientation of a rigid body respect to an observer

Consider a fixed reference frame Σ and a rigid body. A body is called rigid if given two points s belonging to it, their distance does not change over time, i.e. the body experiences no deformation. So, the body is rigid whenever it is possible to identify at least one reference frame, i.e. an observer for which all the coordinates of the points belonging to the body do not change over time. There are infinite reference frames that meet this condition.
To define the orientation of a rigid body with respect to a global reference frame Σ , it is then not necessary to draw the rigid body, as it can be represented by the local reference system S, jointed to it.
Hence, to define the configuration of the body we need to know:

- the position of the origin of the reference frame S with respect to the reference frame Σ, i.e. $(O - \Omega) = (\xi_{10}, \xi_{20}, \xi_{30})$
- the orientation of the axes of the reference frame S with respect to the reference frame Σ.

The information about the way the axes x_1, x_2, x_3 are oriented in space is needed. In general, the orientation of a straight line is provided by the direction cosines. To define the straight line r, we need to know a point that belongs to r e.g. A and the unitary tangent vector $\vec{\tau}$:

$$r : (P - A) = \lambda\vec{\tau}, P \in r \tag{3}$$

To define $\vec{\tau}$ we need to know the projections of $\vec{\tau}$ onto the axes x_1, x_2, x_3.

$$\begin{aligned}
\vec{\tau} &= (\tau_1, \tau_2, \tau_3) \\
\tau_1 &= \vec{\tau} \cdot \vec{e}_1 \\
\tau_2 &= \vec{\tau} \cdot \vec{e}_2 \\
\tau_3 &= \vec{\tau} \cdot \vec{e}_3
\end{aligned}$$

The same reasoning holds for the body S:

$$R_{11} = \vec{u}_1 \cdot \vec{e}_1 \;\; R_{12} = \vec{u}_2 \cdot \vec{e}_1 \;\; R_{13} = \vec{u}_3 \cdot \vec{e}_1 \tag{4}$$

$$R_{21} = \vec{u}_1 \cdot \vec{e}_2 \;\; R_{22} = \vec{u}_2 \cdot \vec{e}_2 \;\; R_{23} = \vec{u}_3 \cdot \vec{e}_2 \tag{5}$$

$$R_{31} = \vec{u}_1 \cdot \vec{e}_3 \;\; R_{32} = \vec{u}_2 \cdot \vec{e}_3 \;\; R_{33} = \vec{u}_3 \cdot \vec{e}_3 \tag{6}$$

where $(\Omega, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ represents the reference frame Σ and $(O, \vec{u}_1, \vec{u}_2, \vec{u}_3)$ the reference frame S, as shown in Figure 6.

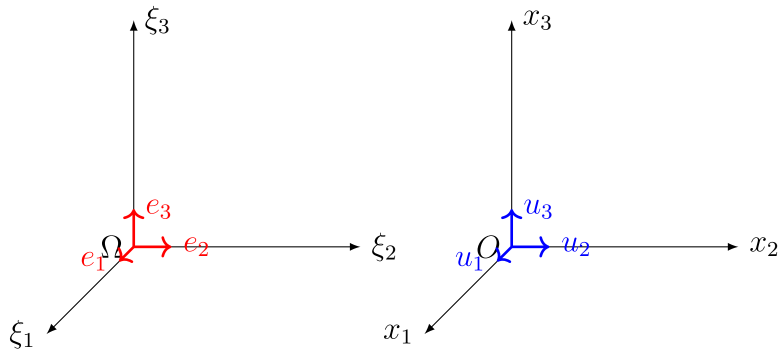


Figure 6: Fixed reference frame Σ and rigid-body jointed reference frame S

It is possible then to collect the elements just found in the so-called *rotation matrix*, which describes the rotation of the body S with respect to Σ

$$R_{S\Sigma} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \tag{7}$$

The columns of the matrix represent the components of $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$ with respect to the reference frame Σ, while the rows represent the components of $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ in the reference frame S.
Moreover, it is possible to notice that $R_{\Sigma S} = R^T_{S\Sigma}$ so that the rotation matrix $R_{S\Sigma}$ is a linear application that rotates the reference frame Σ to make it match with the reference frame S. Also, R is invertible and its inverse matrix corresponds to R^T. Let's focus now on some properties of the matrix $R_{S\Sigma}$; by considering the columns:

$$R^2_{11} + R^2_{21} + R^2_{31} = (\vec{u}_1 \cdot \vec{e}_1)^2 + (\vec{u}_1 \cdot \vec{e}_2)^2 + (\vec{u}_1 \cdot \vec{e}_3)^2 = \mid \vec{u}_1 \mid^2 = 1 \tag{8}$$
$$R^2_{12} + R^2_{22} + R^2_{32} = (\vec{u}_2 \cdot \vec{e}_1)^2 + (\vec{u}_2 \cdot \vec{e}_2)^2 + (\vec{u}_2 \cdot \vec{e}_3)^2 = \mid \vec{u}_2 \mid^2 = 1 \tag{9}$$
$$R^2_{13} + R^2_{23} + R^2_{33} = (\vec{u}_3 \cdot \vec{e}_1)^2 + (\vec{u}_3 \cdot \vec{e}_2)^2 + (\vec{u}_3 \cdot \vec{e}_3)^2 = \mid \vec{u}_3 \mid^2 = 1 \tag{10}$$
$$R_{11}R_{12} + R_{21}R_{22} + R_{31}R_{33} = \vec{u}_1 \cdot \vec{u}_2 = 0 \tag{11}$$
$$R_{11}R_{13} + R_{21}R_{23} + R_{31}R_{33} = \vec{u}_1 \cdot \vec{u}_3 = 0 \tag{12}$$
$$R_{12}R_{13} + R_{22}R_{23} + R_{32}R_{33} = \vec{u}_2 \cdot \vec{u}_3 = 0 \tag{13}$$

By considering the rows:

$$R_{11}^2 + R_{12}^2 + R_{13}^2 = (\vec{u}_1 \cdot \vec{e}_1)^2 + (\vec{u}_2 \cdot \vec{e}_1)^2 + (\vec{u}_3 \cdot \vec{e}_1)^2 =\mid \vec{e}_1 \mid^2 = 1 \tag{14}$$
$$R_{21}^2 + R_{22}^2 + R_{23}^2 = (\vec{u}_1 \cdot \vec{e}_2)^2 + (\vec{u}_2 \cdot \vec{e}_2)^2 + (\vec{u}_3 \cdot \vec{e}_2)^2 =\mid \vec{e}_2 \mid^2 = 1 \tag{15}$$
$$R_{31}^2 + R_{32}^2 + R_{33}^2 = (\vec{u}_1 \cdot \vec{e}_3)^2 + (\vec{u}_2 \cdot \vec{e}_3)^2 + (\vec{u}_3 \cdot \vec{e}_3)^2 =\mid \vec{e}_3 \mid^2 = 1 \tag{16}$$
$$R_{11}R_{12} + R_{21}R_{22} + R_{31}R_{33} = \vec{u}_1 \cdot \vec{u}_2 = 0 \tag{17}$$
$$R_{11}R_{13} + R_{21}R_{23} + R_{31}R_{33} = \vec{u}_1 \cdot \vec{u}_3 = 0 \tag{18}$$
$$R_{12}R_{13} + R_{22}R_{23} + R_{32}R_{33} = \vec{u}_2 \cdot \vec{u}_3 = 0 \tag{19}$$

The last six equations, derived by considering the rows of the rotation matrix, are totally dependent on those derived by considering the columns. We have nine elements of the matrix and six dependent equations so to have three independent elements, which identify the body configuration. It is possible to corroborate this analysis by means of a geometrical representation. If the vector $\vec{u}_1 = (R_{11}, R_{12}, R_{13})$ is known, with R_{11}, R_{12}, R_{13}its projections onto the axes of the reference frame Σ,it is then possible to find the plane π orthogonal to the axis x_1 by the knowledge of the origin point O and the vector orthogonal to π.
The equation of the plane π can be written considering a generic point $P \in \pi$:

$$(P - O) \cdot \vec{u}_1 = 0 \tag{20}$$

Projecting this equation into the axes of the reference frame Σ:

$$(\xi_1 - \xi_{10})R_{11} + (\xi_2 - \xi_{20})R_{21} + (\xi_3 - \xi_{30})R_{31} = 0 \tag{21}$$

Since $\vec{u}_2$ is perpendicular to $\vec{u}_1$, it must belongs to π. To identify the vector $\vec{u}_2$ we only need two components, as the $\vec{u}_2$ projection onto the straight line perpendicular to $\vec{u}_1$ is null vector. Moreover $\vec{u}_2$ has modulus equal to 1 and, then, we have two equations:

$$\vec{u}_1 \cdot \vec{u}_2 = 0 \tag{22}$$
$$\mid \vec{u}_2 \mid = 1 \tag{23}$$

The components of the vector $\vec{u}_2$ in the reference frame Σ are:

$$R_{12} = \vec{u}_2 \cdot \vec{e}_1 \ \ R_{22} = \vec{u}_2 \cdot \vec{e}_2 \ \ R_{32} = \vec{u}_2 \cdot \vec{e}_3 \tag{24}$$

Hence, it comes out, that one component only is necessary to identify the vector $\vec{u}_2$.
The vector $\vec{u}_3$ is already determined and we do not need to look for its components, as the following relation holds:

$$\vec{u}_3 = \vec{u}_1 \wedge \vec{u}_2$$

Therefore, to determine the orientations of the body S four parameters only are needed:

$$R_{11} = \vec{u}_1 \cdot \vec{e}_1 \ \ R_{21} = \vec{u}_1 \cdot \vec{e}_2 \ \ R_{31} = \vec{u}_1 \cdot \vec{e}_3 \ \ R_{32} = \vec{u}_2 \cdot \vec{e}_3$$

but it is known that to determine the rotation of a rigid body, given O fixed, only three components are needed. Where is the problem?
As the vector $\vec{u}_1$ has modulus equal to 1 we just need three parameters: two components of the vector $\vec{u}_1$ and one component of the vector $\vec{u}_2$.
We have three degrees of freedom with regards to the rotation and three with regards to the translation.

2.1 Determinant of the matrix

Let's consider now three vectors resembling a right-handed triplet $(\vec{u}, \vec{v}, \vec{w})$. The plane π is identified by $\vec{u}$ and $\vec{v}$ and the normal positive vector $\vec{n}^+$ is known. The angle between $\vec{w}$ and $\vec{n}^+$ is α and the projection of $\vec{w}$ onto $\vec{n}^+$ is called h. The prism can then be built up, as presented in Figure 7. The volume of the prim is $V = Ah$, where A is the base area of its base

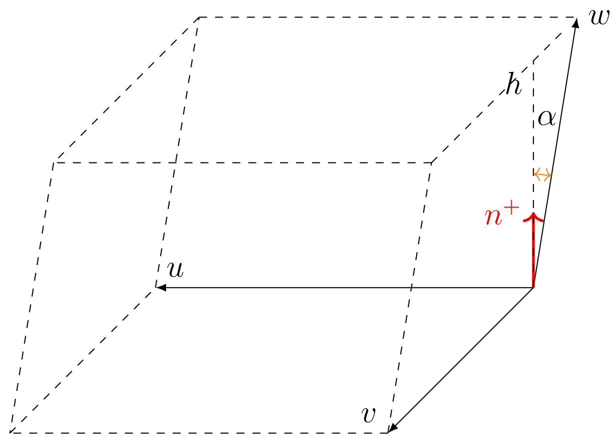
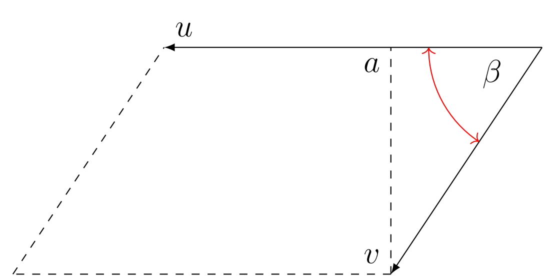


Figure 7: Schematic of the prism, according to the right-handed triplet $(\vec{u}, \vec{v}, \vec{w})$



$$A = \mid \vec{u} \mid a \tag{25}$$
$$a = \mid \vec{v} \mid \sin\beta \tag{26}$$
$$A = \mid \vec{u} \mid\mid \vec{v} \mid \sin\beta = \mid \vec{u} \wedge \vec{v} \mid \tag{27}$$
$$\vec{u} \wedge \vec{v} = \mid \vec{u} \wedge \vec{v} \mid \vec{n}^{+} = A\vec{n}^{+} \tag{28}$$
$$h = \vec{w} \cdot \vec{n}^{+} = \mid \vec{w} \mid\mid \vec{n}^{+} \mid \cos\alpha = \mid \vec{w} \mid \cos\alpha \tag{29}$$
$$V = Ah = (A\vec{n}^{+}) \cdot \vec{w} = (\vec{u} \wedge \vec{v}) \cdot \vec{w} \tag{30}$$

The result of the mixed product is a scalar whose absolute value depends neither on the order of the three vectors nor on the order of the two operations. The absolute value is equal to the volume of the parallelepiped built on the three vectors. The sign of the mixed product depends on the order of the vectors and the two operations. An even or cycle permutation of the three vectors does not affect the sign:

$$(\vec{u} \wedge \vec{v}) \cdot \vec{w} = (\vec{v} \wedge \vec{w}) \cdot \vec{u} = (\vec{w} \wedge \vec{u}) \cdot \vec{v} \tag{31}$$

This property can be formally rendered by making use of the properties of the determinant. Indeed:

$$(\vec{u} \wedge \vec{v}) \cdot \vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{32}$$

The same reasoning is done for the matrix $R_{S\Sigma}$

$$det(R_{S\Sigma}) = det(R_{S\Sigma}^{T}) = (\vec{u}_1 \wedge \vec{u}_2) \cdot \vec{u}_3 = (\vec{e}_1 \wedge \vec{e}_2) \cdot \vec{e}_3 = V = 1 \tag{33}$$

3 Transformation relation of the components of a vector

Let's consider the reference frame Σ, the reference frame S and a vector $\vec{v}$.
The vector $\vec{v}$ components are known with regards to the reference frame S and we want to find the corresponding values with respect to the reference frame Σ. We can then write:

$$\vec{v} = v_1^s \vec{u}_1 + v_2^s \vec{u}_2 + v_3^s \vec{u}_3 \tag{34}$$

$$v_1^\Sigma = \vec{v} \cdot \vec{e}_1 = (v_1^s \vec{u}_1) \cdot \vec{e}_1 + (v_2^s \vec{u}_2) \cdot \vec{e}_1 + (v_3^s \vec{u}_3) \cdot \vec{e}_1 = v_1^s (\vec{u}_1 \cdot \vec{e}_1) + v_2^s (\vec{u}_2 \cdot \vec{e}_1) + v_3^s (\vec{u}_3 \cdot \vec{e}_1) = \tag{35}$$

$$= v_1^s R_{11} + v_2^s R_{12} + v_3^s R_{13} = \sum_{k=1}^{3} R_{1k} v_k^s$$

$$v_2^\Sigma = \vec{v} \cdot \vec{e}_2 = \sum_{k=1}^{3} R_{2k} v_k^s$$

$$v_3^\Sigma = \vec{v} \cdot \vec{e}_3 = \sum_{k=1}^{3} R_{3k} v_k^s$$

$$\begin{bmatrix} v_1^\Sigma \\ v_2^\Sigma \\ v_3^\Sigma \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} v_1^s \\ v_2^s \\ v_3^s \end{bmatrix} \tag{36}$$

which, in turn, can be expressed as:

$$v^\Sigma = R\, v^S \tag{37}$$

multiplying it by R^T and remembering that $R^T R = R\, R^T = I$ the following is obtained:

$$R^T v^\Sigma = R^T R\, v^S = I v^S \Rightarrow \; v^S = R^T v^\Sigma \tag{38}$$

In fact:

$$R\, R^T = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} R_{11} & R_{21} & R_{31} \\ R_{12} & R_{22} & R_{32} \\ R_{13} & R_{23} & R_{33} \end{bmatrix} = \begin{bmatrix} \mid \vec{e}_1 \mid^2 & \vec{e}_1 \cdot \vec{e}_2 & \vec{e}_1 \cdot \vec{e}_3 \\ \vec{e}_2 \cdot \vec{e}_1 & \mid \vec{e}_1 \mid^2 & \vec{e}_2 \cdot \vec{e}_3 \\ \vec{e}_3 \cdot \vec{e}_1 & \vec{e}_3 \cdot \vec{e}_2 & \mid \vec{e}_3 \mid^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \tag{39}$$

It is overt that the components of the same vector $\vec{v}$ in the reference frame Σ and S are different. If $\vec{v}$ is a real vector, it can be transformed through Equations (38) and (39).
It has to borne in mind that the pressure is not a real vector because its components does not depend on the reference frame and the aforementioned relations can not be employed.

4 Transformation relation of the components of a tensor

Let's consider two vectors $\vec{u}$ and $\vec{v}$ and a reference frame $(O, \vec{e}_1, \vec{e}_2, \vec{e}_3)$.

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \; \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{40}$$

We can define the outer product as:

$$\vec{u} \otimes \vec{v} = \begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{bmatrix} \tag{41}$$

where the result is a tensor.

In index notation, the previous expression can be written as:

$$(\vec{u} \otimes \vec{v})_{ij} = u_i v_j \tag{42}$$

The vector $\vec{v}$, in n-dimensional space $(\vec{e}_1, \vec{e}_2, \vec{e}_3, ..., \vec{e}_n)$, can be written as:

$$\vec{v} = \sum_{k=1}^{n} v_k \, \vec{e}_k$$

In fact the following holds:

$$\begin{bmatrix} v_1^\Sigma \\ v_2^\Sigma \\ v_3^\Sigma \end{bmatrix} = v_1^\Sigma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_2^\Sigma \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3^\Sigma \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{43}$$

The same reasoning applies for the tensor:

$$\begin{aligned} \vec{u} \otimes \vec{v} = u_1v_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + u_1v_2 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + u_1v_3 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \\ + u_2v_1 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + u_2v_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + u_2v_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \\ + u_3v_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + u_3v_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + u_3v_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

In index notation, the previous equation takes the following form:

$$\vec{u} \otimes \vec{v} = \sum_{i,j=1}^{3} u_i \, v_j \, \vec{e}_i \otimes \vec{e}_j \tag{44}$$

In general, any tensor can be written as:

$$\bar{\bar{A}} = \sum_{i,j=1}^{3} A_{ij} \, \vec{e}_i \otimes \vec{e}_j \tag{45}$$

$$\bar{\bar{A}} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{46}$$

A scalar product between a vector $\vec{v}$ and a unit vector $\vec{e}_j$ of the basis can be defined then:

$$\vec{v} = \sum_{i=1}^{n} v_i \, \vec{e}_i \tag{47}$$

$$\vec{v} \cdot \vec{e}_j = \sum_{i=1}^{3} v_i \, \vec{e}_i \cdot \vec{e}_j = \sum_{i=1}^{3} v_i \, \delta_{ij} = \sum_{j=1}^{3} v_j \, \delta_{jj} = v_j \tag{48}$$

where δ_{ij} are the elements of the identity matrix.
Hence, for tensors:

$$\bar{\bar{A}}:(\vec{e}_h \otimes \vec{e}_k) = A_{hk} \tag{49}$$

but, before proceeding, the *double scalar product* between two tensors has to be defined.
Let's consider four vectors $\vec{u}, \vec{v}, \vec{p}, \vec{q}$

$$\bar{\bar{A}} = \vec{u} \otimes \vec{v} = \sum_{i,j=1}^{3} u_i \, v_j \, \vec{e}_i \otimes \vec{e}_j$$

$$\bar{\bar{B}} = \vec{p} \otimes \vec{q} = \sum_{i,j=1}^{3} p_i \, q_j \, \vec{e}_i \otimes \vec{e}_j$$

$$\bar{\bar{A}}:\bar{\bar{B}} = \sum_{i,j=1}^{3} A_{ij} B_{ij} = \sum_{i,j=1}^{3} (u_i v_j)(p_i q_j) =$$

$$= \sum_{i,j=1}^{3} (u_i p_i)(v_j q_j) = (\sum_{i=1}^{3} u_i p_i)(\sum_{j=1}^{3} v_j q_j) = (\vec{u} \cdot \vec{p})(\vec{v} \cdot \vec{q})(\vec{u} \otimes \vec{v}):(\vec{p} \otimes \vec{q}) = (\vec{u} \cdot \vec{p})(\vec{v} \cdot \vec{q})$$

$\bar{\bar{A}}:(\vec{e}_h \otimes \vec{e}_k) = A_{hk}$

proof:

$$\bar{\bar{A}}:(\vec{e}_h \otimes \vec{e}_k) = \sum_{i,j=1}^{3} A_{ij}(\vec{e}_i \otimes \vec{e}_j):(\vec{e}_h \otimes \vec{e}_k) = \sum_{i,j=1}^{3} A_{ij}(\vec{e}_i \cdot \vec{e}_h)(\vec{e}_j \cdot \vec{e}_k) = \sum_{i,j=1}^{3} A_{ij}\delta_{ih}\delta_{jk} =$$

$$= \sum_{i=1}^{3} \delta_{ih} \sum_{j=1}^{3} A_{ij}\delta_{jk} = \sum_{i=1}^{3} \delta_{ih} \sum_{j=1}^{3} A_{ik}\delta_{kk} = \sum_{i=1}^{3} \delta_{ih} A_{ik} = \delta_{hh}A_{hk} = A_{hk}$$

Let's consider now two reference frames $\Sigma(\Omega, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ and $S(O, \vec{u}_1, \vec{u}_2, \vec{u}_3)$.
The tensor $\bar{\bar{A}}$ can be projected onto the basis $(\vec{e}_i \otimes \vec{e}_j)$) and onto the basis $(\vec{u}_i \otimes \vec{u}_j)$ for each of the reference frames respectively.

$$\bar{\bar{A}} = \sum_{i,j=1}^{3} A^S_{ij}(\vec{e}_i \otimes \vec{e}_j)$$

$$\bar{\bar{A}} = \sum_{i,j=1}^{3} A^\Sigma_{ij}(\vec{u}_i \otimes \vec{u}_j)$$

Then, we can write A^Σ_{ij} as to be:

$$A^\Sigma_{ij} = \bar{\bar{A}}:(\vec{e}_i \otimes \vec{e}_j) = \sum_{h,k=1}^{3} A^S_{hk}(\vec{u}_h \otimes \vec{u}_k):(\vec{e}_i \otimes \vec{e}_j) = \sum_{h,k=1}^{3} A^S_{hk}(\vec{u}_h \otimes \vec{u}_k):(\vec{e}_i \otimes \vec{e}_j) =$$

$$= \sum_{h,k=1}^{3} A^S_{hk}(\vec{u}_h \cdot \vec{e}_i)(\vec{u}_k \cdot \vec{e}_j) = \sum_{h,k=1}^{3} A^S_{hk}R_{ih}R_{jk} = \sum_{h,k=1}^{3} R_{ih}A^S_{hk}R_{jk}$$

Therefore, we have that:

$$A^\Sigma_{ij} = \sum_{h,k=1}^{3} R_{ih}A^S_{hk}R^T_{kj} \Rightarrow A^\Sigma = R\,A^S R^T \tag{50}$$

and

$$A^S = R^T\,A^\Sigma R \tag{51}$$

Let's suppose now to have a matrix representing a generic object in the reference frame S and another matrix representing the same object in the reference frame Σ. If these two matrices are related to each other through Equations (51) and (52), they are real tensors. We want to know if the rotation matrix R is a tensor. Making use of Equation (52), it is obtained:

$$R^S = R^T R^\Sigma R = (R^T R)R^\Sigma = R^\Sigma \tag{52}$$

This explains the subsequent reasoning: let's consider the vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$, and project them onto the axes of the observer's reference frame, that is Σ. If the observer moves from

Σ to S, the components of $\vec{u}_1, \vec{u}_2, \vec{u}_3$ must be projected onto S so that the matrix is an identity matrix. If the matrix R had been a tensor, it would have satisfied the condition $R^\Sigma = R^S = R$, but this is not true because it is projected onto the axes of the reference frame S and the identity matrix I, is obtained; while, projecting it onto the axes of Σ the result will be different from I. Therefore, the matrix R is not a tensor.
Let's consider a reference frame Σ (Ω, $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$), a vector $\vec{v}$ and two normal vectors $\vec{\tau}$ and $\vec{\mu}$ orthogonal to each other. If we want to evaluate the components of the vector $\vec{v}$ projected onto $\vec{\tau}$:

$$\vec{v} \cdot \vec{\tau} = v_1\tau_1 + v_2\tau_2 + v_3\tau_3 \tag{53}$$

where τ_1, τ_2, τ_3 are the components of $\vec{\tau}$ with respect to the reference frame Σ:

$$\vec{\tau} = \tau_1\vec{e}_1 + \tau_2\vec{e}_2 + \tau_3\vec{e}_3 \tag{54}$$

and the same holds for a tensor:

$$A_{\tau\mu} = \bar{\bar{A}}:(\vec{\tau} \otimes \vec{\mu}) \tag{55}$$

Moreover, the outer product is not commutative. In fact, by substituting Equation (56) into Equation (46), we get:

$$\begin{aligned} A_{\tau\mu} &= \sum_{i,j=1}^{3} A_{ij}(\vec{e}_i \otimes \vec{e}_j):(\vec{\tau} \otimes \vec{\mu}) = \sum_{i,j=1}^{3} A_{ij}(\vec{e}_i \cdot \vec{\tau})(\vec{e}_j \cdot \vec{\mu}) = \\ &= \sum_{i,j=1}^{3} A_{ij}\tau_i\,\mu_j = \sum_{i,j=1}^{3} A_{ij}\,\mu_j\,\tau_i = \sum_{i=1}(\sum_{j=1} A_{ij}\,\mu_j)\tau_i = \sum_{i=1}(\bar{\bar{A}}\vec{\mu})_i\,\tau_i = (\bar{\bar{A}}\vec{\mu}) \cdot \vec{\tau} \end{aligned}$$

Indeed:

$$\bar{\bar{A}}\vec{\mu} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} a_{11}\mu_1 + a_{12}\mu_2 + a_{13}\mu_3 \\ a_{21}\mu_1 + a_{22}\mu_2 + a_{23}\mu_3 \\ a_{31}\mu_1 + a_{32}\mu_2 + a_{33}\mu_3 \end{bmatrix} = \begin{bmatrix} a_{1j}\mu_j \\ a_{2j}\mu_j \\ a_{3j}\mu_j \end{bmatrix} \tag{56}$$

and

$$A_{\tau\mu} = \bar{\bar{A}}:(\vec{\tau} \otimes \vec{\mu}) = (\bar{\bar{A}}\vec{\mu}) \cdot \vec{\tau} \tag{57}$$

For a calculator it is very difficult to write these kind of formulas, thus we want to write them in a different way: each vector can be expressed as a column vector, so that

$$\vec{v} \cdot \vec{v} = [u]^T[v] = [v]^T[u] \tag{58}$$

$$\vec{u} \otimes \vec{v} = [u][v]^T \tag{59}$$

$$\vec{v} \otimes \vec{u} = [v][u]^T \tag{60}$$

5 Poisson theorem for vectors and tensors

Let's consider a reference frame $\Sigma(\Theta, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ and a rigid body $S(O, \vec{u}_1, \vec{u}_2, \vec{u}_3)$, the last one moving with respect to Σ in space (it can translate and rotate). The reference frame Σ or other observers are able to calculate at each time the rotation matrix $R = R_{S\Sigma}$, so that R is a function of time t (each element of the matrix is function of time). The rotation velocity of S is also measured by Σ.
Let's take a vector $\vec{v}$, whose representations in Σ is:

$$v^{\Sigma} = \begin{bmatrix} v_1^{\Sigma} \\ v_2^{\Sigma} \\ v_3^{\Sigma} \end{bmatrix} \tag{61}$$

and in S is:

$$v^{S} = \begin{bmatrix} v_1^{S} \\ v_2^{S} \\ v_3^{S} \end{bmatrix} \tag{62}$$

Then, we can write the time derivative of the vector $\vec{v}$, with respect to both the observers:

$$\frac{dv}{dt}|_{\Sigma} = \begin{bmatrix} \dot{v}_1^{\Sigma} \\ \dot{v}_2^{\Sigma} \\ \dot{v}_3^{\Sigma} \end{bmatrix} \tag{63}$$

$$\frac{dv}{dt}|_{S} = \begin{bmatrix} \dot{v}_1^{S} \\ \dot{v}_2^{S} \\ \dot{v}_3^{S} \end{bmatrix} \tag{64}$$

$$\frac{dv}{dt}|_{\Sigma} \neq \frac{dv}{dt}|_{S} \tag{65}$$

in fact, for each $\vec{v} \in S$

$$\frac{dv}{dt}|_{S} = 0 \tag{66}$$

while in Σ, we have:

$$\frac{dv}{dt}|_{\Sigma} \neq 0 \tag{67}$$

It is important then to highlight that the time derivative of the vector $\vec{v}$ depends on the observer.
Thus, we want to find whether the reference frame Σ can evaluate the time derivative of the same vector $\vec{v}$ with respect to the reference frame S.
The reference frame Σ can calculate the components of the vector $\vec{v}$ in S, as it knows the configuration of the latter via the rotation matrix.

$$\frac{d[v^S]}{dt} = \frac{1}{t_2 - t_1} \begin{bmatrix} v_1^S(t_2) - v_1^S(t_1) \\ v_2^S(t_2) - v_2^S(t_1) \\ v_3^S(t_2) - v_3^S(t_1) \end{bmatrix} \tag{68}$$

with the additional assumption that the time t is the same for the two reference frames.

Recalling that the vector $\vec{v}$ can be expressed as:

$$v^\Sigma = R\, v^S \quad \text{(matrix form)} \tag{69}$$

$$v_i^\Sigma = \sum_{j=1}^{3} R_{ij}\, v_j^S \quad \text{(elements form)} \tag{70}$$

the derivative becomes:

$$\dot{v}_i^\Sigma = \sum_{j=1}^{3} \dot{R}_{ij}\, v_j^S + \sum_{j=1}^{3} R_{ij}\, \dot{v}_j^S \tag{71}$$

Assuming that $\vec{v} \in S, \dot{v}_j^S = 0$, finally we have:

$$\dot{v}_i^\Sigma = \sum_{j=1}^{3} \dot{R}_{ij}\, v_j^S \quad \text{(elements form)} \tag{72}$$

$$\dot{v}^\Sigma = \dot{R}\, v^S \quad \text{(matrix form)} \tag{73}$$

We know that:

$$v^S = R^T\, v^\Sigma \tag{74}$$

putting this in the equation yield:

$$\dot{v}^\Sigma = (\dot{R}\, R^T)\, v^\Sigma \tag{75}$$

where, $\dot{R}\, R^T = \Omega^\Sigma$ is the rotation velocity tensor of the rigid body S and it is the representation of the tensor $\bar{\bar{\Omega}}$ in Σ

Therefore, it can be concluded that: :

$$\frac{d\vec{v}}{dt}|_\Sigma = \bar{\bar{\Omega}}\, \vec{v} \tag{76}$$

$$\frac{d\vec{v}}{dt}|_\Sigma = \vec{\omega} \wedge \vec{v} \tag{77}$$

$$\tag{78}$$

The equation (77,78) is known as the Poisson theorem in tensor and vector form respectively. In addition, considering the expression of $\bar{\bar{\Omega}}$ and multiplying it by R:

$$\dot{R}\,(R^T R) = \Omega^\Sigma R \Rightarrow \quad \dot{R} = \Omega^\Sigma R \tag{79}$$

Indeed, observing the derivative of the first column of the matrix $\dot{R}$, we have:

$$\begin{bmatrix} \dot{R}_{11} \\ \dot{R}_{21} \\ \dot{R}_{31} \end{bmatrix} = \begin{bmatrix} . & . & . \\ . & . & . \\ . & . & . \end{bmatrix} \begin{bmatrix} R_{11} \\ R_{21} \\ R_{31} \end{bmatrix} \tag{80}$$

that can be written as:

$$\frac{d\vec{v}_1}{dt}|_\Sigma = \bar{\bar{\Omega}}\,\vec{v}_1 \tag{81}$$

that can be generalized as follows:

$$\frac{d\vec{v}_i}{dt}|_\Sigma = \bar{\bar{\Omega}}\,\vec{v}_i \tag{82}$$

The tensor $\bar{\bar{\Omega}}$ has a representation in S that is Ω^S.
Making use of the coordinate transformation of tensor components gives:

$$\Omega^S = R^T \Omega^\Sigma R = R^T(\dot{R}\,R^T)R = R^T\dot{R}(R^T R) = R^T\dot{R} \tag{83}$$

Finally, it can be summarized:

$$\Omega^\Sigma = \dot{R}\,R^T \qquad \Omega^S = R^T\dot{R} \tag{84}$$

or

$$\omega^\Sigma = \dot{R}\,R^T \qquad \omega^S = R^T\dot{R} \tag{85}$$

where ω^Σ is the angular velocity measured by Σ in the reference frame Σ. Instead ω^S is the angular velocity measured by Σ in the reference frame S.
The vector angular velocity $\vec{\omega}$ of the body S is measured by Σ as in the reference frame S it is null, so that it is a vector that can be represented both in Σ and S.

Let's consider now the relationship between the second order tensor $\bar{\bar{\Omega}}$ and the vector $\vec{\omega}$, where $\bar{\bar{\Omega}}$ is a skew tensor(antisymmetric tensor). Previously, it has been found

$$\frac{d\vec{v}}{dt}|_\Sigma = \bar{\bar{\Omega}}\,\vec{v} = \vec{\omega} \wedge \vec{v} \tag{86}$$

$$\begin{bmatrix} . & . & . \\ . & . & . \\ . & . & . \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega_x & \omega_y & \omega_z \\ v_x & v_y & v_z \end{vmatrix} = \begin{bmatrix} \omega_y v_z - \omega_z v_y \\ \omega_z v_x - \omega_x v_z \\ \omega_x v_y - \omega_y v_x \end{bmatrix} \tag{87}$$

so that the tensor $\bar{\bar{\Omega}}$ is:

$$\bar{\bar{\Omega}} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{88}$$

$$\vec{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{89}$$

Now let's suppose that the vector $\vec{v}$ is not attached to the body S but it is free to moving with respect to the reference frames Σ and S.
Both Σ and S observe that the components of the vector $\vec{v}$ on their own axes change. Thus, we can just recall Equation (70) and evaluate its time derivative:

$$\dot{v}^{\Sigma} = \dot{R}v^S + R\dot{v}^{\,S} \tag{90}$$

Recalling that the vector $\vec{v}$ representation in the reference frame S can be expressed as $v^S = R^T v^{\Sigma}$

$$\dot{v}^{\Sigma} = \dot{R}R^T v^{\Sigma} + (R\dot{v}^{\,S}) = \Omega^{\Sigma} v^{\Sigma} + (R\dot{v}^{\,S}) \tag{91}$$

that embodies the time derivative of the vector $\vec{v}$ measured by S and expressed with respect to the reference frame S.
Moreover, the derivative of the vector $\vec{v}$ measured by S and expressed in the reference frame Σ is.

$$R\,\dot{v}^{\,S} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} \dot{v}_1^S \\ \dot{v}_2^S \\ \dot{v}_3^S \end{bmatrix} \tag{92}$$

Finally, the generalized Poisson theorem can be presented as:

$$\frac{d\vec{v}}{dt}|_{\Sigma} = \vec{\omega} \wedge \vec{v} + \frac{d\vec{v}}{dt}|_S \tag{93}$$

Multiplying it by R^T:

$$R^T \dot{v}^{\Sigma} = R^T \Omega^{\Sigma} v^{\Sigma} + R^T R\,\dot{v}^S \tag{94}$$

and, being $v^{\Sigma} = R\,v^S$, we can further write:

$$R^T \dot{v}^{\Sigma} = (R^T \Omega^{\Sigma} R)\,v^S + R^T R\,\dot{v}^S = \Omega^S v^S + \dot{v}^S \tag{95}$$

which can be cast as:

$$\dot{v}^S = (-\Omega^S)\,v^S + R^T \dot{v}^{\Sigma} \tag{96}$$

If the observer is located in S, he notices that the reference frame Σ is moving with respect to S and the Poisson theorem can be applied as well, but carefully heeding to consider the angular velocity of Σ with respect to S:

$$\frac{d\vec{v}}{dt}|_S = \bar{\bar{\Omega}}_{\Sigma S}\,\vec{v} + \frac{d\vec{v}}{dt}|_\Sigma \tag{97}$$

and, in matrix form:

$$\dot{v}^S = \Omega^S_{\Sigma S}\,v^S + R_{\Sigma S}\dot{v}^\Sigma$$

$$\dot{v}^S = \Omega^S_{\Sigma S}\,v^S + R^T\dot{v}^\Sigma \tag{98}$$

By comparing Equations (97) and (98):

$$\Omega^S_{S\Sigma}\,v^S + R^T\dot{v}^\Sigma = \Omega^S_{\Sigma S}\,v^S + R^T\dot{v}^\Sigma$$
$$(\Omega^S_{S\Sigma} + \Omega^S_{\Sigma S})\,v^S = 0$$

$$\bar{\bar{\Omega}}_{\Sigma S} = -\bar{\bar{\Omega}}_{S\Sigma} \tag{99}$$

Now, let's suppose to have a second-order tensor that can be described both by Σ and in S. This tensor is moving so as to the inertia tensor of the rigid body S. When S is moving, the tensor is moving too and its components in the reference frame S do not change over time, but it cannot be said the same with regards to Σ. The time derivatives of the components of the tensor in S are null.
We have previously observed that

$$A^S = R^TA^\Sigma R \qquad\qquad A^\Sigma = R\,A^SR^T$$

By evaluating the time derivative of A^Σ we have that

$$\dot{A}^\Sigma = \dot{R}A^SR^T + R\dot{A}^SR^T + RA^S\dot{R}^T \tag{100}$$

then, by exploiting A^S by means of Equation (1.58):

$$\dot{A}^\Sigma = (\dot{R}R^T)A^\Sigma(RR^T) + (RR^T)A^\Sigma(R\dot{R}^T) \tag{101}$$

$$\dot{A}^\Sigma = \Omega^\Sigma A^\Sigma + A^\Sigma(\Omega^\Sigma)^T \tag{102}$$

Equation (103) is the Poisson theorem for a second order tensor and it can be expressed also as:

$$\frac{d\bar{\bar{A}}}{dt}|_\Sigma = \bar{\bar{\Omega}}\,\bar{\bar{A}} + \bar{\bar{A}}\,\bar{\bar{\Omega}}^T \tag{103}$$

Being $\bar{\bar{\Omega}}$ a skew tensor ($\Omega_{ij} = -\Omega_{ji} = -\Omega^T_{ij}$), finally we can write:

$$\frac{d\bar{\bar{A}}}{dt}|_\Sigma = \bar{\bar{\Omega}}\,\bar{\bar{A}} - \bar{\bar{A}}\,\bar{\bar{\Omega}} \tag{104}$$

If we have a tensor that is not attached to the rigid body S instead, we have that the time derivatives of the tensor components in S are not null:

$$\dot{A}^\Sigma = \dot{R}A^SR^T + R\dot{A}^SR^T + RA^S\dot{R}^T = \Omega^\Sigma A^\Sigma - A^\Sigma\Omega^\Sigma + R\dot{A}^SR^T \tag{105}$$

where $\dot{A}^S$ is the time derivative of the tensor $\bar{\bar{A}}$ observed by S and expressed in S, and $R\dot{A}^SR^T$ is the time derivative of the tensor $\bar{\bar{A}}$ observed by S and expressed in Σ

Hence, we find the following expression:

$$\frac{d\bar{\bar{A}}}{dt}|_\Sigma = \frac{d\bar{\bar{A}}}{dt}|_S + \bar{\bar{\Omega}}\,\bar{\bar{A}} - \bar{\bar{A}}\,\bar{\bar{\Omega}} \tag{106}$$

6 relation between rotation matrix

Let's consider a reference frame Σ (Θ, $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$) that is fixed and a reference frame S_1 (O_1, $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$) and a reference frame S_2 (O_2, $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$)
We can define three rotation matrices:

$$\begin{aligned} R_1 &= R_{S_1\Sigma} \\ R_2 &= R_{S_2\Sigma} \\ R_{21} &= R_{S_2S_1} \end{aligned}$$

We want to find the relationship between these matrices. To do that, let's consider a vector $\vec{w}$, whose components can be expressed in Σ, S_1 and S_2 in as follows:

$$w^\Sigma = \begin{bmatrix} w_1^\Sigma \\ w_2^\Sigma \\ w_3^\Sigma \end{bmatrix} \quad w^{S_1} = \begin{bmatrix} w_1^{S_1} \\ w_2^{S_1} \\ w_3^{S_1} \end{bmatrix} \quad w^{S_2} = \begin{bmatrix} w_1^{S_2} \\ w_2^{S_2} \\ w_3^{S_2} \end{bmatrix}$$

We can write:

$$w^{\Sigma} = R_1 w^{S_1}$$
$$w^{S_1} = R_{21} w^{S_2}$$
$$w^{\Sigma} = R_2 w^{S_2}$$

and by their suitable combination, we obtain:

$$w^{\Sigma} = R_1(R_{21} w^{S_2}) = (R_1 R_{21}) w^{S_2} = R_2 w^{S_2}$$

so to have:

$$R_2 = R_1 R_{21}$$

It means that the overall rotation from Σ to S_2 2 is simply the product of the first two rotations.
The rotations are not commutative because of of the non-commutative property of the product of two matrices $A\,B \neq B\,A$.

In general, it holds the following:

$$R = R_1 R_2 R_3 R_4 R_n = \prod_{i=1}^{N} R_i \tag{107}$$

7 Euler angles

Let's focus now on the reference frame $\Sigma(O, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ presented in Figure 1.8.

Rotating the reference about the ξ_3 axis by angle θ we have

$$R_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

After that, a second rotation about the x x_1 axis s by an angle ϕ θ is performed.

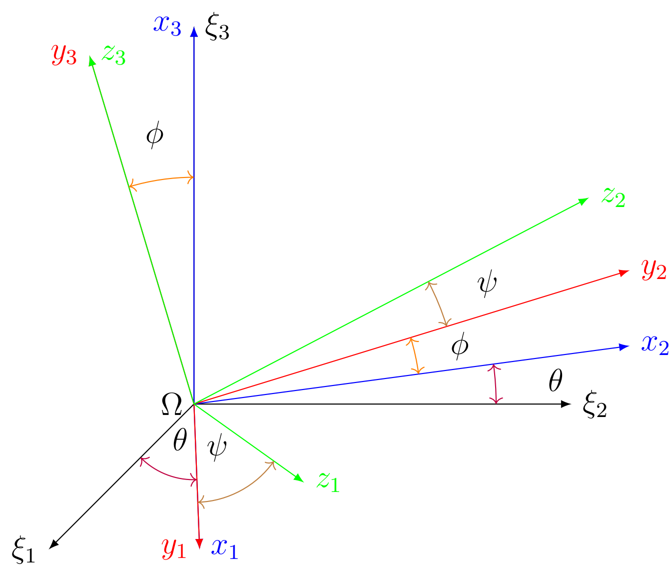
$$R_2 = \begin{bmatrix} 1 & -0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

Finally, a third rotation about the y_3 axis by an angle ψ gives

$$R_3 = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the number of rotational degrees of freedom is three, we need three rotation matrices where the axes of rotation are independent of one another. The total rotation matrix is:

$$R = R_1 R_2 R_3 = \begin{bmatrix} \cos\theta\cos\psi - \cos\phi\sin\theta\sin\psi & -\cos\phi\cos\psi\sin\theta - \cos\theta\sin\psi & \sin\theta\sin\phi \\ \cos\psi\sin\theta + \cos\theta\cos\phi\sin\psi & \cos\theta\cos\phi\cos\psi - \sin\theta\sin\psi & -\cos\theta\sin\phi \\ \sin\phi\sin\psi & \cos\psi\sin\phi & \cos\phi \end{bmatrix}$$



8 eigenvalues and eigenvectors

The Cartesian plane and Euclidean space are particular examples of vector spaces: each point of space can be described by a vector, graphically represented by a segment that connects the origin to the point. In a vector space it is possible to perform linear transformations on the vectors that constitute it: examples of linear transformations are rotations, homothetics (which allow a vector to be amplified or contracted) and reflections (which allow a vector to be transformed into its mirror image with respect to an assigned point, line or plane).

An eigenvector for the linear transformation L is a vector $\vec{v} \neq 0$ which does not change following the application of L its direction, limited to being multiplied by a scalar λ the respective eigenvalue. The vector can therefore only change module (by being amplified or contracted) and direction (by being overturned):

if $\lambda > 0$ the direction of $\vec{v}$ remains unchanged, while if $\lambda < 0$ the direction of $\vec{v}$ changes
if $\mid \lambda \mid = 1$ the modulus of $\vec{v}$ remains unchanged, if $\mid \lambda \mid > 1$ the modulus increases, if $\mid \lambda \mid < 1$ decreases.

Let's try to find the eigenvalue of the matrix R
We have to solve this problem:

$$R\,\vec{v} = \lambda\,\vec{v} \quad \Rightarrow (R - \lambda I)\vec{v} = 0$$

$$\begin{vmatrix} \cos\alpha - \lambda & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha - \lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \quad \Rightarrow (1-\lambda)[(\cos\alpha - \lambda)^2 + \sin^2\alpha] = 0$$

and the eigenvalues are

$$\lambda = 1, \lambda = e^{i\,\alpha}, \lambda = e^{-i\,\alpha}$$