Functional Analysis

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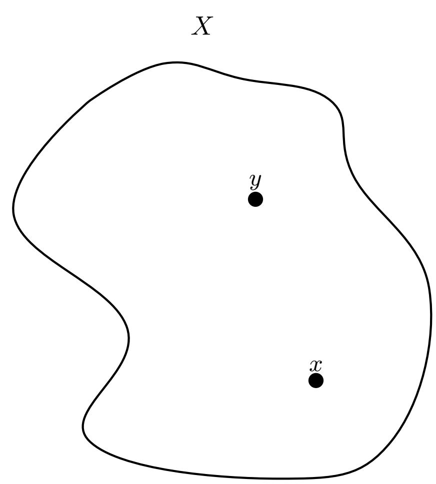
Abstract

Your abstract.

1 Metric Space

Let a set X which can be visualized as a collection of points. For example in the set X we have the element x and the element y as in the picture.



With the concept of set, we cannot say anything about these points other than they are equal or not. We want to give to the set X more structure, to know which is the distance between two chosen points. We have to write down a map called Metric that measure distances:

$$d : X \times X \to [0, \infty)$$

This metric should respect three properties:

1)$d(x, y) = 0 \leftrightarrow x = y$;

2)$d(x, y) = d(y, x)$ Symmetric;

3)$d(x, y) \leq d(x, z) + d(z, y)$ Triangle inequality.

The set X with the metric d is called Metric Space and is denoted with (X, d).

1.1 Example.1

$X = \mathbb{C}$, $d(x, y) = |x - y|$

1.2 Example.2

$X = \mathbb{R}^n$, $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2}$
This standard metric that was chose is called Euclidean metric(Euclidean distance).

1.3 Example.3

$X = \mathbb{R}^n$, $d(x,y) = max\{|x_1 - y_1|, |x_2 - y_2|, ..., |x_n - y_n|\}$

1.4 Example.3

X is any set but not the empty set and

$$d(x,y) = \begin{cases} 0 \ , \ \ x = y \\ 1 \ , \ \ x \neq y \end{cases}$$

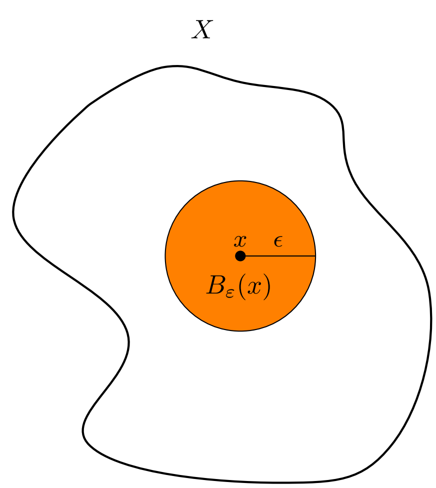
This metric is called discrete metric.

2 Open and Closed Set

Let's considering a metric space (X, d). If we fix a point x inside the metric space, it is possible to look the other points that have the same distance from it. In the common geometry of the plane , this would be a circle around x or in three dimensional space it would be a sphere (or also called ball).
The open epsilon ball of radius $\epsilon > 0$ centered at x is defined in this way

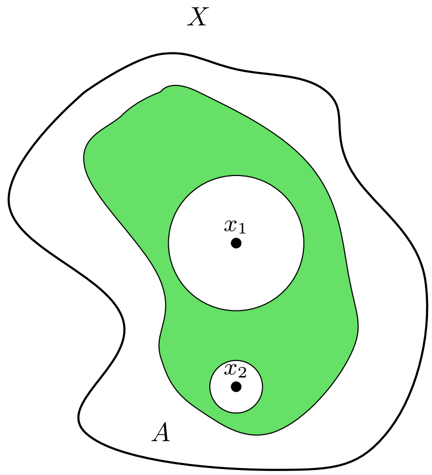
$$B_\epsilon(x) := \{y \in X | d(x,y) < \epsilon\}$$

In the picture can be visualized not considering the border of the ball (in black) but everything is inside it (in orange).
The epsilon ball is never empty because at least the point x lies in this set.



2.1 Open Set

Let's take a sub set $A \subseteq X$(green in the picture). If you fix an arbitrary point $x \in A$, there should be enough points in all directions around this point that also belongs the the set A. So this means that for each $x \in A$ there is an open ball with $B_\epsilon(x) \subseteq A$.
Of course for each x it is possible to chose another ϵ is needed. If you are close to the boundary of A, you need a smaller ϵ. The so-called open balls, are also open with this definition.

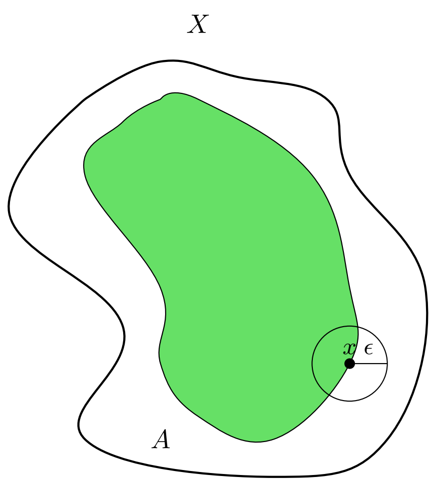


2.2 Boundary Points

Let's take a sub set $A \subseteq X$. $x \in X$ is called a boundary point for A if $\forall \epsilon > 0$:

$$B_\epsilon(x) \cap A \neq \emptyset \ \ and \ \ B_\epsilon(x) \cap A^c \neq \emptyset$$

This means that if the point x is at the boundary and we construct an epsilon ball centered on it, we will hit points that are in A and other points that are not. This should happen no matter small we chose the epsilon ball.



The complement of A is defined as $A^c := X \setminus A$.
The symbol to denote all boundary points is

$$\partial A := \{x \in X | \, x \text{ is boundary point for } A\}$$

2.3 Closed Set

We remember the the open set is exactly such set where all boundary points are outside of A and so $A \cap \partial A = \emptyset$.
A closed set is a set where all the boundary points belongs to this set and so $A \cup \partial A = A$.
We can say also that A is called closed if A^c is open (because all boundary points belongs to A).

2.4 Closure

The closure of A is defined as $\overline{A} := A \cup \partial A$ and is always closed.

3 Sequences, Limits and Closed Sets

Consider a metric space (X, d). It is possible to use sequences to describe the properties of the metric space. The sequence is just an ordered sets of points inside the metric space $(x_1, x_2, ..., x_n)$ or $(x_n)_{n\in\mathbb{N}}$ or we can look it as a map from the natural number into the metric space

$$\begin{aligned} x\colon \mathbb{N} &\to X \\ n &\mapsto x_n \end{aligned}$$

Since that inside the metric space we can measure distances, we can also talk about convergences.
A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X, d) is called convergent if there is $\tilde{x} \in X$ with

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N : d(x_n, \tilde{x}) < \epsilon$$

The members of the sequences get closer and closer to the limit point $\tilde{x}$. We know that to measure the closeness we can use an arbitrary epsilon ball centered at $\tilde{x}$ and no matter how small should be the ϵ, the members of the sequence bigger or equal to a certain number N should be inside that balls. We can write

$$x_n \xrightarrow{n\to\infty} \tilde{x} \quad \text{or} \quad \lim_{n\to\infty} x_n = \tilde{x}$$

3.1 proposition

Let X be a metric space. A set $A \subseteq X$ is closed $\leftrightarrow$ for every convergent sequence $(a_n)_{n\in\mathbb{N}} \subseteq A$, one has

$$\lim_{n\to\infty} a_n \in A$$

3.2 Proof ($\Leftarrow$):

Suppose now that A contains all of its limit points.
Show it by contra-position. Assume A is not closed $\Rightarrow$ A^c is not open $\Rightarrow$ there exist an $\tilde{x} \in A^c$ such that every epsilon ball centered in $\tilde{x}$, contains at least one point in A ($B_\epsilon(\tilde{x}) \cap A \neq \emptyset$ for all $\epsilon > 0$)$\Rightarrow$ there is a sequence $(a_n)_{n\in\mathbb{N}} \in A$ with $a_n \in B_{\frac{1}{n}}(\tilde{x}) \cap A$ with a limit $\tilde{x}$ that is not in A. This contradicts that A contains all of its limit points so A must be closed.

3.3 Proof ($\Rightarrow$):

Suppose that A is closed.
Show it by contra-position. Let $(a_n)_{n\in\mathbb{N}} \in A \ \forall n$ be a convergent sequence and let $\tilde{x} = \lim_{n\to\infty} a_n \notin A \Rightarrow \tilde{x} \in A^c \Rightarrow B_\epsilon(\tilde{x}) \cap A \neq \emptyset \ \forall \epsilon \Rightarrow A^c$ is not open $\Rightarrow A$ is not closed

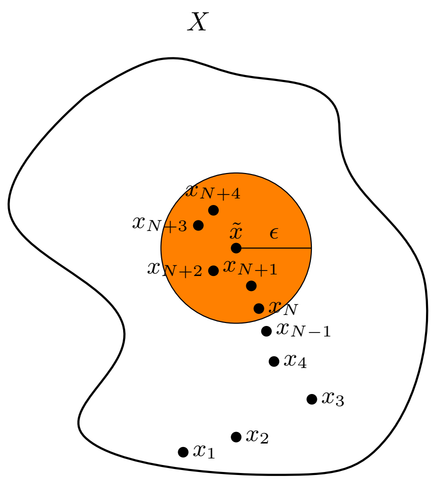
3.4 Cauchy Sequences and Complete Space

3.4.1 Example

3.5 Definition of Cauchy Sequence

Let (X, d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}} \in X$ is called Cauchy Sequence if

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \ \forall n, m \geq N : d(x_n, x_m) < \epsilon$$



3.6 Complete Set

(X, d) is called complete if all Cauchy sequences in X converge in X.

3.7 Example

Consider the metric space $X = \mathbb{R}$ with $d(x, y) = |x - y|$. The sequence $\frac{1}{n}$ is Cauchy. We can proof it showing that

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \ \forall n, m \geq N : \left| \frac{1}{n} - \frac{1}{m} \right| < \epsilon$$

We can use the triangle inequality and says that

$$\left| \frac{1}{n} - \frac{1}{m} \right| = \left| \frac{1}{n} + \frac{-1}{m} \right| \leq \left| \frac{1}{n} \right| + \left| \frac{-1}{m} \right| = \frac{1}{n} + \frac{1}{m}$$

So we need to guarantied that $\frac{1}{n}, \frac{1}{m} < \frac{\epsilon}{2} \Rightarrow n, m > \frac{2}{\epsilon}$. We can say that $N > \frac{2}{\epsilon}$ but since that $N \leq n, m$ we know that

$$\frac{1}{n} + \frac{1}{m} \leq \frac{1}{N} + \frac{1}{N} < \frac{1}{2/\epsilon} + \frac{1}{2/\epsilon} = \epsilon$$

3.8 example

Consider the metric space is $X = (0, 1]$ with $d(x, y) = |x - y|$. Considering the sequence $\frac{1}{n}$ we know that is a cauchy sequence because each x_n lies in X. But we can see that $\lim_{n\to\infty} \frac{1}{n} = 0$ which is not in X and so the metric space X is not complete.

4 Norms and Banach Spaces

Let $\mathbb{F} \in \{\mathbb{R} \text{ or } \mathbb{C}\}$ (field of numbers). Let X be a $\mathbb{F}-$ vector space (the scaling of vector is done with numbers of $\mathbb{F}$).
A map $|| \cdot || : X \to [0, \infty)$ is called norm (which measure the length of the vectors) if respect these properties:

1)$||x|| = 0 \leftrightarrow x = 0$ (positive defined);

2)$||\lambda \cdot x|| = |\lambda| \cdot ||x|| \qquad \forall \lambda \in \mathbb{F} \ , \ x \in X$ (Absolutely homogeneous);

3)$||x + y|| \leq ||x|| + ||y||$ (Triangle inequality).

The vector space X with the norm $||\cdot||$ is called Normed Space and is denoted with $(X, ||\cdot||)$.

If $||\cdot||$ is a norm for the $\mathbb{F}-$ vector space X, then is possible to define a metric

$$d_{||\cdot||}(x,y) := ||x-y||$$

All the definitions that we have for metric spaces are valid also for normed spaces.

4.1 Banach Spaces

If $(X, d_{||\cdot||})$ is a complete metric space, then the normed space $(X, ||\cdot||)$ is called Banach Space.

4.2 Example.1

$\mathbb{R}$ is a one-dimensional real vector space. We can calculate length, if we do the absolute values $|\cdot| : \mathbb{R} \to [0,\infty)$ which is a norm. By this norm we can associate a metric $d_{|\cdot|}(x,y) := |x-y|$. We know that all Cauchy sequences converge in $\mathbb{R}$ and so $(\mathbb{R}, |\cdot|)$ is a Banach space.

4.3 Example.2

$X = 0$ zero dimensional vector space. This means that the zero vector is the only vector in the space. There is just one norm that we can define here $||0|| := 0$. This is a Banach Space.

4.4 Example.3

The l-p space denoted as $l^p(\mathbb{N}, \mathbb{F})$ where $\mathbb{F} \in \{\mathbb{R} \text{ or } \mathbb{C}\}$, $p \in [1,\infty)$ (p is a real number). This space is defines as all $(x_j)_{j\in\mathbb{N}}$ in $\mathbb{F}$ such that

$$\sum_{j=1}^{\infty} |x_j|^p < \infty \quad \text{(Converges)}$$

so are sequences of numbers that are p-power summable. Then $||\cdot||_p : l^p \to [0,\infty)$ with

$$||x||_p := \Big(\sum_{j=1}^{\infty} |x_j|^p\Big)^{\frac{1}{p}} = \Big(|x_1|^p + |x_2|^p + ... + |x_n|^p\Big)^{\frac{1}{p}}$$

is a norm. We can say that $(l^p, ||\cdot||_p)$ is a Banach Space.
We also know that the function $d_p : l^p \times l^p \to [0,\infty)$, given by

$$d_p(x,y) = ||x-y||_p = \Big(\sum_{j=1}^{\infty} |x_j - y_j|^p\Big)^{\frac{1}{p}}$$

is a metric, induced by norm.

4.5 Proof

We have to show the completeness. Let $x^{(n)} = (x_j^{(n)})_{j\in\mathbb{N}}$ be a Cauchy sequence in l^p

$$x^{(1)} = (x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_5^{(1)}, ..., x_j^{(1)})$$
$$x^{(2)} = (x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, x_4^{(2)}, x_5^{(2)}, ..., x_j^{(2)})$$
$$x^{(3)} = (x_1^{(3)}, x_2^{(3)}, x_3^{(3)}, x_4^{(3)}, x_5^{(3)}, ..., x_j^{(3)})$$
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$$x^{(N)} = (x_1^{(N)}, x_2^{(N)}, x_3^{(N)}, x_4^{(N)}, x_5^{(N)}, ..., x_j^{(N)})$$
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$$x^{(n)} = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, x_4^{(n)}, x_5^{(n)}, ..., x_j^{(n)})$$
$$x^{(m)} = (x_1^{(m)}, x_2^{(m)}, x_3^{(m)}, x_4^{(m)}, x_5^{(m)}, ..., x_j^{(m)})$$

Remembering the definition of Cauchy Sequence

$$\forall \epsilon > 0 \;\; \exists N \in \mathbb{N} \;\; \forall n, m \geq N : ||x^{(m)} - x^{(n)}||_p < \epsilon$$

$$|x_j^{(m)} - x_j^{(n)}|^p \leq \sum_{j=1}^{\infty} |x_j^{(m)} - x_j^{(n)}|^p = ||x^{(m)} - x^{(n)}||_p^p < \epsilon^p$$

Since that $\mathbb{R}$ is complete, for each j there exist a $x_j \in \mathbb{R}$ such that

$$\lim_{n\to\infty} x_j^{(n)} = x_j$$

Let fix $k \in \mathbb{N}$, then is similar way

$$\sum_{j=1}^{k} |x_j^{(m)} - x_j^{(n)}|^p \leq \sum_{j=1}^{\infty} |x_j^{(m)} - x_j^{(n)}|^p = ||x^{(m)} - x^{(n)}||_p^p < \epsilon^p$$

Letting $n \to \infty$ we get that from $m \geq N$

$$\sum_{j=1}^{k} |x_j^{(m)} - x_j|^p < \epsilon^p$$

5 Inner Products and Hilbert Spaces

We know that with the metric we can measure distances in the space, the norm measure distances and length of vectors, the inner product measure distances, length and angles between vectors. The inner product is defined as $< x, y >= ||x|| \cdot ||y|| \cdot cos(\alpha)$.
Let $\mathbb{F} \in \{\mathbb{R}$ or $\mathbb{C}\}$. Let X be an $\mathbb{F}-$vector space.
A map $< \cdot, \cdot >: X \times X \to \mathbb{F}$ is called inner product on X if respect these properties:

1)$< x, x > \geq 0 \;\; \forall x \in X$ and $< x, x >= 0 \leftrightarrow x = 0$ (positive defined);

2)$< x, y >=< y, x >$ for $\mathbb{F} = \mathbb{R}$, $< x, y >= \overline{< y, x >}$ for $\mathbb{F} = \mathbb{C} \;\; \forall x, y \in X$((conjucate) symmetric);

3a)$< x, y_1 + y_2 >=< x, y_1 > + < x, y_2 > \quad \forall x, y_1, y_2 \in X$

3b)$< x, \lambda y >= \lambda < x, y > \quad \forall x, y \in X \;,\; \lambda \in \mathbb{F}$

If $< x, x >$ is an inner product, then $||x|||_{<\cdot,\cdot>} := \sqrt{< x, x >}$ defines norm.
If $(X, < \cdot, \cdot >)$ is complete, than the space is called a Hilbert Space. In particular, every Hilbert space is a Banach space with respect to the norm defined before.

5.1 Examples.1

$\mathbb{R}^n, \mathbb{C}^n$ with the following inner product is a finite-dimensional Hilbert Space

$$< x, y >= \sum_{i=1}^{n} \overline{x_i} y_i$$

This give the euclidean geometry in $\mathbb{R}^n, \mathbb{C}^n$

5.2 Examples.2

$l^2(\mathbb{N}, \mathbb{F})$ with the following inner product is an Hilbert Space

$$< x, y >= \sum_{i=1}^{\infty} \overline{x_i} y_i$$

This generalize the euclidean geometry in infinite dimension

5.3 Examples.2

The continuous functions defined on the unit interval $C([0,1], \mathbb{F})$ with the following inner product is not an Hilbert Space because the completeness fails

$$< f, g >= \int_0^1 \overline{f(t)} g(t) dt$$

6 Cauchy-Schwartz inequality

Let $(X, < \cdot, \cdot >)$ be an inner product space and $||x||| := \sqrt{< x, x >}$. Then for all $x, y \in X$:

$$| < x, y > | \leq ||x|| \cdot ||y||$$

and

$$| < x, y > | = ||x|| \cdot ||y|| \leftrightarrow x, y \quad \text{linearly dependent (this means on the same axis)}$$

7 Orthogonality

Let $(X, < \cdot, \cdot >)$ be an inner product space.

1) $x, y \in X$ are called orthogonal if $< x, y >= 0$. Write $x \perp y$.

2)For $U, V \subseteq X$, write $U \perp V$ if $x \perp y \ \forall x \in U, y \in V$.

3)For $U \subseteq X$, the orthogonal complement of U is

$$U^{\perp} := \{x \in X | < x, u >= 0 \ \forall u \in U\}$$

It can be proof that $U^{\perp} \subseteq X$

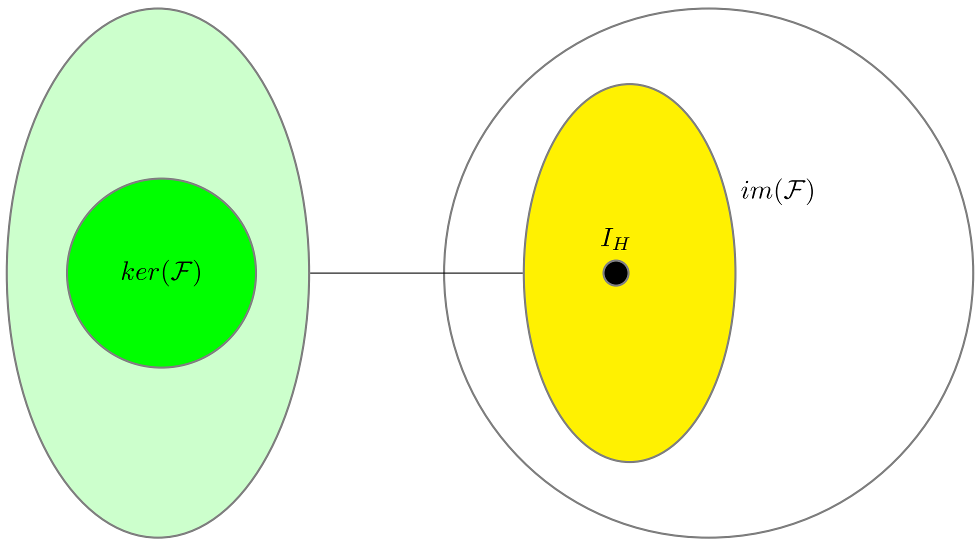
8 Remarks

(1)The orthogonal complement of the zero vector is the whole space $\{0\}^{\perp} = X$ and $X^{\perp} = \{0\}$
(2) $U \subseteq V \Rightarrow U^{\perp} \supseteq V^{\perp}$
(3) If $x \perp y$, then $||x + y||^2_{<\cdot,\cdot>} = ||x||^2_{<\cdot,\cdot>} + ||y||^2_{<\cdot,\cdot>}$ (Pythagorean Theorem)

9 Continuity for Metric Spaces

Let $(X, d_X), (Y, d_Y)$ two metric spaces. A map $f : X \to Y$ is called:

(1) Continuous if the preimage $f^{-1}[B] = \{x \in X | f(x) \in B\}$ is open in X for all open sets $B \subseteq Y$ In the picture bellow we fix a set on the right had side and we look the set on the left hand side



$$G \xrightarrow{\mathcal{F}} H$$

In you do not have boundary points in the set B on the left you will not have the same for the set $f^{-1}[B]$ on the right.

(2) Sequentially continuous if for all $\tilde{x} \in X$ and $(x_n)_{n\in\mathbb{N}} \subseteq X$ with $x_n \xrightarrow{n\to\infty} \tilde{x}$ holds $f(x_n) \xrightarrow{n\to\infty} f(\tilde{x})$. A convergent sequence on the left hand side, correspond a convergent sequences on the right hand side.

In metric spaces, the two definitions are equivalent and we will say just continuous.

10 Bounded Operators

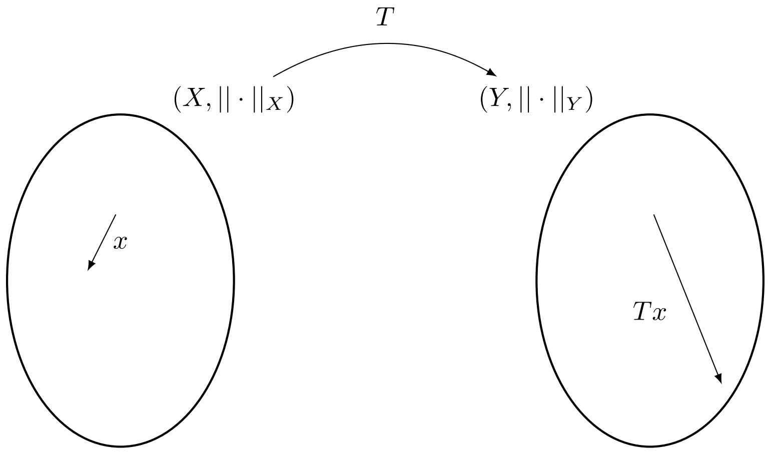
An operator $T : X \to Y$ is just a map that conserves some structures of our space. We call T an operator and not a function because often we have a space of functions as a domain or the co-domain. The first property that T should conserve is the algebraic structure (linear structure given in the vector space). In other words the map should be a linear map. The second property is that we need to conserve the topological structure, because in the metric space we have the notion of open sets and the properties that conserves open sets is the continuity (for normed spaces we define another property called bounded).

10.0.1 Definition

$(X, ||\cdot||_X), (Y, ||\cdot||_Y)$ two normed spaces. $T : X \to Y$ linear (which means $T(x + \tilde{x}) = Tx + T\tilde{x}$ and $T(\lambda x) = \lambda Tx \ \ \forall x, \tilde{x} \in X, \lambda \in \mathbb{F}$).

$$||T|| = ||T||_{X\to Y} := sup\left\{\frac{||T||_Y}{||x||_X} \,\middle|\, x \in X, x \neq 0\right\}$$

is called the operator norm of T If $||T|| < \infty, T$ is called bounded.



The vector x with length $||x||_X$ is transformed into the vector Tx with length $||Tx||_Y$

10.1 Proposition

Let $(X, ||\cdot||_X), (Y, ||\cdot||_Y)$ two normed spaces, $T : X \to Y$ linear. Then the following claims are equivalent:

(1) T is continuous
(2) T is continuous at $x = 0$
(3) T is bounded

10.2 Riesz representation theory

Let $(X, <\cdot, \cdot>)$ be a Hilbert Space. Then for each continuous-linear map $l : X \to \mathbb{F}$ (bounded-linear operator) called also functional, there is one $x_l \in X$ such that

$$l(x) = < x_l, x > \ \ \forall x \in X \quad \text{and} \quad ||l||_{X \to \mathbb{F}} = ||x_l||_X$$