Computing the bias of a coin - Bayesian style

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Let's imagine that we are flipping a coin, but we don't know how biased it is. Let's call the bias p (ie. if it's fair, then $p = 1/2$). How do we determine, through experiment, what p is? If we are able to flip the coin as many times as we want, then we can use the outcomes of this experiment to determine this. In other words, Bayes theorem allows us to quantify the hypothesis of induction! The major assumption made though, is that there exists a real distribution.
Let's define X_N to be the number of heads obtained after N flips of a coin which has bias p. Then X_N has a distribution given by

$$f(X_N = k|p) = \binom{N}{k} p^k (1-p)^{N-k}.$$

But we really want to know what is $f(p|X_n = k)$, ie. we want to learn the bias from the data. Thus we use Baye's theorem:

$$f(p|X_N = k) = \frac{f(X_N = k|p)f(p)}{f(X_n = k)}.$$

What is $f(X_n = k)$? This is summing up all possibilities over all probabilities. In other words, we have

$$f(X_n = k) = \int_0^1 \binom{N}{k} p^k (1-p)^{N-k} f(p)dp.$$

Thus we have

$$f(p|X_N = k) = \frac{\binom{N}{k} p^k (1-p)^{N-k} f(p)}{\int_0^1 \binom{N}{k} p^k (1-p)^{N-k} f(p)dp}.$$

What is $f(p)$? This is just our prior belief about what p is - since we don't know anything, we set it to be the *uniform distribution*, so $f(p) \equiv 1$. Thus we finally have

$$f(p|X_N = k) = \frac{p^k(1-p)^{N-k}}{\int_0^1 p^k(1-p)^{N-k}dp}.$$

For simplicity in the next sections, are going to write $N = n + m$ so that we have

$$f(p|X_n = m) = \frac{p^m(1-p)^n}{\int_0^1 p^m(1-p)^n dp}.$$

Also recall that the Beta function is defined as

$$B(n+1, m+1) = \int_0^1 p^n(1-p)^m dp.$$

0.1 Case 1: $m = 0$ as $n \to +\infty$

In this case we have

$$f(p|m = 0, n) = \frac{p^n}{\int_0^1 p^n dp} = (n+1)p^n.$$

It is easy to verify that the above converges in a distributional sense to $\delta_{p=1}(p)$. This works the same when $m = o(n)$ as $n \to +\infty$. We leave it as an exercise for the reader to ensure the calculations work the same.

0.2 Case 2: $m = n$ as $n \to +\infty$

$$\mathbb{E}(p|X_n = n) = \frac{B(n+2, n+1)}{B(n+1, n+1)} = \frac{n+1}{2n+3}\frac{B(n+1, n+1)}{B(n+1, n+1)} = \frac{n+1}{2n+3}.$$

$$\mathrm{Var}(p|X_n = n) = \frac{1}{B(n+1, n+1)}\int_0^1 (p - \frac{n+1}{2n+3})^2 p^n(1-p)^n dp \tag{0.1}$$

$$\frac{B(n+3, n+1) - B(n+2, n+1)(n+1)/(2n+3) + (n+1)^2/(2n+3)^2 B(n+1, n+1)}{B(n+1, n+1)}.$$

Using the identity $B(m+1, n) = \frac{m}{m+n}B(m, n)$ repeatedly and using the fact that $n \to +\infty$, we have

$$\mathrm{Var}(p|X_n = n) = \frac{1}{2}\left(1 - \frac{1}{2}\right) + o(n) \text{ as } n \to +\infty. \tag{0.2}$$

$$\mathbb{E}(p|X_n = n) = \frac{1}{2} + o(n) \text{ as } n \to +\infty. \tag{0.3}$$

Next, we need to check if $\{Z_n = p|X_n = n\}_{n\in\mathbb{N}}$ form an independent set of random variables. Indeed, interpreting $\{p|X_n = n\}$ as the bias on a coin for which we have observed n heads out of $2n$ trials, we have

$$\mathbb{E}(Z_m Z_n) = \frac{1}{B(m+1,m+1)B(n+1,n+1)} \int_0^1 \int_0^1 p_1 p_2 p_1^m (1-p_1)^m p_2^n (1-p_2)^n dp_1 dp_2 \tag{0.4}$$

$$= \mathbb{E}(Z_m)\mathbb{E}(Z_n) \tag{0.5}$$

ala Fubini's theorem. d
Thus $\{Z_i\}_i$ form an indepdent set of random variables with means converging to 1/2 and variances to $\frac{1}{2}(1-\frac{1}{2})$.
Now define $Z_i = p_i|(X_n = n)$ for $i = 1, \cdots, N$ (noting that n and N are unrelated!), and denote

$$d\mathbb{P}_n(p) = \frac{p^n(1-p)^n}{\int_0^1 p^n(1-p)^n dp}.$$

Then using (0.2) and (0.3) and the Central Limit Theorem applied to Z_i, we have

$$\int_0^1 \frac{1}{N} \sum_{i=1}^{N} f(p_i) d\mathbb{P}_n(p_i) = \frac{1}{N} \sum_{i=1}^{N} \int_0^1 f(p) d\mathbb{P}_n(p) \tag{0.6}$$

$$= \frac{N}{N} \int_0^1 f(p) d\mathbb{P}_n(p) \to \mathcal{N}(1/2 + o(n), 1/4 + o(n)). \tag{0.7}$$

Sending $n \to +\infty$ proves the result.