Se consideran $a, b \in \mathbb{R}$ con $b \neq 0$.

In[1]:=

$$\begin{pmatrix} a & b & b & b & b & b & b & b & b \\ b & a & b & b & b & b & b & b & b \\ b & b & a & b & b & b & b & b & b \\ b & b & b & a & b & b & b & b & b \\ b & b & b & b & a & b & b & b & b \\ b & b & b & b & b & a & b & b & b \\ b & b & b & b & b & b & a & b & b \\ b & b & b & b & b & b & b & a & b \\ b & b & b & b & b & b & b & b & a \end{pmatrix};$$

In[4]:=

```
Dimensions[%1]
```

Out[4]=

```
{9, 9}
```

In[2]:=

```
Det[%1 - t * IdentityMatrix[9]]
```

Out[2]=

$$a^9 - 36\,a^7\,b^2 + 168\,a^6\,b^3 - 378\,a^5\,b^4 + 504\,a^4\,b^5 - 420\,a^3\,b^6 + 216\,a^2\,b^7 - 63\,a\,b^8 + 8\,b^9 - 9\,a^8\,t + 252\,a^6\,b^2\,t - 1008\,a^5\,b^3\,t + 1890\,a^4\,b^4\,t - 2016\,a^3\,b^5\,t + 1260\,a^2\,b^6\,t - 432\,a\,b^7\,t + 63\,b^8\,t + 36\,a^7\,t^2 - 756\,a^5\,b^2\,t^2 + 2520\,a^4\,b^3\,t^2 - 3780\,a^3\,b^4\,t^2 + 3024\,a^2\,b^5\,t^2 - 1260\,a\,b^6\,t^2 + 216\,b^7\,t^2 - 84\,a^6\,t^3 + 1260\,a^4\,b^2\,t^3 - 3360\,a^3\,b^3\,t^3 + 3780\,a^2\,b^4\,t^3 - 2016\,a\,b^5\,t^3 + 420\,b^6\,t^3 + 126\,a^5\,t^4 - 1260\,a^3\,b^2\,t^4 + 2520\,a^2\,b^3\,t^4 - 1890\,a\,b^4\,t^4 + 504\,b^5\,t^4 - 126\,a^4\,t^5 + 756\,a^2\,b^2\,t^5 - 1008\,a\,b^3\,t^5 + 378\,b^4\,t^5 + 84\,a^3\,t^6 - 252\,a\,b^2\,t^6 + 168\,b^3\,t^6 - 36\,a^2\,t^7 + 36\,b^2\,t^7 + 9\,a\,t^8 - t^9$$

In[3]:=

```
Factor[%]
```

Out[3]=

$$(a - b - t)^8\,(a + 8\,b - t)$$

$a - b \neq a + 8\,b$

In[5]:=

```
MatrixForm[%1 - (a - b) * IdentityMatrix[9]]
```

Out[5]//MatrixForm=

$$\begin{pmatrix} b & b & b & b & b & b & b & b & b \\ b & b & b & b & b & b & b & b & b \\ b & b & b & b & b & b & b & b & b \\ b & b & b & b & b & b & b & b & b \\ b & b & b & b & b & b & b & b & b \\ b & b & b & b & b & b & b & b & b \\ b & b & b & b & b & b & b & b & b \\ b & b & b & b & b & b & b & b & b \\ b & b & b & b & b & b & b & b & b \end{pmatrix}$$

In[6]:=

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & -3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & -4 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & -5 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & -6 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -7 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -8 \end{pmatrix};$$

In[7]:=

```
MatrixForm[%1 - (a + 8 b) * IdentityMatrix[9]]
```

Out[7]//MatrixForm=

$$\begin{pmatrix} -8b & b & b & b & b & b & b & b & b \\ b & -8b & b & b & b & b & b & b & b \\ b & b & -8b & b & b & b & b & b & b \\ b & b & b & -8b & b & b & b & b & b \\ b & b & b & b & -8b & b & b & b & b \\ b & b & b & b & b & -8b & b & b & b \\ b & b & b & b & b & b & -8b & b & b \\ b & b & b & b & b & b & b & -8b & b \\ b & b & b & b & b & b & b & b & -8b \end{pmatrix}$$

In[8]:=

```
NullSpace[%1 - (a + 8 b) * IdentityMatrix[9]]
```

Out[8]=

```
{{1, 1, 1, 1, 1, 1, 1, 1, 1}}
```

In[10]:=

```
{{1, -1, 0, 0, 0, 0, 0, 0, 0}, {1, 1, -2, 0, 0, 0, 0, 0, 0},
 {1, 1, 1, -3, 0, 0, 0, 0, 0}, {1, 1, 1, 1, -4, 0, 0, 0, 0},
 {1, 1, 1, 1, 1, -5, 0, 0, 0}, {1, 1, 1, 1, 1, 1, -6, 0, 0},
 {1, 1, 1, 1, 1, 1, 1, -7, 0}, {1, 1, 1, 1, 1, 1, 1, 1, -8},
 {1, 1, 1, 1, 1, 1, 1, 1, 1}};
```

In[11]:=

```
MatrixForm[%]
```

Out[11]//MatrixForm=

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & -3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & -4 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & -5 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & -6 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -7 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

In[12]:=

```
Det[%]
```

Out[12]=

```
362880
```

In[13]:=

```
9!
```

Out[13]=

```
362880
```

In[14]:=

```
<< LinearAlgebra`Orthogonalization`;
```

In[15]:=

```
MatrixForm[GramSchmidt[%11]]
```

Out[15]//MatrixForm=

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{30}} & \frac{1}{\sqrt{30}} & \frac{1}{\sqrt{30}} & \frac{1}{\sqrt{30}} & \frac{1}{\sqrt{30}} & -\sqrt{\frac{5}{6}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{42}} & -\sqrt{\frac{6}{7}} & 0 & 0 \\ \frac{1}{2\sqrt{14}} & \frac{1}{2\sqrt{14}} & \frac{1}{2\sqrt{14}} & \frac{1}{2\sqrt{14}} & \frac{1}{2\sqrt{14}} & \frac{1}{2\sqrt{14}} & \frac{1}{2\sqrt{14}} & -\frac{\sqrt{\frac{7}{2}}}{2} & 0 \\ \frac{1}{6\sqrt{2}} & \frac{1}{6\sqrt{2}} & \frac{1}{6\sqrt{2}} & \frac{1}{6\sqrt{2}} & \frac{1}{6\sqrt{2}} & \frac{1}{6\sqrt{2}} & \frac{1}{6\sqrt{2}} & \frac{1}{6\sqrt{2}} & -\frac{2\sqrt{2}}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

In[16]:=

```
MatrixForm[Simplify[%.Transpose[%]]]
```

Out[16]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[17]:=

```
MatrixForm[Simplify[%15.%1.Transpose[%15]]]
```

Out[17]//MatrixForm=

$$\begin{pmatrix} a-b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a-b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a-b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a-b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a-b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a-b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a-b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a-b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a+8b \end{pmatrix}$$

In[1]:=

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

In[2]:=

$$E_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};$$

In[4]:=

$$E_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$

In[5]:=

$$E_{2,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix};$$

In[6]:=

$$E_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};$$

In[7]:=

```
b1[X_, Y_] := Tr[X.M.Y];
```

In[8]:=

$$\begin{pmatrix} b_1[E_{1,1}, E_{1,1}] & b_1[E_{1,1}, E_{1,2}] & b_1[E_{1,1}, E_{2,1}] & b_1[E_{1,1}, E_{2,2}] \\ b_1[E_{1,2}, E_{1,1}] & b_1[E_{1,2}, E_{1,2}] & b_1[E_{1,2}, E_{2,1}] & b_1[E_{1,2}, E_{2,2}] \\ b_1[E_{2,1}, E_{1,1}] & b_1[E_{2,1}, E_{1,2}] & b_1[E_{2,1}, E_{2,1}] & b_1[E_{2,1}, E_{2,2}] \\ b_1[E_{2,2}, E_{1,1}] & b_1[E_{2,2}, E_{1,2}] & b_1[E_{2,2}, E_{2,1}] & b_1[E_{2,2}, E_{2,2}] \end{pmatrix};$$

In[9]:=

```
MatrixForm[%]
```

Out[9]//MatrixForm=

$$\begin{pmatrix} a & 0 & b & 0 \\ c & 0 & d & 0 \\ 0 & a & 0 & b \\ 0 & c & 0 & d \end{pmatrix}$$

In[10]:=

$$\begin{pmatrix} b_1[E_{1,1}, E_{1,1}] & b_1[E_{1,1}, E_{2,1}] & b_1[E_{1,1}, E_{1,2}] & b_1[E_{1,1}, E_{2,2}] \\ b_1[E_{2,1}, E_{1,1}] & b_1[E_{2,1}, E_{2,1}] & b_1[E_{2,1}, E_{1,2}] & b_1[E_{2,1}, E_{2,2}] \\ b_1[E_{1,2}, E_{1,1}] & b_1[E_{1,2}, E_{2,1}] & b_1[E_{1,2}, E_{1,2}] & b_1[E_{1,2}, E_{2,2}] \\ b_1[E_{2,2}, E_{1,1}] & b_1[E_{2,2}, E_{2,1}] & b_1[E_{2,2}, E_{1,2}] & b_1[E_{2,2}, E_{2,2}] \end{pmatrix}$$

;

In[11]:=

```
MatrixForm[%]
```

Out[11]//MatrixForm=

$$\begin{pmatrix} a & b & 0 & 0 \\ 0 & 0 & a & b \\ c & d & 0 & 0 \\ 0 & 0 & c & d \end{pmatrix}$$

In[12]:=

```
b2[X_, Y_] := Tr[X.M.Transpose[Y]];
```

In[13]:=

$$\begin{pmatrix} b_2[E_{1,1}, E_{1,1}] & b_2[E_{1,1}, E_{1,2}] & b_2[E_{1,1}, E_{2,1}] & b_2[E_{1,1}, E_{2,2}] \\ b_2[E_{1,2}, E_{1,1}] & b_2[E_{1,2}, E_{1,2}] & b_2[E_{1,2}, E_{2,1}] & b_2[E_{1,2}, E_{2,2}] \\ b_2[E_{2,1}, E_{1,1}] & b_2[E_{2,1}, E_{1,2}] & b_2[E_{2,1}, E_{2,1}] & b_2[E_{2,1}, E_{2,2}] \\ b_2[E_{2,2}, E_{1,1}] & b_2[E_{2,2}, E_{1,2}] & b_2[E_{2,2}, E_{2,1}] & b_2[E_{2,2}, E_{2,2}] \end{pmatrix}$$

;

In[14]:=

```
MatrixForm[%]
```

Out[14]//MatrixForm=

$$\begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}$$

In[15]:=

$$\begin{pmatrix} b_2[E_{1,1}, E_{1,1}] & b_2[E_{1,1}, E_{2,1}] & b_2[E_{1,1}, E_{1,2}] & b_2[E_{1,1}, E_{2,2}] \\ b_2[E_{2,1}, E_{1,1}] & b_2[E_{2,1}, E_{2,1}] & b_2[E_{2,1}, E_{1,2}] & b_2[E_{2,1}, E_{2,2}] \\ b_2[E_{1,2}, E_{1,1}] & b_2[E_{1,2}, E_{2,1}] & b_2[E_{1,2}, E_{1,2}] & b_2[E_{1,2}, E_{2,2}] \\ b_2[E_{2,2}, E_{1,1}] & b_2[E_{2,2}, E_{2,1}] & b_2[E_{2,2}, E_{1,2}] & b_2[E_{2,2}, E_{2,2}] \end{pmatrix}$$

;

In[16]:=

```
MatrixForm[%]
```

Out[16]//MatrixForm=

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

In[17]:=

```
b3[X_, Y_] := Tr[Transpose[X].M.Y];
```

In[18]:=

$$\begin{pmatrix} b_3[E_{1,1}, E_{1,1}] & b_3[E_{1,1}, E_{1,2}] & b_3[E_{1,1}, E_{2,1}] & b_3[E_{1,1}, E_{2,2}] \\ b_3[E_{1,2}, E_{1,1}] & b_3[E_{1,2}, E_{1,2}] & b_3[E_{1,2}, E_{2,1}] & b_3[E_{1,2}, E_{2,2}] \\ b_3[E_{2,1}, E_{1,1}] & b_3[E_{2,1}, E_{1,2}] & b_3[E_{2,1}, E_{2,1}] & b_3[E_{2,1}, E_{2,2}] \\ b_3[E_{2,2}, E_{1,1}] & b_3[E_{2,2}, E_{1,2}] & b_3[E_{2,2}, E_{2,1}] & b_3[E_{2,2}, E_{2,2}] \end{pmatrix}$$

```
;
```

In[19]:=

```
MatrixForm[%]
```

Out[19]//MatrixForm=

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

In[20]:=

$$\begin{pmatrix} b_3[E_{1,1}, E_{1,1}] & b_3[E_{1,1}, E_{2,1}] & b_3[E_{1,1}, E_{1,2}] & b_3[E_{1,1}, E_{2,2}] \\ b_3[E_{2,1}, E_{1,1}] & b_3[E_{2,1}, E_{2,1}] & b_3[E_{2,1}, E_{1,2}] & b_3[E_{2,1}, E_{2,2}] \\ b_3[E_{1,2}, E_{1,1}] & b_3[E_{1,2}, E_{2,1}] & b_3[E_{1,2}, E_{1,2}] & b_3[E_{1,2}, E_{2,2}] \\ b_3[E_{2,2}, E_{1,1}] & b_3[E_{2,2}, E_{2,1}] & b_3[E_{2,2}, E_{1,2}] & b_3[E_{2,2}, E_{2,2}] \end{pmatrix}$$

```
;
```

In[21]:=

```
MatrixForm[%]
```

Out[21]//MatrixForm=

$$\begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}$$

In[22]:=

```
b4[X_, Y_] := Tr[Transpose[X].M.Transpose[Y]];
```

In[23]:=

$$\begin{pmatrix} b_4[E_{1,1}, E_{1,1}] & b_4[E_{1,1}, E_{1,2}] & b_4[E_{1,1}, E_{2,1}] & b_4[E_{1,1}, E_{2,2}] \\ b_4[E_{1,2}, E_{1,1}] & b_4[E_{1,2}, E_{1,2}] & b_4[E_{1,2}, E_{2,1}] & b_4[E_{1,2}, E_{2,2}] \\ b_4[E_{2,1}, E_{1,1}] & b_4[E_{2,1}, E_{1,2}] & b_4[E_{2,1}, E_{2,1}] & b_4[E_{2,1}, E_{2,2}] \\ b_4[E_{2,2}, E_{1,1}] & b_4[E_{2,2}, E_{1,2}] & b_4[E_{2,2}, E_{2,1}] & b_4[E_{2,2}, E_{2,2}] \end{pmatrix}$$

```
;
```

In[24]:=

```
MatrixForm[%]
```

Out[24]//MatrixForm=

$$\begin{pmatrix} a & b & 0 & 0 \\ 0 & 0 & a & b \\ c & d & 0 & 0 \\ 0 & 0 & c & d \end{pmatrix}$$

In[25]:=

$$\begin{pmatrix} b_4[E_{1,1}, E_{1,1}] & b_4[E_{1,1}, E_{2,1}] & b_4[E_{1,1}, E_{1,2}] & b_4[E_{1,1}, E_{2,2}] \\ b_4[E_{2,1}, E_{1,1}] & b_4[E_{2,1}, E_{2,1}] & b_4[E_{2,1}, E_{1,2}] & b_4[E_{2,1}, E_{2,2}] \\ b_4[E_{1,2}, E_{1,1}] & b_4[E_{1,2}, E_{2,1}] & b_4[E_{1,2}, E_{1,2}] & b_4[E_{1,2}, E_{2,2}] \\ b_4[E_{2,2}, E_{1,1}] & b_4[E_{2,2}, E_{2,1}] & b_4[E_{2,2}, E_{1,2}] & b_4[E_{2,2}, E_{2,2}] \end{pmatrix}$$

```
;
```

In[26]:=

```
MatrixForm[%]
```

Out[26]//MatrixForm=

$$\begin{pmatrix} a & 0 & b & 0 \\ c & 0 & d & 0 \\ 0 & a & 0 & b \\ 0 & c & 0 & d \end{pmatrix}$$

$$\mathcal{B}_1 = \left\{E_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_{2,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

$$\mathcal{B}_2 = \left\{E_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_{2,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

```
b1[X_, Y_] := Tr[X.M.Y]
```

$$\mathcal{M}(b_1, \mathcal{B}_1) = \begin{pmatrix} a & 0 & b & 0 \\ c & 0 & d & 0 \\ 0 & a & 0 & b \\ 0 & c & 0 & d \end{pmatrix} \quad \mathcal{M}(b_1, \mathcal{B}_2) = \begin{pmatrix} a & b & 0 & 0 \\ 0 & 0 & a & b \\ c & d & 0 & 0 \\ 0 & 0 & c & d \end{pmatrix}$$

```
b2[X_, Y_] := Tr[X.M.Transpose[Y]]
```

$$\mathcal{M}(b_2, \mathcal{B}_1) = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix} \quad \mathcal{M}(b_2, \mathcal{B}_2) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

```
b3[X_, Y_] := Tr[Transpose[X].M.Y]
```

$$\mathcal{M}(b_3, \mathcal{B}_1) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \quad \mathcal{M}(b_3, \mathcal{B}_2) = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}$$

```
b4[X_, Y_] := Tr[Transpose[X].M.Transpose[Y]]
```

$$\mathcal{M}(b_4, \mathcal{B}_1) = \begin{pmatrix} a & b & 0 & 0 \\ 0 & 0 & a & b \\ c & d & 0 & 0 \\ 0 & 0 & c & d \end{pmatrix} \quad \mathcal{M}(b_4, \mathcal{B}_2) = \begin{pmatrix} a & 0 & b & 0 \\ c & 0 & d & 0 \\ 0 & a & 0 & b \\ 0 & c & 0 & d \end{pmatrix}$$

In[1]:

```
p0[x] = 1;
```

In[2]:

```
p1[x] = x;
```

In[3]:

```
p2[x] = x^2;
```

In[4]:

```
p3[x] = x^3;
```

In[5]:

```
b1[p_, q_] := ∫_a^b p*q dx;
```

In[6]:

```
(
| b1[p0[x], p0[x]] | b1[p0[x], p1[x]] | b1[p0[x], p2[x]] | b1[p0[x], p3[x]] |
| b1[p1[x], p0[x]] | b1[p1[x], p1[x]] | b1[p1[x], p2[x]] | b1[p1[x], p3[x]] |
| b1[p2[x], p0[x]] | b1[p2[x], p1[x]] | b1[p2[x], p2[x]] | b1[p2[x], p3[x]] |
| b1[p3[x], p0[x]] | b1[p3[x], p1[x]] | b1[p3[x], p2[x]] | b1[p3[x], p3[x]] |
);
```

In[7]:

```
MatrixForm[%]
```

Out[7]//MatrixForm=

$$\begin{pmatrix} -a+b & -\frac{a^2}{2}+\frac{b^2}{2} & -\frac{a^3}{3}+\frac{b^3}{3} & -\frac{a^4}{4}+\frac{b^4}{4} \\ -\frac{a^2}{2}+\frac{b^2}{2} & -\frac{a^3}{3}+\frac{b^3}{3} & -\frac{a^4}{4}+\frac{b^4}{4} & -\frac{a^5}{5}+\frac{b^5}{5} \\ -\frac{a^3}{3}+\frac{b^3}{3} & -\frac{a^4}{4}+\frac{b^4}{4} & -\frac{a^5}{5}+\frac{b^5}{5} & -\frac{a^6}{6}+\frac{b^6}{6} \\ -\frac{a^4}{4}+\frac{b^4}{4} & -\frac{a^5}{5}+\frac{b^5}{5} & -\frac{a^6}{6}+\frac{b^6}{6} & -\frac{a^7}{7}+\frac{b^7}{7} \end{pmatrix}$$

In[8]:

```
MatrixForm[% /. {a → 0, b → 1}]
```

Out[8]//MatrixForm=

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix}$$

In[9]:=

```
MatrixForm[%% /. {a → -1, b → 1}]
```

Out[9]//MatrixForm=

$$\begin{pmatrix} 2 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{2}{5} \\ \frac{2}{3} & 0 & \frac{2}{5} & 0 \\ 0 & \frac{2}{5} & 0 & \frac{2}{7} \end{pmatrix}$$

In[10]:=

$$\begin{pmatrix} \mathtt{b_1[p_3[x],\ p_3[x]]} & \mathtt{b_1[p_3[x],\ p_2[x]]} & \mathtt{b_1[p_3[x],\ p_1[x]]} & \mathtt{b_1[p_3[x],\ p_0[x]]} \\ \mathtt{b_1[p_2[x],\ p_3[x]]} & \mathtt{b_1[p_2[x],\ p_2[x]]} & \mathtt{b_1[p_2[x],\ p_1[x]]} & \mathtt{b_1[p_2[x],\ p_0[x]]} \\ \mathtt{b_1[p_1[x],\ p_3[x]]} & \mathtt{b_1[p_1[x],\ p_2[x]]} & \mathtt{b_1[p_1[x],\ p_1[x]]} & \mathtt{b_1[p_1[x],\ p_0[x]]} \\ \mathtt{b_1[p_0[x],\ p_3[x]]} & \mathtt{b_1[p_0[x],\ p_2[x]]} & \mathtt{b_1[p_0[x],\ p_1[x]]} & \mathtt{b_1[p_0[x],\ p_0[x]]} \end{pmatrix};$$

In[11]:=

```
MatrixForm[%]
```

Out[11]//MatrixForm=

$$\begin{pmatrix} -\frac{a^7}{7}+\frac{b^7}{7} & -\frac{a^6}{6}+\frac{b^6}{6} & -\frac{a^5}{5}+\frac{b^5}{5} & -\frac{a^4}{4}+\frac{b^4}{4} \\ -\frac{a^6}{6}+\frac{b^6}{6} & -\frac{a^5}{5}+\frac{b^5}{5} & -\frac{a^4}{4}+\frac{b^4}{4} & -\frac{a^3}{3}+\frac{b^3}{3} \\ -\frac{a^5}{5}+\frac{b^5}{5} & -\frac{a^4}{4}+\frac{b^4}{4} & -\frac{a^3}{3}+\frac{b^3}{3} & -\frac{a^2}{2}+\frac{b^2}{2} \\ -\frac{a^4}{4}+\frac{b^4}{4} & -\frac{a^3}{3}+\frac{b^3}{3} & -\frac{a^2}{2}+\frac{b^2}{2} & -a+b \end{pmatrix}$$

In[12]:=

```
MatrixForm[% /. {a → 0, b → 1}]
```

Out[12]//MatrixForm=

$$\begin{pmatrix} \frac{1}{7} & \frac{1}{6} & \frac{1}{5} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 \end{pmatrix}$$

In[13]:=
```
MatrixForm[%% /. {a → -1, b → 1}]
```

Out[13]//MatrixForm=

$$\begin{pmatrix} \frac{2}{7} & 0 & \frac{2}{5} & 0 \\ 0 & \frac{2}{5} & 0 & \frac{2}{3} \\ \frac{2}{5} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & 2 \end{pmatrix}$$

In[14]:=
```
b2[p_, q_] := ∫_a^b p * q * x dx;
```

In[15]:=
```
( b2[p0[x], p0[x]] | b2[p0[x], p1[x]] | b2[p0[x], p2[x]] | b2[p0[x], p3[x]]
  b2[p1[x], p0[x]] | b2[p1[x], p1[x]] | b2[p1[x], p2[x]] | b2[p1[x], p3[x]]
  b2[p2[x], p0[x]] | b2[p2[x], p1[x]] | b2[p2[x], p2[x]] | b2[p2[x], p3[x]]
  b2[p3[x], p0[x]] | b2[p3[x], p1[x]] | b2[p3[x], p2[x]] | b2[p3[x], p3[x]] );
```

In[16]:=
```
MatrixForm[%]
```

Out[16]//MatrixForm=

$$\begin{pmatrix} -\frac{a^2}{2}+\frac{b^2}{2} & -\frac{a^3}{3}+\frac{b^3}{3} & -\frac{a^4}{4}+\frac{b^4}{4} & -\frac{a^5}{5}+\frac{b^5}{5} \\ -\frac{a^3}{3}+\frac{b^3}{3} & -\frac{a^4}{4}+\frac{b^4}{4} & -\frac{a^5}{5}+\frac{b^5}{5} & -\frac{a^6}{6}+\frac{b^6}{6} \\ -\frac{a^4}{4}+\frac{b^4}{4} & -\frac{a^5}{5}+\frac{b^5}{5} & -\frac{a^6}{6}+\frac{b^6}{6} & -\frac{a^7}{7}+\frac{b^7}{7} \\ -\frac{a^5}{5}+\frac{b^5}{5} & -\frac{a^6}{6}+\frac{b^6}{6} & -\frac{a^7}{7}+\frac{b^7}{7} & -\frac{a^8}{8}+\frac{b^8}{8} \end{pmatrix}$$

In[17]:=
```
b3[p_, q_] := ∫_a^b p * q * x^2 dx;
```

In[18]:=

$$\begin{pmatrix} b_3[p_0[x], p_0[x]] & b_3[p_0[x], p_1[x]] & b_3[p_0[x], p_2[x]] & b_3[p_0[x], p_3[x]] \\ b_3[p_1[x], p_0[x]] & b_3[p_1[x], p_1[x]] & b_3[p_1[x], p_2[x]] & b_3[p_1[x], p_3[x]] \\ b_3[p_2[x], p_0[x]] & b_3[p_2[x], p_1[x]] & b_3[p_2[x], p_2[x]] & b_3[p_2[x], p_3[x]] \\ b_3[p_3[x], p_0[x]] & b_3[p_3[x], p_1[x]] & b_3[p_3[x], p_2[x]] & b_3[p_3[x], p_3[x]] \end{pmatrix};$$

In[19]:=

```
MatrixForm[%]
```

Out[19]//MatrixForm=

$$\begin{pmatrix} -\frac{a^3}{3}+\frac{b^3}{3} & -\frac{a^4}{4}+\frac{b^4}{4} & -\frac{a^5}{5}+\frac{b^5}{5} & -\frac{a^6}{6}+\frac{b^6}{6} \\ -\frac{a^4}{4}+\frac{b^4}{4} & -\frac{a^5}{5}+\frac{b^5}{5} & -\frac{a^6}{6}+\frac{b^6}{6} & -\frac{a^7}{7}+\frac{b^7}{7} \\ -\frac{a^5}{5}+\frac{b^5}{5} & -\frac{a^6}{6}+\frac{b^6}{6} & -\frac{a^7}{7}+\frac{b^7}{7} & -\frac{a^8}{8}+\frac{b^8}{8} \\ -\frac{a^6}{6}+\frac{b^6}{6} & -\frac{a^7}{7}+\frac{b^7}{7} & -\frac{a^8}{8}+\frac{b^8}{8} & -\frac{a^9}{9}+\frac{b^9}{9} \end{pmatrix}$$

In[20]:=

$$b_4[p\_, q\_] := \int_a^b p*q*\frac{1}{x}\,dx;$$

In[21]:=

$$\begin{pmatrix} b_4[p_0[x], p_0[x]] & b_4[p_0[x], p_1[x]] & b_4[p_0[x], p_2[x]] & b_4[p_0[x], p_3[x]] \\ b_4[p_1[x], p_0[x]] & b_4[p_1[x], p_1[x]] & b_4[p_1[x], p_2[x]] & b_4[p_1[x], p_3[x]] \\ b_4[p_2[x], p_0[x]] & b_4[p_2[x], p_1[x]] & b_4[p_2[x], p_2[x]] & b_4[p_2[x], p_3[x]] \\ b_4[p_3[x], p_0[x]] & b_4[p_3[x], p_1[x]] & b_4[p_3[x], p_2[x]] & b_4[p_3[x], p_3[x]] \end{pmatrix};$$

In[22]:=

```
MatrixForm[%]
```

Out[22]//MatrixForm=

$$\begin{pmatrix} -\text{Log}[a]+\text{Log}[b] & -a+b & -\frac{a^2}{2}+\frac{b^2}{2} & -\frac{a^3}{3}+\frac{b^3}{3} \\ -a+b & -\frac{a^2}{2}+\frac{b^2}{2} & -\frac{a^3}{3}+\frac{b^3}{3} & -\frac{a^4}{4}+\frac{b^4}{4} \\ -\frac{a^2}{2}+\frac{b^2}{2} & -\frac{a^3}{3}+\frac{b^3}{3} & -\frac{a^4}{4}+\frac{b^4}{4} & -\frac{a^5}{5}+\frac{b^5}{5} \\ -\frac{a^3}{3}+\frac{b^3}{3} & -\frac{a^4}{4}+\frac{b^4}{4} & -\frac{a^5}{5}+\frac{b^5}{5} & -\frac{a^6}{6}+\frac{b^6}{6} \end{pmatrix}$$

In[23]:

```
b5[p_, q_] := ∫_a^b p*q* 1/x^2 dx;
```

In[24]:

$$\begin{pmatrix} b_5[p_0[x], p_0[x]] & b_5[p_0[x], p_1[x]] & b_5[p_0[x], p_2[x]] & b_5[p_0[x], p_3[x]] \\ b_5[p_1[x], p_0[x]] & b_5[p_1[x], p_1[x]] & b_5[p_1[x], p_2[x]] & b_5[p_1[x], p_3[x]] \\ b_5[p_2[x], p_0[x]] & b_5[p_2[x], p_1[x]] & b_5[p_2[x], p_2[x]] & b_5[p_2[x], p_3[x]] \\ b_5[p_3[x], p_0[x]] & b_5[p_3[x], p_1[x]] & b_5[p_3[x], p_2[x]] & b_5[p_3[x], p_3[x]] \end{pmatrix};$$

In[25]:

```
MatrixForm[%]
```

Out[25]//MatrixForm=

$$\begin{pmatrix} \frac{1}{a}-\frac{1}{b} & -\text{Log}[a]+\text{Log}[b] & -a+b & -\frac{a^2}{2}+\frac{b^2}{2} \\ -\text{Log}[a]+\text{Log}[b] & -a+b & -\frac{a^2}{2}+\frac{b^2}{2} & -\frac{a^3}{3}+\frac{b^3}{3} \\ -a+b & -\frac{a^2}{2}+\frac{b^2}{2} & -\frac{a^3}{3}+\frac{b^3}{3} & -\frac{a^4}{4}+\frac{b^4}{4} \\ -\frac{a^2}{2}+\frac{b^2}{2} & -\frac{a^3}{3}+\frac{b^3}{3} & -\frac{a^4}{4}+\frac{b^4}{4} & -\frac{a^5}{5}+\frac{b^5}{5} \end{pmatrix}$$

In[42]:

```
b6[p_, q_] := ∫_(-π/2)^(π/2) p*q*Sin[x] dx;
```

In[43]:

$$\begin{pmatrix} b_6[p_0[x], p_0[x]] & b_6[p_0[x], p_1[x]] & b_6[p_0[x], p_2[x]] & b_6[p_0[x], p_3[x]] \\ b_6[p_1[x], p_0[x]] & b_6[p_1[x], p_1[x]] & b_6[p_1[x], p_2[x]] & b_6[p_1[x], p_3[x]] \\ b_6[p_2[x], p_0[x]] & b_6[p_2[x], p_1[x]] & b_6[p_2[x], p_2[x]] & b_6[p_2[x], p_3[x]] \\ b_6[p_3[x], p_0[x]] & b_6[p_3[x], p_1[x]] & b_6[p_3[x], p_2[x]] & b_6[p_3[x], p_3[x]] \end{pmatrix};$$

In[44]:=

```
MatrixForm[%]
```

Out[44]//MatrixForm=

$$\begin{pmatrix} 0 & 2 & 0 & -12+\frac{3\pi^2}{2} \\ 2 & 0 & -12+\frac{3\pi^2}{2} & 0 \\ 0 & -12+\frac{3\pi^2}{2} & 0 & 240-30\pi^2+\frac{5\pi^4}{8} \\ -12+\frac{3\pi^2}{2} & 0 & 240-30\pi^2+\frac{5\pi^4}{8} & 0 \end{pmatrix}$$

In[45]:=

```
b7[p_, q_] := ∫_a^b D[p, x] * q dx;
```

In[46]:=

$$\begin{pmatrix} b_7[p_0[x], p_0[x]] & b_7[p_0[x], p_1[x]] & b_7[p_0[x], p_2[x]] & b_7[p_0[x], p_3[x]] \\ b_7[p_1[x], p_0[x]] & b_7[p_1[x], p_1[x]] & b_7[p_1[x], p_2[x]] & b_7[p_1[x], p_3[x]] \\ b_7[p_2[x], p_0[x]] & b_7[p_2[x], p_1[x]] & b_7[p_2[x], p_2[x]] & b_7[p_2[x], p_3[x]] \\ b_7[p_3[x], p_0[x]] & b_7[p_3[x], p_1[x]] & b_7[p_3[x], p_2[x]] & b_7[p_3[x], p_3[x]] \end{pmatrix};$$

In[47]:=

```
MatrixForm[%]
```

Out[47]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ -a+b & -\frac{a^2}{2}+\frac{b^2}{2} & -\frac{a^3}{3}+\frac{b^3}{3} & -\frac{a^4}{4}+\frac{b^4}{4} \\ 2\left(-\frac{a^2}{2}+\frac{b^2}{2}\right) & 2\left(-\frac{a^3}{3}+\frac{b^3}{3}\right) & 2\left(-\frac{a^4}{4}+\frac{b^4}{4}\right) & 2\left(-\frac{a^5}{5}+\frac{b^5}{5}\right) \\ 3\left(-\frac{a^3}{3}+\frac{b^3}{3}\right) & 3\left(-\frac{a^4}{4}+\frac{b^4}{4}\right) & 3\left(-\frac{a^5}{5}+\frac{b^5}{5}\right) & 3\left(-\frac{a^6}{6}+\frac{b^6}{6}\right) \end{pmatrix}$$

In[48]:=

```
b8[p_, q_] := ∫_a^b D[p, x] * D[q, {x, 2}] dx;
```

In[49]:=

```
( {{b8[p0[x], p0[x]], b8[p0[x], p1[x]], b8[p0[x], p2[x]], b8[p0[x], p3[x]]},
  {b8[p1[x], p0[x]], b8[p1[x], p1[x]], b8[p1[x], p2[x]], b8[p1[x], p3[x]]},
  {b8[p2[x], p0[x]], b8[p2[x], p1[x]], b8[p2[x], p2[x]], b8[p2[x], p3[x]]},
  {b8[p3[x], p0[x]], b8[p3[x], p1[x]], b8[p3[x], p2[x]], b8[p3[x], p3[x]]} } );
```

In[50]:=

```
MatrixForm[%]
```

Out[50]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2a+2b & 6\left(-\frac{a^2}{2}+\frac{b^2}{2}\right) \\ 0 & 0 & 4\left(-\frac{a^2}{2}+\frac{b^2}{2}\right) & 12\left(-\frac{a^3}{3}+\frac{b^3}{3}\right) \\ 0 & 0 & 6\left(-\frac{a^3}{3}+\frac{b^3}{3}\right) & 18\left(-\frac{a^4}{4}+\frac{b^4}{4}\right) \end{pmatrix}$$

In[52]:=

```
b9[p_, q_] := (p /. x → a) * (q /. x → b);
```

In[53]:=

```
( {{b9[p0[x], p0[x]], b9[p0[x], p1[x]], b9[p0[x], p2[x]], b9[p0[x], p3[x]]},
  {b9[p1[x], p0[x]], b9[p1[x], p1[x]], b9[p1[x], p2[x]], b9[p1[x], p3[x]]},
  {b9[p2[x], p0[x]], b9[p2[x], p1[x]], b9[p2[x], p2[x]], b9[p2[x], p3[x]]},
  {b9[p3[x], p0[x]], b9[p3[x], p1[x]], b9[p3[x], p2[x]], b9[p3[x], p3[x]]} } );
```

In[54]:

```
MatrixForm[%]
```

Out[54]//MatrixForm=

$$\begin{pmatrix} 1 & b & b^2 & b^3 \\ a & a\,b & a\,b^2 & a\,b^3 \\ a^2 & a^2\,b & a^2\,b^2 & a^2\,b^3 \\ a^3 & a^3\,b & a^3\,b^2 & a^3\,b^3 \end{pmatrix}$$

In[55]:

```
b10[p_, q_] := (p /. x → a) * (D[q, x] /. x → b);
```

In[56]:

$$\begin{pmatrix} b_{10}[p_0[x], p_0[x]] & b_{10}[p_0[x], p_1[x]] & b_{10}[p_0[x], p_2[x]] & b_{10}[p_0[x], p_3[x]] \\ b_{10}[p_1[x], p_0[x]] & b_{10}[p_1[x], p_1[x]] & b_{10}[p_1[x], p_2[x]] & b_{10}[p_1[x], p_3[x]] \\ b_{10}[p_2[x], p_0[x]] & b_{10}[p_2[x], p_1[x]] & b_{10}[p_2[x], p_2[x]] & b_{10}[p_2[x], p_3[x]] \\ b_{10}[p_3[x], p_0[x]] & b_{10}[p_3[x], p_1[x]] & b_{10}[p_3[x], p_2[x]] & b_{10}[p_3[x], p_3[x]] \end{pmatrix};$$

In[57]:

```
MatrixForm[%]
```

Out[57]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 2\,b & 3\,b^2 \\ 0 & a & 2\,a\,b & 3\,a\,b^2 \\ 0 & a^2 & 2\,a^2\,b & 3\,a^2\,b^2 \\ 0 & a^3 & 2\,a^3\,b & 3\,a^3\,b^2 \end{pmatrix}$$

In[1]:=

$$\begin{pmatrix} -4 & -1 & 0 & 1 & 0 & 1 & 0 & 4 & 0 \\ -1 & -3 & 1 & -3 & 0 & -3 & -1 & 1 & -1 \\ 0 & 1 & 1 & -1 & 1 & -3 & 3 & -1 & -1 \\ 1 & -3 & -1 & -4 & 0 & 3 & -1 & -2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & -3 & 3 & 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 3 & -1 & 0 & -1 & 0 & -2 & 0 \\ 4 & 1 & -1 & -2 & 0 & -1 & -2 & 0 & -2 \\ 0 & -1 & -1 & -1 & 0 & 1 & 0 & -2 & 0 \end{pmatrix};$$

In[2]:=

```
Dimensions[%]
```

Out[2]=

```
{9, 9}
```

In[3]:=

```
MatrixForm[%1 - Transpose[%1]]
```

Out[3]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[4]:=

```
Det[%1 - t * IdentityMatrix[9]]
```

Out[4]=

$$14980\,t + 20785\,t^2 - 20591\,t^3 - 13334\,t^4 - 2\,t^5 + 768\,t^6 + 60\,t^7 - 10\,t^8 - t^9$$

In[5]:=

```
CoefficientList[%, t]
```

Out[5]=

```
{0, 14980, 20785, -20591, -13334, -2, 768, 60, -10, -1}
```

$F_1 \leftrightarrow F_3$; $C_1 \leftrightarrow C_3$

In[6]:=

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

In[7]:=

```
MatrixForm[%.%1.Transpose[%]]
```

Out[7]//MatrixForm=

$$\begin{pmatrix} 1 & 1 & 0 & -1 & 1 & -3 & 3 & -1 & -1 \\ 1 & -3 & -1 & -3 & 0 & -3 & -1 & 1 & -1 \\ 0 & -1 & -4 & 1 & 0 & 1 & 0 & 4 & 0 \\ -1 & -3 & 1 & -4 & 0 & 3 & -1 & -2 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & -3 & 1 & 3 & 0 & 0 & -1 & -1 & 1 \\ 3 & -1 & 0 & -1 & 0 & -1 & 0 & -2 & 0 \\ -1 & 1 & 4 & -2 & 0 & -1 & -2 & 0 & -2 \\ -1 & -1 & 0 & -1 & 0 & 1 & 0 & -2 & 0 \end{pmatrix}$$

$$\begin{aligned} &F_2 \longrightarrow F_2 - F_1;\ C_2 \longrightarrow C_2 - C_1 \\ &F_4 \longrightarrow F_4 + F_1;\ C_4 \longrightarrow C_4 + C_1 \\ &F_5 \longrightarrow F_5 - F_1;\ C_5 \longrightarrow C_5 - C_1 \\ &F_6 \longrightarrow F_6 + 3\,F_1;\ C_6 \longrightarrow C_6 + 3\,C_1 \\ &F_7 \longrightarrow F_7 - 3\,F_1;\ C_7 \longrightarrow C_7 - 3\,C_1 \\ &F_8 \longrightarrow F_8 + F_1;\ C_8 \longrightarrow C_8 + C_1 \\ &F_9 \longrightarrow F_9 + F_1;\ C_9 \longrightarrow C_9 + C_1 \end{aligned}$$

In[8]:=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

In[9]:

```
MatrixForm[%.%7.Transpose[%]]
```

Out[9]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & -1 & -2 & -1 & 0 & -4 & 2 & 0 \\ 0 & -1 & -4 & 1 & 0 & 1 & 0 & 4 & 0 \\ 0 & -2 & 1 & -5 & 1 & 0 & 2 & -3 & -2 \\ 0 & -1 & 0 & 1 & -1 & 3 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 3 & -9 & 8 & -4 & -2 \\ 0 & -4 & 0 & 2 & -3 & 8 & -9 & 1 & 3 \\ 0 & 2 & 4 & -3 & 1 & -4 & 1 & -1 & -3 \\ 0 & 0 & 0 & -2 & 1 & -2 & 3 & -3 & -1 \end{pmatrix}$$

$F_2 \leftrightarrow F_5 ; C_2 \leftrightarrow C_5$

In[10]:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

In[11]:

```
MatrixForm[%.%9.Transpose[%]]
```

Out[11]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 3 & -3 & 1 & 1 \\ 0 & 0 & -4 & 1 & -1 & 1 & 0 & 4 & 0 \\ 0 & 1 & 1 & -5 & -2 & 0 & 2 & -3 & -2 \\ 0 & -1 & -1 & -2 & -4 & 0 & -4 & 2 & 0 \\ 0 & 3 & 1 & 0 & 0 & -9 & 8 & -4 & -2 \\ 0 & -3 & 0 & 2 & -4 & 8 & -9 & 1 & 3 \\ 0 & 1 & 4 & -3 & 2 & -4 & 1 & -1 & -3 \\ 0 & 1 & 0 & -2 & 0 & -2 & 3 & -3 & -1 \end{pmatrix}$$

$F_4 \longrightarrow F_4 + F_2;\ C_4 \longrightarrow C_4 + C_2$
$F_5 \longrightarrow F_5 - F_2;\ C_5 \longrightarrow C_5 - C_2$
$F_6 \longrightarrow F_6 + 3\,F_2;\ C_6 \longrightarrow C_6 + 3\,C_2$
$F_7 \longrightarrow F_7 - 3\,F_2;\ C_7 \longrightarrow C_7 - 3\,C_2$
$F_8 \longrightarrow F_8 + F_2;\ C_8 \longrightarrow C_8 + C_2$
$F_9 \longrightarrow F_9 + F_2;\ C_9 \longrightarrow C_9 + C_2$

In[12]:=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

In[13]:=

```
MatrixForm[%.%11.Transpose[%]]
```

Out[13]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 1 & -1 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & -4 & -3 & 3 & -1 & -2 & -1 \\ 0 & 0 & -1 & -3 & -3 & -3 & -1 & 1 & -1 \\ 0 & 0 & 1 & 3 & -3 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 4 & -2 & 1 & -1 & -2 & 0 & -2 \\ 0 & 0 & 0 & -1 & -1 & 1 & 0 & -2 & 0 \end{pmatrix}$$

$F_5 \longrightarrow F_5 + F_8 \; ; \; C_5 \longrightarrow C_5 + C_8$

In[14]:=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} ;$$

In[15]:=

```
MatrixForm[%.%13.Transpose[%]]
```

Out[15]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 1 & 3 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & -4 & -5 & 3 & -1 & -2 & -1 \\ 0 & 0 & 3 & -5 & -1 & -4 & -3 & 1 & -3 \\ 0 & 0 & 1 & 3 & -4 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 & -3 & -1 & 0 & -2 & 0 \\ 0 & 0 & 4 & -2 & 1 & -1 & -2 & 0 & -2 \\ 0 & 0 & 0 & -1 & -3 & 1 & 0 & -2 & 0 \end{pmatrix}$$

$F_3 \longleftrightarrow F_5 \; ; \; C_3 \longleftrightarrow C_5$

In[16]:=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} ;$$

In[17]:=

```
MatrixForm[%.%15.Transpose[%]]
```

Out[17]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -5 & 3 & -4 & -3 & 1 & -3 \\ 0 & 0 & -5 & -4 & 1 & 3 & -1 & -2 & -1 \\ 0 & 0 & 3 & 1 & -4 & 1 & 0 & 4 & 0 \\ 0 & 0 & -4 & 3 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & -3 & -1 & 0 & -1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -2 & 4 & -1 & -2 & 0 & -2 \\ 0 & 0 & -3 & -1 & 0 & 1 & 0 & -2 & 0 \end{pmatrix}$$

$$\begin{aligned} &F_4 \longrightarrow F_4 - 5\,F_3;\ C_4 \longrightarrow C_4 - 5\,C_3 \\ &F_5 \longrightarrow F_5 + 3\,F_3;\ C_5 \longrightarrow C_5 + 3\,C_3 \\ &F_6 \longrightarrow F_6 - 4\,F_3;\ C_6 \longrightarrow C_6 - 4\,C_3 \\ &F_7 \longrightarrow F_7 - 3\,F_3;\ C_7 \longrightarrow C_7 - 3\,C_3 \\ &F_8 \longrightarrow F_8 + F_3;\ C_8 \longrightarrow C_8 + C_3 \\ &F_9 \longrightarrow F_9 - 3\,F_3;\ C_9 \longrightarrow C_9 - 3\,C_3 \end{aligned}$$

In[18]:=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

In[19]:=

```
MatrixForm[%.%17.Transpose[%]]
```

Out[19]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 21 & -14 & 23 & 14 & -7 & 14 \\ 0 & 0 & 0 & -14 & 5 & -11 & -9 & 7 & -9 \\ 0 & 0 & 0 & 23 & -11 & 16 & 11 & -5 & 13 \\ 0 & 0 & 0 & 14 & -9 & 11 & 9 & -5 & 9 \\ 0 & 0 & 0 & -7 & 7 & -5 & -5 & 1 & -5 \\ 0 & 0 & 0 & 14 & -9 & 13 & 9 & -5 & 9 \end{pmatrix}$$

$$F_4 \leftrightarrow F_8 \,;\; C_4 \leftrightarrow C_8$$

In[20]:=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} ;$$

In[21]:=

```
MatrixForm[%.%19.Transpose[%]]
```

Out[21]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 7 & -5 & -5 & -7 & -5 \\ 0 & 0 & 0 & 7 & 5 & -11 & -9 & -14 & -9 \\ 0 & 0 & 0 & -5 & -11 & 16 & 11 & 23 & 13 \\ 0 & 0 & 0 & -5 & -9 & 11 & 9 & 14 & 9 \\ 0 & 0 & 0 & -7 & -14 & 23 & 14 & 21 & 14 \\ 0 & 0 & 0 & -5 & -9 & 13 & 9 & 14 & 9 \end{pmatrix}$$

$$F_5 \longrightarrow F_5 - 7\,F_4 \,;\; C_5 \longrightarrow C_5 - 7\,C_4$$
$$F_6 \longrightarrow F_6 + 5\,F_4 \,;\; C_6 \longrightarrow C_6 + 5\,C_4$$
$$F_7 \longrightarrow F_7 + 5\,F_4 \,;\; C_7 \longrightarrow C_7 + 5\,C_4$$
$$F_8 \longrightarrow F_8 + 7\,F_4 \,;\; C_8 \longrightarrow C_8 + 7\,C_4$$
$$F_9 \longrightarrow F_9 + 5\,F_4 \,;\; C_9 \longrightarrow C_9 + 5\,C_4$$

In[22]:=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

In[23]:

```
MatrixForm[%.%21.Transpose[%]]
```

Out[23]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -44 & 24 & 26 & 35 & 26 \\ 0 & 0 & 0 & 0 & 24 & -9 & -14 & -12 & -12 \\ 0 & 0 & 0 & 0 & 26 & -14 & -16 & -21 & -16 \\ 0 & 0 & 0 & 0 & 35 & -12 & -21 & -28 & -21 \\ 0 & 0 & 0 & 0 & 26 & -12 & -16 & -21 & -16 \end{pmatrix}$$

$F_7 \longrightarrow F_7 - F_9;\ C_7 \longrightarrow C_7 - C_9$

In[24]:=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

In[25]:=

```
MatrixForm[%.%23.Transpose[%]]
```

Out[25]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -44 & 24 & 0 & 35 & 26 \\ 0 & 0 & 0 & 0 & 24 & -9 & -2 & -12 & -12 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 35 & -12 & 0 & -28 & -21 \\ 0 & 0 & 0 & 0 & 26 & -12 & 0 & -21 & -16 \end{pmatrix}$$

$F_6 \longrightarrow F_6 - 2\,F_7;\ C_6 \longrightarrow C_6 - 2\,C_7$

In[26]:=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

In[27]:=

```
MatrixForm[%.%25.Transpose[%]]
```

Out[27]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -44 & 24 & 0 & 35 & 26 \\ 0 & 0 & 0 & 0 & 24 & -1 & -2 & -12 & -12 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 35 & -12 & 0 & -28 & -21 \\ 0 & 0 & 0 & 0 & 26 & -12 & 0 & -21 & -16 \end{pmatrix}$$

$F_5 \leftrightarrow F_6;\ C_5 \leftrightarrow C_6$

In[28]:=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

In[29]:=
```
MatrixForm[%.%27.Transpose[%]]
```

Out[29]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 24 & -2 & -12 & -12 \\ 0 & 0 & 0 & 0 & 24 & -44 & 0 & 35 & 26 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -12 & 35 & 0 & -28 & -21 \\ 0 & 0 & 0 & 0 & -12 & 26 & 0 & -21 & -16 \end{pmatrix}$$

$$\begin{aligned} &F_6 \longrightarrow F_6 + 24\, F_5;\ C_6 \longrightarrow C_6 + 24\, C_5 \\ &F_7 \longrightarrow F_7 - 2\, F_5;\ C_7 \longrightarrow C_7 - 2\, C_5 \\ &F_8 \longrightarrow F_8 - 12\, F_5;\ C_8 \longrightarrow C_8 - 12\, C_5 \\ &F_9 \longrightarrow F_9 - 12\, F_5;\ C_9 \longrightarrow C_9 - 12\, C_5 \end{aligned}$$

In[30]:=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 24 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -12 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -12 & 0 & 0 & 0 & 1 \end{pmatrix};$$

In[31]:=
```
MatrixForm[%.%29.Transpose[%]]
```

Out[31]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 532 & -48 & -253 & -262 \\ 0 & 0 & 0 & 0 & 0 & -48 & 4 & 24 & 24 \\ 0 & 0 & 0 & 0 & 0 & -253 & 24 & 116 & 123 \\ 0 & 0 & 0 & 0 & 0 & -262 & 24 & 123 & 128 \end{pmatrix}$$

$$F_7 \leftrightarrow \frac{1}{2} F_7 ; C_7 \leftrightarrow \frac{1}{2} C_7$$

In[32]:=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} ;$$

In[33]:=

```
MatrixForm[%.%31.Transpose[%]]
```

Out[33]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 532 & -24 & -253 & -262 \\ 0 & 0 & 0 & 0 & 0 & -24 & 1 & 12 & 12 \\ 0 & 0 & 0 & 0 & 0 & -253 & 12 & 116 & 123 \\ 0 & 0 & 0 & 0 & 0 & -262 & 12 & 123 & 128 \end{pmatrix}$$

$$F_6 \leftrightarrow F_7 ; C_6 \leftrightarrow C_7$$

In[34]:=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

In[35]:=

```
MatrixForm[%.%33.Transpose[%]]
```

Out[35]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -24 & 12 & 12 \\ 0 & 0 & 0 & 0 & 0 & -24 & 532 & -253 & -262 \\ 0 & 0 & 0 & 0 & 0 & 12 & -253 & 116 & 123 \\ 0 & 0 & 0 & 0 & 0 & 12 & -262 & 123 & 128 \end{pmatrix}$$

$$\begin{aligned} &F_7 \longrightarrow F_7 + 24\, F_6;\ C_7 \longrightarrow C_7 + 24\, C_6 \\ &F_8 \longrightarrow F_8 - 12\, F_6;\ C_8 \longrightarrow C_8 - 12\, C_6 \\ &F_9 \longrightarrow F_9 - 12\, F_6;\ C_9 \longrightarrow C_9 - 12\, C_6 \end{aligned}$$

In[36]:=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 24 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -12 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -12 & 0 & 0 & 1 \end{pmatrix};$$

In[37]:=

```
MatrixForm[%.%35.Transpose[%]]
```

Out[37]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -44 & 35 & 26 \\ 0 & 0 & 0 & 0 & 0 & 0 & 35 & -28 & -21 \\ 0 & 0 & 0 & 0 & 0 & 0 & 26 & -21 & -16 \end{pmatrix}$$

$F_7 \longrightarrow F_7 + F_8;\ C_7 \longrightarrow C_7 + C_8$

In[38]:=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

In[39]:=

```
MatrixForm[%.%37.Transpose[%]]
```

Out[39]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 7 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & -28 & -21 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & -21 & -16 \end{pmatrix}$$

$F_8 \longrightarrow F_8 - F_9;\ C_8 \longrightarrow C_8 - C_9$

In[40]:=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

In[41]:=

```
MatrixForm[%.%39.Transpose[%]]
```

Out[41]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & -5 & -16 \end{pmatrix}$$

$$\boxed{F_8 \longrightarrow F_8 + F_7;\ C_8 \longrightarrow C_8 + C_7}$$

In[42]:=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

In[43]:=

```
MatrixForm[%.%41.Transpose[%]]
```

Out[43]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & -16 \end{pmatrix}$$

$$F_9 \longrightarrow F_9 + \frac{5}{2} F_7;\ C_9 \longrightarrow C_9 + \frac{5}{2} C_7$$

In[44]:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{2} & 0 & 1 \end{pmatrix};$$

In[45]:

```
MatrixForm[%.%43.Transpose[%]]
```

Out[45]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{7}{2} \end{pmatrix}$$

$$F_7 \leftrightarrow \frac{1}{\sqrt{2}} F_7 ; C_7 \leftrightarrow \frac{1}{\sqrt{2}} C_7$$

$$F_9 \leftrightarrow \frac{\sqrt{2}}{\sqrt{7}} F_9 ; C_9 \leftrightarrow \frac{\sqrt{2}}{\sqrt{7}} C_9$$

In[46]:=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{\sqrt{7}} \end{pmatrix} ;$$

In[47]:=

```
MatrixForm[%.%45.Transpose[%]]
```

Out[47]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$F_2 \leftrightarrow F_4 ; C_2 \leftrightarrow C_4$$

$$F_3 \leftrightarrow F_6 ; C_3 \leftrightarrow C_6$$

$$F_8 \leftrightarrow F_9 ; C_8 \leftrightarrow C_9$$

In[48]:=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix};$$

In[49]:=
```
MatrixForm[%.%47.Transpose[%]]
```

Out[49]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[50]:=
```
MatrixForm[
 Simplify[%48.%46.%44.%42.%40.%38.%36.%34.%32.%30.
   %28.%26.%24.%22.%20.%18.%16.%14.%12.%10.%8.%6]]
```

Out[50]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & -18 & -1 & \frac{5}{2} & -6 & -\frac{5}{2} \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 16 & 1 & -2 & 6 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & -\sqrt{2} & 0 & \frac{1}{\sqrt{2}} & -\frac{23}{\sqrt{2}} & 0 & 3\sqrt{2} & -\sqrt{2} & -3\sqrt{2} \\ \frac{5}{\sqrt{14}} & -3\sqrt{\frac{2}{7}} & 0 & \frac{5}{\sqrt{14}} & -\frac{55}{\sqrt{14}} & 0 & 9\sqrt{\frac{2}{7}} & 2\sqrt{\frac{2}{7}} & -8\sqrt{\frac{2}{7}} \\ 1 & -2 & 0 & 2 & -21 & 0 & 6 & 0 & -7 \end{pmatrix}$$

In[51]:=
```
MatrixForm[Simplify[%50.%1.Transpose[%50]]]
```

Out[51]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$