AMATH741/CM750/CS778 Winter 2023: Assignment 3

Instr.: Hans De Sterck e-mail: *hdesterck@uwaterloo.ca* Office hrs: Wednesdays 2-3pm

Assignment due date: Friday March 17, 5pm (dual submission, on Crowdmark and on LEARN)

Submission instructions:

- **All questions (theoretical (1–2) and computational (3–4)):** Please submit your assignment electronically via Crowdmark (you will receive a Crowdmark invitation). Your assignment package should contain the full answers to all questions, showing all your work. You can handwrite your answers on paper and then submit photos of your answers on Crowdmark. Another option is to type up your answers (e.g. using latex) and submit the electronic document. In either case, please start your answer for each question on a new page; Crowdmark requires that you submit a separate file or files for each question.

- **Computational questions (3–4):** The Crowdmark submission for Questions 3–4 should include written or typed answers to all the actual questions asked in the problems, including any computer output (tables or plots) that is required to answer the questions. In addition, a pdf or screenshot of all computer code for the programming Questions 3–4 should also be included in the Crowdmark submission (to facilitate marking feedback on your computer code). In addition to the Crowdmark submission, you must also separately submit all computer code files for the programming Questions 3–4 to the LEARN dropbox for Assignment 1. The LEARN computer code submission is due at the same time as the Crowdmark submission.

1. **Riemann problem for a linear hyperbolic system.** (10 marks)

   Consider linear hyperbolic system $U_t + A\,U_x = 0$ with

   $$A = \begin{bmatrix} 10 & 6 \\ -18 & -11 \end{bmatrix}.$$

   What is the solution of the Riemann problem with

   $$U_l = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad U_r = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \qquad ?$$

   Give a full description of the solution. What is U^*? (The solution state between U_l and U_r). To illustrate the solution, give plots of u_1 and u_2 as a function of x, for $t = 0$ and for $t = 1$, and give a plot of the solution in the xt plane (indicate in which parts of the plane the solution is U_l, U^* and U_r).

2. **Advection of density profile in the Euler equations.** (10 marks)

The Euler Equations in one spatial dimension x and with one vector component u (in the x direction) are given by

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\,u\\ e\end{bmatrix}+\frac{\partial}{\partial x}\begin{bmatrix}\rho\,u\\ \rho\,u^2+p\\ (e+p)\,u\end{bmatrix}=0,$$

where $e=p/(\gamma-1)+\rho u^2/2$. Show that

$$U=\begin{bmatrix}\rho\\ \rho\,\overline{u}\\ \overline{p}/(\gamma-1)+\rho\overline{u}^2/2\end{bmatrix}$$

with $\overline{u}$ and $\overline{p}$ constant in space and time, and $\rho(x,t)$ given by any function $f(x-\overline{u}\,t)$, is a solution of the equations. This means that any density profile advected with constant velocity $\overline{u}$ in a background of constant pressure $\overline{p}$ is a solution of the Euler equations. (Note that the existence of this type of advective solution is the consequence of the second eigenvalue of the Jacobian, $\lambda_2=u$, being associated with a wave field that is 'linearly degenerate'. Since the second wave field has this property, there exist 'simple wave' solutions for it that have straight and parallel λ_2-characteristics, just like in linear advection.)

3. **Dam breaking problem in 1D.** (15 marks)

Consider the Shallow Water equations in 1D:

$$\frac{\partial}{\partial t}\begin{bmatrix}h\\ h\,u\\ h\,v\end{bmatrix}+\frac{\partial}{\partial x}\begin{bmatrix}h\,u\\ h\,u^2+g\,h^2/2\\ h\,u\,v\end{bmatrix}=0, \tag{1}$$

where h is the water height and (u,v) is the 2D velocity vector. Assume that the gravitational acceleration $g=1$. Consider the following Riemann IVP:

$$\begin{array}{ll}
\text{PDE (1)} & \text{on } \Omega,\\
\Omega\text{: } x\in[-1,1],\ t\in[0,1/2], & \\
\vec{u}(x,0)=[2\ 0\ 0]^T & \text{for } x\in[-1,0),\\
\vec{u}(x,0)=[1\ 0\ 0]^T & \text{for } x\in[0,1].
\end{array}$$

Here $\vec{u}$ is the vector $[h\ hu\ hv]^T$. This Riemann problem models the breaking of a dam with initial water height $h=2$ on the upstream side of the dam, and $h=1$ on the downstream side.

Write a matlab function that simulates this dam breaking problem using a Finite Volume method with numerical flux function of Lax-Friedrichs type:

$$\vec{v}_i^{\,n+1}=\vec{v}_i^{\,n}-\frac{\Delta t}{\Delta x}\left(\vec{f}^{*n}_{i+\frac{1}{2}}-\vec{f}^{*n}_{i-\frac{1}{2}}\right). \tag{2}$$

with

$$\vec{f}^{*}_{i+\frac{1}{2}}=\frac{1}{2}\left[\vec{f}(\vec{v}_i)+\vec{f}(\vec{v}_{i+1})\right]-\frac{1}{2}\overline{|\lambda|_{max}}\left(\vec{v}_{i+1}-\vec{v}_i\right), \tag{3}$$

where $\overline{|\lambda|_{max}}=\max(|\lambda_i|_{max},|\lambda_{i+1}|_{max})$ is the maximum of the largest eigenvalues in absolute value of the Jacobian matrix of the hyperbolic system in states i and $i+1$ ($|\lambda|_{max}=|u|+\sqrt{gh}$ in this case).

You can use ghost cells at the spatial boundaries, but there is no need to change the ghost cell values as the simulation progresses (no waves reach the boundaries before $t = 1/2$). The time step should be chosen dynamically in every step according to

$$\Delta t = c \ \min\left(\frac{\Delta x}{|\lambda|_{max}}\right),$$

where the minimum can be taken over all physical cells. The CFL safety constant c may have to be chosen smaller than 1 for this nonlinear system in order to avoid oscillations (for example, $c = 0.5$).

Submit plots of h, u and v for $t = 0.25$ and $t = 0.5$, using $m = 100$ physical cells in your simulations (which means $\Delta x = 2/100$). Explain in a few sentences what types of waves you see in the simulation results (shocks and/or rarefactions), and give some physical interpretation to relate the results to what happens when a dam breaks.

Submit your program files to the LEARN drop box. The code submitted should use $m = 100$ and should produce plots of h, u and v at $t = 0.5$. Also, include a text copy of the code in your assignment package to be submitted on Crowdmark.

4. **Extension of the dam breaking problem to 2D.** (15 marks)

   Consider the Shallow Water equations in 2D:

$$\frac{\partial}{\partial t}\begin{bmatrix} h \\ h\,u \\ h\,v \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} h\,u \\ h\,u^2 + g\,h^2/2 \\ h\,u\,v \end{bmatrix} + \frac{\partial}{\partial y}\begin{bmatrix} h\,v \\ h\,u\,v \\ h\,v^2 + g\,h^2/2 \end{bmatrix} = 0. \tag{4}$$

   Assume again that the gravitational acceleration $g = 1$. Consider the following IVBVP:

PDE (4)	on Ω,
Ω: $(x, y) \in [-1, 1]^2$, $t \in [0, 3]$,	
$\vec{u}(x, y, 0) = [2\ 0\ 0]^T$	for $(x, y) \in [-1/2, 1/2]^2$,
$\vec{u}(x, y, 0) = [1\ 0\ 0]^T$	otherwise
spatial boundaries are perfect walls.	

   At a perfect wall no fluid is allowed to penetrate the wall.

   Write a matlab function that simulates this IVBVP problem using a FV method of Lax-Friedrichs type in 2D.

   Use ghost cells at the spatial boundaries. Before every timestep, copy the values of $\vec{v}$ from the nearest physical cells to the ghost cells, except for the velocity component normal to the boundary, which has to be copied over with the sign reversed. By this trick, the effective normal velocity at the interface between the last physical cell and the ghost cell will be close to zero, as required for a perfect wall.

   The time step should be chosen dynamically in every step according to

$$\Delta t = c/2 \ \min\left(\min\left(\frac{\Delta x}{|\lambda_x|_{max}}\right), \min\left(\frac{\Delta y}{|\lambda_y|_{max}}\right)\right),$$

   where the two inner minima can be taken over all physical cells, and $|\lambda_x|_{max} = |u| + \sqrt{gh}$ and $|\lambda_y|_{max} = |v| + \sqrt{gh}$. The CFL safety constant c may have to be chosen smaller than 1 for this nonlinear system in order to avoid oscillations (for example, $c = 0.8$).

Submit 3D plots of h as a function of x and y (e.g., in Matlab you can use 'mesh') for $t = 1.35$ and $t = 3$, using $m = 60$ for each spatial direction (which means that you will have $m^2 = 3600$ physical cells in your simulations, with $\Delta x = \Delta y = 2/60$). Explain in a few sentences what types of waves you see in the simulation results. Also, provide a plot of h integrated over the spatial domain as a function of time. Is the total amount of water in the box conserved over time?

You can also experiment with other ways to visualize the results. (E.g., make a Matlab movie, ...)

Submit your program files to the dropbox on LEARN. The code submitted should use $m = 60$ and should produce a plot of h at $t = 3$. Also, include a copy of the code in your assignment package submitted on Crowdmark.