3.3 Second-order accurate methods

(for scalar conservation laws in 1D)

3.3.1 Second-order time integration:

recall semi-discrete FV method:

$$\frac{d v_i(t)}{dt} + \frac{\hat{f}_{i+\frac{1}{2}}(t) - \hat{f}_{i-\frac{1}{2}}(t)}{\Delta x} = 0$$

residual vector as a function of $\vec{v}$

or $$\boxed{\frac{d\,\vec{v}(t)}{dt} + \vec{r}\,(\vec{v}(t)) = 0}$$

with $$\vec{v}(t) = \begin{bmatrix} v_1(t) \\ \vdots \\ v_{i-1}(t) \\ v_i(t) \\ v_{i+1}(t) \\ \vdots \\ v_N(t) \end{bmatrix}$$

this is an N-dimensional system of coupled ~~P~~ODEs, and we can use any numerical ODE method to integrate the ODE system in time (this approach is called the "method of lines")

$$\boxed{\frac{d\vec{v}(t)}{dt} + \vec{\tau}(\vec{v}(t)) = 0}$$

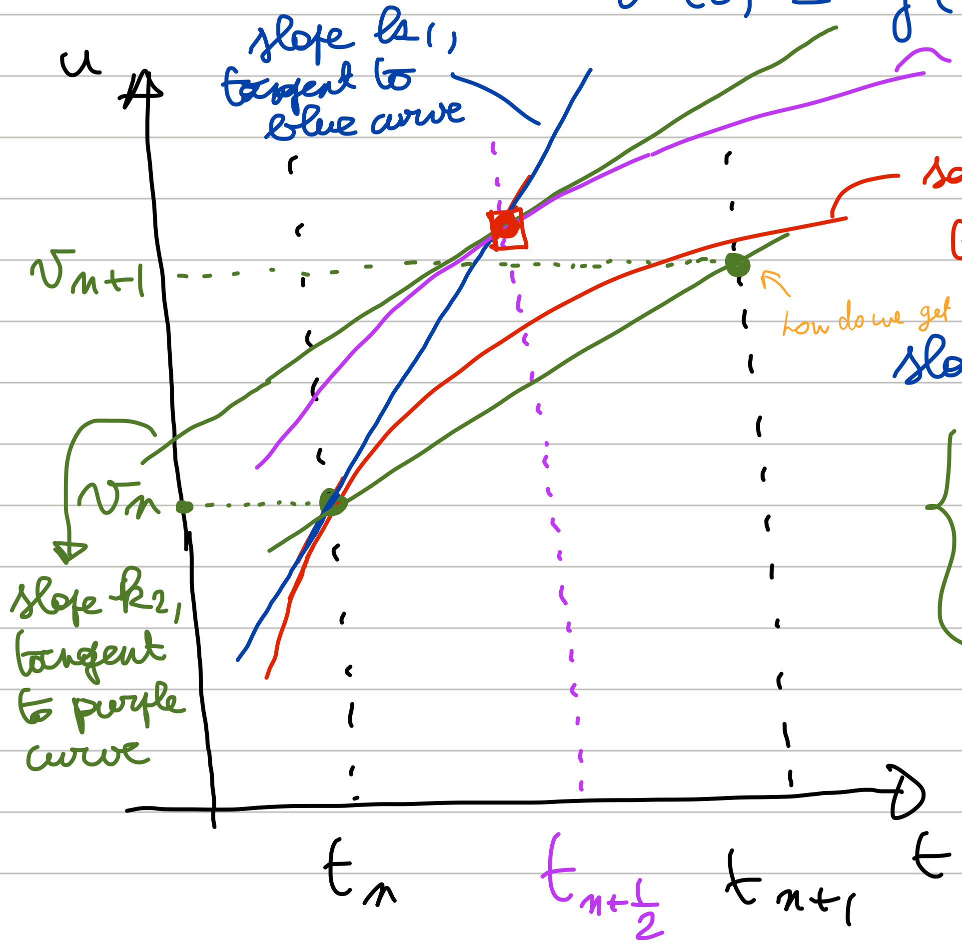
2nd order accuracy in time (derived by Taylor series)

for example, we can use a 2-stage Runge-Kutta (RK) method with order of accuracy 2:

$$\boxed{\begin{aligned} \vec{v}^{\,n+\frac{1}{2}} &= \overbrace{\vec{v}^{\,n} - \frac{\Delta t}{2}\,\vec{\tau}(\vec{v}^{\,n})}^{g(v_n,\,t_n)} \\ \vec{v}^{\,n+1} &= \vec{v}^{\,n} - \Delta t\,\vec{\tau}(\vec{v}^{\,n+\frac{1}{2}}) \end{aligned}} \quad \text{RK2}$$

graphically: for the case of a scalar ODE,

$$v'(t) = g(v(t), t)$$



slopes:

$$\begin{cases} k_1 = g(v_n, t_n) & \text{tangent to exact solution that is unknown} \\ k_2 = g\Big(\underbrace{\boxed{v_n + k_1 \frac{\Delta t}{2}}}_{v_{n+\frac{1}{2}}}, \underbrace{t_n + \frac{\Delta t}{2}}_{t_{n+\frac{1}{2}}}\Big) \end{cases}$$

$$\boxed{v_{n+1} = v_n + k_2\,\Delta t}$$