Numerical solution of partial differential equations (PDEs)

part ① : Finite difference methods (FD)

part ② : Finite volume methods (FV)

part ③ : Finite element methods (FE)

(roughly, 4 weeks each)

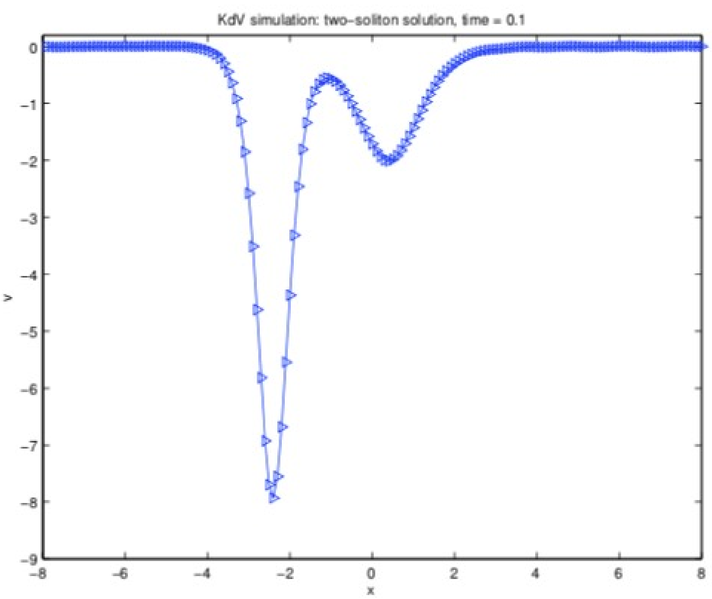
We will consider various applications, for example:

example (1): FD method for the Korteweg-Devries equation

$$\boxed{\frac{\partial u(x,t)}{\partial t} - 6\,u(x,t)\,\frac{\partial u(x,t)}{\partial x} + \frac{\partial^3 u(x,t)}{\partial x^3} = 0}$$

→ one-dimensional (1D) in space (x), nonlinear

→ can describe solitary waves in fluids, fiber optics, biology



ex. (2): FV method for the shallow water equations in 2D

find fluid height $h(x,y,t)$, and fluid velocity components $u(x,y,t)$ and $v(x,y,t)$ such that

$$\boxed{\frac{\partial}{\partial t}\begin{bmatrix} h \\ hu \\ hv \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} hu \\ hu^2 + gh^2/2 \\ huv \end{bmatrix} + \frac{\partial}{\partial y}\begin{bmatrix} hv \\ huv \\ hv^2 + gh^2/2 \end{bmatrix} = 0}$$

($g = 9.8\ m/s^2$, gravitational acceleration)

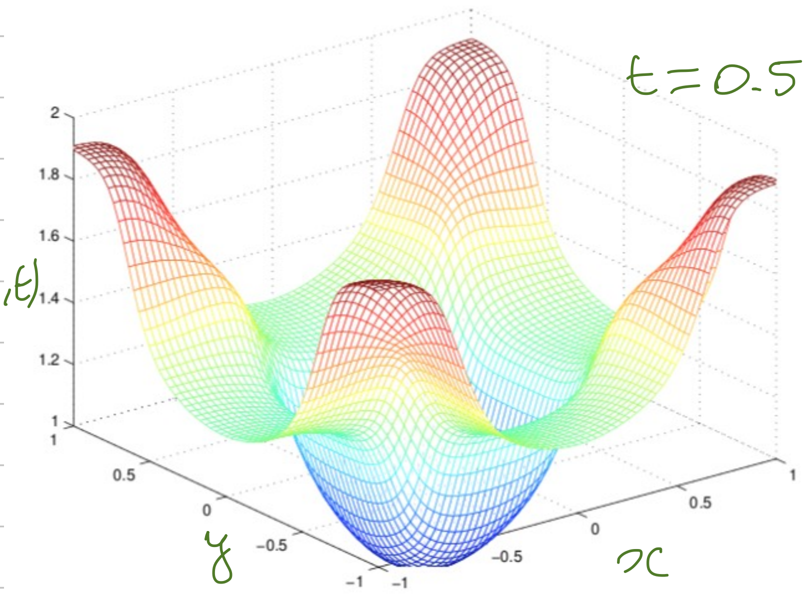
"HYPERBOLIC PDE SYSTEM"

→ system of PDEs,

2D in space,

nonlinear

→ can describe sloshing water in a bathtub



ex. (3): FE method for the stationary heat equation in 2D

"PARABOLIC PDE"

find temperature $u(x,y,t)$ such that

$$\cancel{\frac{\partial u}{\partial t}} - D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f(x,y)$$

$= 0$, stationary

heat conduction coefficient known

known heat source

Laplacian PDE operator: $\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

→ linear, scalar PDE, 2D in space

→ can describe the temperature distribution in an engine block with piston chamber (C_1) and cooling tubes (C_2, C_3, C_4, C_5)

"ELLIPTIC PDE"

$$\begin{array}{lll}
-\Delta u = f, & \forall (x,y) \in (-1,1)\times(-1,1), & f = 0.01 \\
u = g_1, & \forall (x,y) \in \mathcal{C}_1 & 700 \text{ degrees} \\
u = g_2, & \forall (x,y) \in \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5 & 10 \\
u = g_3, & \forall (x,y) \in \Gamma, & 50
\end{array}$$

we need boundary conditions to get a unique solution