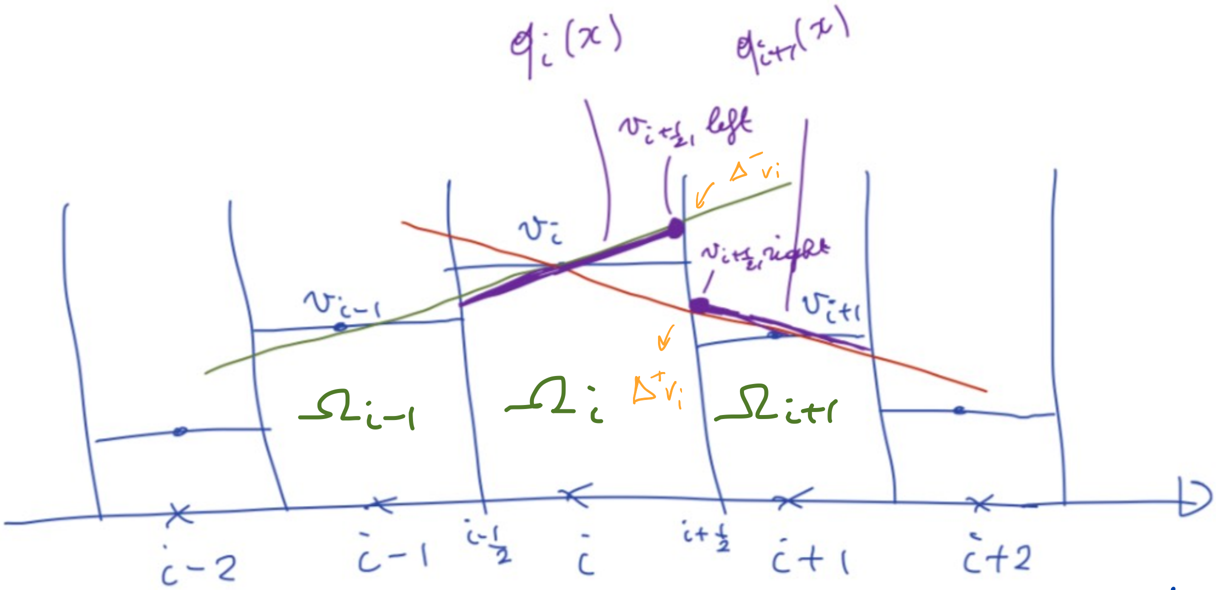
3.3.2 Second-order accuracy in space:

recall semi-discrete FV method:

$$\frac{dv_i(t)}{dt} + \frac{f^*\left(v^-_{i+\frac{1}{2}}(t), v^+_{i+\frac{1}{2}}(t)\right) - f^*\left(v^-_{i-\frac{1}{2}}(t), v^+_{i-\frac{1}{2}}(t)\right)}{\Delta x} = 0$$

where we can use the LF flux function:

$$f^*(u^-,u^+) = (f(u^-)+f(u^+))/2 - \alpha/2\,(u^+ - u^-)$$



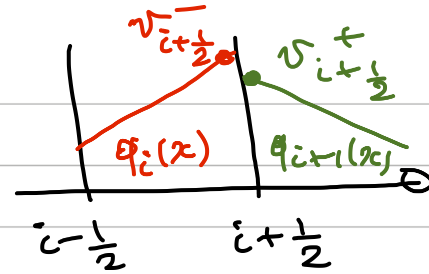
to increase spatial accuracy, we consider linear reconstruction $q_i(x)$ in cell Ω_i of the averages v_j:

$$\boxed{q_i(x) = v_i + \frac{S_i}{\Delta x}(x - x_i)}$$

where we can choose $S_i = v_i - v_{i-1} = \Delta^- v_i$

or $S_i = v_{i+1} - v_i = \Delta^+ v_i$

(or a $\overbrace{\text{convex combination}}^{\text{2 coefficients sum up to one}}$ of the two)



$$\boxed{q_i(x) = v_i + \frac{\delta_i}{\Delta x}(x - x_i)}$$

so

$$\boxed{v^-_{i+\frac{1}{2}} = q_i(x_{i+\frac{1}{2}}) = v_i + \frac{\delta_i}{2}}$$

$$\boxed{v^+_{i+\frac{1}{2}} = q_{i+1}(x_{i+\frac{1}{2}}) = v_{i+1} - \frac{\delta_{i+1}}{2}}$$

↳ second order accurate but with oscillations

by Taylor expansion, it can be shown that this gives second-order spatial accuracy for smooth flows

<u>however</u>: we get oscillations at shocks (like Lax-Wendroff)

<u>but</u>: when $\delta_i = 0$ is chosen in the reconstruction

<u>idea</u>: "limit" the slope $\frac{\delta_i}{\Delta x}$ to 0 when a discontinuity is detected, adaptively

$$\boxed{v^-_{i+\frac{1}{2}} = q_i(x_{i+\frac{1}{2}}) = v_i + \frac{\delta_i}{2}}$$

$$\boxed{v^+_{i+\frac{1}{2}} = q_{i+1}(x_{i+\frac{1}{2}}) = v_{i+1} - \frac{\delta_{i+1}}{2}}$$

($\delta_i = \Delta^- v_i$ or $\Delta^+ v_i$, or something in between)

let $r_i = \dfrac{v_i - v_{i-1}}{v_{i+1} - v_i} = \dfrac{\Delta^- v_i}{\Delta^+ v_i}$ (used to "detect" oscillations)

consider "limiting function" ("limiter")

$$\boxed{\phi(r) = \max(0, \min(r, 1))}$$

↳ zero when r is negative

this called the "minmod" limiter

implies oscillation

different sign

$V_i \to V_{i-1}$

$V_i \to V_{i+1}$

limited reconstruction using minmod limiter:

$$\boxed{v^-_{i+\frac{1}{2}} = v_i + \frac{1}{2}\phi(r_i)(v_{i+1} - v_i)}$$

$$\boxed{v^+_{i+\frac{1}{2}} = v_{i+1} - \frac{1}{2}\phi(r_{i+1})(v_{i+2} - v_{i+1})}$$

interpretation:

if $r_i < 0$: $\Delta^- v_i$ and $\Delta^+ v_i$ have different signs: indicates oscillation (at shock ...)

then $\phi(r_i) = 0$, so $\delta_i = 0$: revert to first-order at oscillation

if $r_i \geq 0$: $\phi(r_i) = \min(r_i, 1)$, so

$\phi(r_i)\Delta^+ v_i$ chooses the |smallest| slope $\Delta^- v_i$ or $\Delta^+ v_i$ ("safest" choice for oscillations) notes: – this retains second-order spatial accuracy "away from shocks"

– it can be shown that this approach eliminates spurious oscillations at shocks (see section 3.3.5, on "total variation diminishing" second-order methods)

– combine this with second-order time integration

But it is fundamentally first order accurate in presence of discontinuities because of minmod.