Q1)

$u_t + (u^3)_x = 0 \qquad u_t + (f(u))_x = 0$

$f(u) = u^3 \qquad f'(u) = 3u^2$

a) On a characteristic curve:

$$\frac{dx(t)}{dt} = \frac{df(u)}{du} = 3u^2$$

Along a characteristic curve the value of $u(x,t)$ is constant $\therefore \frac{dx(t)}{dt} = 3u^2$ is constant for a given value of u. $\therefore$ The charachteristic curves are straight lines

When $u=2$

$$\frac{dx(t)}{dt} = 3u^2 = 3\times 2^2 = 3\times 4 = 12$$

b) $u_l = 2$, $f(u_l) = 2^3 = 8$, $u_r = 1$, $f(u_r) = 1^3 = 1$

$$\frac{dx_l(t)}{dt} = 3\times 2^2 = 12 \qquad \frac{dx_r(t)}{dt} = 3\times 1^2 = 3$$

$$\frac{d\hat{x}(t)}{dt} = \frac{f(u_r) - f(u_l)}{u_r - u_l} = \frac{1-8}{1-2} = \frac{-7}{-1} = 7$$

Q2) $u(x,t) = \frac{x}{t}$

① $u_t + \left(\frac{u^2}{2}\right)_x = 0$

② $\left(\frac{u^2}{2}\right)_t + \left(\frac{u^3}{3}\right)_x = 0$

for ①

$u_t = -\frac{x}{t^2}$

$f(u) = \frac{u^2}{2}$

$f'(u) = u = \frac{x}{t}$

$u_x = \frac{1}{t}$

i) Substitute $u(x,t) = \frac{x}{t}$ in ①

$$\underbrace{-\frac{x}{t^2}}_{} + \underbrace{\left(\frac{x}{t}\right)}_{f'(u)}\underbrace{\left(\frac{1}{t}\right)}_{u_x} = \cancel{-\frac{x}{t^2}} + \cancel{\frac{x}{t^2}} = 0$$

$\therefore$ $u(x,t) = \frac{x}{t}$ is a solution of ①

Let $V = \frac{u^2}{2}$ $\qquad u = \sqrt{2V}$

$V_t = v'(u)\,u_t$

$= \frac{2u}{2} \times u_t = u\,u_t = \left(\frac{x}{t}\right)\left(\frac{-x}{t^2}\right) = -\frac{x^2}{t^3}$

Let $f(V) = \frac{\left(\sqrt{2V}\right)^3}{3}$

$= \frac{(2V)^{\frac{3}{2}}}{3}$

$V_x = v'(u)\,u_x$

$= \frac{2u}{2} \times u_x = u\,u_x$

$= \left(\frac{x}{t}\right)\left(\frac{1}{t}\right) = \frac{x}{t^2}$

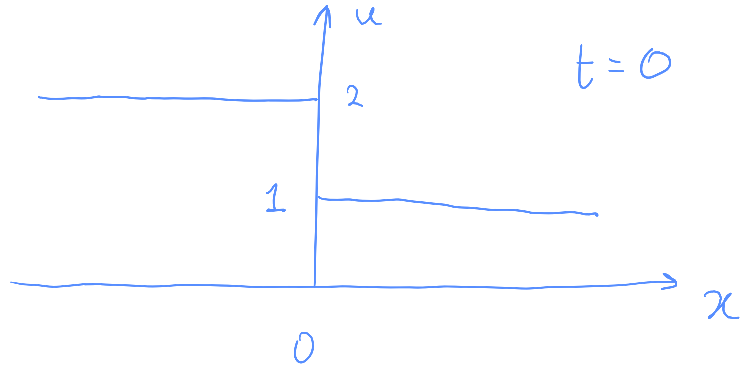
$f'(v) = \left(\frac{3}{\cancel{2}}\right)\frac{(2v)^{\frac{1}{2}}}{\cancel{3}} \times \cancel{2}$

$= \sqrt{2v} = \sqrt{\frac{\cancel{2}u^2}{\cancel{2}}} = u = \frac{x}{t}$

Substitute $u(x,t) = \frac{x}{t}$ in ②

$$\underbrace{\left(-\frac{x^2}{t^3}\right)}_{V_t} + \underbrace{\left(\frac{x}{t}\right)}_{f'(v)}\underbrace{\left(\frac{x}{t^2}\right)}_{V_x} = \cancel{-\frac{x^2}{t^3}} + \cancel{\frac{x^2}{t^3}} = 0$$

$\therefore$ $u = \frac{x}{t}$ is a solution of ② ii)



for ① :

$$u_t + f'(u)\,u_x = 0$$

$$f(u) = \frac{u^2}{2}, \quad f'(u) = u$$

characteristic curves for $x \leq 0$ are given by:

$$u_l = 2, \quad f(u_l) = \frac{2^2}{2} = 2$$

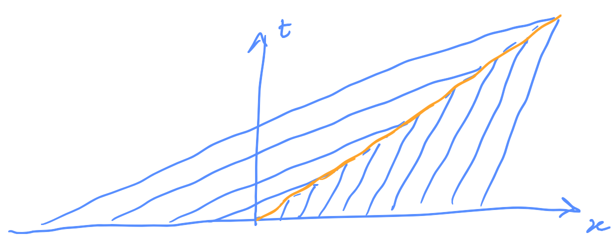
$$\frac{dx(t)}{dt} = \frac{df(u)}{du} = u = 2$$

characteristic curves for $x > 0$ are given by:

$$u_r = 1, \quad f(u_r) = \frac{1}{2}$$

$$\frac{dx(t)}{dt} = \frac{df(u)}{du} = u = 1$$

since $u_l > u_r$ we have a shock



shock speed

$$\frac{d\hat{x}(t)}{dt} = \frac{\frac{1}{2} - 2}{1 - 2} = \frac{-\frac{3}{2}}{-1} = \frac{3}{2}$$

$\therefore$ $u = 2$ for $x \leq \frac{3}{2}t$

$u = 1$ for $x > \frac{3}{2}t$

for ②

Let $v = \frac{u^2}{2}$

$$v_t + f'(v)\, v_x = 0$$

$$f(v) = \frac{(2v)^{3/2}}{3} \qquad f'(v) = \sqrt{2v} = u$$

$$= \frac{(u^2)^{3/2}}{3} = \frac{u^3}{3}$$

Charachteristic curves for $x \leq 0$ are given by :

$$u_l = 2, \quad f(u_l) = \frac{2^3}{3} = \frac{8}{3}$$

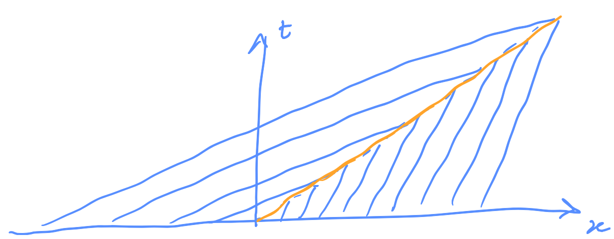
$$\frac{dx(t)}{dt} = \frac{df(v)}{dv} = u_l = 2$$

characteristic curves for $x > 0$ are given by :

$$u_r = 1, \quad f(u_r) = \frac{1^3}{3} = \frac{1}{3}$$

$$\frac{dx(t)}{dt} = \frac{df(v)}{dv} = u_r = 1$$

Since $u_l > u_r$ we have a shock



$$v_l = \frac{2^2}{2} = 2 \qquad v_r = \frac{1}{2}$$

shock speed:

$$\frac{d\hat{x}(t)}{dt} = \frac{\left(\frac{1}{3}\right) - \left(\frac{8}{3}\right)}{\frac{1}{2} - 2} = \frac{\left(\frac{-7}{3}\right)}{\left(-\frac{3}{2}\right)} = \frac{7}{3} \times \frac{2}{3} = \frac{14}{9}$$

$\therefore$ $u = 2$ for $x \leq \frac{14}{9} t$

$u = 1$ for $x > \frac{14}{9} t$

iii) The quantity conserved in ② is not u but rather $v = \frac{u^2}{2}$

and the flux function is

$f(v) = \frac{(2v)^{3/2}}{3} = \frac{u^3}{3}$ rather than $f(u) = \frac{u^2}{2}$ as in 1

At $t=0$ we have

for ①: $f(u) = \frac{u^2}{2}$

$u_l = 2$, $u_r = 1$, $f(u_l) = 2$, $f(u_r) = \frac{1}{2}$, $\frac{dx(t)}{dt} = u$

for ②

$u_l = 2$, $u_r = 1$, $f(v) = \frac{(2v)^{3/2}}{3} = \frac{u^3}{3}$

$v_l = 2$, $v_r = \frac{1}{2}$, $f(v_l) = \frac{8}{3}$, $f(v_r) = \frac{1}{3}$, $\frac{dx(t)}{dt} = \sqrt{2v} = u$

Here we see that the characteristic curves are identical for ① and ②. However the quantities being conserved and the fluxes are different. These differences result in a different shock speed and thus a different discontinuous solution.

Note: The quantity being conserved in ② is $\frac{u^2}{2} + \cancel{C_1} = \int u \, du$

where C_1 is zero.

Note that u is the quantity conserved in ①

The flux function for ② is $\frac{u^3}{3} + \cancel{C_2} = 2\int \frac{u^2}{2} du$

where C_2 is zero

Note that $\left(\frac{u^2}{2}\right)$ is the flux function of ①