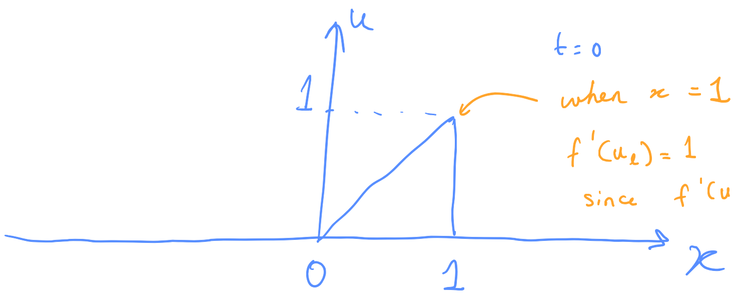
Q 3) $u_t + \left(\frac{1}{2}u^2\right)_x = 0$

a) $f(u) = \frac{1}{2}u^2 \qquad f'(u) = u$



$t = 0$

when $x = 1$

$f'(u_l) = 1$ and $f'(u_r) = 0$

since $f'(u_l) > f'(u_r)$ there is a shock here

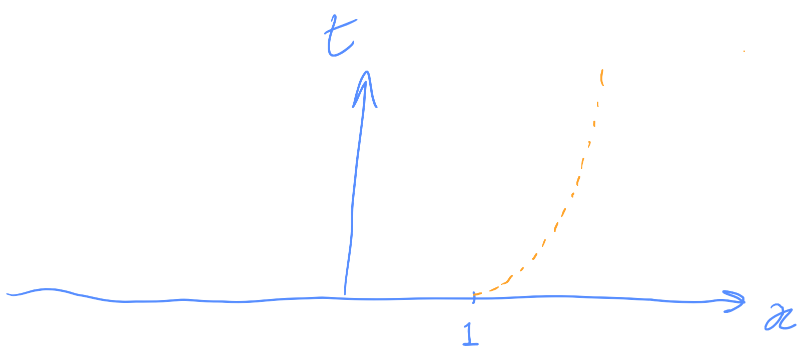
for $x < 1$

$u(x)$ is either flat or increasing $\therefore$ there are no shocks for $x < 1$

shock speed

$$\hat{x}'(t) = \frac{f(u_r) - f(u_l)}{u_r - u_l} = \frac{0 - \frac{x^2}{2}}{0 - x} = \frac{x}{2}$$

$\therefore$ The slope of the characteristic increases with x



b) at $t = 0$, $x = 1$

$$\hat{x}'(t) = \frac{x}{2} = \frac{1}{2}$$

c) Change of notation:

Let $\frac{d x_s(t)}{dt} = x'(t)$

since
$f(u(-\infty,t)) = 0$
$f(u(\infty,t)) = 0$

$$\frac{d}{dt}\left[\int_{-\infty}^{0} \cancel{u}^{\,0}\, dx + \underbrace{\int_0^{x_s^-(t)} u\, dx}_{G^-(x_s^-(t),t)} + \underbrace{\int_{x_s^+(t)}^{\infty} u\, dx}_{G^+(x_s^+(t),t)}\right] + \cancel{\int_{-\infty}^{-\infty} \frac{d f(u)}{dx} dx}^{\,0} = 0$$

Since $\frac{d}{dt} G^-(x_s^-(t),t) = \frac{\partial G^-}{\partial x_s}\frac{d x_s}{dt} + \frac{\partial G^-}{\partial t}$

NOTE: $\cancel{u(-\infty,t)}^{0}$ $\cancel{u(\infty,t)}^{0}$

$= u(x_s^-(t),t)\frac{d x_s}{dt} + \int_0^{x_s^-(t)} \frac{\partial u}{\partial t} dx$, Since $u_t + (f(u))_x = 0$

$u_t = -f(u)_x$

$\frac{du}{dt} = -\frac{\partial f(u)}{\partial x}$

$= u(x_s^-(t),t)\frac{d x_s}{dt} + \left[-f(u(x_s^-(t),t)) + \cancel{f(u(0,t))}^{0}\right]$

$= u(x_s^-(t),t)\frac{d x_s}{dt} - f(u(x_s^-(t),t))$

Similarly $\frac{d}{dt} G^+(x_s^+(t),t) = -u(x_s^+(t),t)\frac{d x_s}{dt} + f(u(x_s^+(t),t))$

So $\frac{d}{dt} G^-(x_s^-(t),t) + \frac{d}{dt} G^+(x_s^+(t),t) = 0$

Let $u^- = u(x_s^-(t),t)$ and $u^+ = u(x_s^+(t),t)$

Then

$$u^- \frac{d x_s}{dt} - f(u^-,t) - u^+ \frac{d x_s}{dt} + f(u^+,t) = 0$$

$$\frac{d x_s}{dt} = \frac{f(u^-) - \cancel{f(u^+)}^{0}}{u^- - \cancel{u^+}^{0}} = \frac{\left(\frac{(u^-)^2}{2}\right)}{u^-} = \frac{u^-}{2}$$

Since $u^- = x$, then $\frac{d x_s}{dt} = \frac{x_s}{2}$

$\frac{d x_s(t)}{dt} = \frac{x_s(t)}{2}$

$\int \frac{1}{x_s} dx_s = \int \frac{1}{2} dt$

$\ln(x_s) = \frac{1}{2}t + c$

$x_s = C e^{\frac{1}{2}t}$

when $t=0$, $x_s = 1$

$\therefore C = 1$

$x_s = e^{\frac{1}{2}t}$

$$\frac{dx_s}{dt} = \frac{x_s}{2}$$

$$\int \left(\frac{1}{x_s}\right) dx_s = \int \frac{1}{2} dt$$

$$\ln(x_s) = \frac{1}{2}t + C$$

$$x_s = e^{\frac{1}{2}t + c} = e^{\frac{1}{2}t} e^{c} \quad , \quad \text{let } a = e^{c}$$

$$= a e^{\frac{1}{2}t}$$

when $t = 0$ $x_s = 1$ so:

$$a = 1$$

$$\therefore \quad x_s = e^{\frac{1}{2}t}$$

Q 4) $u_t + f(u)_x = 0$, $u_t + f'(u) u_x = 0$

u is constant along characteristic curves $x(t)$ i.e. $u_t = 0$

$$x'(t) = f'(u)$$

Consider a particular characteristic curve $x_\xi(t)$ and its neighboring curve $x_{\xi + \delta\xi}(t)$ such that

$$x_\xi(0) = \xi \quad \text{and} \quad x_{\xi+\delta\xi}(0) =$$