3.2.3 Numerical flux functions for $u_t + \alpha u_x = 0$

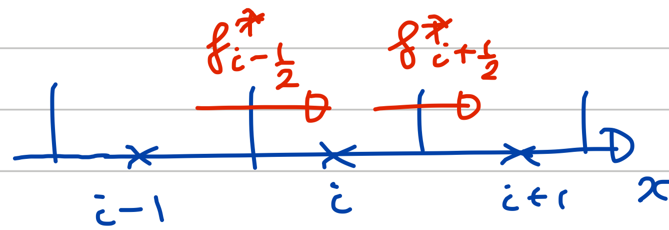
$(f(u) = \alpha u)$

FV method (low order, explicit)

$$\boxed{\frac{v_i^{n+1} - v_i^n}{\Delta t} + \frac{f^*(v_i^n, v_{i+1}^n) - f^*(v_{i-1}^n, v_i^n)}{\Delta x} = 0 \qquad (1)}$$

$\alpha > 0$

① Forward Upwind method:



assume $\alpha > 0$: upwind FD method

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} + \alpha \underbrace{\frac{v_i^n - v_{i-1}^n}{\Delta x}}_{\text{upwind derivative}} = 0$$

this FD method is also a FV method !

$$f^*_{i+\frac{1}{2}} = \alpha v_i^n \qquad \left(= f^*(v_i^n, v_{i+1}^n) \right)$$

$$f^*_{i-\frac{1}{2}} = \alpha v_{i-1}^n \qquad \left(= f^*(v_{i-1}^n, v_i^n) \right)$$

$\Rightarrow$ $f^*(u, v)$ is the flux $f(u) = \alpha u$

determined by the upstream cell average

$\left(f^*(u, v) = \alpha u \quad \text{when } \alpha > 0 \right)$

average in left cell

$a < 0$

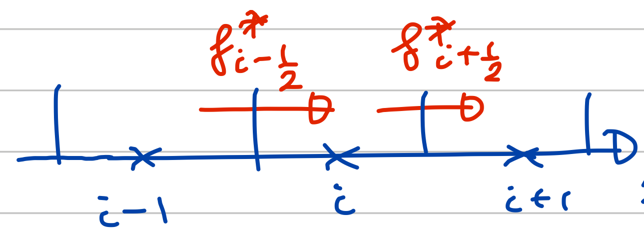
assume $a<0$: upwind FD method

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} + a \underbrace{\frac{v_{i+1}^n - v_i^n}{\Delta x}}_{\text{upwind derivative}} = 0$$

this FD method is also a FV method!

$$f^*_{i+\frac{1}{2}} = a\, v_{i+1}^n$$

$$f^*_{i-\frac{1}{2}} = a\, v_i^n$$



"upwind numerical flux" now uses the cell average to the right of interface $i+\frac{1}{2}$

$f^*(u,v) = av$

we can also treat both cases together:

average of flux on the left and right + some correction

recall: numerical diffusion

$$f^*_{i+\frac{1}{2}} = \underbrace{\frac{a\, v_i^n + a\, v_{i+1}^n}{2}}_{\frac{f(v_i^n) + f(v_{i+1}^n)}{2}} - \frac{|a|}{2}\left(v_{i+1}^n - v_i^n\right)$$

$\frac{f(v_i^n) + f(v_{i+1}^n)}{2}$ → central FD method, unstable

$f^*(v_i^n, v_{i+1}^n)$

$$= a^+ v_i^n + a^- v_{i+1}^n$$

If we just had the average of the fluxes we get

$$\left(\text{with } a^+ = \frac{a+|a|}{2} = \max(a,0) \right.$$

$\frac{F(v_{i+1}^n) + F(v_{i-1}^n)}{2\Delta x}$ } looks like a central discretization (numerically unstable)

$$\left. a^- = \frac{a-|a|}{2} = \min(a,0) \right)$$

The correction term makes it upwind and so it becomes stable

and recall: stability: $\Delta t \leq \frac{\Delta x}{|a|} = \frac{\Delta x}{|f'(u)|}$

② Lax-Wendroff method: $u_t + a u_x = 0$

$$\boxed{\frac{v_j^{n+1} - v_j^n}{\Delta t} + a \frac{v_{j+1}^n - v_{j-1}^n}{2\Delta x} - \frac{a^2}{2} \Delta t \frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{\Delta x^2} = 0}$$

this is also a FV method !

$$f^*_{i+\frac{1}{2}} = f^*(v_i^n, v_{i+1}^n)$$

$$= \frac{a v_i^n + a v_{i+1}^n}{2} - \frac{a^2}{2} \frac{\Delta t}{\Delta x} (v_{i+1}^n - v_i^n)$$

correction term to make it stable (2nd order in space and time)

<u>But</u>: oscillations for discontinuous

$u(x, 0)$

FV methods are useful for non-linear cases when there may be shockwaves etc.

Take a leap of faith from the linear case to the non-linear case