2.3 FD methods for parabolic PDEs:

heat equation in 1D:

$$u_t = \eta\, u_{xx} + f(x,t) \qquad (\eta \geq 0)$$

① Forward Central: (FC)

central

$$\frac{v_j^{n+1} - v_j^{n}}{\Delta t} = \eta \frac{v_{j+1}^{n} - 2 v_j^{n} + v_{j-1}^{n}}{\Delta x^2} + f(x_j, t_n)$$

forward

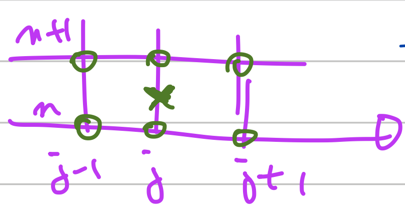
typo in PDF Notes

truncation error $t_j^n = O(\Delta t) + O(\Delta x^2)$

② Crank-Nicolson: (CN)

(this can be thought of as an average between forward central and backward central methods)

$$\frac{v_j^{n+1} - v_j^{n}}{\Delta t} = \frac{\eta}{2}\left(\frac{v_{j+1}^{n} - 2 v_j^{n} + v_{j-1}^{n}}{\Delta x^2} + \frac{v_{j+1}^{n+1} - 2 v_j^{n+1} + v_{j-1}^{n+1}}{\Delta x^2} \right) + f(x_j, t_n)$$



truncation error $t_j^n = O(\Delta t^2) + O(\Delta x^2)$

truncation first order errors cancel out due to symmetry, we are only left with second order errors

(can be checked with taylor series)

numerical stability: Von Neumann

$$v_j^n = \hat{v}_n \exp(i j k \Delta x)$$

① FC:

$$\frac{\hat{v}_{n+1} - \hat{v}_n}{\Delta t} = \frac{\eta}{\Delta x^2} \hat{v}_n \left(\exp(ik\Delta x) - 2 + \exp(-ik\Delta x) \right)$$

or

$$\frac{\hat{v}_{n+1}}{\hat{v}_n} = 1 + \eta \frac{\Delta t}{\Delta x^2} \underbrace{\left(2\cos(k\Delta x) - 2\right)}_{\in [-4,0]}$$

$$= S(k)$$

choose $k = -4$ for most negative result

$$1 - 4\eta \frac{\Delta t}{\Delta x^2} \geq -1$$

$$-4\eta \frac{\Delta t}{\Delta x^2} \geq -2, \quad 4\eta \frac{\Delta t}{\Delta x^2} \leq 2, \quad \eta \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

$$\Delta t \leq \frac{\Delta x^2}{2\eta}$$

stability:

$$\max_{k \in \mathbb{R}} |S(k)| \leq 1$$

$$\Updownarrow$$

$$\Delta t \leq \frac{\Delta x^2}{2\eta} \iff \eta \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

→ more restrictive than $\Delta t \leq \frac{\Delta x}{|a|}$ for $u_t + a u_x = 0$, when Δx is small (information propagates "fast" in parabolic PDEs)

→ large η requires small timestep (information travels "faster" for large diffusion)

② CN:

for parabolic PDEs implicit methods are better since the information is propagated faster → CN method is valuable

similarly,

$$S(\xi) = \frac{1 + \frac{\eta \Delta t}{2 \Delta x^2} \underbrace{2(\cos(\xi x) - 1)}_{\in [-2,0]}}{1 - \frac{\eta \Delta t}{2 \Delta x^2} 2(\cos(\xi x) - 1)} = \frac{1-s}{1+s}$$

$(s \geq 0)$

stability: $|S(\xi)| \leq 1$

$$\Updownarrow$$

$$-1 \leq \frac{1-s}{1+s} \leq 1$$

$$\Updownarrow$$

$-1-s \leq 1-s$ and $1-s \leq 1+s$

$$\Updownarrow$$

$-2 \leq 0$ and $2s \geq 0$

OK OK $\forall s \geq 0$

so CN is stable for any Δt

(note: since the time step limitation for explicit methods is so restrictive, one normally uses implicit methods for parabolic PDEs)