3.2 Finite volume methods for scalar conservation laws

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \qquad (\text{and } u_t + f'(u)\, u_x = 0)$$

nonlinear wave speed
shape of characteristic curve

examples:

1) Burgers equation

$$f(u) = \frac{u^2}{2} \qquad (u_t + u\, u_x = 0)$$

nonlinear, shock formation etc.

2) linear advection:

$$f(u) = \alpha u \qquad (u_t + \alpha\, u_x = 0)$$

recall: FD methods worked fine for many PDEs

but: there are issues for nonlinear conservation laws:

FD works with linear conservation laws

1. incorrect shock speed
2. oscillations at shocks

$\Longrightarrow$ we need Finite Volume Methods

3.2.1 Problems with applying FD methods to nonlinear conservation laws:

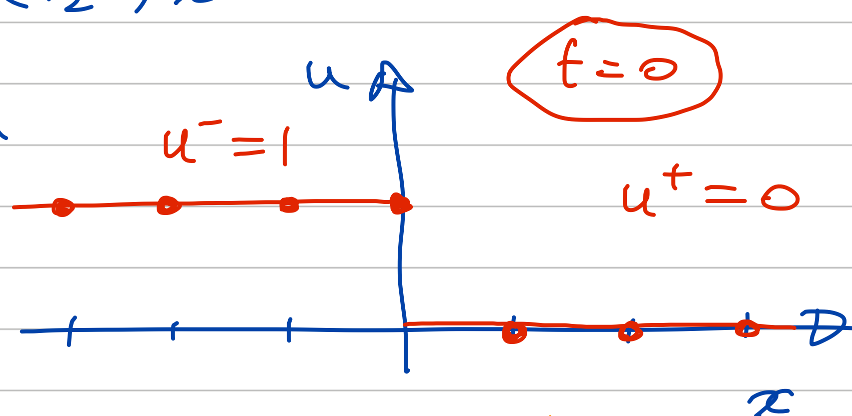
① incorrect shock speed

we consider an example:

Burgers: $u_t + \left(\frac{u^2}{2}\right)_x = 0$ $\quad \left(f(u) = \frac{u^2}{2} \right)$

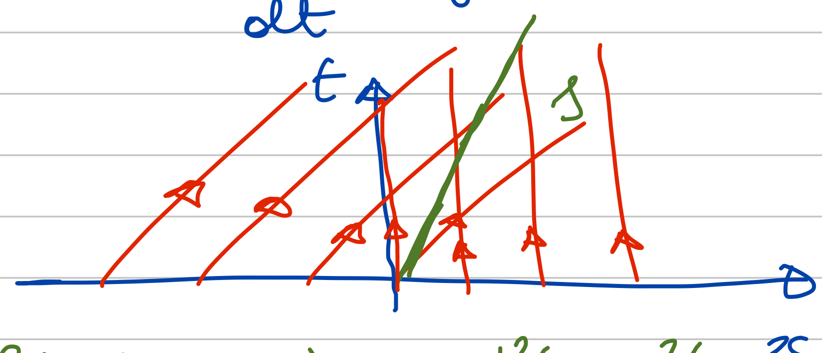
Riemann problem

$$\begin{cases} u(x,0) = 1 & \text{if } x \leq 0 \\ u(x,0) = 0 & \text{if } x > 0 \end{cases}$$



characteristic curve

Characteristic curves: $\frac{dx(t)}{dt} = f'(u) = u$



$$s = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \frac{1^2/2 - 0^2/2}{1 - 0} = \frac{1}{2}$$

weak solution: shock moving with speed

$$s = \frac{1}{2}$$

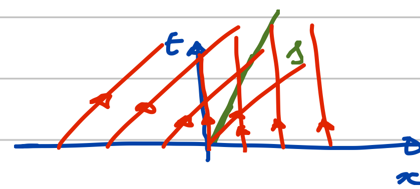
consider a naive, explicit "upwind" FD method for Burgers:

$u_t + u\, u_x = 0$, assume $u(x,t) \geq 0$

then $$\frac{v_i^{n+1} - v_i^n}{\Delta t} + v_i^n \frac{v_i^n - v_{i-1}^n}{\Delta x} = 0$$

at $t=0$: $v_i^0 = 1 \quad i \leq 0$

$v_i^0 = 0 \quad i > 0$



at $t = \Delta t$: $$v_i^1 = v_i^0 - \frac{\Delta t}{\Delta x} v_i^0 \left(v_i^0 - v_{i-1}^0\right)$$

$$i \leq 0: \quad v_i^1 = v_i^0 - \frac{\Delta t}{\Delta x}\, 1\, \underbrace{(1-1)}_{0} = v_i^0$$

Despite there being a flux difference the shock doesn't move

$$i > 0: \quad v_i^1 = v_i^0 - \frac{\Delta t}{\Delta x} \underbrace{0}\left(v_i^0 - v_{i-1}^0\right) = v_i^0$$

so the shock does not move!

even as Δt & $\Delta x \to 0$ we get an incorrect solution

and: same for all $t_n = n\,\Delta t$

↳ we do not get weak solutions

Conclusion: v_i^n for this naive FD method converges to a stationary shock as $\Delta x, \Delta t \to 0$; this is NOT a weak solution of the conservation law

notes:

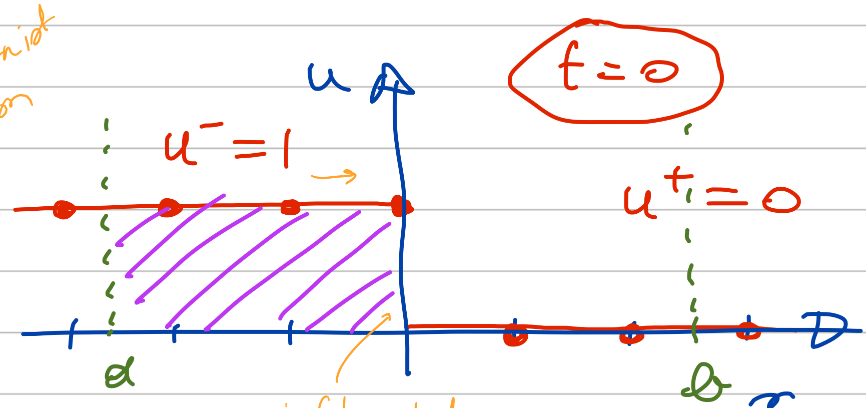
- the shock moves at incorrect speed because conservation of u is not properly maintained

$$\frac{d}{dt}\left(\int_a^b u(x,t)\,dx\right) + f(u(b,t)) - f(u(a,t)) = 0$$

(first integral form)

If there is a flux difference the shock must move

Rankine Hugoniot relation



inflow but no out flow

$$\underbrace{f(u^-) = \frac{1^2}{2}}_{\text{flux is non-zero}} \neq \underbrace{f(u^+) = \frac{0^2}{2}}_{\text{flux is zero}}$$

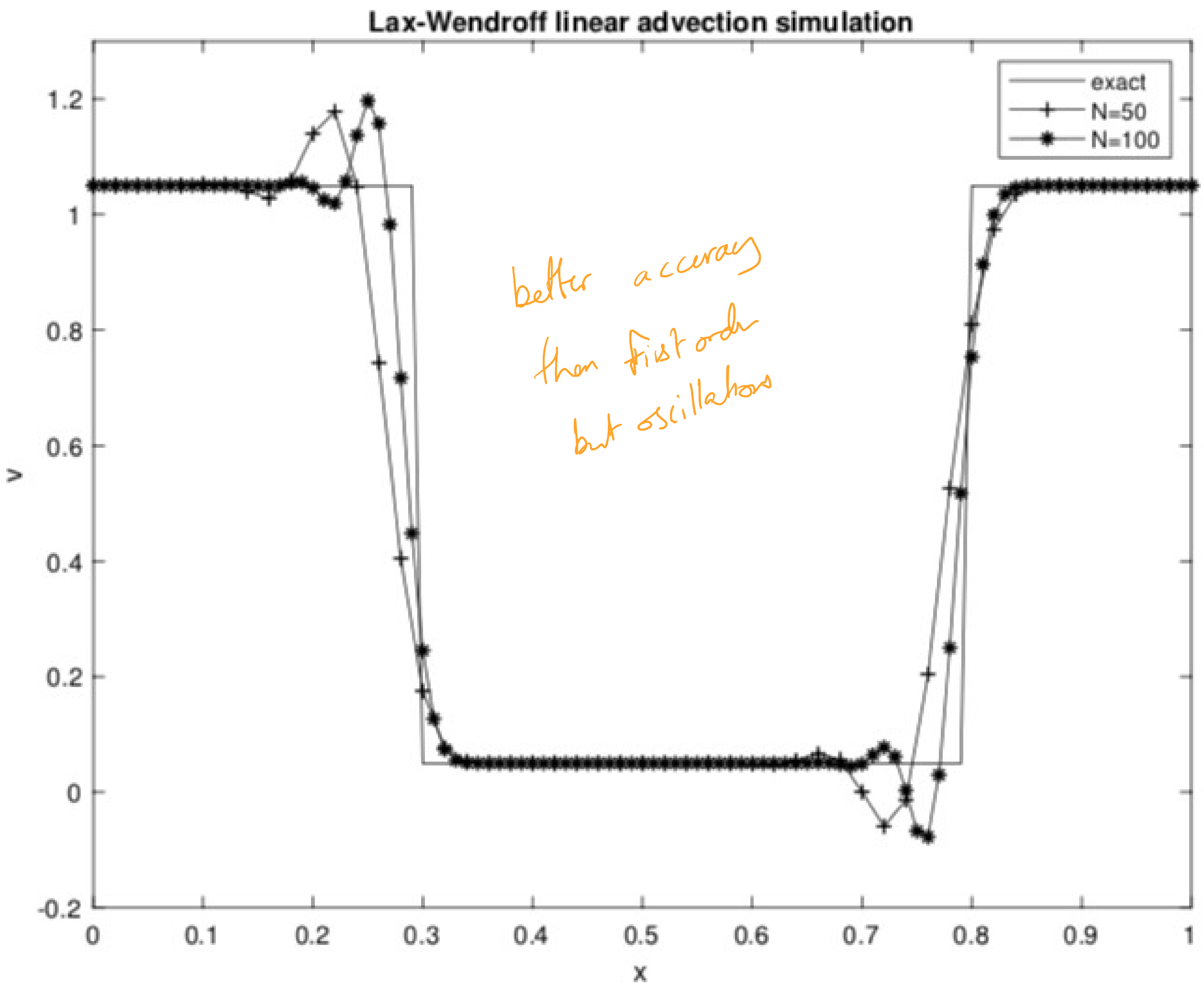
so shock has to move

numerically, the naive FD method does not conserve u (FV methods will be constructed to conserve u)

- this issue also occurs when $u^+ \neq 0$, for this naive FD method (convergence to a function with incorrect shock speed)

② Unphysical oscillations at discontinuity with second-order FD method:

recall: Lax–Wendroff applied to linear advection of discontinuity
└ 2nd order accurate, dispersion errors dominate



better accuracy than first order but oscillations

dispesion: the components of the solution travel at different speeds and cause distortion (avg error is kept low)

dissipation – The solution weakens as it progresss (error ↑ with time)

this can be a real problem, especially when $u(x,t)$ represents a quantity that should remain nonnegative, e.g., a gas density