3.3.4 TVD methods for conservation laws:

def: Total Variation of a grid function

consider grid function $\vec{v}^{\,n} = \begin{bmatrix} v_1^n \\ \vdots \\ v_m^n \end{bmatrix}$ on a grid with m spatial grid points at time t_n, and assume periodic boundary conditions ($v_m^n = v_0^n$)

then the total variation of $\vec{v}^{\,n}$ is defined by

$$TV(\vec{v}^{\,n}) = \sum_{i=1}^{m} |v_i^n - v_{i-1}^n|$$

idea: if we want to preclude spurious numerical oscillations, we can require a TVD property for the numerical method:

$$TV(\vec{v}^{\,n+1}) \leq TV(\vec{v}^{\,n})$$

(since the exact solution also satisfies such a property) when is a two-level scheme TVD?

consider the general form:

$$v_i^{n+1} = v_i^n - C_{i-\frac{1}{2}} \left(v_i^n - v_{i-1}^n \right) + D_{i+\frac{1}{2}} \left(v_{i+1}^n - v_i^n \right)$$

first order in time & space

use notation: $\Delta^- v_i^n = v_i^n - v_{i-1}^n$

$\Delta^+ v_i^n = v_{i+1}^n - v_i^n$

then we can derive conditions on the coefficients $C_{i-\frac{1}{2}}$ and $D_{i+\frac{1}{2}}$ for TVD:

Theorem 3.22: TVD Conditions

Consider numerical method

$$v_i^{n+1} = v_i^n - C_{i-\frac{1}{2}} \Delta^- v_i^n + D_{i+\frac{1}{2}} \Delta^+ v_i^n, \tag{3.66}$$

with periodic boundary conditions. If, for all i,

$$\begin{cases} C_{i+\frac{1}{2}} \geq 0, \\ D_{i+\frac{1}{2}} \geq 0, \\ C_{i+\frac{1}{2}} + D_{i+\frac{1}{2}} \leq 1 \end{cases} \tag{3.67}$$

then the numerical method is TVD.

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$$\Delta^- v_i^n = v_i^n - v_{i-1}^n$$

$$\Delta^+ v_i^n = v_{i+1}^n - v_i^n$$

proof: $TV(\vec{v}^{\,n+1}) = \sum_i |v_{i+1}^{n+1} - v_i^{n+1}|$

$$= \sum_i \Big| \Delta^+ v_i^n - C_{i+\frac{1}{2}} \underbrace{\Delta^- v_{i+1}^n}_{\Delta^+ v_i^n} + D_{i+\frac{3}{2}} \Delta^+ v_{i+1}^n + C_{i-\frac{1}{2}} \underbrace{\Delta^- v_i^n}_{\Delta^+ v_{i-1}^n} - D_{i+\frac{1}{2}} \Delta^+ v_i^n \Big|$$

$$\leq \sum_i |(1 - C_{i+\frac{1}{2}} - D_{i+\frac{1}{2}}) \Delta^+ v_i^n| + |D_{i+\frac{3}{2}} \Delta^+ v_{i+1}^n| + |C_{i-\frac{1}{2}} \Delta^+ v_{i-1}^n|$$

$$= \sum_i |(1 - C_{i+\frac{1}{2}} - D_{i+\frac{1}{2}}) \Delta^+ v_i^n| + |D_{i+\frac{1}{2}} \Delta^+ v_i^n| + |C_{i+\frac{1}{2}} \Delta^+ v_i^n| \quad \text{(by periodicity)}$$

because we sum over all i's, by periodicity the sum is invariant to shifts in i

$$= \sum_i (1 - C_{i+\frac{1}{2}} - D_{i+\frac{1}{2}} + D_{i+\frac{1}{2}} + C_{i+\frac{1}{2}}) |\Delta^+ v_i^n|$$

(by the assumptions on the coefficients)

$$= \sum_i |v_{i+1}^n - v_i^n| = TV(\vec{v}^{\,n})$$

$\square$

example: first-order upwind FV method for linear advection is TVD

$$u_t + (a u)_x = 0 \qquad a > 0$$

FOU: $$\frac{v_j^{n+1} - v_j^n}{\Delta t} + \frac{a\, v_j^n - a\, v_{j-1}^n}{\Delta x} = 0$$

is also FV method with LF flux function

so $$v_j^{n+1} = v_j^n - a \frac{\Delta t}{\Delta x} \left(v_j^n - v_{j-1}^n \right)$$

or $$C_{j-\frac{1}{2}} = a \frac{\Delta t}{\Delta x}$$

$$D_{j+\frac{1}{2}} = 0$$

TVD: This is another way to measure stability → no oscillations at discontinuities

TVD conditions:

$C \geq 0$: OK

$D \geq 0$: OK

$C + D \leq 1$: or $C \leq 1$

$$C = a\frac{\Delta t}{\Delta x} \leq 1 \quad \Leftrightarrow \quad \boxed{\Delta t \leq \frac{\Delta x}{a}} \text{ OK!}$$

(CFL condition!)

⇒ this explains why we see no spurious oscillations at shocks for this method

Can we have linear FV methods that are TVD?

def: linear FV scheme

A FV method is called linear if, when the scheme is applied to a linear PDE, all the coefficients c_i in

$$v_j^{n+1} = \sum_i c_i v_{j-i}^n$$

are constant (i.e., do not depend on $\vec{v}^n$)

The following theorem can be proved:

Thm: Godunov's theorem

Linear TVD schemes are at most first-order accurate.

TVD is useful to avoid oscillations but bad for higher order methods

$\Longrightarrow$ so if we want second-order accurate schemes, we need schemes with non-constant coefficients $\rightarrow$ Think of minmod slope limiter

(see section 3.3.5)