3.2.5 Numerical conservation:

$$u_t + f(u)_x = 0 \quad (1)$$

<u>def:</u> $u(x,t)$ is called a weak solution of PDE (1) if, $\forall\ \varphi(x,t) \in C_0^1(\mathbb{R}^2)$,

$$-\int_0^{\infty}\int_{-\infty}^{\infty} (u\,\varphi_t + f(u)\,\varphi_x)\,dx\,dt - \int_{-\infty}^{\infty} u(x,0)\,\varphi(x,0)\,dx = 0$$

Theorem 3.16: Lax-Wendroff theorem

If the numerical approximation $\{v_j^n\}$ obtained by Finite Volume method

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} + \frac{f^*_{i+\frac{1}{2}} - f^*_{i-\frac{1}{2}}}{\Delta x} = 0,$$

for conservation law (3.15) converges, as $\Delta t \to 0$ and $\Delta x \to 0$, boundedly almost everywhere to a function $u(x,t)$, then $u(x,t)$ is a weak solution of the conservation law.

<u>proof:</u> we will use "summation by parts":

$$\boxed{\sum_{k=1}^{N} a_k(\phi_k - \phi_{k-1}) + \sum_{k=1}^{N} \phi_k(a_{k+1} - a_k) = -a_1\phi_0 + \phi_N a_{N+1}}$$

$\int u'v\,dx = uv - \int v'u\,dx$

OR

letting

$u = u(x)$ & $du = u'(x)dx$

$v = v(x)$ & $dv = v'(x)dx$

$\int v\,du = uv - \int u\,dv$

$\int v\,du + \int u\,dv = uv$

(compare: $\displaystyle\int_a^b u\,dv + \int_a^b v\,du = u(b)v(b) - u(a)v(a)$)

let $\phi_i^n = \phi(x_i, t_n)$ with $\phi(x,t) \in C_0^1(\mathbb{R}^2)$

then
$$0 = \sum_{n=1}^{\infty}\sum_{i=-\infty}^{\infty}\left(\frac{v_i^{n+1} - v_i^n}{\Delta t} + \frac{f^*_{i+\frac{1}{2}} - f^*_{i-\frac{1}{2}}}{\Delta x}\right)\phi_i^n\,\Delta x\,\Delta t$$

$$\boxed{\begin{aligned}&\sum_{q=1}^{N} a_q(\phi_q-\phi_{q-1})+\sum_{q=1}^{N}\phi_q(a_{q+1}-a_q)\\&=-a_1\phi_0+\phi_N a_{N+1}\end{aligned}}$$

$$0=\sum_{n=1}^{\infty}\sum_{i=-\infty}^{\infty}\left(\frac{v_i^{n+1}-v_i^n}{\Delta t}+\frac{f^*_{i+\frac{1}{2}}-f^*_{i-\frac{1}{2}}}{\Delta x}\right)\phi_i^n\,\Delta x\,\Delta t$$

(exact) Riemann integral of step functions

then $$0=-\underbrace{\sum_{n=1}^{\infty}\sum_{i=-\infty}^{\infty}}\left(v_i^n\,\frac{\phi_i^n-\phi_i^{n-1}}{\Delta t}+f^*_{i-\frac{1}{2}}\,\frac{\phi_i^n-\phi_{i-1}^n}{\Delta x}\right)\Delta x\,\Delta t$$

$$-\sum_{i=-\infty}^{\infty}v_i^1\,\phi_i^0\,\Delta x$$

step functions on $\mathbb{R}\times\mathbb{R}^+$, bounded, converge to u, ϕ_t, $f(u)$, ϕ_x

converge to $u(x,0)$, $\phi(x,0)$

using the Lebesgue Dominated Convergence Theorem,

it can be shown that this implies

$$-\int_0^{\infty}\int_{-\infty}^{\infty}\Big(u(x,t)\,\phi_t(x,t)+f(u(x,t))\,\phi_x(x,t)\Big)$$

$$-\int_{-\infty}^{\infty}\phi(x,0)\,u(x,0)\,dx=0\qquad\forall\phi\in C_0^1(\mathbb{R}^2)$$

i.e., $u(x,t)$ is a weak solution

DCT: if $f_n(x)\to f(x)$ pointwise and
$|f_n(x)|\le g(x)$ $\forall n$, with $g(x)$ integrable
then $f(x)$ is integrable and
$\int f_n(x)\,dx\to\int f(x)\,dx$

$\square$