3.2.4 Numerical flux functions for nonlinear conservation laws

$$u_t + f(u)_x = 0$$

① the (local) Lax-Friedrichs flux function:

recall the FU flux function for linear advection:

numerical diffusion

for stability

$$f^*_{i+\frac{1}{2}} = \underbrace{\frac{a v_i^n + a v_{i+1}^n}{2}}_{\frac{f(v_i^n) + f(v_{i+1}^n)}{2}} - \frac{|a|}{2}\left(v_{i+1}^n - v_i^n\right)$$

the larger the $|a|$ the more diffusion

$\frac{f(v_i^n) + f(v_{i+1}^n)}{2}$ → central FD method, unstable

with $f(u) = a u$, $f'(u) = a$ polynomial (interpolation)

generalize to the nonlinear case:

for second order accurate use estimate of $f(v_i^n)$ further away from the interface

$$\boxed{f^*_{i+\frac{1}{2}} = \frac{f(v_i^n) + f(v_{i+1}^n)}{2} - \frac{\alpha}{2}\left(v_{i+1}^n - v_i^n\right) \quad \text{with } \alpha = \max\big(\underbrace{|f'(v_i^n)|}_{\text{left state}}, \underbrace{|f'(v_{i+1}^n)|}_{\text{right state}}\big)}$$

the larger value results in more diffusion → more stability

→ (local) Lax-Friedrichs flux function (LF)

in the linear case: $\alpha = |a|$

first-order FV method with LF flux function:

$$\boxed{\frac{v_i^{n+1} - v_i^n}{\Delta t} + \frac{f^*(v_i^n, v_{i+1}^n) - f^*(v_{i-1}^n, v_i^n)}{\Delta x} = 0} \quad (1)$$

$$\boxed{f^*_{i+\frac{1}{2}} = \frac{f(v_i^n) + f(v_{i+1}^n)}{2} - \frac{\alpha}{2}(v_{i+1}^n - v_i^n)}$$

with $\underbrace{\alpha = \max\left(|f'(v_i^n)|, |f'(v_{i+1}^n)|\right)}_{\text{guarantees the physically relevant solution}}$

some properties

→ consistency holds $\Rightarrow$ $t_i^n = O(\Delta t) + O(\Delta x)$
$(f^*(u,u) = f(u))$

→ stable if $\Delta t_n \leq \dfrac{\Delta x}{\max_j |f'(v_j^n)|}$

(approximately, can be shown)

Burgers: $f''(u)$

→ if $f(u)$ is convex (i.e., $f''(u) > 0 \ \forall u$), then LF converges to the unique entropy solution (= vanishing viscosity solution) (can be shown)

but: other choices for α, e.g. $\alpha = \left|f'\left(\frac{v_i^n + v_{i+1}^n}{2}\right)\right|$, may converge to a solution with an entropy-violating shock

② nonlinear Lax-Wendroff flux function:

similar derivation as linear LW scheme:

$$u^{n+1} = u^n + u_t^n \Delta t + u_{tt}^n \frac{\Delta t^2}{2} + O(\Delta t^3)$$

then use

$$u_t = -f(u)_x$$

$$u_{tt} = -f(u)_{xt} = -(f(u)_t)_x$$

$$= -(f'(u)\,u_t)_x = (f'(u)\,f(u)_x)_x$$

and use central discretizations in space, retaining terms up to second order:

$$\boxed{\begin{aligned} &\frac{v_i^{n+1} - v_i^n}{\Delta t} + \frac{f(v_{i+1}^n) - f(v_{i-1}^n)}{2\Delta x} \\ &= \frac{\Delta t}{2\Delta x}\left(f'(v_{i+\frac{1}{2}}^n)\,\frac{f(v_{i+1}^n) - f(v_i^n)}{\Delta x} - f'(v_{i-\frac{1}{2}}^n)\,\frac{f(v_i^n) - f(v_{i-1}^n)}{\Delta x} \right) \\ &\text{where } v_{i+\frac{1}{2}}^n = \frac{v_i^n + v_{i+1}^n}{2} \end{aligned}}$$

this is a FV method with flux function

$$\boxed{f^*_{i+\frac{1}{2}} = \frac{f(v_i) + f(v_{i+1})}{2} - \frac{\Delta t}{2\Delta x} f'(v_{i+\frac{1}{2}}) \cdot \left(f(v_{i+1}) - f(v_i)\right) \quad \text{with } v_{i+\frac{1}{2}} = \frac{v_i + v_{i+1}}{2}}$$

note: this FV method is 2nd-order accurate, but suffers from spurious oscillations at discontinuities