3.3.3 Total variation diminishing properties of scalar conservation laws

consider $\boxed{u_t + f(u)_x = 0}$ (1)

and its quasi-linear form

$$u_t + f'(u)\, u_x = 0$$

recall characteristic curves $x(t)$:

$$\frac{dx(t)}{dt} = f'(u(x,t))$$

such that $\dfrac{du(x(t),t)}{dt} = 0$: $u(x,t)$ is constant on characteristics

$\Rightarrow$ no new local minima or maxima appear over time for solutions of (1)!

this intuitive realization is captured using the mathematical property of "total variation" (TV), which is non-increasing over time for solutions of (1)

def: Total Variation (TV) of $u(x)$ over interval Ω:

$$TV(u(x);\Omega) = \int_\Omega \left| \frac{du(x)}{dx} \right| dx$$

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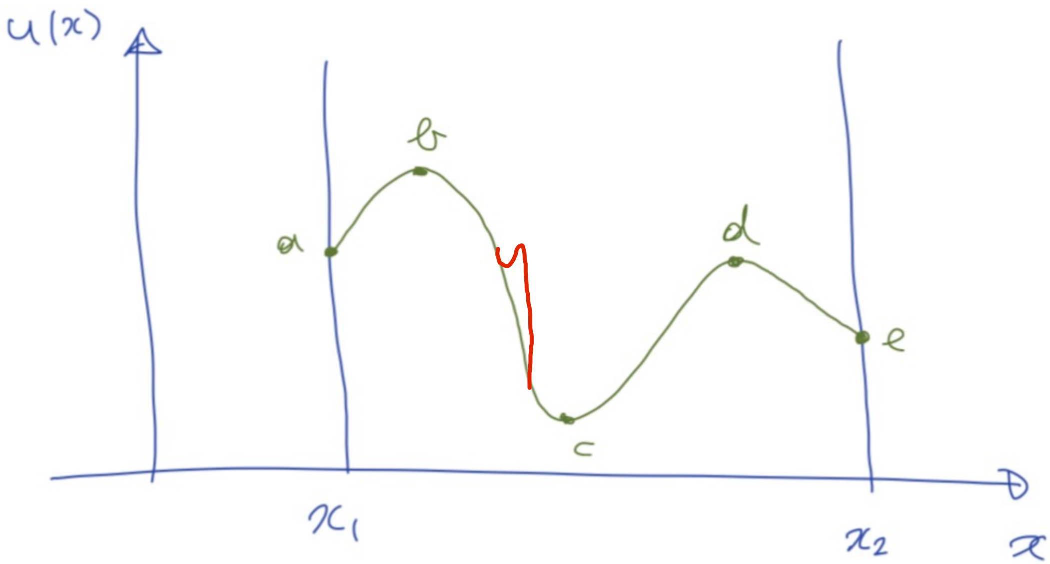
(note: this def. also holds, in the distributional sense, for functions $u(x)$ with jump discontinuities)

example:

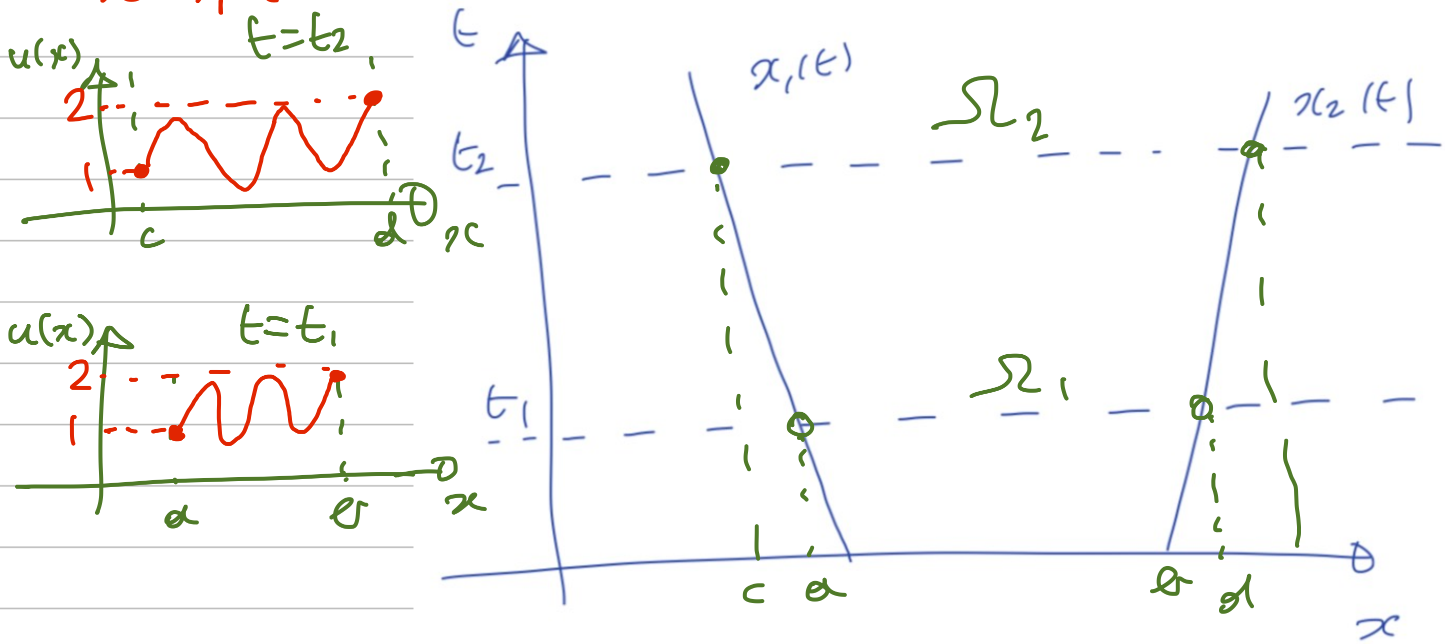


$$TV(u(x);[x_1,x_2]) = |u_b - u_a| + |u_c - u_b| + |u_d - u_c| + |u_e - u_d|$$

note: if new local extrema arise, TV goes up

Total Variation Diminishing (TVD) property:

example:



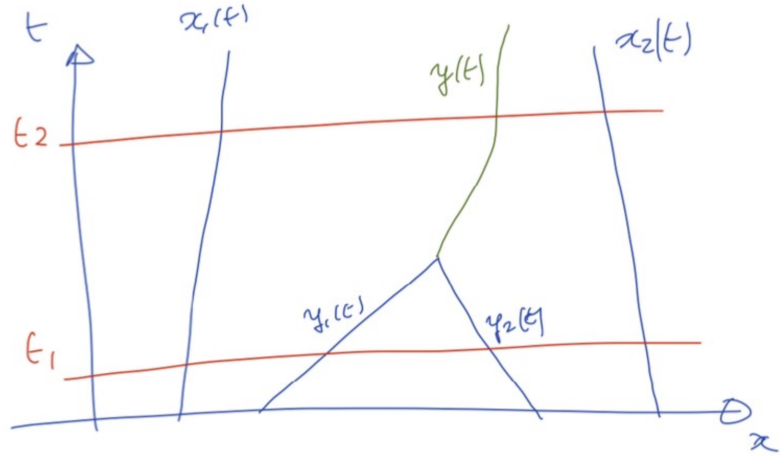
consider smooth flow between characteristics $x_1(t)$ and $x_2(t)$, then it is easy to see that

$$\underbrace{TV(u(x,t_2))}_{\text{on } \Omega_1} = \underbrace{TV(u(x,t_1))}_{\text{on } \Omega_2}$$

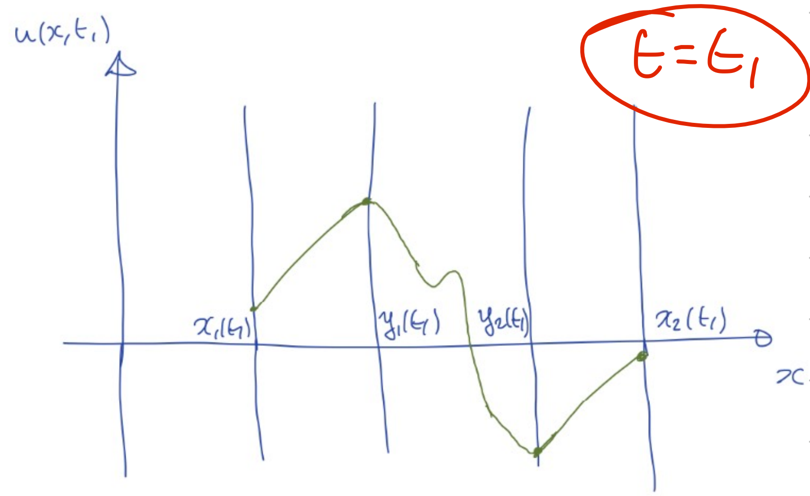
and

$$\frac{d}{dt} TV(u(x,t);[x_1(t),x_2(t)]) = 0$$

example: $u(x,t)$ steepening into shock
(entropy-satisfying, assume convex flux function)

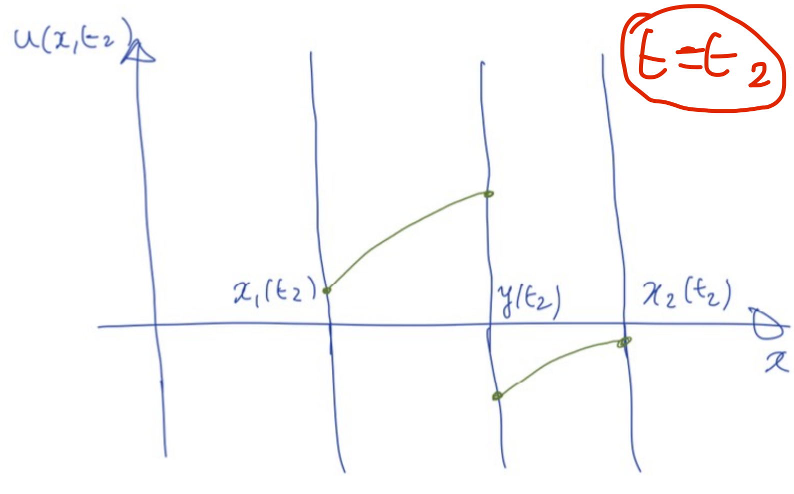


the variation between the characteristics $y_1(t)$ and $y_2(t)$ is "swept up" into the shock!



$\Downarrow$

the total variation decreases:



$TV(u(x,t_2);\Omega(t_2)) <$

$TV(u(x,t_1);\Omega(t_1))$

(typo in pdf notes!)

in general: (for scalar conservation law solutions)

$$\boxed{\frac{d}{dt} TV(u(x,t);[x_1(t),x_2(t)]) = \frac{d}{dt}\int_{x_1(t)}^{x_2(t)} \left|\frac{\partial u(x,t)}{\partial x}\right| dx \leq 0}$$