1 Black-Scholes equation for option pricing

1.1 The Black-Scholes equation

The Black-Scholes equation is a *backwards* parabolic PDE that models the price of a financial option. Consider the boundary value problem

given that element in Cartesian product

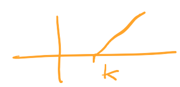
$$\begin{cases} u_t + \frac{1}{2}\sigma^2 x^2\, u_{xx} + r\,x\,u_x - r\,u = 0 & \text{on } \Omega = \{(x,t) | (x,t) \in (0, x_{\max}) \times (0,T)\} \\ u(x,T) = \max(x-K, 0) & \text{(end condition, European call option)} \\ u(0,t) = 0 & \text{(boundary condition)} \\ u(x_{\max}, t) = x_{\max} - K\exp\left(-r(T-t)\right) & \text{(boundary condition)}, \end{cases} \tag{1}$$

$e^{-r(T-t)}$

- call options go up as the underlying asset goes above the strike price
- put options go down as the underlying asset goes below the strike price

which models the price, $u(x,t)$, of a *European call option,* where x is the price of stock S that is *underlying* the option and t is time. In a European call option, at time $t = 0$, the holder acquires the right, but not the obligation, to buy stock S for the *strike price* K at the *expiry time* T. Here, r is the risk-free interest rate (e.g., 5% per year) and σ is the volatility of the stock (e.g., 30% per year). The European call option has value for the holder, because the holder will be able to make a profit $x - K$ if the price x of stock S at time T is greater than the strike price K; the holder would then exercise the option to buy the stock at price K, and sell at price $x > K$ for a profit of $x - K$. If $x < K$ at $t = T$ the option is not exercised and has zero value. Hence, at the expiry time T, the value of the European call option is given by the *pay-off function*

$$g(x) = \max(x - K, 0),$$



ReLU shifted

where x is the value at time T of stock S. Other types of options, e.g. the European put option, can be modeled by considering different pay-off functions.
The value of the option varies over time and, at any given time $t \in [0, T]$, depends on the price x of stock S at time t. The Black-Scholes PDE in problem Equation (1) provides a model for computing the value, $u(x,t)$, of the option for any price x of stock S at any time $t \in [0,T]$. The Black-Scholes PDE can be derived by considering a random price $X(t)$ for stock S that follows a geometric Brownian motion process, subject to volatility σ. The PDE is derived for a hedging scenario where a portfolio of stock S is made risk-free by adding options, assuming there are no arbitrage opportunities, i.e., all risk-free portfolios must earn the risk-free rate of return, r. An important characteristic of the Black-Scholes equation is that it is solved *backwards* in time. That is, that we know the value of stock option at time $T > 0$, and we wish to compute the value of the option at time $t = 0$, in order to assess whether the option is worth its cost of purchase at the present time.

1.2 End condition and boundary conditions

Clearly, at time $t = T$ the value of the option is given by the pay-off function $g(x)$, and this gives the *end condition* for the backwards parabolic equation in problem Equation (1). To derive *boundary conditions* for the PDE, we consider two scenarios. First, if the price, x, of stock S is zero at any time t, then the value of the option is $u(0,t) = 0$, since the potential for profit is greater if we buy the stock at zero price rather than an option. This provides the first boundary condition in problem Equation (1). On the other hand, for a large stock price, x, a reasonable approximation of the option price at time t is given by $u(x,t) \approx x - K \exp(-r(T-t))$ because, for large x, it becomes increasingly likely that the holder will be able to exercise the option for a profit, paying K to get x (or likely more), and the cost K that will be needed to exercise the option at time T can be discounted using the risk-free rate r (because the holder can put $K \exp(-r(T-t))$ in the bank at time t to obtain K at the expiry time T). This provides the second boundary condition in problem Equation (1).

1.3 Crank-Nicolson Method for the Black-Scholes Equation

We now explain how the Crank-Nicolson finite-difference method can be used to compute option prices modeled by the Black-Scholes equation.
Considering the Black-Scholes problem Equation (1), we can approximate the option price on a Cartesian grid in $\Omega = \{(x,t)|(x,t) \in (0, x_{\max}) \times (0,T)\}$ by

$$u_{k,l} \approx u(x_k, t_l).$$

Since explicit finite-difference methods would usually lead to stringent time-step limitations for the parabolic Black-Scholes PDE, implicit methods are often preferred. The Crank-Nicolson method is a popular method for the Black-Scholes PDE, because there is no time-step limitation for stability, and the approximation is second-order accurate. The Crank-Nicolson discretization for the Black-Scholes PDE is given by

$$\frac{u_{k,l+1} - u_{k,l}}{h_t} + \frac{1}{2} q_{k,l} + \frac{1}{2} q_{k,l+1} = 0, \tag{2}$$

where

$$q_{k,l} = \frac{1}{2}\sigma^2 x_k^2 \, \frac{u_{k+1,l} - 2u_{k,l} + u_{k-1,l}}{h_x^2} + r\, x_k \, \frac{u_{k+1,l} - u_{k-1,l}}{2\, h_x} - r\, u_{k,l}. \tag{3}$$

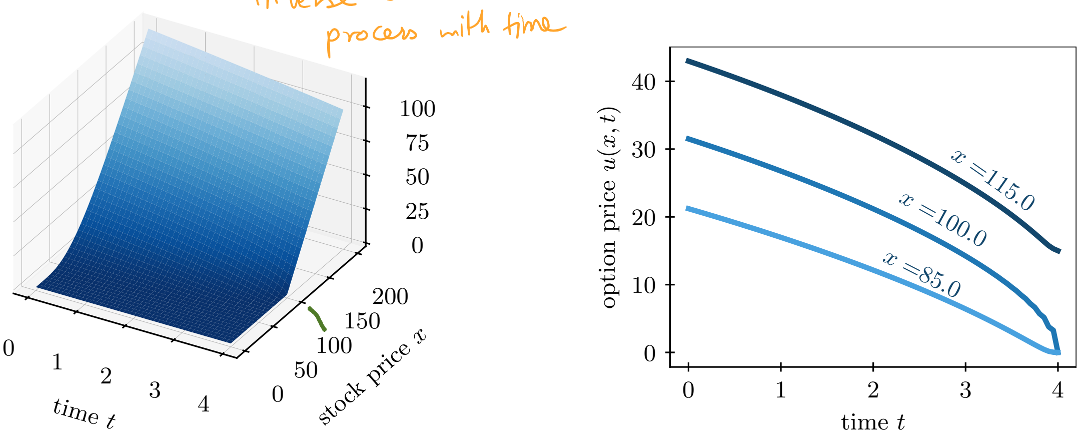
1.4 Example problem and interpretation

Figure 1 shows numerical results for Crank-Nicolson applied to the European call option problem of Equation (1), for strike price $K = \$100$ and expiry time $T = 4$ years. As desired, the option price $u(x,t)$ equals the pay-off function at time $t = T$. If the initial stock price $x = 85$ at time $t = 0$, the price of the option at $t = 0$ is approximately \$20. This option price reflects the risk-free value of the option to the holder, and is influenced by the risk-free rate r (a larger rate corresponds to a larger expected increase over time in the value of the risk-free portfolio) and by the volatility σ of the underlying stock S (a larger volatility increases the option value). If the stock price $x = 85$ were to remain constant over time, the value of the option would steadily decrease to 0 over time, because it would become increasingly likely that the holder would not be able to exercise the option for a profit. Considering now, for example, the value of the option at $t = 2$, we can see that the option would increase in value if the stock price x were to rise to, say, $x = 100$, reflecting the higher probability for a larger payout at $t = T$. If the stock price rises to $x = 115$, the option price increases further, and the option value at $t = T$ indeed equals the profit $115 - K = 15$. In the Black-Scholes PDE, the constant r reflects the expected increase in value of the portfolio, and the constant σ

reflects a backward diffusion process resulting from the volatility in the price of the underlying stock S.

Note, finally, that the Crank-Nicolson approximation shows some small spurious oscillations near the expiry time $T = 4$, especially for stock price x close to the strike price $K = 100$. While these oscillations are small and reduce in amplitude as the grid is refined, they are undesirable. In this problem, they occur due to the nonsmooth pay-off function, and they can be reduced by replacing the Crank-Nicolson time integration by Backward Euler.

inverse diffusion
process with time



(a) Option price $u(x,t)$ as a function of stock price x and time t.

(b) Option price as a function of time at stock price $x = 85$, 100, and 115.

Figure 1: Option price $u(x,t)$ for a European call option with strike price $K = \$100$ and expiry time $T = 4$ years. The risk-free cash rate $r = 5\%$ /year and the volatility $\sigma = 30\%$ /year. The option price is obtained by solving the Black-Scholes equation backward in time using the Crank-Nicolson method on a grid with 80 time intervals and 80 intervals of the stock price x over a range [0,200].