AMATH741/CM750/CS778 Winter 2023: Assignment 2

Instr.: Hans De Sterck e-mail: *hdesterck@uwaterloo.ca* Office hrs: Wednesdays 2-3pm

Assignment due date: Friday March 3, 5pm (dual submission, on Crowdmark and on LEARN)

Submission instructions:

- **All questions (theoretical (1–4) and computational (5–6)):** Please submit your assignment electronically via Crowdmark (you will receive a Crowdmark invitation). Your assignment package should contain the full answers to all questions, showing all your work. You can handwrite your answers on paper and then submit photos of your answers on Crowdmark. Another option is to type up your answers (e.g. using latex) and submit the electronic document. In either case, please start your answer for each question on a new page; Crowdmark requires that you submit a separate file or files for each question.
- **Computational questions (5–6):** The Crowdmark submission for Questions 5–6 should include written or typed answers to all the actual questions asked in the problems, including any computer output (tables or plots) that is required to answer the questions. In addition, a pdf or screenshot of all computer code for the programming Questions 5–6 should also be included in the Crowdmark submission (to facilitate marking feedback on your computer code). In addition to the Crowdmark submission, you must also separately submit all computer code files for the programming Questions 5–6 to the LEARN dropbox for Assignment 1. The LEARN computer code submission is due at the same time as the Crowdmark submission.

1. Characteristic curves and shock speed. (5 marks)

   Consider the nonlinear hyperbolic conservation law

   $$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} u^3 = 0.$$

   (a) Let curve $x(t)$ be a characteristic curve of the PDE. Find an expression for $dx(t)/dt$ as a function of u for this characteristic curve. Are the characteristic curves straight lines? Explain why. What is the slope of the characteristic curve that passes through a point in the xt-plane where $u = 2$?

   (b) Consider a discontinuity with left and right states $u_l = 2$ and $u_r = 1$. What is the shock speed?

2. Weak solutions of hyperbolic conservation laws. (10 marks)
   Consider the two PDEs

   $$\begin{cases} (1) \qquad u_t + \left(\frac{u^2}{2}\right)_x = 0, \\ (2) \qquad \left(\frac{u^2}{2}\right)_t + \left(\frac{u^3}{3}\right)_x = 0. \end{cases}$$

i. Show that $u(x,t) = x/t$ is a solution of both (1) and (2).

ii. Consider a Riemann problem with initial conditions $u_l = 2$ and $u_r = 1$ at $t = 0$. With these initial conditions, what is the solution for $t \geq 0$ for equation (1)? What is the solution for equation (2)?

iii. It appears that (1) and (2) have the same smooth solutions, but different discontinuous solutions. Resolve this apparent contradiction by explaining why the smooth solutions are the same (hint: how does equation (1) relate to (2)?), but the discontinuous solutions are not the same.

3. Burgers shock. (10 marks)
   Consider the Burgers equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0,$$

with initial conditions:

$$\begin{aligned} u(x,0) &= 0 \quad && \text{if} \quad x < 0 \\ u(x,0) &= x \quad && \text{if} \quad 0 \leq x \leq 1 \\ u(x,0) &= 0 \quad && \text{if} \quad x > 1. \end{aligned}$$

(a) Draw the characteristic curves in the xt plane, and explain why these initial conditions result in a solution with a shock that has a curved shock front.

make the figure
↳ write
→ you get a shock front if the left and right states are the same
→ otherwise it is different

(b) What is the shock speed at $t = 0$?

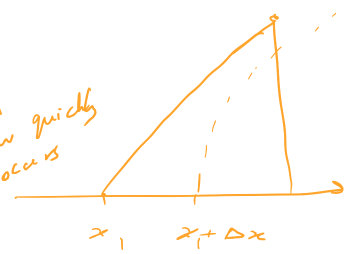
(c) Let $x_s(t)$ describe the location of the shock front as a function of time. Then

$$s(t) = \frac{dx_s(t)}{dt}$$

is the shock speed, which changes as a function of time, and is determined by the Rankine-Hugoniot relation.
Find the location $x_s(t)$ of the shock as a function of time t.

4. Shock formation time. (10 marks)

slope of the initial condition determines how quickly steepening occurs



Consider the scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0,$$

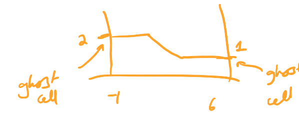
Find the time when the characteristics intersect

with $(x,t) \in \mathbb{R} \times [0,\infty)$. Assume that $f(u)$ is a smooth function of u and $f''(u) \neq 0 \, \forall u$. Assume that the initial condition $u(x, t = 0)$ is a smooth function of x. Find an expression for the time T at which the first discontinuity forms. (Hint: The expression contains derivatives of u and $f(u)$, and can be obtained by Taylor series expansion using the fact that shocks form when neighbouring characteristics intersect.)

5. Computational question: Finite Volume method for the 1D Burgers equation. (15 marks)
   Consider the Burgers equation

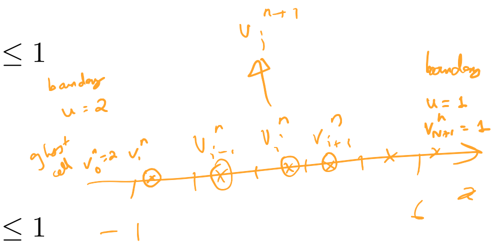
$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0,$$

with the following two sets of initial conditions.
Problem 1:



$$
\begin{array}{lll}
u_1(x,0) = 1 & \text{if} & x < 0 \\
u_1(x,0) = 1 + x & \text{if} & 0 \le x \le 1 \\
u_1(x,0) = 2 & \text{if} & x > 1.
\end{array}
$$

Problem 2:



$$
\begin{array}{lll}
u_2(x,0) = 2 & \text{if} & x < 0 \\
u_2(x,0) = 2 - x & \text{if} & 0 \le x \le 1 \\
u_2(x,0) = 1 & \text{if} & x > 1.
\end{array}
$$

(a) For each of the two problems, draw an xt diagram with characteristic curves. Do you obtain a solution with a shock or a rarefaction wave? In case of a shock solution, derive the shock speed from the Rankine-Hugoniot relation.

(b) Write a program in which you simulate Burgers problems 1 and 2 on a domain $(x,t) \in [-1,6] \times [0,2]$, using the (local) Lax-Friedrichs method (which is a conservative method). Verify that you obtain the solutions predicted by your analysis of the problem in (a). Submit your program file to the LEARN drop box. Also, include a copy of the code in your pdf assignment document. Include plots of the solution for problems 1 and 2, at times $t = 0.5$ and $t = 2.0$. Use $N = 70$ grid cells and $\Delta t = 0.5\,\Delta x / \max_i |v_i|$ (where you can either determine Δt dynamically in every timestep, or use the knowledge that $u_{max} = 2$).

(Note: It may be easiest to consider the use of *ghost cells* at the boundaries of the domain, to impose boundary conditions. You can add one extra FV cell on each side of the domain, containing a solution value that you set to an appropriate value to impose the boundary condition. In this way, you can apply the usual FV scheme to every physical cell in the interior of the domain (because every physical cell now has a left and a right neighbouring cell). Since in the problems we solve here the waves emanating from the centre of the domain do not reach the boundary before the final simulation time, the ghost cell values can just be kept identical to the initial conditions on the left and right edges of the physical domain; that way nothing will propagate into the domain from the boundaries, as desired.)

need 2 ghost cells

(c) Extend your code to second-order accuracy using linear reconstruction with the minmod limiter in space, and the second-order Runge-Kutta method in time. Provide plots of the solution with this scheme for problems 1 and 2, at time $t = 2.0$. (You will now need two ghost cells on each side of the spatial domain.)

(d) Provide a plot for problem 1 where you overlay the solutions with the schemes from (b) and (c), at $t = 2$ in the spatial interval [1,6]. Which solution is more accurate and why?

(e) Provide a plot for problem 2 where you overlay the solutions with the schemes from (b) and (c), at $t = 2$ in the spatial interval [2.5,4.5]. Which solution is more accurate and why?

6. Computational question: Crank-Nicolson finite difference method for the Black-Scholes equation. (15 marks)

Consider the Black-Scholes boundary value problem

$$
\begin{cases}
u_t + \frac{1}{2}\sigma^2 x^2\, u_{xx} + r\,x\,u_x - r\,u = 0 & \text{on } \Omega = \{(x,t) | (x,t) \in (0, x_{\max}) \times (0,T)\} \\
u(x,T) = \max(x - K, 0) & \text{(end condition, European call option)} \\
u(0,t) = 0 & \text{(boundary condition)} \\
u(x_{\max}, t) = x_{\max} - K \exp\left(-r(T-t)\right) & \text{(boundary condition)},
\end{cases}
\tag{1}
$$

which models the price, $u(x,t)$, of a *European call option* with pay-off function $g(x) = \max(x - K, 0)$, where x is the price of stock S that is *underlying* the option and t is time. In a European call option, at time $t = 0$, the holder acquires the right, but not the obligation, to buy stock S for the *strike price* K at the *expiry time* T. Here, r is the risk-free interest rate (e.g., 5% per year) and σ is the volatility of the stock (e.g., 30% per year).

(a) Write a MATLAB m-file `myBlackScholes.m` that implements the Crank-Nicolson method for boundary value problem (3).

(b) Reproduce the two plots in the pdf handout `Black-Scholes-handout.pdf` that was distributed in class, for strike price $K = \$100$ and expiry time $T = 4$ years, with risk-free cash rate $r = 5\%$ /year and volatility $\sigma = 30\%$ /year. Use a grid with 80 time intervals and 80 intervals of the stock price x over the range [0,200].

(c) Assume the stock price $x = \$85$ at $t = 0$ years. Make a table with the value of the option at $x = \$85$, $t = 0$, for different risk-free cash rates $r = 3\%$ /year, $r = 5\%$ /year, and $r = 7\%$ /year, and for different volatilities $\sigma = 20\%$ /year, $\sigma = 30\%$ /year, and $\sigma = 40\%$ /year. How does the option value change as r increases, and as σ increases? Explain intuitively why.