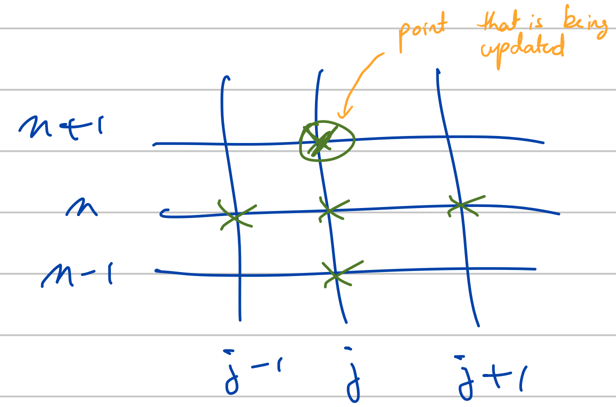
2.2.4 FD methods for the wave equation:

$$u_{tt} - a^2 u_{xx} = 0$$

we consider a standard method: (central in space and time)

$$\frac{v_j^{n+1} - 2v_j^n + v_j^{n-1}}{\Delta t^2} = a^2 \frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{\Delta x^2}$$

stencil:



explicit method

truncation error: $t_j^n = O(\Delta t^2) + O(\Delta x^2)$

stability: $\Delta t \leq \dfrac{\Delta x}{|a|}$ (CFL condition)

(can be derived by Von Neumann method:

$$v_j^n = \hat{v}_n \exp(i j k \Delta x)$$)

physical interpretation of the CFL condition:

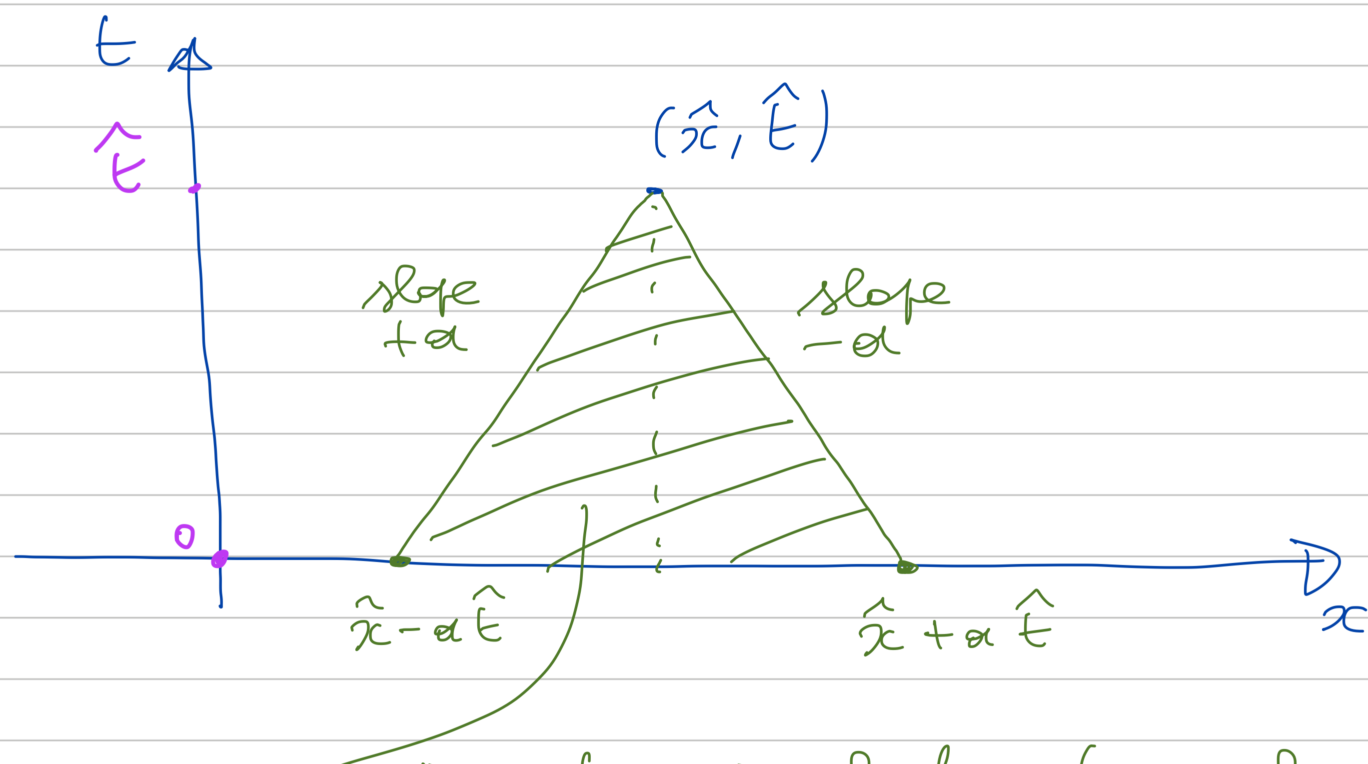
consider IVP

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0 & \text{(PDE)} \\ \quad \text{(assume } a > 0) & \\ (x,t) \in (-\infty,\infty) \times (0,\infty) & \text{(domain)} \\ \left.\begin{aligned} u(x,0) &= \phi_0(x) \\ u_t(x,0) &= \phi_1(x) \end{aligned}\right\} & \text{initial conditions (ICs)} \end{cases}$$

general solution: d'Alembert solution

$$u(x,t) = \frac{1}{2}\left(\phi_0(x+at) + \phi_0(x-at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} \phi_1(y)\,dy$$

average of ICs advected to the right & left



D : domain of dependence of the solution at $(\hat{x}, \hat{t})$

derivation of d'Alembert solution:

(section 1.2.3)

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0 & (1) \\ (x,t) \in (-\infty,\infty)\times(0,\infty) & \\ u(x,0) = \phi_0(x) & (2) \\ u_t(x,0) = \phi_1(x) & (3) \end{cases}$$

observe:

$$\boxed{u(x,t) = f(x-at) + g(x+at)} \quad (4)$$

is a solution of (1) for any suitable $f(x)$ and $g(x)$

since (1) can be written as

$$(\partial_t + a\partial_x)(\partial_t - a\partial_x)\, u(x,t) = 0$$

(check by using chain rule)

now substitute (4) into ICs (2) and (3):

$$f(x) + g(x) = \phi_0(x) \qquad (5)$$

$$-a f'(x) + a g'(x) = \phi_1(x)$$

$$-a\big(f(x) - f(c)\big) + a\big(g(x) - g(c)\big) = \int_c^x \phi_1(y)\,dy \qquad (6)$$

$a(5)+(6)$: $\quad 2a\,g(x) = a\,\phi_0(x) + \int_c^x \phi_1(y)\,dy$

$$- a\big(f(c) - g(c)\big)$$

$a(5)-(6)$: $\quad 2a\,f(x) = a\,\phi_0(x) - \int_c^x \phi_1(y)\,dy$

$$+ a\big(f(c) - g(c)\big)$$

$$2\alpha g(x) = \alpha\phi_0(x) + \int_c^x \phi_1(y)\,dy$$
$$-\alpha\left(f(c) - g(c)\right)$$
$$2\alpha f(x) = \alpha\phi_0(x) - \int_c^x \phi_1(y)\,dy$$
$$+\alpha\left(f(c) - g(c)\right)$$

so $$f(x-\alpha t) = \frac{1}{2}\phi_0(x-\alpha t) - \frac{1}{2\alpha}\int_c^{x-\alpha t} \phi_1(y)\,dy$$
$$+ f(c) - g(c)$$
$$g(x+\alpha t) = \frac{1}{2}\phi_0(x+\alpha t) + \frac{1}{2\alpha}\int_c^{x+\alpha t} \phi_1(y)\,dy$$
$$-\left(f(c) - g(c)\right)$$

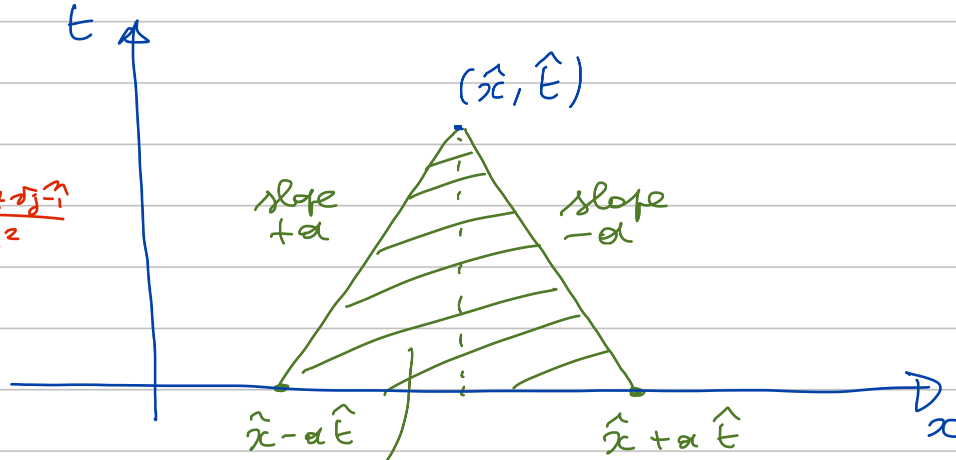
and $$u(x,t) = \frac{1}{2}\left(\phi_0(x-\alpha t) + \phi_0(x+\alpha t)\right)$$
$$+\frac{1}{2\alpha}\int_{x-\alpha t}^{x+\alpha t} \phi_1(y)\,dy$$

$\square$

$$u(x,t) = \frac{1}{2}\left(\phi_0(x+at) + \phi_0(x-at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} \phi_1(y)\,dy$$

$$u_{tt} - a^2 u_{xx} = 0$$

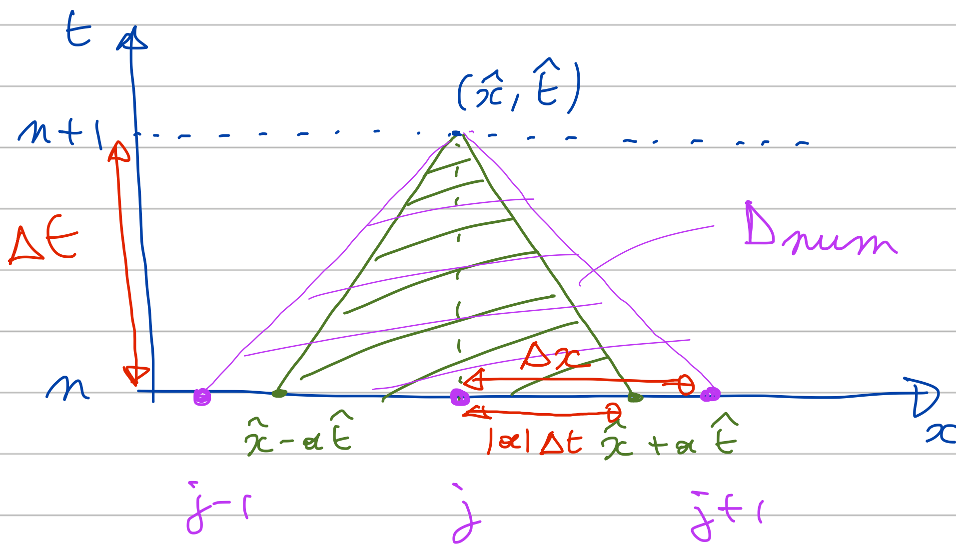
$$\frac{v_j^{n+1} - 2v_j^n + v_j^{n-1}}{\Delta t^2} = a^2 \frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{\Delta x^2}$$



D : domain of dependence of the solution at $(\hat{x}, \hat{t})$

① $\Delta t \leq \dfrac{\Delta x}{|a|}$

$(\Delta x \geq |a| \Delta t)$

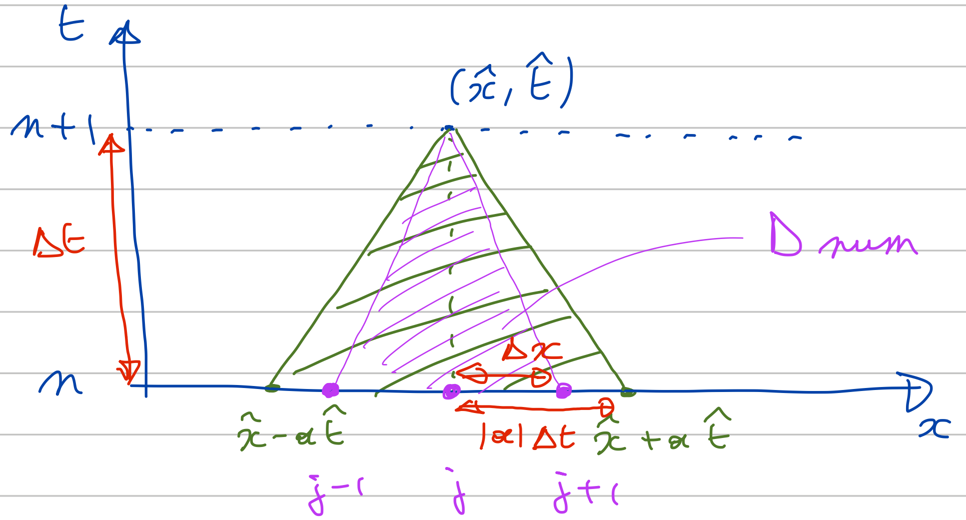


numerical domain of dependence, D_{num}, contains physical domain of dependence, D : numerically stable

(the numerical scheme has access to all the data it needs, in the entire physical domain of dependence)

② $\Delta t \geq \frac{\Delta x}{|\alpha|}$

$(\Delta x \leq |\alpha| \Delta t)$



numerical domain of dependence, D num, does not contain physical domain of dependence, D : numerically unstable

(the numerical scheme does not have access to the physically relevant data at time level n)