Chapter 2: Finite Difference Methods

2.1 FD methods for elliptic PDEs

prototype: $u_{xx} + u_{yy} = f(x,y)$

2.1.1 1D elliptic model problem

Consider the boundary value problem (BVP)

$$\text{BVP} \begin{cases} u''(x) = f(x) & \to \text{PDE (ODE here)} \\ \Omega = (0,1) & \to \text{domain} \\ u(0) = \alpha & \to \text{boundary conditions} \\ u(1) = \beta & \end{cases}$$

note: it can be shown that elliptic BVPs of this type have a unique solution (for suitable functions $f(x,y)$)

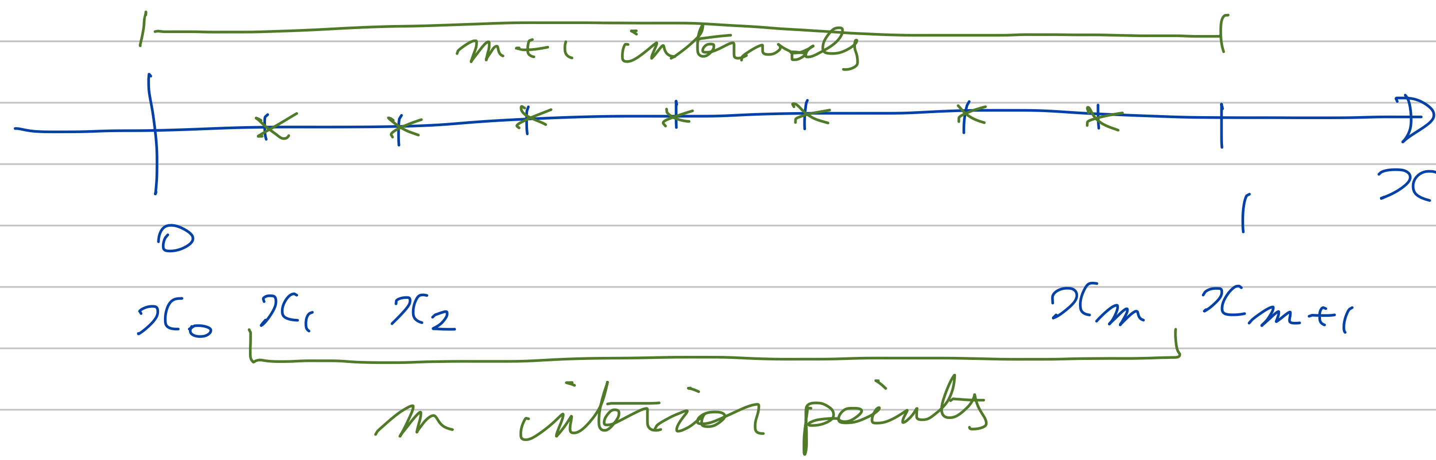
how to find a numerical approximation for $u(x)$?

BVP $\begin{cases} u''(x) = f(x) \\ \Omega = (0,1) \\ u(0) = \alpha \\ u(1) = \beta \end{cases}$

step 1: discretize the domain

divide $\Omega = (0,1)$ into $m+1$ intervals (of equal size, for now)

using $m+2$ discrete points



$$x_i = x_0 + i h \qquad (i = 0, \dots, m+1)$$

grid spacing $\Delta x = \dfrac{1}{m+1} = h$

step 2: discretize the equation

$$\boxed{u''(x) = f(x)}$$

let $u(x)$ be the exact solution

let $\underbrace{u_i = u(x_i)}_{\text{exact solution in the grid points}} \qquad (i = 0, \dots, m+1)$

$f_i = f(x_i)$

$$\boxed{u''(x) = f(x)} \qquad (1)$$

use "finite differences" to discretize (1)

<u>notation:</u> let $u_i' = u'(x_i)$

$u_i'' = u''(x_i)$

Fundamental theorem of calculus

slope

$$\lim_{h \to 0} \frac{\left(\frac{u_{i+1} - u_i}{h}\right) - \left(\frac{u_i - u_{i-1}}{h}\right)}{h} = \frac{u_{i+1} - 2u_i + u_{i+1}}{h^2}$$

then we have

$$u_i'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + O(h^2)$$

Taylor Series:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$$

where $f^{(n)}(a) \equiv u_i^{(n)}$

$a \equiv x_i$ $\quad u_i \equiv u(x_i)$ $\quad u_{i\pm1} \equiv u(x_{i\pm1})$

$x \equiv x_{i\pm1}$ $\quad (x - a) \equiv h$

by Taylor series expansion:

$$u(x_{i+1}) = u_{i+1} = u_i + u_i' h + u_i'' \frac{h^2}{2} + u_i''' \frac{h^3}{6} + u_i'''' \frac{h^4}{24} + O(h^5)$$

$$u(x_{i-1}) = u_{i-1} = u_i - u_i' h + u_i'' \frac{(-h)^2}{2} + u_i''' \frac{(-h)^3}{6} + u_i'''' \frac{(-h)^4}{24} + O(h^5)$$

or $u_{i+1} + u_{i-1} = 2u_i + u_i'' h^2 + u_i'''' \dfrac{h^4}{12} + O(h^5)$

so

$$\boxed{u_i'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - u_i'''' \frac{h^2}{12} + O(h^3)}$$

$$\boxed{u''(x) = f(x)} \qquad (1)$$

So we discretize (1):

we know $\dfrac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \approx f_i$

exact solution

now we seek an approximation

$v_i \approx u_i = u(x_i)$

If this linear system is non-singular we have our solution

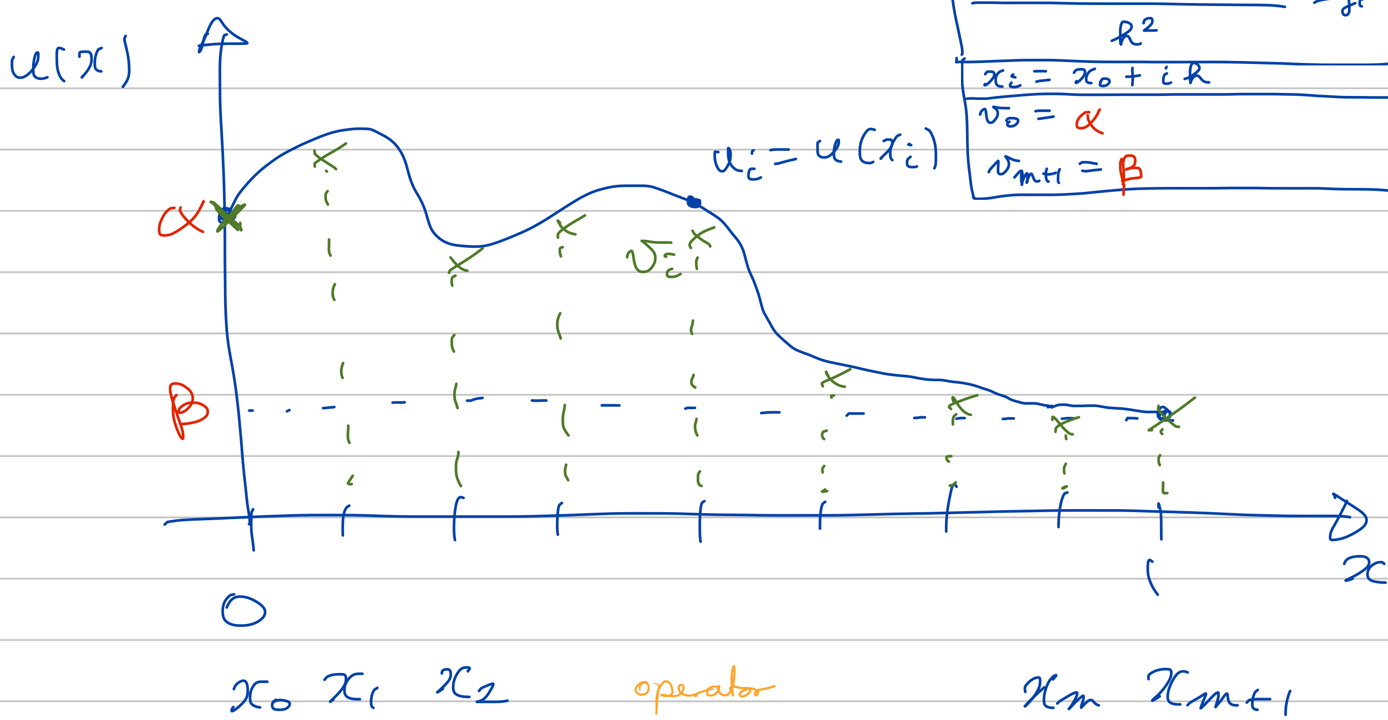
such that

$$\boxed{\begin{aligned} &\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} = f_i \quad (i = 1, \dots, m) \\ &x_i = x_0 + ih \quad (i = 0, \dots, m+1) \\ &v_0 = \alpha \\ &v_{m+1} = \beta \end{aligned}}$$

discretize BVP

linear system of equations!

$$\boxed{\begin{aligned} &\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} = f_i \quad (i = 1, \dots, m) \\ &x_i = x_0 + ih \quad (i = 0, \dots, m+1) \\ &v_0 = \alpha \\ &v_{m+1} = \beta \end{aligned}}$$



operator

superscript indicates discretized h

in matrix form: $A^h V^h = F^h$

$$V^h = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} \in \mathbb{R}^m$$

note: V^h is called a "grid function" (discrete approximation of the continuous function $u(x)$)

this matrix is specific to the 1d laplace equation

(the h in V^h is a generic superscript indicating a "grid function")

tridiagonal matrices are sparse (many zeros) and easy to solve.

$$F^h = \begin{bmatrix} f_1 - \frac{\alpha}{h^2} \\ f_2 \\ \vdots \\ f_{m-1} \\ f_m - \frac{\beta}{h^2} \end{bmatrix} \in \mathbb{R}^m$$

boundary conditions (exterior points) are moved to the other side of the equality and appear here

computational cost of order m

- usually matrices have a cost of order m^3 (Gaussian elimination)

$$A^h = \frac{1}{h^2} \begin{bmatrix} -2 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{bmatrix} \in \mathbb{R}^{m \times m}$$

m interior points

how accurate is the approximation?

def: actual error

$$E^h = U^h - V^h$$ (exact − approximate)

per component: $e_i = u_i - v_i$

later on, we will study convergence of V^h to U^h:

→ we desire $E^h \to 0$ as $h \to 0$
$n \to \infty$

→ we know $u_i'' = \dfrac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + O(h^2)$

for every reduction of h by factor 2, the approx improves by factor 4

→ therefore, we expect

$$e_i = O(h^2) \text{ as } h \to 0$$

(convergence with order 2)

For polynomials that have an order less than the order of the truncation error, the numerical solution is exact ($u_{(n)x} = 0$)
n^{th} derivative of u

(we will show this later)