Part 2: Finite Volume (FV) methods for hyperbolic conservation laws

↳ in second set of pdf course notes

Chapter 3: Scalar conservation laws in 1D

Chapter 4: Systems of conservation laws in 2D and 3D

1.2.1 Scalar first-order conservation laws:

scalar function $u(\vec{x}, t)$

where $\vec{x} = (x, y, z)$

scalar conservation law: (in 3D)

$$\boxed{u_t + \nabla \cdot \vec{f}(u) = 0}$$

where $\vec{f}(u) = (f(u), g(u), h(u))$

$$\nabla \cdot \vec{f}(u) = \frac{\partial f(u)}{\partial x} + \frac{\partial g(u)}{\partial y} + \frac{\partial h(u)}{\partial z}$$

$\vec{f}(u)$ is called the "flux vector"

(1) $\boxed{u_t + \nabla \cdot \vec{f}(u) = 0}$ (conservation law) (differential form)

integrate over spatial domain $\Omega \subset \mathbb{R}^3$:

$$\iiint_{\Omega} (u_t + \nabla \cdot \vec{f}(u))\, dV = 0 \qquad (*)$$

let $\displaystyle\iiint_{\Omega} u(x,t)\, dV = \hat{u}(t)$

then

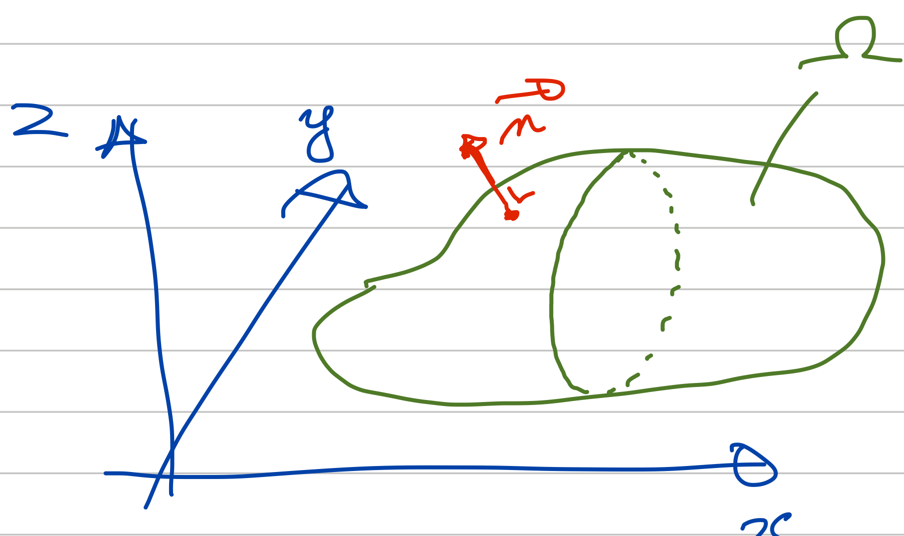
integral over a closed surface

$$(*) \Longrightarrow (2) \quad \boxed{\hat{u}_t + \oiint_{\partial\Omega} \vec{f}(u)\cdot\vec{n}\, dS = 0}$$

(first integral form of the conservation law)

we have used Gauss' theorem :

$$\boxed{\iiint_{\Omega} \nabla\cdot\vec{f}\, dV = \oiint_{\partial\Omega} \vec{f}\cdot\vec{n}\, dS}$$



$\vec{n}$ = unit outward normal

$$(1) \quad \boxed{u_t + \nabla \cdot \vec{f}(u) = 0}$$

$\iiint_{\Omega} \vec{u}(x,t)\, dV = \hat{u}(t)$

$$(2) \quad \boxed{\hat{u}_t + \oiint_{\partial\Omega} \vec{f}(u) \cdot \vec{n}\, dS = 0}$$

observe: $u(\vec{x},t)$ is called a "conserved quantity"

since the total amount of $u(\vec{x},t)$ in a domain Ω can only change if there is a net flux through the boundary $\partial\Omega$ of Ω

now integrate over time interval $[t_1, t_2]$

$$(3) \quad \boxed{\hat{u}(t_2) - \hat{u}(t_1) + \int_{t_1}^{t_2} \oiint_{\partial\Omega} \vec{f}(u) \cdot \vec{n}\, dS\, dt = 0}$$

(second integral form)

note: we will later also consider systems of conservation laws