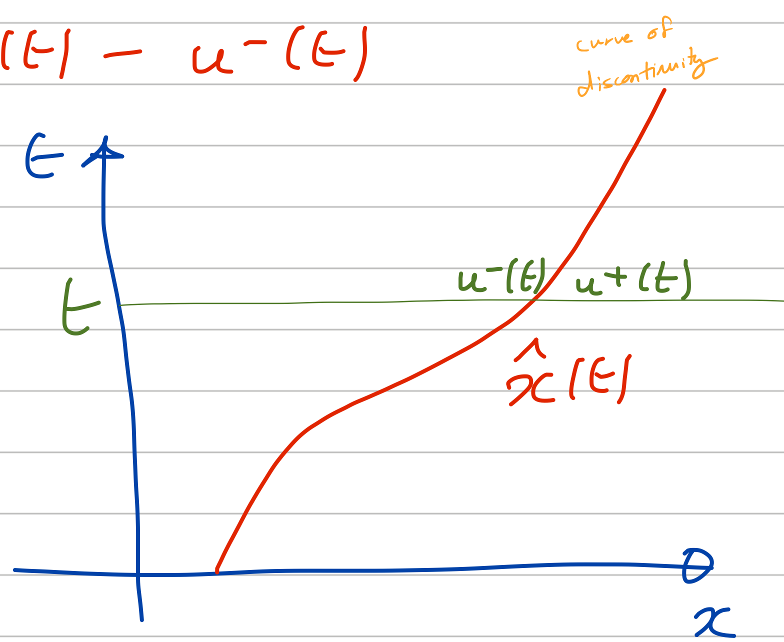
3.1.3 Shock speed:

Thm. 3.7: Let $\hat{x}(t)$ be a curve along which a weak solution of $u_t + f(u)_x = 0$ has a jump discontinuity. Then

$$\hat{x}'(t) = \frac{f(u^+(t)) - f(u^-(t))}{u^+(t) - u^-(t)}$$

Charachteristic curve is given by:

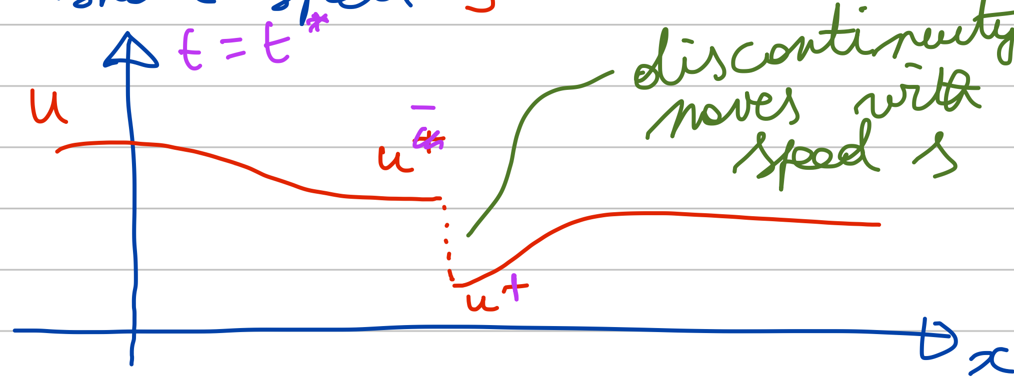
$$\frac{d x(t)}{dt} = \frac{df(u)}{du}$$



note:

$$s = \hat{x}' = \frac{f(u^+) - f(u^-)}{u^+ - u^-}$$

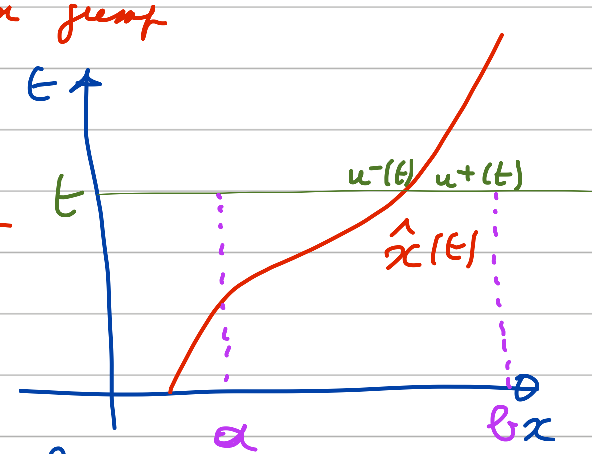
is called the Rankine – Hugoniot relation for shock speed s



Thm. 3.7: Let $\hat{x}(t)$ be a curve along which a weak solution of $u_t + f(u)_x = 0$ has a jump discontinuity. Then

$$\boxed{u_t + f(u)_x = 0}$$

$$\hat{x}'(t) = \frac{f(u^+(t)) - f(u^-(t))}{u^+(t) - u^-(t)}$$



Proof:

consider first integral form

$$\frac{d}{dt}\left(\underbrace{\int_a^{\hat{x}(t)^-} u(x,t)\,dx}_{G_a(\hat{x}(t)^-,\,t)} + \underbrace{\int_{\hat{x}(t)^+}^{b} u(x,t)\,dx}_{G_b(\hat{x}(t)^+,\,t)}\right) + f(u(b,t)) - f(u(a,t)) = 0$$

since
$$\frac{d}{dt} G_a(\hat{x}(t)^-,t) = \frac{\partial G_a}{\partial \hat{x}}\frac{d\hat{x}}{dt} + \frac{\partial G_a}{\partial t}$$
$$= u(\hat{x}(t)^-,t)\,\hat{x}'(t) + \int_a^{\hat{x}(t)^-} \underbrace{u_t}_{-f(u)_x}\,dx$$
$$= u(\hat{x}(t)^-,t)\,\hat{x}'(t) - \left(f(u(\hat{x}(t)^-,t)) - f(u(a,t))\right)$$

respects conservation law.

$$\frac{d}{dt} G_b(\hat{x}(t)^+,t) = -u(\hat{x}(t)^+,t)\,\hat{x}'(t) + f(u(\hat{x}(t)^+,t)) - f(u(b,t))$$

so:
$$u^-\hat{x}' - f(u^-) + \cancel{f(u(a,t))} - u^+\hat{x}' + f(u^+) - \cancel{f(u(b,t))} + \cancel{f(u(b,t))} - \cancel{f(u(a,t))} = 0$$

or
$$(u^- - u^+)\,\hat{x}' = f(u^-) - f(u^+) \qquad \square$$

shock speed: $s = \hat{x}' = \dfrac{f(u^+) - f(u^-)}{u^+ - u^-}$ (Rankine–Hugoniot)

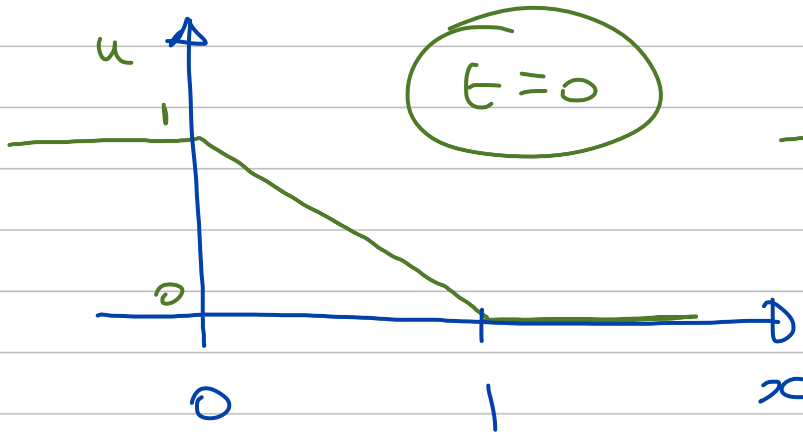
E2 shock wave: (steepening wave)

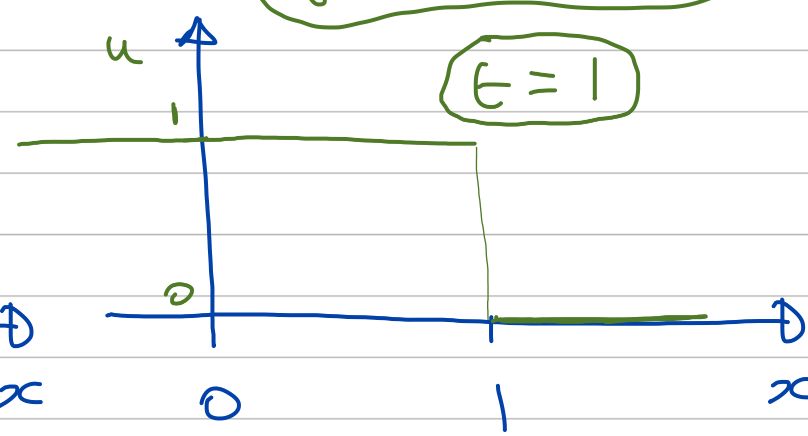
initial condition

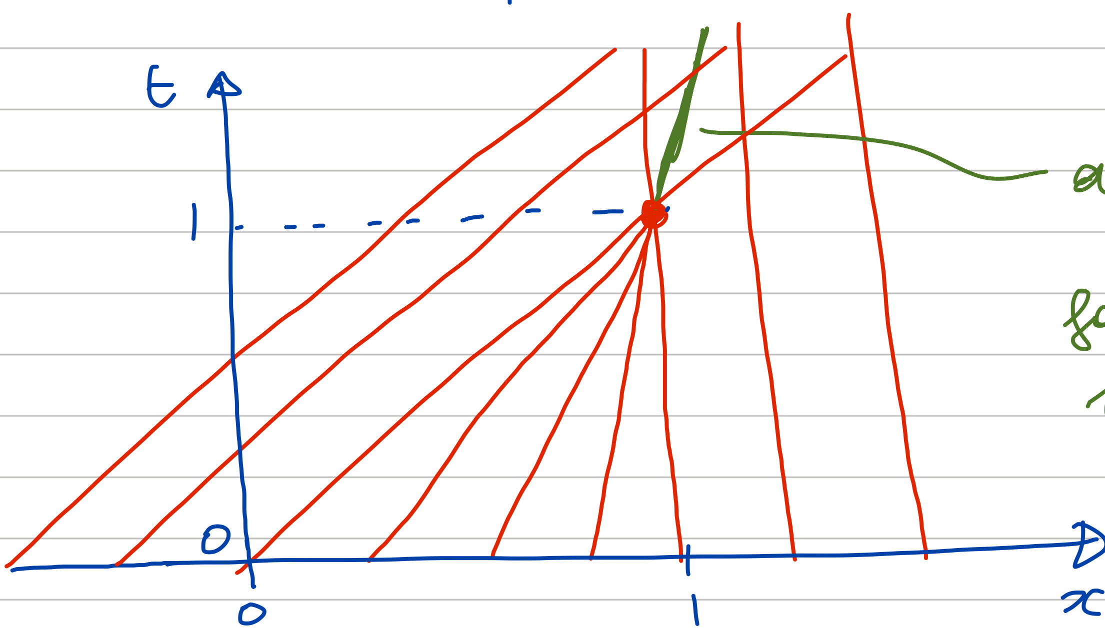
$$\begin{cases} u(x,0) = 1 & (x \le 0) \\ u(x,0) = 1 - x & (0 \le x \le 1) \\ u(x,0) = 0 & (1 \le x) \end{cases}$$

Burgers: $u_t + \left(\dfrac{u^2}{2}\right)_x = 0$

$f(u) = u^2/2$







a shock (discontinuity) forms at $t = 1$; what is the shock speed?

$$s = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \frac{\frac{0^2}{2} - \frac{1^2}{2}}{0 - 1} = \frac{1}{2}$$

$\Rightarrow$ shock moves with speed $1/2$