AMATH741/CM750/CS778 Winter 2023: Assignment 2

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Assignment due date: Friday March 3, 5pm (dual submission, on Crowdmark and on LEARN)

Submission instructions:

- **All questions (theoretical (1–4) and computational (5–6)):** Please submit your assignment electronically via Crowdmark (you will receive a Crowdmark invitation). Your assignment package should contain the full answers to all questions, showing all your work. You can handwrite your answers on paper and then submit photos of your answers on Crowdmark. Another option is to type up your answers (e.g. using latex) and submit the electronic document. In either case, please start your answer for each question on a new page; Crowdmark requires that you submit a separate file or files for each question.
- **Computational questions (5–6):** The Crowdmark submission for Questions 5–6 should include written or typed answers to all the actual questions asked in the problems, including any computer output (tables or plots) that is required to answer the questions. In addition, a pdf or screenshot of all computer code for the programming Questions 5–6 should also be included in the Crowdmark submission (to facilitate marking feedback on your computer code). In addition to the Crowdmark submission, you must also separately submit all computer code files for the programming Questions 5–6 to the LEARN dropbox for Assignment 1. The LEARN computer code submission is due at the same time as the Crowdmark submission.

1. Characteristic curves and shock speed. (5 marks)

   Consider the nonlinear hyperbolic conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} u^3 = 0.$$

   (a) Let curve $x(t)$ be a characteristic curve of the PDE. Find an expression for $dx(t)/dt$ as a function of u for this characteristic curve. Are the characteristic curves straight lines? Explain why. What is the slope of the characteristic curve that passes through a point in the xt-plane where $u = 2$?

   (b) Consider a discontinuity with left and right states $u_l = 2$ and $u_r = 1$. What is the shock speed?

Solution: (a) Recall that $x(t)$ is a characteristic curve of the PDE if it satisfies

$$\frac{d}{dt} u(x(t), t) = 0,$$

i.e. if u is constant along the curve. Applying chain rule, this expression can be rewritten as

$$\frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} = 0. \tag{1}$$

Similarly, we can rewrite the conservation law as

$$\frac{\partial u}{\partial t} + 3u^2 \frac{\partial u}{\partial x} = 0. \tag{2}$$

Comparing (**??**) and (**??**), we have

$$\frac{dx}{dt} = 3u^2.$$

Since u is constant along the characteristic curve, this is equivilant to saying

$$\frac{dx}{dt} = \text{constant},$$

and so the characteristic curve is a straight line with slope $3u^2$. When $u = 2$, the slope is $3 \cdot 2^2 = 12$.

(b) The shock speed is given by

$$s = \frac{f(u_r) - f(u_l)}{u_r - u_l} = \frac{1^3 - 2^3}{1 - 2} = 7.$$

2. Weak solutions of hyperbolic conservation laws. (10 marks)
Consider the two PDEs

$$\begin{cases} (1) \qquad u_t + \left(\dfrac{u^2}{2}\right)_x = 0, \\ (2) \qquad \left(\dfrac{u^2}{2}\right)_t + \left(\dfrac{u^3}{3}\right)_x = 0. \end{cases}$$

i. Show that $u(x,t) = x/t$ is a solution of both (1) and (2).

ii. Consider a Riemann problem with initial conditions $u_l = 2$ and $u_r = 1$ at $t = 0$. With these initial conditions, what is the solution for $t \geq 0$ for equation (1)? What is the solution for equation (2)?

iii. It appears that (1) and (2) have the same smooth solutions, but different discontinuous solutions. Resolve this apparent contradiction by explaining why the smooth solutions are the same (hint: how does equation (1) relate to (2)?), but the discontinuous solutions are not the same.

Solution:

6 (a)

α (1) $\left(\frac{x}{t}\right)_t = \frac{-x}{t^2}$, $\left(\frac{x^2}{t^2}\frac{1}{2}\right)_x = \frac{2x}{2t^2} = \frac{x}{t^2}$

$$u_t + \left(\frac{u^2}{2}\right)_x = -\frac{x}{t^2} + \frac{x}{t^2} = 0$$

(2) $\left(\frac{x^2}{2t^2}\right)_t = \frac{x^2}{2}\frac{(-2)}{t^3} = \frac{-x^2}{t^3}$, $\left(\frac{x^3}{t^3}\frac{1}{3}\right)_x = \frac{3x^2}{3t^3} = \frac{x^2}{t^3}$

$$u_t + \left(\frac{u^2}{2}\right)_x = -\frac{x^2}{t^3} + \frac{x^2}{t^3} = 0$$

β (1) $f(u) = \frac{u^2}{2}$

$$s = \frac{f_r - f_e}{u_r - u_e} = \frac{\frac{1}{2} - \frac{4}{2}}{1 - 2} = \boxed{\frac{3}{2}}$$

$f' = u \qquad f'(u_e) = 2 \qquad f'(u_r) = 1$

$2 > \frac{3}{2} > 1$ shock

(2) $f(u) = \frac{u^3}{3} \qquad v = \frac{u^2}{2} \qquad u = \sqrt{2v}$

$$\cancel{s = \frac{f_r - f_e}{u_r - u_e}} = \frac{\frac{1}{3} - \frac{8}{3}}{\frac{1}{2} - \frac{4}{2}} = \frac{7}{3}\,\frac{2}{3} = \boxed{\frac{14}{9}}$$

$s = \frac{f(v_r) - f(v_e)}{v_r - v_e} \qquad 2 > \frac{14}{9} > 1$: shock of course.

$$f(v) = \frac{u^3}{3} = \frac{(2v)^{3/2}}{3} \qquad f'(v) = \frac{2^{3/2}}{3}\,\frac{3}{2}\,v^{\frac{1}{2}} = \frac{2^{3/2}}{2}\,\frac{u}{\sqrt{2}} = u$$

(γ) same smooth solutions:

(1) $u_t + u\,u_x = 0$

(2) $u\,u_t + u^2 u_x = 0$

multiplication by u does not change solution

different weak solution:

(1) $u_t + \left(\frac{u^2}{2}\right)_x = 0$ $\quad$ u is conserved

(2) $v_t + \left(\frac{(2v)^{3/2}}{3}\right)_x = 0$ $\quad$ v is conserved

different shock speeds $\quad$ $v = \frac{u^2}{2}$

3. Burgers shock. (10 marks)
Consider the Burgers equation
$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0,$$
with initial conditions:
$$\begin{array}{lll} u(x,0) = 0 & \text{if} & x < 0 \\ u(x,0) = x & \text{if} & 0 \le x \le 1 \\ u(x,0) = 0 & \text{if} & x > 1. \end{array}$$

(a) Draw the characteristic curves in the xt plane, and explain why these initial conditions result in a solution with a shock that has a curved shock front.

(b) What is the shock speed at $t = 0$?

(c) Let $x_s(t)$ describe the location of the shock front as a function of time. Then
$$s(t) = \frac{dx_s(t)}{dt}$$
is the shock speed, which changes as a function of time, and is determined by the Rankine-Hugoniot relation.
Find the location $x_s(t)$ of the shock as a function of time t.

Solution:

(a)

(b)
$$\frac{\frac{1}{2}(1^2 - 0^2)}{1-0} = 1/2$$

(c) On the characteristic emanating from $x = x_0 \in [0,1]$ at $t = 0$, the value of u equals $u = x_0$, so the slope of the characteristic is x_0. The x-coordinate of this characteristic, as a function of t, is given by
$$x = x_0 + x_0\, t.$$
Conversely, at any position (x,t) in the rarefaction, the point x_0 to which the characteristic going through (x,t) is traced back, is
$$x_0 = \frac{x}{1+t}.$$
So we have, at (x,t) in the rarefaction,
$$u(x,t) = x_0 = \frac{x}{1+t}.$$
With $x_s(t)$ the location of the shock front as a function of time, we have
$$s(t) = \frac{dx_s}{dt} = \frac{f(u_R(t)) - f(u_L(t))}{u_R(t) - u_L(t)} = \frac{1}{2}\frac{u_R^2 - u_L^2}{u_R - u_L} = \frac{u_R + u_L}{2} = \frac{1}{2}(0 + \frac{x_s}{1+t}).$$

Integrating the ODE gives

$$
\begin{aligned}
\frac{dx_s}{dt} &= \frac{1}{2}\frac{x_s}{1+t} \\
2\frac{dx_s}{x_s} &= \frac{dt}{1+t} \\
2d\ln x_s &= d\ln(1+t) \\
\ln x_s^2 &= \ln(1+t) + c \\
x_s^2 &= 1+t \\
x_s &= \sqrt{1+t}
\end{aligned}
$$

4. Shock formation time. (10 marks)

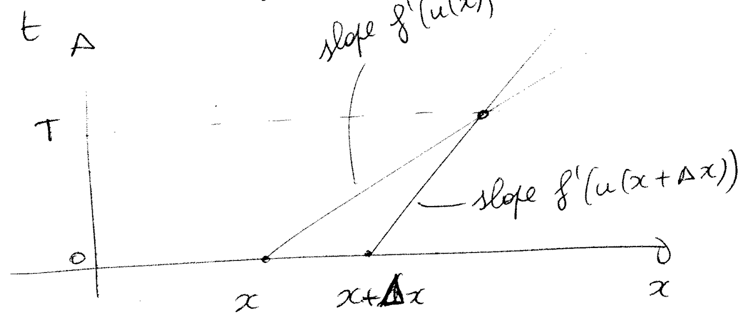
Consider the scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0,$$

with $(x,t) \in \mathbb{R} \times [0, \infty)$. Assume that $f(u)$ is a smooth function of u and $f''(u) \neq 0 \, \forall u$. Assume that the initial condition $u(x, t = 0)$ is a smooth function of x. Find an expression for the time T at which the first discontinuity forms. (Hint: The expression contains derivatives of u and $f(u)$, and can be obtained by Taylor series expansion using the fact that shocks form when neighbouring characteristics intersect.)

Solution:

(4) $\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$ or $\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} = 0$



consider two neighbouring points at $t=0$, x and $x+\Delta x$

shock forms at $t=T$ if characteristic curves from x and $x+\Delta x$ intersect at time T:

$$\cancel{x} + f'(u(x))T = \cancel{x} + \Delta x + f'(u(x+\Delta x))T$$

$$\Rightarrow T = \frac{-\Delta x}{f'(u(x+\Delta x)) - f'(u(x))}$$

take limit $\Delta x \to 0$:

$$T = -\left(\frac{d}{dx}(f'(u(x))\right)^{-1}$$

$$\text{or } T = \frac{-1}{f''(u)\frac{\partial u}{\partial x}}$$

but: no shock formation for $t>0$ if $T<0$.
and: take minimum T over whole interval (earliest shock formation)

$$\Rightarrow \hat{T} = \min_x \left(\max\left(0, \frac{-1}{f'(u)\frac{\partial u}{\partial x}}\right)\right)$$

note: $\hat{T}=0$ does mean ~~that~~ no shock is formed for $t>0$