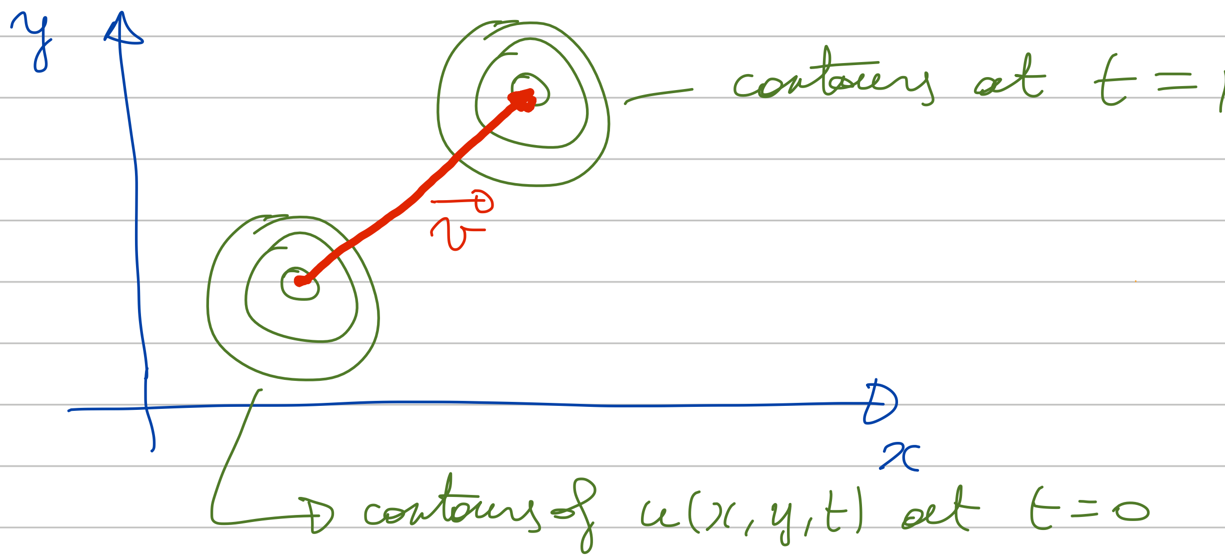
2.2.5 FD methods in 2D:

example: linear advection equation in 2 spatial dimensions

find $u(x,y,t)$ satisfying

$$\boxed{u_t + a\, u_x + b\, u_y = 0} \qquad (\text{assume } a>0, b>0)$$

$\vec{v} = (a, b)$ is the advection speed vector in the (x,y)-plane

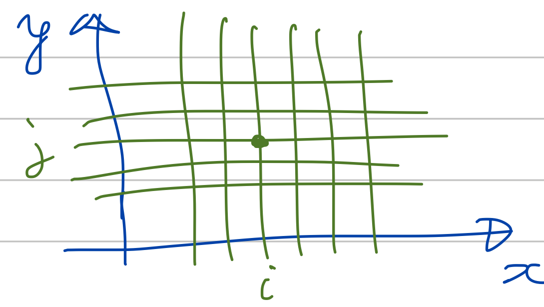


note: general solution: $u(x,y,t) = f(x-at, y-bt)$

Forward Upwind:

$$\frac{v_{ij}^{n+1} - v_{ij}^{n}}{\Delta t} + a\,\frac{v_{ij}^{n} - v_{i-1,j}^{n}}{\Delta x} + b\,\frac{v_{ij}^{n} - v_{i,j-1}^{n}}{\Delta y} = 0$$

$$t_{ij}^{n} = O(\Delta t) + O(\Delta x) + O(\Delta y)$$



numerical stability: Von Neumann

$$v^n_{j_1 j_2} = \hat{v}_n \exp\big(i\,(\overbrace{j_1 k_1 \Delta x}^{x_{j_1}} + \overbrace{j_2 k_2 \Delta y}^{y_{j_2}})\big)$$

and $\hat{v}_{n+1} = S(k_1, k_2)\,\hat{v}_n$

for stability: we require $\max\limits_{k_1, k_2 \in \mathbb{R}} |S(k_1,k_2)| \leq 1$

this results in the conditions:

$$\begin{cases} 0 \leq a \dfrac{\Delta t}{\Delta x} + b \dfrac{\Delta t}{\Delta y} \leq 1 & (1) \\ a \dfrac{\Delta t}{\Delta x} \geq 0 \\ b \dfrac{\Delta t}{\Delta y} \geq 0 \end{cases}$$

The derivation is in one of the reference textbooks

(1): $$\Delta t \leq \frac{1}{\dfrac{a}{\Delta x} + \dfrac{b}{\Delta y}}$$

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note : $\Delta t \leq \frac{1}{2} \min\left(\frac{\Delta x}{a}, \frac{\Delta y}{b}\right)$ is sufficient for stability

(because $\frac{1}{2} \min\left(\frac{\Delta x}{a}, \frac{\Delta y}{b}\right) \leq \frac{1}{\frac{a}{\Delta x} + \frac{b}{\Delta y}}$)

<u>proof:</u> $\frac{1}{2} \min(c, d) \leq \frac{cd}{c+d}$ $(c \geq 0, d \geq 0)$

$$\Updownarrow$$

$$c \leq 2\frac{cd}{c+d} \text{ or } d \leq 2\frac{cd}{c+d}$$

$$\Updownarrow$$

$$c^2 + cd \leq 2cd \text{ or } d^2 + cd \leq 2cd$$

$$\Updownarrow$$

$$c^2 \leq cd \text{ or } d^2 \leq cd$$

$$\Updownarrow \; c \geq 0 \text{ and } d \geq 0$$

$$c \leq d \text{ or } d \leq c \text{ OK}$$