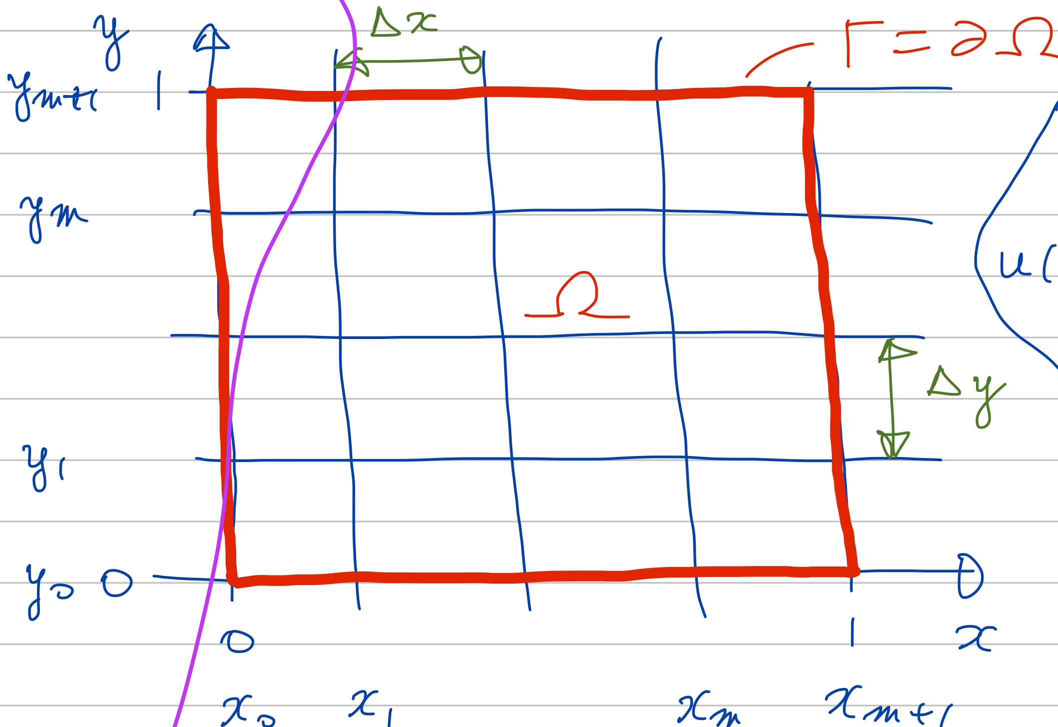
## 2.1.2 2D elliptic model problem:

BVP
$$\begin{cases} u_{xx} + u_{yy} = f(x,y) \text{ in } \Omega & (1) \quad \text{(PDE)} \\ \Omega : (x,y) \in (0,1) \times (0,1) & \text{(domain)} \\ u(x,y) = 0 \text{ on } \Gamma := \partial\Omega & \text{(boundary condition)} \end{cases}$$

"boundary of $\Omega$"




example:

$u(x,y) = \sin(\pi x)\sin(\pi y)$

if $f(x,y) = -2\pi^2 \sin(\pi x)\sin(\pi y)$

note: it can be shown that elliptic BVPs of this type have a unique solution (for suitable functions $f(x,y)$ and domains $\Omega$)

specify $u(x,y)$ on $\Gamma$: Dirichlet BC

we may also have: Neumann BC

BVP
$$\begin{cases} u_{xx} + u_{yy} = f(x,y) \text{ in } \Omega \\ \Omega : (x,y) \in (0,1) \times (0,1) \\ u(x,y) = 0 \text{ on } \Gamma := \partial\Omega \end{cases}$$

we seek a numerical approximation:

**step ①:**

discretize domain into squares (of equal size, for now) with $h = \Delta x = \Delta y = \frac{1}{m+1}$

so, $m+1$ intervals per direction
$m+2$ points
$m$ interior points

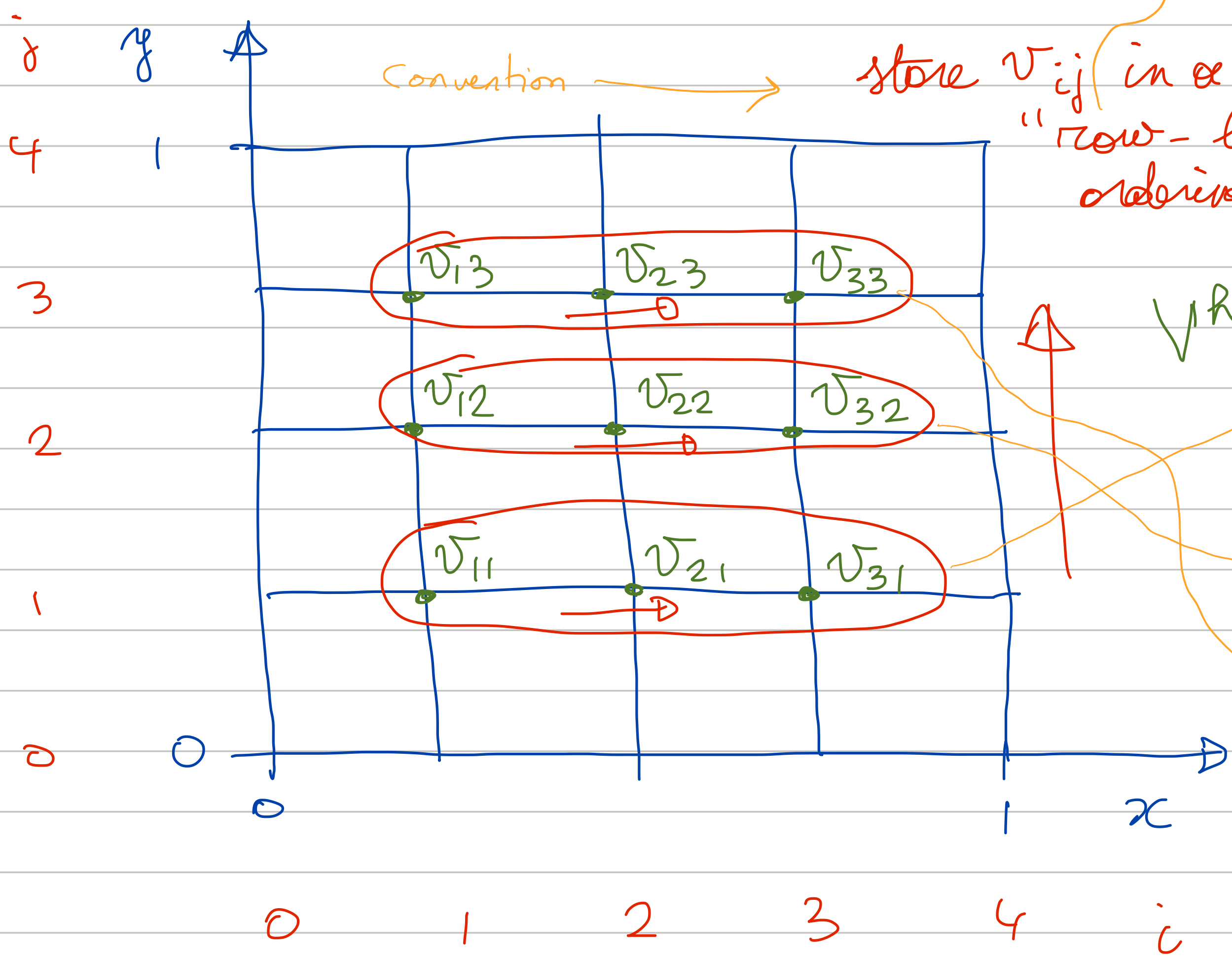
**step ②:** discretize the equations

$$v_{ij} \approx u(x_i, y_j) =: u_{ij}$$

$m = 3$

incorrect in pdf course notes this is correct

Convention




store $v_{ij}$ in a vector in "row-lexicographic ordering":

$$V^h = \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \\ v_{12} \\ v_{22} \\ v_{32} \\ v_{13} \\ v_{23} \\ v_{33} \end{bmatrix}$$

with $u_{ij} = u(x_i, y_j)$ (exact solution)

$f_{ij} = f(x_i, y_j)$

we know: $$u_{xx}(x_i, y_j) = \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{\Delta x^2} + O(\Delta x^2)$$

$$u_{yy}(x_i, y_j) = \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{\Delta y^2} + O(\Delta y^2)$$

so with $u_{xx} + u_{yy} = f$,

we seek approximation $v_{ij}$ s.t.

discrete BVP

$$\frac{v_{i+1,j} + v_{i-1,j} - 4v_{ij} + v_{i,j+1} + v_{i,j-1}}{h^2} = f_{ij}$$

$(i = 1, \dots, m;\ j = 1, \dots, m)$

$x_i = x_0 + ih \qquad i = 0, \dots, m+1$

$y_j = y_0 + jh \qquad j = 0, \dots, m+1$

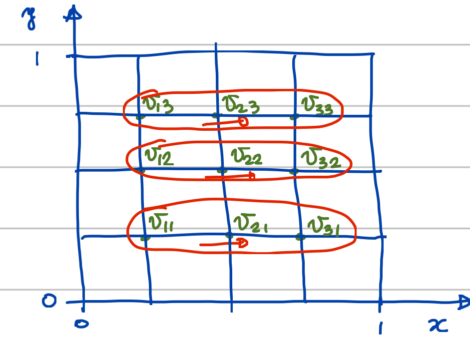
$v_{0,j} = 0 \qquad j = 0, \dots, m+1$

$v_{m+1,j} = 0 \qquad j = 0, \dots, m+1$

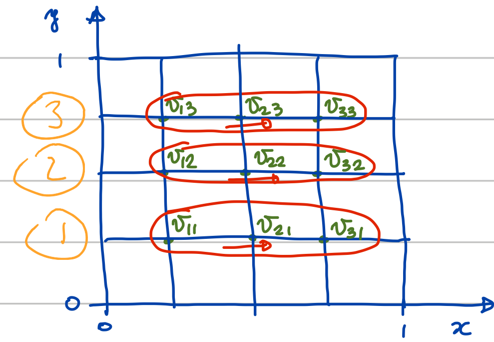
$v_{i,0} = 0 \qquad i = 0, \dots, m+1$

$v_{i,m+1} = 0 \qquad i = 0, \dots, m+1$

finite difference becomes harder if the domain isn't rectangular



$$\begin{cases} \dfrac{v_{i+1,j} + v_{i-1,j} - 4v_{ij} + v_{i,j+1} + v_{i,j-1}}{h^2} = f_{ij} \qquad (i = 1,\dots,m \;;\; j = 1,\dots,m) \\[2ex] x_i = x_0 + ih \qquad i = 0,\dots,m+1 \\ y_j = y_0 + jh \qquad j = 0,\dots,m+1 \\[2ex] v_{0j} = 0 \qquad j = 0,\dots,m+1 \\ v_{m+1,j} = 0 \qquad j = 0,\dots,m+1 \\ v_{i0} = 0 \qquad i = 0,\dots,m+1 \\ v_{i\,m+1} = 0 \qquad i = 0,\dots,m+1 \end{cases}$$




write as linear system: $A^h V^h = F^h$

$$V^h = \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \\ v_{12} \\ v_{22} \\ v_{32} \\ v_{13} \\ v_{23} \\ v_{33} \end{bmatrix}$$

This block corresponds to the first row on the grid

i,j diagonal

This block corresponds to the second row on the grid

distance m

$$A^h = \frac{1}{h^2} \left[\begin{array}{ccc|ccc|ccc} -4 & 1 & & 1 & & & & & \\ 1 & -4 & 1 & & 1 & & & & \\ & 1 & -4 & 0 & & 1 & & & \\ \hline 1 & & 0 & -4 & 1 & & 1 & & \\ & 1 & & 1 & -4 & 1 & & 1 & \\ & & 1 & & 1 & -4 & 0 & & 1 \\ \hline & & & 1 & & 0 & -4 & 1 & \\ & & & & 1 & & 1 & -4 & 1 \\ & & & & & 1 & & 1 & -4 \end{array}\right]$$

boundary point (in general this should be moved to the right hand side)

third row on the grid

$$F^h = \begin{bmatrix} f_{11} \\ f_{21} \\ f_{31} \\ f_{12} \\ f_{22} \\ f_{32} \\ f_{13} \\ f_{23} \\ f_{33} \end{bmatrix}$$

$m = 3$

$m^2$ blocks $(m \times m)$

$$A^h = \frac{1}{h^2}\left[\begin{array}{ccc|ccc|ccc} -4 & 1 & & 1 & & & & & \\ 1 & -4 & 1 & & 1 & & & & \\ & 1 & -4 & & & 1 & & & \\ \hline 1 & & & -4 & 1 & & 1 & & \\ & 1 & & 1 & -4 & 1 & & 1 & \\ & & 1 & & 1 & -4 & & & 1 \\ \hline & & & 1 & & & -4 & 1 & \\ & & & & 1 & & 1 & -4 & 1 \\ & & & & & 1 & & 1 & -4 \end{array}\right]$$

<u>in general:</u>

$$A^h V^h = F^h$$

let $$T = \begin{bmatrix} -4 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -4 \end{bmatrix} \in \mathbb{R}^{m \times m}$$

"tri-diagonal sparse matrix"

$$I = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \in \mathbb{R}^{m \times m}$$

$i-1,j \,:\, i,j \,:\, i+1,j$

$i,j+1$

$i,j-1$

$$A^h = \frac{1}{h^2}\left[\begin{array}{c|c|c|c} T & I & & \\ \hline I & T & I & \\ \hline & I & T & I \\ \hline & & I & T \end{array}\right] \in \mathbb{R}^{m^2 \times m^2}$$

"block-tridiagonal sparse matrix"

$\rightarrow$ $m \times m$ blocks of size $m \times m$

<u>note</u>: if $u(x,y) = g(x,y)$ on $\Gamma = \partial\Omega$

then we can put $g_{ij} := g(x_i, y_j)$ on $\Gamma$

into the right-hand side vector $F^h$,

as before

<u>note</u>: actual error: $E^h = U^h - V^h$

$$e_{ij} = u_{ij} - v_{ij}$$

we expect $e_{ij} = O(h^2)$

(convergence with order 2)

(see later)