Appendix A: Norms of vectors, functions and matrices

(see Appendix A in course notes)

to investigate convergence of FD methods for elliptic PDEs, we need norms of vectors, functions and matrices

(A1) Vector and function norms

[1] vector norms: $\vec{x} \in \mathbb{R}^n$

$$\|\vec{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

$$\|\vec{x}\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|\vec{x}\|_\infty = \max\left(|x_1|, |x_2|\right)$$

$$\|\vec{x}\|_p = \left(|x_1|^p + |x_2|^p\right)^{1/p} \qquad (p \geq 1)$$

[2] function norms:

1D $\quad \Omega = (a, b)$

$$\|u\|_2 = \sqrt{\int_a^b (u(x))^2 \, dx}$$

$$\|u\|_1 = \int_a^b |u(x)| \, dx$$

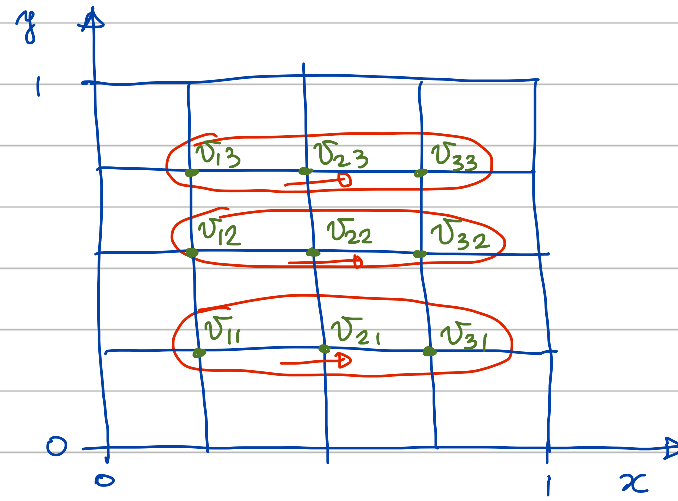
2D $\quad \Omega$

$$\|u\|_2 = \sqrt{\iint_\Omega u^2(x,y) \, dx \, dy}$$

$$\|u\|_1 = \iint_\Omega |u(x,y)| \, dx \, dy$$

$$\|u\|_2 = \sqrt{\iint_\Omega u^2(x,y)\,dx\,dy}$$

$$\|u\|_1 = \iint_\Omega |u(x,y)|\,dx\,dy$$



(A2) Grid function norms:

consider a function $u(x,y)$ that is sampled on the grid: $u_{ij} = u(x_i, y_j)$

→ this is called a grid function

(also: v_{ij} that approximates u_{ij} is a grid function)

that we refer to by u^h — indicates grid function

→ we can store the grid function in a row-lexicographic vector U^h

we know: $\|u(x,y)\|_2 = \sqrt{\int_0^1\int_0^1 u^2(x,y)\,dx\,dy} \approx \sqrt{\sum_{i=0}^{m+1}\sum_{j=0}^{m+1} u_{ij}^2\,\Delta x\,\Delta y}$

def: 2-norm of 2D grid function u^h (or U^h)

$$\|u^h\|_2 = \sqrt{\sum_{i=0}^{m+1}\sum_{j=0}^{m+1} u_{ij}^2\,\underbrace{\Delta x\,\Delta y}_{\sqrt{h^2}\,=\,h}}$$

def: 1-norm of 2D grid function u^h

$$\|u^h\|_1 = \sum_{\substack{i=0 \\ j=0}}^{m+1} |u_{ij}| \underbrace{\Delta x \, \Delta y}_{= h^2}$$

(similar in 1D, 3D) $\quad \|U^h\|_1$

(A3) Matrix norms:

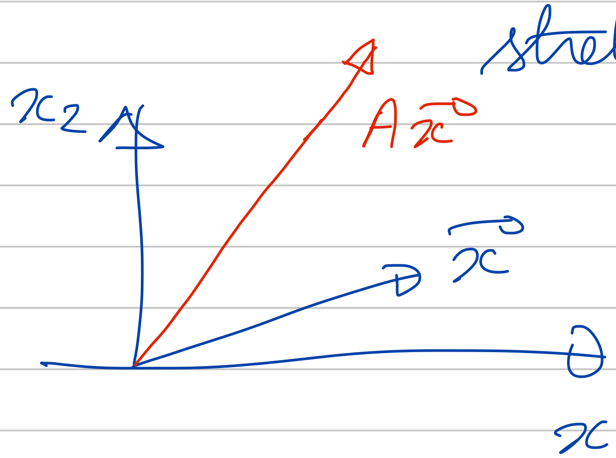
let $A \in \mathbb{R}^{n \times n}$, $\vec{x} \in \mathbb{R}^n$

def: vector-induced matrix norm

$$\|A\|_p = \max_{\substack{\vec{x} \in \mathbb{R}^n \\ \vec{x} \neq 0}} \frac{\|A\vec{x}\|_p}{\|\vec{x}\|_p}$$

note: also, $\|A\|_p = \max\limits_{\|\vec{x}\|_p = 1} \|A\vec{x}\|_p$

interpretation: operator A rotates and stretches $\vec{x}$

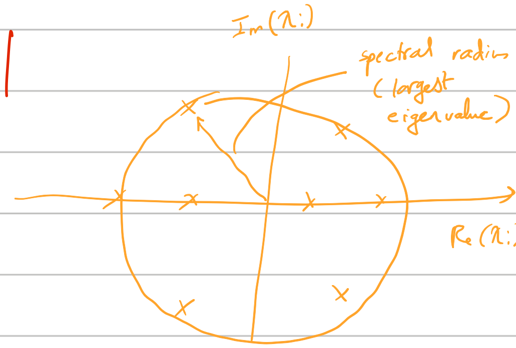


$\|A\|_p$ = maximal stretching of a unit vector

def: spectral radius of a matrix A

let $A \in \mathbb{R}^{n \times n}$ with eigenvalues λ_i $(i = 1, \ldots, n)$

then $$\rho(A) = \max_{1 \le i \le n} |\lambda_i|$$



properties of matrix norms:

① $\|A\vec{x}\|_p \le \|A\|_p \, \|\vec{x}\|_p$

② $\|A + B\|_p \le \|A\|_p + \|B\|_p$

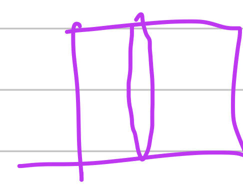
③ $\|A\|_p \ge \rho(A)$ for any p-norm

④ $\|A\|_2 = \sqrt{\rho(AA^T)} = \sqrt{\rho(A^TA)}$

$= $ largest singular value of A

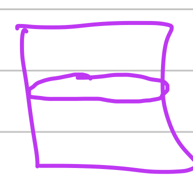
⑤ if $A = A^T$, then $\|A\|_2 = \rho(A)$

⑥ $$\|A\|_1 = \max_{1 \le j \le n} \left(\sum_{i=1}^{n} |a_{ij}| \right)$$



(maximum absolute column sum)

⑦ $$\|A\|_\infty = \max_{1 \le i \le n} \left(\sum_{j=1}^{n} |a_{ij}| \right)$$



(maximum absolute row sum)