5.4: Building the linear system:

Matrix Assembly for Nodal Linear Finite Elements in 1D

Let u solve the weak form.

We seek Ritz-Galerkin approximation $u^h(x) \in V_1^h$

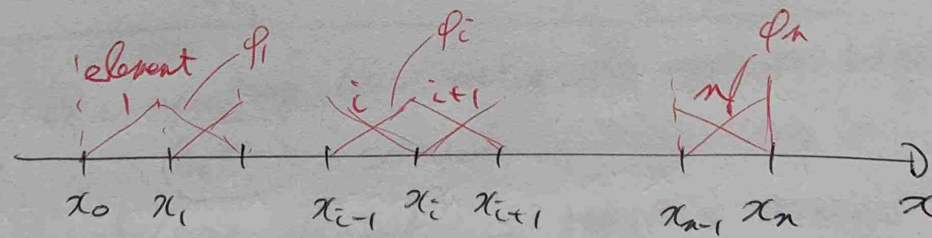
$$u^h(x) = \sum_{i=1}^{n} c_i \varphi_i(x)$$

where $c_i = u^h(x_i)$ since $\varphi_i(x_j) = \delta_{ij}$

for $i = 1, 2, \dots, n$

$j = 0, 1, \dots, n$

Here, we have used a grid with n elements, $n+1$ nodes



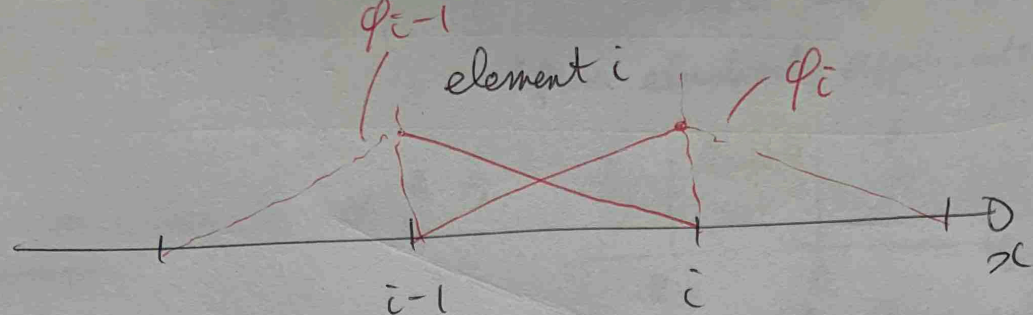
node set: $\{x_0, x_1, \dots, x_n\}$

approximation space: $V_1^h = \text{span}\{\varphi_1, \varphi_2, \dots, \varphi_n\}$

element set: $\{\tau_1, \tau_2, \dots, \tau_n\} = T^h$

where element $\tau_i = [x_{i-1}, x_i]$

in element i:



restricted weak form: (Ritz-Galerkin)

[find $u^h \in V_1^h$ s.t. $\boxed{a(u^h, \varphi_i) = \langle f, \varphi_i \rangle \ \forall \varphi_i \quad (i = 1, 2, \dots, n)}$

where $a(u^h, \varphi_i) = \int_0^1 u^{h\prime}(x)\, \varphi_i'(x)\, dx$

and $u^h(x) = \sum_{j=1}^{n} c_j \, \varphi_j(x)$

$A\vec{c} = \vec{f}$

so we have n equations in n unknowns:

$$\boxed{\sum_{j=1}^{n} c_j \underbrace{\left(\int_0^1 \varphi_j'(x)\, \varphi_i'(x)\, dx \right)}_{a_{ij}} = \underbrace{\int_0^1 f\, \varphi_i(x)\, dx}_{f_i} \qquad i = 1, 2, \dots, n}$$

instead of computing these integrals over the whole of [0,1], it is better to compute contributions to these integrals per element := MATRIX ASSEMBLY

because φ_i and φ_j are 0 in most elements!

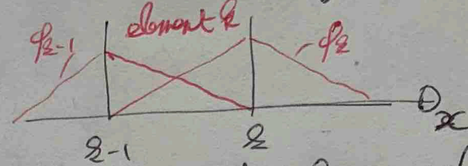
rewrite the equations using element-wise integrals:

superscript

eq. i: $$\sum_{j=1}^{n} c_j \underbrace{\left(\sum_{\tau_h \in T^h} \overbrace{\int_{\tau_h} \varphi_j' \varphi_i' \, dx}^{a_{ij}^h} \right)}_{a_{ij}} = \underbrace{\sum_{\tau_h \in T^h} \overbrace{\left(\int_{\tau_h} f \varphi_i \, dx \right)}^{f_i^h}}_{f_i}$$

in $A\vec{c} = \vec{f}$, A is called the (global) "stiffness matrix"
(from elasticity application of $-u'' = f$)

how to compute a_{ij}^{k} ?



~~consider~~

in element k, $a_{ij}^{k} = 0$ except ~~for~~ when
$(i = k-1$ or $k)$ and $(j = k-1$ or $k)$
4 cases

this defines the "local stiffness matrix" A^k:

$$A^k = \begin{bmatrix} \int_{\tau_k} \varphi'_{k-1}\,\varphi'_{k-1}\,dx & \int_{\tau_k} \varphi'_{k-1}\,\varphi'_{k}\,dx \\ \int_{\tau_k} \varphi'_{k}\,\varphi'_{k-1}\,dx & \int_{\tau_k} \varphi'_{k}\,\varphi'_{k}\,dx \end{bmatrix}$$

By symmetry some elements will be the same

similarly, the "local right-hand side" $\vec{f}^{k}$:

$$\vec{f}^{k} = \begin{bmatrix} \int_{\tau_k} f(x)\varphi_{k-1}(x)\,dx \\ \int_{\tau_k} f(x)\varphi_{k}(x)\,dx \end{bmatrix}$$

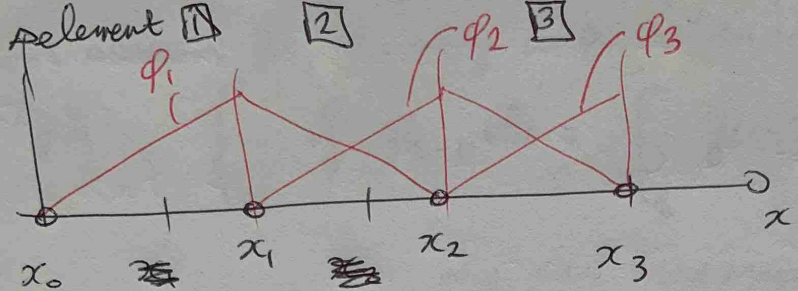
the "local" A^k and f^k (computed per element k)
can then be "assembled" into the global A and $\vec{f}$
(see the next example)

~~how to compute~~

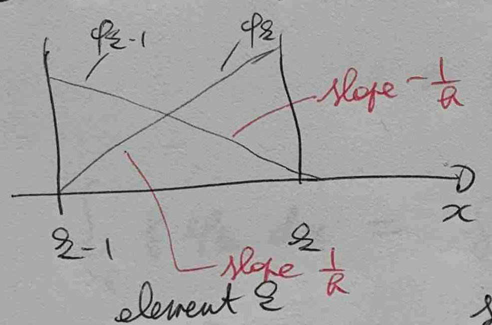
$$\begin{cases} -u'' = f \\ u(0) = 0 \\ u'(0) = 1 \end{cases}$$

example 5.20:

$n = 3$ $\qquad h_i = h = \frac{1}{3}$



for a generic element τ_k $\quad (k = 2, 3)$:



$$\varphi_k(x)\Big|_{\tau_k} = \frac{x - x_{k-1}}{x_k - x_{k-1}} = \frac{x - x_{k-1}}{h}$$

so $\boxed{\varphi_k'(x) = \frac{1}{h}}$ in τ_k

then $$\int_{\tau_k} \varphi_k' \varphi_k' \, dx = \int_{\tau_k} +\frac{1}{h^2} \, dx = \frac{h}{h^2} = h^{-1}$$

also $$\varphi_{k-1}(x)\Big|_{\tau_k} = \frac{x_k - x}{x_k - x_{k-1}} = \frac{x_k - x}{h}$$

$$\varphi_{k-1}'(x)\Big|_{\tau_k} = -\frac{1}{h}$$ in τ_k

$$\int_{\tau_k} \varphi_{k-1}' \varphi_{k-1}' \, dx = \int_{\tau_k} \left(-\frac{1}{h}\right)^2 dx = \frac{h}{h^2} = h^{-1}$$

$$\int_{\tau_k} \varphi_k' \varphi_{k-1}' \, dx = \int_{\tau_k} \left(-\frac{1}{h}\right)\left(\frac{1}{h}\right) dx = -\frac{h}{h^2} = -h^{-1}$$

so $$A^{k} = \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\frac{1}{h}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(note: for element τ_1, the only contribution is $a^1_{11} = \frac{1}{h}$)

similarly:
$$\int_{\tau_k} f\,\varphi_{k-1}\,dx = \int_{\tau_k} f(x)\frac{(x_k - x)}{h}\,dx$$

may need numerical integration for generic $f(x)$

$$\int_{x_{k-1}}^{x_k} f\,\frac{(x_k - x)}{h}\,dx$$

$$\frac{f}{h}\int_{x_{k-1}}^{x_k}(x_k - x)\,dx = \frac{f}{h}\left[x_k x - \frac{x^2}{2}\Big|_{x_{k-1}}^{x_k}\right] = \frac{f}{h}\left[\left(x_k^2 - \frac{x_k^2}{2}\right) - \left(x_k x_{k-1} - \frac{x_{k-1}^2}{2}\right)\right]$$

$$= \frac{f}{h}\left[x_k^2 - x_k x_{k-1} - \frac{x_k^2}{2} + \frac{x_{k-1}^2}{2}\right] = \frac{f}{h}\left[x_k\underbrace{(x_k - x_{k-1})}_{h} - \frac{1}{2}\left(x_k^2 - x_{k-1}^2\right)\right] = \frac{f}{h}\left[x_k h - \frac{1}{2}\underbrace{(x_k - x_{k-1})}_{h}(x_k + x_{k-1})\right]$$

$$= \frac{f}{h}\left[x_k h - \frac{h}{2}(x_k + x_{k-1})\right] = \frac{f}{h}\left[x_k h - \frac{h}{2}(2x_k - h)\right] = \frac{f}{\cancel{h}}\left[\cancel{x_k h} - \cancel{x_k h} + \frac{h^{\cancel{2}}}{2}\right] = \frac{fh}{2}$$

assume $f(x)$ constant, then

$$= f\,\frac{h}{2}$$

$$\int_{\tau_k} f\,\varphi_k\,dx = f\,\frac{h}{2}$$

so $$\vec{f}^{\,k} = \frac{h}{2} f \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(note: for element 1, the only contribution is $f^1_1 = f\,h/2$)

We can now assemble the global stiffness matrix from the ~~global~~ local stiffness matrices:

$\begin{bmatrix} 0 & 0 \\ 0 & \end{bmatrix}$

the only contribution from element τ_1

A^1: element 1 contributes in rows/columns 1 and 2 (corresponding to φ_1 and φ_2)

$$A = \frac{1}{h}\begin{array}{c} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}\begin{array}{l} \Rightarrow \varphi_1 \\ \Rightarrow \varphi_2 \\ \Rightarrow \varphi_3 \end{array} + \frac{1}{h}\begin{bmatrix} \boxed{\begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{bmatrix} + \frac{1}{h}\begin{bmatrix} 0 & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \boxed{\begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix}} \end{bmatrix}$$

A^2: element 2 contributes in rows/columns 2 and 3 so $A = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ (similar to FD method, except at right boundary, where we want $u'(1)=0$)

assemble $\vec{f}$:

$$\vec{f} = \begin{bmatrix} \boxed{fh/2} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \boxed{\begin{matrix} fh/2 \\ fh/2 \end{matrix}} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \boxed{\begin{matrix} fh/2 \\ fh/2 \end{matrix}} \end{bmatrix} = \frac{hf}{2}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$\vec{f}^{2}$ $\vec{f}^{3}$

element 1 contribution

contribution from element 2

contribution from element 3

or $$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \frac{f}{2}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

□

5.5 extensions: different boundary conditions

→ we can mix Dirichlet and Neumann BCs

→ we can make BCs non-homogeneous

brief examples:

ex $\boxed{1}$:
$$\begin{cases} -u'' = f & (x \in (0,1)) \quad (1) \\ u(0) = g_1 \\ u(1) = g_2 \end{cases}$$
Dirichlet twice, non-homogeneous

derive weak form:

use $u''v + u'v' = (u'v)'$

multiply (1) by a test function v and integrate:

$$\begin{cases} -\int u''v \, dx = \int f v \, dx \quad \forall v \in V_0 \\ u(0) = g_1 \\ u(1) = g_2 \end{cases}$$

we require test functions satisfy boundary conditions

choose $V_0 = \{ v \in L^2([0,1]) \mid a(v,v) < \infty, \ v(0) = 0, \ v(1) = 0 \}$

now $-\int u''v \, dx = \int f v \, dx \quad \forall v \in V_0$

$\Updownarrow$ integration by parts

$$\int u'v' \, dx - \underbrace{u'v|_0^1}_{=0} = \int f v \, dx \quad \forall v \in V_0$$

Note that now u is no longer in V

$=0$ since we have chosen V_0 such that $v(0) = 0$ and $v(1) = 0$

weak form:

$$\boxed{\text{Find } u \in V_g \text{ s.t. } a(u,v) = \int f v\,dx \quad \forall v \in V_0}$$

where $V_g = \{ v \in L^2([0,1]) \mid a(v,v) < \infty,\ v(0) = g_1,\ v(1) = g_2 \}$

It can be shown this weak form is equivalent to the strong form. (when u smooth enough)

general recipe: two Dirichlet BCs require two BCs in V_0 and V_g: homogeneous ~~for~~ BCs for $v \in V_0$, non-homogeneous for $u \in V_g$

Dirichlet = essential BC, needs to be imposed in V_0 and V_g

4 x 4 = 16 combination of
Dirichlet homogeneous
Dirichlet non homogeneous
Neumann homogeneous
Neumann non homogeneous

energy norm becomes a semi-norm

ex. $\boxed{2}$:

$$(2)\quad \begin{cases} -u'' = f & (x \in (0,1)) \\ u'(0) = g_1 \\ u'(1) = g_2 \end{cases}$$

Neumann twice, non-homogeneous

similar to before:

$$\int u' v'\,dx - u'v\Big|_0^1 = \int f v\,dx \quad \forall v \in V$$

$u'(0) = g_1,\ u'(1) = g_2$

we can now impose the BC directly in the weak form!

$$\boxed{\text{find } u \in V \text{ s.t. } \int u' v'\,dx - g_2 v(1) + g_1 v(0) = \int f v\,dx \quad \forall v \in V}$$

since the BCs are now imposed in the weak form, no need to impose BCs directly on u or v,

so we now use $V = \{ v \in L^2([0,1]) \mid a(v,v) < \infty \}$

no BC needed in V for u or v: Neumann BCs are natural note on model problem with two Neumann BCs:

$\longrightarrow$ if u solves (2), then $w = u + c$ also solves (2) for any $c \in \mathbb{R}$ (since $w'' = u''$ and $w' = u'$)

(the solution is not unique; we can make it unique by specifying $u(x)$ at a point, or its average, or ...)

(note: A in $A\vec{c} = \vec{f}$ will be singular) (discrete case)

$\longrightarrow$ a solution only exists if the following compatibility condition between f, g_1 and g_2 is satisfied:

$$-u'' = f$$

$$\Downarrow$$

$$-\int_0^1 u''\,dx = \int_0^1 f(x)\,dx$$

$$-(u'(1) - u'(0)) = \int_0^1 f(x)\,dx$$

$$\boxed{g_1 - g_2 = \int_0^1 f(x)\,dx}$$

(otherwise, no solution)

(discrete case: $\vec{f} \in$ Range(A) is required for $A\vec{c} = \vec{f}$ to have a solution, when A is singular)

columnspace

energy norm becomes a semi-norm

$a(u,u) = 0$ when $u = c$, $c \in \mathbb{R}$

$\int (u')^2\,dx = 0$

$\rightarrow$ we didn't have this issue before because V was defined such that $v(0) = 0$ and so c can only be 0