Chapter 3: FV methods for scalar conservation laws in 1D

3.1 Some properties of scalar hyperbolic conservation laws

3.1.1 Nonlinear scalar conservation laws and characteristic curves

$$\boxed{u_t + \nabla \cdot \vec{f}(u) = 0}$$

in 1D: $$\boxed{\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u(x,t))}{\partial x} = 0}$$

or $$\boxed{\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0}$$

or $$\boxed{\frac{\partial u}{\partial t} + \underbrace{\frac{df(u)}{du}}_{f'(u)} \frac{\partial u}{\partial x} = 0}$$

("quasi-linear form")

f may be a non-linear function of u

$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0} \quad (1)$$

$$\boxed{\frac{\partial u}{\partial t} + \frac{d f(u)}{du}\frac{\partial u}{\partial x} = 0} \quad (2)$$

$$\frac{\partial u}{\partial t}(x(t),t) = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial t}$$

Consider the curves where u is constant with time: $\frac{\partial u}{\partial t} = 0$

then $\frac{\partial u}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial u}{\partial x} = 0$

Compare to (2) $\longrightarrow$ $\frac{\partial f(u)}{\partial u} = \frac{\partial x}{\partial t}$

def: characteristic curve

let $u(x,t)$ be a solution of conservation law (1)

then the curve

$$\boxed{x(t) : \frac{dx}{dt}(t) = \frac{df}{du}(u(x(t),t))}$$

or: $x'(t) = f'(u)$

is called a characteristic curve of the PDE for solution $u(x,t)$

when the PDE becomes an ODE

observe: along the $\overbrace{\text{characteristic curve}}^{\text{when the PDE becomes an ODE}}$ $x(t)$ we have:

ODE

$$\frac{du(x(t),t)}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t}$$

ODE that governs changes in u along curve $x(t)$

conservation Law equation

$$= \frac{\partial u}{\partial x}\frac{df}{du} + \frac{\partial u}{\partial t} = 0$$

so $u(x,t)$ is constant along the characteristic curve $x(t)$

when $\frac{\partial x}{\partial t} = \frac{\partial f}{\partial u}$

$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0} \quad (1) \qquad \boxed{\frac{\partial u}{\partial t} + \frac{df(u)}{du}\frac{\partial u}{\partial x} = 0} \quad (2)$$

$$\boxed{x(t): \quad \frac{dx}{dt}(t) = \frac{df}{du}(u(x(t),t))}$$

or: $x'(t) = f'(u)$

$$\boxed{\frac{d\, u(x(t),t)}{dt} = 0}$$

so $u(x,t)$ is constant along the curve $x(t)$

observe: since u is constant along characteristic, and $f'(u)$ is the slope of the characteristic, the characteristics are straight lines

u is constant therefore $\frac{\partial f}{\partial u}(u(x(t),t))$ is also a constant $\rightarrow$ fixed slope

note: the PDE is called hyperbolic because characteristic curves exist along which the PDE reduces to an ODE (see also later, for hyperbolic systems)

$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0}$$

$$\boxed{\frac{\partial u}{\partial t} + f'(u)\,\frac{\partial u}{\partial x} = 0}$$

$$\boxed{x(t):\ \frac{dx}{dt} = \frac{df}{du}}$$

$$\boxed{\frac{du(x(t),t)}{dt} = 0}$$

(A) linear advection:

$$\boxed{\frac{\partial u}{\partial t} + \alpha\,\frac{\partial u}{\partial x} = 0}$$

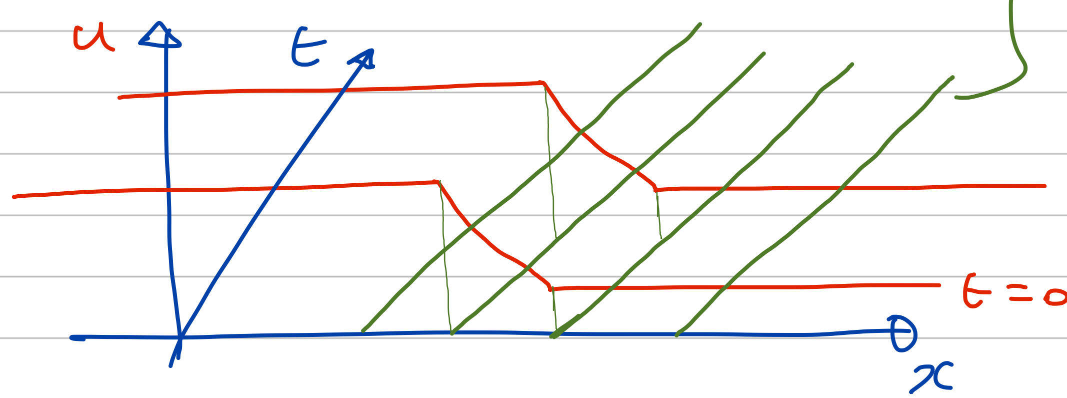
this is a hyperbolic conservation law !

with: $f(u) = \alpha\, u$ ("linear advection flux")

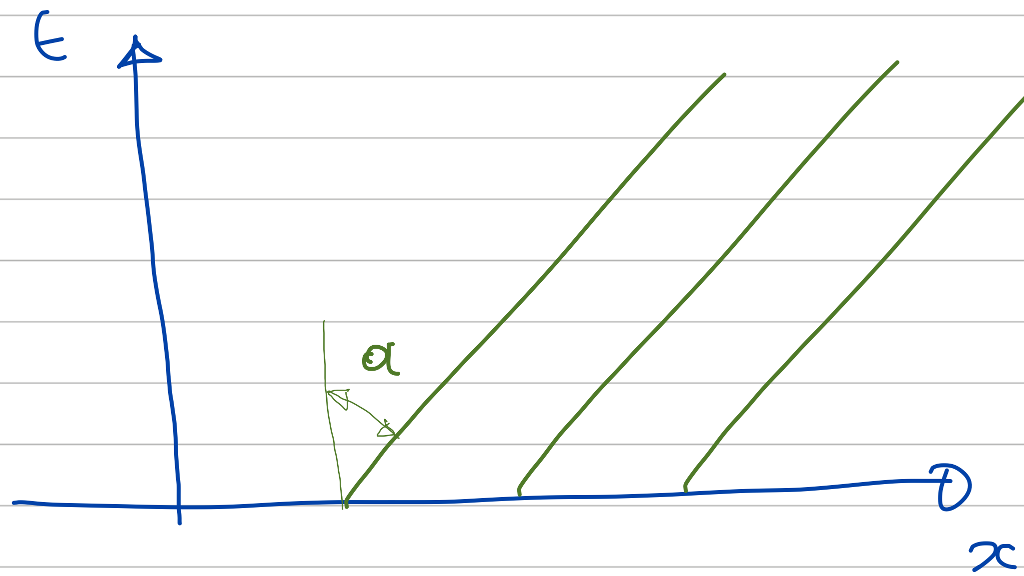
$f'(u) = \alpha$

characteristic curves:

characteristics with slope α



where is the other set of characteristic curves??



$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0}$$

$$\boxed{\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} = 0}$$

$$\boxed{x(t):\ \frac{dx}{dt} = \frac{df}{du}}$$

$$\boxed{\frac{du(x(t),t)}{dt} = 0}$$

(B) the Burgers equation: (inviscid)

def: $$\boxed{\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0}$$

viscous Burgers: $u_t + \left(\frac{u^2}{2}\right)_x = \eta\, u_{xx}$

so $f(u) = \frac{u^2}{2}$, $f'(u) = u$

and $$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0}$$

observe: $f'(u) = u$ is like a nonlinear wave speed

characteristics:

$$\boxed{\frac{dx(t)}{dt} = u(x(t),t)}$$

(constant in t, straight line!)

Burgers examples:

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0}$$

$f(u) = u^2/2$

$f'(u) = u$

char: $\boxed{\frac{dx}{dt} = u}$

$$\boxed{\frac{du(x(t),t)}{dt} = 0}$$

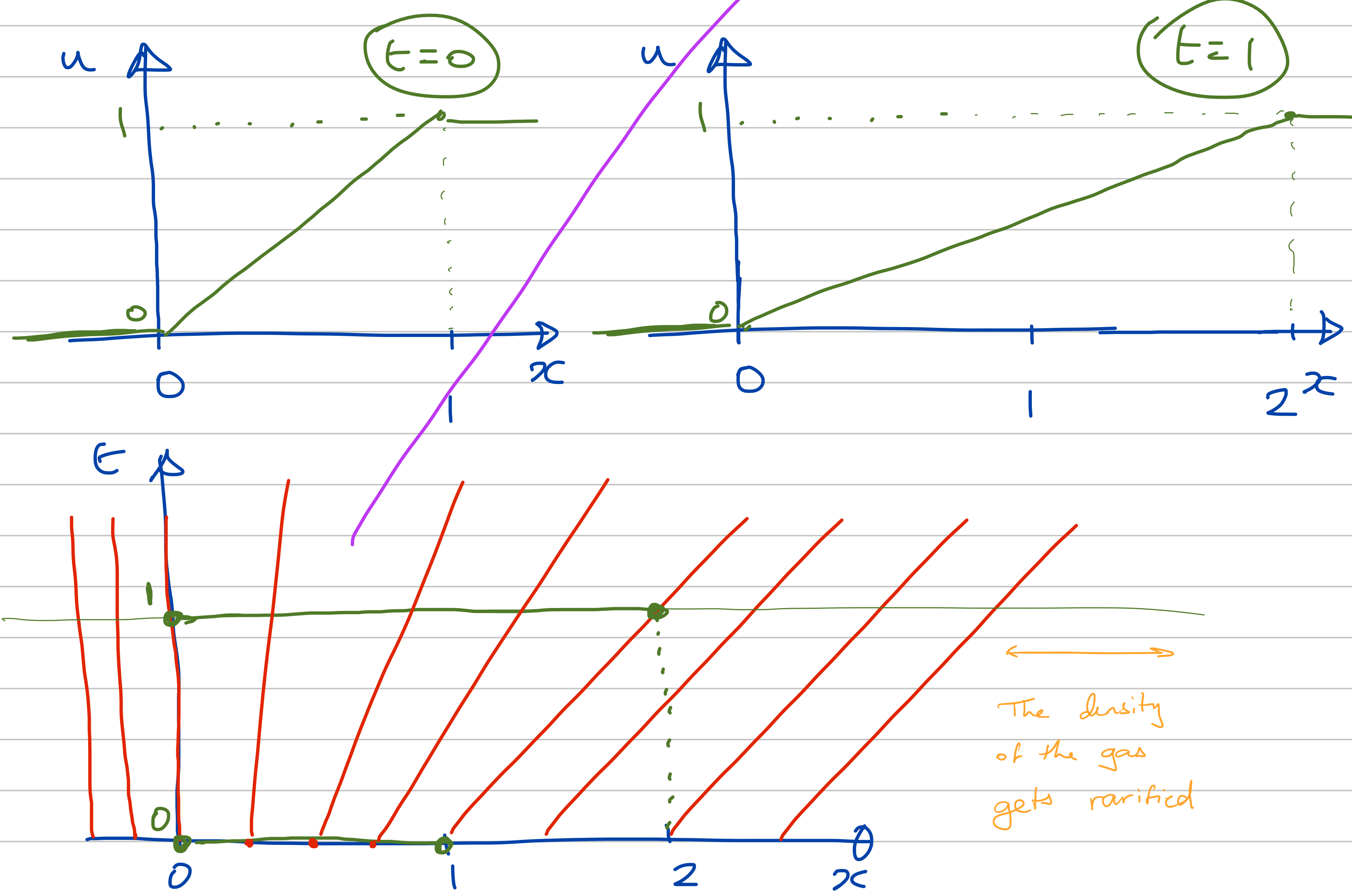
(E1) rarefaction wave:

initial condition

$$\begin{cases} u(x,0) = 0 & (x \le 0) \\ u(x,0) = x & (0 \le x \le 1) \\ u(x,0) = 1 & (1 \le x) \end{cases}$$

~~$u(x,t) = \frac{x}{t}$~~ not correct

$x(t) = x_0 + ut$



The density of the gas gets rarified

Burgers examples:

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0}$$

char: $\boxed{\frac{dx}{dt} = u}$

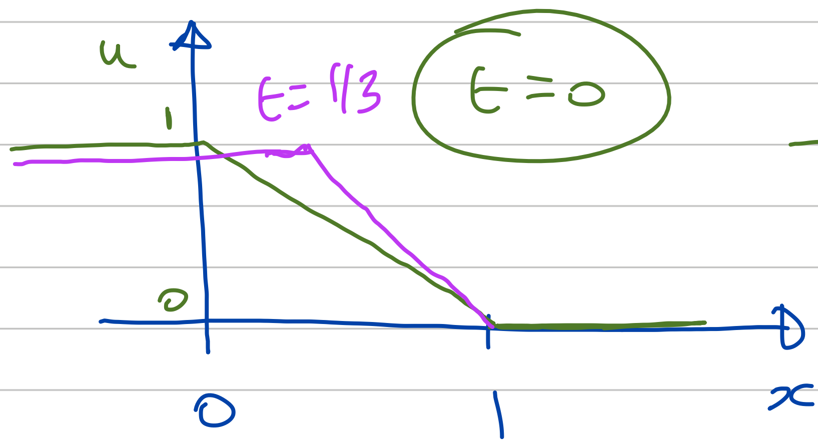
$$\boxed{\frac{du(x(t),t)}{dt} = 0}$$

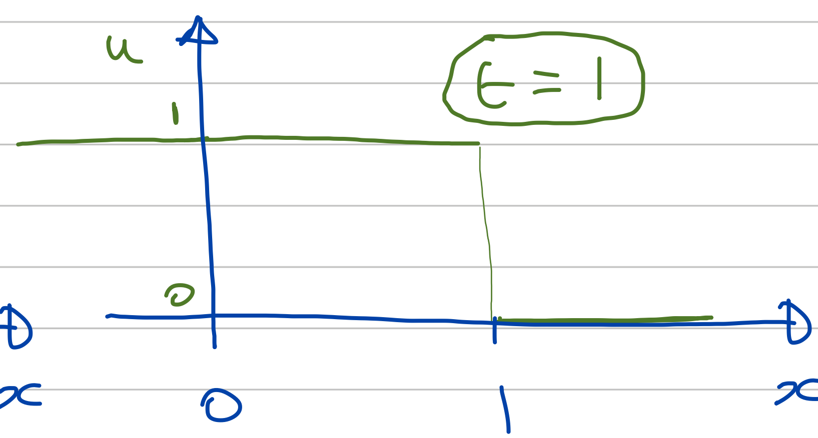
E2 Shock wave: (steepening wave)

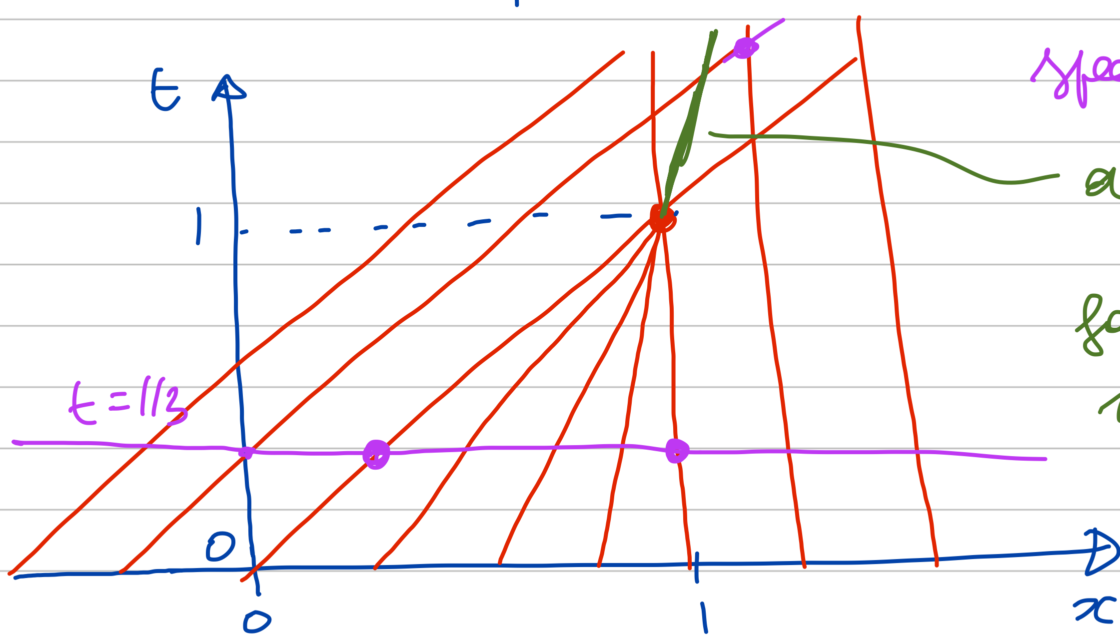
initial condition

$$\begin{cases} u(x,0) = 1 & (x \le 0) \\ u(x,0) = 1-x & (0 \le x \le 1) \\ u(x,0) = 0 & (1 \le x) \end{cases}$$

This is a discontinuity tha arises due to steepening of the wave







speed $s = \frac{1}{2}$

a shock (discontinuity) forms at $t=1$; what is the shock speed?