3.12 Weak solutions

Functional analysis [edit]

Functional analysis is organized around adequate techniques to bring function spaces as topological vector spaces within reach of the ideas that would apply to normed spaces of finite dimension. Here we use the real line as an example domain, but the spaces below exist on suitable open subsets $\Omega \subseteq \mathbb{R}^n$

- $C(\mathbb{R})$ continuous functions endowed with the uniform norm topology
- $C_c(\mathbb{R})$ continuous functions with compact support
- $B(\mathbb{R})$ bounded functions
- $C_0(\mathbb{R})$ continuous functions which vanish at infinity
- $C^r(\mathbb{R})$ continuous functions that have continuous first r derivatives.
- $C^\infty(\mathbb{R})$ smooth functions
- $C_c^\infty(\mathbb{R})$ smooth functions with compact support
- $C^\omega(\mathbb{R})$ real analytic functions
- $L^p(\mathbb{R})$, for $1 \le p \le \infty$, is the L^p space of measurable functions whose p-norm $\|f\|_p = \left(\int_{\mathbb{R}} |f|^p\right)^{1/p}$ is finite
- $\mathcal{S}(\mathbb{R})$, the Schwartz space of rapidly decreasing smooth functions and its continuous dual, $\mathcal{S}'(\mathbb{R})$ tempered distributions
- $D(\mathbb{R})$ compact support in limit topology
- $W^{k,p}$ Sobolev space of functions whose weak derivatives up to order k are in L^p
- $\mathcal{O}_U$ holomorphic functions
- linear functions
- piecewise linear functions
- continuous functions, compact open topology
- all functions, space of pointwise convergence
- Hardy space
- Hölder space
- Càdlàg functions, also known as the Skorokhod space
- $\mathrm{Lip}_0(\mathbb{R})$, the space of all Lipschitz functions on $\mathbb{R}$ that vanish at zero.

consider
$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0} \qquad (1)$$

for $(x,t) \in \Omega \subset \mathbb{R}^2$

def: $C^1(\Omega) = \{ u(x,t) \mid \frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$ exist and are continuous for all $(x,t) \in \Omega \}$

def: $u(x,t)$ is a classical solution of PDE (1) if $u(x,t) \in C^1$ and $u(x,t)$ satisfies (1)

but: solutions $u(x,t)$ with discontinuities
(e.g., Burgers shock, or linear advection of discontinuous profile)
are not classical solutions; we need the concept of weak solutions

def: $u(x,t)$ is called a weak solution of PDE (1) if $u(x,t)$ satisfies

$$\frac{d}{dt}\left(\int_a^b u(x,t)\,dx\right) + f(u(b,t)) - f(u(a,t)) = 0$$
(first integral form)

for "almost all" intervals (a,b) and times t

measure zero (points where PDE cannot satisfy) we can also consider the following more formal definition of weak solution

<u>def:</u> $C_0^1(\mathbb{R}^2) = \{ \varphi(x,t) \mid \varphi(x,t) \in C^1(\mathbb{R}^2)$ and $\varphi(x,t) = 0$ outside a bounded subset of $\mathbb{R}^2 \}$

C^1 functions with bounded (or compact) support

part of the domain where the function is non-zero

<u>def:</u> $u(x,t)$ is called a weak solution of PDE (1) if, $\forall\ \varphi(x,t) \in C_0^1(\mathbb{R}^2)$,

$$-\int_0^\infty \int_{-\infty}^\infty (u\,\varphi_t + f(u)\,\varphi_x)\,dx\,dt$$

$$-\int_{-\infty}^\infty u(x,0)\,\varphi(x,0)\,dx = 0$$

are there not infinitely many bump functions that fail to satisfy this at a discontinuity??
- Perhaps this infinity → zero with regard to the set of bumps that do satisfy the conservation law.

Integration by parts

<u>note:</u> the relevant weak solutions are piecewise C^1

def: $u(x,t)$ is called a weak solution of PDE (1) if, $\forall\ \varphi(x,t) \in C_0^1(\mathbb{R}^2)$,

$$-\int_0^{\infty}\int_{-\infty}^{\infty} (u\,\varphi_t + f(u)\,\varphi_x)\,dx\,dt$$

$$- \int_{-\infty}^{\infty} u(x,0)\,\varphi(x,0)\,dx = 0$$

link with classical solutions:

consider IVP
$$\begin{cases} u_t + f(u)_x = 0 \\ \Omega: (x,t) \in \underbrace{\mathbb{R}\times\mathbb{R}^+}_{\text{half plane}} \\ u(x,0) = u_0(x) \end{cases}$$

for a classical solution $u(x,t)$:

$$u_t + f(u)_x = 0 \iff \varphi\, u_t + \varphi\, f(u)_x = 0 \quad \underbrace{\forall \varphi \in C_0^1(\mathbb{R}^2)}_{\text{test functions}}$$

$$\iff \iint_\Omega \varphi\, u_t\, dx\,dt + \iint_\Omega \varphi\, f(u)_x\, dx\,dt = 0 \quad \forall\varphi$$

integration by parts

$$\iff -\iint_\Omega (\varphi_t u + \varphi_x f(u))\,dx\,dt$$

$$+ \int_{-\infty}^{\infty} (\varphi u)\Big|_{t=0}^{t=\infty} dx + \int_0^{\infty} (\varphi f(u))\Big|_{x=-\infty}^{x=\infty} dt = 0 \quad \forall\varphi$$

bump function evaluates to zero

$$\iff -\iint_\Omega (\varphi_t u + \varphi_x f(u))\,dx\,dt$$

$$- \int_{-\infty}^{\infty} \varphi(x,0)\,u(x,0)\,dx = 0 \quad \forall\varphi$$