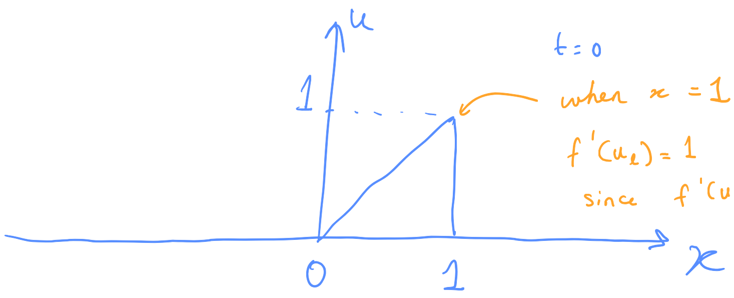
Q3) $u_t + \left(\frac{1}{2}u^2\right)_x = 0$

a) $f(u) = \frac{1}{2}u^2 \quad f'(u) = u \quad \therefore$ all characteristic curves are straight lines



$t=0$

when $x = 1$

$f'(u_l) = 1$ and $f'(u_r) = 0$

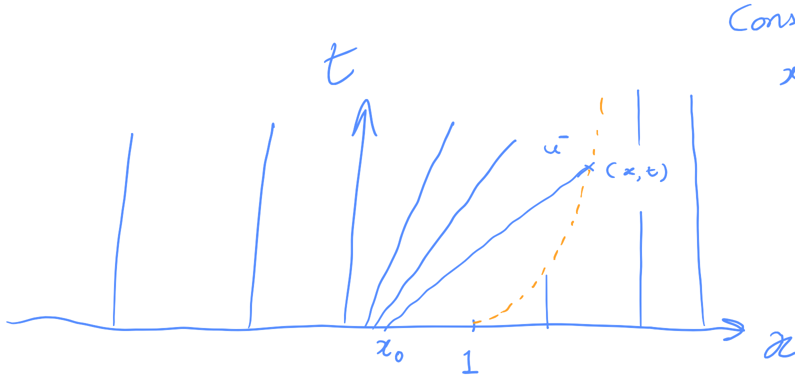
since $f'(u_l) > f'(u_r)$ there is a shock here

for $x < 1$

$u(x)$ is either flat or increasing $\therefore$ there are no shocks for $x < 1$

shock speed

$$\hat{x}'(t) = \frac{f(u_r) - f(u_l)}{u_r - u_l} \quad (1)$$



On characteristic curves $x(t)$:

$$\frac{dx(t)}{dt} = \frac{df(u)}{du} = u$$

Consider the range $0 < x < 1$, let $x(t=0) = x_0$

$$x(t) = x_0 + u(x = x_0, t=0)\,t$$

$$= x_0 + x_0 t$$

Since $u(x,t)$ is constant on a characteristic curve $u(x(t),t) = u(x_0, 0) = x_0$

$\therefore x(t) = u + ut$

$$u(x(t),t) = \frac{x(t)}{1+t} \qquad f(u) = \frac{1}{2}\frac{x^2(t)}{(1+t)^2}$$

when $0 < x < 1$

b) at $t = 0$, $x = 1$

$$\hat{x}'(t) = \frac{0 - \left(\frac{1}{2}\right)}{0 - 1} = \frac{1}{2}$$

Using (1)

$$\hat{x}'(t) = \frac{0 - \frac{\hat{x}^2(t)}{2(1+t)^2}}{0 - \frac{\hat{x}(t)}{(1+t)}}$$

$\hat{x}$ is the x coordinate of the shock at time t

$$= \frac{\hat{x}(t)}{2(1+t)}$$

c) Change of notation:

Let $\frac{dx_s(t)}{dt} = x''(t)$

since
$f(u(-\infty,t))=0$
$f(u(\infty,t))=0$

$$\frac{d}{dt}\Bigg[\underbrace{\int_{-\infty}^{0} u\,dx}_{0} + \underbrace{\int_{0}^{x_s^-(t)} u\,dx}_{G^-(x_s^-(t),t)} + \underbrace{\int_{x_s^+(t)}^{\infty} u\,dx}_{G^+(x_s^+(t),t)}\Bigg] + \underbrace{\int_{-\infty}^{\infty} \frac{df(u)}{dx}\,dx}_{0} = 0$$

REMOVE

Since $\frac{d}{dt}G^-(x_s^-(t),t) = \frac{\partial G^-}{\partial x_s}\frac{dx_s}{dt} + \frac{\partial G^-}{\partial t}$

NOTE:
$u(-\infty,t) \to 0$
$u(\infty,t) \to 0$

$= u(x_s^-(t),t)\frac{dx_s}{dt} + \int_0^{x_s^-(t)} \frac{\partial u}{\partial t}\,dx$, Since

$u_t + (f(u))_x = 0$
$u_t = -f(u)_x$
$\frac{\partial u}{\partial t} = -\frac{\partial f(u)}{\partial x}$

$= u(x_s^-(t),t)\frac{dx_s}{dt} + \left[-f(u(x_s^-(t),t)) + \underbrace{f(u(0,t))}_{0}\right]$

$= u(x_s^-(t),t)\frac{dx_s}{dt} - f(u(x_s^-(t),t))$

Similarly $\frac{d}{dt}G^+(x_s^+(t),t) = -u(x_s^+(t),t)\frac{dx_s}{dt} + f(u(x_s^+(t),t))$

So $\frac{d}{dt}G^-(x_s^-(t),t) + \frac{d}{dt}G^+(x_s^+(t),t) = 0$

Let $u^- = u(x_s^-(t),t)$ and $u^+ = u(x_s^+(t),t)$

Then

$$u^-\frac{dx_s}{dt} - f(u^-,t) - u^+\frac{dx_s}{dt} + f(u^+,t) = 0$$

$$\frac{dx_s}{dt} = \frac{f(u^-) - \underbrace{f(u^+)}_{0}}{u^- - \underbrace{u^+}_{0}} = \frac{\left(\frac{(u^-)^2}{2}\right)}{u^-} = \frac{u^-}{2}$$

Since $u^- = x$, then $\frac{dx_s}{dt} = \frac{x_s}{2}$

$\frac{dx_s(t)}{dt} = \frac{x_s(t)}{2}$

$\int \frac{1}{x_s}\,dx_s = \int \frac{1}{2}\,dt$

$\ln(x_s) = \frac{1}{2}t + c$

$x_s = Ce^{\frac{1}{2}t}$

when $t=0$, $x_s = 1$

$\therefore$ $C = 1$

$x_s = e^{\frac{1}{2}t}$

c) $$\frac{dx_s}{dt} = \frac{x_s}{2(1+t)}$$

let $m = 1+t$

then $t = m-1$

$$\frac{dt}{dm} = 1$$

$$\int\left(\frac{1}{x_s}\right)dx_s = \int \frac{1}{2(1+t)}\, dt$$

$$\ln(x_s) = \int \frac{1}{2m}\, dm$$

$$\ln(x_s) = \frac{1}{2}\ln(m) + c$$

$$x_s = e^{\frac{1}{2}\ln(m) + c}$$

$$x_s = e^{\frac{1}{2}\ln(m)} e^{c}$$

$$x_s = e^{\ln(m^{1/2})} e^{c}$$

$$x_s = m^{1/2} e^{c}$$

$$x_s = (1+t)^{1/2} e^{c}$$

at $t = 0$, $x_s = 1$

so, $1 = e^{c}$

$$\therefore\ x_s = (1+t)^{1/2}$$

$$t = x_s^2 - 1$$

Q4) $u_t + f(u)_x = 0$, $u_t + f'(u)u_x = 0$

u is constant along characteristic curves $x(t)$ i.e. $u_t = 0$

$$x'(t) = f'(u)$$

Consider a particular characteristic curve $x_\xi(t)$ and its neighboring curve $x_{\xi+\Delta\xi}(t)$ such that

not a partial derivative!!

$$x_\xi(0) = \xi, \quad \text{and} \quad x_{\xi+\Delta\xi}(0) = \xi + \Delta\xi$$

Considering curves close together implies taking the limit $\Delta\xi$ approaching zero

i.e. $\Delta\xi \to \delta\xi$

Take the integral with respect to time over a characteristic curves $x_{\xi+\delta\xi}(t)$ and $x_\xi(t)$

$$\int_0^t x'_{\xi+\delta\xi}(t)\,dt = \int_0^t f'(u_{\xi+\delta\xi})\,dt \qquad \int_0^t x'_\xi(t)\,dt = \int_0^t f'(u_\xi)\,dt$$

$$x_{\xi+\delta\xi}(t) - (\xi + \delta\xi) = \int_0^t f'(u_{\xi+\delta\xi})\,dt \qquad x_\xi(t) - \xi = \int_0^t f'(u_\xi)\,dt$$

Note: $u_\xi = u(x_\xi(t))$

since u is constant over characteristic curves for any t

$$x_{\xi+\delta\xi}(t) = \xi + \delta\xi + f'(u_{\xi+\delta\xi})t \quad ① \qquad x_\xi(t) = \xi + f'(u_\xi)t \quad ②$$

Characteristic curves intersect when:

$$x_{\xi+\delta\xi}(t) = x_\xi(t)$$

This results in shock formation.

By Taylor series: ③

$$f'(u_{\xi+\delta\xi}) = f'(u_\xi) + f''(u_\xi)\frac{du_\xi}{dx_\xi}\delta\xi + \dots$$

Setting ① = ②

$$\xi + \delta\xi + f'(u_{\xi+\delta\xi})t = \xi + f'(u_\xi)t$$

Substituting ③

$$\cancel{\xi} + \delta\xi + \cancel{f'(u_\xi)t} + f''(u_\xi)\frac{du_\xi}{dx_\xi}\delta\xi t + \dots = \cancel{\xi} + \cancel{f'(u_\xi)t}$$

$$\delta\xi + f''(u_\xi)\frac{du_\xi}{dx_\xi}\delta\xi t + \dots = 0$$

$$\delta\xi + f''(u_\xi)\frac{du_\xi}{dx_\xi}\delta\xi t + \ldots = 0$$

Since a subsequent terms of the Taylor series will have $(\delta\xi)^n$ with $n \geq 1$ $(\delta\xi)$ can be factored out:

$$\cancel{\delta\xi}\left[1 + f''(u_\xi)\frac{du_\xi}{dx_\xi} t + \ldots\right] = 0$$

$$\therefore\ f''(u_\xi)\frac{du_\xi}{dx_\xi} t + \ldots = -1$$

$u(x,t) =$

Factoring t and rearranging gives:

$$t_\xi = \frac{-1}{f''(u_\xi)\frac{du_\xi}{dx_\xi} + \ldots}$$

$$u_\xi = u(\xi, 0)$$

$$\frac{du_\xi}{dx_\xi} = \left.\frac{\delta u}{\delta x}\right|_{(\xi,0)}$$

Considering: $-\infty < \xi < \infty$ some t_ξ may be negative

shock formation $t_s = \underbrace{\min_\xi}_{\text{Since we want the earliest shock time}}\left[\underbrace{\max(0, t_\xi)}\right]$

Since we don't want negative shock times for a given $x(t=0) = \xi$

References used:

Salsa S. Partial Differential Equations in Action: From Modelling to Theory
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Chapter 4 Section 4.4