part I : Finite difference methods (weeks 1-4)

Chapter 1: Overview of PDEs

└ we will only cover sections 1.1 and 1.2, (reading the rest can be helpful for those who are new to PDEs)

1.1 linear second-order PDEs with two independent variables

1.1.1 Definitions and examples

ex. 1: 1D heat equation (prototype PARABOLIC PDE)

$$\boxed{\frac{\partial u(x,t)}{\partial t} - D\frac{\partial^2 u(x,t)}{\partial x^2} = 0} \qquad (D>0)$$

alternative notation:

$$\boxed{\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0}$$

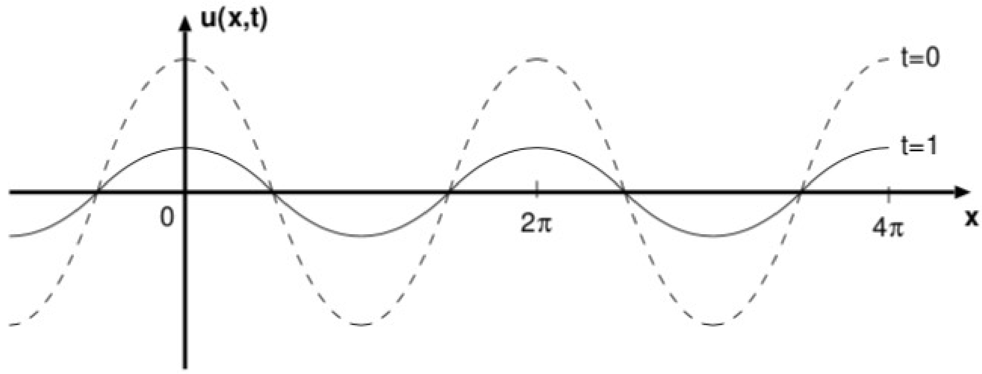
or $\boxed{u_t - u_{xx} = 0}$

thermal diffusivity

$$\boxed{\frac{\partial u(x,t)}{\partial t} - D \frac{\partial^2 u(x,t)}{\partial x^2} = 0} \qquad (D > 0)$$

e.g., $D = 1$: $u(x,t) = \exp(-t)\cos(x)$

is a solution (check this!)



u	$e^{-t}\cos(x)$
u_t	$-e^{-t}\cos(x)$
u_x	$-e^{-t}\sin(x)$
u_{xx}	$-e^{-t}\cos(x)$

$u_t - u_{xx} = 0$

$\cancel{-e^{-t}\cos x} + \cancel{e^{-t}\cos(x)} = 0$

heat differences are smoothed out in time by "diffusion", exponentially

ex. ② : 1D wave equation (prototype HYPERBOLIC PDE)

$$\boxed{\frac{\partial^2 u(x,t)}{\partial t^2} - a^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0}$$

or $u_{tt} - a^2 u_{xx} = 0$

($\pm a$ = wave speeds)

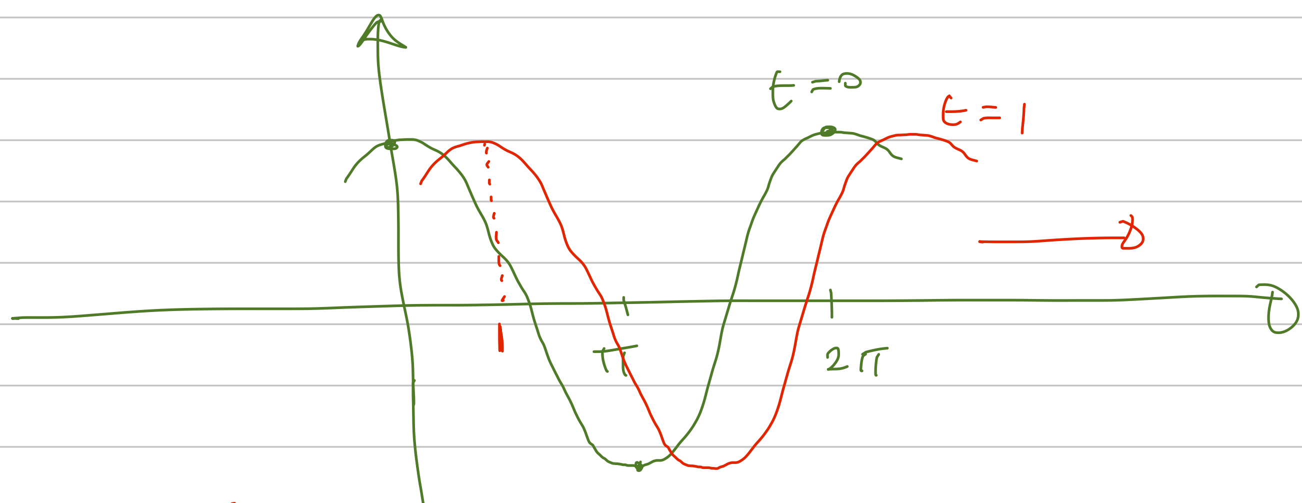
e.g., $a = 1$: $u(x,t) = \cos(x-t)$ is a solution (check this !)

$\cos(x+t)$

u	$\cos(x-t)$
u_t	$\sin(x-t)$
u_{tt}	$-\cos(x-t)$
u_x	$-\sin(x-t)$
u_{xx}	$-\cos(x-t)$

$u_{tt} - u_{xx} = 0$

$-\cancel{\cos(x-t)} + \cancel{\cos(x-t)} = 0$



↳ wave solution is advected to the right with speed 1

(note: amplitude does not decay, no diffusion) ex. (3): 2D Laplace equation (prototype ELLIPTIC PDE)

$$\boxed{\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0}$$

or $\quad u_{xx} + u_{yy} = 0$

or $\quad \Delta u = 0$

e.g.: $\quad u(x,y) = x + y$

is a solution

in general:

Definition: second-order PDE in two variables x and y

$$F(u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}, x, y) = 0 \quad (1)$$

def: PDE (1) is linear if the function F is linear in u and its partial derivatives

$\boxed{\text{def: PDE (1) is linear if the function } F \text{ is linear in } u \text{ and its partial derivatives}}$

note: this means the following

let $G(u,\cancel{x,y}) = F(u, \cancel{u_x, u_y,} u_{xx}, u_{yy}, \cancel{u_{xy}, x, y})$

$F(u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}, x, y) = G(u,x,y) + g(x,y)$ with $G(0,x,y)=0$ $\forall x,y$

$\leftarrow$ G is homogeneous in u

then linearity requires

$$G(\alpha u + b v, x, y) = \alpha G(u,x,y) + b\, G(v,x,y)$$

$\forall\, \alpha, b \in \mathbb{R}$ and $\forall$ functions u, v

ex: $u_t - u_{xx} + 3 = 0$ is linear

$u u_t - u_{xx} + x = 0$ is nonlinear

$\underbrace{x u_t - u_{xx}}_{G(u,x,t)} \underbrace{- x t^2}_{g(x,t)} = 0$ is linear

Theorem: superposition principle

If PDE (1) is linear and homogeneous (i.e., $g(x,y)=0$), then any linear combination of solutions $u(x,y)$ and $v(x,y)$ of (1) is a solution of (1)

proof: we know $G(u,x,y)=0$ and $G(v,x,y)=0$

then $G(\alpha u + b v, x, y) = \alpha G(u,x,y) + b\, G(v,x,y) = 0$

$\forall\, \alpha, b \in \mathbb{R}$, since (1) is linear $\square$

1.1.2 Classification of linear 2nd-order PDEs with constant coefficients

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} = f(x,y) + \alpha\,u_x + \beta\,u_y + \gamma\,u$$

def: let $D = B^2 - 4AC$

if $D = 0$: the PDE is parabolic

$D > 0$: hyperbolic

$D < 0$: elliptic

parabolic PDEs have 1 family of characteristic curves (t = const for heat eqn)

hyperbolic PDEs have 2 families of charateristic curves (t = const & x = const for wave eqn)

elliptic PDEs have no real characteristic curves

note: this classification is related to the existence of characteristic curves

ex: Laplace eq.: $u_{xx} + u_{yy} = 0$

$A = 1 \quad B = 0 \quad C = 1 \quad D = B^2 - 4AC = -4 < 0$

ELLIPTIC

heat eq.: $u_t - \eta\,u_{xx} = 0$ ($\eta > 0$) (t plays role of y)

$A = -\eta \quad B = 0 \quad C = 0$

$D = 0$: PARABOLIC

wave eq.: $u_{tt} - a^2\,u_{xx} = 0$

$A = -a^2 \quad B = 0 \quad C = 1$

$D = 4a^2 > 0$: HYPERBOLIC