STAT844: SATELLITE ORBIT PREDICTION USING AUTOREGRESSION

BY ANGELO RAJENDRAM[1,a],

[1]*Math, University of Waterloo,* [a]*aa3rajen@uwaterloo,ca*

1. Introduction. Development in the commercial space sector has seen rapid growth in the number of satellites placed into orbit, nearly doubling over the course of a decade. The growth in the number of satellites increases the risk of collisions between these objects and has raised concerns about Kessler syndrome, wherein the density of objects in LEO (low-earth orbit) is high enough to cause cascading collisions, and eventually render future space activities in this orbital range difficult. To mitigate this risk, orbit determination and prediction are essential to planning satellite missions and tracking satellites. Upon estimating the actual kinematic state of a satellite through orbit determination, sophisticated physics-informed mathematical models are used to predict its future trajectory over short horizons. For this project, we consider only the problem of orbit prediction given a time series of kinematic states for several satellites.

The dataset used was sourced from the 2020 International Data Analysis Olympiad (IDAO 2020) competition. It must be noted that this dataset is synthetic, generated using an established and sophisticated mathematical model that accepts an initial satellite kinematic state to generate a prediction of its trajectory. For this project, we naively apply a classical statistical model used for time series forecasting and demonstrate that it is capable of generating accurate forecasts over short horizons despite its simplicity and lack of reliance on problem-specific information. We provide intuition on the connection between the statistical models and the underlying equations of motion. Finally, we discuss where similar statistical approaches may be useful, in light of the fact that using them naively cannot guarantee physically consistent solutions and so would not generally be used in practice to simulate such systems.

2. Exploratory Data Analysis. The IDAO dataset consists of the trajectories of 600 satellites over a period of one month. Each trajectory consists of six separate time series, each associated with a satellite's kinematic state variables (position (x, y, z) and velocity (v_x, v_y, v_z) provided in Cartesian coordinates with origin at earth's center). The dataset was generated using NASA's GMAT (General Mission Analysis Tool) which uses a sophisticated physics-informed mathematical model to generate the spacecraft trajectories, taking into account orbital dynamics, gravitational models of an oblate and non-smooth earth, as well effects due to solar radiation pressure and thin atmospheric drag (non-linear functions of the satellite state). Consequently, the data is deterministic and is not generated by a stochastic process. In attempting to learn a model from this data, all uncertainty is associated with an underlying model rather than any intrinsic variance in the data itself.

Keywords and phrases: autoregression, time series.

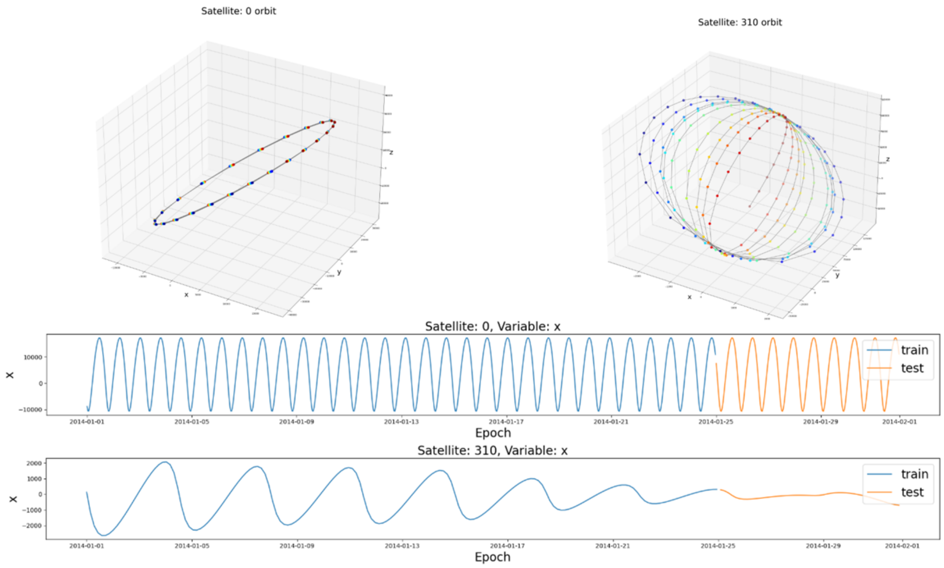

FIG 1. ***Top left:*** *Orbit of sat_id 0.* ***Top Right:*** *Orbit of sat_id 310, we observe that the satellite orbit precesses over the course of one month (cooler colours occur earlier than warmer colours).* ***Middle:*** *Time series of x for sat_id 0. The last five days, shown by the orange curve are used as a test set to evaluate the model forecast.* ***Bottom:*** *Time series of x for sat_id 310, we observe that the orbit has precessed to the point where the orbital plane is nearly orthogonal to the x axis.*

Figure 1 shows a 3D plot of the position of sat_ids 0 and 310 over the course of one month. Also shown are the time series associated with the x state variable for two satellites. The variation of the other state variables exhibits similar patterns. In general, the data is very periodic, and the state is sampled at equally spaced time intervals. This sampling interval is fixed for each satellite but varies across satellites, resulting in time series of varying lengths. In this project, we naively treat each state variable as an independent time series and fit models to each of these, i.e. we do not attempt to learn a globally valid model.

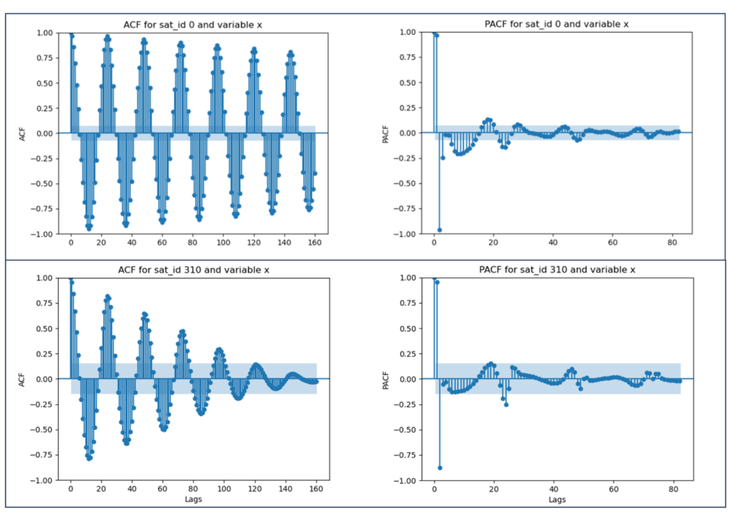

FIG 2. ***Top left:*** *Autocorrelation function of x for sat_id 0.* ***Top right:*** *Autocorrelation function of x for sat_id 310.* ***Bottom left:*** *Partial autocorrelation function of x for sat_id 0.* ***Bottom right:*** *Partial autocorrelation function of x for sat_id 310.*

For time series forecasting, a desirable property is stationarity. A time series is said to be generated by a stationary process when three conditions are satisfied: constant mean with time, constant variance with time, and an autocovariance function between random variables X_{t_1} and X_{t_2} that only depends on the interval $(t_1 - t_2)$. Although the time series here is deterministic, and thus cannot be described as stationary, it nevertheless has a relatively constant frequency and level content over time. Autocorrelation and partial autocorrelation plots are provided in Figure 2 for the x state variable of sat_ids 0 and 310. We observe that the autocorrelation is trailing and has a trigonometric form of decreasing magnitude, and the partial autocorrelation is truncated after a few lags. Autoregressive models are able to parsimoniously represent processes that meet these criteria (Box and Jenkins (1976), Chen and Wang (2019)).

3. Autoregressive Model. We fit a univariate autoregressive (AR) model to forecast the trajectory of each satellite in the dataset. The autoregressive model is a classical statistical model used for time series forecasting and predicts x_t by taking a linear combination of the p previous lags $x_{t-i}, i \in 1...p$ and some deterministic trend component.

$$x_t = \alpha + \beta t + \sum_{i=1}^{p} \phi_i x_{t-i} + \epsilon_t$$

The deterministic trend component was naively selected here as a linear function of time, and the number of lags p is selected using cross-validation. The same choice of p is used for all state variable time series for a particular satellite, as they are expected to have the same period and thus similar autocorrelation properties. Given a training dataset, the AR model parameters (α, β, and ϕ_i) are fit using ordinary least squares (OLS).

The simple autoregressive model belongs to a broader class of statistical models for time series forecasting, namely ARIMA (autoregressive integrated moving average). The "I" (integrated) and "MA" (moving average) components of this class of models are not used for this problem as we do not seek to remove trends from the time series prior to fitting the model, nor do we seek to average out any noise or variance in the data.

Deeper intuition as to why the AR(p) model may be able to capture the dynamics underlying the data can be arrived at by considering what domain knowledge we have of the problem. The evolution of the satellite trajectory with time (its equations of motion) is a system of ordinary differential equations in the state variables. Upon discretizing the independent variable (time) these differential equations become difference equations. As the equations of motion are derived from Newton's second law $\boldsymbol{F} = m\ddot{\boldsymbol{x}}$ we expect that this would be a second-order differential/difference equation.

Now consider the AR(p) model, the deterministic nature of the data generation process allows for the noise term ϵ_t to be ignored, which then effectively turns the AR(p) model into a deterministic p-th order difference equation. As the evolution of the satellite trajectory in discrete time is itself described by difference equations, it is reasonable to expect that the AR model is capable of capturing some aspect of the underlying dynamics. However, it must be noted that by treating each state variable as an independent time series we do not account for the coupled dynamics between the state variables. By allowing p to be greater than 2 (the order of the underlying true equations), it may be possible to improve our forecast to a limited extent despite this.

As a final remark, in typical time series forecasting approaches, the observed data is decomposed into level, trend, and seasonal components. Despite the periodicity and trends observed in IDAO 2020 dataset, we do not explicitly attempt such a decomposition. We find that the AR model is generally able to capture the observed dynamics so long as a sufficient number of lags are used.

4. Model Selection and Scoring.

4.1. *Time Series Cross-validation.* Performance estimation seeks to estimate the loss that a predictive model will incur on unseen data and facilitates model selection. For independent and identically distributed (i.i.d.) data a common approach is cross-validation, which estimates the average generalization error of a model over training datasets of fixed size. However, in the context of time series, causal dependencies with preceding observations violate the i.i.d. assumption and raise questions about the most appropriate way to estimate performance. Currently, there is no consensual approach for performance estimation with time series data (Cerqueira, Torgo and Mozetič (2020)).

When the time series are non-stationary, estimates can be produced by out-of-sample (OOS) methods that use a hold-out validation set. For stationary time series, Cerqueira, Torgo and Mozetič (2020) show empirically that blocked cross-validation, which preserves sequential ordering, can be applied. Bergmeir, Hyndman and Koo (2018), show that a normal K-fold cross-validation procedure can be used if the residuals of the fitted model are uncorrelated, which is typically the case when the fitted model nests an appropriate model.

In this report, we try prequential sliding blocks cross-validation, prequential blocks cross-validation (also known as expanding window CV), as well as OOS validation to select the appropriate hyperparameter p. For each state variable of each satellite, we fit AR(p) models with p ranging from 1 to 50 lags and use cross-validation to evaluate the model and select p. The prequential schemes used are illustrated in Figure 3.

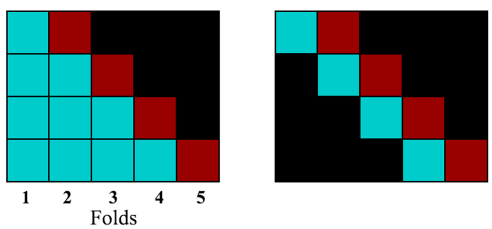

FIG 3. ***Left:****Prequential blocks CV (expanding window CV).* ***Right:*** *Prequential siding blocks CV.*

For assessment, and to save computational time we restricted ourselves to use only 5 CV folds for the prequential methods and a 75% split train/validation split for the OOS method. After selecting the best AR(p) models with CV a separate test set was reserved for final assessment in the Results section of this report. We find that the different validation methods show preferences towards different values of p. However, plots of the validation score (Figure 4) show that it is quite flat for sufficiently large p.

4.2. *Scoring.* To evaluate the fit of the model we use a variant of SMAPE (Symmetric Mean Absolute Percentage Error) that accepts vector-valued targets and predictions instead of scalars as shown below,

$$\text{SMAPE} = \frac{100}{n} \sum_{i=1}^{n} \frac{\|\boldsymbol{x_i} - \boldsymbol{\hat{x}_i}\|}{\|\boldsymbol{x_i}\| + \|\boldsymbol{\hat{x}_i}\|} + \frac{\|\boldsymbol{v}_i - \boldsymbol{\hat{v}_i}\|}{\|\boldsymbol{v}_i\| + \|\boldsymbol{\hat{v}_i}\|}$$

Here, $\boldsymbol{\hat{x}_i}$ represents the prediction of the position (x, y, z) at the i^{th} test point and $\boldsymbol{x_i}$ is the corresponding target. Similarly for the velocities, $\boldsymbol{\hat{v}}$ and $\boldsymbol{v}$. This variant of SMAPE was selected as it offers a normalized metric to compare accuracy across the various satellites in the dataset which range anywhere from ~ 100 km to $\sim 70,000$ km in orbital altitude.

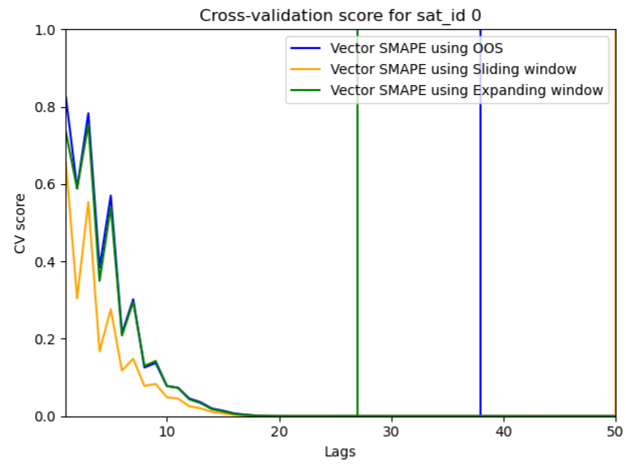

FIG 4. *CV scores for p from 0 to 50 for sat_id 0. Vertical lines indicate p with the minimum score. (Note that the sliding window CV selected $p = 50$ and so is not clearly visible.)*

5. Results. Using the approach described, without explicitly modeling coupling between the state variables, we obtain a surprisingly accurate forecast over the test horizon (the last five days of the month). Figure 5 shows a histogram of the accuracy (100 - SMAPE) over the 600 satellites in the dataset.

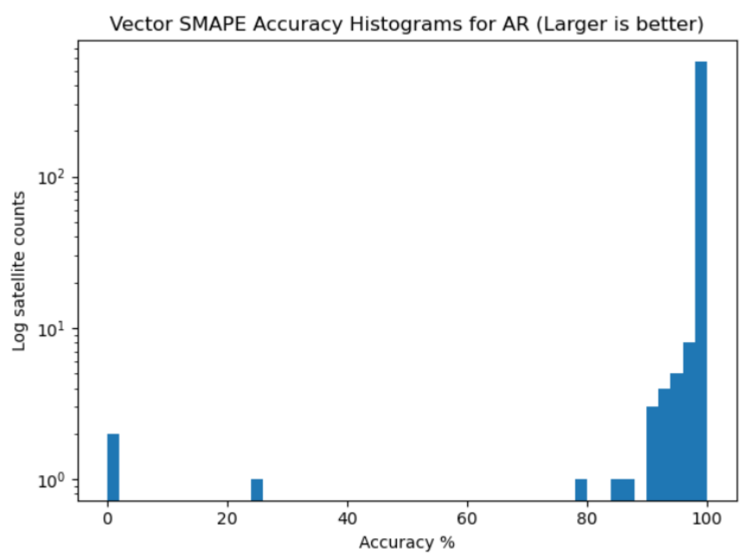

FIG 5. *Histogram showing the Vector SMAPE accuracy (higher is better) for the 600 satellites in the dataset. The y-axis shows the log of the satellite counts to make the poorer cases more evident.*

We see that out of the 600 forecasted trajectories, 574 achieve a score of between 98-100%. A few satellite trajectory forecasts scored poorly, but closer examination revealed that in several of these cases, the sampling interval was large and as a result, the training data was scarce (on the order of 100 data points or less over the whole month). In a more interesting failure, a satellite's orbit was found to have precessed such that the variation of one of the state variables approached zero i.e. one of the cartesian axes was orthogonal to the orbital plane (see Figures 1 and 6). Without modeling couplings between the state variables and taking the geometry of the problem into account the univariate AR(p) model cannot capture these dynamics solely from the lagged state values, resulting in a physically inconsistent forecast and a poorer score.

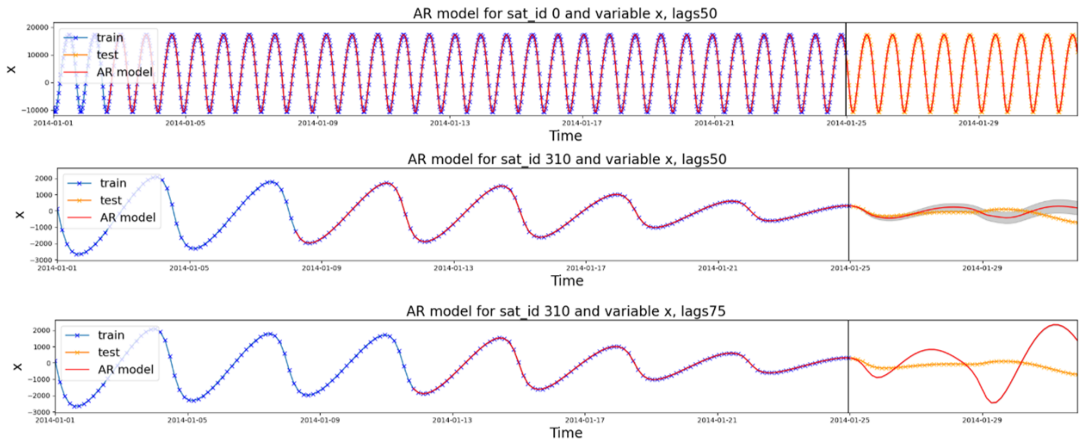

FIG 6. *Training and test time series data and model forecasts.* ***Top:*** *sat_id 0 shows a perfect match between the forecast and test sequence.* ***Middle:*** *For sat_id 310, the model does not correctly forecast the state with p=50. Confidence intervals are shown but they assume normally distributed residuals in the training data which is not the case here.* ***Bottom:*** *Increasing p to 75 collapses the confidence intervals but the predictions are still wrong despite the increased model complexity.*

Overall, the forecasts yielded high accuracy over the five-day test horizon, though in a few cases the predictions are physically inconsistent as the model does not take any prior physics knowledge into account.

6. Conclusion. Natural enhancements to the approach used here include using a vector autoregressive model (VAR) which considers all state variables together for the forecast. Also, applying coordinate transformations based on the available data prior to fitting a model would likely be beneficial. For example, in the instance of linear ordinary differential equations, a transformation to the eigenbasis would decouple the system dynamics which would improve the performance of univariate models fit to the data. However, based on the performance of the AR(p) model on the IDAO dataset there is little improvement to be gained on this particular dataset.

In practice, care must be taken to avoid relying on naive statistical models to make predictions without taking domain-specific information into account. Despite the high accuracy of the dataset, the AR(p) model is liable to produce physically inconsistent forecasts as it does not represent the true underlying dynamics. Nevertheless, statistical approaches that take data into account can complement deterministic first-principles models which use little to no data.

For the problem of modeling satellite trajectories, we know that forces are additive, and our existing gravitational models are based on proven physical laws. At the same time, the influence of force components due to solar radiation pressure, or interaction with the magnetosphere cannot be easily modeled from first principles. In this case, statistical approaches could potentially be used to enhance predictions when measurements of the system are available by incorporating exogenous predictor variables or historical observations.

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