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Revenue Management of Callable Products

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A callable product is a unit of capacity sold to self-selected low-fare customers who willingly grant the capacity provider the option to "call" the capacity at a prespecified recall price. We analyze callable products in a finite-capacity setting with two fare classes where low-fare customers book first, and show that callable products provide a riskless source of additional revenue to the capacity provider. An optimal recall price and an optimal discount-fare booking limit for the two-period problem are obtained. Numerical examples show the benefits from offering callable products can be significant, especially when high-fare demand uncertainty is large. Extensions to multifare structures, network models, overbooking, and to other industries are discussed.

Key words: revenue management; product design; overbooking; capacity allocation
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1. Introduction

The classic mark-up revenue management model studies a finite capacity, two-period problem with two exogenous fares under the assumption that low-fare customers book in the first period and high-fare customers book in the second period. This model was developed after the passenger airline industry was deregulated as an attempt to sell excess capacity to price-sensitive travelers. Airlines did this by offering discounted fares to customers willing to book early and willing to abide by traveling restriction such as Saturday night stays. A major function of airline revenue management systems is to calculate and apply booking limits on low-fare bookings. Using newsvendor-like logic to calculate profit-maximizing booking limits is at the heart of classical revenue management and has been credited with generating hundreds of millions of dollars in additional revenue for airlines and industries with similar characteristics such as hotels and rental cars.

Setting a limit on low-fare bookings is, however, an imperfect mechanism for hedging against uncertainty in future high-fare demand. Despite heavy investment in sophisticated revenue management systems, airlines lose millions of dollars a year in potential revenue; both when low-fare bookings displace higher than expected high-fare bookings ("cannibalization") and when airlines fly empty seats protected for high-fare bookings that do not materialize ("spoilage"). Airlines could avoid cannibalization and spoilage if they could forecast future full-fare demand with certainty or, failing that, could convince business passengers to book earlier and leisure passengers to book later. Because neither of these is likely to occur, airlines are motivated to find other ways to better hedge against full-fare demand uncertainty. Overviews of the theory and application of revenue management can be found in Phillips (2005), Smith et al. (1992), and Talluri and van Ryzin (2004).

In this paper, we analyze the potential of "callable products" as a way for an airline to increase revenue when full-fare demand is uncertain. The concept behind a callable product is quite simple. An airline would offer both callable and standard (noncallable) products at the same low-fare during the first period. A customer purchasing the callable product would grant the airline an option to recall his seat at some future time before departure at a prespecified recall price. Customers whose product is called would be notified by the airline sometime before departure that their seat had been called and the airline would pay the recall price. The recall price will be above the low-fare purchase price but below the full fare. The airline would call the option only if it finds that full-fare demand exceeds available capacity—that is, the remaining seats after low-fare bookings. In that case, the airline pays the recall price for a seat that it sells at the (higher) full fare.

Callable products are appealing because actions are voluntary on both sides: customers individually determine whether they wish to purchase the standard product or the callable product. The airline determines how many (if any) of the callable products it wishes to call. We show that offering callable products can generate significant, riskless, additional revenue.

Various forms of callable products have been used to some degree in a number of different industries. Some companies use a "callback" option by which they can pay a predetermined amount to recall previously committed advertisement time.[1] The callable concept is also used in a business-to-business setting to reduce inventory risk; see Sheffi (2005, pp. 229–231).[2] As a further example, Jay Walker (an early investor in Priceline) and some of his colleagues at Walker Digital were granted a U.S. patent in 1998 (Walker et al. 1998) on the concept of callable airline tickets. However, the patent filing does not specify how to determine booking limits or recall prices.

Biyalogorsky et al. (1999) discussed the concept of overselling with opportunitistic cancellations in an airline context. Biyalogorsky and Gerstner (2004, p. 147) extended the idea to a more general setting under the name of *contingent pricing*, which they define as "...an arrangement to sell a product at a low price if the seller does not succeed in obtaining a higher price during a specified period." They consider the special case of a seller with a single unit of inventory facing two types of customers—those with low willingness-to-pay who purchase in an initial period and those with a high willingness-to-pay who do not arrive until the second period. In a two-period model with independent customer demands and a single product for sale, they show that contingent pricing "...(a) mitigates the expected losses from price risks, (b) can be profitable regardless of buyers' risk attitudes even if buyers are more risk averse than sellers are, (c) benefits buyers as well as sellers, and (d) improves economic efficiency" (Biyalogorsky and Gerstner 2004, p. 153). They also derive optimal pricing policies for the seller under assumptions of different risk preferences on the part of buyers.

Our paper differs from and extends Biyalogorsky and Gerstner (2004) in several ways. First, we take the point of view of a seller with multiple units of capacity (or inventory) who wishes to maximize his expected revenue. For such a seller, as we shall show, it can well be optimal to sell both callable and standard (noncallable) products in the first period. This extends the analysis in Biyalogorsky and Gerstner (2004) which consider sales of only a single unit. Second, we do not assume common willingness-to-pay among buyers nor do we assume that the willingness-to-pay of any specific buyer is known to the seller as in Biyalogorsky and Gerstner (2004). Rather, we assume that the seller faces uncertain demand for callable products that depends on the recall price. Third, our approach allows us to compute the optimal recall price, and to develop bounds on the number of both callable and standard products that it is optimal for the seller to offer under various demand conditions. Our analysis also leads to an optimization method to evaluate the magnitude of the expected revenue gain from the introduction of callable products. We believe that these quantitative aspects are crucial to ascertain whether or not the expected revenue gains justify the introduction of callable products in practice.

Another contribution of this paper is to show the magnitude of the riskless gain. Although it is certainly good to have a "riskless" revenue gain, it is important to understand the magnitude of the gain. If the expected additional profit is small, the cost of implementing callable products may be larger than the benefit. We show that by choosing the units allocated to the low-fare customers, a, and the recall price p in an optimal way, the revenue gain can be indeed substantial. Our numerical results in §5 show that the riskless gain may be as high as 10%, especially when there is high uncertainty of high-fare demand.

In §2, we describe the concept of callable products and compare the callable product approach with other mechanisms in the context of a two-period market in which low-fare customers book before high-fare customers. The effect of introducing callable products in the basic two-period revenue management model is analyzed in §3. We show under very mild conditions that offering callable products can generate a riskless increase in revenue—that is, they never reduce revenue and can increase it with positive probability. We present first-order conditions for a globally optimal low-fare booking limit and recall price and show how these can be calculated in §4. Section 5 describes numerical studies that illustrate the expected revenue gains from offering callable products. In §6, we discuss extensions to multifare structures, network models, overbooking, and to other industries.

[1] We thank one of the referees for calling this example to our attention.

[2] Sheffi (2005, pp. 229–231) describes how Caterpillar sells its products to dealers who input in a database items in their inventory they are willing to share with others. When a customer comes to a dealer and the dealer finds himself without the required part, the dealer checks the database and sees where such an item would be available. Caterpillar then buys back the item from the dealer that has the item at a 10% premium and ships it to the dealer/customer who needs it.

2. Callable Products

2.1. The Concept

When an airline offers callable products, the sequence of events is as follows:

1. The airline publishes its fares for both periods, p_L and p_H, and the recall price p.

2. Based on anticipated demands, the airline chooses a booking limit for low-fare customers.

3. First-period low-fare booking requests arrive. Each low-fare booking request is offered the choice of a standard low-fare product or a callable product. Low-fare bookings are accepted until the booking limit is reached or the end of the first period, whichever comes first. Each low-fare booking pays a fare p_L whether choosing the standard or the callable product.

4. At the end of the first period, the airline no longer offers low-fare bookings. At this point it has accepted S_L total low-fare sales of which some (possibly zero) are callable. Let V_L be the number of callable bookings and $\bar{V}_L = S_L - V_L$ the number of standard (noncallable) bookings that the airline has taken.

5. During the second period, full-fare booking requests arrive. The capacity available for full-fare bookings is $c - \bar{V}_L$, where c is the total capacity. The airline accepts full-fare booking requests during the second period until this limit is reached.

6. If the number of full-fare bookings exceeds $c - S_L$, then the airline will call some, or all, of the callable products.

7. The airline collects p_H from every high-fare customer and pays p to every customer whose option was called. Note that the net payment to a customer whose booking gets called is $p - p_L \geq 0$.

We believe that airlines could easily implement callable products via the Internet.[3] Each time a low-fare booking is made on an airline's website, the customer would be informed of the terms and conditions of the callable product (including the recall price, p), and asked if he would like his booking to be callable. If he agrees, then the airline would notify him (e.g., 24 hours prior to departure) via e-mail whether or not his booking had been called. If the airline chooses to exercise its call, the call price p could be credited to the customer's credit card. Of course, callable products could be offered via other channels as well.

2.2. Comparison to Alternatives

Callable products are a mechanism for suppliers to hedge against uncertainty in future high-fare demand. In addition to setting limits on discount bookings, suppliers have used a number of other mechanisms to hedge against high-fare demand uncertainty:

1. *Stand-bys*: A stand-by booking is one sold at a deep discount that gives a customer access to capacity only on a "space-available" basis. Customers with stand-by tickets arrive at the airport and are told at the gate whether or not they will be accommodated on their flight. If they cannot be accommodated, the airline books them on a future flight (possibly also on a stand-by basis). This strategy helps reduce spoilage but it may cannibalize demand from either the low or the high fare.

2. *Bumping*: If the fares for late-booking passengers are sufficiently high, an airline could pursue a bumping strategy—that is, if unexpected high-fare demand materializes, the airline would overbook with the idea that it can deny boardings to low-fare bookings to accommodate the high-fare passengers. For a bumping strategy to make sense, the revenue gain from the full-fare passenger must outweigh the loss from bumping the low-fare booking, including all penalties and "ill-will" cost. Historically, airlines were reluctant to overbook with the conscious intent of bumping low-fare passengers to accommodate high-fare passengers.[4] However, with the average full fare now equal to seven times or more the lowest discount fare on many routes (Donofrio 2002), the bumping strategy is beginning to make more and more economic sense. For more details on the bumping strategy, the reader is referred to Gallego and Lee (2004).

3. *The replane concept*: Under the replane idea, an airline that sees higher than anticipated full-fare demand will contact customers with discount-fare bookings (via the Internet or phone, for example) on the same flight and offer them some level of compensation to give up their seats.

4. *Flexible products*: With flexible products, passengers can purchase discount tickets that ensured a seat on one of a set of flights to the same destination, with the airline having the freedom to choose which flight the customer will actually be booked on. Gallego and Phillips (2004) show that offering flexible products can increase revenue by both enabling better capacity utilization and inducing additional demand.

5. *Last-minute discounts*: The price of airline tickets generally increases as departure approaches because airlines exploit the fact that later-booking customers tend to be less price sensitive than early-booking customers. However, increasingly airlines have been using last-minute deep discounts to sell capacity that would otherwise go unused.[5]

6. *Auctions*: An alternative mechanism to recover previously sold capacity is to hold an auction toward the end of the booking process. An auction would

[3] There is a sufficient volume of transactions for this to make sense because 40% of all airline ticket sales in North America were made online in 2003 with 27% of all bookings taking place on airline websites (SITA 2003).

[4] Of course, airlines have long overbooked as a means of hedging against cancellations and no-shows (Rothstein 1985).

[5] For example, the company Last-Minute Travel (www.lastminutetravel.com) specializes in selling deeply discounted capacity for flights that are nearing departure. To limit cannibalization, many airlines, hotels, and rental car companies only offer last-minute discounts through disguised ("opaque") channels such as Priceline (www.priceline.com) or Hotwire (www.hotwire.com).

allow customers to learn more about their valuation before agreeing on a price to sell back their capacity. While this information may enable the capacity provider to extract more of the ex-post consumer surplus, there are a number of complications. First, it is difficult to conceive of a practical method that would allow a large majority of the low-fare customers to participate unless the auction was held at the airport very shortly before departure (this would be even more difficult to implement in other industries such as hotels and car rentals). In addition, the valuations may end up higher than predicted and the capacity provider may need to pay more than anticipated. Moreover, the capacity provider would have to make full-fare overbooking decisions before holding the auction so there is no riskless profit. Finally, there are low-fare customers that are willing to forfeit their capacity if their travel plans change. Under an auction, these customers would need to be paid.

None of these approaches is perfect. Management of stand-bys and bumped passengers adds operational complexity and can create flight delays. Bumping is unpopular with passengers. Replaning requires searching for passengers who are willing to change flights. Flexible products require customers that are more or less indifferent to the actual flight on which they travel. Last-minute discounts risk cannibalizing high fares and can train customers to wait rather than book early. Auctions are difficult to implement. By and large, callable products avoid these shortcomings.

We also note that—unlike callable products—stand-by bookings, replane, and flexible products require accommodating all booked passengers. These approaches are most effective when there is a wide disparity in capacity utilization among flights serving the same market. They allow an airline to move demand from highly utilized flights to less utilized flights, thereby freeing up capacity. These approaches are much less effective when all flights are highly utilized. In this situation, we would anticipate that callable products would be more effective because they would allow the airline to free up capacity in the market to sell to high-fare customers. All of these approaches listed above have their place. There is no reason why airlines cannot use any or all of them in combinations with callable products to maximize revenue for a particular flight. In particular, callable products can be used in a network where instead of paying the recall price p, the passenger is sent in an alternative, prespecified route, and given a prespecified compensation.

Callable products can be abused if a speculator buys a callable ticket and then makes phantom full-fare bookings with the purpose of triggering the recall of his ticket. For this strategy to work, the speculator may need to book a large number of full-fare, fully refundable, tickets without restrictions and then cancel them as soon as his low-fare ticket is called by the airline. The uncertainty surrounding the number of full-fare tickets needed to trigger a recall and the need to cancel full-fare tickets in a very short time span makes this strategy unattractive. In addition, airlines can impose time-window restrictions on fully refundable fares or offer partially refundable fares to deter speculators.

3. A Two-Period Model with Callable Products

We analyze callable products in the context of a two-period, two-fare model with low-fare customers booking in the first period and high-fare customers booking in the second period. The two fares are exogenous and bookings are firm: that is, there are no cancellations or no-shows. This is a classic revenue management model, first studied by Littlewood (1972) and extended to multiple periods and fare classes by a number of others including Belobaba (1987, 1989), Curry (1990), and Wollmer (1992). We present our analysis in the context of an airline although the results are general across industries with similar characteristics (see §6).

An airline accepts bookings for a flight with fixed capacity c during two periods. Each booking request is for a single unit of capacity (e.g., a single seat). Bookings occur during two periods. First-period and second-period demands are integer-valued random variables denoted by D_L and D_H, respectively. We do not initially assume that D_L and D_H are independent. First-period and second-period fares are denoted by p_L and p_H, respectively, with $p_H > p_L > 0$.

When an airline offers callable products, low-fare customers are given the opportunity at the time of purchase to grant the capacity provider the option of recalling their booking at a known *recall price* $p \in [p_L, p_H]$. There is no additional charge (or discount) to customers for choosing this alternative. If $p > p_L$, then customers whose reservation price for a seat is between p_L and p may choose the callable product. At the end of the second period, the provider has the opportunity to meet some of the high-fare demand in excess of residual capacity by recalling callable capacity at p and selling it for p_H. For the capacity provider, the recall price p is a decision variable. For each customer, the decision whether or not to purchase a callable product is based on the recall price p and her reservation price. We initially assume that total low-fare demand (that is, callable demand plus standard low-fare demand) is independent of the value of p. In this case, the only effect of changing p is to change the allocation of total low-fare demand between standard and callable customers. We relax this assumption in §5.3 to discuss the implication of demand induction.

The seller's decision variables are the recall price $p \in [p_L, \bar{p}_H]$ and the low-fare booking limit $a \in \{0, \dots, c\}$. Sales at the low fare are denoted by $S_L(a) = \min(D_L, a)$. Of these, $V_L(a)$ are callable and $\bar{V}_L(a) = S_L(a) - V_L(a)$ are not. The capacity available for sale at the high fare is $c - \bar{V}_L(a)$, the number of units sold at the high fare is $\min(D_H, c - \bar{V}_L(a))$, and the number of units called is $\min((S_L(a) + D_H - c)^+, V_L(a))$. This means that, for any choice of the decision variables a and p, expected revenue is given by $r(a, p) = E[R(a, p)]$, where

$$R(a,p) = p_L S_L(a) + p_H \min(D_H, c - \bar{V}_L(a)) - p\min((S_L(a) + D_H - c)^+, V_L(a)). \quad (1)$$

Let $R(a)$ be the revenue corresponding to the traditional strategy without the callable product. Because $\bar{V}_L(a) = S_L(a)$ and $V_L(a) = 0$, it follows that $r(a) \equiv E[R(a)]$, where $R(a) = p_L S_L(a) + p_H[\min(D_H, c - S_L(a))]$.

Proposition 1. *For any feasible values of a and p, the revenue realized with callable products is at least as large as the corresponding revenue without callable products with probability one. More precisely, $R(a,p) = R(a) + W(a,p)$, where $W(a,p) \geq 0$ denotes the additional revenue gained from callable bookings and,*

$$W(a,p) = (p_H - p)\min(G(a), V_L(a)) \geq 0, \quad G(a) = (S_L(a) + D_H - c)^+, \quad (2)$$

that is, $W(a,p)$ is the margin on called tickets times the number of tickets called, and $G(a)$ is the excess high-fare demand.

Proof. From (1), we have

$$\begin{aligned} R(a,p) &= p_L S_L(a) + p_H(\min(D_H, c - S_L(a))) \\ &\quad - p_H(\min(D_H, c - S_L(a))) \\ &\quad + p_H \min(D_H, c - (S_L(a) - V_L(a))) \\ &\quad - p\min((S_L(a) + D_H - c)^+, V_L(a)) \\ &= R(a) + p_H\{\min(D_H, c - (S_L(a) - V_L(a))) \\ &\qquad - \min(c - S_L(a), D_H)\} \\ &\quad - p\min((S_L(a) + D_H - c)^+, V_L(a)) \\ &= R(a) + (p_H - p)\min((S_L(a) + D_H - c)^+, V_L(a)) \\ &= R(a) + W(a,p) \geq R(a), \end{aligned}$$

where $W(a,p) \geq 0$ follows from $\min(G(a), V_L(a)) \geq 0$ and $p_H - p \geq 0$. □

On the surface, Proposition 1 may seem to be surprising because the revenue gain by adding the callable feature is nonnegative with probability one ("riskless"). However, this is a consequence of the fact that the capacity provider will only exercise his option when it is worthwhile to do so. He can always realize the same revenue as the traditional model, with a booking limit a, by simply not calling any of the options.

Proposition 1 does not require any assumptions on $V_L(a)$. However, in what follows, we assume that low-fare customers decide whether or not to purchase the callable product independently with probability $q = g(p)$. We further assume that $q = g(p)$ is a continuous increasing function of p.[6]

Let $\bar{p} = \inf\{p \geq p_L: g(p) = 1\}$ be the smallest call price at which low-fare customers will purchase the callable product with probability one and let $\bar{p}_H = \min(\bar{p}, p_H)$. Clearly, the capacity provider will limit his choice of p to $p \in [p_L, \bar{p}_H]$ because it is clearly suboptimal to use a recall price above $\bar{p}_H$. We assume that g is strictly increasing over the interval $[p_L, \bar{p}_H]$. Denote its inverse by $p = h(q)$, where $h(q)$ is strictly increasing and continuous. $h(q) \equiv g^{-1}(q)$ is defined for values of $q \in [q_L, \bar{q}_H]$, where $0 \leq q_L = g(p_L) < \bar{q}_H = g(\bar{p}_H) \leq 1$. Because customers make the decision to grant the call independently with the same probability q, the number of callable units, $V_L(a)$, is conditionally binomial with parameters $S_L(a)$ and q; i.e. $V_L(a) = \text{bino}(S_L(a), q)$. In this case, we can show that when there is a nonzero chance that at least one low-fare booking is sold and there is excess total demand for the flight, the probability of positive gain from offering callable products is greater than zero.

Proposition 2. *If customers make independent decisions to grant the call with equal probability $q > 0$, $p < p_H$, and*

$$P\{S_L(a) + D_H > c, S_L(a) > 0\} > 0, \quad (3)$$

then

$$P\{W(a,p) > 0\} > 0. \quad (4)$$

Proof.

$$\begin{aligned} &P\{W(a,p) > 0\} \\ &\geq P\{V_L(a) > 0 \text{ and } S_L(a) + D_H > c\} \\ &\geq P\{V_L(a) > 0 \text{ and } S_L(a) + D_H > c \mid S_L(a) > 0\} P\{S_L(a) > 0\} \\ &\geq qP\{S_L(a) + D_H > c \mid S_L(a) > 0\} P\{S_L(a) > 0\} \\ &= qP\{S_L(a) + D_H > c, S_L(a) > 0\} > 0, \end{aligned}$$

where the third inequality follows from the fact that $q = P\{V_L(a) > 0 \mid S_L(a) = 1\}$ and $q \leq P\{V_L(a) > 0 \mid S_L(a) > 1\}$, which implies that $q \leq P\{V_L(a) > 0 \mid S_L(a) > 0\}$. □

[6] We will use the terms "increasing" and "decreasing" in the weak sense unless stated otherwise.

Note that (3) is not particularly restrictive: it specifies that the joint probability that total bookings exceed capacity and low-fare bookings is positive. Offering the callable product results in a win-win-win situation because the capacity provider increases his revenues, low-fare customers increase their utility, and high-fare customers have additional available capacity.[7]

The riskless revenue gain shown in Equation (4) is similar to the concept of *arbitrage* in finance because the revenue gain $W(a, p)$ is always nonnegative and is positive with nonzero probability. A common assumption in financial analysis is that arbitrage opportunities cannot exist in standard well-behaved markets. The reason that an arbitrage opportunity can exist in our markets is due to asymmetric information and operational advantage. First, the seller, unlike the customers, is able to observe second-period full-fare demand before calling fares. Second, airline tickets are typically nontransferable; even if a low-fare customer finds another customer willing to pay a higher price, the low-fare customer cannot transfer the ticket. We should point out, however, that airlines often sell blocks of transferable capacity to consolidators at very low fares leading to the possibility of riskless revenue gain.

4. Optimization

4.1. First-Order Condition for a

We assume that D_L and D_H are independent from now on and recall that $r(a, p)$ is the expected gain from offering products, that is, $r(a, p) \equiv E[R(a, p)]$. To derive the first-order conditions for a, we will use the following result describing how $R(a, p)$ changes as a is incremented.

Lemma 1. *For* $a \in [0, c-1]$,

$$\Delta r(a, p) \equiv r(a+1, p) - r(a, p) = P\{D_L > a\}[p_L - \psi(a, p)], \tag{5}$$

where, with $\bar{q} = 1 - q = 1 - g(p)$,

$$\psi(a, p) = (p_H - p)\bar{q}P\{D_H \geq c - \text{bino}(a, \bar{q})\} + pP\{D_H \geq c - a\}. \tag{6}$$

Proof. See the online supplement (provided in the e-companion).[8] □

The expression $p_L - \psi(a, p)$ inside the square brackets in Equation (5) is decreasing in a. Therefore, Equation (5) admits at most one sign change in a for any fixed p, and this must be from positive to negative. Thus, $r(a, p)$ is unimodal in a for fixed p and the *largest maximizer* of $r(a, p)$ is given by

$$a(p) \equiv \min\{a \in [0, c]: \psi(a, p) > p_L\}, \tag{7}$$

where the minimization is over the set of integers, with $a(p) = c$ if the set is empty.[9]

The largest optimal booking limit for the traditional revenue management problem without callable products is given by

$$a_T^* \equiv \min\{a \in [0, c]: p_H P\{D_H \geq c - a\} > p_L\},$$

with $a_T^* = c$ if the set is empty.[10] Let b be the essential supremum of D_H, i.e., the smallest integer such that $P\{D_H \geq b\} = 0$. If D_H is unbounded, then $b = \infty$.

We can now relate the optimal booking limit under traditional revenue management to the optimal booking limit with callable products.

Lemma 2. $(c - b)^+ \leq a_T^* \leq a(p)$ *for all* $p \in [p_L, \bar{p}_H]$.

Proof. See the online supplement. □

It would be natural to conjecture that $a(p)$ is monotone increasing in p. However, this is not necessarily true because the function $\psi(a, p)$ is not necessarily decreasing in p. To see this, consider the case where $q_L = 0$ and note that $\psi(a, p_L) = \psi(a, p_H) = p_H P\{D_H \geq c - a\} \geq \psi(a, p)$. As a result, we cannot say that the function $r(a, p)$ is submodular, and therefore we cannot invoke Topkis' Monotone Optimal Selection Theorem (Topkis 1998) to claim that $a(p)$ is monotone.[11]

[7] Because buyers can decide whether or not they grant the provider the callable option, their utility is at least as large as when the callable option is not available. Therefore, there is no need to reduce the low fare to induce customers to accept the callable product; this conclusion would change if there was a secondary market allowing customers to trade both low-fare tickets and callable tickets. Because airline tickets are not transferable, it is unlikely that such a market would ever develop in that industry. However, it would be a consideration in sporting events or Broadway shows where tickets are transferable.

[8] An electronic companion to this paper is available as part of the online version that can be found at http://mansci.journal.informs.org/.

[9] Note that if k is a positive integer such that $\psi(a(p) - k, p) = p_L$, then all the elements in the set $\{a(p) - k, \ldots, a(p)\}$ maximize $r(a, p)$.

[10] This solution to the two-period revenue management problem without callables was proposed by Littlewood (1972). Bhatia and Parekh (1973) and Richter (1982) demonstrate the optimality of Littlewood's formula.

[11] Intuitively, when p is close to p_L, as p increases we can allow a larger number of low-fare bookings, knowing that we can recall them and get almost the full margin $p_H - p_L$ on each recalled unit. As p approaches p_H, we increasingly cannibalize high-fare sales. Thus, in most applications, we would expect $a(p)$ to initially increase and then decrease with p, but more complicated behavior is also possible.

4.2. First-Order Condition for q

The following assumptions will be needed for various results in this subsection.

Assumption 1. *$P\{c - S_L(a) < D_H \le c-1\} > 0$. This assumption is not restrictive: if $P\{D_H \le c-1\} = 0$, then the airline could fill the plane with high-fare customers with probability one. In this case, there is no reason for the airline to offer any low-fare product. On the other hand, if $p\{c - S_L(a) < D_H\} = 0$, then the airline is expecting to have empty seats on the flight with probability one. In this case, it would adjust a or possibly lower the fare of one of the products to better utilize capacity. Note that this assumption implies condition* (3).

Assumption 2. *$h'(q) > 0$ is an increasing function of q for $q \in [0, \bar{q}_H]$. In other words, h is increasing and convex in q for $q \in [0, \bar{q}_H]$. The implication of the assumption is that it is increasingly more difficult to attract additional customers to grant the call option.*

Assumption 3. *$q_L = 0$. In other words, low-fare customers will not participate in the program if the recall price is $p = p_L$.*

Note that $R(a,p) = R(a) + W(a,p)$, where $R(a)$ does not involve q. By (2), we have

$$\begin{aligned}\frac{d}{dq}r(a,p) &= \frac{d}{dq}E[W(a,p)]\\ &= -h'(q)E[\min(G(a), V_L(a))]\\ &\quad + (p_H - p)\frac{d}{dq}E[\min(G(a), V_L(a))], \end{aligned}\tag{8}$$

yielding the first-order condition

$$\begin{aligned}&h'(q)E[\min(G(a), V_L(a))]\\ &\quad = (p_H - p)\frac{d}{dq}E[\min(G(a), V_L(a))].\end{aligned}\tag{9}$$

To derive the first-order conditions for q, we make use of the following lemma, which itself uses the regularized incomplete beta function (see Chapter 6 in Abramowitz and Stegun 1972), which is defined for $0 \le x \le 1$ by

$$I_x(a,b) := \frac{B_x(a,b)}{B(a,b)},$$

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt \quad \text{if } a, b > 0,$$

$$I_x(a,b) \equiv 1 \quad \text{if } a = 0, \qquad I_x(a,b) \equiv 0 \quad \text{if } b = 0,$$

where $B(a,b) \equiv B_1(a,b)$ is the standard beta function. In addition,

$$0 < I_x(a,b) < 1 \quad \text{and} \quad I_x(a,b) \text{ is strictly increasing in } x \quad \text{if } 0 < x < 1,\ a \ne 0,\ b \ne 0. \tag{10}$$

Lemma 3. *Suppose that $X = \text{bino}(n, q)$. Then, for any integer $y \ge 0$,*

$$E[\min(X,y)] = \begin{cases} 0, & n = 0 \text{ or } y = 0,\\ nq, & y \ge n \text{ and } n \ge 1,\\ nq\{1 - I_q(y, n-y)\} + yI_q(y+1, n-y), & 1 \le y \le n-1 \text{ and } n \ge 2.\end{cases}$$

$$\frac{d}{dq}E[\min(X,y)] = \begin{cases} 0, & n = 0 \text{ or } y = 0,\\ n, & y \ge n \text{ and } n \ge 1,\\ n\{1 - I_q(y, n-y)\}, & 1 \le y \le n-1 \text{ and } n \ge 2.\end{cases}$$

In particular, $(d/dq)E[\min(X,y)]$ is a decreasing function of q. For any $y \ge 0$ and any $x \in [0, n]$,

$$P\{\min(X,y) > x\} = \begin{cases} 0, & n = 0 \text{ or } y = 0,\\ I_q(\lfloor x\rfloor + 1, n - \lfloor x\rfloor), & y \ge x \text{ and } n \ge 1,\\ 0, & y < x \text{ and } n \ge 1,\end{cases}$$

where $\lfloor x\rfloor$ is the integer part of x.

Proof. See the online supplement. □

Proposition 3. *For any feasible value of a and p, the expected revenue gain from offering callable products is*

$$\begin{aligned}E[W(a,p)] &= (p_H - p)E[\min(G(a), V_L(a))]\\ &= (p_H - p)qE\big[S_L(a)\{1 - I_q(G(a), S_L(a) - G(a))\};\\ &\qquad\qquad S_L(a) + D_H > c, D_H \le c-1\big]\\ &\quad + (p_H - p)E\big[G(a)I_q(G(a)+1, S_L(a) - G(a));\\ &\qquad\qquad S_L(a) + D_H > c, D_H \le c-1\big]\\ &\quad + (p_H - p)qE[S_L(a); D_H \ge c, S_L(a) \ge 1],\end{aligned}$$

and, for any $x \ge 0$,

$$\begin{aligned}P\{W(a,p) > x\} &= P\left\{\min(G(a), V_L(a)) > \frac{x}{p_H - p}\right\}\\ &= E\left[I_q\left(\left\lfloor\frac{x}{p_H - p}\right\rfloor + 1, S_L(a) - \left\lfloor\frac{x}{p_H - p}\right\rfloor\right);\right.\\ &\qquad\qquad \left. G(a) \ge \frac{x}{p_H - p}, S_L(a) \ge 1\right].\end{aligned}$$

Proof. Because

$$\begin{aligned}\{0 < G(a) \le S_L(a) - 1, S_L(a) \ge 2\} &= \{S_L(a) + D_H > c, D_H \le c - 1, S_L(a) \ge 2\} \\ &= \{S_L(a) + D_H > c, D_H \le c - 1\},\end{aligned}$$

applying Lemma 3 immediately yields the proposition. □

The following lemma gives a condition for the existence of the solution to the first-order condition (9). It also discusses the concavity in q of the function $r(a, h(q))$.

Lemma 4. *Under Assumptions* 1 *and* 2, $E[W(a, h(q))]$ *and* $r(a, h(q))$ *are strictly concave in* $q \in [0, \bar{q}_H]$. *In addition, suppose that Assumptions* 1, 2, *and* 3 *hold. If*

$$h'(q)E[\min(G(a), V_L(a))] \ge (p_H - p)\frac{d}{dq}E[\min(G(a), V_L(a))] \quad at\ q = \bar{q}_H,$$

then (9) *has a unique solution within* $[0, \bar{q}_H]$; *otherwise,* $q = \bar{q}_H$ *is optimal and is defined to be the solution of* (9).

Proof. See the online supplement. □

By Lemma 3, we can write the optimality Equation (9) for q as

$$\begin{aligned}h'(q)\{&qE[S_L(a)\{1 - I_q(G(a), S_L(a) - G(a))\}; \\ &\qquad S_L(a) + D_H > c, D_H \le c - 1] \\ &+ E[G(a)I_q(G(a) + 1, V_L(a) - G(a)); \\ &\qquad S_L(a) + D_H > c, D_H \le c - 1] \\ &+ qE[S_L(a); D_H \ge c, S_L(a) \ge 1]\} \\ = (p_H - p)\{&E[S_L(a)\{1 - I_q(G(a), S_L(a) - G(a))\}; \\ &\qquad S_L(a) + D_H > c, D_H \le c - 1] \\ &+ E[S_L(a); D_H \ge c, S_L(a) \ge 1]\}. \quad (11)\end{aligned}$$

For programming purposes, the terms in (11) can all be computed easily as shown in the online supplement.

4.3. Global Optimality

It is not immediately obvious that a value of (a, p) that satisfies the first-order conditions is necessarily a global optimum for three reasons: (1) The function $r(a, p)$ is not concave in a, although from Lemma 4 it is concave in q. This can be seen easily because $r(a, 0)$ is not concave in a. (2) There may be more than one global optimal solution for $r(a, p)$, as can be seen from the case of $r(a, 0)$. (3) The two parameters a and p are, respectively, discrete and continuous. However, the following proposition sheds light on the issue of global optimality.

Proposition 4. *There must be at least one global maximizer for* $r(a, p)$. *Furthermore, if* (a^*, p^*) *is the global maximizer with the largest* a^*, *then we must have* $a_T^* \le a^*$. *In other words, the largest optimal solution has a more generous booking limit than any traditional solution.* (*Note that Assumptions* 1, 2, *and* 3 *are not needed for this result.*)

Proof. Because the domain $a \in [0, c]$ and $p \in [p_L, \bar{p}_H]$ is compact, $r(a, p)$ must have at least one global maximum. Next, we prove $a_T^* \le a^*$ by contradiction. Suppose that $a_T^* > a^*$. Because a_T^* is optimal for the traditional revenue management without callables, we have $E[R(a_T^*)] \ge E[R(a^*)]$. Because $E[W(a, p)]$ is increasing in a, it follows that $E[W(a_T^*, p^*)] \ge E[W(a^*, p^*)]$. Therefore, we have

$$\begin{aligned}r(a_T^*, p^*) &= E[R(a_T^*)] + E[W(a_T^*, p^*)] \\ &\ge E[R(a^*)] + E[W(a^*, p^*)] = r(a^*, p^*),\end{aligned}$$

which contradicts the fact that (a^*, p^*) is the global optimizer with the largest booking limit. □

Proposition 5. *Under Assumptions* 1, 2, *and* 3, *the global maximizer* (a^*, p^*) *with the largest* a^* *must satisfy the first-order condition* (9) *with* $p^* \in (p_L, \bar{p}_H]$ *and* $a^* = a(g(p^*))$.

Proof. By Assumption 1 and Proposition 1, we have that $E[W(a, p)] > 0$ for $p \in (p_L, p_H)$, and, by Assumption 3, $E[W(a, p)] = 0$ at $p = p_L$ and $p = p_H$, so it follows that any optimal solution must be in the set (p_L, p_H). Moreover, because the profit at $\bar{p}_H$ is at least as large as the profit in $(\bar{p}_H, p_H)$, it follows that any optimal solution must be in the set $(p_L, \bar{p}_H]$. If $\bar{p}_H$ is the global optimum, then the statement is clearly true. Now let (a^*, p^*) be the global maximizer with the largest a^*, $p_L < p^* < \bar{p}_H$. Then, clearly $(a^*, g(p^*))$ must satisfy the first-order condition in q (Equation (9)); otherwise, if the derivative is nonzero, one can always decrease (respectively, increase) p^* if the derivative is less than zero (respectively, greater than zero) and achieve a higher value for the objective function because q^* is in the interior. The rest of the result follows easily from Lemma 2. □

Suppose that Assumptions 1, 2, and 3 hold. Then, one approach to optimization would be to find the unique $p \in (p_L, \bar{p}_H]$ satisfying (9) for every fixed $a \in [a_T^*, c]$, and then find the best a through exhaustive search.

5. Numerical Results

In this section, we present simulation results that illustrate the effects of offering callable products under different assumptions. We start by introducing the models that we will use for choosing the demands D_H and D_L and for the participation function g. We then extend the analysis to the case in which offering a callable product may actually induce additional demand.

5.1. Modelling Demand

Our demand model is based on the assumption that low- and high-fare customers are drawn from disjoint populations and that all customers have a reservation price for the flight being sold. A low-fare customer will seek to book if and only if her reservation price $R_L \geq p_L$, and a high-fare customer will seek to book if and only if her reservation price $R_H \geq p_H$. Initially, we assume that a low-fare customer's decision to seek a booking is based only on her reservation price R_L and the fare p_L, but not on the recall price p. We relax this assumption in §5.3.

We assume that low-fare and high-fare customers arrive according to Poisson processes with rates $\lambda_{L,0}$ and $\lambda_{H,0}$, respectively. Then, D_L is Poisson with mean $\lambda_L := \lambda_{L,0}P\{R_L \geq p_L\}$, and D_H is Poisson with parameter $\lambda_H := \lambda_{H,0}P\{R_H \geq p_H\}$.

We consider two possibilities. In the first case (called the *Poisson case*), the parameters $\lambda_{L,0}$ and $\lambda_{H,0}$ are known with certainty (i.e., they are fixed constants). In the second case, the parameters $\lambda_{L,0}$ and $\lambda_{H,0}$ are random. It is well known (see p. 204 in Johnson et al. 1992) that if the parameter λ of a Poisson random variable has a gamma distribution with density

$$f(\lambda) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda/\beta}, \quad \lambda > 0,\ \alpha > 0,\ \beta > 0,$$

then the resulting random variable is negative binomial with

$$P(X = x) = \binom{\alpha + x - 1}{\alpha - 1} \left(\frac{\beta}{\beta+1}\right)^x \left(\frac{1}{1+\beta}\right)^\alpha, \quad x = 0, 1, 2, \ldots.$$

This is the second case (called the *negative binomial case*). The negative binomial case can also arise if customers arrive according to Poisson processes, each customer may purchase more than one ticket, and the number of tickets purchased follows a logarithmic distribution (see Johnson et al. 1992).

5.2. The Participation Function g

Each first-period (low-fare) customer decides whether or not to purchase by comparing the low fare to his reservation price. Once he has decided to purchase, he then decides whether or not to grant the call option by comparing his reservation price to the recall price. Specifically, a first-period customer will purchase if $R_L \geq p_L$ and will grant the call option if $p > R_L \geq p_L$.[12] In this case,

$$g(p) = P\{R_L < p \mid R_L \geq p_L\} = 1 - P\{R_L \geq p\}/P\{R_L \geq p_L\}.$$

We extend this simple consumer-choice model in §5.3 to the more realistic case in which potential low-fare customers use both the low fare p_L and the recall price p to determine whether or not they will purchase. We also note that, in practice, the participation function will most likely not be derived from a structural model but estimated directly from historical data.

First, we consider the case when the reservation price is uniformly distributed. For R_L uniformly distributed in the interval $[a, b]$ with $p_L \in [a, b)$, we have $g(p) = 1 - (b - p)/(b - p_L)$ for $p_L \leq p \leq b$, resulting in $p = h(q) = b - (1 - q)(b - p_L)$ for $0 \leq q \leq 1$, which is linearly increasing and hence convex in q.

For all the examples, we set the capacity of the plane $c = 100$. Suppose that R_L is uniformly distributed in the interval [\$0, \$300] and that $p_L = \$150$. Then, $h(q) = 300 - 150(1 - q) = 150 + 150q$ for $0 \leq q \leq 1$, $h'(q) = 150$ and $p = h(q) = 150 + 150q$. In this case, $\bar{q}_H = 1$, and the range of q is $q \in [0, 1]$.

Table 1 shows the results with and without callable products for various settings of p_H, λ_{H0}, and λ_{L0} for both the Poisson and negative binomial cases, where R_H is uniformly distributed in the interval [\$0, \$600] when $p_H \in \{\$400, \$500, \$550\}$ and in the interval [\$1,000, \$1,300] when $p_H = \$1,100$. The expected revenue increase from offering callable products ranges from 0.25% to 2.81% with an average of 2.15% in the Poisson case and from 2.85% to 10.12% with an average of 6.98% in the negative binomial case. The increase is especially pronounced when the difference between low and high fares is significant, when high-fare demand is uncertain (the negative binomial case), and when expected demand exceeds capacity. In the case of very high demand, the increase due to the callable products dips because the optimal allocation to low-fare bookings is small.

We want to ensure that the substantial gain from callables in Table 1 is not due to the fact that p_L and p_H were arbitrarily chosen. To reduce this risk, we use a fluid limit heuristic to calculate p_L and p_H (see Gallego and van Ryzin 1994):

$$\begin{aligned} \max_{p_L, p_H} \ & \lambda_L p_L + \lambda_H p_H = \lambda_{L0} P\{R_L \geq p_L\} p_L \\ & \qquad + \lambda_{H0} P\{R_H \geq p_H\} p_H \\ \text{subject to } \ & \lambda_{L0} P\{R_L \geq p_L\} + \lambda_{H0} P\{R_H \geq p_H\} \leq c. \end{aligned} \tag{12}$$

Table 2 shows the results using p_L and p_H calculated according to (12) for both uniform and exponential reservation prices. Note that this significantly

[12] In general, the customer's decision to grant the call option may include a calculation for the potential cost related to giving back the product. More precisely, the customer will buy the low-fare ticket if $R_L \geq p_L$, but will only grant the call option if $p > R_L + c$ for some cost parameter c. However, if upon calling back the customer is given an alternative service (such as being rescheduled on an alternative flight), the cost c may be quite small. Furthermore, the impact of this additional cost c may be diminished in the case of demand induction, where more customers participate in granting the call option.

Table 1 Comparison of Low-Fare Booking Limits and Expected Revenues With and Without Callable Products

				Optimal low-fare seats				Expected revenue		
p_H ($)	λ_{H0}	λ_{L0}	Case	Base	Call.	p^*	q^*	Base	Call.	Ex. rev. increase (%)
400	80	160	P	72	78	178.70	0.191	20,815.2	21,263.6	2.15
			NB	71	83	210.72	0.405	18,915.8	19,982.4	5.64
	100	200	P	65	71	182.58	0.217	22,452.1	23,027.9	2.57
			NB	64	76	215.88	0.439	20,379.6	21,730.6	6.63
	150	300	P	48	54	193.49	0.290	26,421.4	27,036.2	2.33
			NB	46	59	229.81	0.532	23,872.5	25,169.2	5.43
500	80	160	P	85	90	171.81	0.145	18,273.2	18,424.9	0.83
			NB	84	94	206.06	0.374	16,865.1	17,622.2	4.49
	100	200	P	81	87	174.68	0.165	20,102.8	20,637.5	2.66
			NB	79	91	208.56	0.390	18,314.3	19,736.1	7.76
	150	300	P	73	79	179.85	0.199	22,864.8	23,506.6	2.81
			NB	69	84	217.61	0.451	20,705.0	22,422.5	8.30
550	80	160	P	92	96	167.48	0.117	15,569.8	15,609.3	0.25
			NB	92	99	199.81	0.332	14,754.8	15,175.7	2.85
	100	200	P	90	95	169.70	0.131	17,710.0	18,078.4	2.08
			NB	89	98	200.09	0.334	16,300.6	17,370.7	6.57
	150	300	P	85	91	172.67	0.151	19,334.9	19,863.7	2.74
			NB	84	95	205.40	0.369	17,702.0	19,276.1	8.89
1,100	50	200	P	60	70	191.77	0.278	45,236.8	46,315.8	2.39
			NB	46	73	259.67	0.731	41,301.4	44,578.2	7.93
	20	200	P	83	90	176.41	0.176	26,734.7	27,456.7	2.70
			NB	74	93	224.35	0.496	24,080.2	26,516.9	10.12
	10	200	P	90	96	170.66	0.138	20,570.7	21,034.5	2.26
			NB	85	99	211.43	0.410	18,619.6	20,322.1	9.14

Notes. Reservation prices for the low fare are uniformly distributed $U[\$0, \$300]$ and $p_L = \$150$. "P" refers to the Poisson case and "NB" to the negative binomial case. The negative binomial case has $\beta = 2$ and $\beta = 10$ for the low-fare and high-fare customers and α's are chosen to match the mean of the Poisson case. "Base" refers to results without callable products and "Call." to results with. Total capacity $c = 100$. p^* is the optimal recall price and $q^* = g(p^*)$ is the corresponding participation probability. The high fare is $U[\$0, \$600]$ for $p_H \in \{\$400, \$500, \$550\}$ and $U[\$1{,}000, \$1{,}300]$ when $p_H = \$1{,}100$. The last column is the expected percentage revenue increase from offering callables.

improves revenue for all choices of λ_{L0} and λ_{H0}. Nevertheless, the expected revenue increase from offering callable products remains high. For the exponential distribution, we assume that R_L and R_H have means 150 and 300, respectively. Then, $g(p) = 1 - e^{(p_L - p)/150} = q$, and

$$p = h(q) = p_L - 150 \log(1-q), \qquad h'(q) = p_L + \frac{150}{1-q}, \qquad 0 \le q \le q_{\max},$$

where $q_{\max} = 1 - e^{(p_L - p_H)/150}$. Note also that $\lambda_L = \lambda_{L0} e^{-p_L/150}$ and $\lambda_H = \lambda_{H0} e^{-p_H/300}$. Note that the expected revenue is lower when reservation prices are exponentially distributed. This is due to the fact that the exponential distribution results in more probability being associated with lower reservation prices than the uniform distribution. However, the incremental benefits from offering callable products are still significant.

Our analysis so far has assumed that low-fare customers make their decisions to book independently of the existence of the callable product and its recall price. That is, we have assumed that a low-fare customer will seek to book if and only if his reservation price R_L is greater than or equal to the low fare, p_L. However, the option to purchase the callable product may also induce new customers who would not seek to book if the callable product was not available.

To see this, consider a low-fare customer who has a subjective probability s that his unit will be called if he purchases a callable product. His expected surplus will be $R_L - p_L$ if he purchases the standard product and $s(p - p_L) + (1-s)(R_L - p_L)$ if he purchases a callable product. This quantity is nonnegative as long as $R_L \ge p_L - (p - p_L)s/(1-s)$. Assuming that all customers are risk neutral, demand is induced from customers with $R_L \in [p_L - (p - p_L)s/(1-s), p_L)$, and the probability that a buyer will agree to a call given that he makes a purchase has to be adjusted accordingly. Note that the induced customers will all purchase the callable product. The model without demand induction corresponds to the pessimistic prior $s = 0$.

Table 2 Comparison of Results With and Without Callable Products for Uniformly Distributed (Low Fare $U[0, 300]$ and High Fare $U[0, 600]$) and Exponential Distributed Reservation Prices, When p_L and p_H Are Chosen According to (12)

	Case	Optimal low-fare seats Base	Call.	p^*	q^*	Expected revenue Base	Call.	Ex. rev. increase (%)
Uniform distribution								
$p_L = \$150$ $\lambda_{H0} = 60, \lambda_{L0} = 120$	P	70	74	\$174.54	0.164	\$17,798.7	\$17,865.6	0.38
$p_H = \$300$ $(\lambda_H = 30, \lambda_L = 60)$	NB	73	81	199.09	0.327	16,525.2	16,866.4	2.07
$p_L = 180$ $\lambda_{H0} = 80, \lambda_{L0} = 160$	P	65	70	205.15	0.210	22,304.0	22,577.6	1.23
$p_H = 330$ $(\lambda_H = 36, \lambda_L = 64)$	NB	70	78	227.20	0.393	20,446.2	21,071.2	3.06
$p_L = 210$ $\lambda_{H0} = 100, \lambda_{L0} = 200$	P	61	67	231.54	0.249	25,747.1	26,066.3	1.24
$p_H = 360$ $(\lambda_H = 40, \lambda_L = 60)$	NB	67	77	253.49	0.483	23,617.3	24,360.6	3.15
$p_L = 250$ $\lambda_{H0} = 150, \lambda_{L0} = 300$	P	52	59	267.32	0.346	31,026.5	31,424.6	1.28
$p_H = 400$ $(\lambda_H = 50, \lambda_L = 50)$	NB	60	74	286.89	0.738	28,406.6	29,373.7	3.40
Exponential distribution								
$p_L = \$150$ $\lambda_{H0} = 60, \lambda_{L0} = 120$	P	78	81	\$164.95	0.095	\$13,243.63	\$13,243.64	0.00
$p_H = 300$ $(\lambda_H = 22, \lambda_L = 44)$	NB	81	84	185.69	0.212	13,024.9	13,073.2	0.37
$p_L = 150$ $\lambda_{H0} = 80, \lambda_{L0} = 160$	P	71	75	170.35	0.127	17,517.5	17,564.3	0.27
$p_H = 300$ $(\lambda_H = 29, \lambda_L = 59)$	NB	74	80	186.28	0.215	16,335.1	16,609.0	1.68
$p_L = 168$ $\lambda_{H0} = 100, \lambda_{L0} = 200$	P	66	70	191.01	0.142	20,958.3	21,191.5	1.11
$p_H = 318$ $(\lambda_H = 35, \lambda_L = 65)$	NB	70	76	204.40	0.215	19,203.4	19,671.8	2.44
$p_L = 243$ $\lambda_{H0} = 150, \lambda_{L0} = 300$	P	62	65	266.12	0.143	28,957.5	29,222.7	0.92
$p_H = 393$ $(\lambda_H = 41, \lambda_L = 59)$	NB	69	74	276.83	0.202	26,590.9	27,097.8	1.91

Notes. The negative binomial case (NB) has $\beta = 2$ and $\beta = 10$ for low-fare and high-fare customers, respectively, and α's are chosen to match the mean of the Poisson case (P). "Base" refers to results without callable products and "Call." to results with callable products. Total capacity $c = 100$.

5.3. Demand Induction

Demand induction has two possible impacts: it may increase the arrival rate for the low-fare demand D_L, and it may increase the participation probability. For example, in the case of Poisson demand, with $s = 6\%$, $a = 0, b = \$300$ for the low fare and $a = 0, b = \$600$ for the high fare, the expected revenue for the case $\lambda_{L0} = 200$ and $\lambda_{H0} = 100$ (with an optimal choice of p_L and p_H) is \$26,131. This number was obtained by simulation using the booking limit and the recall price that were found optimal for the case without demand induction. This represents an increment of about \$65 relative to the case without demand induction. Of the \$65 increase, \$6 is due to the increase in the participation function and the rest is due to the increase in D_L. After optimizing over a and p using Crystal Ball, we obtained an estimated expected revenue equal to \$26,154 with $a = 69$ and $p = \$241.54$.

We note that the numerical results in this section have assumed that potential customers are risk neutral. Everything else being equal, we would expect that the benefits would be somewhat lower in the case in which some or all customers showed some degree of risk aversion. In particular, the effect of risk aversion is to reduce the value of $g(p)$ for each value of p. However, the same basic analysis would be used to calculate the p^*, a, and expected revenue. In practice, we anticipate that the function $g(p)$ would be empirically derived from historical experience, just as airlines forecast demand for various booking classes as described in Chapter 9 of Talluri and van Ryzin (2004).

6. Discussion and Extensions

We have introduced the concept of callable products and have shown how they can generate riskless revenue improvement in the case of a simple two-period revenue management setting. There are a number of ways that this model simplifies reality. Real-world revenue management involves multiple fare classes offered through multiple channels over many time periods, often across a network. Extension to multiple time periods and multiple fares raises a number of important design issues and potential analytical complications. However, we believe that they would not alter the fundamental insight—that offering callable products to early-booking low-fare customers would increase expected revenue when there are also later booking high-fare passengers.[13] We have presented our analysis of the callable products in the traditional revenue management model in which low-fare passengers book prior to high-fare

[13] Of course, calculation of the booking limits for all fares would need to be modified in the presence of callable products. It is unlikely that a closed-form solution for the general problem of setting booking limits in the presence of callable products will be possible—rather, we would expect that airlines would use modifications of existing booking limit heuristics such as EMSR-a and EMSR-b (see Talluri and van Ryzin 2004 for a discussion of these heuristics).

passengers. We should note that this assumption simplifies analysis and provides easy comparison with standard results, such as Littlewood's rule. However, for callable products to improve revenue, it is not critical that all low-fare demand books before high-fare demand; it is only important that *some* low-fare demand books before *some* high-fare demand.

6.1. The Timing Issue

A more serious issue involves timing. We have assumed in this analysis that the airline can observe all high-fare demand before determining the number of calls to issues. In reality, the calls would need to be issued some time before departure (say 24 hours). There is then the probability of additional high-fare demand materializing after the decision has been made on how many bookings to call. In fact, airlines experience significant walk-up demand (sometimes called "go-shows") that does not appear until just prior to departure. With the possibility of such late-booking high-fare demand, the decision of how many calls to issue is no longer trivial. However, even in this case, offering callable products can provide additional revenue.

To see this, consider the case in which the high-fare booking period is divided into two subperiods. Calls need to be issued at the end of the first subperiod when some, but not all, of the high-fare bookings have arrived. Specifically, let D_{H1} be demand during the first subperiod and D_{H2} be demand during the second subperiod. It is easy to see that during the first subperiod, it is optimal for the airline to first sell inventory until it is exhausted and then exercise calls until they are exhausted. Let s be the number of seats and let z be the number of calls remaining at the end of the first subperiod, after observing D_{H1} but before observing D_{H2}. The problem is to decide how many, if any, of the calls to exercise *before* the beginning of the second subperiod. We can write this problem as $\pi_2(s,z) = \min_{0\le x\le z}[p_H E\min(D_{H2}, s+x) - px]$. The expected marginal value of exercising the $(x+1)$st option is $p_H Pr\{D_{H2} > s+x\} - p$. Conceptually, we could find the optimal value x^* by starting at zero and increasing x by one until we find the smallest integer at which increasing to $x+1$ would lead to a decrease in value. It follows that $x^* = \min[z, (y^*-s)^+]$, where y^* is the smallest integer such that $p_H P\{D_{H2} > y\} < p$.

We have assumed so far that the demand in the two subperiods is independent. In fact, we would anticipate that the high-fare demand observed in the first subperiod would be highly correlated with the high-fare demand in the second subperiod. In this case, $y^*(d_{H1})$ would be the smallest integer such that $p_H P\{D_{H2} > y \mid D_{H1} = d_{h1}\} < p$ and $x^*(d_{H1})$ would be modified accordingly. The problem of setting a booking limit and an optimal recall price becomes

$$\max_{a,p} E\big[R(a) + W(a,p) + \pi_2((c - S_L(a) - D_{H1})^+, \\ (V_L(a) - (S_L(a) + D_{H1} - c)^+)^+)\big],$$

where now $R(a)$ and $W(a,p)$ are defined relative to the low-fare demand and the high-fare demand during the first subperiod. Gallego and Lee (2004) have developed this extension allowing for cancellations and no-shows. They show that selling callable products is an effective revenue enhancing and overbooking reduction tool even when calls have to be made before departure.

6.2. Hedging Against Cancellations and No-Shows

We have presented the concept of callable tickets as a hedge against high-fare demand uncertainty. However, even in industries in which late-booking demand does not pay a higher fare, there is the opportunity to use callable products to hedge against no-shows and late cancellations. Optimal calculation of total booking limits in the face of cancellations and no-shows has been estimated to increase revenue at American Airlines by 8% or more (Smith et al. 1992). A typical approach to setting total booking levels is for an airline to estimate a denied boarding cost B and determine the level of total bookings at which the marginal expected denied boarding cost of accepting an additional booking is equal to the expected increase in fare revenue from the additional booking (see Phillips 2005 for a discussion of different overbooking approaches). In this spirit, we consider overbooking in the following simple model. An airline offers seats at a single fare p_L. The ticket price is entirely refundable, so no-shows pay no penalty. Each "show" pays the fare but the airline must pay $B > p_L$ to each denied boarding. Let the random variable $T(y(a))$ denote the number of shows given y bookings at departure when the airline has set a total booking limit a. Then, the net revenue that the airline will receive is $R(a) = p_L T(y(a)) - B(T(y(a)) - c)^+$, where D is demand and $y(a) = \min(D, a)$ is the number of bookings at departure given the booking limit a.

Now assume that the airline has sold $V(a)$ units of callable product with recall price p with $B > p > p_L$, and the airline is able to observe shows before deciding which customers to call. In this case, the airline would call $x = \min(T(y(a) - c)^+, V(a))$ units and revenue would be $R'(a) = p_L T(y(a)) - px - B(T(y(a)) - x - c)^+$. Because $R'(a) \ge R(a)$ for all values of $T(y(a))$, this approach would generate a riskless savings to the airline. However, it is more realistic that the airline would need to exercise the calls some time before departure. In this case, the airline would not know whether or not the passenger being called would actually show and the revenue to the airline would be $\hat{R}(a) = p_L T(y(a)) - px - BE[T(y(a) - x) - c]^+$. The airline may have the opportunity to improve revenue by calling some bookings, however, the revenue is not riskless because products need to be called prior to observing actual shows. Furthermore, using the call option in this fashion raises the issue of adverse selection—customers who choose the callable products may tend to be those who would be most likely to no-show. This means that the recall price, the total booking limit, and the number of products called would need to be jointly optimized to maximize revenue. Callables would also provide revenue benefits to the seller in the case where there is a chance that a flight (or other event) might be canceled. In this case, the seller would only need pay a cancellation penalty of p to holders of a callable product but a higher penalty of B to ordinary ticket holders.

6.3. Pure Callable Products

We have considered the case where low-fare customers decide whether or not to grant the call option to the capacity provider. We can also consider the case where the capacity provider offers a pure callable product in which the callable option is an intrinsic part of the product. If the provider is selling both standard and pure callable products at the same price, then from the customer's point of view the situation is the same as the model we have already analyzed.

Alternatively, the provider may decide to discount the pure callable product relative to the standard low-fare product. The following proposition shows that, under certain conditions, the provider can achieve the same expected revenue using a single low-fare product with a callable option.

Proposition 6. *Assume that a provider is offering (noncallable) discount product at a fare p_L along with pure callable product at a price $p_C < p_L$ and a call price p. If first-period buyers are risk neutral and share a common probability q that their product will be called where $q > 0$ is equal to the fraction of callable products that the airline will call, then the airline can achieve the same revenue by selling a single discount product at p_L and offering a free callable option with recall price $\hat{p} = p + (p_L - p_C)/q$.*

Proof. See the online supplement. □

The equivalence principal in Proposition 6 holds for any distribution of first-period customer willingness-to-pay. However, it does not necessarily hold if customers vary in their levels of risk aversion and/or their ex ante estimates of the probability that their product will be called. Any one of these variations could motivate the provider to offer one or more flavors of callable product. In addition, offering pure callable products would allow the supplier to, at any point, choose to sell only callable products. In this case, a potential customer will find that he can only purchase at the low fare if he grants the callable option. This could discourage speculators from buying tickets for shows and sporting events with the anticipation of selling them later at a higher price. Using an intensity control model, Feng and Gallego (2004) show that it is never optimal to close the callable product before closing the standard product (without the call option). They also investigate the problem of dynamically selecting the recall price as a function of the number of available calls.

6.4. Callable Products with Reaccommodation

Callable products can be generalized to include passenger reaccommodation when an airline offers several different routes between the same origin and destination. Currently airlines manually reschedule the flights of customers that are bumped at the airport and of customers whose travel plans are changed by the replane mechanism. The same manual mechanism could be used to reschedule passengers whose seats are called, but we envision the development of more sophisticated reservation systems that would do this automatically. The strategy of selling callable products can be complemented by the introduction of flexible products in Gallego et al. (2004). While callable products pay only if the call option is exercised, flexible products offer an up-front discount to customers willing to grant the provider the option of assigning them, at a later date, to a specific product from a known prespecified set. As an example, a customer could buy a morning flight from New York to San Francisco at a discount with the airline assigning the customer to an actual flight at some later time. In a competitive environment it is also possible, maybe even desirable, to combine the features of flexible and callable products. One possibility would be to give a small up-front discount at the time of sale, as in flexibles, and an additional discount, as in callables, if the option of reallocating the customer is exercised. Another option is to sell a flexible product as a callable product that can be recalled at a preagreed price. Gallego et al. (2004) provide a network fluid model of callable products that include elements of flexible products.

6.5. Further Application

This paper has focused on the application of callable products on a single leg in the case of two fare classes. However, the basic concept can be extended in a number of ways. One extension would be to offer multiple callable products on a leg with different conditions, strike prices, etc. While this might be of theoretical interest, we suspect that it might prove consuming in practice as well as highly complex to implement. Furthermore, some recent research has suggested that offering customers too many complex choices can be demotivating and actually lead to lower overall sales (see Iyengar and Lepper 2000). Another extension would be to apply the concept to a network of flight legs—Gallego et al. (2004) provide some results on using a column generation approach to choose the optimal mix of callable and flexible products to offer on a network. Also, as noted by Biyalogorsky et al. (2005), a similar approach could be used for upgrading—for example, selling a firm a coach ticket that includes an option to be upgraded to first class.

Although we have focused on airlines as the primary example, the analysis is applicable to any company selling constrained and perishable capacity in which some customers booking later tend to pay more than some of those booking earlier. Such companies include any of those in the traditional revenue management industries such as hotels, rental cars, cruise lines, freight carriers, and tour operators. We believe that callable products are also potentially applicable for tickets to sporting events, concerts, or shows. Secondary markets such as ebay.com and stubhub.com are currently used as a forum in which tickets for popular events are sold at prices that are often well above face value. Offering callable tickets in such markets would allow a seller to keep list price constant but call some tickets if the event proves more popular than expected.

The examples that we have discussed so far have involved consumer markets. However, we believe that there are also business-to-business selling situations in which low-margin demand tends to arrive early and there is supply uncertainty. In these cases, a call option could provide a useful addition to the various business-to-business contract types surveyed in Kleindorfer and Wu (2003). The concept of a callable product could also apply in electric-power markets. Currently, "interruptible" contracts provide the supplier the right to interrupt service to certain customers in cases of very high demand in return for compensation. The callable alternative would be to pay willing customers a preagreed amount whenever service is interrupted.

Finally, we note that callable products could be communicated and implemented in a number of different ways. In some industries, it might be most appropriate to implement "short-selling" as discussed in Biyalogorsky et al. (2000). A similar approach was taken by a now defunct company, iDerive.com, that enabled sellers to purchase "put options" from consumers (*Businesswire* 2000). A seller would make a small payment to a willing customer prior to the release of a new product. If demand for the product turned out to be lower than anticipated, then the seller could exercise the put—in which case the customer would be obliged to buy the product at a discount. If the product proved to be more popular than anticipated, the seller could choose to sell the product at list price to another customer and the holder of the "put" would keep the payment. This provided a way for companies to hedge against the demand uncertainty intrinsic in introducing a high-risk new product such as a game player or other consumer electronic product.

Ultimately, the form in which callable products would be communicated and managed would depend on the industry and the setting. While there are obvious challenges, the benefits could be substantial. This is particularly true for airlines in the current environment in which load factors are extremely high and, for many carriers, the difference between full-fare and discount tickets is at record levels. This is the perfect environment for callable products, and several airlines are currently investigating ways to implement the concept.

7. Electronic Companion

An electronic companion to this paper is available as part of the online version that can be found at http://mansci.journal.informs.org/.

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