NONCONFORMING FINITE ELEMENT METHODS ON QUADRILATERAL MESHES

JUN HU AND SHANGYOU ZHANG

ABSTRACT. It is well-known that it is comparatively difficult to design nonconforming finite elements on quadrilateral meshes by using Gauss-Legendre points on each edge of triangulations. One reason lies in that these degrees of freedom associated to these Gauss-Legendre points are not all linearly independent for usual expected polynomial spaces, which explains why only several lower order nonconforming quadrilateral finite elements can be found in literature. The present paper proposes two families of nonconforming finite elements of any odd order and one family of nonconforming finite elements of any even order on quadrilateral meshes. Degrees of freedom are given for these elements, which are proved to be well-defined for their corresponding shape function spaces in a unifying way. These elements generalize three lower order nonconforming finite elements on quadrilaterals to any order. In addition, these nonconforming finite element spaces are shown to be full spaces which is somehow not discussed for nonconforming finite elements in literature before.

1. INTRODUCTION

Because of their flexibility and stability when compared with conforming finite element methods, nonconforming finite element methods have become very important and effective discretization methods for numerically solving, among others, high order elliptic problems, Stokes-like problems and Reissner-Mindlin plate bending problems.

Quadrilateral meshes are very important in scientific and engineering computing. Indeed, many popular softwares for computations of the fluid mechanics in two dimensions are defined on quadrilateral meshes. However, most of nonconforming finite element methods for second order problems are defined on triangles [1, 3, 4, 5, 7] while there are only a few nonconforming finite element methods on quadrilaterals.

Compared with nonconforming triangular finite elements, it is more difficult to construct nonconforming quadrilateral finite elements. In fact, a sufficient condition for convergence of consistency error terms is to require nonconforming functions to be continuous at Gauss-Legendre points on interior edges of the triangulation used [18, 19, 20, 21]. Hence, for m order nonconforming finite elements, there are $4m$

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Gauss-Legendre points on the boundary of each element. However, these degrees of freedom on $4m$ Gauss-Legendre points are not all linearly independent for the space of polynomials whose restrictions on four edges are polynomials of degree $\leq m$, see (2.1) for details. This means that at least one relation holds, which motivates the use of higher order monomials of one variable in shape function spaces. In such a spirit, a class of nonconforming quadrilateral elements is proposed in literature, which includes, the Han element [8], the nonconforming rotated Q_1 element due to Rannacher and Turek [16], the Douglas-Santos-Sheen-Ye (DSSY) element [6], the enriched nonconforming rotated Q_1 element due to Lin, Tobiska and Zhou [13]. All of these nonconforming quadrilateral finite element methods are of first order and are stable for the Stokes problem with pure Dirichlet boundary conditions for the velocity. However, since the usual Korn inequality does not hold for them, they can not be applied to the Stokes problem with mixed boundary conditions for the velocity. Moreover, these nonconforming elements are somehow not extended to higher order nonconforming elements in literature so far .

In [15], a nonconforming linear element is introduced on quadrilateral meshes, which is motivated by a key observation that any linear function on a quadrilateral can be uniquely determined at any three of the four midpoints of edges, which leads to a set of nodal basis functions whose values at four edges satisfies the aforementioned relation. A similar element (but different on general quadrilateral meshes) is designed in [9], which is based on an observation that a frame of the linear function space on the reference element can be mapped and glued to form a basis of the interpolated space of the conforming bilinear element space by the canonical interpolation operator of the nonconforming rotated Q_1 element [16]. This explains why the element therein is named as "constrained quadrilateral nonconforming rotated Q_1 element" by its authors. Very recently, the ideas of [9, 15] are extended to design a nonconforming cubic element on quadrilaterals in [14]. We refer interested readers to [10, 12] for the nonconforming quadratic element on quadrilaterals, which can be regarded as the quadrilateral counterpart of the triangular Fortin-Soulie element [7].

The purpose paper is to propose nonconforming finite elements of any order on quadrilaterals in a unifying way, which is somehow missed in literature. Compared with nonconforming finite elements on triangles [1, 3, 4, 5, 7], there are at least two more difficulties on quadrilaterals: (1) what shape function spaces should be used for nonconforming finite elements of any order; (2) how to prove unisolvency of these degrees of freedom which consist of values of polynomials at aforementioned $4m$ Gauss-Legendre points and other degrees of freedom in the interior of elements. It should be stressed that nonconforming finite elements on quadrilaterals in literature are defined and analyzed one by one, see [6, 8, 9, 10, 12, 13, 14, 15, 16].

In order to overcome the first difficulty, we propose to use the following two families of shape function spaces

$$R_m(\hat{K}) := P_m(\hat{K}) + \text{span}\{\hat{x}^m\hat{y} - \hat{x}\hat{y}^m\}, \tag{1.1}$$

$$ER_m(\hat{K}) := P_m(\hat{K}) + \text{span}\{\hat{x}^m\hat{y} - \hat{x}\hat{y}^m, \hat{x}^{m+1} - \hat{y}^{m+1}\}, \tag{1.2}$$

for odd integer $m \geq 1$. Here and throughout this paper, $P_m(M)$ denotes the space of polynomials of degree $\leq m$ over the domain M; $Q_m(M)$ denotes the space of polynomials of degree $\leq m$ in each variable. For even m, we propose to use the same shape function spaces as the serendipity elements of [2, 4], namely,

$$R_m^+(\hat{K}) := P_m(\hat{K}) + \text{span}\{\hat{x}^m\hat{y}, \hat{x}\hat{y}^m\}. \tag{1.3}$$

These three families of elements generalize the P_1 nonconforming element of [9, 15] and the nonconforming cubic element of [14], the nonconforming rotated Q_1 element of [16], and the nonconforming quadratic element of [10, 12] to any order, respectively.

Degrees of freedom are given for these shape function spaces, which are proved to be well defined based on some observation concerning Legendre polynomials on four edges. In addition, these finite element spaces are proved to be full spaces.

The rest of the paper is organized as follows. Sections 2, 3 and 4 present three families of shape function spaces and their corresponding degrees of freedom, which are proved to be well-defined. Section 5 defines nonconforming finite element spaces which are proved to be full spaces, and shows approximations of these spaces. Section 6 analyzes consistency errors, which is followed by numerical examples in the final section.

2. The first family of Shape function spaces

Since Gauss-Legendre points on each edge will be used to define degrees of freedom and consequently continuity for new elements under consideration, given an integer $k \geq 0$, let g_i, $i = -k, \cdots, -1, 0, 1, \cdots, k$, denote zeros of Legendre polynomials of degrees $2k+1$ on the interval $[-1, 1]$. By skew symmetry, $g_i = -g_{-i}$, $i = 1, \cdots, k$, and $g_0 = 0$.

Before we present shape function spaces on $\hat{K} := [-1, 1]^2$ and corresponding degrees of freedom, we investigate a special relation of values at $8k + 4$ Gauss-Legendre points $G_{r,i} = (1, g_i)$, $G_{l,i} = (-1, g_i)$, $G_{t,i} = (g_i, 1)$, $G_{b,i} = (g_i, -1)$, $i = -k, \cdots, k$ on four edges of $\hat{K}$, of polynomials over $\hat{K}$ whose restrictions on four edges are polynomials of degree $\leq 2k + 1$.

Lemma 2.1. *Let $\hat{v}$ be a polynomial over $\hat{K}$ such that its restrictions on four edges are polynomials of degree $\leq 2k+1$. Then it holds that*

$$\sum_{i=-k}^{k} \gamma_i \Big(\hat{v}(1, g_i) + \hat{v}(-1, g_i) \Big) = \sum_{i=-k}^{k} \gamma_i \Big(\hat{v}(g_i, 1) + \hat{v}(g_i, -1) \Big), \tag{2.1}$$

where

$$\gamma_i = \prod_{i \neq j = -(k+1)}^{k} \frac{1 - g_j}{g_i - g_j} + \prod_{i \neq j = -k}^{k+1} \frac{-1 - g_j}{g_i - g_j} = \frac{2}{g_i^2} \prod_{|i| \neq j = 1}^{k} \frac{1 - g_j^2}{g_i^2 - g_j^2}, \tag{2.2}$$

for $i = -k, \cdots, -1, 1, \cdots, k$*, and*

$$\gamma_0 = 4 \prod_{j=1}^{k} \frac{g_j^2 - 1}{g_j^2}. \tag{2.3}$$

Remark 2.2. *For k=0, the relation* (2.1) *is the constraint used in* [9, 15]*; for k=1, the relation* (2.1) *recovers that of* [14].

Proof. The main idea is: Since the restrictions of $\hat{v}$ on four edges are polynomials of degree $\leq 2k+1$, they can be exactly represented by $2k+2$ Lagrange interpolation basis functions of degree $\leq 2k+1$ in one dimension, which gives two representations of values of $\hat{v}$ at each corner of $\hat{K}$, then the desired result follows. In order to accomplish this, first add the point $g_{-k-1} = -1$ to these $2k+1$ Gauss-Legendre points, and define a first set of Lagrange interpolation basis functions:

$$\mathcal{L}_i^0(\hat{x}) = \prod_{i \neq j = -(k+1)}^{k} \frac{\hat{x} - g_j}{g_i - g_j} \text{ for } \hat{x} \in [-1, 1], i = -(k+1), -k, \cdots, k.$$

Then, add the point $g_{k+1} = 1$ to these $2k+1$ Gauss-Legendre points, and define another set of Lagrange interpolation basis functions:

$$\mathcal{L}_i^1(\hat{x}) = \prod_{i \neq j = -k}^{k+1} \frac{\hat{x} - g_j}{g_i - g_j} \text{ for } \hat{x} \in [-1, 1], i = -k, \cdots, k+1.$$

Since $\text{span}\{\mathcal{L}^0_{-(k+1)}(\hat{x}), \cdots, \mathcal{L}^0_k(\hat{x})\} = \text{span}\{\mathcal{L}^1_{-k}(\hat{x}), \cdots, \mathcal{L}^1_{k+1}(\hat{x})\} = P_{2k+1}([-1, 1])$, and the restriction of $\hat{v}$ on each edge of $\hat{K}$ is a polynomial of degree $\leq 2k+1$, these restrictions can be expressed as

$$\begin{aligned} \hat{v}(\pm 1, \hat{y}) &= \sum_{i=-(k+1)}^{k} \mathcal{L}_i^0(\hat{y}) \hat{v}(\pm 1, g_i), \quad & \hat{v}(\hat{x}, \pm 1) &= \sum_{i=-(k+1)}^{k} \mathcal{L}_i^0(\hat{x}) \hat{v}(g_i, \pm 1), \\ \hat{v}(\pm 1, \hat{y}) &= \sum_{i=-k}^{k+1} \mathcal{L}_i^1(\hat{y}) \hat{v}(\pm 1, g_i), & \hat{v}(\hat{x}, \pm 1) &= \sum_{i=-k}^{k+1} \mathcal{L}_i^1(\hat{x}) \hat{v}(g_i, \pm 1). \end{aligned} \tag{2.4}$$

Let $\alpha_i = \mathcal{L}_i^0(1)$, $i = -(k+1), \cdots, k$, and $\beta_i = \mathcal{L}_i^1(-1)$, $i = -k, \cdots, k+1$. The skew symmetry of Gauss-Legendre points, and definitions of $\alpha_{-(k+1)}$ and β_{k+1}, yield

$$\alpha_{-(k+1)} = \prod_{j=-k}^{k} \frac{1 - g_j}{-1 - g_j} = \prod_{j=-k}^{k} \frac{-1 - g_j}{1 - g_j} = \beta_{k+1}. \tag{2.5}$$

Since values $\hat{v}(1,1)$ (resp. $\hat{v}(-1,1)$, $\hat{v}(1,-1)$ and $\hat{v}(-1,-1)$) of two representations are identical, it follows from (2.4) that

$$\sum_{i=-(k+1)}^{k} \alpha_i \hat{v}(1,g_i) = \sum_{i=-(k+1)}^{k} \alpha_i \hat{v}(g_i,1), \quad \sum_{i=-k}^{k+1} \beta_i \hat{v}(1,g_i) = \sum_{i=-(k+1)}^{k} \alpha_i \hat{v}(g_i,-1),$$
$$\sum_{i=-(k+1)}^{k} \alpha_i \hat{v}(-1,g_i) = \sum_{i=-k}^{k+1} \beta_i \hat{v}(g_i,1), \quad \sum_{i=k}^{k+1} \beta_i \hat{v}(-1,g_i) = \sum_{i=-k}^{k+1} \beta_i \hat{v}(g_i,-1).$$

By (2.5), this gives

$$\sum_{i=-k}^{k} (\alpha_i+\beta_i)\big(\hat{v}(1,g_i)+\hat{v}(-1,g_i)\big) = \sum_{i=-k}^{k} (\alpha_i+\beta_i)\big(\hat{v}(g_i,1)+\hat{v}(g_i,-1)\big).$$

Then a direct calculation completes the proof. □

By canceling a common factor, (2.1) can be simplified slightly that

$$\sum_{i=-k}^{k} \frac{\gamma_i'}{\prod_{|i|\neq j=1}^{k}(g_j^2-g_i^2)} \big(\hat{v}(1,g_i)+\hat{v}(-1,g_i)-\hat{v}(g_i,1)-\hat{v}(g_i,-1)\big) = 0,$$

where

$$\gamma_i' = \begin{cases} 2 & \text{if } i=0, \\ 1/g_i^2 & \text{if } i\neq 0. \end{cases}$$

Given an odd integer $m = 2k+1 \geq 0$, recall shape function spaces:

$$R_m(\hat{K}) := P_m(\hat{K}) + \operatorname{span}\{\hat{x}^m\hat{y} - \hat{x}\hat{y}^m\} \text{ for any } (\hat{x},\hat{y}) \in \hat{K} := [-1,1]^2. \tag{2.6}$$

Note that for $k=0$, $R_m(\hat{K})$ is the shape function space of [9], i.e.,

$$R_1 = \operatorname{span}\{1, \hat{x}, \hat{y}\}.$$

For $k=1$, $R_m(\hat{K})$ is the shape function space of [14].

To define degrees of freedom for this space, let G denote the set of Gauss-Legendre points on four edges of $\hat{K}$, namely,

$$G := \{(1,g_i), (-1,g_i), (g_i,1), (g_i,-1), i=-k,\cdots,k\}. \tag{2.7}$$

To define other degrees of freedom in the interior of $\hat{K}$, let

$$I := \{(\hat{x}_\ell, \hat{y}_\ell), \ell = 1,\cdots,(2k-1)(k-1)\} \tag{2.8}$$

be a set of interior points of $\hat{K}$, the standard Lagrange points inside the reference element $\hat{K}$, so that any polynomial $\hat{q}(\hat{x},\hat{y}) \in P_{2k-3}(\hat{K})$ can be uniquely defined by its values at points in I.

Theorem 2.3. *If $\hat{v}(\hat{x},\hat{y}) \in R_m(\hat{K})$ vanishes at all points in $G \cup I$, cf.* (2.7) *and* (2.8)*, then $\hat{v}(\hat{x},\hat{y}) \equiv 0$.*

Proof. The function $\hat{v}(\hat{x}, \hat{y}) \in R_m(\hat{K})$ can be expressed as

$$\hat{v}(\hat{x}, \hat{y}) = a_0(\hat{x}^{2k+1}\hat{y} - \hat{x}\hat{y}^{2k+1}) + \sum_{i+j\le 2k+1} c_{i,j}\hat{x}^i\hat{y}^j. \tag{2.9}$$

As $\hat{v}(1, \hat{y}) = 0$ at $2k+1$ Gauss-Legendre points, $\hat{v}(1, \hat{y})$ is a multiple of the $(2k+1)$-st Legendre polynomial, $\hat{v}(1, \hat{y}) = c_0 L_{2k+1}(\hat{y})$ for some constant c_0. We will show that the constant c_0 is zero. In fact, by the continuity of $\hat{v}(\hat{x}, \hat{y})$ at four vertexes of the square,

$$\hat{v}(-1, \hat{y}) = -c_0 L_{2k+1}(\hat{y}) = -\hat{v}(1, \hat{y}) \text{ for } \hat{y} \in [-1, 1], \tag{2.10}$$

$$\hat{v}(\hat{x}, -1) = -c_0 L_{2k+1}(\hat{x}) = -\hat{v}(1, \hat{x}) \text{ for } \hat{x} \in [-1, 1]. \tag{2.11}$$

This indicates that

$$\begin{cases} c_{0,0} + c_{2,0} + c_{4,0} + c_{6,0} + \cdots + c_{2k,0} & = 0, \quad (\text{ for } j = 0) \\ c_{0,1} + c_{2,1} + c_{4,1} + c_{6,1} + \cdots + c_{2k,1} & = 0, \quad (\text{ for } j = 1) \\ c_{0,2} + c_{4,2} + c_{6,2} + c_{8,2} + \cdots + c_{2k-2,2} & = 0, \quad (\text{ for } j = 2) \\ \vdots & \\ c_{0,2k-2} + c_{2,2k-2} & = 0, \quad (\text{ for } j = 2k-2) \\ c_{0,2k-1} + c_{2,2k-1} & = 0, \quad (\text{ for } j = 2k-1) \\ c_{0,2k} & = 0, \quad (\text{ for } j = 2k) \\ c_{0,2k+1} & = 0. \quad (\text{ for } j = 2k+1) \end{cases}$$

$$\begin{cases} c_{0,0} + c_{0,2} + c_{0,4} + c_{0,6} + \cdots + c_{0,2k} & = 0, \quad (\text{ for } i = 0) \\ c_{1,0} + c_{1,2} + c_{1,4} + c_{1,6} + \cdots + c_{1,2k} & = 0, \quad (\text{ for } i = 1) \\ c_{2,0} + c_{2,4} + c_{2,6} + c_{2,8} + \cdots + c_{2,2k-2} & = 0, \quad (\text{ for } i = 2) \\ \vdots & \\ c_{2k-2,2} + c_{2k-2,2} & = 0, \quad (\text{ for } i = 2k-2) \\ c_{2k-1,0} + c_{2k-1,2} & = 0, \quad (\text{ for } i = 2k-1) \\ c_{2k,0} & = 0, \quad (\text{ for } i = 2k) \\ c_{2k+1,0} & = 0. \quad (\text{ for } i = 2k+1) \end{cases}$$

Since $c_{0,2k+1} = 0$, by comparing the coefficients of $\hat{y}^{2k+1}$ in (2.10), a_0 is a multiple of the leading coefficient of the Legendre polynomial L_{2k+1}:

$$c_0 \frac{(4k+2)!}{2^{2k+1}(2k+1)!} = -a_0.$$

Again, comparing the coefficients of $\hat{x}^{2k+1}$ in (2.11),

$$c_0 \frac{(4k+2)!}{2^{2k+1}(2k+1)!} = a_0.$$

Hence $c_0 = 0$. Thus $\hat{v}(\hat{x}, \hat{y})$ vanishes on the whole boundary $\partial \hat{K}$ and can be expressed as

$$\hat{v}(\hat{x}, \hat{y}) = \hat{b}(\hat{x}, \hat{y})\hat{q}(\hat{x}, \hat{y}) \text{ with } \hat{q}(\hat{x}, \hat{y}) \in P_{2k-3}(\hat{K}),$$

where the bubble function $\hat{b}(\hat{x}, \hat{y}) = (1 - \hat{x}^2)(1 - \hat{y}^2)$. Finally, since $\hat{v}(\hat{x}, \hat{y})$ vanishes at points in I, $\hat{q}(\hat{x}, \hat{y}) \equiv 0$. This completes the proof. $\square$

The above theorem implies that any $\hat{v}(\hat{x}, \hat{y}) \in R_m(\hat{K})$ can be uniquely determined by its values at points in $G \cup I$. Though the number of points in $G \cup I$ is $(2k + 3)(k+1)+2$, which is 1 greater than the dimension $(2k+3)(k+1)+1$ of $R_m(\hat{K})$, the relation (2.1) implies that the number of linearly independent functionals defined for the space $R_m(\hat{K})$ is equal to the dimension of the shape function space. This motivates the following degrees of freedom for the shape function space $R_m(\hat{K})$:

- values at points in G which satisfy the relation (2.1);
- values at points in I.

Remark 2.4. *We can take the following shape function spaces*

$$\tilde{R}_m(\hat{K}) := P_m(\hat{K}) + \mathit{span}\{\hat{x}^m \hat{y}\}, \text{ and } \bar{R}_m(\hat{K}) := P_m(\hat{K}) + \mathit{span}\{\hat{x}\hat{y}^m\}. \tag{2.12}$$

3. The second family of shape function spaces

As we see in the previous section, functionals defined by values at points in $G \cup I$ are not all linearly independent for shape function spaces $R_m(\hat{K})$ defined in the previous section, there is a relation (2.1) for all functions in $R_m(\hat{K})$. To make these functionals linearly independent for some shape function spaces, we propose to use higher order monomials of one variable, say $\hat{x}^{2k+2}$ and $\hat{y}^{2k+2}$, which motivates to enrich $R_m(\hat{K})$ by $\mathrm{span}\{\hat{x}^{2k+2} - \hat{y}^{2k+2}\}$. This leads to the following shape function spaces:

$$ER_m(\hat{K}) := P_m(\hat{K}) + \mathrm{span}\{\hat{x}^{2k+1}\hat{y} - \hat{x}\hat{y}^{2k+1}, \hat{x}^{2k+2} - \hat{y}^{2k+2}\}. \tag{3.1}$$

Remark 3.1. *For $k = 0$, the space $ER_m(\hat{K})$ is the shape function space of the nonconforming rotated Q_1 element from* [16]*:*

$$ER_1(\hat{K}) = \mathit{span}\{1, \hat{x}, \hat{y}, \hat{x}^2 - \hat{y}^2\}.$$

For the space $ER_m(\hat{K})$, the degrees of freedom are:

- values at points in G, defined in (2.7);
- values at points in I, defined in (2.8).

Theorem 3.2. *If $\hat{v}(\hat{x}, \hat{y}) \in ER_m(\hat{K})$ vanishes at points in $G \cup I$, then $\hat{v}(\hat{x}, \hat{y}) \equiv 0$.*

Proof. By the definition of $ER_m(\hat{K})$, $\hat{v}(\hat{x}, \hat{y})$ can be expressed as

$$\hat{v}(\hat{x}, \hat{y}) = \hat{v}_1(\hat{x}, \hat{y}) + c_1\hat{x}^{2k+1} + c_2\hat{y}^{2k+1} + c_3(\hat{x}^{2k+1}\hat{y} - \hat{x}\hat{y}^{2k+1}) + c_4(\hat{x}^{2k+2} - \hat{y}^{2k+2}), \tag{3.2}$$

where $\hat{v}_1(\hat{x}, \hat{y}) \in P_m(\hat{K}) \backslash \mathrm{span}\{\hat{x}^{2k+1}, \hat{y}^{2k+1}\}$, and c_i, $i = 1, \cdots, 4$, are four interpolation parameters. This implies that the restrictions of $\hat{v}$ on four edges of $\hat{K}$ are polynomials of degree $\leq 2k+2$. Since $\hat{v}(\hat{x},\hat{y})$ vanishes at $m = 2k+1$ Gauss-Legendre points on four edges of $\hat{K}$, these restrictions on four edges can be written as

$$\begin{aligned} &\hat{v}(\hat{x},1) = L_{2k+1}(\hat{x})(a_1\hat{x}+b_1), \quad \hat{v}(\hat{x},-1) = L_{2k+1}(\hat{x})(a_2\hat{x}+b_2),\\ &\hat{v}(1,\hat{y}) = L_{2k+1}(\hat{y})(a_3\hat{y}+b_3), \quad \hat{v}(-1,\hat{y}) = L_{2k+1}(\hat{y})(a_4\hat{y}+b_4), \end{aligned} \tag{3.3}$$

where a_i, b_i, $i = 1, \cdots, 4$, are some constants which will be shown to be zero next. To this end, let $c = (4k+2)!/(2^{2k+1}(2k+1)!)$ be the coefficient of the monomial $\hat{x}^{2k+1}$ of $L_{2k+1}(\hat{x})$. A comparison of coefficients from (3.2) and (3.3) for monomials $\hat{x}^{2k+2}$, $\hat{y}^{2k+2}$, $\hat{x}^{2k+1}$, and $\hat{y}^{2k+1}$ leads to

$$a_1 = a_2 = -a_3 = -a_4 = c_4/c, \tag{3.4}$$

and

$$\begin{aligned} c_1 + c_3 = c\, b_1, \quad c_1 - c_3 = c\, b_2,\\ c_2 - c_3 = c\, b_3, \quad c_2 + c_3 = c\, b_4. \end{aligned} \tag{3.5}$$

Note that $\hat{v}(\hat{x},1)$ and $\hat{v}(1,\hat{y})$ are equal at corner $(1,1)$, $\hat{v}(\hat{x},1)$ and $\hat{v}(-1,\hat{y})$ are equal at corner $(-1,1)$, $\hat{v}(\hat{x},-1)$ and $\hat{v}(1,\hat{y})$ are equal at corner $(1,-1)$, $\hat{v}(\hat{x},-1)$ and $\hat{v}(-1,\hat{y})$ are equal at corner $(-1,-1)$. Since $L_{2k+1}(1) = -L_{2k+1}(-1)$, these observations give

$$\begin{aligned} a_1 + b_1 = a_3 + b_3, \quad a_1 - b_1 = a_4 + b_4,\\ a_2 + b_2 = a_3 - b_3, \quad a_2 - b_2 = a_4 - b_4. \end{aligned} \tag{3.6}$$

It follows from (3.4), (3.5) and (3.6) that

$$a_1 = a_2 = a_3 = a_4 = b_1 = b_2 = b_3 = b_4 = c_1 = c_2 = c_3 = 0. \tag{3.7}$$

This implies that $c_4 = 0$ and $\hat{v}(\hat{x},\hat{y})$ vanishes on the boundary of $\hat{K}$. Hence,

$$\hat{v}(\hat{x},\hat{y}) = \hat{b}(\hat{x},\hat{y})\hat{q}(\hat{x},\hat{y}) \text{ with } \hat{q}(\hat{x},\hat{y}) \in P_{2k-3}(\hat{K}),$$

where the bubble function $\hat{b}(\hat{x},\hat{y}) = (1-\hat{x}^2)(1-\hat{y}^2)$. Finally, since $\hat{v}(\hat{x},\hat{y})$ vanishes at points in I, $\hat{q}(\hat{x},\hat{y}) \equiv 0$. This completes the proof. □

For the space $ER_m(\hat{K})$, we define another set of degrees of freedom as follows

- moments of order $\leq 2k$ on each edge of $\hat{K}$;
- values at points in I.

We note that these two sets of degrees of freedom are not equivalent as $(2k+2)$ polynomials are involved in $ER_m(\hat{K})$.

Theorem 3.3. *For any $\hat{v}(\hat{x},\hat{y}) \in ER_m(\hat{K})$, suppose its moments of order $\leq 2k$ on each edge of $\hat{K}$ and values at points in I vanish. Then $\hat{v}(\hat{x},\hat{y}) \equiv 0$.*

Proof. By the definition of $ER_m(\hat{K})$, $\hat{v}(\hat{x},\hat{y})$ can be expressed as

$$\hat{v}(\hat{x},\hat{y}) = \hat{v}_1(\hat{x},\hat{y}) + c_1\hat{x}^{2k+1} + c_2\hat{y}^{2k+1} + c_3(\hat{x}^{2k+1}\hat{y} - \hat{x}\hat{y}^{2k+1}) + c_4(\hat{x}^{2k+2} - \hat{y}^{2k+2}),$$

where $\hat{v}_1(\hat{x},\hat{y}) \in P_m(\hat{K})\backslash\text{span}\{\hat{x}^{2k+1},\hat{y}^{2k+1}\}$, and c_i, $i=1,\cdots,4$, are four interpolation parameters. Consider the restriction on edge $\hat{e}_2$ of $\hat{v}$, denoted by $\hat{w}$, which is a polynomial of degree $\leq 2k+2$. The function $\hat{w}$ can be decomposed as

$$\hat{w} = \hat{w}_1 + \hat{w}_2,$$

where

$$\hat{w}_1 = \sum_{i=0}^{k+1} d_{2i}\hat{x}^{2i}, \text{ and } \hat{w}_2 = \sum_{i=0}^{k} d_{2i+1}\hat{x}^{2i+1},$$

where d_i, $i=0,1,\cdots,2(k+1)$, are interpolation constants for $\hat{w}$. From degrees of freedom it follows

$$\int_{-1}^{1} \hat{w}_1\hat{x}^{2i}d\hat{x} = 0, i=0,\cdots,k, \text{ and } \int_{-1}^{1} \hat{w}_2\hat{x}^{2i+1}d\hat{x} = 0, i=0,\cdots,k-1.$$

Since $\hat{w}_1$ (resp. $\hat{w}_2$) is an even (odd) function on $[-1,1]$, it holds that

$$\int_{-1}^{1} \hat{w}_1\hat{x}^{2i+1}d\hat{x} = 0, i=0,\cdots,k, \text{ and } \int_{-1}^{1} \hat{w}_2\hat{x}^{2i}d\hat{x} = 0, i=0,\cdots,k.$$

Note that the degree of polynomial $\hat{w}_1$ (resp. $\hat{w}_2$) is not more than $2k+2$ (resp. $2k+1$). Therefore both $\hat{w}_1$ and $\hat{w}_2$ are Legendre polynomials (up to multiplication constants), namely,

$$\hat{w}_1 = a_2L_{2k+2}(\hat{x}), \text{ and } \hat{w}_2 = b_2L_{2k+1}(\hat{x}),$$

for two constants a_2 and b_2. Similar arguments apply to the restrictions on the other three edges of $\hat{K}$, which leads to

$$\begin{aligned}
\hat{v}|_{\hat{e}_1} &= a_1L_{2k+2}(\hat{y}) + b_1L_{2k+1}(\hat{y}),\\
\hat{v}|_{\hat{e}_2} &= a_2L_{2k+2}(\hat{x}) + b_2L_{2k+1}(\hat{x}),\\
\hat{v}|_{\hat{e}_3} &= a_3L_{2k+2}(\hat{y}) + b_3L_{2k+1}(\hat{y}),\\
\hat{v}|_{\hat{e}_4} &= a_4L_{2k+2}(\hat{x}) + b_4L_{2k+1}(\hat{x}).
\end{aligned}$$

Since the coefficient before monomial $\hat{x}^{2k+2}$ is opposite to that before monomial $\hat{y}^{2k+2}$ for $\hat{v}$, this gives

$$a_1 = a_3 = -a_2 = -a_4. \tag{3.8}$$

Let $c \neq 0$ be the coefficient of monomial $\hat{x}^{2k+1}$ of $L_{2k+1}(\hat{x})$. A comparison of coefficients for monomials $\hat{x}^{2k+1}$, and $\hat{y}^{2k+1}$ in $\hat{v}|_{\hat{e}_i}$, $i=1,\cdots,4$, and those in $\hat{v}$, leads to

$$\begin{aligned}
c_1 + c_3 &= cb_4, \quad & c_1 - c_3 &= cb_2,\\
c_2 - c_3 &= cb_3, \quad & c_2 + c_3 &= cb_1.
\end{aligned} \tag{3.9}$$

Note that $\hat{v}(\hat{x},1)$ and $\hat{v}(1,\hat{y})$ are equal at corner $(1,1)$, $\hat{v}(\hat{x},1)$ and $\hat{v}(-1,\hat{y})$ are equal at corner $(-1,1)$, $\hat{v}(\hat{x},-1)$ and $\hat{v}(1,\hat{y})$ are equal at corner $(1,-1)$, $\hat{v}(\hat{x},-1)$

and $\hat{v}(-1,\hat{y})$ are equal at corner $(-1,-1)$. Since $e = L_{2k+1}(1) = -L_{2k+1}(-1)$ and $d = L_{2k+2}(1) = L_{2k+2}(-1)$, this leads to

$$\begin{aligned} da_1 + eb_1 = da_4 - eb_4, \quad da_3 + eb_3 = da_4 + eb_4, \\ da_3 - eb_3 = da_2ecb_2, \quad da_1 - eb_1 = da_2 - eb_2. \end{aligned} \tag{3.10}$$

It follows from (3.8), (3.9) and (3.10) that

$$a_1 = a_2 = a_3 = a_4 = b_1 = b_2 = b_3 = b_4 = c_1 = c_2 = c_3 = 0.$$

This implies that $c_4 = 0$ and $\hat{v}(\hat{x},\hat{y})$ vanishes on the boundary of $\hat{K}$. Hence,

$$\hat{v}(\hat{x},\hat{y}) = \hat{b}(\hat{x},\hat{y})\hat{q}(\hat{x},\hat{y}) \text{ with } \hat{q}(\hat{x},\hat{y}) \in P_{2k-3}(\hat{K}),$$

where the bubble function $\hat{b}(\hat{x},\hat{y}) = (1-\hat{x}^2)(1-\hat{y}^2)$. Finally, since $\hat{v}(\hat{x},\hat{y})$ vanishes at points of I, $\hat{q}(\hat{x},\hat{y}) \equiv 0$. This completes the proof.

□

4. The third family of shape function spaces

For nonconforming elements of even order, there is always a discrete bubble function which vanishes at $4m$ Gauss-Legendre points. Thus, in additional to one extra term for R_m of odd m, we need another extra term enriching P_m polynomials so that the finite element function has exactly $4m$ degrees of freedom on the element boundary. For $m = 2k$, we define the third family of nonconforming elements by

$$R_m^+(\hat{K}) := P_m(\hat{K}) + \operatorname{span}\{\hat{x}^m\hat{y}, \hat{x}\hat{y}^m\}. \tag{4.1}$$

For the continuity requirement, we need use the $m = 2k$ Gauss-Legendre points,

$$g_{-k}, \ldots, g_{-1}, g_1, \ldots, g_k.$$

Lemma 4.1. *Let $\hat{v} \in R_m^+$ be a polynomial over $\hat{K}$ such that its restrictions on four edges are polynomials of degree $\le 2k$. Then it holds that*

$$\sum_{0\neq i=-k}^{k} \frac{\hat{v}(1,g_i) - \hat{v}(-1,g_i) - \hat{v}(g_i,1) + \hat{v}(g_i,-1)}{g_i(1-g_i^2)\prod_{|i|\neq j=1}^{k}(g_i^2 - g_j^2)} = 0. \tag{4.2}$$

Proof. We add the point $g_{-k-1} = -1$ to the $2k$ Gauss-Legendre points, and define a first set of Lagrange interpolation basis functions:

$$\mathcal{L}_i^0(\hat{x}) = \prod_{0,i\neq j=-(k+1)}^{k} \frac{\hat{x}-g_j}{g_i - g_j}, \quad i = -(k+1), -k, \cdots, -1, 1, \cdots, k.$$

At the other end, adding a point $g_{k+1} = 1$, we define another set of Lagrange interpolation basis functions:

$$\mathcal{L}_i^1(\hat{x}) = \prod_{0,i\neq j=-k}^{k+1} \frac{\hat{x}-g_j}{g_i - g_j}, \quad i = -k, \cdots, -1, 1, \cdots, k+1.$$

Since

$$P_{2k}([-1,1]) = \mathrm{span}\{\mathcal{L}_i^0(\hat{x}),\ i=-k-1,\cdots,-1,1,\cdots,k\}$$
$$= \mathrm{span}\{\mathcal{L}_i^1(\hat{x}),\ i=-k,\cdots,-1,1,\cdots,k+1\},$$

and the restriction of $\hat{v}$ on each edge of $\hat{K}$ is a polynomial of degree $\le 2k$, we have the four equations in (2.4). Again, let

$$\alpha_i = \mathcal{L}_i^0(1) = \begin{cases} \frac{1-(-1)}{g_i-(-1)} \prod_{0,i\neq j=-k}^k \frac{1-g_j}{g_i-g_j} & \text{if } i \neq -k-1, \\ \prod_{0\neq j=-k}^k \frac{1-g_j}{-1-g_j} & \text{if } i=-k-1, \end{cases}$$

$$\beta_i = \mathcal{L}_i^1(-1) = \begin{cases} \frac{-1-1}{g_i-1} \prod_{0,i\neq j=-k}^k \frac{-1-g_j}{g_i-g_j} & \text{if } i \neq k+1, \\ \prod_{0\neq j=-k}^k \frac{-1-g_j}{1-g_j} & \text{if } i=k+1. \end{cases}$$

Note that, because the Gauss-Legendre points are symmetric,

$$\alpha_{-k-1} = \beta_{k+1}, \quad \text{and} \quad \alpha_i = -\beta_i \ \text{ if } \ i=-k,\cdots,-1,1,\dots,k.$$

By the continuity of $\hat{v}$ at the four corner vertexes, it yields

$$\alpha_{-k-1}\hat{v}(-1,-1) + \sum_{0\neq i=-k}^{k} \alpha_i \hat{v}(g_i,-1) = \beta_{k+1}\hat{v}(1,1) + \sum_{0\neq i=-k}^{k} \beta_i \hat{v}(1,g_i),$$

$$\alpha_{-k-1}\hat{v}(-1,1) + \sum_{0\neq i=-k}^{k} \alpha_i \hat{v}(g_i,1) = \alpha_{-k-1}\hat{v}(1,-1) + \sum_{0\neq i=-k}^{k} \alpha_i \hat{v}(1,g_i),$$

$$\beta_{k+1}\hat{v}(1,1) + \sum_{0\neq i=-k}^{k} \beta_i \hat{v}(g_i,1) = \alpha_{-k-1}\hat{v}(-1,-1) + \sum_{0\neq i=-k}^{k} \alpha_i \hat{v}(-1,g_i),$$

$$\beta_{k+1}\hat{v}(1,-1) + \sum_{0\neq i=-k}^{k} \beta_i \hat{v}(g_i,-1) = \beta_{k+1}\hat{v}(-1,1) + \sum_{0\neq i=-k}^{k} \beta_i \hat{v}(-1,g_i).$$

Eliminating four corner values of $\hat{v}$, we get

$$\sum_{0\neq i=-k}^{k} \alpha_i \big(\hat{v}(1,g_i) - \hat{v}(-1,g_i) - \hat{v}(g_i,1) + \hat{v}(g_i,-1)\big) = 0.$$

This is simplified to (4.2). □

To define degrees of freedom for this space, let the set of even Gauss-Legendre points on four edges of $\hat{K}$ be

$$G^+ := \{(1,g_i),(-1,g_i),(g_i,1),(g_i,-1),\ 0\neq i=-k,\cdots,k\}.$$

To define other degrees of freedom in the interior of $\hat{K}$, let

$$I^+ := \{(\hat{x}_\ell,\hat{y}_\ell), \ell=1,\cdots,(2k-3)(k-1)\}$$

be a set of interior points of $\hat{K}$, the standard Lagrange points inside the reference element $\hat{K}$, so that any polynomial $\hat{q}(\hat{x},\hat{y}) \in P_{2k-4}(\hat{K})$ can be uniquely defined by its values at points in I^+.

Theorem 4.2. *If $\hat{v}(\hat{x},\hat{y}) \in R_m^+(\hat{K})$ vanishes at all points in $G^+ \cup I^+ \cup \{(1,1)\}$, then $\hat{v}(\hat{x},\hat{y}) \equiv 0$.*

Proof. The function $\hat{v}(\hat{x},\hat{y}) \in R_m^+(\hat{K})$ is a P_{2k} polynomial when restricted to $\hat{y}=1$. On the edge $\hat{y}=1$, $\hat{v}$ vanishes at $2k$ Gauss-Legendre points plus a corner point $\{(1,1)\}$. So $\hat{v} \equiv 0$ on $\hat{y}=1$. Repeating the argument, as $\hat{v}$ is continuous at the four corners, we find $\hat{v} \equiv 0$ on the whole boundary.

$$\hat{v}(\hat{x},\hat{y}) = \hat{b}(\hat{x},\hat{y})\hat{q}(\hat{x},\hat{y}) \text{ with } \hat{q}(\hat{x},\hat{y}) \in P_{2k-4}(\hat{K}),$$

where $\hat{b}(\hat{x},\hat{y}) = (1-\hat{x}^2)(1-\hat{y}^2)$. Here with a careful division, we find the coefficients for $x^m y$ and $y^m x$ in $\hat{v}$ are zero. So $\hat{q}(\hat{x},\hat{y}) \in P_{2k-4}(\hat{K})$. Finally, since $\hat{v}(\hat{x},\hat{y})$ vanishes at points of I, $\hat{q}(\hat{x},\hat{y}) \equiv 0$. This completes the proof. □

The above theorem implies that any $\hat{v}(\hat{x},\hat{y}) \in R_m^+(\hat{K})$ can be uniquely determined by its values at points in $G^+ \cup I^+ \cup \{(1,1)\}$. But the number of points is one more than the dimension of $R_m^+(\hat{K})$. The relation (4.2) implies that the number of linearly independent functionals defined for the space $R_m^+(\hat{K})$ is equal to the dimension of the space. This motivates the following degrees of freedom for the shape function space $R_m^+(\hat{K})$:

- values at points in G^+ which satisfy the relation (4.2);
- value at point $(1,1)$;
- values at points in I^+.

Note that the value at point $(1,1)$ is to determine the coefficient of the discrete bubble function in $\hat{v}$:

$$\hat{b}_0(\hat{x},\hat{y}) = \prod_{i=1}^{k} (\hat{x}^2 + \hat{y}^2 - 1 - g_i^2).$$

5. Nonconforming finite element spaces

This section defines nonconforming quadrilateral element spaces.

5.1. **Quadrilateral Mesh.** Let $\mathcal{T}_h := \{K_i, i = 1, \cdots, Ne\}$ be a shape regular quadrilateral partition of Ω with $\operatorname{diam}(K_i) \le h$. We assume that the partition $\mathcal{T}_h$ satisfies the bisection condition of [17]: The distance d_K between the midpoints of two diagonals of each element K is of order $\mathcal{O}(h^2)$.

For a given element $K \in \mathcal{T}_h$, its four nodes are denoted by $A_i(x_i, y_i), i = 1, \cdots, 4$ in the counterclockwise order. Let $\hat{K} := [-1,1]^2$ denote the reference element with

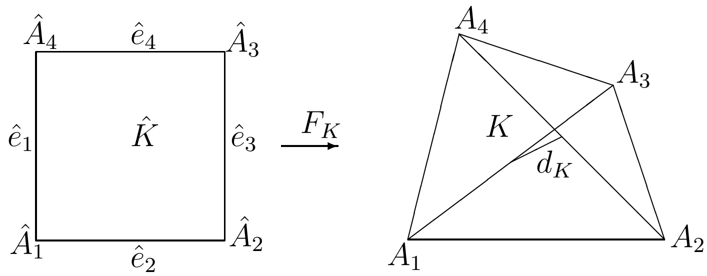

FIGURE 1. The reference element $\hat{K}$ and a quadrilateral element K.

nodes $\hat{A}_i(\hat{x}_i, \hat{y}_i)$, $i = 1, \cdots, 4$, shown in Figure 1. Define the bilinear transformation $F_K : \hat{K} \to K$ by

$$x = \sum_{i=1}^{4} x_i N_i(\hat{x}, \hat{y}), \quad y = \sum_{i=1}^{4} y_i N_i(\hat{x}, \hat{y}), \quad (\hat{x}, \hat{y}) \in \hat{K},$$

where $N_i(\hat{x}, \hat{y}), i = 1, 2, 3, 4$ are the bilinear basis functions, which can be written as

$$\begin{aligned} N_1(\hat{x}, \hat{y}) &= \frac{1}{4}(1-\hat{x})(1-\hat{y}), \quad N_2(\hat{x}, \hat{y}) = \frac{1}{4}(1+\hat{x})(1-\hat{y}), \\ N_3(\hat{x}, \hat{y}) &= \frac{1}{4}(1+\hat{x})(1+\hat{y}), \quad N_4(\hat{x}, \hat{y}) = \frac{1}{4}(1-\hat{x})(1+\hat{y}). \end{aligned}$$

5.2. **Nonconforming finite element spaces and dimensions.** For an odd integer $m = 2k + 1 > 0$, define nonconforming finite element spaces by

$$\begin{aligned} R_h := \{v \in L^2(\Omega), v|_K \circ F_K^{-1} \in R_m(\hat{K}) \text{ for any } K \in \mathcal{T}_h, v \text{ is continuous} \\ \text{at } m \text{ Gauss-Legendre points of each interior edge of } \mathcal{T}_h\}. \end{aligned} \tag{5.1}$$

$$\begin{aligned} ER_h^P := \{v \in L^2(\Omega), v|_K \circ F_K^{-1} \in ER_m(\hat{K}) \text{ for any } K \in \mathcal{T}_h, v \text{ is continuous} \\ \text{at } m \text{ Gauss-Legendre points of each interior edge of } \mathcal{T}_h\}. \end{aligned} \tag{5.2}$$

$$\begin{aligned} ER_h^M := \{v \in L^2(\Omega), v|_K \circ F_K^{-1} \in ER_m(\hat{K}) \text{ for any } K \in \mathcal{T}_h, \int_e [v] q ds = 0 \\ \text{for any } q \in P_{2k}(e) \text{ for each interior edge } e \text{ of } \mathcal{T}_h\}, \end{aligned} \tag{5.3}$$

where $[v]$ denotes the jump of v across edge e. We note again that ER_h^M is different from ER_h^P due to the higher order polynomial term $(x^{2k+2} - y^{2k+2})$. For an even integer $m = 2k$, the nonconforming finite element space is defined by

$$\begin{aligned} R_h^+ := \{v \in L^2(\Omega), v|_K \circ F_K^{-1} \in R_m^+(\hat{K}) \text{ for any } K \in \mathcal{T}_h, v \text{ is continuous} \\ \text{at } m \text{ Gauss-Legendre points of each interior edge of } \mathcal{T}_h\}. \end{aligned} \tag{5.4}$$

The corresponding homogeneous spaces are defined, respectively,

$$R_{h,0} := \{v \in R_h, v \text{ vanishes at } m \text{ Gauss-Legendre points} \\ \text{of each boundary edge } e \text{ of } \mathcal{T}_h\}. \tag{5.5}$$

$$ER^P_{h,0} := \{v \in ER^P_h, v \text{ vanishes at } m \text{ Gauss-Legendre points} \\ \text{of each boundary edge } e \text{ of } \mathcal{T}_h\}. \tag{5.6}$$

$$ER^M_{h,0} := \{v \in ER^M_h, \int_e vq ds = 0 \text{ for any } q \in P_{2k}(e) \\ \text{for each boundary edge } e \text{ of } \mathcal{T}_h\}. \tag{5.7}$$

$$R^+_{h,0} := \{v \in R^+_h, v \text{ vanishes at } m \text{ Gauss-Legendre points} \\ \text{of each boundary edge } e \text{ of } \mathcal{T}_h\}. \tag{5.8}$$

5.3. **Approximations of nonconforming finite element spaces.** Given $K \in \mathcal{T}_h$, define

$$G_K := G \circ F_K, \text{ and } I_K := I \circ F_K. \tag{5.9}$$

Then, define the canonical interpolation operator $\Pi^P_{ER} : H^2(\Omega) \to ER^P_h$ by

$$(\Pi^P_{ER} v|_K)(p) = v|_K(p) \text{ for any } p \in G_K \cup I_K \text{ and } K \in \mathcal{T}_h \tag{5.10}$$

for any $v \in H^2(\Omega)$.

Define the canonical interpolation operator $\Pi^M_{ER} : H^2(\Omega) \to ER^P_h$ by

$$\begin{aligned} &(\Pi^M_{ER} v|_K)(p) = v|_K(p) \quad \text{for any } p \in I_K, \\ &\int_e \Pi^M_{ER} v|_e ds = \int_e v ds \quad \text{for any } e \subset \partial K, \end{aligned} \tag{5.11}$$

for any $K \in \mathcal{T}_h$ and $v \in H^2(\Omega)$.

To define an interpolation operator for R_h, let Π^Q be the canonical interpolation operator of the conforming Q_m element space $Q_{m,h} := \{v \in H^1(\Omega), v|_K \circ F_K^{-1} \in Q_m(\hat{K}), K \in \mathcal{T}_h\}$. Then, define an interpolation operator $\Pi_R : H^2(\Omega) \to R_h$ by

$$(\Pi_R v|_K)(p) = \Pi^Q v|_K(p) \text{ for any } p \in G_K \cup I_K \text{ and } K \in \mathcal{T}_h \tag{5.12}$$

for any $v \in H^2(\Omega)$. Since the values at points in G_K of $\Pi^Q v$ satisfy the relation (2.1), this operator is well-defined.

For the even order nonconforming finite elements, we define

$$(\Pi_{R^+} v|_K)(p) = \Pi^Q v|_K(p) \quad \text{for any } p \in G^+_K \cup I^+_K \cup \{F_K(1,1)\} \text{ and } K \in \mathcal{T}_h \tag{5.13}$$

for any $v \in H^2(\Omega)$. Since the values of Q_m polynomial $\Pi^Q v$ at points in G^+_K satisfy the constraint (4.2), the operator $\Pi_{R^+} : H^2(\Omega) \to R^+_h$ is well-defined.

An immediate consequence of these interpolation operators is the following approximation property

$$\inf_{v_h \in V_h} \|\nabla_h (u - v_h)\|_0 \le C h^m \|u\|_{m+1}, \tag{5.14}$$

provided that $u \in H^{m+1}(\Omega)$ and the mesh satisfies the bisection condition where $V_h = R_h, ER_h^P, ER_h^M, R_h^+$.

5.4. **The full space of nonconforming finite elements.** For the finite element spaces $ER_{h,0}^P$ and $ER_{h,0}^M$, local nodal basis functions are uniquely determined, which are glued together to form the global nodal basis functions. Thus, the interpolation operators Π_{ER}^P and Π_{ER}^M, cf. (5.10) and (5.11), are on-to mappings, by which the dimension of the finite element spaces can be counted.

Let N_V, N_S, and N_E denote the numbers of vertexes, edges and elements of the partition $\mathcal{T}_h$, respectively. Let N_V^i, N_V^b, N_S^i and N_S^b denote the numbers of interior vertexes, interior edges, boundary vertexes and boundary edges, respectively. The dimensions are

$$\dim ER_{h,0}^P = \dim ER_{h,0}^M = N_E(2k-1)(k-1) + N_S^i(2k+1). \tag{5.15}$$

However, for nonconforming spaces $R_{h,0}$ and $R_{h,0}^+$, the local nodal basis is not unique. The canonical basis functions are not linearly independent but subject to the constraints (2.1) and (4.2). When gluing these functions together to form the global space, it is not clear if they can form a basis for the full space. In other words, the local constraints (2.1) (on each element) might be linearly dependent globally, that is, some constraints may be automatically satisfied when the neighboring elements are subject to the constraints. If so, the interpolation operators Π_R and Π_{R^+} in (5.12) and (5.13) are not on-to mappings. Then, the computational finite element spaces are not the full spaces defined mathematically in (5.1) and (5.4), but subspaces. This problem is not well-addressed in previous research. We give a rigorous analysis.

Theorem 5.1. *The interpolation operators are on-to mappings that*

$$\Pi_R(Q_{m,h} \cap H_0^1(\Omega)) = R_{h,0}, \tag{5.16}$$

$$\Pi_{R^+}(Q_{m,h} \cap H_0^1(\Omega)) = R_{h,0}^+. \tag{5.17}$$

The dimensions of these spaces are

$$\dim R_{h,0} = N_E((2k-1)(k-1)) + N_V^i + N_S^i(2k), \tag{5.18}$$

$$\dim R_{h,0}^+ = N_E((2k-3)(k-1)+1) + N_V^i + N_S^i(2k-1). \tag{5.19}$$

Proof. We analyze the space $R_{h,0}$ only as that for $R_{h,0}^+$ is the same. $R_{h,0}$ is defined as the full space. But the range of the interpolation operator is a subspace

$$\Pi_R(Q_{m,h} \cap H_0^1(\Omega)) \subset R_{h,0}.$$

To show that the two spaces are equal, we show that the dimension of the range of Π_R is no less than that of the full space $R_{h,0}$. The former is easily counted by the definition of the operator Π_R in (5.12),

$$\dim \operatorname{Range}(\Pi_R) = N_E((2k-1)(k-1)) + N_V^i + N_S^i(2k). \tag{5.20}$$

We define a completely discontinuous space

$$R_h^d = \{v \in L^2(\Omega) \mid v|_K \circ F_K^{-1} \in R_m(\hat{K}) \quad \forall K \in \mathcal{T}_h\}.$$

Then

$$\dim R_h^d = N_E \cdot (\dim P_{2k+1} + 1) = N_E((2k+3)(k+1)+1). \tag{5.21}$$

Now, we introduce linear functionals on the space R_h^d:

$$f_i \ : \ R_h^d \to R^1, \tag{5.22}$$

$$f_i(v) = \begin{cases} v(g_i) & \text{for any } g_i \in G_K \cap \partial\Omega, \\ v(g_i^+) - v(g_i^-) & \text{for rest } g_i \in G_K \cap \Omega^i, \end{cases} \tag{5.23}$$

for any $K \in \mathcal{T}_h$, where we randomly choose a plus side for each edge, and $v(g_i^+)$ and $v(g_i^+)$ are values of v on the two sides of an edge where g_i is a Gauss-Legendre point, defined in (2.7). The nonconforming finite element space R_h is the kernel space of the space of above linear functionals.

Assume the domain is simply connected. Let g_i be any one boundary Gauss-Legendre point. Let the linear functional evaluating at this point be f_0. We show next that value of $f_0(v) = v(g_i)$ is completely determined by the rest nodal values for all $v \in R_h$. For example, if the domain consists of one element K, then $v(g_i) = \gamma_i^{-1} \sum_{j \neq i} -\gamma_j v(g_j)$, cf. (2.1). We define a set for the rest linear functionals in (5.23)

$$\mathcal{F} = \{f_i \mid i = 1, 2, \cdots, N_S(2k+1) - 1\}. \tag{5.24}$$

Assume

$$\sum_{i=1}^{\dim \mathcal{F}} c_i f_i = 0. \tag{5.25}$$

That is

$$\sum_{i=1}^{\dim \mathcal{F}} c_i f_i(v_h) = 0 \quad \forall v_h \in R_h^d.$$

For convenience, we let $c_0 = 0$. Let $v_h \in R_h^d$ and $v_h = 0$ on all elements except the element where f_0 is defined. We have then

$$0 = \sum_{i=0}^{\dim \mathcal{F}} c_i f_i(v_h) = \sum_{i=0}^{4m-1} c_i(\pm v_h(g_i)).$$

By the constraint (2.1),

$$c_i = c\gamma_i \quad \text{or} \quad -c\gamma_i$$

for some uniform constant c. But above $\{c_i\}$ contains c_0 which is 0. Thus all c_i related to the element K are 0. In this fashion, we know that all c_i in (5.25) are 0 and the set $\mathcal{F}$ is linearly independent. Then the dimension of its kernel is $\dim \mathcal{F}$. By (5.21) and (5.24), (5.18) holds,

$$\begin{aligned}\dim R_{h,0} &= \dim R_h^d - \dim \mathcal{F}\\ &= N_E((2k+3)(k+1)+1) - (N_S(2k+1)-1)\\ &= N_E((2k-1)(k-1)) + N_S^i(2k) + N_V^i,\end{aligned}$$

where we used the relations $4N_E = 2N_S^i + N_S^b$, $2N_E = 2N_V^i + N_V^b - 2$ and $N_V^b = N_S^b$. (5.16) follows (5.20) and (5.18). □

6. The analysis of consistency errors

To analyze consistency errors, we need some additional interpolation operators. On the interval $[-1,1]$, define two L^2 projection operators $\mathcal{P}_{m-i} : L^2([-1,1]) \to P_{m-i}([-1,1])$, for $i=1,2$, respectively,

$$\int_{-1}^{1} \mathcal{P}_{m-i} w\, q ds = \int_{-1}^{1} w\, q ds \text{ for any } q \in P_{m-i}([-1,1]), i=1,2. \tag{6.1}$$

for any $w \in L^2([-1,1])$. Define the interpolation $\Pi_{m-1} : C^0([-1,1]) \to P_{m-1}([-1,1])$ by

$$(\Pi_{m-1} v)(g_i) = v(g_i), i = -k, \cdots, k, \text{ for any } v \in C^0([-1,1]), \tag{6.2}$$

where g_i are Gauss-Legendre points on the interval $[-1,1]$. We recall that $\hat{e}_1$ and $\hat{e}_3$ be two edge of $\hat{K}$ that parallel to the $\hat{y}$ axis, see Figure 1.

Since the analysis of consistency errors for both $R_{h,0}$ and $ER_{h,0}^P$ is similar, only details for $ER_{h,0}^P$ are presented as follows.

Lemma 6.1. *Let Π_{m-1} be the interpolation operator defined in* (6.2). *Then it holds that*

$$\left| \int_{\hat{e}_3} \hat{u} \left(\hat{v} - \Pi_{m-1}\hat{v}|_{\hat{e}_3}\right) d\hat{y} - \int_{\hat{e}_1} \hat{u} \left(\hat{v} - \Pi_{m-1}\hat{v}|_{\hat{e}_1}\right) d\hat{y} \right| \le C|\hat{u}|_{H^m(\hat{K})} |\hat{v}|_{H^m(\hat{K})},$$

for any $\hat{u} \in H^m(\hat{K})$ and $\hat{v} \in ER_m(\hat{K})$.

Proof. Let $\mathcal{P}_{m-1}$ be the L^2 projection operator defined in (6.1). By which, we use the following decomposition

$$\int_{\hat{e}_3} \hat{u} \left(\hat{v} - \Pi_{m-1}\hat{v}|_{\hat{e}_3}\right) d\hat{y} - \int_{\hat{e}_1} \hat{u} \left(\hat{v} - \Pi_{m-1}\hat{v}|_{\hat{e}_1}\right) d\hat{y} =: I_1 + I_2$$

where

$$I_1 = \int_{-1}^{1} (I - \mathcal{P}_{m-1})\hat{u}|_{\hat{e}_3} \left(\hat{v}|_{\hat{e}_3} - \Pi_{m-1}\hat{v}|_{\hat{e}_3}\right) d\hat{y}$$
$$\qquad - \int_{-1}^{1} (I - \mathcal{P}_{m-1})\hat{u}|_{\hat{e}_1} \left(\hat{v}|_{\hat{e}_1} - \Pi_{m-1}\hat{v}|_{\hat{e}_1}\right) d\hat{y},$$
$$I_2 = \int_{-1}^{1} \mathcal{P}_{m-1}\hat{u}|_{\hat{e}_3} \left(\hat{v}|_{\hat{e}_3} - \Pi_{m-1}\hat{v}|_{\hat{e}_3}\right) d\hat{y}$$
$$\qquad - \int_{-1}^{1} \mathcal{P}_{m-1}\hat{u}|_{\hat{e}_1} \left(\hat{v}|_{\hat{e}_1} - \Pi_{m-1}\hat{v}|_{\hat{e}_1}\right) d\hat{y}.$$

The first term I_1 can be bounded by the Cauchy-Schwarz inequality, the trace theorem and the usual Bramble-Hilbert lemma,

$$|I_1| \leq C|\hat{u}|_{H^m(\hat{K})}|\hat{v}|_{H^m(\hat{K})}.$$

To analyze the second term I_2, introduce the following decomposition for $\hat{v}$

$$\hat{v} = \hat{v}_1 + \hat{v}_2,$$

where $\hat{v}_1 \in P_m(\hat{K}) + \text{span}\{\hat{x}^{2k+1}\hat{y} - \hat{x}\hat{y}^{2k+1}\}$, and $\hat{v}_2 = c_0(\hat{x}^{2k+2} - \hat{y}^{2k+2})$ with c_0, an interpolation constant. Since $\hat{v}_1|_{\hat{e}_i} - \Pi_{m-1}\hat{v}_1|_{\hat{e}_i}$, $i = 1, 3$, are polynomials of degree at most $2k + 1$, and vanish at $2k + 1$ Gauss-Legendre points of $\hat{e}_i$,

$$\hat{v}_1|_{\hat{e}_i} - \Pi_{m-1}\hat{v}_1|_{\hat{e}_i} = c_i L_{2k+1}(\hat{y}),$$

with two constants c_i, $i = 1, 3$. Since degrees of polynomials $\mathcal{P}_{m-1}\hat{u}|_{\hat{e}_i}$, $i = 1, 3$, are not more than $2k$, this gives

$$\int_{-1}^{1} \mathcal{P}_{m-1}\hat{u}|_{\hat{e}_3} \left(I - \Pi_{m-1}\right)\hat{v}_1|_{\hat{e}_3} d\hat{y} - \int_{-1}^{1} \mathcal{P}_{m-1}\hat{u}|_{\hat{e}_1} \left(I - \Pi_{m-1}\right)\hat{v}_1|_{\hat{e}_1} d\hat{y} = 0. \tag{6.3}$$

Since $\hat{v}_2|_{\hat{e}_3} = \hat{v}_2|_{\hat{e}_1}$, it holds that $(I - \Pi_{m-1})\hat{v}_2|_{\hat{e}_3} = (I - \Pi_{m-1})\hat{v}_2|_{\hat{e}_1}$. Hence,

$$\begin{aligned} &\int_{-1}^{1} \mathcal{P}_{m-1}\hat{u}|_{\hat{e}_3} \left(I - \Pi_{m-1}\right)\hat{v}_2|_{\hat{e}_3} d\hat{y} - \int_{-1}^{1} \mathcal{P}_{m-1}\hat{u}|_{\hat{e}_1} \left(I - \Pi_{m-1}\right)\hat{v}_2|_{\hat{e}_1} d\hat{y} \\ &= \int_{-1}^{1} \mathcal{P}_{m-1}\left(\hat{u}|_{\hat{e}_3} - \hat{u}|_{\hat{e}_1}\right) \left(I - \Pi_{m-1}\right)\hat{v}_2|_{\hat{e}_3} d\hat{y}. \end{aligned} \tag{6.4}$$

From facts that $(I - \Pi_{m-1})\hat{v}_2|_{\hat{e}_3}$ is a polynomial of degree $\leq 2k + 2$ and vanishes at $2k + 1$ Gauss-Legendre points of $\hat{e}_3$, it follows

$$\begin{aligned} &\int_{-1}^{1} \mathcal{P}_{m-1}\left(\hat{u}|_{\hat{e}_3} - \hat{u}|_{\hat{e}_1}\right) \left(I - \Pi_{m-1}\right)\hat{v}_2|_{\hat{e}_3} d\hat{y} \\ &= \int_{-1}^{1} (I - \mathcal{P}_{m-2})\mathcal{P}_{m-1}\left(\hat{u}|_{\hat{e}_3} - \hat{u}|_{\hat{e}_1}\right) \left(I - \Pi_{m-1}\right)\hat{v}_2|_{\hat{e}_3} d\hat{y}. \end{aligned} \tag{6.5}$$

It is straightforward to see that the right hand side of (6.5) vanishes for all $\hat{u} \in P_{m-1}(\hat{K})$, which leads to

$$
\begin{aligned}
(6.6) \qquad & \left| \int_{-1}^{1} \mathcal{P}_{m-1}\big(\hat{u}|_{\hat{e}_3} - \hat{u}|_{\hat{e}_1}\big) \, \big(I - \Pi_{m-1}\big)\hat{v}_2|_{\hat{e}_3} d\hat{y} \right| \\
& \leq C\,|c_0||\hat{u}|_{H^m(\hat{K})} \leq C|\hat{u}|_{H^m(\hat{K})}|\hat{v}|_{H^{m+1}(\hat{K})}.
\end{aligned}
$$

A summary of (6.3), (6.4), (6.5), and (6.6) proves

$$
|I_2| \leq C|\hat{u}|_{H^m(\hat{K})}|\hat{v}|_{H^m(\hat{K})},
$$

which completes the proof. □

Let $\hat{e}_2$ and $\hat{e}_4$ be two edge of $\hat{K}$ that parallel to the $\hat{x}$ axis, see figure 1. A similar argument of the above lemma can prove the following result.

Lemma 6.2. *Let Π_{m-1} be the interpolation operator defined in* (6.2)*. Then it holds that*

$$
\left| \int_{\hat{e}_4} \hat{u}\big(\hat{v} - \Pi_{m-1}\hat{v}|_{\hat{e}_4}\big) d\hat{x} - \int_{\hat{e}_2} \hat{u}\big(\hat{v} - \Pi_{m-1}\hat{v}|_{\hat{e}_2}\big) d\hat{x} \right| \leq C|\hat{u}|_{H^m(\hat{K})}|\hat{v}|_{H^m(\hat{K})}.
$$

for any $\hat{u} \in H^m(\hat{K})$ and $\hat{v} \in ER_m(\hat{K})$.

Theorem 6.3. *Suppose that the quadrilateral mesh $\mathcal{T}_h$ satisfies the bi-section condition. Then it holds that*

$$
(6.7) \qquad \sup_{0 \neq v_h \in ER^P_{h,0}} \frac{\sum\limits_{e \in \mathcal{E}_h} \int_e \frac{\partial u}{\partial n}[v_h] ds}{\|\nabla_h v_h\|_0} \leq Ch^m |u|_{m+1},
$$

for any $u \in H^{m+1}(\Omega)$.

Proof. Based on Lemmas 6.1 and 6.2, the proof follows from similar procedures used in [9, 17]. □

Remark 6.4. *Similar arguments can show similar estimates for consistency error for spaces $R_{h,0}$, $ER^M_{h,0}$ and $R^+_{h,0}$.*

Theorem 6.5. *Let u and u_h be the exact solution and finite element solution of the Poisson equation, respectively,*

$$
\begin{aligned}
(\nabla u, \nabla v) &= (f, v) \quad \forall v \in H^{r+1}(\Omega) \cap H^1_0(\Omega), \\
(\nabla u_h, \nabla v) &= (f, v) \quad \forall v \in V_h,
\end{aligned}
$$

where $V_h = R_{h,0}$, $ER^P_{h,0}$, $ER^M_{h,0}$ or $R^+_{h,0}$. Then

$$
|u - u_h|_{H^1_h} \leq Ch^{\min\{r,m\}}|u|_{H^r(\Omega)},
$$

where $m = 2k - 1$ for $V_h = R_{h,0}$, $ER^P_{h,0}$ or $ER^M_{h,0}$, and $m = 2k$ for $V_h = R^+_{h,0}$.

Proof. It is standard, by applying the Strang lemma, cf. [4]. □

7. Numerical test

We compute the P_k nonconforming finite element solutions on uniform rectangular grids for the following Poisson equation:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega = (0,1)^2, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

In computation, the exact solution is,

$$u(x,y) = 2^4(x - x^6)(y - y^2). \tag{7.1}$$

The first level grid is the unit square. Each subsequent grid is a refinement of the last one by dividing each square into 4, denoted by $\{\mathcal{T}_h\}$. We use the following 6 nonconforming finite elements:

$$V_h^{(3E)} = \{v \in L^2 \mid v|_K \in P_3 \oplus \{xy^3 - x^3y, x^4 - y^4\},\ [v]_e \perp_P P_2\}, \tag{7.2}$$

$$V_h^{(4)} = \{v \in L^2 \mid v|_K \in P_4 \oplus \{xy^4, x^4y\},\ [v]_e \perp P_3\}, \tag{7.3}$$

$$V_h^{(5)} = \{v \in L^2 \mid v|_K \in P_5 \oplus \{xy^5\},\ [v]_e \perp P_4\}, \tag{7.4}$$

$$V_h^{(5E)} = \{v \in L^2 \mid v|_K \in P_5 \oplus \{xy^5 - x^5y, x^6 - y^6\},\ [v]_e \perp_P P_4\}, \tag{7.5}$$

$$V_h^{(6)} = \{v \in L^2 \mid v|_K \in P_6 \oplus \{xy^6, x^6y\},\ [v]_e \perp P_5\}, \tag{7.6}$$

$$V_h^{(7)} = \{v \in L^2 \mid v|_K \in P_7 \oplus \{xy^7\},\ [v]_e \perp P_6\}, \tag{7.7}$$

where K is any square element in $\mathcal{T}_h$ and e is any edge in the grid $\mathcal{T}_h$. To be precise,

$$\begin{aligned} V_h^{(3E)} &= ER_{h,0}^P, && \text{defined in (5.6) with } m = 3, \\ V_h^{(4)} &= R_{h,0}^+, && \text{defined in (5.8) with } m = 4, \\ V_h^{(5)} &= R_{h,0}, && \text{defined in (5.5) with } m = 5, \\ V_h^{(5E)} &= ER_{h,0}^P, && \text{defined in (5.6) with } m = 5, \\ V_h^{(6)} &= R_{h,0}^+, && \text{defined in (5.8) with } m = 6, \\ V_h^{(7)} &= R_{h,0}, && \text{defined in (5.5) with } m = 7. \end{aligned}$$

We list the computation results in Tables 1–6. In all cases except the case of polynomial degree 3, we stop the computation when the machine accuracy is reached, i.e., the relative error of computed solutions is about 10^{-15}. We can see, for all the finite element spaces, the computational solutions converge at the optimal order of rate. This is proved in paper.

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TABLE 1. The error and order of convergence by the 2D C^{-1}-P_3 element (7.2) for (7.1).

	$\|u-u_h\|_{L^2}$	h^r	$\|u-u_h\|_{H_h^1}$	h^r
1	0.000000000	0.0	0.00000000	0.0
2	0.172089821	0.0	1.15734862	0.0
3	0.012510804	3.8	0.16038663	2.9
4	0.000823397	3.9	0.02155950	2.9
5	0.000052434	4.0	0.00280349	2.9
6	0.000003300	4.0	0.00035752	3.0
7	0.000000207	4.0	0.00004514	3.0
8	0.000000013	4.0	0.00000567	3.0

TABLE 2. The error and order of convergence by the 2D C^{-1}-P_4 element (7.3) for (7.1).

	$\|u-u_h\|_{L^2}$	h^r	$\|u-u_h\|_{H_h^1}$	h^r
1	1.520882827023	0.0	10.1100205010	0.0
2	0.073186215065	4.4	0.9404045783	3.4
3	0.002467158503	4.9	0.0655417961	3.8
4	0.000078057209	5.0	0.0042062381	4.0
5	0.000002441049	5.0	0.0002645249	4.0
6	0.000000076219	5.0	0.0000165552	4.0
7	0.000000002381	5.0	0.0000010349	4.0

TABLE 3. The error and order of convergence by the 2D C^{-1}-P_5 element (7.4) for (7.1).

	$\|u-u_h\|_{L^2}$	h^r	$\|u-u_h\|_{H_h^1}$	h^r
1	0.162340154	0.0	0.94025897	0.0
2	0.003652811	5.5	0.05082935	4.2
3	0.000061390	5.9	0.00176166	4.9
4	0.000000983	6.0	0.00005729	4.9
5	0.000000016	6.0	0.00000182	5.0
6	0.000000000	6.0	0.00000006	5.0

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TABLE 4. The error and order of convergence by the 2D C^{-1}-P_5 element (7.5) for (7.1).

	$\|u-u_h\|_{L^2}$	h^r	$\|u-u_h\|_{H_h^1}$	h^r
1	0.160588210	0.0	0.93096655	0.0
2	0.003712721	5.4	0.05204506	4.2
3	0.000062480	5.9	0.00180745	4.8
4	0.000000999	6.0	0.00005869	4.9
5	0.000000016	6.0	0.00000186	5.0
6	0.000000000	6.0	0.00000006	5.0

TABLE 5. The error and order of convergence by the 2D C^{-1}-P_6 element (7.6) for (7.1).

	$\|u-u_h\|_{L^2}$	h^r	$\|u-u_h\|_{H_h^1}$	h^r
1	0.051014969254	0.0	0.4290029114	0.0
2	0.000428314992	6.9	0.0080196383	5.7
3	0.000003402349	7.0	0.0001296547	6.0
4	0.000000026614	7.0	0.0000020505	6.0
5	0.000000000209	7.0	0.0000000322	6.0
6	0.000000000021	3.3	0.0000000012	4.8

TABLE 6. The error and order of convergence by the 2D C^{-1}-P_7 element (7.7) for (7.1).

	$\|u-u_h\|_{L^2}$	h^r	$\|u-u_h\|_{H_h^1}$	h^r
1	0.012193263916	0.0	0.1177979140	0.0
2	0.000046859707	8.0	0.0009292948	7.0
3	0.000000182695	8.0	0.0000072707	7.0
4	0.000000000839	7.8	0.0000000571	7.0

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LMAM and School of Mathematical Sciences, Peking University, Beijing 100871, P. R. China. hujun@math.pku.edu.cn

Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA. szhang@udel.edu