SEMINORMAL FORMS AND CYCLOTOMIC QUIVER HECKE ALGEBRAS OF TYPE A

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Abstract. This paper shows that the cyclotomic quiver Hecke algebras of type A, and the gradings on these algebras, are intimately related to the classical seminormal forms. We start by classifying all seminormal bases and then give an explicit "integral" closed formula for the Gram determinants of the Specht modules in terms of the combinatorics associated with the KLR grading. We then use seminormal forms to give a deformation of the KLR algebras of type A. This makes it possible to study the cyclotomic quiver Hecke algebras in terms of the semisimple representation theory and seminormal forms. As an application we construct a new distinguished graded cellular basis of the cyclotomic KLR algebras of type A.

Contents

1. Introduction

The quiver Hecke algebras are a remarkable family of algebras that were introduced independently by Khovanov and Lauda [21, 22] and Rouquier [31]. These algebras are attached to an arbitrary oriented quiver, they are $\mathbb{Z}$-graded and they categorify the negative part of the associated quantum group. Over a field, Brundan and Kleshchev showed that the cyclotomic quiver Hecke algebras of type A, which are certain quotients of the quiver Hecke algebras of type A, are isomorphic to the cyclotomic Hecke algebras of type A.

The quiver Hecke algebras have a homogeneous presentation by generators and relations. As a consequence they have well-defined integral forms. Unlike Hecke algebras, which are generically semisimple, the cyclotomic quiver Hecke algebras are typically not semisimple even over the rational field. As a result the cyclotomic quiver Hecke algebras are rarely isomorphic to the cyclotomic Hecke algebras over an arbitrary ring.

The first main result of this paper shows that the cyclotomic quiver Hecke algebras of type A admit a one-parameter deformation. Moreover, this deformation is

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isomorphic to cyclotomic Hecke algebra defined over the corresponding ring. Before we can state this result we need some notation.

Fix integers $n \geq 0$ and $e > 1$ and let Γ_e be the oriented quiver with vertex set $I = \mathbb{Z}/e\mathbb{Z}$ and edges $i \to i+1$, for $i \in I$. Given $i \in I$ let $\hat{\imath} \geq 0$ be the smallest non-negative integer such that $i = \hat{\imath} + e\mathbb{Z}$. For each dominant weight Λ for the corresponding Kac-Moody algebra $\mathfrak{g}(\Gamma_e)$, there exists a cyclotomic quiver Hecke algebra $\mathcal{R}_n^\Lambda$ and a cyclotomic Hecke algebra $\mathcal{H}_n^\Lambda$. To each tuple $\mathbf{i} \in I^n$ we associate the set of standard tableaux $\operatorname{Std}(\mathbf{i})$ with residue sequence $\mathbf{i}$. All of these terms are defined in Section 3.1.

Like the cyclotomic quiver Hecke algebra, our deformation of $\mathcal{R}_n^\Lambda$ is adapted to the choice of e through the choice of base ring $\mathcal{O}$ that must be an *e-idempotent subring* (Definition 4.1). This definition ensures that the cyclotomic Hecke algebras are semisimple over $\mathscr{K}$, the field of fractions of $\mathcal{O}$, and that $\mathcal{H}_n^\Lambda(\mathcal{O}) \otimes_{\mathcal{O}} K$ is a cyclotomic quiver Hecke algebra whenever $K = \mathcal{O}/\mathfrak{m}$, for $\mathfrak{m}$ a maximal ideal of $\mathcal{O}$. For $t \in \mathcal{O}$ and $d \in \mathbb{Z}$ let $[d] = [d]_t$ be the corresponding quantum integer, so that $[d] = (t^d - 1)/(t-1)$ if $t \neq 1$ or $[d] = d$ if $t = 1$.

We can now state our first main result.

Theorem A. *Suppose that $1 < e < \infty$ and that $(\mathcal{O}, t)$ is an e-idempotent subring of a field $\mathscr{K}$. Then the algebra $\mathcal{H}_n^\Lambda(\mathcal{O})$ is generated as an $\mathcal{O}$-algebra by the elements*

$$\{\, f_{\mathbf{i}}^{\mathcal{O}} \mid \mathbf{i} \in I^n \,\} \cup \{\, \psi_r^{\mathcal{O}} \mid 1 \leq r < n \,\} \cup \{\, y_r^{\mathcal{O}} \mid 1 \leq r \leq n \,\}$$

subject only to the following relations:

$$\prod_{\substack{1 \leq l \leq \ell \\ \kappa_l \equiv i_1 \ (mod\ e)}} (y_1^{\mathcal{O}} - [\kappa_l - \hat{\imath}_1]) f_{\mathbf{i}}^{\mathcal{O}} = 0,$$

$$\begin{aligned}
f_{\mathbf{i}}^{\mathcal{O}} f_{\mathbf{j}}^{\mathcal{O}} &= \delta_{\mathbf{i}\mathbf{j}} f_{\mathbf{i}}^{\mathcal{O}}, \qquad \textstyle\sum_{\mathbf{i} \in I^n} f_{\mathbf{i}}^{\mathcal{O}} = 1, \qquad y_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = f_{\mathbf{i}}^{\mathcal{O}} y_r^{\mathcal{O}}, \\
\psi_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} &= f_{s_r \cdot \mathbf{i}}^{\mathcal{O}} \psi_r^{\mathcal{O}}, \qquad y_r^{\mathcal{O}} y_s^{\mathcal{O}} = y_s^{\mathcal{O}} y_r^{\mathcal{O}}, \\
\psi_r^{\mathcal{O}} y_{r+1}^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} &= (y_r^{\mathcal{O}} \psi_r^{\mathcal{O}} + \delta_{i_r i_{r+1}}) f_{\mathbf{i}}^{\mathcal{O}}, \qquad y_{r+1}^{\mathcal{O}} \psi_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = (\psi_r^{\mathcal{O}} y_r^{\mathcal{O}} + \delta_{i_r i_{r+1}}) f_{\mathbf{i}}^{\mathcal{O}}, \\
\psi_r^{\mathcal{O}} y_s^{\mathcal{O}} &= y_s^{\mathcal{O}} \psi_r^{\mathcal{O}}, \qquad \textit{if } s \neq r, r+1, \\
\psi_r^{\mathcal{O}} \psi_s^{\mathcal{O}} &= \psi_s^{\mathcal{O}} \psi_r^{\mathcal{O}}, \qquad \textit{if } |r-s| > 1,
\end{aligned}$$

$$(\psi_r^{\mathcal{O}})^2 f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases}
(y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\mathcal{O}})(y_{r+1}^{\langle 1-\rho_r(\mathbf{i})\rangle} - y_r^{\mathcal{O}}) f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } i_r \leftrightarrows i_{r+1}, \\
(y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\mathcal{O}}) f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } i_r \rightarrow i_{r+1}, \\
(y_{r+1}^{\langle 1-\rho_r(\mathbf{i})\rangle} - y_r^{\mathcal{O}}) f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } i_r \leftarrow i_{r+1}, \\
0, & \textit{if } i_r = i_{r+1}, \\
f_{\mathbf{i}}^{\mathcal{O}}, & \textit{otherwise,}
\end{cases}$$

and where $\big(\psi_r^{\mathcal{O}} \psi_{r+1}^{\mathcal{O}} \psi_r^{\mathcal{O}} - \psi_{r+1}^{\mathcal{O}} \psi_r^{\mathcal{O}} \psi_{r+1}^{\mathcal{O}}\big) f_{\mathbf{i}}^{\mathcal{O}}$ *is equal to*

$$\begin{cases}
(y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} + y_{r+2}^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\langle 1-\rho_r(\mathbf{i})\rangle}) f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } i_{r+2} = i_r \rightleftarrows i_{r+1}, \\
-t^{1+\rho_r(\mathbf{i})} f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } i_{r+2} = i_r \rightarrow i_{r+1}, \\
f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } i_{r+2} = i_r \leftarrow i_{r+1}, \\
0, & \textit{otherwise,}
\end{cases}$$

where $\rho_r(\mathbf{i}) = \hat{\imath}_r - \hat{\imath}_{r+1}$ and $y_r^{\langle d\rangle} = t^d y_r^{\mathcal{O}} + [d]$, for $d \in \mathbb{Z}$.

As we explain in Corollary 2.15 by taking e large enough this result also applies when $e = 0$. The appendix gives a direct treatment of this case.

To help the reader interpret Theorem A we include the following special case of this result that gives a new presentation of the group algebra of the symmetric group over the ring $\mathbb{Z}_{(p)}$, where p is an integer prime and $\mathbb{Z}_{(p)}$ is the localisation of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$.

1.1. **Corollary.** *Suppose that $e = p$ be an odd prime number and let $I = \mathbb{Z}/p\mathbb{Z}$ and $\Lambda = \Lambda_0$. Then the group algebra $\mathbb{Z}_{(p)}\mathfrak{S}_n$ is generated as an $\mathbb{Z}_{(p)}$-algebra by the elements*

$$\{\, f^{\mathcal{O}}_{\mathbf{i}} \mid \mathbf{i}\in I^n \,\} \cup \{\, \psi^{\mathcal{O}}_r \mid 1 \le r < n \,\} \cup \{\, y^{\mathcal{O}}_s \mid 1 \le s \le n \,\}$$

subject only to the relations:

$$\begin{gathered}
(y^{\mathcal{O}}_1)^{(\Lambda,\alpha_{i_1})} e(\mathbf{i}) = 0, \qquad f^{\mathcal{O}}_{\mathbf{i}} f^{\mathcal{O}}_{\mathbf{j}} = \delta_{\mathbf{i}\mathbf{j}} f^{\mathcal{O}}_{\mathbf{i}}, \qquad \textstyle\sum_{\mathbf{i}\in I^n} f^{\mathcal{O}}_{\mathbf{i}} = 1, \qquad y^{\mathcal{O}}_r f^{\mathcal{O}}_{\mathbf{i}} = f^{\mathcal{O}}_{\mathbf{i}} y^{\mathcal{O}}_r,\\
\psi^{\mathcal{O}}_r f^{\mathcal{O}}_{\mathbf{i}} = f^{\mathcal{O}}_{s_r\cdot\mathbf{i}} \psi^{\mathcal{O}}_r, \qquad y^{\mathcal{O}}_r y^{\mathcal{O}}_s = y^{\mathcal{O}}_s y^{\mathcal{O}}_r,\\
\psi^{\mathcal{O}}_r y^{\mathcal{O}}_{r+1} f^{\mathcal{O}}_{\mathbf{i}} = (y^{\mathcal{O}}_r \psi^{\mathcal{O}}_r + \delta_{i_r i_{r+1}}) f^{\mathcal{O}}_{\mathbf{i}}, \qquad y^{\mathcal{O}}_{r+1} \psi^{\mathcal{O}}_r f^{\mathcal{O}}_{\mathbf{i}} = (\psi^{\mathcal{O}}_r y^{\mathcal{O}}_r + \delta_{i_r i_{r+1}}) f^{\mathcal{O}}_{\mathbf{i}},\\
\psi^{\mathcal{O}}_r y^{\mathcal{O}}_s = y^{\mathcal{O}}_s \psi^{\mathcal{O}}_r, \qquad \text{if } s \ne r, r+1,\\
\psi^{\mathcal{O}}_r \psi^{\mathcal{O}}_s = \psi^{\mathcal{O}}_s \psi^{\mathcal{O}}_r, \qquad \text{if } |r-s| > 1,\\
(\psi^{\mathcal{O}}_r)^2 f^{\mathcal{O}}_{\mathbf{i}} = \begin{cases}
(y^{\mathcal{O}}_r - y^{\mathcal{O}}_{r+1}) f^{\mathcal{O}}_{\mathbf{i}}, & \text{if } i_r \to i_{r+1} \ne 0,\\
(y^{\mathcal{O}}_r + p - y^{\mathcal{O}}_{r+1}) f^{\mathcal{O}}_{\mathbf{i}}, & \text{if } i_r \to i_{r+1} = 0,\\
(y^{\mathcal{O}}_{r+1} - y^{\mathcal{O}}_r) f^{\mathcal{O}}_{\mathbf{i}}, & \text{if } 0 \ne i_r \leftarrow i_{r+1},\\
(y^{\mathcal{O}}_{r+1} + p - y^{\mathcal{O}}_r) f^{\mathcal{O}}_{\mathbf{i}}, & \text{if } 0 = i_r \leftarrow i_{r+1},\\
0, & \text{if } i_r = i_{r+1},\\
f^{\mathcal{O}}_{\mathbf{i}}, & \text{otherwise},
\end{cases}\\
\big(\psi^{\mathcal{O}}_r \psi^{\mathcal{O}}_{r+1} \psi^{\mathcal{O}}_r - \psi^{\mathcal{O}}_{r+1} \psi^{\mathcal{O}}_r \psi^{\mathcal{O}}_{r+1}\big) f^{\mathcal{O}}_{\mathbf{i}} = \begin{cases}
-f^{\mathcal{O}}_{\mathbf{i}}, & \text{if } i_{r+2} = i_r \to i_{r+1},\\
f^{\mathcal{O}}_{\mathbf{i}}, & \text{if } i_{r+2} = i_r \leftarrow i_{r+1},\\
0, & \text{otherwise},
\end{cases}
\end{gathered}$$

for all admissible r, s and $\mathbf{i} \in I^n$.

Except for the cyclotomic relation and the last two relations (that is, the quadratic relations and the braid relations for $\psi^{\mathcal{O}}_1, \dots \psi^{\mathcal{O}}_{n-1}$), all of the relations in Theorem A coincide with the corresponding KLR-relations in $\mathcal{R}^\Lambda_n$. Interestingly, only the "Jucys-Murphy like elements" $y^{\mathcal{O}}_r$ need to be modified in order to define a deformation of $\mathcal{R}^\Lambda_n$. Over a field $K = \mathcal{O}/\mathfrak{m}$, the presentation in Theorem A collapses to give the KLR algebra $\mathcal{R}^\Lambda_n$ because the definition of an idempotent subring ensures that $t^{1+\rho_r(\mathbf{i})} \otimes 1_K = 1$ and $y_r^{\langle 1\pm\rho_r(\mathbf{i})\rangle} \otimes 1_K = y^{\mathcal{O}}_r \otimes 1_K$, for $1 \le r \le n$.

As a first application of Theorem A, Corollary 4.37 gives what appears to be tight upper bounds on the nilpotency indices of the elements $y_1, \dots, y_n$ in the cyclotomic quiver Hecke algebras of type A. Previously such a result was known only in the special case of the linear quiver or, equivalently, when $e = 0$.

To prove Theorem A we work almost entirely inside the semisimple representation theory of the cyclotomic Hecke algebras $\mathcal{H}^\Lambda_n$. We show that definition of the quiver Hecke algebra $\mathcal{R}^\Lambda_n$, and its grading, is implicit in *Young's seminormal form*. With hindsight, using the perspective afforded by this paper, it is not too much of an exaggeration to say that Murphy could have discovered the cyclotomic quiver Hecke algebras in 1983 soon after writing his paper on the Nakayama conjecture [29].

Our proof of Theorem A gives another explanation for the KLR relations and a more conceptual proof of one direction in Brundan and Kleshchev's isomorphism theorem [6] (see Theorem 2.14). In fact, we give a new proof of the Brundan-Kleshchev isomorphism theorem by using the Ariki-Brundan-Kleshchev categorification theorem [2, 7] to bound the dimension of the algebras defined by the presentation in Theorem A.

For the algebras of type A the authors have constructed a graded cellular basis $\{\, \psi_{\mathfrak{s}\mathfrak{t}} \mid (\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}^\Lambda_n) \,\}$ for $\mathcal{R}^\Lambda_n$ [14]. Here $\operatorname{Std}^2(\mathcal{P}^\Lambda_n)$ is the set of all pairs of standard tableaux of the same shape, where the shape is a multipartition of n. The element $\psi_{\mathfrak{s}\mathfrak{t}}$ is homogeneous of degree $\deg_e \mathfrak{s} + \deg_e \mathfrak{t}$, where $\deg_e : \operatorname{Std}(\mathcal{P}^\Lambda_n) \longrightarrow \mathbb{Z}$ is the combinatorial degree function introduced by Brundan, Kleshchev and Wang [8]. Li [24] has shown that $\{\psi_{\mathfrak{st}}\}$ is a graded cellular basis of $\mathcal{R}_n^\Lambda$ over an arbitrary ring. In particular, the KLR algebra $\mathcal{R}_n^\Lambda$ is always free of rank $\dim \mathcal{H}_n^\Lambda(K)$, for K a field.

One of the problems with the basis $\{\psi_{\mathfrak{st}}\}$ is that, because the KLR generators ψ_r, for $1 \le r < n$, do not satisfy the braid relations, the basis elements $\psi_{\mathfrak{st}}$ depend upon a choice of reduced expression for certain permutations $d(\mathfrak{s}), d(\mathfrak{t}) \in \mathfrak{S}_n$ associated with the tableaux $\mathfrak{s}$ and $\mathfrak{t}$; see Section 2.4. As a consequence, the results of [14] constructs different ψ-bases for different choices of reduced expressions for the elements of $\mathfrak{S}_n$. The different ψ-bases constructed in this way are closely related and it would be advantageous to be able to make a canonical choice of basis, however, until now it has not been clear how to do this.

Fix a modular system $(\mathscr{K}, \mathcal{O}, K)$ as in Chapter 6 and consider the corresponding cyclotomic Hecke algebras $(\mathcal{H}_n^\Lambda(\mathscr{K}), \mathcal{H}_n^\Lambda(\mathcal{O}), \mathcal{H}_n^\Lambda)$, where $\mathcal{H}_n^\Lambda = \mathcal{H}_n^\Lambda(K)$. The algebra $\mathcal{H}_n^\Lambda(\mathscr{K})$ is semisimple and has a seminormal basis $\{\, f_{\mathfrak{st}} \mid (\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda) \,\}$, $\mathcal{H}_n^\Lambda(\mathcal{O})$ is a free $\mathcal{O}$-subalgebra of $\mathcal{H}_n^\Lambda(\mathscr{K})$ and $\mathcal{H}_n^\Lambda \cong \mathcal{H}_n^\Lambda(\mathcal{O}) \otimes_{\mathcal{O}} K$. Finally, we recall that the set $\operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ comes equipped with a naturally partial order $\blacktriangleright$; see Section 2.4.

Theorem B. *Suppose that K is a field of characteristic zero and that $(\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$. Then there is a unique element $B_{\mathfrak{st}}^{\mathcal{O}} \in \mathcal{H}_n^\Lambda(\mathcal{O})$ such that*

a) $B_{\mathfrak{st}}^{\mathcal{O}} = f_{\mathfrak{st}} + \sum_{(\mathfrak{u},\mathfrak{v})\blacktriangleright(\mathfrak{s},\mathfrak{t})} p_{\mathfrak{uv}}^{\mathfrak{st}}(x^{-1}) f_{\mathfrak{uv}}$, *where if* $(\mathfrak{u},\mathfrak{v}) \blacktriangleright (\mathfrak{s},\mathfrak{t})$ *then* $p_{\mathfrak{uv}}^{\mathfrak{st}}(x) \in xK[x]$ *and* $\deg p_{\mathfrak{uv}}^{\mathfrak{st}}(x) \le \frac{1}{2}(\deg \mathfrak{u} - \deg \mathfrak{s} + \deg \mathfrak{v} - \deg \mathfrak{t})$.

b) $B_{\mathfrak{st}}^{\mathcal{O}} \otimes_{\mathcal{O}} 1_K = B'_{\mathfrak{st}} + C_{\mathfrak{st}}$, *where $B'_{\mathfrak{st}}$ is homogeneous of degree* $\deg \mathfrak{s} + \deg \mathfrak{t}$ *and $C_{\mathfrak{st}}$ is a sum of homogeneous terms of degree strictly larger than* $\deg B_{\mathfrak{st}}$.

Moreover, $\{\, B'_{\mathfrak{st}} \mid (\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}(\mathcal{P}_n^\Lambda) \,\}$ is a graded cellular basis of $\mathcal{H}_n^\Lambda$ and

$$B'_{\mathfrak{st}} = \psi_{\mathfrak{st}} + \sum_{(\mathfrak{u},\mathfrak{v})\blacktriangleright(\mathfrak{s},\mathfrak{t})} r_{\mathfrak{uv}} \psi_{\mathfrak{uv}},$$

for some $r_{\mathfrak{uv}} \in K$.

There is a similar graded cellular basis of $\mathcal{H}_n^\Lambda(\mathcal{O})$ when K is a field of positive characteristic, however, its' description is more complicated because the corresponding polynomial $p_{\mathfrak{uv}}^{\mathfrak{st}}(x)$ do not necessarily satisfy the degree bound in Theorem B(a). The construction of the B-basis is reminiscent of the Kazhdan-Lusztig basis [20]. The B-basis can depend on the choice of multicharge.

As remarked above, the basis element $\psi_{\mathfrak{st}}$ depends upon choices of reduced expression for the permutations $d(\mathfrak{s}), d(\mathfrak{t}) \in \mathfrak{S}_n$, yet for any choice $B_{\mathfrak{st}}$ is equal to $\psi_{\mathfrak{st}}$ plus a linear combination of more dominant terms by Theorem B. The B-basis elements depend only on the indexing tableaux, and not on choices of reduced expressions. In this sense, the B-basis corrects for a deficiency in the definition of the ψ-bases.

To prove the two theorems above, we define a seminormal basis of a semisimple Hecke algebra to be a basis of $\mathcal{H}_n^\Lambda$ of simultaneous eigenvectors for the Gelfand-Zetlin subalgebra of $\mathcal{H}_n^\Lambda$. Seminormal bases are classical objects that are ubiquitous in the literature, having been rediscovered many times since were first introduced for the symmetric groups by Young in 1900 [37].

Seminormal basis elements are a basis of eigenvectors for the action of the Jucys-Murphy elements on the regular representation of $\mathcal{H}_n^\Lambda(K)$. Eigenbases are, of course, only unique up to scalar multiplication. This paper starts by introducing *seminormal coefficient systems* that gives a combinatorial framework for describing the structure constants of the algebra in terms of the choice of eigenvectors. The real surprise is that seminormal coefficient systems encode the KLR grading.

The close connections between the semisimple representation theory and the KLR gradings is made even more explicit in the third main result of this paper that gives a closed formula for the Gram determinants of the semisimple Specht modules of these algebras. Closed formulas for these determinants already exist in the literature [4, 16–18], however, all of these formulas describe these determinants as rational functions (or rational numbers in the degenerate case). The theorem below gives the first *integral* formula for these determinants.

In order to state the closed integral formulas for the Gram determinant of the Specht module $S^{\boldsymbol{\lambda}}$, for a multipartition $\boldsymbol{\lambda}$ define

$$\deg_e(\boldsymbol{\lambda}) = \sum_{\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})} \deg_e(\mathfrak{t}) \in \mathbb{Z},$$

where $\operatorname{Std}(\boldsymbol{\lambda})$ is the set of standard $\boldsymbol{\lambda}$-tableaux. Let $\Phi_e(t) \in \mathbb{Z}[t]$ be the eth cyclotomic polynomial for $e > 1$. We prove the following (see Theorem 3.21 for a more precise statement).

Theorem C. *Suppose that $\mathcal{H}_n^\Lambda$ is a semisimple cyclotomic Hecke algebra over $\mathbb{Q}(t)$, with Hecke parameter t. Let $\boldsymbol{\lambda}$ be a multipartition of n. Then the Gram determinant of the Specht module $S^{\boldsymbol{\lambda}}$ is equal to*

$$t^N \prod_{e>1} \Phi_e(t)^{\deg_e(\boldsymbol{\lambda})},$$

for a known integer N. In particular, $\deg_e(\boldsymbol{\lambda}) \geq 0$, for all $e \in \{0, 2, 3, 4, \dots\}$.

As the integers $\deg_e(\boldsymbol{\lambda})$ are defined combinatorially, it should be possible to give a purely combinatorial proof that $\deg_e(\boldsymbol{\lambda}) \geq 0$. In Section 3.3 we give two representation theoretic proofs of this result. The first proof is elementary but not very enlightening. The second proof uses deep positivity properties of the graded decomposition numbers of $\mathcal{H}_n^\Lambda(\mathbb{C})$ to show that the tableaux combinatorics of $\mathcal{H}_n^\Lambda$ provides a framework for giving purely combinatorial formulas for the graded dimensions of the simple $\mathcal{H}_n^\Lambda$-modules and for the graded decomposition numbers of $\mathcal{H}_n^\Lambda$. Interestingly, we show that there is a close connection between the graded dimensions of the simple $\mathcal{H}_n^\Lambda$-modules and the graded decomposition numbers for $\mathcal{H}_n^\Lambda$. Note that in characteristic zero, the graded decomposition numbers of $\mathcal{H}_n^\Lambda$ are parabolic Kazhdan-Lusztig polynomials of type A [7], so our results show that the tableaux combinatorics leads to combinatorial formulas for these polynomials. Unfortunately, we are only able to prove that such formulas exist and we are not able to make them explicit or to show that they are canonical in any way.

The outline of this paper is as follows. Chapter 2 defines the cyclotomic Hecke algebras of type A, giving a uniform presentation for the degenerate and non-degenerate algebras. Previously these algebras have been treated separately in the literature. We then recall the basic results about these algebras that we need from the literature, including Brundan and Kleshchev's isomorphism theorem [6]. Chapter 3 develops the theory of seminormal bases for these algebras in full generality. We completely classify the seminormal bases of $\mathcal{H}_n^\Lambda$ and then use them to prove Theorem C, thus establishing a link between the semisimple representation theory of $\mathcal{H}_n^\Lambda$ and the quiver Hecke algebra $\mathcal{R}_n^\Lambda$. Using this we prove the existence of combinatorial formulas for the graded dimensions of the simple modules and the graded decomposition numbers of $\mathcal{H}_n^\Lambda$. In Chapter 4 we use the theory of seminormal forms to construct a deformation of the cyclotomic quiver Hecke algebras of type A, culminating with the proof of Theorem A. Chapter 5 builds on Theorem A to give a quicker construction of the graded cellular basis of $\mathcal{H}_n^\Lambda(K)$, over a field K, which was one of the main results of [14]. Finally, in Chapter 6 we use Theorem A to show that $\mathcal{H}_n^\Lambda(K)$ has the distinguished graded cellular basis described in Theorem B.

2. Cyclotomic Hecke algebras

This chapter defines the cyclotomic Hecke and quiver Hecke algebras of type A and it introduces some of the basic machinery that we need for understanding these algebras. We give a new presentation for the *cyclotomic Hecke algebras of type A*, which simultaneously captures the degenerate and non-degenerate cyclotomic Hecke algebras that currently appear in the literature, and then we recall the results from the literature that we need, including Brundan and Kleshchev's isomorphism theorem [6].

2.1. **Quiver combinatorics.** Fix an integer $e \in \{0, 2, 3, 4 \dots\}$ and let Γ_e be the oriented quiver with vertex set $I = \mathbb{Z}/e\mathbb{Z}$ and edges $i \longrightarrow i+1$, for $i \in I$. If $i, j \in I$ and i and j are not connected by an edge in Γ_e then we write $i \not\!\!- j$.

To the quiver Γ_e we attach the Cartan matrix $(c_{ij})_{i,j\in I}$, where

$$c_{i,j} = \begin{cases} 2, & \text{if } i = j,\\ -1, & \text{if } i \to j \text{ or } i \leftarrow j,\\ -2, & \text{if } i \leftrightarrows j,\\ 0, & \text{otherwise,}\end{cases}$$

Let $\widehat{\mathfrak{sl}}_e$ be the corresponding Kac-Moody algebra [19] with fundamental weights $\{\, \Lambda_i \mid i \in I \,\}$, positive weight lattice $P^+_e = \sum_{i\in I} \mathbb{N}\Lambda_i$ and positive root lattice $Q^+ = \bigoplus_{i\in I} \mathbb{N}\alpha_i$. Let $(\cdot,\cdot)$ be the bilinear form determined by

$$(\alpha_i, \alpha_j) = c_{ij} \qquad \text{and} \qquad (\Lambda_i, \alpha_j) = \delta_{ij}, \qquad \text{for } i, j \in I.$$

More details can be found, for example, in [19, Chapter 1].

Fix, once and for all, a **multicharge** $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_\ell) \in \mathbb{Z}^\ell$ that is a sequence of integers such that if $e \neq 0$ then $\kappa_l - \kappa_{l+1} \geq n$ for $1 \leq l < \ell$. Define $\Lambda = \Lambda_e(\boldsymbol{\kappa}) = \Lambda_{\bar\kappa_1} + \dots + \Lambda_{\bar\kappa_\ell}$, where $\bar\kappa = \kappa \pmod{e}$. Equivalently, Λ is the unique element of P^+_e such that

$$(\Lambda, \alpha_i) = \#\{\, 1 \leq l \leq \ell \mid \kappa_l \equiv i \pmod{e} \,\}, \qquad \text{for all } i \in I. \tag{2.1}$$

All of the bases for the modules and algebras in this paper depend implicitly on the choice of $\boldsymbol{\kappa}$ even though the algebras themselves depend only on Λ.

2.2. **Cyclotomic Hecke algebras.** This section defines the cyclotomic Hecke algebras of type A and explains the connection between these algebras and the degenerate and non-degenerate Hecke algebras of type $G(\ell, 1, n)$.

Fix an integral domain $\mathcal{O}$ that contains an invertible element $\xi \in \mathcal{O}^\times$.

2.2. **Definition.** *Fix integers $n \geq 0$ and $\ell \geq 1$. Then the* ***cyclotomic Hecke algebra of type** A with Hecke parameter $\xi \in \mathcal{O}^\times$ and cyclotomic parameters $Q_1, \dots, Q_\ell \in \mathcal{O}$ is the unital associative $\mathcal{O}$-algebra $\mathcal{H}_n = \mathcal{H}_n(\mathcal{O}, \xi, Q_1, \dots, Q_\ell)$ with generators $L_1, \dots, L_n$, $T_1, \dots, T_{n-1}$ that are subject to the relations*

$$\prod_{l=1}^{\ell}(L_1 - Q_l) = 0, \qquad (T_r + 1)(T_r - \xi) = 0,$$
$$L_r L_t = L_t L_r, \qquad T_r T_s = T_s T_r \quad \text{if } |r - s| > 1,$$
$$T_s T_{s+1} T_s = T_{s+1} T_s T_{s+1}, \qquad T_r L_t = L_t T_r, \quad \text{if } t \neq r, r+1,$$
$$L_{r+1}(T_r - \xi + 1) = T_r L_r + 1,$$

where $1 \leq r < n$, $1 \leq s < n - 1$ and $1 \leq t \leq n$.

2.3. *Remark.* If $\xi = 1$ then, by definition, $\mathcal{H}_n$ is a degenerate cyclotomic Hecke algebra of type $G(\ell, 1, n)$. If $\xi \neq 1$ then $\mathcal{H}_n$ is (isomorphic to) an integral cyclotomic Hecke algebra of type $G(\ell, 1, n)$. To see this define $L'_k = (\xi - 1)L_k + 1$, for $1 \le k \le n$, and observe that $\mathcal{H}_n$ is generated by $L'_1, T_1, \dots, T_{n-1}$ subject to the usual relations for these algebras as originally defined by Ariki and Koike [3]. It is now easy to verify our claim. For each $1 \le m \le n$, an eigenvector for L_m of eigenvalue $[k]_\xi$ is the same as an eigenvector for L'_m of eigenvalue ξ^k. The presentation of $\mathcal{H}_n$ in Definition 2.2 unifies the definition of the 'degenerate' and 'non-degenerate' Hecke algebras, which corresponds to the cases where $\xi = 1$ or $\xi \neq 1$, respectively.

Let $\mathfrak{S}_n$ be the **symmetric group** on n letters. For $1 \le r < n$ let $s_r = (r, r+1)$ be the corresponding simple transposition. Then $\{s_1, \dots, s_{n-1}\}$ is the standard set of Coxeter generators for $\mathfrak{S}_n$. A **reduced expression** for $w \in \mathfrak{S}_n$ is a word $w = s_{r_1}, \dots s_{r_k}$ with k minimal and $1 \le r_j < n$ for $1 \le j \le k$. If $w = s_{r_1} \dots s_{r_k}$ is reduced then set $T_w = T_{r_1} \dots T_{r_k}$. Then T_w is independent of the choice of reduced expression since the braid relations hold in $\mathcal{H}_n$. It follows arguing as in [3, Theorem 3.3] that $\mathcal{H}_n$ is free as an $\mathcal{O}$-module with basis

$$\{\, L_1^{a_1} \dots L_n^{a_n} T_w \mid 0 \le a_1, \dots, a_n < \ell \text{ and } w \in \mathfrak{S}_n \,\}.$$

Consequently, $\mathcal{H}_n$ is free as an $\mathcal{O}$-module of rank $\ell^n n!$, which is the order of the complex reflection group of type $G(\ell, 1, n)$.

We now restrict our attention to the case of *integral* cyclotomic parameters. Recall that for any integer d and $t \in \mathcal{O}$ the **quantum integer** $[d]_t$ is

$$[d]_t = \begin{cases} 1 + t + \dots + t^{d-1}, & \text{if } d \ge 0, \\ -(t^{-1} + t^{-2} + \dots + t^d), & \text{if } d < 0. \end{cases}$$

When t is understood we write $[d] = [d]_t$. Set $[d]_t^! = [d]^! = [1][2] \dots [d]$ when $d > 0$.

An **integral cyclotomic Hecke algebra** is a cyclotomic Hecke algebra $\mathcal{H}_n$ with cyclotomic parameters of the form $Q_r = [\kappa_r]_\xi$, for $\kappa_1, \dots, \kappa_\ell \in \mathbb{Z}$. The sequence of integers $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_\ell) \in \mathbb{Z}^\ell$ is the **multicharge** of $\mathcal{H}_n$.

Translating the Morita equivalence theorems of [11, Theorem 1.1] and [5, Theorem 5.19] into the current setting, every cyclotomic Hecke algebras of type A is Morita equivalent to a direct sum of tensor products of integral cyclotomic Hecke algebras. Therefore, there is no loss of generality in restricting our attention to the integral cyclotomic Hecke algebras of type A.

Recall that $\Lambda \in P_e^+$ and that we have fixed an integer $e \in \{0, 2, 3, 4, \dots\}$. We assume that $\mathcal{O}$ contains an invertible element $\xi \in \mathcal{O}^\times$ such that $[e]_\xi = 0$ if $e > 0$ and $[f]_\xi \neq 0$ for all $f \ge 1$ if $e = 0$. Hence, either:

a) $\xi = 1$ and e is prime and equal to the characteristic of $\mathcal{O}$,
b) $e > 0$ and ξ is a primitive eth root of unity, or,
c) $e = 0$ and ξ is not a root of unity.

In addition, fix a multicharge $\boldsymbol{\kappa}$ so that $\Lambda = \Lambda_e(\boldsymbol{\kappa})$ as in (2.1).

Let $\mathcal{H}_n^\Lambda = \mathcal{H}_n^\Lambda(\mathcal{O})$ be the integral cyclotomic Hecke algebra $\mathcal{H}_n(\mathcal{O}, \xi, \boldsymbol{\kappa})$. Using the definitions it is easy to see that, up to isomorphism, $\mathcal{H}_n^\Lambda$ depends only on ξ and Λ. In fact, by Theorem 2.14 below, it depends only on e and Λ. Nonetheless, many of the constructions that follow, particularly the definitions of bases, depend upon the choice of $\boldsymbol{\kappa}$.

2.3. **Graded algebras and cellular bases.** This section recalls the definitions and results from the representation theory of (graded) cellular algebras that we need.

Let A be a unital associative $\mathcal{O}$-algebra that is free and of finite rank as an $\mathcal{O}$-module. In this paper a **graded module** will always mean a $\mathbb{Z}$-graded module.

That is, an $\mathcal{O}$-module M that has a decomposition $M = \bigoplus_{n\in\mathbb{Z}} M_d$ as an $\mathcal{O}$-module. If $m \in M_d$, for $d \in \mathbb{Z}$, then m is **homogeneous** of **degree** d and we set $\deg m = d$. If M is a graded $\mathcal{O}$-module and $s \in \mathbb{Z}$ let $M\langle s\rangle$ be the graded $\mathcal{O}$-module obtained by shifting the grading on M up by s; that is, $M\langle s\rangle_d = M_{d-s}$, for $d \in \mathbb{Z}$.

Similarly a **graded algebra** is a unital associative $\mathcal{O}$-algebra $A = \bigoplus_{d\in\mathbb{Z}} A_d$ that is a graded $\mathcal{O}$-module such that $A_dA_e \subseteq A_{d+e}$, for all $d, e \in \mathbb{Z}$. It follows that $1 \in A_0$ and that A_0 is a graded subalgebra of A. A graded (right) A-module is a graded $\mathcal{O}$-module M such that $\underline{M}$ is an $\underline{A}$-module and $M_dA_e \subseteq M_{d+e}$, for all $d, e \in \mathbb{Z}$, where $\underline{M}$ and $\underline{A}$ mean forgetting the $\mathbb{Z}$-grading structures on M and A respectively. Graded submodules, graded left A-modules and so on are all defined in the obvious way.

The following definition extends Graham and Lehrer's [12] definition of cellular algebras to the graded setting.

2.4. **Definition** (Graded cellular algebras [12,14])**.** *Suppose that A is an $\mathcal{O}$-algebra that is free of finite rank over $\mathcal{O}$. A* ***cell datum*** *for A is an ordered triple $(\mathcal{P}, T, C)$, where $(\mathcal{P}, \rhd)$ is the* ***weight poset****, $T(\lambda)$ is a finite set for $\lambda \in \mathcal{P}$, and*

$$C\colon \coprod_{\lambda\in\mathcal{P}} T(\lambda)\times T(\lambda) \longrightarrow A;\, (\mathfrak{s},\mathfrak{t}) \mapsto c_{\mathfrak{s}\mathfrak{t}},$$

is an injective function such that:

(GC$_1$) *$\{\, c_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \in T(\lambda)$ for $\lambda \in \mathcal{P}\,\}$ is an $\mathcal{O}$-basis of A.*

(GC$_2$) *If $\mathfrak{s},\mathfrak{t} \in T(\lambda)$, for some $\lambda \in \mathcal{P}$, and $a \in A$ then there exist scalars $r_{\mathfrak{t}\mathfrak{v}}(a)$, which do not depend on $\mathfrak{s}$, such that*

$$c_{\mathfrak{s}\mathfrak{t}}a = \sum_{\mathfrak{v}\in T(\lambda)} r_{\mathfrak{t}\mathfrak{v}}(a) c_{\mathfrak{s}\mathfrak{v}} \pmod{A^{\rhd\lambda}},$$

where $A^{\rhd\lambda}$ is the $\mathcal{O}$-submodule of A spanned by $\{\, c_{\mathfrak{a}\mathfrak{b}} \mid \mu \rhd \lambda$ and $\mathfrak{a},\mathfrak{b} \in T(\mu)\,\}$.

(GC$_3$) *The $\mathcal{O}$-linear map $*\colon A \longrightarrow A$ determined by $(c_{\mathfrak{s}\mathfrak{t}})^* = c_{\mathfrak{t}\mathfrak{s}}$, for all $\lambda \in \mathcal{P}$ and all $\mathfrak{s},\mathfrak{t} \in T(\lambda)$, is an anti-isomorphism of A.*

A ***cellular algebra*** *is an algebra that has a cell datum. If A is a cellular algebra with cell datum $(\mathcal{P}, T, C)$ then the basis $\{\, c_{\mathfrak{s}\mathfrak{t}} \mid \lambda \in \mathcal{P}$ and $\mathfrak{s},\mathfrak{t} \in T(\lambda\,\}$ is a* ***cellular basis*** *of A with $*$ its cellular algebra anti-automorphism.*

If, in addition, A is a $\mathbb{Z}$-graded algebra then a ***graded cell datum*** *for A is a cell datum $(\mathcal{P}, T, C)$ together with a degree function*

$$\deg\colon \coprod_{\lambda\in\mathcal{P}} T(\lambda) \longrightarrow \mathbb{Z}$$

such that

(GC$_d$) *the element $c_{\mathfrak{s}\mathfrak{t}}$ is homogeneous of degree $\deg c_{\mathfrak{s}\mathfrak{t}} = \deg(\mathfrak{s}) + \deg(\mathfrak{t})$, for all $\lambda \in \mathcal{P}$ and $\mathfrak{s},\mathfrak{t} \in T(\lambda)$.*

In this case, A is a ***graded cellular algebra*** *with* ***graded cellular basis*** *$\{c_{\mathfrak{s}\mathfrak{t}}\}$.*

Fix a (graded) cellular algebra A with graded cellular basis $\{c_{\mathfrak{s}\mathfrak{t}}\}$. If $\lambda \in \mathcal{P}$ then the graded **cell module** is the $\mathcal{O}$-module C^λ with basis $\{\, c_{\mathfrak{t}} \mid \mathfrak{t} \in T(\lambda)\,\}$ and with A-action

$$c_{\mathfrak{t}}a = \sum_{\mathfrak{v}\in T(\lambda)} r_{\mathfrak{t}\mathfrak{v}}(a) c_{\mathfrak{v}},$$

where the scalars $r_{\mathfrak{t}\mathfrak{v}}(a) \in \mathcal{O}$ are the same scalars appearing in (GC$_2$). One of the key properties of the graded cell modules is that by [14, Lemma 2.7] they come equipped with a homogeneous bilinear form $\langle\ ,\ \rangle$ of degree zero that is determined by the equation

$$\langle c_{\mathfrak{t}}, c_{\mathfrak{u}}\rangle c_{\mathfrak{s}\mathfrak{v}} \equiv c_{\mathfrak{s}\mathfrak{t}} c_{\mathfrak{u}\mathfrak{v}} \pmod{A^{\rhd\lambda}}, \tag{2.5}$$

for $\mathfrak{s},\mathfrak{t},\mathfrak{u},\mathfrak{v}\in T(\lambda)$. The radical of this form

$$\operatorname{rad} C^\lambda = \{\, x\in C^\lambda \mid \langle x,y\rangle = 0 \text{ for all } y\in C^\lambda \,\}$$

is a graded A-submodule of C^λ so that $D^\lambda = C^\lambda/\operatorname{rad} C^\lambda$ is a graded A-module. It is shown in [14, Theorem 2.10] that

$$\{\, D^\lambda\langle k\rangle \mid \lambda\in\mathcal{P}, D^\lambda\ne 0 \text{ and } k\in\mathbb{Z}\,\}$$

is a complete set of pairwise non-isomorphic irreducible (graded) A-modules when $\mathcal{O}$ is a field.

2.4. **Multipartitions and tableaux.** A **partition** of d is a weakly decreasing sequence $\lambda=(\lambda_1,\lambda_2,\dots)$ of non-negative integers such that $|\lambda|=\lambda_1+\lambda_2+\dots=d$. An **$\ell$-multipartition** of n is an ℓ-tuple $\boldsymbol{\lambda}=(\lambda^{(1)},\dots,\lambda^{(\ell)})$ of partitions such that $|\lambda^{(1)}|+\dots+|\lambda^{(\ell)}|=n$. We identify the multipartition $\boldsymbol{\lambda}$ with its **diagram** that is the set of **nodes** $[\![\boldsymbol{\lambda}]\!]=\{\,(l,r,c) \mid 1\le c\le\lambda^{(l)}_r \text{ for } 1\le l\le\ell\,\}$, which we think of as an ordered ℓ-tuple of arrays of boxes in the plane. For example, if $\boldsymbol{\lambda}=(3,1^2|2,1|3,2)$ then

$$[\![\boldsymbol{\lambda}]\!]=\left(\begin{array}{|c|c|c|}\hline & & \\\hline \\\cline{1-1} \\\cline{1-1}\end{array}\;\middle|\;\begin{array}{|c|c|}\hline & \\\hline \\\cline{1-1}\end{array}\;\middle|\;\begin{array}{|c|c|c|}\hline & & \\\hline & \\\cline{1-2}\end{array}\right).$$

In this way we talk of the **rows**, **columns** and **components** of $\boldsymbol{\lambda}$.

Given two nodes $\alpha=(l,r,c)$ and $\beta=(l',r',c')$ then β is **below** α, or α is **above** β, if $(l,r,c)<(l',r',c')$ in the lexicographic order.

The set of multipartitions of n becomes a poset ordered by **dominance** where $\boldsymbol{\lambda}$ dominates $\boldsymbol{\mu}$, or $\boldsymbol{\lambda}\trianglerighteq\boldsymbol{\mu}$, if

$$\sum_{k=1}^{l-1}|\lambda^{(k)}|+\sum_{j=1}^{i}\lambda^{(l)}_j\ge\sum_{k=1}^{l-1}|\mu^{(k)}|+\sum_{j=1}^{i}\mu^{(l)}_j,$$

for $1\le l\le\ell$ and $i\ge1$. If $\boldsymbol{\lambda}\trianglerighteq\boldsymbol{\mu}$ and $\boldsymbol{\lambda}\ne\boldsymbol{\mu}$ then write $\boldsymbol{\lambda}\triangleright\boldsymbol{\mu}$. Let $\mathcal{P}^\Lambda_n=(\mathcal{P}^\Lambda_n,\trianglerighteq)$ be the poset of multipartitions of n ordered by dominance.

Fix a multipartition $\boldsymbol{\lambda}$. Then a **$\boldsymbol{\lambda}$-tableau** is a bijective map $\mathfrak{t}\colon[\![\boldsymbol{\lambda}]\!]\longrightarrow\{1,2,\dots,n\}$, which we identify with a labelling of $[\![\boldsymbol{\lambda}]\!]$ by $\{1,2,\dots,n\}$. For example,

$$\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3\\\hline 4 \\\cline{1-1} 5 \\\cline{1-1}\end{array}\;\middle|\;\begin{array}{|c|c|}\hline 6 & 7\\\hline 8\\\cline{1-1}\end{array}\;\middle|\;\begin{array}{|c|c|c|}\hline 9 & 10 & 11\\\hline 12 & 13\\\cline{1-2}\end{array}\right)\quad\text{and}\quad\left(\begin{array}{|c|c|c|}\hline 9 & 12 & 13\\\hline 10 \\\cline{1-1} 11 \\\cline{1-1}\end{array}\;\middle|\;\begin{array}{|c|c|}\hline 6 & 8\\\hline 7\\\cline{1-1}\end{array}\;\middle|\;\begin{array}{|c|c|c|}\hline 1 & 3 & 5\\\hline 2 & 4\\\cline{1-2}\end{array}\right)$$

are both $\boldsymbol{\lambda}$-tableaux when $\boldsymbol{\lambda}=(3,1^2|2,1|3,2)$ as above. In this way we speak of the rows, columns and components of tableaux. If $\mathfrak{t}$ is a tableau and $1\le k\le n$ set $\operatorname{comp}_{\mathfrak{t}}(k)=l$ if k appears in the lth component of $\mathfrak{t}$.

A $\boldsymbol{\lambda}$-tableau is **standard** if its entries increase along rows and columns in each component. Both of the tableaux above are standard. Let $\operatorname{Std}(\boldsymbol{\lambda})$ be the set of standard $\boldsymbol{\lambda}$-tableaux and let $\operatorname{Std}(\mathcal{P}^\Lambda_n)=\bigcup_{\boldsymbol{\lambda}\in\mathcal{P}^\Lambda_n}\operatorname{Std}(\boldsymbol{\lambda})$. Similarly set $\operatorname{Std}^2(\boldsymbol{\lambda})=\{\,(\mathfrak{s},\mathfrak{t}) \mid \mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})\,\}$ and $\operatorname{Std}^2(\mathcal{P}^\Lambda_n)=\{\,(\mathfrak{s},\mathfrak{t}) \mid \mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda}) \text{ for some } \boldsymbol{\lambda}\in\mathcal{P}^\Lambda_n\,\}$.

If $\mathfrak{t}$ is a $\boldsymbol{\lambda}$-tableau set $\operatorname{Shape}(\mathfrak{t})=\boldsymbol{\lambda}$ and let $\mathfrak{t}_{\downarrow m}$ be the subtableau of $\mathfrak{t}$ that contains the numbers $\{1,2,\dots,m\}$. If $\mathfrak{t}$ is a standard $\boldsymbol{\lambda}$-tableau then $\operatorname{Shape}(\mathfrak{t}_{\downarrow m})$ is a multipartition for all $m\ge0$. We extend the dominance ordering to the set of all standard tableaux by defining $\mathfrak{s}\trianglerighteq\mathfrak{t}$ if

$$\operatorname{Shape}(\mathfrak{s}_{\downarrow m})\trianglerighteq\operatorname{Shape}(\mathfrak{t}_{\downarrow m}),$$

for $1\le m\le n$. As before, we write $\mathfrak{s}\triangleright\mathfrak{t}$ if $\mathfrak{s}\trianglerighteq\mathfrak{t}$ and $\mathfrak{s}\ne\mathfrak{t}$. We extend the dominance to $\operatorname{Std}^2(\mathcal{P}^\Lambda_n)$ by declaring that $(\mathfrak{s},\mathfrak{t})\trianglerighteq(\mathfrak{u},\mathfrak{v})$ if $\mathfrak{s}\trianglerighteq\mathfrak{u}$ and $\mathfrak{t}\trianglerighteq\mathfrak{v}$. Similarly, $(\mathfrak{s},\mathfrak{t})\triangleright(\mathfrak{u},\mathfrak{v})$ if $(\mathfrak{s},\mathfrak{t})\trianglerighteq(\mathfrak{u},\mathfrak{v})$ and $(\mathfrak{s},\mathfrak{t})\ne(\mathfrak{u},\mathfrak{v})$

It is easy to see that there are unique standard $\boldsymbol{\lambda}$-tableaux $\mathfrak{t}^{\boldsymbol{\lambda}}$ and $\mathfrak{t}_{\boldsymbol{\lambda}}$ such that $\mathfrak{t}^{\boldsymbol{\lambda}} \trianglerighteq \mathfrak{t} \trianglerighteq \mathfrak{t}_{\boldsymbol{\lambda}}$, for all $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. The tableau $\mathfrak{t}^{\boldsymbol{\lambda}}$ has the numbers $1, 2, \dots, n$ entered in order from left to right along the rows of $\mathfrak{t}^{\lambda^{(1)}}$, and then $\mathfrak{t}^{\lambda^{(2)}}, \dots, \mathfrak{t}^{\lambda^{(\ell)}}$ and similarly, $\mathfrak{t}_{\boldsymbol{\lambda}}$ is the tableau with the numbers $1, \dots, n$ entered in order down the columns of $\mathfrak{t}^{\lambda^{(\ell)}}, \dots, \mathfrak{t}^{\lambda^{(2)}}, \mathfrak{t}^{\lambda^{(1)}}$. When $\boldsymbol{\lambda} = (3, 1^2|2, 1|3, 2)$ then the two $\boldsymbol{\lambda}$-tableaux displayed above are $\mathfrak{t}^{\boldsymbol{\lambda}}$ and $\mathfrak{t}_{\boldsymbol{\lambda}}$.

Given a standard $\boldsymbol{\lambda}$-tableau $\mathfrak{t}$ define $d(\mathfrak{t}) \in \mathfrak{S}_n$ to be the permutation such that $\mathfrak{t} = \mathfrak{t}^{\boldsymbol{\lambda}} d(\mathfrak{t})$. Let $\leq$ be the Bruhat order on $\mathfrak{S}_n$ with the convention that $1 \leq w$ for all $w \in \mathfrak{S}_n$. By a well-known result of Ehresmann and James, if $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ then $\mathfrak{s} \trianglerighteq \mathfrak{t}$ if and only if $d(\mathfrak{s}) \leq d(\mathfrak{t})$; see, for example, [26, Theorem 3.8].

Recall from Section 2.1 that we have fixed a multicharge $\kappa \in \mathbb{Z}^\ell$. The **residue** of the node $A = (l, r, c)$ is $\operatorname{res}(A) = \kappa_l + c - r \pmod e$ (where we adopt the convention that $i \equiv i \pmod 0$, for $i \in \mathbb{Z}$). Thus, $\operatorname{res}(A) \in I$. A node A is an **i-node** if $\operatorname{res}(A) = i$. If $\mathfrak{t}$ is a $\boldsymbol{\mu}$-tableaux and $1 \leq k \leq n$ then the **residue** of k in $\mathfrak{t}$ is $\operatorname{res}_{\mathfrak{t}}(k) = \operatorname{res}(A)$, where $A \in \boldsymbol{\mu}$ is the unique node such that $\mathfrak{t}(A) = k$. The **residue sequence** of $\mathfrak{t}$ is

$$\operatorname{res}(\mathfrak{t}) = (\operatorname{res}_{\mathfrak{t}}(1), \operatorname{res}_{\mathfrak{t}}(2), \dots, \operatorname{res}_{\mathfrak{t}}(n)) \in I^n.$$

As an important special case we set $\mathbf{i}^\mu = \operatorname{res}(\mathfrak{t}^{\boldsymbol{\mu}})$, for $\boldsymbol{\mu} \in \mathcal{P}_n^\Lambda$.

Refine the dominance ordering on the set of standard tableaux by defining $\mathfrak{s} \underline{\blacktriangleright} \mathfrak{t}$ if $\mathfrak{s} \trianglerighteq \mathfrak{t}$ and $\operatorname{res}(\mathfrak{s}) = \operatorname{res}(\mathfrak{t})$. Similarly, we write $(\mathfrak{s}, \mathfrak{t}) \underline{\blacktriangleright} (\mathfrak{u}, \mathfrak{v})$ if $(\mathfrak{s}, \mathfrak{t}) \trianglerighteq (\mathfrak{u}, \mathfrak{v})$, $\operatorname{res}(\mathfrak{s}) = \operatorname{res}(\mathfrak{u})$ and $\operatorname{res}(\mathfrak{t}) = \operatorname{res}(\mathfrak{v})$ and $(\mathfrak{s}, \mathfrak{t}) \blacktriangleright (\mathfrak{u}, \mathfrak{v})$ now has the obvious meaning.

Following Brundan, Kleshchev and Wang [8, Definition. 3.5] we now define the degree of a standard tableau. Suppose that $\boldsymbol{\mu} \in \mathcal{P}_n^\Lambda$. A node A is an **addable node** of $\boldsymbol{\mu}$ if $A \notin \boldsymbol{\mu}$ and $\boldsymbol{\mu} \cup \{A\}$ is (the diagram of) a multipartition of $n+1$. Similarly, a node B is a **removable node** of $\boldsymbol{\mu}$ if $B \in \boldsymbol{\mu}$ and $\boldsymbol{\mu} \setminus \{B\}$ is a multipartition of $n-1$. Suppose that A is an i-node and define integers

$$d_A(\boldsymbol{\mu}) = \#\left\{ \begin{matrix} \text{addable } i\text{-nodes of } \boldsymbol{\mu} \\ \text{strictly below } A \end{matrix} \right\} - \#\left\{ \begin{matrix} \text{removable } i\text{-nodes of } \boldsymbol{\mu} \\ \text{strictly below } A \end{matrix} \right\}.$$

If $\mathfrak{t}$ is a standard $\boldsymbol{\mu}$-tableau define its **degree** inductively by setting $\deg_e(\mathfrak{t}) = 0$, if $n = 0$, and if $n > 0$ then

$$\deg_e(\mathfrak{t}) = \deg_e(\mathfrak{t}_{\downarrow(n-1)}) + d_A(\boldsymbol{\mu}), \tag{2.6}$$

where $A = \mathfrak{t}^{-1}(n)$. When e is understood we write $\deg(\mathfrak{t})$.

The following result shows that the degrees of the standard tableau are almost completely determined by the Cartan matrix (c_{ij}) of Γ_e.

2.7. **Lemma** (Brundan, Kleshchev and Wang [8, Proposition 3.13])**.** *Suppose that $\mathfrak{s}$ and $\mathfrak{t}$ are standard tableaux such that $\mathfrak{s} \triangleright \mathfrak{t} = \mathfrak{s}(r, r+1)$, where $1 \leq r < n$ and $\mathbf{i} \in I^n$. Let $\mathbf{i} = \operatorname{res}(\mathfrak{s})$. Then $\deg_e(\mathfrak{s}) = \deg_e(\mathfrak{t}) + c_{i_r i_{r+1}}$.*

2.5. **The Murphy basis and cyclotomic Specht modules.** The cyclotomic Hecke algebra $\mathcal{H}_n^\Lambda$ is a cellular algebra with several different cellular bases. This section introduces one of these bases, the Murphy basis, and uses it to define the Specht modules and simple modules of $\mathcal{H}_n^\Lambda$.

Fix a multipartition $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$. Following [10, Definition 3.14] and [4, §6], if $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ define $m_{\mathfrak{s}\mathfrak{t}} = T_{d(\mathfrak{s})^{-1}} m_{\boldsymbol{\lambda}} T_{d(\mathfrak{t})}$, where $m_{\boldsymbol{\lambda}} = u_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}}$ where

$$u_{\boldsymbol{\lambda}} = \prod_{1 \leq l < \ell} \prod_{r=1}^{|\lambda^{(1)}| + \dots + |\lambda^{(l)}|} \xi^{-\kappa_{l+1}}(L_r - [\kappa_{l+1}]) \quad \text{and} \quad x_{\boldsymbol{\lambda}} = \sum_{w \in \mathfrak{S}_{\boldsymbol{\lambda}}} T_w.$$

Let $*$ be the unique anti-isomorphism of $\mathcal{H}_n^\Lambda$ that fixes each of the generators $T_1, \dots, T_{n-1}, L_1, \dots, L_n$ of Definition 2.2.

2.8. **Theorem** ([10, Theorem 3.26] and [4, Theorem 6.3]). *The cyclotomic Hecke algebra $\mathcal{H}_n^\Lambda$ is free as an $\mathcal{O}$-module with cellular basis*

$$\{\, m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \,\}$$

with respect to the weight poset $(\mathcal{P}_n^\Lambda, \trianglerighteq)$ and automorphism $*$.

Proof. This theorem can be proved uniformly in all cases by modifying the argument of [10, Theorem 3.26], however, for future reference we explain how to deduce this result from the literature for the degenerate and non-degenerate algebras.

First suppose that $\xi = 1$. Then the element $m_{\boldsymbol{\lambda}}$, for $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$, coincides exactly with the corresponding elements defined for the non-degenerate cyclotomic Hecke algebras in [4, §6]. It follows that $\{\, m_{\mathfrak{s}\mathfrak{t}} \mid (\mathfrak{s}, \mathfrak{t}) \in \mathcal{P}_n^\Lambda \,\}$ is the Murphy basis of the degenerate cyclotomic Hecke algebra $\mathcal{H}_n^\Lambda$ defined in [4, §6] and that the theorem is just a restatement of [4, Theorem 6.3] when $\xi = 1$.

Now suppose that $\xi \neq 1$ and, as in Remark 2.3, let $L'_r = (\xi - 1)L_r + 1$ be the 'non-degenerate' Jucys-Murphy elements for $\mathcal{H}_n^\Lambda$, for $1 \le r \le n$. An application of the definitions shows that if $\kappa \in \mathbb{Z}$ then

$$\xi^{-\kappa}(L_r - [\kappa]) = \frac{\xi^{-\kappa}}{\xi - 1}(L'_r - \xi^\kappa).$$

Therefore, $u_{\boldsymbol{\lambda}}$ is a scalar multiple of the element $u^+_{\boldsymbol{\lambda}}$ given by [10, Definition 3.1,3.5]. Consequently, if $(\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ then $m_{\mathfrak{s}\mathfrak{t}}$ is a scalar multiple of the corresponding Murphy basis element from [10, Definition 3.14]. Hence, the theorem is an immediate consequence of [10, Theorem 3.26] in the non-degenerate case. □

Suppose that $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$. The (cyclotomic) **Specht module** $\underline{S}^{\boldsymbol{\lambda}}$ is the cell module associated to $\boldsymbol{\lambda}$ using the (ungraded) cellular basis $\{\, m_{\mathfrak{s}\mathfrak{t}} \mid (\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda) \,\}$. We underline $\underline{S}^{\boldsymbol{\lambda}}$ to emphasize that $\underline{S}^{\boldsymbol{\lambda}}$ is not graded. When $\mathcal{O}$ is a field let $\underline{D}^{\boldsymbol{\lambda}} = \underline{S}^{\boldsymbol{\lambda}} / \operatorname{rad} \underline{S}^{\boldsymbol{\lambda}}$ and set $\mathcal{K}_n^\Lambda = \{\, \boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \mid \underline{D}^{\boldsymbol{\lambda}} \neq 0 \,\}$. Ariki [2] has given a combinatorial description of the set $\mathcal{K}_n^\Lambda$. By the theory of cellular algebras [12], $\{\, \underline{D}^{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathcal{K}_n^\Lambda \,\}$ is a complete set of pairwise non-isomorphic irreducible $\mathcal{H}_n^\Lambda$-modules.

The following well-known fact is fundamental to all of the results in this paper.

2.9. **Lemma.** *Suppose that $1 \le r \le n$ and that $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$. Then*

$$m_{\mathfrak{s}\mathfrak{t}} L_r \equiv [c_r(\mathfrak{t})] m_{\mathfrak{s}\mathfrak{t}} + \sum_{\substack{\mathfrak{v} \triangleright \mathfrak{t} \\ \mathfrak{v} \in \operatorname{Std}(\boldsymbol{\lambda})}} r_{\mathfrak{v}} m_{\mathfrak{s}\mathfrak{v}} \ (\mathit{mod}\ \mathcal{H}_n^{\triangleright \boldsymbol{\lambda}}),$$

for some $r_{\mathfrak{v}} \in \mathcal{O}$.

Proof. If $\xi = 1$ then this is a restatement of [4, Lemma 6.6]. If $\xi \neq 1$ then

$$m_{\mathfrak{s}\mathfrak{t}} L'_r = \xi^{c_r(\mathfrak{t})} m_{\mathfrak{s}\mathfrak{t}} + \sum_{\mathfrak{v} \triangleright \mathfrak{t}} r'_{\mathfrak{v}} m_{\mathfrak{s}\mathfrak{t}} \qquad (\operatorname{mod} \mathcal{H}_n^{\triangleright \boldsymbol{\lambda}}),$$

for some $r'_v \in \mathcal{O}$, by [17, Proposition 3.7] (and the notational translations given in the proof of Theorem 2.8). As $L_r = (L'_r - 1)/(\xi - 1)$ the result follows. □

2.6. **Cyclotomic quiver Hecke algebras.** Brundan and Kleshchev [6] have given a very different presentation of $\mathcal{H}_n^\Lambda$. This presentation is more difficult to work with but it has the advantage of showing that $\mathcal{H}_n^\Lambda$ is a $\mathbb{Z}$-graded algebra.

2.10. **Definition** (Brundan-Kleshchev [6]). *Suppose that $n \ge 0$ and $e \in \{0, 2, 3, 4, \dots\}$. The **cyclotomic quiver Hecke algebra**, or **cyclotomic Khovanov-Lauda–Rouquier algebra**, of weight Λ and type Γ_e is the unital associative $\mathcal{O}$-algebra $\mathcal{R}_n^\Lambda = \mathcal{R}_n^\Lambda(\mathcal{O})$ with generators*

$$\{\psi_1, \dots, \psi_{n-1}\} \cup \{y_1, \dots, y_n\} \cup \{\, e(\mathbf{i}) \mid \mathbf{i} \in I^n \,\}$$

and relations

$$y_1^{(\Lambda,\alpha_{i_1})}e(\mathbf{i})=0,\qquad e(\mathbf{i})e(\mathbf{j})=\delta_{\mathbf{ij}}e(\mathbf{i}),\qquad \textstyle\sum_{\mathbf{i}\in I^n}e(\mathbf{i})=1,$$

$$y_re(\mathbf{i})=e(\mathbf{i})y_r,\qquad \psi_re(\mathbf{i})=e(s_r{\cdot}\mathbf{i})\psi_r,\qquad y_ry_s=y_sy_r,$$

$$\psi_ry_{r+1}e(\mathbf{i})=(y_r\psi_r+\delta_{i_ri_{r+1}})e(\mathbf{i}),\qquad y_{r+1}\psi_re(\mathbf{i})=(\psi_ry_r+\delta_{i_ri_{r+1}})e(\mathbf{i}),\tag{2.11}$$

$$\psi_ry_s=y_s\psi_r,\qquad \text{if } s\neq r,r+1,\tag{2.12}$$

$$\psi_r\psi_s=\psi_s\psi_r,\qquad \text{if } |r-s|>1,$$

$$\psi_r^2e(\mathbf{i})=\begin{cases}0, & \text{if } i_r=i_{r+1},\\ (y_r-y_{r+1})e(\mathbf{i}), & \text{if } i_r\to i_{r+1},\\ (y_{r+1}-y_r)e(\mathbf{i}), & \text{if } i_r\leftarrow i_{r+1},\\ (y_{r+1}-y_r)(y_r-y_{r+1})e(\mathbf{i}), & \text{if } i_r\rightleftarrows i_{r+1}\\ e(\mathbf{i}), & \text{otherwise,}\end{cases}$$

$$\psi_r\psi_{r+1}\psi_re(\mathbf{i})=\begin{cases}(\psi_{r+1}\psi_r\psi_{r+1}-1)e(\mathbf{i}), & \text{if } i_{r+2}=i_r\to i_{r+1},\\ (\psi_{r+1}\psi_r\psi_{r+1}+1)e(\mathbf{i}), & \text{if } i_{r+2}=i_r\leftarrow i_{r+1},\\ \big(\psi_{r+1}\psi_r\psi_{r+1}+y_r-2y_{r+1}+y_{r+2}\big)e(\mathbf{i}), & \\ & \text{if } i_{r+2}=i_r\rightleftarrows i_{r+1},\\ \psi_{r+1}\psi_r\psi_{r+1}e(\mathbf{i}), & \text{otherwise,}\end{cases}$$

for $\mathbf{i},\mathbf{j}\in I^n$ *and all admissible* r *and* s*. Moreover,* $\mathcal{R}_n^\Lambda$ *is naturally* $\mathbb{Z}$*-graded with degree function determined by*

$$\deg e(\mathbf{i})=0,\qquad \deg y_r=2\qquad \textit{and}\qquad \deg\psi_se(\mathbf{i})=-c_{i_s,i_{s+1}},$$

for $1\le r\le n$*,* $1\le s<n$ *and* $\mathbf{i}\in I^n$*.*

2.13. *Remark.* The presentation of $\mathcal{R}_n^\Lambda$ given in Definition 2.10 differs by a choice of signs with the definition given in [6, Theorem 1.1]. The presentation of $\mathcal{R}_n^\Lambda$ given above agrees with that used in [23] as the orientation of the quiver is reversed in [23].

The connection between the cyclotomic quiver Hecke algebras of type Γ_e and the cyclotomic Hecke algebras of type $G(\ell,1,n)$ is given by the following remarkable result of Brundan and Kleshchev.

2.14. **Theorem** (Brundan-Kleshchev's isomorphism theorem [6, Theorem 1.1])**.** *Suppose that* $\mathcal{O}=K$ *is a field,* $\xi\in K$ *as above, and that* $\Lambda=\Lambda(\boldsymbol{\kappa})$*. Then there is an isomorphism of algebras* $\underline{\mathcal{R}}_n^\Lambda\cong\mathcal{H}_n^\Lambda$*.*

Rouquier [31, Corollary 3.20] has, independently, given a quick proof of Theorem 2.14.

We prove a stronger version of Theorem 2.14 in Theorem 4.32 below. For now we note the following simple corollary of Theorem 2.14. Recall that a choice of multicharge $\boldsymbol{\kappa}$ determines a dominant weight $\Lambda_e(\boldsymbol{\kappa})$.

2.15. **Corollary.** *Suppose that* $n\ge 0$*,* $\boldsymbol{\kappa}=(\kappa_1,\dots,\kappa_\ell)\in\mathbb{Z}^\ell$ *and that*

$$e>\max\{\,n+\kappa_k-\kappa_l\mid 1\le k,l\le\ell\,\}.$$

Fix invertible scalars $\xi_0\in K$ *and* $\xi_e\in K$ *such that* ξ_0 *is not a root of unity and* ξ_e *is a primitive eth root of unity. Then the cyclotomic Hecke algebras* $\mathcal{H}_{K,\xi_0}^{\Lambda_0(\kappa)}$ *and* $\mathcal{H}_{\mathscr{K},\xi_e}^{\Lambda_e(\boldsymbol{\kappa})}$ *are isomorphic* $\mathbb{Z}$*-graded* K*-algebras.*

Proof. Let $\mathcal{R}_n^\Lambda(0)\cong\mathcal{H}_n(K,\xi_0,\boldsymbol{\kappa})$ and $\mathcal{R}_n^\Lambda(e)\cong\mathcal{H}_n(K,\xi_e,\boldsymbol{\kappa})$ be the corresponding cyclotomic quiver Hecke algebras as in Theorem 2.14. By [14, Lemma 4.1], $e(\mathbf{i})\neq 0$ if and only if $\mathbf{i}=\operatorname{res}(\mathfrak{t})$, for some standard tableau $\mathfrak{t}\in\operatorname{Std}(\mathcal{P}_n^\Lambda)$. The definition of e ensures that if $\mathbf{i}=\mathbf{i}^{\mathfrak{t}}$ then $i_r=i_{r+1}$ or $i_r=i_{r+1}\pm1$ if and only if $i_r\equiv i_{r+1}\pmod e$

or $i_r \equiv i_{r+1} \pm 1 \pmod e$. Therefore, $\mathcal{R}_n^\Lambda(0) \cong \mathcal{R}_n^\Lambda(e)$ arguing directly from the presentations of the cyclotomic quiver Hecke algebras given in Definition 2.10. Hence, the result follows by Theorem 2.14. □

Therefore, without loss of generality, we may assume that $e > 0$. In the appendix we show how to modify the results and definitions in this paper to cover the case when $e = 0$ directly.

Under the assumptions of the Corollary we note that the algebras $\mathcal{H}_{K,\xi_0}^\Lambda$ and $\mathcal{H}_{K,\xi_e}^\Lambda$ are Morita equivalent by the main result of [11]. That these algebras are actually isomorphic is another miracle provided by Brundan and Kleshchev's isomorphism theorem.

3. Seminormal forms for Hecke algebras

In this chapter we develop the theory of seminormal forms in a slightly more general context than appears in the literature. In particular, in this paper a seminormal basis will be a basis for $\mathcal{H}_n^\Lambda$ rather than a basis of a Specht module of $\mathcal{H}_n^\Lambda$. We also treat all of the variations of the seminormal bases simultaneously as this will give us the flexibility to change seminormal forms when we use them in the next chapter to study the connections between $\mathcal{H}_n^\Lambda$ and the cyclotomic quiver Hecke algebra $\mathcal{R}_n^\Lambda$.

3.1. Content functions and the Gelfand-Zetlin algebra. Underpinning Brundan and Kleshchev's isomorphism theorem (Theorem 2.14) is the decomposition of any $\mathcal{H}_n^\Lambda$-module into a direct sum of generalised eigenspaces for the Jucys-Murphy elements $L_1, \dots, L_n$. This section studies the action of the Jucys-Murphy elements on $\mathcal{H}_n^\Lambda$. The results in this section are well-known, at least to experts, but they are needed in the sequel.

The **content** of the node $\gamma = (l, r, c)$ is the integer

$$\mathrm{c}_\gamma = \kappa_l - r + c$$

If $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ is a standard $\boldsymbol{\lambda}$-tableau and $1 \le k \le n$ then the **content** of k in $\mathfrak{t}$ is $\mathrm{c}_k(\mathfrak{t}) = \mathrm{c}_\gamma$, where $\mathfrak{t}(\gamma) = k$ for $\gamma \in [\![\boldsymbol{\lambda}]\!]$.

3.1. **Definition.** *Let $\mathcal{O}$ be a commutative integral domain and suppose that $t \in \mathcal{O}^\times$ is an invertible element of $\mathcal{O}$. The pair $(\mathcal{O}, t)$* ***separates*** $\operatorname{Std}(\mathcal{P}_n^\Lambda)$ *if*

$$[n]_t^! \prod_{1 \le l < m \le \ell} \prod_{-n < d < n} [\kappa_l - \kappa_m + d]_t \in \mathcal{O}^\times .$$

Fix a multicharge $\boldsymbol{\kappa} \in \mathbb{Z}^\ell$ and let $\mathcal{H}_n^\Lambda(\mathcal{O})$ be the Hecke algebra defined over $\mathcal{O}$ with parameter t. In spite of our notation, note that $\mathcal{H}_n^\Lambda(\mathcal{O})$ depends only on $\boldsymbol{\kappa}$ and not directly on $\Lambda = \Lambda_e(\boldsymbol{\kappa})$. Let $\mathscr{K}$ be a field that contains the field of fractions of $\mathcal{O}$. Then $\mathcal{H}_n^\Lambda(\mathscr{K}) = \mathcal{H}_n^\Lambda(\mathcal{O}) \otimes_{\mathcal{O}} \mathscr{K}$.

Throughout this chapter we are going to work with the Hecke algebras $\mathcal{H}_n^\Lambda(\mathcal{O})$ and $\mathcal{H}_n^\Lambda(\mathscr{K}) = \mathcal{H}_n^\Lambda(\mathcal{O}) \otimes_{\mathcal{O}} \mathscr{K}$, however, we have in mind the situation of Theorem 2.14. By assumption $e > 0$, so we can replace the multicharge $\boldsymbol{\kappa}$ with $(\kappa_1 + a_1 e, \kappa_2 + a_2 e, \dots, \kappa_\ell + a_\ell e)$, for any integers $a_1, \dots, a_\ell \in \mathbb{Z}$, without changing the dominant weight $\Lambda = \Lambda_e(\boldsymbol{\kappa})$. In view of Definition 3.1 we therefore assume that

$$\kappa_l - \kappa_{l+1} \ge n, \qquad \text{for } 1 \le l < \ell. \tag{3.2}$$

Until further notice, we fix a multicharge $\boldsymbol{\kappa} \in \mathbb{Z}^\ell$ satisfying (3.2) and consider the algebra $\mathcal{H}_n^\Lambda(\mathcal{O})$ with parameter t.

Although we do not need this, we remark that it follows from [1] and [4, Theorem 6.11] that $\mathcal{H}_n^\Lambda(\mathscr{K}, t)$ is semisimple if and only if $(\mathscr{K}, t)$ separates $\operatorname{Std}(\mathcal{P}_n^\Lambda)$.

Our main use of the separation condition is the following fundamental fact that is easily proved by induction on n; see, for example, [17, Lemma 3.12].

3.3. **Lemma.** *Suppose that $\mathcal{O}$ is an integral domain and $t \in \mathcal{O}^\times$ is invertible. Then the following are equivalent:*

a) *$(\mathcal{O}, t)$ separates $\mathrm{Std}(\mathcal{P}_n^\Lambda)$,*
b) *If $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\mathcal{P}_n^\Lambda)$ then $\mathfrak{s} = \mathfrak{t}$ if and only if $[c_r(\mathfrak{s})] = [c_r(\mathfrak{t})]$, for $1 \le r \le n$.*

Following [30], define the **Gelfand-Zetlin subalgebra** of $\mathcal{H}_n^\Lambda$ to be the algebra $\mathscr{L}(\mathcal{O}) = \langle L_1, \dots, L_n \rangle$. The aim of this section is to understand the semisimple representation theory of $\mathscr{L} = \mathscr{L}(\mathcal{O})$. It follows from Definition 2.2 that $\mathscr{L}$ is a commutative subalgebra of $\mathcal{H}_n^\Lambda$.

If $\mathcal{O}$ is an integral domain then it follows from Lemma 2.9 that, as an $(\mathscr{L}, \mathscr{L})$-bimodule, $\mathcal{H}_n^\Lambda(\mathcal{O})$ has a composition series with composition factors that are $\mathcal{O}$-free of rank 1 upon which L_r acts as multiplication by $[c_r(\mathfrak{s})]$ from the left and as multiplication by $[c_r(\mathfrak{t})]$ from the right. Obtaining a better description of $\mathscr{L}$, and of $\mathcal{H}_n^\Lambda$ as an $(\mathscr{L}, \mathscr{L})$-bimodule, in the non-semisimple case is likely to be important. For example, the dimension of $\mathscr{L}$ over a field is not known in general.

3.4. **Proposition** (cf. [3, Proposition 3.17])**.** *Suppose that $(\mathscr{K}, t)$ separates $\mathrm{Std}(\mathcal{P}_n^\Lambda)$, where $\mathscr{K}$ is a field and $0 \ne t \in \mathscr{K}$. Then $\mathcal{H}_n^\Lambda(\mathscr{K})$ is a semisimple $\big(\mathscr{L}, \mathscr{L}\big)$-bimodule with decomposition*

$$\mathcal{H}_n^\Lambda(\mathscr{K}) = \bigoplus_{\substack{\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \\ \mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})}} H_{\mathfrak{s}\mathfrak{t}},$$

where $H_{\mathfrak{s}\mathfrak{t}} = \{\, h \in \mathcal{H}_n^\Lambda \mid L_r h = [c_r(\mathfrak{s})]h \text{ and } hL_r = [c_r(\mathfrak{t})]h, \text{ for } 1 \le r \le n \,\}$ is one dimensional.

Proof. By Lemma 2.9, the Jucys-Murphy elements $L_1, \dots, L_n$ are a family of JM-elements for $\mathcal{H}_n^\Lambda$ in the sense of [28, Definition 2.4]. Therefore, the result is a special case of [28, Theorem 3.7]. □

Key to the proof of the results in [28] are the following elements that have their origins in the work of Murphy [29]. For $\mathfrak{t} \in \mathrm{Std}(\mathcal{P}_n^\Lambda)$ define

$$F_{\mathfrak{t}} = \prod_{k=1}^{n} \prod_{\substack{c \in \mathscr{C} \\ [c_k(\mathfrak{t})] \ne [c]}} \frac{L_k - [c]}{[c_k(\mathfrak{t})] - [c]} \tag{3.5}$$

where $\mathscr{C} = \{\, c_r(\mathfrak{t}) \mid 1 \le r \le n \text{ and } \mathfrak{t} \in \mathrm{Std}(\mathcal{P}_n^\Lambda) \,\}$ is the set of the possible contents that can appear in a standard tableau of size n. By definition, $F_{\mathfrak{t}} \in \mathscr{L}(\mathscr{K})$ and it follows directly from Proposition 3.4 that if $h_{\mathfrak{u}\mathfrak{v}} \in H_{\mathfrak{u}\mathfrak{v}}$ then

$$F_{\mathfrak{s}} h_{\mathfrak{u}\mathfrak{v}} F_{\mathfrak{t}} = \delta_{\mathfrak{s}\mathfrak{u}} \delta_{\mathfrak{v}\mathfrak{t}} h_{\mathfrak{s}\mathfrak{t}}, \tag{3.6}$$

for all $(\mathfrak{s}, \mathfrak{t}), (\mathfrak{u}, \mathfrak{v}) \in \mathrm{Std}^2(\mathcal{P}_n^\Lambda)$. Therefore, $H_{\mathfrak{s}\mathfrak{t}} = F_{\mathfrak{s}} \mathcal{H}_n^\Lambda F_{\mathfrak{t}}$.

By Proposition 3.4 we can write $1 = \sum_{\mathfrak{s},\mathfrak{t}} e_{\mathfrak{s}\mathfrak{t}}$ for unique $e_{\mathfrak{s}\mathfrak{t}} \in H_{\mathfrak{s}\mathfrak{t}}$. Since $F_{\mathfrak{t}} = F_{\mathfrak{t}}^*$, the last displayed equation implies that $F_{\mathfrak{t}} = e_{\mathfrak{t}\mathfrak{t}} \in H_{\mathfrak{t}\mathfrak{t}}$ is an idempotent. Consequently,

$$\mathscr{L}(\mathscr{K}) = \bigoplus_{\mathfrak{t} \in \mathrm{Std}(\mathcal{P}_n^\Lambda)} H_{\mathfrak{t}\mathfrak{t}} = \bigoplus_{\mathfrak{t} \in \mathrm{Std}(\mathcal{P}_n^\Lambda)} \mathscr{K} F_{\mathfrak{t}}.$$

In particular, $F_{\mathfrak{t}}$ is a primitive idempotent in $\mathscr{L}(\mathscr{K})$. If follows that $\mathscr{L}(\mathscr{K})$ is a split semisimple algebra of dimension $\#\,\mathrm{Std}(\mathcal{P}_n^\Lambda)$.

3.2. **Seminormal forms.** Seminormal bases for $\mathcal{H}_n^\Lambda$ are well-known in the literature, having their origins in the work of Young [37]. Many examples of "seminormal bases" appear in the literature. In this section we classify the seminormal bases of $\mathcal{H}_n^\Lambda$. This characterisation of seminormal forms appears to be new, even in the special case of the symmetric groups, although some of the details will be familiar to experts.

Throughout this section we assume that $\mathscr{K}$ is a field, $0 \neq t \in \mathscr{K}$ and that $(\mathscr{K}, t)$ separates $\mathrm{Std}(\mathcal{P}_n^\Lambda)$. Recall the decomposition $\mathcal{H}_n^\Lambda = \bigoplus_{(\mathfrak{s},\mathfrak{t})\in\mathrm{Std}^2(\mathcal{P}_n^\Lambda)} H_{\mathfrak{s}\mathfrak{t}}$ from Proposition 3.4.

Define an **anti-involution** on an algebra A to be an algebra anti-automorphism of A of order 2.

3.7. **Definition.** *Suppose that $(\mathscr{K}, t)$ separates $\mathrm{Std}(\mathcal{P}_n^\Lambda)$ and let ι be an anti-involution on $\mathcal{H}_n^\Lambda(\mathscr{K})$. An **$\iota$-seminormal basis** of $\mathcal{H}_n^\Lambda(\mathscr{K})$ is a basis of the form $\{\, f_{\mathfrak{s}\mathfrak{t}} \mid f_{\mathfrak{s}\mathfrak{t}} = \iota(f_{\mathfrak{t}\mathfrak{s}}) \in H_{\mathfrak{s}\mathfrak{t}}$ for $(\mathfrak{s},\mathfrak{t}) \in \mathrm{Std}$*

Recall that $*$ is the unique anti-involution of $\mathcal{H}_n^\Lambda(\mathscr{K})$ that fixes each of the generators $T_1, \dots, T_{n-1}, L_1, \dots, L_n$. Then $m_{\mathfrak{s}\mathfrak{t}}^* = m_{\mathfrak{t}\mathfrak{s}}$, for all $(\mathfrak{s},\mathfrak{t}) \in \mathrm{Std}^2(\mathcal{P}_n^\Lambda)$. The assumption that $f_{\mathfrak{s}\mathfrak{t}}^* = f_{\mathfrak{t}\mathfrak{s}}$ is not essential for what follows but it is natural because we want to work within the framework of cellular algebras.

In order to describe the action of $\mathcal{H}_n^\Lambda$ on its seminormal bases, if $\mathfrak{t} \in \mathrm{Std}(\mathcal{P}_n^\Lambda)$ then define the integers

$$\rho_r(\mathfrak{t}) = c_r(\mathfrak{t}) - c_{r+1}(\mathfrak{t}), \qquad \text{for } 1 \le r < n. \tag{3.8}$$

Then $\rho_r(\mathfrak{t})$ is the 'axial distance' between r and $r+1$ in the tableau $\mathfrak{t}$.

3.9. **Definition.** *A **$*$-seminormal coefficient system** for $\mathcal{H}_n^\Lambda(\mathscr{K})$ is a set of scalars $\boldsymbol{\alpha} = \{\, \alpha_r(\mathfrak{s}) \mid 1 \le r < n$ and $\mathfrak{s} \in \mathrm{Std}(\mathcal{P}_n^\Lambda) \,\}$ in $\mathscr{K}$ such that if $\mathfrak{t} \in \mathrm{Std}(\mathcal{P}_n^\Lambda)$ and $1 \le r < n$ then:*

a) $\alpha_r(\mathfrak{t})\alpha_{r+1}(\mathfrak{t}s_r)\alpha_r(\mathfrak{t}s_rs_{r+1}) = \alpha_{r+1}(\mathfrak{t})\alpha_r(\mathfrak{t}s_{r+1})\alpha_{r+1}(\mathfrak{t}s_{r+1}s_r)$ *if* $r < n-1$,
b) $\alpha_r(\mathfrak{t})\alpha_k(\mathfrak{t}s_r) = \alpha_k(\mathfrak{t})\alpha_r(\mathfrak{t}s_k)$ *if* $1 \le k < n$ *and* $|r-k| > 1$,
c) *if* $\mathfrak{v} = \mathfrak{t}(r, r+1)$ *then* $\alpha_r(\mathfrak{v}) = 0$ *if* $\mathfrak{v} \notin \mathrm{Std}(\mathcal{P}_n^\Lambda)$ *and otherwise*

$$\alpha_r(\mathfrak{t})\alpha_r(\mathfrak{v}) = \frac{[1+\rho_r(\mathfrak{t})][1+\rho_r(\mathfrak{v})]}{[\rho_r(\mathfrak{t})][\rho_r(\mathfrak{v})]}.$$

We will see that conditions (a) and (b) correspond to the braid relations satisfied by $T_1, \dots, T_{n-1}$ and that (c) corresponds to the quadratic relations. Quite surprisingly, as the proof of Theorem 3.21 below shows, Definition 3.9(c) also encodes the KLR grading on $\mathcal{H}_n^\Lambda$.

Usually, we omit the $*$ and simply call $\boldsymbol{\alpha}$ a seminormal coefficient system.

3.10. **Example** A nice 'rational' seminormal coefficient system is given by

$$\alpha_r(\mathfrak{t}) = \begin{cases} \frac{[1+\rho_r(\mathfrak{t})]}{[\rho_r(\mathfrak{t})]}, & \text{if } \mathfrak{t}(r,r+1) \text{ is standard,} \\ 0, & \text{otherwise,} \end{cases}$$

for $\mathfrak{t} \in \mathrm{Std}(\mathcal{P}_n^\Lambda)$ and $1 \le r < n$. ◇

3.11. **Example** By Proposition 3.17 below, the following seminormal coefficient system is associated with the Murphy basis of $\mathcal{H}_n^\Lambda$: if $\mathfrak{t} \in \mathrm{Std}(\mathcal{P}_n^\Lambda)$ set $\mathfrak{v} = \mathfrak{t}(r, r+1)$ and define

$$\alpha_r(\mathfrak{t}) = \begin{cases} 1 & \text{if } \mathfrak{v} \text{ is standard and } \mathfrak{t} \rhd \mathfrak{v}, \\ \frac{[1+\rho_r(\mathfrak{t})][1+\rho_r(\mathfrak{v})]}{[\rho_r(\mathfrak{t})][\rho_r(\mathfrak{v})]}, & \text{if } \mathfrak{v} \text{ is standard and } \mathfrak{v} \rhd \mathfrak{t}, \\ 0, & \text{otherwise,} \end{cases}$$

for $1 \le r < n$. ◇

Another seminormal coefficient system, which is particularly well adapted to Brundan and Kleshchev's Graded Isomorphism Theorem 2.14, is given in Section 5.1.

3.12. **Lemma.** *Suppose that* $(\mathscr{K},t)$ *separates* $\operatorname{Std}(\mathcal{P}_n^\Lambda)$ *and that* $\{f_{\mathfrak{s}\mathfrak{t}}\}$ *is a seminormal basis of* $\mathcal{H}_n^\Lambda$*. Then there exists a unique seminormal coefficient system* $\boldsymbol{\alpha}$ *such that if* $1\le r<n$ *and* $(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ *then*

$$f_{\mathfrak{s}\mathfrak{t}}T_r=\alpha_r(\mathfrak{t})f_{\mathfrak{s}\mathfrak{v}}-\frac{1}{[\rho_r(\mathfrak{t})]}f_{\mathfrak{s}\mathfrak{t}},$$

where $\mathfrak{v}=\mathfrak{t}(r,r+1)$ *and* $f_{\mathfrak{s}\mathfrak{t}}=0$ *if* $(\mathfrak{s},\mathfrak{t})\notin\operatorname{Std}^2(\mathcal{P}_n^\Lambda)$.

Proof. The uniqueness statement is automatic, since $\{f_{\mathfrak{s}\mathfrak{t}}\}$ is a basis of $\mathcal{H}_n^\Lambda(\mathscr{K})$, so we need to prove that such a seminormal coefficient system $\boldsymbol{\alpha}$ exists.

Fix $(\mathfrak{s},t)\in\operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ and $1\le r<n$ and write

$$f_{\mathfrak{s}\mathfrak{t}}T_r=\sum_{(\mathfrak{u},\mathfrak{v})\in\operatorname{Std}^2(\mathcal{P}_n^\Lambda)}a_{\mathfrak{u}\mathfrak{v}}f_{\mathfrak{u}\mathfrak{v}},$$

for some $a_{\mathfrak{u}\mathfrak{v}}\in\mathscr{K}$. Multiplying on the left by $F_{\mathfrak{s}}$ it follows that $a_{\mathfrak{u}\mathfrak{v}}\ne0$ only if $\mathfrak{u}=\mathfrak{s}$. If $k\ne r,r+1$ then L_k commutes with T_r so it follows $a_{\mathfrak{s}\mathfrak{v}}\ne0$ only if $[c_k(\mathfrak{v})]=[c_k(\mathfrak{t})]$, for $k\ne r,r+1$. Using Definition 3.1, and arguing as in Lemma 3.3, this implies that $a_{\mathfrak{s}\mathfrak{v}}\ne0$ only if $\mathfrak{v}\in\{\mathfrak{t},\mathfrak{t}(r,r+1)\}$. Therefore, we can write

$$f_{\mathfrak{s}\mathfrak{t}}T_r=\alpha_r(\mathfrak{t})f_{\mathfrak{s}\mathfrak{v}}+\alpha_r'(\mathfrak{t})f_{\mathfrak{s}\mathfrak{t}},$$

for some $\alpha_r(\mathfrak{t}),\alpha_r'(\mathfrak{t})\in\mathscr{K}$, where $\mathfrak{v}=\mathfrak{t}(r,r+1)$. (Here, and below, we adopt the convention that $f_{\mathfrak{s}\mathfrak{v}}=0$ if either of $\mathfrak{s}$ or $\mathfrak{v}$ is not standard.) By Definition 2.2, $T_rL_r=L_{r+1}(T_r-t+1)-1$, so multiplying both sides of the last displayed equation on the right by L_r and comparing the coefficient of $f_{\mathfrak{s}\mathfrak{t}}$ on both sides shows that

$$[c_{r+1}(\mathfrak{t})]\big(\alpha_r'(\mathfrak{t})-t+1\big)-1=\alpha_r'(\mathfrak{t})[c_r(\mathfrak{t})].$$

Hence, $\alpha_r'(\mathfrak{t})=-1/[\rho_r(\mathfrak{t})]$ as claimed. If $\mathfrak{v}$ is not standard then we set $\alpha_r(\mathfrak{t})=0$. If $\mathfrak{v}$ is standard then comparing the coefficient of $f_{\mathfrak{s}\mathfrak{t}}$ on both sides of

$$\Big(\alpha_r(\mathfrak{t})f_{\mathfrak{s}\mathfrak{v}}-\frac{1}{[\rho_r(\mathfrak{t})]}f_{\mathfrak{s}\mathfrak{t}}\Big)T_r=f_{\mathfrak{s}\mathfrak{t}}T_r^2=f_{\mathfrak{s}\mathfrak{t}}\big((t-1)T_r+t\big)$$

shows that $\alpha_r(\mathfrak{t})\alpha_r(\mathfrak{v})=\frac{[1+\rho_r(\mathfrak{t})][1+\rho_r(\mathfrak{v})]}{[\rho_r(\mathfrak{t})][\rho_r(\mathfrak{v})]}$ in accordance with Definition 3.9(c).

If $1\le r<s-1<n-1$ and $(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ then $(f_{\mathfrak{s}\mathfrak{t}}T_r)T_s=f_{\mathfrak{s}\mathfrak{t}}(T_rT_s)=f_{\mathfrak{s}\mathfrak{t}}(T_sT_r)=(f_{\mathfrak{s}\mathfrak{t}}T_s)T_r$. By a direct calculation, we can deduce that $\alpha_r(\mathfrak{t})\alpha_s(\mathfrak{t}(r,r+1))=\alpha_s(\mathfrak{t})\alpha_r(\mathfrak{t}(s,s+1))$.

Finally, it remains to show that Definition 3.9(a) holds. If $1\le r<n$ then $T_rT_{r+1}T_r=T_{r+1}T_rT_{r+1}$ by Definition 2.2. On the other hand, if we set $\mathfrak{t}_1=\mathfrak{t}(r,r+1)$, $\mathfrak{t}_2=\mathfrak{t}(r+1,r+2)$, $\mathfrak{t}_{12}=\mathfrak{t}_1(r+1,r+2)$, $\mathfrak{t}_{21}=\mathfrak{t}_2(r,r+1)$ and $\mathfrak{t}_{121}=\mathfrak{t}_{212}=\mathfrak{t}(r,r+2)$ then direct calculation shows that $0=f_{\mathfrak{s}\mathfrak{t}}(T_rT_{r+1}T_r-T_{r+1}T_rT_{r+1})$

is equal to

$$
\begin{aligned}
-\Big(\frac{1}{[\rho_r(\mathfrak{t})]^2[\rho_{r+1}(\mathfrak{t})]}-\frac{1}{[\rho_r(\mathfrak{t})][\rho_{r+1}(\mathfrak{t})]^2}+\frac{\alpha_r(\mathfrak{t})\alpha_r(\mathfrak{t}_1)}{[\rho_{r+1}(\mathfrak{t}_1)]}-\frac{\alpha_{r+1}(\mathfrak{t})\alpha_{r+1}(\mathfrak{t}_2)}{[\rho_r(\mathfrak{t}_2)]}\Big)f_{\mathfrak{s}\mathfrak{t}}&\\
+\alpha_r(\mathfrak{t})\Big(\frac{1}{[\rho_r(\mathfrak{t}_1)][\rho_{r+1}(\mathfrak{t}_1)]}+\frac{1}{[\rho_r(\mathfrak{t})][\rho_{r+1}(\mathfrak{t})]}-\frac{1}{[\rho_{r+1}(\mathfrak{t})][\rho_{r+1}(\mathfrak{t}_1)]}\Big)f_{\mathfrak{s}\mathfrak{t}_1}&\\
-\alpha_{r+1}(\mathfrak{t})\Big(\frac{1}{[\rho_r(\mathfrak{t}_2)][\rho_{r+1}(\mathfrak{t}_2)]}+\frac{1}{[\rho_r(\mathfrak{t})][\rho_{r+1}(\mathfrak{t})]}-\frac{1}{[\rho_r(\mathfrak{t})][\rho_r(\mathfrak{t}_2)]}\Big)f_{\mathfrak{s}\mathfrak{t}_2}&\\
-\alpha_r(\mathfrak{t})\alpha_{r+1}(\mathfrak{t}_1)\Big(\frac{1}{[\rho_r(\mathfrak{t}_{12})]}-\frac{1}{[\rho_{r+1}(\mathfrak{t})]}\Big)f_{\mathfrak{s}\mathfrak{t}_{12}}&\\
+\alpha_{r+1}(\mathfrak{t})\alpha_r(\mathfrak{t}_2)\Big(\frac{1}{[\rho_{r+1}(\mathfrak{t}_{21})]}-\frac{1}{[\rho_r(\mathfrak{t})]}\Big)f_{\mathfrak{s}\mathfrak{t}_{21}}&\\
+\big(\alpha_r(\mathfrak{t})\alpha_{r+1}(\mathfrak{t}_1)\alpha_r(\mathfrak{t}_{12})-\alpha_{r+1}(\mathfrak{t})\alpha_r(\mathfrak{t}_2)\alpha_{r+1}(\mathfrak{t}_{21})\big)f_{\mathfrak{s}\mathfrak{t}_{121}}&.
\end{aligned}
$$

By our conventions, if any tableau $\mathfrak{t}_?$ is not standard then $f_{\mathfrak{s}\mathfrak{t}_?}$ and the corresponding $\boldsymbol{\alpha}$-coefficient are both zero. As the coefficient of $f_{\mathfrak{s}\mathfrak{t}_{121}}$ in the last displayed equation is zero it follows that Definition 3.9(a) holds. Consequently, $\boldsymbol{\alpha}=\{\alpha_r(\mathfrak{t})\}$ is a seminormal coefficient system, completing the proof. (It is not hard to see, using Definition 3.9 and identities like $\rho_r(\mathfrak{t}_1)=-\rho_r(\mathfrak{t})$ and $\rho_r(\mathfrak{t}_{12})=\rho_{r+1}(\mathfrak{t})$, that the remaining coefficients in the last displayed equation are automatically zero.) □

Lemma 3.12 really says that acting from the right on a seminormal basis determines a seminormal coefficient system. Similarly, the left action on a seminormal basis determines a seminormal coefficient system. In general, the seminormal coefficient systems attached to the left and right actions will be different, however, because we are assuming that our seminormal bases are $*$-invariant these left and right coefficient systems coincide. Thus, for $(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ and $1\le r<n$ we also have $T_rf_{\mathfrak{s}\mathfrak{t}}=\alpha_r(\mathfrak{s})f_{\mathfrak{u}\mathfrak{t}}-\frac{1}{[\rho_r(\mathfrak{s})]}f_{\mathfrak{s}\mathfrak{t}}$, where $\mathfrak{u}=\mathfrak{s}(r,r+1)$.

Exactly as eigenvectors are not uniquely determined by their eigenvalues, seminormal bases are not uniquely determined by seminormal coefficient systems. We now fully characterize seminormal bases — and prove a converse to Lemma 3.12.

Recall that a set of idempotents in an algebra is **complete** if they sum to 1.

3.13. **Theorem** (The Seminormal Basis Theorem)**.** *Suppose that $(\mathscr{K},t)$ separates $\operatorname{Std}(\mathcal{P}_n^\Lambda)$ and that $\boldsymbol{\alpha}$ is a seminormal coefficient system for $\mathcal{H}_n^\Lambda(\mathscr{K})$. Then $\mathcal{H}_n^\Lambda(\mathscr{K})$ has a $*$-seminormal basis $\{\,f_{\mathfrak{s}\mathfrak{t}}\mid(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathcal{P}_n^\Lambda)\,\}$ such that if $(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ then*

$$f_{\mathfrak{s}\mathfrak{t}}^*=f_{\mathfrak{t}\mathfrak{s}},\quad f_{\mathfrak{s}\mathfrak{t}}L_k=[c_k(\mathfrak{t})]f_{\mathfrak{s}\mathfrak{t}}\quad\text{and}\quad f_{\mathfrak{s}\mathfrak{t}}T_r=\alpha_r(\mathfrak{t})f_{\mathfrak{s}\mathfrak{v}}-\frac{1}{[\rho_r(\mathfrak{t})]}f_{\mathfrak{s}\mathfrak{t}},\tag{3.14}$$

where $\mathfrak{v}=\mathfrak{t}(r,r+1)$ and $f_{\mathfrak{s}\mathfrak{v}}=0$ if $\mathfrak{v}$ is not standard. Moreover, there exist non-zero scalars $\gamma_\mathfrak{t}\in\mathscr{K}$, for $\mathfrak{t}\in\operatorname{Std}(\mathcal{P}_n^\Lambda)$, such that

$$F_\mathfrak{u}f_{\mathfrak{s}\mathfrak{t}}F_\mathfrak{v}=\delta_{\mathfrak{u}\mathfrak{s}}\delta_{\mathfrak{t}\mathfrak{v}}f_{\mathfrak{s}\mathfrak{t}},\quad f_{\mathfrak{s}\mathfrak{t}}f_{\mathfrak{u}\mathfrak{v}}=\delta_{\mathfrak{t}\mathfrak{u}}\gamma_\mathfrak{t}f_{\mathfrak{s}\mathfrak{v}},\quad\text{and}\quad F_\mathfrak{t}=\frac{1}{\gamma_\mathfrak{t}}f_{\mathfrak{t}\mathfrak{t}}.\tag{3.15}$$

Furthermore, $\{\,F_\mathfrak{t}\mid\mathfrak{t}\in\operatorname{Std}(\mathcal{P}_n^\Lambda)\,\}$ is a complete set of pairwise orthogonal primitive idempotents. In particular, every irreducible $\mathcal{H}_n^\Lambda(\mathscr{K})$-module is isomorphic to $F_\mathfrak{s}\mathcal{H}_n^\Lambda(\mathscr{K})$, for some $\mathfrak{s}\in\operatorname{Std}(\mathcal{P}_n^\Lambda)$, and $F_\mathfrak{s}\mathcal{H}_n^\Lambda(\mathscr{K})\cong F_\mathfrak{u}\mathcal{H}_n^\Lambda(\mathscr{K})$ if and only if $\operatorname{Shape}(\mathfrak{s})=\operatorname{Shape}(\mathfrak{u})$.

Finally, the basis $\{\,f_{\mathfrak{s}\mathfrak{t}}\mid\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda}\in\mathcal{P}_n^\Lambda\,\}$ is uniquely determined by the choice of seminormal coefficient system $\boldsymbol{\alpha}$ and the scalars $\{\,\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}}\mid\boldsymbol{\lambda}\in\mathcal{P}_n^\Lambda\,\}\subseteq\mathscr{K}^\times$.

Proof. For each $\boldsymbol{\lambda}\in\mathcal{P}_n^\Lambda$ fix an arbitrary pair of tableaux and a non-zero element $f_{\mathfrak{s}\mathfrak{t}}\in H_{\mathfrak{s}\mathfrak{t}}$. Then $f_{\mathfrak{s}\mathfrak{t}}$ is a simultaneous eigenvector for all of the elements of $\mathscr{L}$, where they act from the left and from the right.

Now, suppose that $1 \le r < n$ and that $\mathfrak{v} = \mathfrak{t}(r, r+1)$ is standard. Then $\alpha_r(\mathfrak{t}) \neq 0$ so we can set $f_{\mathfrak{s}\mathfrak{v}} = \frac{1}{\alpha_r(\mathfrak{t})} f_{\mathfrak{s}\mathfrak{t}}(T_r + \frac{1}{[\rho_r(\mathfrak{t})]})$. Equivalently, $f_{\mathfrak{s}\mathfrak{t}} T_r = \alpha_r(\mathfrak{t}) f_{\mathfrak{s}\mathfrak{v}} - \frac{1}{[\rho_r(\mathfrak{t})]} f_{\mathfrak{s}\mathfrak{t}}$. Then using the relations in $\mathcal{H}_n^\Lambda(\mathscr{K})$ and the defining properties of the seminormal coefficient system $\boldsymbol{\alpha}$, it is straightforward to check that $f_{\mathfrak{s}\mathfrak{v}} L_k = [c_k(\mathfrak{v})] f_{\mathfrak{s}\mathfrak{v}}$, so that $f_{\mathfrak{s}\mathfrak{v}} \in H_{\mathfrak{s}\mathfrak{v}}$. Moreover, $f_{\mathfrak{s}\mathfrak{v}} \neq 0$ since $f_{\mathfrak{s}\mathfrak{t}} = \frac{1}{\alpha_r(\mathfrak{v})} f_{\mathfrak{s}\mathfrak{v}}(T_r + \frac{1}{[\rho_r(\mathfrak{v})]})$.

More generally, it is easy to see that if $\mathfrak{v}$ is any $\boldsymbol{\lambda}$-tableau then there is a sequence of standard tableaux $\mathfrak{v}_0 = \mathfrak{s}, \mathfrak{v}_1, \dots, \mathfrak{v}_z = \mathfrak{v}$ such that $\mathfrak{v}_{i+1} = \mathfrak{v}_i(r_i, r_i + 1)$, for some integers $1 \le r_i < n$. Therefore, continuing in this way it follows that given two tableaux $\mathfrak{u}, \mathfrak{v} \in \operatorname{Std}(\boldsymbol{\lambda})$ we can define non-zero elements $f_{\mathfrak{u}\mathfrak{v}} \in H_{\mathfrak{u}\mathfrak{v}}$ that satisfy (3.14). It follows that, once $f_{\mathfrak{s}\mathfrak{t}}$ is fixed, there is at most one choice of elements $\{ f_{\mathfrak{u}\mathfrak{v}} \mid \mathfrak{u}, \mathfrak{v} \in \operatorname{Std}(\boldsymbol{\lambda}) \}$, such that (3.14) holds.

To complete the proof that the seminormal coefficient system determines a seminormal basis we need to check that the elements $f_{\mathfrak{u}\mathfrak{v}}$ from the last paragraph are well-defined. That is, we need to show that $f_{\mathfrak{u}\mathfrak{v}}$ is independent of the choice of the sequences of simple transpositions that link $\mathfrak{u}$ and $\mathfrak{v}$ to $\mathfrak{s}$ and $\mathfrak{t}$, respectively. Equivalently, we need to prove that the action of $\mathcal{H}_n^\Lambda(\mathscr{K})$ given by (3.14) respects the relations of $\mathcal{H}_n^\Lambda(\mathscr{K})$. Using (3.14), all of the relations in Definition 2.2 are easy to check except for the braid relations of length three that hold by virtue of the argument of Lemma 3.12. Hence, by choosing elements $f_{\mathfrak{s}\mathfrak{t}} \in H_{\mathfrak{s}\mathfrak{t}}$, for $(\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\boldsymbol{\lambda})$ and $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$, the seminormal coefficient system determines a unique seminormal basis.

Using (3.6) it is straightforward to prove (3.15) so we leave these details to the reader; cf. [28, Theorem 3.16]. In particular, this shows that $F_{\mathfrak{s}} = \frac{1}{\gamma_{\mathfrak{s}}} f_{\mathfrak{s}\mathfrak{s}}$ is an idempotent. To show that $F_{\mathfrak{s}}$ is primitive, suppose that a is a non-zero element of $F_{\mathfrak{s}} \mathcal{H}_n^\Lambda(\mathscr{K})$. By (3.14), $a = \sum_{\mathfrak{v} \in \operatorname{Std}(\boldsymbol{\lambda})} r_{\mathfrak{v}} f_{\mathfrak{s}\mathfrak{v}}$, for some $r_{\mathfrak{v}} \in \mathscr{K}$. Fix $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ such that $r_{\mathfrak{t}} \neq 0$. Then $f_{\mathfrak{s}\mathfrak{t}} = 1/r_{\mathfrak{t}} a F_{\mathfrak{t}} \in F_{\mathfrak{s}} \mathcal{H}_n^\Lambda(\mathscr{K})$. Using (3.14) we deduce that $F_{\mathfrak{s}} \mathcal{H}_n^\Lambda(\mathscr{K})$ has basis $\{ f_{\mathfrak{s}\mathfrak{v}} \mid \mathfrak{v} \in \operatorname{Std}(\boldsymbol{\lambda}) \}$. Consequently, $a\mathcal{H}_n^\Lambda = F_{\mathfrak{s}} \mathcal{H}_n^\Lambda(\mathscr{K})$, showing that $F_{\mathfrak{s}} \mathcal{H}_n^\Lambda(\mathscr{K})$ is irreducible. Therefore, $F_{\mathfrak{s}}$ is a primitive idempotent in $\mathcal{H}_n^\Lambda(\mathscr{K})$.

The last paragraph, together with Definition 3.9(c), implies that if $\mathfrak{s}, \mathfrak{u} \in \operatorname{Std}(\boldsymbol{\lambda})$ then $F_{\mathfrak{s}} \mathcal{H}_n^\Lambda \cong F_{\mathfrak{u}} \mathcal{H}_n^\Lambda$ where an isomorphism is given by $f_{\mathfrak{s}\mathfrak{t}} \mapsto f_{\mathfrak{u}\mathfrak{t}}$, for $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. Consequently, if $\mathfrak{s}$ and $\mathfrak{u}$ are standard tableaux of different shape then $F_{\mathfrak{s}} \mathcal{H}_n^\Lambda \ncong F_{\mathfrak{u}} \mathcal{H}_n^\Lambda$ because the multiplicity of $S^{\boldsymbol{\lambda}} \cong F_{\mathfrak{s}} \mathcal{H}_n^\Lambda(\mathscr{K})$ in $\mathcal{H}_n^\Lambda(\mathscr{K})$ is $\# \operatorname{Std}(\boldsymbol{\lambda})$ by the Wedderburn theorem.

Finally, it remains to show that the basis $\{f_{\mathfrak{s}\mathfrak{t}}\}$ is uniquely determined by $\boldsymbol{\alpha}$ and the choice of the γ-coefficients $\{ \gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} \mid \boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \}$. If $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ then we have shown that, once $f_{\mathfrak{s}\mathfrak{t}}$ is fixed, there is a unique seminormal basis $\{ f_{\mathfrak{u}\mathfrak{v}} \mid \mathfrak{u}, \mathfrak{v} \in \operatorname{Std}(\boldsymbol{\lambda}) \}$ satisfying (3.14). In particular, taking $\mathfrak{s} = \mathfrak{t}^{\boldsymbol{\lambda}} = \mathfrak{t}$ and fixing $f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$ determines these basis elements. By (3.15) the choice of $f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$ also uniquely determines $\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}}$. Conversely, by setting $f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} = \gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} F_{\mathfrak{t}^{\boldsymbol{\lambda}}}$ for any choice of non-zero scalars $\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} \in \mathscr{K}$, for $\boldsymbol{\lambda} \in \mathscr{K}$, the seminormal coefficient system $\boldsymbol{\alpha}$ determines a unique seminormal basis. $\square$

The results that follow are independent of the choice of seminormal coefficient system $\boldsymbol{\alpha}$, however, the choice of γ-coefficients will be important — and in what follows it will be useful to be able to vary both the seminormal coefficient system $\boldsymbol{\alpha}$ and the γ-coefficients.

The proof of Theorem 3.13 implies that the choice of $\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}}$ determines all of the scalars $\gamma_{\mathfrak{s}}$, for $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda})$. In what follows we need the following result that makes the relationship between these coefficients more explicit.

3.16. **Corollary.** *Suppose that $\mathfrak{t} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$ and that $\mathfrak{v} = \mathfrak{t}(r, r+1)$ is standard, where $1 \le r < n$. Then $\alpha_r(\mathfrak{v})\gamma_{\mathfrak{t}} = \alpha_r(\mathfrak{t})\gamma_{\mathfrak{v}}$.*

Proof. Applying (3.14) and (3.15) several times each,

$$\begin{aligned}\gamma_{\mathfrak{v}} f_{\mathfrak{v}\mathfrak{v}} &= f_{\mathfrak{v}\mathfrak{v}} f_{\mathfrak{v}\mathfrak{v}} = \frac{1}{\alpha_r(\mathfrak{t})} f_{\mathfrak{v}\mathfrak{t}}\Big(T_r + \frac{1}{[\rho_r(\mathfrak{t})]}\Big) f_{\mathfrak{v}\mathfrak{v}} = \frac{1}{\alpha_r(\mathfrak{t})} f_{\mathfrak{v}\mathfrak{t}} T_r f_{\mathfrak{v}\mathfrak{v}} \\ &= \frac{1}{\alpha_r(\mathfrak{t})} f_{\mathfrak{v}\mathfrak{t}}\Big(\alpha_r(\mathfrak{v}) f_{\mathfrak{t}\mathfrak{v}} - \frac{1}{[\rho_r(\mathfrak{v})]} f_{\mathfrak{v}\mathfrak{v}}\Big) = \frac{\alpha_r(\mathfrak{v})}{\alpha_r(\mathfrak{t})} f_{\mathfrak{v}\mathfrak{t}} f_{\mathfrak{t}\mathfrak{v}} \\ &= \frac{\alpha_r(\mathfrak{v})}{\alpha_r(\mathfrak{t})} \gamma_{\mathfrak{t}} f_{\mathfrak{v}\mathfrak{v}}.\end{aligned}$$

Comparing coefficients, $\alpha_r(\mathfrak{t})\gamma_{\mathfrak{v}} = \alpha_r(\mathfrak{v})\gamma_{\mathfrak{t}}$ as required. □

3.3. **Seminormal bases and the Murphy basis.** In this section we compute the Gram determinant of the Specht modules of $\mathcal{H}_n^\Lambda$, with respect to the Murphy basis, as a product of cyclotomic polynomials when $\xi \neq 1$ or as a product of primes when $\xi = 1$. These determinants are already explicitly known [4, 16–18] but all existing formulas describe them as products of rational functions, or of rational numbers in the degenerate case.

By Theorem 2.8, the Murphy basis $\{m_{\mathfrak{s}\mathfrak{t}}\}$ is a cellular basis for $\mathcal{H}_n^\Lambda$ over an arbitrary ring. In this section we continue to work with the generic Hecke algebra $\mathcal{H}_n^\Lambda = \mathcal{H}_n^\Lambda(\mathcal{O})$ with parameter t and multicharge $\boldsymbol{\kappa}$ satisfying (3.2).

As $(\mathscr{K}, t)$ separates $\operatorname{Std}(\mathcal{P}_n^\Lambda)$, for $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ we can define

$$f_{\mathfrak{s}\mathfrak{t}} = F_{\mathfrak{s}} m_{\mathfrak{s}\mathfrak{t}} F_{\mathfrak{t}}.$$

By Lemma 2.9, $f_{\mathfrak{s}\mathfrak{t}} \equiv m_{\mathfrak{s}\mathfrak{t}} + \sum r_{\mathfrak{u}\mathfrak{v}} m_{\mathfrak{u}\mathfrak{v}} \pmod{\mathcal{H}_n^{\rhd\boldsymbol{\lambda}}}$, for some $r_{\mathfrak{u}\mathfrak{v}} \in \mathscr{K}$ where $r_{\mathfrak{u}\mathfrak{v}} \neq 0$ only if $(\mathfrak{u}, \mathfrak{v}) \rhd (\mathfrak{s}, \mathfrak{t})$. It follows that $\{f_{\mathfrak{s}\mathfrak{t}}\}$ is a seminormal basis of $\mathcal{H}_n^\Lambda(\mathscr{K})$ in the sense of Definition 3.7.

For $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$ set $[\boldsymbol{\lambda}]_t^! = \prod_{l=1}^{\ell} \prod_{r\geq 1} [\lambda_r^{(l)}]_t^! \in \mathbb{N}[t]$.

3.17. **Proposition.** *The basis $\{ f_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \}$ is the $*$-seminormal basis of $\mathcal{H}_n^\Lambda(\mathscr{K})$ determined by the seminormal coefficient system defined in Example 3.11 and the choices*

$$\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} = [\boldsymbol{\lambda}]_t^! \prod_{1\leq l<m\leq \ell} \prod_{(l,r,c)\in[\boldsymbol{\lambda}]} [\kappa_l - r + c - \kappa_m],$$

for $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$.

Proof. This is equivalent to [27, Theorem 2.11] in the non-degenerate case and to [4, Proposition 6.8] in the degenerate case, however, rather than translating the notation from these two papers it is easier to prove this directly.

As noted above, $(\mathcal{O}, t)$ separates $\operatorname{Std}(\mathcal{P}_n^\Lambda)$ and $f_{\mathfrak{s}\mathfrak{t}} \equiv m_{\mathfrak{s}\mathfrak{t}} + \sum r_{\mathfrak{u}\mathfrak{v}} m_{\mathfrak{u}\mathfrak{v}} \pmod{\mathcal{H}_n^{\rhd\boldsymbol{\lambda}}}$, for some $r_{\mathfrak{u}\mathfrak{v}} \in \mathscr{K}$ where $r_{\mathfrak{u}\mathfrak{v}} \neq 0$ only if $(\mathfrak{u}, \mathfrak{v}) \rhd (\mathfrak{s}, \mathfrak{t})$. Therefore, in view of (3.15), $\{ f_{\mathfrak{s}\mathfrak{t}} \mid (\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda) \}$ is a $*$-seminormal basis of $\mathcal{H}_n^\Lambda(\mathscr{K})$. By Theorem 3.13, this basis is determined by a seminormal coefficient system $\boldsymbol{\alpha}$ and by a choice of scalars $\{ \gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} \mid \boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \}$. If $\mathfrak{t} \rhd \mathfrak{v} = \mathfrak{t}(r, r+1)$ then, by definition, $m_{\mathfrak{s}\mathfrak{t}} T_r = m_{\mathfrak{s}\mathfrak{v}}$. The transition matrix between the $\{m_{\mathfrak{s}\mathfrak{t}}\}$ and $\{f_{\mathfrak{s}\mathfrak{t}}\}$ is unitriangular so, in view of Theorem 3.13, $f_{\mathfrak{s}\mathfrak{t}} T_r = f_{\mathfrak{s}\mathfrak{v}} - \frac{1}{[\rho_r(\mathfrak{t})]} f_{\mathfrak{s}\mathfrak{t}}$. Therefore, by Definition 3.9(c), the seminormal coefficient system corresponding to the basis $\{f_{\mathfrak{s}\mathfrak{t}}\}$ is the one appearing in Example 3.11.

It remains to determine the scalars $\{ \gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} \mid \boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \}$ corresponding to $\{f_{\mathfrak{s}\mathfrak{t}}\}$. It is well-known, and easy to prove using the relations in $\mathcal{H}_n^\Lambda$, that $x_{\boldsymbol{\lambda}}^2 = [\boldsymbol{\lambda}]_t^! x_{\boldsymbol{\lambda}}$. Therefore, by Lemma 2.9,

$$f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}^2 \equiv [\boldsymbol{\lambda}]_t^! m_{\boldsymbol{\lambda}} u_{\boldsymbol{\lambda}} \equiv [\boldsymbol{\lambda}]_t^! \prod_{1\leq l<m\leq \ell} \prod_{(l,r,c)\in[\boldsymbol{\lambda}]} [\kappa_l - r + c - \kappa_m] \cdot m_{\boldsymbol{\lambda}} \pmod{\mathcal{H}_n^{\rhd\boldsymbol{\lambda}}}.$$

Hence, $\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} = [\boldsymbol{\lambda}]_t^! \prod_{1\leq l<m\leq \ell} \prod_{(l,r,c)\in[\boldsymbol{\lambda}]} [\kappa_l - r + c - \kappa_m]$ by (3.15). □

As noted after Theorem 2.8, the Murphy basis $\{\, m_{\mathfrak{s}\mathfrak{t}} \mid (\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)\,\}$ of $\mathcal{H}_n^\Lambda$ gives a basis $\{\, m_\mathfrak{t} \mid \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})\,\}$ of each Specht module $\underline{S}^{\boldsymbol{\lambda}}$, for $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$. For example, we can set $m_\mathfrak{t} = m_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} + \mathcal{H}_n^{\rhd\boldsymbol{\lambda}}$, for $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. By (2.5), the cellular basis equips the Specht module $\underline{S}^{\boldsymbol{\lambda}}$ with an inner product $\langle\ ,\ \rangle$. The matrix

$$\underline{\mathcal{G}}^{\boldsymbol{\lambda}} = \big(\langle m_\mathfrak{s}, m_\mathfrak{t}\rangle\big)_{\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})}$$

is the **Gram matrix** of $\underline{S}^{\boldsymbol{\lambda}}$ with respect to the Murphy basis. Similarly, the seminormal basis yields a second basis $\{\, f_\mathfrak{t} \mid \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})\,\}$ of $\underline{S}^{\boldsymbol{\lambda}}(\mathscr{K})$, where $f_\mathfrak{t} = m_\mathfrak{t} F_\mathfrak{t} = f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} + \mathcal{H}_n^{\rhd\boldsymbol{\lambda}}$, for $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. The transition matrix between these two bases is unitriangular, so by (3.15) we have

$$\det \underline{\mathcal{G}}^{\boldsymbol{\lambda}} = \det\big(\langle f_\mathfrak{s}, f_\mathfrak{t}\rangle\big) = \prod_{\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})} \gamma_\mathfrak{t}. \tag{3.18}$$

This 'classical' formula for $\det \underline{\mathcal{G}}^{\boldsymbol{\lambda}}$ is well-known as it is the cornerstone used to prove the formula for $\det \underline{\mathcal{G}}^{\boldsymbol{\lambda}}$ as a rational function in [17, Theorem 3.35]. The following definition will allow us to give an 'integral' closed formula for $\det \underline{\mathcal{G}}^{\boldsymbol{\lambda}}$.

3.19. **Definition.** *Suppose that $e \in \{0, 2, 3, 4, \dots\}$, p is a prime integer and that $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$ is a multipartition of n. Define*

$$\deg_e(\boldsymbol{\lambda}) = \sum_{\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})} \deg_e(\mathfrak{t}) \qquad \textit{and} \qquad \operatorname{Deg}_p(\boldsymbol{\lambda}) = \sum_{k\ge1} \deg_{p^k}(\boldsymbol{\lambda}).$$

By definition, $\deg_e(\boldsymbol{\lambda})$ and $\operatorname{Deg}_p(\boldsymbol{\lambda})$ are integers that, *a priori*, could be positive, negative or zero. In fact, the next result shows that they are always non-negative integers, although we do not know of a direct combinatorial proof of this. By definition, the integers $\deg_e(\boldsymbol{\lambda})$ and $\operatorname{Deg}_p(\boldsymbol{\lambda})$ depend on $\boldsymbol{\kappa}$ and e. Our definitions ensure that the tableau degrees $\deg_e(\mathfrak{t})$, for $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$, coincide with (2.6) when $\Lambda = \Lambda_e(\boldsymbol{\kappa})$.

For $k \in \mathbb{N}$, let $\Phi_k = \Phi_k(t)$ be the kth cyclotomic polynomial in t. As is well-known, these polynomials are pairwise distinct irreducible polynomials in $\mathbb{Z}[t]$ and

$$[n] = \prod_{1<d\mid n} \Phi_d(t), \tag{3.20}$$

whenever $n \ge 1$.

3.21. **Theorem.** *Suppose that $\kappa_l - \kappa_{l+1} > n$, for $1 \le l < \ell$, and that $\mathcal{O} = \mathbb{Z}[t, t^{-1}]$. Then*

$$\det \underline{\mathcal{G}}^{\boldsymbol{\lambda}} = t^{\ell(\boldsymbol{\lambda})} \prod_{e\ge2} \Phi_e(t)^{\deg_e(\boldsymbol{\lambda})},$$

where $\ell(\boldsymbol{\lambda}) = \sum_{\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})} \ell(d(\mathfrak{t}))$.

Proof. As remarked above, $\det \underline{\mathcal{G}}^{\boldsymbol{\lambda}} = \prod_\mathfrak{t} \gamma_\mathfrak{t}$. Therefore, to prove the theorem it is enough to show that if $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ then

$$\gamma_\mathfrak{t} = t^{\ell(d(\mathfrak{t}))} \prod_{e>1} \Phi_e(t)^{\deg_e(\mathfrak{t})}.$$

We prove this by induction on the dominance ordering.

Suppose first that $\mathfrak{t} = \mathfrak{t}^{\boldsymbol{\lambda}}$. Then Proposition 3.17 gives an explicit formula for $\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}}$ and, using (2.6), it is straightforward to check by induction on n that our claim is true in this case. Suppose then that $\mathfrak{t}^{\boldsymbol{\lambda}} \rhd \mathfrak{t}$. Then we can write $\mathfrak{t} = \mathfrak{s}(r, r+1)$ for some $\mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda})$ such that $\mathfrak{s} \rhd \mathfrak{t}$, and where $1 \le r < n$. Therefore, using induction, Corollary 3.16 and the seminormal coefficient system of Proposition 3.17,

$$\gamma_\mathfrak{t} = t^{\ell(d(\mathfrak{s}))} \frac{[1 + \rho_r(\mathfrak{s})][1 + \rho_r(\mathfrak{t})]}{[\rho_r(\mathfrak{s})][\rho_r(\mathfrak{t})]} \prod_{e>1} \Phi_e(t)^{\deg_e(\mathfrak{s})}.$$

By definition, $[k] = -t^k[-k]$, for any $k \in \mathbb{Z}$. Now $\rho_r(\mathfrak{s}) = -\rho_r(\mathfrak{t}) > 0$ by (3.2), so

$$\frac{[1+\rho_r(\mathfrak{s})][1+\rho_r(\mathfrak{t})]}{[\rho_r(\mathfrak{s})][\rho_r(\mathfrak{t})]} = t\,\frac{[1+\rho_r(\mathfrak{s})][-\rho_r(\mathfrak{t})-1]}{[\rho_r(\mathfrak{s})][-\rho_r(\mathfrak{t})]} = t\prod_{e>1}\Phi_e(t)^{d_e},$$

where, according to (3.20), the integer d_e is given in terms of the quiver Γ_e by

$$d_e = \begin{cases} -2, & \text{if } i_r = i_{r+1},\\ 2, & \text{if } i_r \leftrightarrows i_{r+1},\\ 1, & \text{if } i_r \leftarrow i_{r+1} \text{ or } i_r \rightarrow i_{r+1},\\ 0, & \text{otherwise.}\end{cases}$$

Applying Lemma 2.7 now completes the proof of our claim — and hence proves the theorem. ∎

3.22. *Remark.* We can remove the factor $t^{\ell(\boldsymbol{\lambda})}$ from Theorem 3.21 by rescaling the generators $T_1, \dots, T_{n-1}$ so that the quadratic relations in Definition 2.2 become $(T_r - t^{\frac12})(T_r + t^{-\frac12})$, for $1 \le r < n$. Note that the integer d_e in the proof of Theorem 3.13 is equal to the degree of the homogeneous generator $\psi_r e(\mathbf{i})$ in the cyclotomic KLR algebra $\mathcal{R}_n^\Lambda$.

Setting $t = 1$ gives the degenerate cyclotomic Hecke algebras. As a special case, the next result gives an integral closed formula for the Gram determinants of the Specht modules of the symmetric groups.

3.23. **Corollary.** *Suppose that $\kappa_l - \kappa_{l+1} > n$, for $1 \le l < \ell$, and that $\mathcal{O} = \mathbb{Z}$ and $t = 1$. Then*

$$\det \underline{\mathcal{G}}^{\boldsymbol{\lambda}} = \prod_{\substack{0<p\in\mathbb{Z}\\ p \text{ prime}}} p^{\operatorname{Deg}_p(\boldsymbol{\lambda})},$$

for $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$.

Proof. This follows by setting $t = 1$ in Theorem 3.21 and using the following well-known property of the cyclotomic polynomials:

$$\Phi_e(1) = \begin{cases} p, & \text{if } e = p^k \text{ for some } k \ge 1,\\ 1, & \text{otherwise.}\end{cases}$$

∎

3.24. **Corollary.** *Suppose that $e \in \{0, 2, 3, 4, 5, \dots\}$ and that $p > 0$ is an integer prime. Then $\deg_e(\boldsymbol{\lambda}) \ge 0$ and $\operatorname{Deg}_p(\boldsymbol{\lambda}) \ge 0$, for all $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$.*

Proof. As the Murphy basis is defined over $\mathbb{Z}[t, t^{-1}]$, the Gram determinant $\det \underline{\mathcal{G}}^{\boldsymbol{\lambda}}$ belongs to $\mathbb{Z}[t, t^{-1}]$. Therefore, $\deg_e(\boldsymbol{\lambda}) \ge 0$ whenever $e > 1$ by Theorem 3.21. Consequently, $\operatorname{Deg}_p(\boldsymbol{\lambda}) \ge 0$. Finally, if $e \gg 0$ then $\deg_0(\mathfrak{t}) = \deg_e(\mathfrak{t})$ for any $\mathfrak{t} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$, so $\deg_e(\boldsymbol{\lambda}) \ge 0$ for $e \in \{0, 2, 3, 4, \dots\}$ as claimed. ∎

The statement of Corollary 3.24 is purely combinatorial so it should have a direct combinatorial proof. We now give a second representation theoretic proof of this result that suggests that a combinatorial proof may be difficult.

A **graded set** is a set D equipped with a **degree** function $\deg : D \longrightarrow \mathbb{Z}$. Let q be an indeterminate over $\mathbb{Z}$ and define the **q-cardinality** and **degree** of D to be

$$|D|_q = \sum_{d\in D} q^{\deg d} \in \mathbb{N}[q, q^{-1}] \quad\text{and}\quad \deg D = \sum_{d\in D} \deg d \in \mathbb{Z}.$$

If D is a graded set and $z \in \mathbb{Z}$ let $q^z D$ be the graded set with the same elements as D but where the shifted degree function is shifted so that $d \in D$ now has degree $z + \deg d$. More generally, if $f(q) \in \mathbb{N}[q, q^{-1}]$ let $f(q)D$ be the graded set that is the disjoint union of the appropriate number of shifted copies of D. For example $(2+q)D = D \sqcup D \sqcup qD$. By definition, $|f(q)D|_q = f(q)|D|_q$.

If $e \in \{0, 2, 3, 4, \dots\}$ let $\mathrm{Std}_e(\boldsymbol{\lambda})$ be the graded set with elements $\mathrm{Std}(\boldsymbol{\lambda})$ and degree function $\mathfrak{t} \mapsto \deg_e(\mathfrak{t})$, for $\mathfrak{t} \in \mathrm{Std}_e(\boldsymbol{\lambda})$.

Fix $e \in \{0, 2, 3, 4, \dots\}$ and consider the Hecke algebra $\mathcal{H}_n^\Lambda(\mathbb{C})$ over $\mathbb{C}$ with Hecke parameter ξ, a primitive eth root of unity if $e > 0$ or a non-root of unity if $e = 0$. Let $S^{\boldsymbol{\lambda}}$ be the graded Specht module introduced in [8] (see Section 5.2), and let $D^{\boldsymbol{\mu}} = S^{\boldsymbol{\mu}}/\operatorname{rad} S^{\boldsymbol{\mu}}$ be the graded simple quotient of $S^{\boldsymbol{\mu}}$, as in [14]. Let $\mathcal{K}_n^\Lambda$ be the set of **Kleshchev multipartitions** so that $\{\, D^{\boldsymbol{\mu}}\langle k\rangle \mid \boldsymbol{\mu} \in \mathcal{K}_n^\Lambda \text{ and } k \in \mathbb{Z}\,\}$ is a complete set of non-isomorphic graded simple $\mathcal{H}_n^\Lambda$-modules. As recalled in Section 5.2, $S^{\boldsymbol{\lambda}}$ comes equipped with a homogeneous basis $\{\, \psi_{\mathfrak{t}} \mid \mathfrak{t} \in \mathrm{Std}_e(\boldsymbol{\lambda})\,\}$. Let $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) = [S^{\boldsymbol{\lambda}}{:}D^{\boldsymbol{\mu}}]_q$ be the corresponding graded decomposition number.

Fix a total ordering $\prec$ on $\mathrm{Std}_e(\boldsymbol{\lambda})$ that extends the dominance ordering, such as the lexicographic ordering. Suppose that $\boldsymbol{\mu} \in \mathcal{K}_n^\Lambda$. By Gaussian elimination, there exists a graded subset $\mathrm{DStd}_e(\boldsymbol{\mu})$ of $\mathrm{Std}_e(\boldsymbol{\mu})$ and a homogeneous basis $\{\, C_{\mathfrak{t}} \mid \mathfrak{t} \in \mathrm{DStd}_e(\boldsymbol{\mu})\,\}$ of $D^{\boldsymbol{\mu}}$ such that

$$C_{\mathfrak{t}} = \psi_{\mathfrak{t}} + \sum_{\mathfrak{v} \prec \mathfrak{t}} c_{\mathfrak{t}\mathfrak{v}} \psi_{\mathfrak{v}} + \operatorname{rad} S^{\boldsymbol{\mu}},$$

for some $c_{\mathfrak{t}\mathfrak{v}} \in \mathbb{C}$ such that $c_{\mathfrak{t}\mathfrak{v}} \neq 0$ only if $\deg \mathfrak{v} = \deg \mathfrak{t}$ and $\mathrm{res}(\mathfrak{v}) = \mathrm{res}(\mathfrak{t})$. In particular, $\dim_{\mathrm{q}} D^{\boldsymbol{\lambda}} = |\,\mathrm{DStd}_e(\boldsymbol{\lambda})|_q$. Repeating this argument, with the composition factors that appear in successive layers of the radical filtration of $S^{\boldsymbol{\lambda}}$, shows that there exists a bijection of graded sets

$$\Theta_{\boldsymbol{\lambda}} : \mathrm{Std}_e(\boldsymbol{\lambda}) \xrightarrow{\sim} \bigsqcup_{\boldsymbol{\mu} \in \mathcal{K}_n^\Lambda} d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)\, \mathrm{DStd}_e(\boldsymbol{\mu}).$$

Now if $\boldsymbol{\mu} \in \mathcal{K}_n^\Lambda$ then $D^{\boldsymbol{\mu}} \cong (D^{\boldsymbol{\mu}})^{\circledast}$, so that $\deg \mathrm{DStd}_e(\boldsymbol{\mu}) = 0$. It follows that $\deg q^z\, \mathrm{DStd}_e(\boldsymbol{\mu}) = z \dim \underline{D}^{\boldsymbol{\mu}}$, for $z \in \mathbb{Z}$. Therefore, using the bijection $\Theta_{\boldsymbol{\lambda}}$,

$$\deg_e(\boldsymbol{\lambda}) = \deg \mathrm{Std}_e(\boldsymbol{\lambda}) = \sum_{\boldsymbol{\mu} \in \mathcal{K}_n^\Lambda} d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) \deg \mathrm{DStd}_e(\boldsymbol{\mu}) = \sum_{\boldsymbol{\mu} \in \mathcal{K}_n^\Lambda} d'_{\boldsymbol{\lambda}\boldsymbol{\mu}}(1) \dim \underline{D}^{\boldsymbol{\mu}},$$

where $d'_{\boldsymbol{\lambda}\boldsymbol{\mu}}(1)$ is the derivative of the graded decomposition number $d_{\lambda\boldsymbol{\mu}}(q)$ evaluated at $q = 1$. As we are working with the Hecke algebra $\mathcal{H}_n^\Lambda(\mathbb{C})$ in characteristic zero, $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) \in \mathbb{N}[q]$ by [7, Corollary 5.15]. Consequently, $\deg_e(\boldsymbol{\lambda}) \geq 0$. Hence, the (deep) fact that $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) \in \mathbb{N}[q]$ leads to an alternative proof of Corollary 3.24.

In characteristic zero the graded cyclotomic Schur algebras is Koszul by [15, Theorem C] when $e = 0$ and by [25] and [32, Proposition 7.8,7.9] in general. This implies that the Jantzen and grading filtrations of the graded Weyl modules, and hence of the graded Specht modules, coincide. Therefore, Corollary 3.24 is compatible with this Koszulity Conjecture via Ryom-Hansen's [33, Theorem 1] description of the Jantzen sum formula; see also [38, Theorem 2.11].

The construction of the sets $\mathrm{DStd}_e(\boldsymbol{\mu})$ given above is not unique because it involves many choices. It natural to ask if there is a canonical choice of basis for $S^{\boldsymbol{\lambda}}$ that uniquely determines the sets $\mathrm{DStd}_e(\boldsymbol{\mu})$ and the bijections $\Theta_{\boldsymbol{\lambda}}$. For level 2 such bijections are implicit in [9, §9] when $e = 0$ and [15, Appendix] generalizes this to include the cases when $e > n$. It is interesting to note that the sets $\mathrm{DStd}_e(\boldsymbol{\mu})$, together with the bijections $\Theta_{\boldsymbol{\lambda}}$, determine the graded decomposition numbers. More explicitly, if $\mathfrak{s} \in \mathrm{DStd}_e(\boldsymbol{\mu})$ then

$$d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) = \sum_{\mathfrak{t} \in \Theta_{\boldsymbol{\lambda}}^{-1}(\mathfrak{s})} q^{\deg \mathfrak{t} - \deg \mathfrak{s}},$$

where we abuse notation and let $\Theta_{\boldsymbol{\lambda}}^{-1}(\mathfrak{s})$ be the set of tableaux in $\mathrm{Std}_e(\boldsymbol{\lambda})$ that are mapped onto a (shifted) copy of $\mathfrak{s}$ by $\Theta_{\boldsymbol{\lambda}}$. In particular, we can take $\mathfrak{s} = \mathfrak{t}^{\boldsymbol{\mu}}$ because it is easy to see that we must have $\mathfrak{t}^{\boldsymbol{\mu}} \in \mathrm{DStd}_e(\boldsymbol{\mu})$ whenever $\boldsymbol{\mu} \in \mathcal{K}_n^\Lambda$. Hence, we have shown that the KLR tableau combinatorics leads to closed combinatorial formulas for the parabolic Kazhdan-Lusztig polynomials $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q)$, and the graded simple dimensions $\dim_q D^{\boldsymbol{\mu}}$: both families of polynomials can be described as the q-cardinalities of graded sets of tableaux.

4. Integral Quiver Hecke algebras

The Seminormal Basis Theorem 3.13 compactly describes much of the semisimple representation theory of $\mathcal{H}_n^\Lambda(\mathscr{K})$. For symmetric groups, Murphy [29] showed that seminormal bases can also be used to study the non-semisimple representation theory. Murphy's ideas were extended to the cyclotomic Hecke algebras in [27,28]. In this section we further extend Murphy's ideas to connect seminormal bases and the KLR grading on $\mathcal{H}_n^\Lambda$.

4.1. **Lifting idempotents.** As Section 3.2, we continue to assume that $\boldsymbol{\kappa}$ satisfies (3.2) and that $(\mathscr{K}, t)$ separates $\mathrm{Std}(\mathcal{P}_n^\Lambda)$, where $\mathscr{K}$ is a field and $0 \neq t \in \mathscr{K}$. If $\mathcal{O}$ is a subring of $\mathscr{K}$ then we identify $\mathcal{H}_n^\Lambda(\mathcal{O})$ with the obvious $\mathcal{O}$-subalgebra of $\mathcal{H}_n^\Lambda(\mathscr{K})$ so that $\mathcal{H}_n^\Lambda(\mathscr{K}) \cong \mathcal{H}_n^\Lambda(\mathcal{O}) \otimes_{\mathcal{O}} \mathscr{K}$ as $\mathscr{K}$-algebras.

Let $J(\mathcal{O})$ be the **Jacobson radical** of $\mathcal{O}$, the intersection of all of the maximal ideals of $\mathcal{O}$.

4.1. **Definition.** *Suppose that $\mathcal{O}$ is a subring of $\mathscr{K}$ and $t \in \mathcal{O}^\times$. Then $(\mathcal{O}, t)$ is an e-**idempotent subring** of $\mathscr{K}$ if the following hold:*

a) *$(\mathcal{O}, t)$ separates $\mathrm{Std}(\mathcal{P}_n^\Lambda)$;*
b) *$[k]_t$ is invertible in $\mathcal{O}$ whenever $k \not\equiv 0 \pmod e$, for $k \in \mathbb{Z}$; and*
c) *$[k]_t \in J(\mathcal{O})$ whenever $k \in e\mathbb{Z}$.*

When e and t are understood, we simply call $\mathcal{O}$ an idempotent subring. Note that $\mathscr{K}$ contains the field of fractions of $\mathcal{O}$, so Definition 4.1(a) ensures that $\mathcal{H}_n^\Lambda(\mathscr{K})$ is semisimple and that it has a seminormal basis. Until further notice, we fix such a $*$-seminormal basis $\{f_{\mathfrak{s}\mathfrak{t}}\}$, together with the corresponding seminormal coefficient system $\boldsymbol{\alpha}$ and γ-coefficients.

Let $(\mathcal{O}, t)$ be an e-idempotent subring and suppose $c \not\equiv d \pmod e$, for $c, d \in \mathbb{Z}$. Then $[c] - [d] = t^d[c-d]$ is invertible in $\mathcal{O}$. We use this fact below without mention.

4.2. **Examples** The following local rings are all examples of idempotent subrings.

a) Suppose that $\mathscr{K} = \mathbb{Q}$ and $t = 1$. Then $(\mathscr{K}, t)$ separates $\mathrm{Std}(\mathcal{P}_n^\Lambda)$ and $\mathcal{O} = \mathbb{Z}_{(p)}$ is a p-idempotent subring of $\mathbb{Q}$ for any prime p.
b) Let K be any field and set $\mathscr{K} = K(x)$, where x is an indeterminate over K, and $t = x+\xi$, where ξ is a primitive eth root of unity in K. Then $\mathcal{O} = K[x]_{(x)}$ is an e-idempotent subring of $\mathscr{K}$.
c) Let $\mathscr{K} = \mathbb{Q}(x, \xi)$, where x is an indeterminate over $\mathbb{Q}$ and $\xi = \exp(2\pi i/e)$ is a primitive eth root of unity in $\mathbb{C}$. Let $t = x + \xi$. Then $(\mathscr{K}, t)$ separates $\mathrm{Std}(\mathcal{P}_n^\Lambda)$ and $\mathcal{O} = \mathbb{Z}[x, \xi]_{(x)}$ is an e-idempotent subring of $\mathscr{K}$.
d) Maintain the notation of the last example and let $p > 1$ be a prime not dividing e. Let $\Phi_{e,p}(x)$ be a polynomial in $\mathbb{Z}[x]$ whose reduction modulo p is the minimum polynomial of a primitive eth root of unity in an algebraically closed field of characteristic p. Then $\mathcal{O} = \mathbb{Z}[x, \xi]_{(x,p,\Phi_{e,p}(\xi))}$ is an e-idempotent subring of $\mathbb{C}(x)$.

◇

Suppose that $\mathbf{i} \in I^n$ and set $\mathrm{Std}(\mathbf{i}) = \{\, \mathfrak{t} \in \mathrm{Std}(\mathcal{P}_n^\Lambda) \mid \mathrm{res}(\mathfrak{t}) = \mathbf{i} \,\}$. Define the **residue idempotent** $f_{\mathbf{i}}^{\mathcal{O}}$ by

$$f_{\mathbf{i}}^{\mathcal{O}} = \sum_{\mathfrak{t} \in \mathrm{Std}(\mathbf{i})} F_{\mathfrak{t}}. \tag{4.3}$$

By Theorem 3.13, $f^{\mathcal{O}}_{\mathbf{i}}$ is an idempotent in $\mathcal{H}^\Lambda_n(\mathscr{K})$. In the rest of this section, we fix a seminormal basis $\{f_{\mathfrak{s}\mathfrak{t}}\}$ of $\mathcal{H}^\Lambda_n(\mathscr{K})$ that is determined by a seminormal coefficient system $\{\alpha_r(\mathfrak{s})\}$ and a choice of $\gamma_{\mathfrak{t}^\lambda}$. Then we have that $f^{\mathcal{O}}_{\mathbf{i}} = \sum_{\mathfrak{t}\in\operatorname{Std}(\mathbf{i})} \frac{1}{\gamma_\mathfrak{t}} f_{\mathfrak{t}\mathfrak{t}}$.

4.4. **Lemma.** *Suppose that $\mathcal{O}$ is an idempotent subring of $\mathscr{K}$ and that $\mathbf{i}\in I^n$. Then $f^{\mathcal{O}}_{\mathbf{i}} \in \mathscr{L}(\mathcal{O})$. In particular, $f^{\mathcal{O}}_{\mathbf{i}}$ is an idempotent in $\mathcal{H}^\Lambda_n(\mathcal{O})$.*

Proof. This result is proved when $\mathcal{O}$ is a discrete valuation ring in [28, Lemma 4.2], however, our weaker assumptions necessitate a different proof. Motivated, in part, by the proof of [29, Theorem 2.1], if $\mathfrak{t}\in\operatorname{Std}(\mathbf{i})$ define

$$F'_\mathfrak{t} = \prod_{k=1}^n \prod_{\substack{c\in\mathscr{C}\\ \mathrm{c}_k(\mathfrak{t})\not\equiv c \pmod e}} \frac{L_k-[c]}{[\mathrm{c}_k(\mathfrak{t})]-[c]}.$$

Since $\mathcal{O}$ is an e-idempotent subring, $F'_\mathfrak{t} \in \mathscr{L}(\mathcal{O}) \subset \mathcal{H}^\Lambda_n(\mathcal{O})$. By Theorem 3.13, $\sum_{\mathfrak{s}\in\operatorname{Std}(\mathcal{P}^\Lambda_n)} F_\mathfrak{s}$ is the identity element of $\mathcal{H}^\Lambda_n(\mathscr{K})$ so, using (3.14), we see that

$$F'_\mathfrak{t} = \sum_{\mathfrak{s}\in\operatorname{Std}(\mathcal{P}^\Lambda_n)} F'_\mathfrak{t} F_\mathfrak{s} = \sum_{\mathfrak{s}\in\operatorname{Std}(\mathcal{P}^\Lambda_n)} a_{\mathfrak{s}\mathfrak{t}} F_\mathfrak{s},$$

where $a_{\mathfrak{s}\mathfrak{t}} = \prod_{k,c}([\mathrm{c}_k(\mathfrak{s})]-[c])/([\mathrm{c}_k(\mathfrak{t})]-[c]) \in \mathcal{O}$. In particular, $a_{\mathfrak{t}\mathfrak{t}}=1$. If $\mathfrak{s}\notin\operatorname{Std}(\mathbf{i})$ then there exists an integer k such that $\operatorname{res}_k(\mathfrak{s})\ne\operatorname{res}_k(\mathfrak{t})$, so $[\mathrm{c}_k(\mathfrak{s})]-[\mathrm{c}_k(\mathfrak{t})]\in\mathcal{O}^\times$ and $a_{\mathfrak{s}\mathfrak{t}}=0$. Therefore, $F'_\mathfrak{t} = \sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})} a_{\mathfrak{s}\mathfrak{t}} F_\mathfrak{s}$. Consequently, $f^{\mathcal{O}}_{\mathbf{i}} F'_\mathfrak{t} = F'_\mathfrak{t} = F'_\mathfrak{t} f^{\mathcal{O}}_{\mathbf{i}}$ by (3.15). Notice that $F'_\mathfrak{t} F'_\mathfrak{s} = F'_\mathfrak{s} F'_\mathfrak{t}$ because $\mathscr{L}(\mathscr{K})$ is a commutative subalgebra of $\mathcal{H}^\Lambda_n(\mathscr{K})$. Therefore,

$$\prod_{\mathfrak{t}\in\operatorname{Std}(\mathbf{i})} (f^{\mathcal{O}}_{\mathbf{i}} - F'_\mathfrak{t}) = f^{\mathcal{O}}_{\mathbf{i}} \qquad + \sum_{\substack{\mathfrak{t}_1,\dots,\mathfrak{t}_k\in\operatorname{Std}(\mathbf{i})\\ \text{distinct with } k>0}} (-1)^k F'_{\mathfrak{t}_1}F'_{\mathfrak{t}_2}\dots F'_{\mathfrak{t}_k}.$$

On the other hand, since $f^{\mathcal{O}}_{\mathbf{i}} = \sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})} F_\mathfrak{s}$ and $a_{\mathfrak{t}\mathfrak{t}}=1$,

$$\prod_{\mathfrak{t}\in\operatorname{Std}(\mathbf{i})} (f^{\mathcal{O}}_{\mathbf{i}} - F'_\mathfrak{t}) = \prod_{\mathfrak{t}\in\operatorname{Std}(\mathbf{i})} \sum_{\substack{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})\\ \mathfrak{s}\ne\mathfrak{t}}} (1-a_{\mathfrak{s}\mathfrak{t}})F_\mathfrak{s} = 0,$$

because $F_\mathfrak{s}F_\mathfrak{t}=0$ whenever $\mathfrak{s}\ne\mathfrak{t}$ by (3.15). Combining the last two equations,

$$f^{\mathcal{O}}_{\mathbf{i}} = \sum_{\substack{\mathfrak{t}_1,\dots,\mathfrak{t}_k\in\operatorname{Std}(\mathbf{i})\\ \text{distinct with } k>0}} (-1)^{k+1} F'_{\mathfrak{t}_1}F'_{\mathfrak{t}_2}\dots F'_{\mathfrak{t}_k}.$$

In particular, $f^{\mathcal{O}}_{\mathbf{i}}\in\mathscr{L}(\mathcal{O})$ as we wanted to show. □

4.5. **Corollary.** *Suppose that $\mathcal{O}$ is an idempotent subring of $\mathscr{K}$. Then $\{\, f^{\mathcal{O}}_{\mathbf{i}} \mid \mathbf{i}\in I^n \,\}$ is a complete set of pairwise orthogonal idempotents in $\mathcal{H}^\Lambda_n(\mathcal{O})$.*

Proof. By Theorem 3.13, $\{\, F_\mathfrak{t} \mid \mathfrak{t}\in\operatorname{Std}(\mathcal{P}^\Lambda_n) \,\}$ is a complete set of pairwise orthogonal idempotents in $\mathcal{H}^\Lambda_n(\mathscr{K})$. Hence, the result follows from Lemma 4.4. □

If $\phi\in\mathcal{O}[X_1,\dots,X_n]$ is a polynomial in indeterminates $X_1,\dots,X_n$ over $\mathcal{O}$ then set $\phi(L)=\phi(L_1,\dots,L_n)\in\mathscr{L}(\mathcal{O})$. If $\mathfrak{s}$ is a tableau let $\phi(\mathfrak{s}) = \phi([c_1(\mathfrak{s})],\dots,[c_n(\mathfrak{s})])$ be the scalar in $\mathcal{O}$ obtained by evaluating the polynomial ϕ on the contents of $\mathfrak{s}$; that is, setting $X_1=[c_1(\mathfrak{s})],\dots,X_n=[c_n(\mathfrak{s})]$. Then, $\phi(L)f_{\mathfrak{s}\mathfrak{t}} = \phi(\mathfrak{s})f_{\mathfrak{s}\mathfrak{t}}$, for all $(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathcal{P}^\Lambda_n)$.

Ultimately, the next result will allow us to 'renormalise' intertwiners of the residue idempotents $f^{\mathcal{O}}_{\mathbf{i}}$, for $\mathbf{i}\in I^n$, so that they depend only on e rather than on ξ.

4.6. **Proposition.** *Suppose that* $\mathbf{i} \in I^n$ *and* $\phi \in \mathcal{O}[X_1, \dots, X_n]$ *is a polynomial such that* $\phi(\mathfrak{t})$ *is invertible in* $\mathcal{O}$*, for all* $\mathfrak{t} \in \operatorname{Std}(\mathbf{i})$*. Then*

$$f_{\mathbf{i}}^{\phi} = \sum_{\mathfrak{t} \in \operatorname{Std}(\mathbf{i})} \frac{1}{\phi(\mathfrak{t})} F_{\mathfrak{t}} \in \mathscr{L}(\mathcal{O}).$$

In particular, $f_{\mathbf{i}}^{\phi} \in \mathcal{H}_n^{\Lambda}(\mathcal{O})$.

Proof. By assumption, $\phi(\mathfrak{s})$ is invertible in $\mathcal{O}$ for all $\mathfrak{s} \in \operatorname{Std}(\mathbf{i})$. In particular, $f_{\mathbf{i}}^{\phi}$ is a well-defined element of $\mathscr{L}(\mathscr{K})$. It remains to show that $f_{\mathbf{i}}^{\phi} \in \mathscr{L}(\mathcal{O})$.

As in Lemma 4.4, for each $\mathfrak{t} \in \operatorname{Std}(\mathbf{i})$ define

$$F'_{\mathfrak{t}} = \prod_{\substack{c \in \mathscr{C} \\ c_k(\mathfrak{t}) \not\equiv c \pmod e}} \frac{L_k - [c]}{[c_k(\mathfrak{t})] - [c]} \in \mathscr{L}(\mathcal{O}),$$

and write $F'_{\mathfrak{t}} = \sum_{\mathfrak{s} \in \operatorname{Std}(\mathbf{i})} a_{\mathfrak{s}\mathfrak{t}} F_{\mathfrak{s}}$ for some $a_{\mathfrak{s}\mathfrak{t}} \in \mathcal{O}$. Recall from the proof of Lemma 4.4 that $a_{\mathfrak{t}\mathfrak{t}} = 1$.

Motivated by the definition of $F'_{\mathfrak{t}}$, set $F_{\mathfrak{t}}^{\phi} = \frac{\phi(L)}{\phi(\mathfrak{t})} F'_{\mathfrak{t}}$. Then $F_{\mathfrak{t}}^{\phi} \in \mathscr{L}(\mathcal{O})$ and

$$F_{\mathfrak{t}}^{\phi} = \sum_{\mathfrak{s} \in \operatorname{Std}(\mathbf{i})} a_{\mathfrak{s}\mathfrak{t}} \frac{\phi(L)}{\phi(\mathfrak{t})} F_{\mathfrak{s}} = F_{\mathfrak{t}} + \sum_{\substack{\mathfrak{s} \in \operatorname{Std}(\mathbf{i}) \\ \mathfrak{s} \neq \mathfrak{t}}} \frac{a_{\mathfrak{s}\mathfrak{t}} \phi(\mathfrak{s})}{\phi(\mathfrak{t})} F_{\mathfrak{s}}$$

by (3.14). Consequently, $F_{\mathfrak{t}}^{\phi} f_{\mathbf{i}}^{\mathcal{O}} = F_{\mathfrak{t}}^{\phi} = f_{\mathbf{i}}^{\mathcal{O}} F_{\mathfrak{t}}^{\phi}$. The idempotents $\{\, F_{\mathfrak{s}} \mid \mathfrak{s} \in \operatorname{Std}(\mathbf{i}) \,\}$ are pairwise orthogonal, so

$$f_{\mathbf{i}}^{\phi} F_{\mathfrak{t}}^{\phi} = \Big(\sum_{\mathfrak{s} \in \operatorname{Std}(\mathbf{i})} \frac{1}{\phi(\mathfrak{s})} F_{\mathfrak{s}} \Big) \Big(\sum_{\mathfrak{s} \in \operatorname{Std}(\mathbf{i})} \frac{a_{\mathfrak{s}\mathfrak{t}} \phi(\mathfrak{s})}{\phi(\mathfrak{t})} F_{\mathfrak{s}} \Big) = \sum_{\mathfrak{s} \in \operatorname{Std}(\mathbf{i})} \frac{a_{\mathfrak{s}\mathfrak{t}}}{\phi(\mathfrak{t})} F_{\mathfrak{s}} = \frac{1}{\phi(\mathfrak{t})} F'_{\mathfrak{t}}.$$

Therefore, $f_{\mathbf{i}}^{\phi} F_{\mathfrak{t}}^{\phi} \in \mathscr{L}(\mathcal{O})$, for all $\mathfrak{t} \in \operatorname{Std}(\mathbf{i})$. By (3.14), $f_{\mathbf{i}}^{\phi} f_{\mathbf{i}}^{\mathcal{O}} = f_{\mathbf{i}}^{\phi} = f_{\mathbf{i}}^{\mathcal{O}} f_{\mathbf{i}}^{\phi}$, so this implies that $f_{\mathbf{i}}^{\phi}(f_{\mathbf{i}}^{\mathcal{O}} - F_{\mathfrak{t}}^{\phi}) \equiv f_{\mathbf{i}}^{\phi} \pmod{\mathscr{L}(\mathcal{O})}$. Hence, working modulo $\mathscr{L}(\mathcal{O})$,

$$f_{\mathbf{i}}^{\phi} \equiv f_{\mathbf{i}}^{\phi} \prod_{\mathfrak{t} \in \operatorname{Std}(\mathbf{i})} (f_{\mathbf{i}}^{\mathcal{O}} - F_{\mathfrak{t}}^{\phi}) = f_{\mathbf{i}}^{\phi} \prod_{\mathfrak{t} \in \operatorname{Std}(\mathbf{i})} \sum_{\substack{\mathfrak{s} \in \operatorname{Std}(\mathbf{i}) \\ \mathfrak{s} \neq \mathfrak{t}}} \frac{a_{\mathfrak{s}\mathfrak{t}} \phi(\mathfrak{s})}{\phi(\mathfrak{t})} F_{\mathfrak{s}} = 0,$$

where the last equality follows using the orthogonality of the idempotents $F_{\mathfrak{s}}$ once again. Therefore, $f_{\mathbf{i}}^{\phi} \in \mathscr{L}(\mathcal{O})$, completing the proof. □

Let ϕ be a polynomial in $\mathcal{O}[X_1, \dots, X_n]$ satisfying the assumptions of Proposition 4.6. Then $\phi(L) f_{\mathbf{i}}^{\phi} = f_{\mathbf{i}}^{\mathcal{O}} = f_{\mathbf{i}}^{\phi} \phi(L)$ by (3.14). Abusing notation, in this situation we write

$$\frac{1}{\phi(L)} f_{\mathbf{i}}^{\mathcal{O}} = f_{\mathbf{i}}^{\phi} = \sum_{\mathfrak{s} \in \operatorname{Std}(\mathbf{i})} \frac{1}{\phi(\mathfrak{s})} F_{\mathfrak{s}} = f_{\mathbf{i}}^{\mathcal{O}} \frac{1}{\phi(L)} \in \mathscr{L}(\mathcal{O}).$$

Note that, either by direction calculation or because $\mathscr{L}$ is commutative, we are justified in writing $f_{\mathbf{i}}^{\mathcal{O}} \frac{1}{\phi(L)} = \frac{1}{\phi(L)} f_{\mathbf{i}}^{\mathcal{O}}$.

We need the following three special cases of Proposition 4.6. For $1 \le r < n$ define $M_r = 1 - L_r + tL_{r+1}$ and $M'_r = 1 + tL_r - L_{r+1}$, for $1 \le r < n$. Applying the definitions, if $(\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^{\Lambda})$ then

$$M_r f_{\mathfrak{s}\mathfrak{t}} = t^{c_r(\mathfrak{s})}[1 - \rho_r(\mathfrak{s})] f_{\mathfrak{s}\mathfrak{t}} \quad \text{and} \quad M'_r f_{\mathfrak{s}\mathfrak{t}} = t^{c_{r+1}(\mathfrak{s})}[1 + \rho_r(\mathfrak{s})] f_{\mathfrak{s}\mathfrak{t}}. \tag{4.7}$$

Our main use of Proposition 4.6 is the following application that corresponds to taking $\phi(L)$ be to $L_r - L_{r+1}$, M_r and M'_r, respectively.

4.8. **Corollary.** *Suppose that* $\mathcal{O}$ *is an* e*-idempotent subring,* $1 \le r < n$ *and* $\mathbf{i} \in I^n$*.*

a) *If* $i_r \ne i_{r+1}$ *then* $\dfrac{1}{L_r - L_{r+1}} f^{\mathcal{O}}_{\mathbf{i}} = \displaystyle\sum_{\mathfrak{t}\in\operatorname{Std}(\mathbf{i})} \frac{t^{-c_{r+1}(\mathfrak{t})}}{[\rho_r(\mathfrak{t})]} F_{\mathfrak{t}} \in \mathscr{L}(\mathcal{O})$.

b) *If* $i_r \ne i_{r+1}+1$ *then* $\dfrac{1}{M_r} f^{\mathcal{O}}_{\mathbf{i}} = \displaystyle\sum_{\mathfrak{t}\in\operatorname{Std}(\mathbf{i})} \frac{t^{-c_r(\mathfrak{t})}}{[1-\rho_r(\mathfrak{t})]} F_{\mathfrak{t}} \in \mathscr{L}(\mathcal{O})$.

c) *If* $i_r \ne i_{r+1}-1$ *then* $\dfrac{1}{M'_r} f^{\mathcal{O}}_{\mathbf{i}} = \displaystyle\sum_{\mathfrak{t}\in\operatorname{Std}(\mathbf{i})} \frac{t^{-c_{r+1}(\mathfrak{t})}}{[1+\rho_r(\mathfrak{t})]} F_{\mathfrak{t}} \in \mathscr{L}(\mathcal{O})$.

4.2. **Intertwiners.** By Theorem 2.14, if K is a field then the KLR generators of $\mathcal{H}^\Lambda_n(K)$ satisfy $\psi_r e(\mathbf{i}) = e(s_r\cdot\mathbf{i})\psi_r$. This section defines analogous elements in $\mathcal{H}^\Lambda_n(\mathcal{O})$ that intertwine the residue idempotents $f^{\mathcal{O}}_{\mathbf{i}}$, for $\mathbf{i}\in I^n$.

4.9. **Lemma.** *Suppose that* $i_r = i_{r+1}$, *for some* $\mathbf{i}\in I^n$ *and* $1\le r<n$. *Then* $T_r f^{\mathcal{O}}_{\mathbf{i}} = f^{\mathcal{O}}_{\mathbf{i}} T_r$.

Proof. This follows directly from the Seminormal Basis Theorem 3.13. In more detail, note that if $\mathfrak{t}\in\operatorname{Std}(\mathbf{i})$ then r and $r+1$ cannot appear in the same row or in the same column of $\mathfrak{t}$. Therefore,

$$T_r f^{\mathcal{O}}_{\mathbf{i}} - f^{\mathcal{O}}_{\mathbf{i}} T_r = \sum_{\mathfrak{t}\in\operatorname{Std}(\mathbf{i})} \frac{1}{\gamma_{\mathfrak{t}}}\Big(T_r f_{\mathfrak{t}\mathfrak{t}} - f_{\mathfrak{t}\mathfrak{t}} T_r\Big) = \sum_{\substack{\mathfrak{t},\mathfrak{v}\in\operatorname{Std}(\mathbf{i})\\ \mathfrak{v}=\mathfrak{t}(r,r+1)}} \Big(\frac{\alpha_r(\mathfrak{t})}{\gamma_{\mathfrak{t}}} - \frac{\alpha_r(\mathfrak{v})}{\gamma_{\mathfrak{v}}}\Big) f_{\mathfrak{v}\mathfrak{t}},$$

by (3.14). By Corollary 3.16, if $\mathfrak{v} = \mathfrak{t}(r,r+1)$ then $\alpha_r(\mathfrak{t})\gamma_{\mathfrak{v}} = \alpha_r(\mathfrak{v})\gamma_{\mathfrak{t}}$. Hence, $T_r f^{\mathcal{O}}_{\mathbf{i}} = f^{\mathcal{O}}_{\mathbf{i}} T_r$ as claimed. □

4.10. *Remark.* In the special case of the symmetric groups, Ryom-Hansen [34, §3] has proved an analogue of Lemma 4.9.

Using (3.14), it is easy to verify that $T_r f^{\mathcal{O}}_{\mathbf{i}} \ne f^{\mathcal{O}}_{\mathbf{j}} T_r$ if $\mathbf{j} = s_r\cdot\mathbf{i}\ne\mathbf{i}$, for $1\le r<n$ and $\mathbf{i}\in I^n$. The following elements will allow us to correct for this.

4.11. **Lemma.** *Suppose that* $(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathcal{P}^\Lambda_n)$ *and* $1\le r<n$. *Let* $\mathfrak{u}=\mathfrak{s}(r,r+1)$. *Then* $(T_rL_r - L_rT_r)f_{\mathfrak{s}\mathfrak{t}} = \alpha_r(\mathfrak{s})t^{c_{r+1}(\mathfrak{s})}[\rho_r(\mathfrak{s})]f_{\mathfrak{u}\mathfrak{t}}$.

Proof. Using (3.14) we obtain

$$(T_rL_r - L_rT_r)f_{\mathfrak{s}\mathfrak{t}} = \alpha_r(\mathfrak{s})\big([c_r(\mathfrak{s})] - [c_{r+1}(\mathfrak{s})]\big)f_{\mathfrak{u}\mathfrak{t}} = \alpha_r(\mathfrak{s})t^{c_{r+1}(\mathfrak{s})}[\rho_r(\mathfrak{s})]f_{\mathfrak{u}\mathfrak{t}},$$

where, as usual, we set $f_{\mathfrak{u}\mathfrak{t}}=0$ if $\mathfrak{u}$ is not standard. □

Applying the $*$ anti-involution, $f_{\mathfrak{s}\mathfrak{t}}(T_rL_r - L_rT_r) = -\alpha_r(\mathfrak{t})t^{c_{r+1}(\mathfrak{t})}[\rho_r(\mathfrak{t})]f_{\mathfrak{s}\mathfrak{v}}$, where $\mathfrak{v}=\mathfrak{t}(r,r+1)$.

4.12. **Lemma.** *Suppose that* $i_r\ne i_{r+1}$, *for some* $\mathbf{i}\in I^n$ *and* $1\le r<n$. *Set* $\mathbf{j}=s_r\cdot\mathbf{i}$. *Then* $\big(T_rL_r - L_rT_r\big)f^{\mathcal{O}}_{\mathbf{i}} = f^{\mathcal{O}}_{\mathbf{j}}\big(T_rL_r - L_rT_r\big)$.

Proof. By definition, $f^{\mathcal{O}}_{\mathbf{i}} = \sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})}\frac{1}{\gamma_{\mathfrak{s}}}f_{\mathfrak{s}\mathfrak{s}}$ so, by Lemma 4.11,

$$\begin{aligned}(T_rL_r - L_rT_r)f^{\mathcal{O}}_{\mathbf{i}} &= \sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})}\frac{1}{\gamma_{\mathfrak{s}}}(T_rL_r - L_rT_r)f_{\mathfrak{s}\mathfrak{s}}\\ &= \sum_{\substack{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})\\ \mathfrak{u}=\mathfrak{s}(r,r+1)\in\operatorname{Std}(\mathcal{P}^\Lambda_n)}} \frac{\alpha_r(\mathfrak{s})t^{c_{r+1}(\mathfrak{s})}[\rho_r(\mathfrak{s})]}{\gamma_{\mathfrak{s}}}f_{\mathfrak{u}\mathfrak{s}}.\end{aligned}$$

Note that if $\mathfrak{s}\in\operatorname{Std}(\mathbf{i})$ and $\mathfrak{u}=\mathfrak{s}(r,r+1)$ is standard then $\mathfrak{s}\in\operatorname{Std}(\mathbf{j})$. Similarly,

$$f^{\mathcal{O}}_{\mathbf{j}}(T_rL_r - L_rT_r) = \sum_{\substack{\mathfrak{u}\in\operatorname{Std}(\mathbf{j})\\ \mathfrak{s}=\mathfrak{u}(r,r+1)\in\operatorname{Std}(\mathbf{i})}} -\frac{\alpha_r(\mathfrak{u})t^{c_{r+1}(\mathfrak{u})}[\rho_r(\mathfrak{u})]}{\gamma_{\mathfrak{u}}}f_{\mathfrak{u}\mathfrak{s}}.$$

By (3.14), the tableaux in $\operatorname{Std}(\mathbf{i})$ and $\operatorname{Std}(\mathbf{j})$ that have r and $r+1$ in the same row or in the same column do not contribute to the right hand sides of either of the last two equations. Moreover, the map $\mathfrak{s} \mapsto \mathfrak{u} = \mathfrak{s}(r, r+1)$ defines a bijection from the set of tableaux in $\operatorname{Std}(\mathbf{i})$ such that r and $r+1$ appear in different rows and columns to the set of tableaux in $\operatorname{Std}(\mathbf{j})$ that have r and $r+1$ in different rows and columns. In particular, $(T_rL_r - L_rT_r)f_{\mathbf{i}}^{\mathcal{O}} = 0$ if and only if $f_{\mathbf{j}}^{\mathcal{O}}(T_rL_r - L_rT_r) = 0$.

To complete the proof suppose that $\mathfrak{s} \in \operatorname{Std}(\mathbf{i})$ and that $\mathfrak{u} = \mathfrak{s}(r, r+1) \in \operatorname{Std}(\mathbf{j})$. Now, $\alpha_r(\mathfrak{u})\gamma_{\mathfrak{s}} = \alpha_r(\mathfrak{s})\gamma_{\mathfrak{u}}$, by Corollary 3.16, and $\rho_r(\mathfrak{u}) = -\rho_r(\mathfrak{s})$, by definition. So

$$\frac{-\alpha_r(\mathfrak{u})t^{c_{r+1}(\mathfrak{u})}[\rho_r(\mathfrak{u})]}{\gamma_{\mathfrak{u}}} = \frac{-\alpha_r(\mathfrak{s})t^{c_r(\mathfrak{s})}[-\rho_r(\mathfrak{s})]}{\gamma_{\mathfrak{s}}} = \frac{\alpha_r(\mathfrak{s})t^{c_{r+1}(\mathfrak{s})}[\rho_r(\mathfrak{s})]}{\gamma_{\mathfrak{s}}}.$$

Hence, comparing the equations above, $(T_rL_r - L_rT_r)f_{\mathbf{i}}^{\mathcal{O}} = f_{\mathbf{j}}^{\mathcal{O}}(T_rL_r - L_rT_r)$ as required. □

Recall the definitions of M_r and M_r' from (4.7), for $1 \le r < n$. We finish this section by giving the commutation relations for the elements M_r, M_r', $(1+T_r)$ and $(T_rL_r - L_rT_r)$. These will be important later.

4.13. **Lemma.** *Suppose that $1 \le r < n$. Then*

$$(T_rL_r - L_rT_r)M_r = M_r'(T_rL_r - L_rT_r) \quad \textit{and} \quad (T_r - t)M_r = M_r'(1+T_r).$$

Proof. Both formulas can be proved by applying the relations in Definition 2.2. Alternatively, suppose that $(\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ and set $\mathfrak{v} = \mathfrak{t}(r, r+1)$. Then, by (4.7) and Lemma 4.11,

$$\begin{aligned} f_{\mathfrak{s}\mathfrak{t}}(T_rL_r - L_rT_r)M_r &= -\alpha_r(\mathfrak{t})t^{2c_r(\mathfrak{v})}[\rho_r(\mathfrak{t})][1+\rho_r(\mathfrak{t})]f_{\mathfrak{s}\mathfrak{v}} \\ &= f_{\mathfrak{s}\mathfrak{t}}M_r'(T_rL_r - L_rT_r), \end{aligned}$$

where the last equality follows because $c_r(\mathfrak{v}) = c_{r+1}(\mathfrak{t})$ and $c_{r+1}(\mathfrak{v}) = c_r(\mathfrak{t})$. As the regular representation is a faithful, this implies the first formula. The second formula can be proved similarly. □

4.3. **The integral KLR generators.** In Lemma 4.9 and Lemma 4.12, we have found elements in $\mathcal{H}_n^\Lambda(\mathcal{O})$ that intertwine the residue idempotents $f_{\mathbf{i}}^{\mathcal{O}}$. These intertwiners are not quite the elements that we need, however, because they still depend on t, rather than just on e. To remove this dependence on t we will use Proposition 4.6 to renormalise these elements.

By Lemma 4.4, if $h \in \mathcal{H}_n^\Lambda(\mathcal{O})$ then $h = \sum_{\mathbf{i}\in I^n} hf_{\mathbf{i}}^{\mathcal{O}}$, so that h is completely determined by its projections onto the spaces $\mathcal{H}_n^\Lambda(\mathcal{O})f_{\mathbf{i}}^{\mathcal{O}}$. We use this observation to define analogues of the KLR generators in $\mathcal{H}_n^\Lambda(\mathcal{O})$.

Recall from (4.7) that $M_r = 1 - L_r + tL_{r+1}$. By Corollary 4.8, if $i_r \ne i_{r+1}+1$ then M_r acts invertibly on $f_{\mathbf{i}}^{\mathcal{O}}\mathcal{H}_n^\Lambda(\mathcal{O})$ so $\frac{1}{M_r}f_{\mathbf{i}}^{\mathcal{O}}$ is a well-defined element of $\mathcal{H}_n^\Lambda(\mathcal{O})$.

As in the introduction, define an embedding $I \hookrightarrow \mathbb{Z}; i \mapsto \hat{\imath}$ by defining $\hat{\imath}$ to be the smallest non-negative integer such that $i = \hat{\imath} + e\mathbb{Z}$, for $i \in I$.

4.14. **Definition.** *Suppose that $1 \le r < n$. Define elements $\psi_r^{\mathcal{O}} = \sum_{\mathbf{i}\in I^n} \psi_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}$ in $\mathcal{H}_n^\Lambda(\mathcal{O})$ by*

$$\psi_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} (T_r+1)\frac{t^{\hat{\imath}_r}}{M_r}f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r = i_{r+1}, \\ (T_rL_r - L_rT_r)t^{-\hat{\imath}_r}f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r = i_{r+1}+1, \\ (T_rL_r - L_rT_r)\frac{1}{M_r}f_{\mathbf{i}}^{\mathcal{O}}, & \text{otherwise.} \end{cases}$$

If $1 \le r \le n$ then define $y_r^{\mathcal{O}} = \sum_{\mathbf{i}\in I^n} t^{-\hat{\imath}_r}(L_r - [\hat{\imath}_r])f_{\mathbf{i}}^{\mathcal{O}}$.

The order of the terms in the definition of $\psi_r^{\mathcal{O}}$ matters because M_r does not commute with T_r+1 or with $T_rL_r - L_rT_r$ (see Lemma 4.13), although M_r does commute with $f^{\mathcal{O}}_{\mathbf{i}}$. Notice that $\psi^{\mathcal{O}}_r$ is independent of the choice of seminormal coefficient system because the residue idempotents $f^{\mathcal{O}}_{\mathbf{i}}$ are independent of this choice.

One subtlety of Definition 4.14, which we will pay for later, is that it makes use of the embedding $I \hookrightarrow \mathbb{Z}$ in order to give meaning to expressions like $t^{\pm\hat{\imath}_r}$.

4.15. *Remark.* Unravelling the definitions, the element $\psi^{\mathcal{O}}_r \otimes_{\mathcal{O}} 1_K$ is a scalar multiple of the choice of KLR generators for $\mathcal{H}^\Lambda_n(\mathscr{K})$ made by Stroppel and Webster [36, (27)]. Similarly, $y^{\mathcal{O}}_r \otimes_{\mathcal{O}} 1_K$ is a multiple of the KLR generator y_r defined by Brundan and Kleshchev [6, (4.21)].

4.16. **Proposition.** *The algebra $\mathcal{H}^\Lambda_n(\mathcal{O})$ is generated by the elements*

$$\{\, f^{\mathcal{O}}_{\mathbf{i}} \mid \mathbf{i} \in I^n \,\} \cup \{\, \psi^{\mathcal{O}}_r \mid 1 \le r < n \,\} \cup \{\, y^{\mathcal{O}}_r \mid 1 \le r \le n \,\}.$$

Proof. Let H be the $\mathcal{O}$-subalgebra of $\mathcal{H}^\Lambda_n(\mathcal{O})$ generated by the elements in the statement of the proposition. We need to show that $H = \mathcal{H}^\Lambda_n(\mathcal{O})$. Directly from the definitions, if $1 \le r \le n$ then $L_r = \sum_{\mathbf{i}} (t^{\hat{\imath}_r} y^{\mathcal{O}}_r + [i_r]) f^{\mathcal{O}}_{\mathbf{i}} \in H$. Therefore, the Gelfand-Zetlin algebra $\mathscr{L}(\mathcal{O})$ is contained in H. Consequently, $M_r \in H$, for $1 \le r < n$. By Definition 2.2, $L_rT_r - T_rL_r = T_r(L_{r+1} - L_r) - 1 + (1-t)L_{r+1}$. By Corollary 4.8(a), if $i_r \ne i_{r+1}$ then $\frac{1}{L_r - L_{r+1}} f^{\mathcal{O}}_{\mathbf{i}} \in \mathscr{L}(\mathcal{O}) \subseteq H$. Therefore, since M_r and $f^{\mathcal{O}}_{\mathbf{i}}$ commute, we can write

$$T_r f^{\mathcal{O}}_{\mathbf{i}} = \begin{cases} \big(t^{-\hat{\imath}_r} \psi^{\mathcal{O}}_r M_r - 1\big) f^{\mathcal{O}}_{\mathbf{i}}, & \text{if } i_r = i_{r+1}, \\ \big(-t^{\hat{\imath}_r} \psi^{\mathcal{O}}_r + 1 + (t-1)L_{r+1}\big) \frac{1}{L_{r+1} - L_r} f^{\mathcal{O}}_{\mathbf{i}}, & \text{if } i_r = i_{r+1} + 1 \\ \big(-\psi^{\mathcal{O}}_r M_r + 1 + (t-1)L_{r+1}\big) \frac{1}{L_{r+1} - L_r} f^{\mathcal{O}}_{\mathbf{i}}, & \text{otherwise.} \end{cases}$$

by Definition 4.14. Hence, $T_r = \sum_{\mathbf{i}} T_r f^{\mathcal{O}}_{\mathbf{i}} \in H$. As $T_1, \dots, T_{n-1}, L_1, \dots, L_n$ generate $\mathcal{H}^\Lambda_n(\mathcal{O})$ this implies that $H = \mathcal{H}^\Lambda_n(\mathcal{O})$, completing the proof. □

We now use the seminormal form to show that the elements in the statement of Proposition 4.16 satisfy most of the relations of Definition 2.10.

4.17. **Lemma.** *Suppose that $1 \le r < n$ and $\mathbf{i} \in I^n$. Then $\psi^{\mathcal{O}}_r f^{\mathcal{O}}_{\mathbf{i}} = f^{\mathcal{O}}_{\mathbf{j}} \psi^{\mathcal{O}}_r$, where $\mathbf{j} = s_r \cdot \mathbf{i}$.*

Proof. By Lemma 4.4 and Proposition 4.6, respectively, M_r and $f^{\mathcal{O}}_{\mathbf{i}}$ both belong to $\mathscr{L}(\mathcal{O})$, which is a commutative algebra. Therefore, $\frac{1}{M_r} f^{\mathcal{O}}_{\mathbf{i}}$ and $f^{\mathcal{O}}_{\mathbf{i}}$ commute. If $i_r = i_{r+1}$ then

$$\psi^{\mathcal{O}}_r f^{\mathcal{O}}_{\mathbf{i}} = (T_r + 1)\frac{t^{\hat{\imath}_r}}{M_r} f^{\mathcal{O}}_{\mathbf{i}} = (T_r + 1) f^{\mathcal{O}}_{\mathbf{i}} \frac{t^{\hat{\imath}_r}}{M_r} f^{\mathcal{O}}_{\mathbf{i}} = f^{\mathcal{O}}_{\mathbf{i}} (T_r + 1)\frac{t^{\hat{\imath}_r}}{M_r} f^{\mathcal{O}}_{\mathbf{i}} = f^{\mathcal{O}}_{\mathbf{i}} \psi^{\mathcal{O}}_r,$$

where the third equality comes from Lemma 4.9. The remaining cases follow similarly using Lemma 4.12. □

As we will work with right modules we need the right-handed analogue of Definition 4.14. Note that if $i_r \ne i_{r+1} + 1$ then $f^{\mathcal{O}}_{\mathbf{i}} \frac{1}{M_r} = \frac{1}{M_r} f^{\mathcal{O}}_{\mathbf{i}} \in \mathcal{H}^\Lambda_n(\mathcal{O})$ by Proposition 4.6. Similarly, if $i_r \ne i_{r+1} - 1$ then $f^{\mathcal{O}}_{\mathbf{i}} \frac{1}{M'_r} = \frac{1}{M'_r} f^{\mathcal{O}}_{\mathbf{i}} \in \mathcal{H}^\Lambda_n(\mathcal{O})$. It follows that all of the expressions in the next lemma make sense.

4.18. **Lemma.** *Suppose $1 \le r < n$ and $\mathbf{i} \in I^n$. Then*

$$f^{\mathcal{O}}_{\mathbf{i}} \psi^{\mathcal{O}}_r = \begin{cases} f^{\mathcal{O}}_{\mathbf{i}} \frac{t^{\hat{\imath}_{r+1}}}{M'_r}(T_r - t), & \text{if } i_i = i_{r+1}, \\ f^{\mathcal{O}}_{\mathbf{i}} (T_rL_r - L_rT_r) t^{-\hat{\imath}_{r+1}}, & \text{if } i_r = i_{r+1} - 1, \\ f^{\mathcal{O}}_{\mathbf{i}} \frac{1}{M'_r}(T_rL_r - L_rT_r), & \text{otherwise.} \end{cases}$$

Proof. By Lemma 4.17, $f_{\mathbf{i}}^{\mathcal{O}}\psi_r^{\mathcal{O}} = f_{\mathbf{i}}^{\mathcal{O}}\psi_r^{\mathcal{O}}f_{\mathbf{j}}^{\mathcal{O}}$ where $\mathbf{j} = s_r\cdot\mathbf{i}$. Therefore,

$$f_{\mathbf{i}}^{\mathcal{O}}\psi_r^{\mathcal{O}} = \begin{cases} f_{\mathbf{i}}^{\mathcal{O}}(1+T_r)\frac{t^{\hat{\imath}_{r+1}}}{M_r}f_{\mathbf{j}}^{\mathcal{O}}, & \text{if } i_i = i_{r+1},\\ f_{\mathbf{i}}^{\mathcal{O}}(T_rL_r - L_rT_r)t^{-\hat{\imath}_{r+1}}f_{\mathbf{j}}^{\mathcal{O}}, & \text{if } i_r = i_{r+1}-1,\\ f_{\mathbf{i}}^{\mathcal{O}}(T_rL_r - L_rT_r)\frac{1}{M_r}f_{\mathbf{j}}^{\mathcal{O}}, & \text{otherwise.}\end{cases}$$

To complete the proof apply Lemma 4.13. □

4.19. **Lemma.** *Suppose that* $\mathbf{i},\mathbf{j}\in I^n$ *and* $1\le r,s\le n$. *Then*

$$\sum_{\mathbf{i}\in I^n} f_{\mathbf{i}}^{\mathcal{O}} = 1,\quad f_{\mathbf{i}}^{\mathcal{O}}f_{\mathbf{j}}^{\mathcal{O}} = \delta_{\mathbf{ij}}f_{\mathbf{i}}^{\mathcal{O}},\quad y_r^{\mathcal{O}}f_{\mathbf{i}}^{\mathcal{O}} = f_{\mathbf{i}}^{\mathcal{O}}y_r^{\mathcal{O}}\ \ \textit{and}\ \ y_r^{\mathcal{O}}y_s^{\mathcal{O}} = y_s^{\mathcal{O}}y_r^{\mathcal{O}}.$$

Moreover, if $s\ne r,r+1$ *then* $\psi_r^{\mathcal{O}}y_s^{\mathcal{O}} = y_s^{\mathcal{O}}\psi_r^{\mathcal{O}}$, *for* $1\le r<n$ *and* $1\le s\le n$.

Proof. The elements $f_{\mathbf{i}}^{\mathcal{O}}$, for $\mathbf{i}\in I^n$, form a complete set of pairwise orthogonal idempotents by Lemma 4.4, which gives the first two relations. Since $y_r, f_{\mathbf{i}}^{\mathcal{O}}\in\mathscr{L}(\mathcal{O})$ and $\mathscr{L}(\mathcal{O})$ is a commutative algebra, all of the elements $f_{\mathbf{i}}^{\mathcal{O}}$, $y_r^{\mathcal{O}}$ and $y_s^{\mathcal{O}}$ commute.

Now suppose that $s\ne r,r+1$. Then $y_s^{\mathcal{O}}$ commutes with $\frac{1}{M_r}f_{\mathbf{i}}^{\mathcal{O}}$ and with T_r. Hence, $\psi_r^{\mathcal{O}}f_{\mathbf{i}}^{\mathcal{O}}y_s^{\mathcal{O}} = y_s^{\mathcal{O}}\psi_r^{\mathcal{O}}f_{\mathbf{i}}^{\mathcal{O}}$, for any $\mathbf{i}\in I^n$. Therefore, $\psi_r^{\mathcal{O}}y_s^{\mathcal{O}} = y_s^{\mathcal{O}}\psi_r^{\mathcal{O}}$. □

4.20. **Lemma.** *Suppose that* $\mathbf{i}\in I^n$. *Then*

$$\prod_{\substack{1\le l\le\ell\\ \kappa_l\equiv i_1 \pmod e}}(y_1^{\mathcal{O}} - [\kappa_l - \hat{\imath}_1])f_{\mathbf{i}}^{\mathcal{O}} = 0.$$

Proof. By Definition 2.2, $\prod_{l=1}^{\ell}(L_1 - [\kappa_l]) = 0$ so that $\prod_{l=1}^{\ell}(L_1-[\kappa_l])f_{\mathbf{i}}^{\mathcal{O}} = 0$, for all $\mathbf{i}\in I$. If $\kappa_l\not\equiv i_1 \pmod e$ then $[\hat{\imath}_1]\ne[\kappa_l]$ so that $(L_1-[\kappa_l])$ acts invertibly on $f_{\mathbf{i}}^{\mathcal{O}}\mathcal{H}_n^{\Lambda}$ by Proposition 4.6. Consequently, by Definition 4.14,

$$0 = \prod_{\substack{1\le l\le\ell\\ \kappa_l\equiv i_1 \pmod e}}(t^{\hat{\imath}_1}y_1^{\mathcal{O}} + [\hat{\imath}_1] - [\kappa_l])f_{\mathbf{i}}^{\mathcal{O}} = t^{\hat{\imath}_1\langle\Lambda,\alpha_{i_1}\rangle}\prod_{\substack{1\le l\le\ell\\ \kappa_l\equiv i_1 \pmod e}}(y_1^{\mathcal{O}} - [\kappa_l-\hat{\imath}_1])f_{\mathbf{i}}^{\mathcal{O}}.$$

As t is invertible in $\mathcal{O}$, the lemma follows. □

Suppose that $\mathfrak{s}$ is a standard tableau, $\mathbf{i} = \mathrm{res}(\mathfrak{s})\in I^n$ and $1\le r<n$. Define

$$\beta_r(\mathfrak{s}) = \begin{cases} \dfrac{t^{\hat{\imath}_r - c_r(\mathfrak{s})}\alpha_r(\mathfrak{s})}{[1-\rho_r(\mathfrak{s})]}, & \text{if } i_r = i_{r+1},\\ t^{c_{r+1}(\mathfrak{s})-\hat{\imath}_r}\alpha_r(\mathfrak{s})[\rho_r(\mathfrak{s})], & \text{if } i_r = i_{r+1}+1,\\ \dfrac{t^{-\rho_r(\mathfrak{s})}\alpha_r(\mathfrak{s})[\rho_r(\mathfrak{s})]}{[1-\rho_r(\mathfrak{s})]}, & \text{otherwise,}\end{cases}\tag{4.21}$$

and

$$\widehat{\beta}_r(\mathfrak{s}) = \begin{cases} \dfrac{t^{\hat{\imath}_{r+1} - c_{r+1}(\mathfrak{s})}\alpha_r(\mathfrak{s})}{[1+\rho_r(\mathfrak{s})]}, & \text{if } i_r = i_{r+1},\\ -t^{c_{r+1}(\mathfrak{s})-\hat{\imath}_{r+1}}\alpha_r(\mathfrak{s})[\rho_r(\mathfrak{s})], & \text{if } i_r = i_{r+1}-1,\\ -\dfrac{\alpha_r(\mathfrak{s})[\rho_r(\mathfrak{s})]}{[1+\rho_r(\mathfrak{s})]}, & \text{otherwise.}\end{cases}\tag{4.22}$$

These scalars describe the action of $\psi_r^{\mathcal{O}}$ upon the seminormal basis.

4.23. **Lemma.** *Suppose that* $1\le r<n$ *and that* $(\mathfrak{s},\mathfrak{t})\in\mathrm{Std}^2(\mathcal{P}_n^{\Lambda})$. *Set* $\mathbf{i} = \mathrm{res}(\mathfrak{s})$, $\mathbf{j} = \mathrm{res}(\mathfrak{t})$, $\mathfrak{u} = \mathfrak{s}(r,r+1)$ *and* $\mathfrak{v} = \mathfrak{t}(r,r+1)$. *Then*

$$\psi_r^{\mathcal{O}}f_{\mathfrak{st}} = \beta_r(\mathfrak{s})f_{\mathfrak{ut}} - \delta_{i_ri_{r+1}}\frac{t^{\hat{\imath}_{r+1}-c_{r+1}(\mathfrak{s})}}{[\rho_r(\mathfrak{s})]}f_{\mathfrak{st}},$$

and

$$f_{\mathfrak{s}\mathfrak{t}}\psi_r^{\mathcal{O}} = \widehat{\beta}_r(\mathfrak{t})f_{\mathfrak{s}\mathfrak{v}} - \delta_{j_r j_{r+1}}\frac{t^{\hat{\jmath}_{r+1}-c_{r+1}(\mathfrak{t})}}{[\rho_r(\mathfrak{t})]}f_{\mathfrak{s}\mathfrak{t}}.$$

Similarly, $y_r^{\mathcal{O}}f_{\mathfrak{s}\mathfrak{t}} = [c_r(\mathfrak{s}) - \hat{\imath}_r]f_{\mathfrak{s}\mathfrak{t}}$, *and* $f_{\mathfrak{s}\mathfrak{t}}y_r^{\mathcal{O}} = [c_r(\mathfrak{t}) - \hat{\jmath}_r]f_{\mathfrak{s}\mathfrak{t}}$, *for* $1 \le r \le n$.

Proof. Applying Definition 4.14 and (3.14),

$$y_r^{\mathcal{O}}f_{\mathfrak{s}\mathfrak{t}} = t^{-\hat{\imath}_r}([c_r(\mathfrak{s})] - [\hat{\imath}_r])f_{\mathfrak{s}\mathfrak{t}} = [c_r(\mathfrak{s}) - \hat{\imath}_r]f_{\mathfrak{s}\mathfrak{t}}.$$

The proof that $f_{\mathfrak{s}\mathfrak{t}}y_r^{\mathcal{O}} = [c_r(\mathfrak{t}) - \hat{\jmath}_r]f_{\mathfrak{s}\mathfrak{t}}$ is similar. We now consider $\psi_r^{\mathcal{O}}$.

By (3.15), if $\mathbf{k} \in I^n$ then $f_{\mathbf{k}}^{\mathcal{O}}f_{\mathfrak{s}\mathfrak{t}} = \delta_{\mathbf{i}\mathbf{k}}f_{\mathfrak{s}\mathfrak{t}}$. We use this observation below without mention. By Lemma 4.11, $(T_rL_r - L_rT_r)f_{\mathfrak{s}\mathfrak{t}} = \alpha_r(\mathfrak{s})t^{c_{r+1}(\mathfrak{s})}[\rho_r(\mathfrak{s})]f_{\mathfrak{u}\mathfrak{t}}$. Hence, $\psi_r^{\mathcal{O}}f_{\mathfrak{s}\mathfrak{t}} = \beta_r(\mathfrak{s})f_{\mathfrak{u}\mathfrak{t}}$ when $i_r \ne i_{r+1}$ by Definition 4.14 and (4.7). Now suppose that $i_r = i_{r+1}$. Then, using (4.7) and (3.14),

$$\begin{aligned}\psi_r^{\mathcal{O}}f_{\mathfrak{s}\mathfrak{t}} &= (1+T_r)\frac{t^{\hat{\imath}_r}}{M_r}f_{\mathfrak{s}\mathfrak{t}} = \frac{t^{\hat{\imath}_r - c_r(\mathfrak{s})}}{[1-\rho_r(\mathfrak{s})]}\Big(\alpha_r(\mathfrak{s})f_{\mathfrak{u}\mathfrak{t}} + \big(1 - \frac{1}{[\rho_r(\mathfrak{s})]}\big)f_{\mathfrak{s}\mathfrak{t}}\Big)\\ &= \beta_r(\mathfrak{s})f_{\mathfrak{u}\mathfrak{t}} - \frac{t^{\hat{\imath}_{r+1}-c_{r+1}(\mathfrak{s})}}{[\rho_r(\mathfrak{s})]}f_{\mathfrak{s}\mathfrak{t}},\end{aligned}$$

as required. The formula for $f_{\mathfrak{s}\mathfrak{t}}\psi_r^{\mathcal{O}}$ is proved similarly using Lemma 4.18 in place of Definition 4.14. □

Note that, in general, $\psi_r^{\mathcal{O}}f_{\mathfrak{s}\mathfrak{t}} \ne (f_{\mathfrak{t}\mathfrak{s}}\psi_r^{\mathcal{O}})^*$.

The next relation can also be proved using Lemma 4.13 and Lemma 4.18.

4.24. **Corollary.** *Suppose that* $|r-t| > 1$, *for* $1 \le r, t < n$. *Then* $\psi_r^{\mathcal{O}}\psi_t^{\mathcal{O}} = \psi_t^{\mathcal{O}}\psi_r^{\mathcal{O}}$.

Proof. It follows easily from Lemma 4.23 that $\psi_r\psi_t f_{\mathfrak{s}\mathfrak{t}} = \psi_t\psi_r f_{\mathfrak{s}\mathfrak{t}}$, for all $(\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^{\Lambda})$. Hence, by Lemma 4.4, $\psi_r^{\mathcal{O}}\psi_t^{\mathcal{O}}f_{\mathbf{i}}^{\mathcal{O}} = \psi_t^{\mathcal{O}}\psi_r^{\mathcal{O}}f_{\mathbf{i}}^{\mathcal{O}}$, for all $\mathbf{i} \in I^n$. □

4.25. **Lemma.** *Suppose that* $1 \le r < n$ *and* $\mathbf{i} \in I^n$. *Then*

$$\psi_r^{\mathcal{O}}y_{r+1}^{\mathcal{O}}f_{\mathbf{i}}^{\mathcal{O}} = (y_r^{\mathcal{O}}\psi_r^{\mathcal{O}} + \delta_{i_r i_{r+1}})f_{\mathbf{i}}^{\mathcal{O}} \quad \text{and} \quad y_{r+1}^{\mathcal{O}}\psi_r^{\mathcal{O}}f_{\mathbf{i}}^{\mathcal{O}} = (\psi_r^{\mathcal{O}}y_r^{\mathcal{O}} + \delta_{i_r i_{r+1}})f_{\mathbf{i}}^{\mathcal{O}}.$$

Proof. Both formulas can be proved similarly, so we consider only the first one. We prove the stronger result that $\psi_r^{\mathcal{O}}y_{r+1}^{\mathcal{O}}f_{\mathfrak{s}\mathfrak{t}} = (y_r^{\mathcal{O}}\psi_r^{\mathcal{O}} + \delta_{i_r i_{r+1}})f_{\mathfrak{s}\mathfrak{t}}$, whenever $(\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^{\Lambda})$ and $\operatorname{res}(\mathfrak{s}) = \mathbf{i}$. By (4.3) this implies the lemma.

Suppose first that $i_r = i_{r+1}$. Then, using Lemma 4.23,

$$\psi_r^{\mathcal{O}}y_{r+1}^{\mathcal{O}}f_{\mathfrak{s}\mathfrak{t}} = [c_{r+1}(\mathfrak{s}) - \hat{\imath}_{r+1}]\Big(\beta_r(\mathfrak{s})f_{\mathfrak{u}\mathfrak{t}} - \frac{t^{\hat{\imath}_{r+1}-c_{r+1}(\mathfrak{s})}}{[\rho_r(\mathfrak{s})]}f_{\mathfrak{s}\mathfrak{t}}\Big).$$

On the other hand, by Lemma 4.23 and (4.21),

$$\begin{aligned}(y_r^{\mathcal{O}}\psi_r^{\mathcal{O}} + 1)f_{\mathfrak{s}\mathfrak{t}} &= [c_r(\mathfrak{u}) - \hat{\imath}_{r+1}]\beta_r(\mathfrak{s})f_{\mathfrak{u}\mathfrak{t}} + \Big(1 - \frac{t^{\hat{\imath}_r - c_{r+1}(\mathfrak{s})}[c_r(\mathfrak{s}) - \hat{\imath}_r]}{[\rho_r(\mathfrak{s})]}\Big)f_{\mathfrak{s}\mathfrak{t}}\\ &= [c_r(\mathfrak{u}) - \hat{\imath}_{r+1}]\beta_r(\mathfrak{s})f_{\mathfrak{u}\mathfrak{t}} + \frac{[\hat{\imath}_{r+1} - c_{r+1}(\mathfrak{s})]}{[\rho_r(\mathfrak{s})]}f_{\mathfrak{s}\mathfrak{t}}.\end{aligned}$$

Therefore, $\psi_r^{\mathcal{O}}y_{r+1}^{\mathcal{O}}f_{\mathfrak{s}\mathfrak{t}} = (y_r^{\mathcal{O}}\psi_r^{\mathcal{O}} + 1)f_{\mathfrak{s}\mathfrak{t}}$ since $c_r(\mathfrak{u}) = c_{r+1}(\mathfrak{s})$ and $i_r = i_{r+1}$.

If $i_r \ne i_{r+1}$ then the calculation is easier because

$$\psi_r^{\mathcal{O}}y_{r+1}^{\mathcal{O}}f_{\mathfrak{s}\mathfrak{t}} = [c_{r+1}(\mathfrak{s}) - \hat{\imath}_{r+1}]\beta_r(\mathfrak{s})f_{\mathfrak{u}\mathfrak{t}} = y_r^{\mathcal{O}}\psi_r^{\mathcal{O}}f_{\mathfrak{s}\mathfrak{t}},$$

where, for the last equality, we again use the fact that $c_r(\mathfrak{u}) = c_{r+1}(\mathfrak{s})$. □

The following simple combinatorial identity largely determines both the quadratic and the (deformed) braid relations for the $\psi_r^{\mathcal{O}}$, for $1 \le r < n$. This result can be viewed as a graded analogue of the defining property Definition 3.9(c) of a seminormal coefficient system.

4.26. **Lemma.** *Suppose that* $1 \le r < n$ *and* $\mathfrak{s}, \mathfrak{u} \in \operatorname{Std}(\boldsymbol{\lambda})$ *with* $\mathfrak{u} = \mathfrak{s}(r, r+1)$ *and* $\operatorname{res}(\mathfrak{s}) = \mathbf{i} \in I^n$, *for* $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$. *Then*

$$\beta_r(\mathfrak{s})\beta_r(\mathfrak{u}) = \begin{cases} t^{c_r(\mathfrak{s})+c_{r+1}(\mathfrak{s})-\hat{\imath}_r-\hat{\imath}_{r+1}}[1-\rho_r(\mathfrak{s})][1+\rho_r(\mathfrak{s})], & \text{if } i_r \leftrightarrows i_{r+1},\\ t^{c_{r+1}(\mathfrak{s})-\hat{\imath}_{r+1}}[1+\rho_r(\mathfrak{s})], & \text{if } i_r \rightarrow i_{r+1},\\ t^{c_r(\mathfrak{s})-\hat{\imath}_r}[1-\rho_r(\mathfrak{s})], & \text{if } i_r \leftarrow i_{r+1},\\ -\frac{t^{2\hat{\imath}_r-2c_{r+1}(\mathfrak{s})}}{[\rho_r(\mathfrak{s})]^2}, & \text{if } i_r = i_{r+1},\\ 1, & \text{otherwise.} \end{cases}$$

Proof. The lemma follows directly from the definition of $\beta_r(\mathfrak{s})$ using Definition 3.9(c). □

It is time to pay the price for the failure of the embedding $I \hookrightarrow \mathbb{Z}$ to extend to an embedding of quivers. Together with the cyclotomic relation, this is place where the KLR grading fails to lift to the algebra $\mathcal{H}_n^\Lambda(\mathcal{O})$. Recall from Definition 4.14 that $y_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = t^{-\hat{\imath}_r}(L_r - [\hat{\imath}_r])f_{\mathbf{i}}^{\mathcal{O}}$, where $1 \le r \le n$ and $\mathbf{i} \in I^n$. For $d \in \mathbb{Z}$ define

$$y_r^{\langle d\rangle} f_{\mathbf{i}}^{\mathcal{O}} = t^{d-\hat{\imath}_r}(L_r - [\hat{\imath}_r - d])f_{\mathbf{i}}^{\mathcal{O}} = (t^d y_r^{\mathcal{O}} + [d])f_{\mathbf{i}}^{\mathcal{O}}. \tag{4.27}$$

In particular, $y_r^{\langle 0\rangle} = y_r^{\mathcal{O}}$ and $y_r^{\langle d\rangle} \otimes_{\mathcal{O}} 1_K = y_r^{\mathcal{O}} \otimes_{\mathcal{O}} 1_K$ whenever e divides $d \in \mathbb{Z}$,

As a final piece of notation, set $\rho_r(\mathbf{i}) = \hat{\imath}_r - \hat{\imath}_{r+1} \in \mathbb{Z}$, for $\mathbf{i} \in I^n$ and $1 \le r < n$.

4.28. **Proposition.** *Suppose that* $1 \le r < n$ *and* $\mathbf{i} \in I^n$. *Then*

$$(\psi_r^{\mathcal{O}})^2 f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} (y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\mathcal{O}})(y_{r+1}^{\langle 1-\rho_r(\mathbf{i})\rangle} - y_r^{\mathcal{O}})f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r \leftrightarrows i_{r+1},\\ (y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\mathcal{O}})f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r \rightarrow i_{r+1},\\ (y_{r+1}^{\langle 1-\rho_r(\mathbf{i})\rangle} - y_r^{\mathcal{O}})f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r \leftarrow i_{r+1},\\ 0, & \text{if } i_r = i_{r+1},\\ f_{\mathbf{i}}^{\mathcal{O}}, & \text{otherwise.} \end{cases}$$

Proof. Once again, by (4.3) it is enough to prove the corresponding formulas for $(\psi_r^{\mathcal{O}})^2 f_{\mathfrak{st}}$, where $(\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ and $\mathbf{i} = \operatorname{res}(\mathfrak{s})$.

Suppose that $i_r = i_{r+1}$. Let $\mathfrak{u} = \mathfrak{s}(r, r+1)$ and $\mathbf{j} = \operatorname{res}(\mathfrak{u})$. By Lemma 4.23,

$$(\psi_r^{\mathcal{O}})^2 f_{\mathfrak{st}} = \Big(\frac{t^{2\hat{\imath}_r-2c_{r+1}(\mathfrak{s})}}{[\rho_r(\mathfrak{s})]^2} + \beta_r(\mathfrak{s})\beta_r(\mathfrak{u})\Big)f_{\mathfrak{st}} - \Big(\frac{\beta_r(\mathfrak{s})t^{\hat{\imath}_r-c_r(\mathfrak{s})}}{[\rho_r(\mathfrak{u})]} + \frac{\beta_r(\mathfrak{s})t^{\hat{\jmath}_r-c_r(\mathfrak{u})}}{[\rho_r(\mathfrak{s})]}\Big)f_{\mathfrak{ut}}.$$

Note that $\rho_r(\mathfrak{s}) = -\rho_r(\mathfrak{u})$ and $i_r = j_r$, so that $t^{\hat{\jmath}_r-c_r(\mathfrak{u})}[\rho_r(\mathfrak{u})] = -t^{\hat{\imath}_r-c_r(\mathfrak{s})}[\rho_r(\mathfrak{s})]$. Hence, using Lemma 4.26, $(\psi_r^{\mathcal{O}})^2 f_{\mathfrak{st}} = 0$ when $i_r = i_{r+1}$ as claimed.

Now suppose that $i_r \ne i_{r+1}$. Then, by Lemma 4.23 and Lemma 4.26,

$$\begin{aligned}(\psi_r^{\mathcal{O}})^2 f_{\mathfrak{st}} &= \beta_r(\mathfrak{s})\beta_r(\mathfrak{u})f_{\mathfrak{st}}\\ &= \begin{cases} t^{c_r(\mathfrak{s})+c_{r+1}(\mathfrak{s})-\hat{\imath}_r-\hat{\imath}_{r+1}}[1-\rho_r(\mathfrak{s})][1+\rho_r(\mathfrak{s})]f_{\mathfrak{st}}, & \text{if } i_r \leftrightarrows i_{r+1},\\ t^{c_{r+1}(\mathfrak{s})-\hat{\imath}_{r+1}}[1+\rho_r(\mathfrak{s})]f_{\mathfrak{st}}, & \text{if } i_r \rightarrow i_{r+1},\\ t^{c_r(\mathfrak{s})-\hat{\imath}_r}[1-\rho_r(\mathfrak{s})]f_{\mathfrak{st}}, & \text{if } i_r \leftarrow i_{r+1},\\ f_{\mathfrak{st}}, & \text{otherwise.} \end{cases}\end{aligned}$$

As in Lemma 4.23, if $d \in \mathbb{Z}$ then $y_r^{\langle d\rangle} f_{\mathfrak{st}} = [c_r(\mathfrak{s}) - \hat{\imath}_r + d]f_{\mathfrak{st}}$. So, if $i_r \rightarrow i_{r+1}$ then

$$\begin{aligned}(y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\mathcal{O}})f_{\mathfrak{st}} &= \big([c_r(\mathfrak{s}) + 1 - \hat{\imath}_{r+1}] - [c_{r+1}(\mathfrak{s}) - \hat{\imath}_{r+1}]\big)f_{\mathfrak{st}}\\ &= t^{c_{r+1}(\mathfrak{s})-\hat{\imath}_{r+1}}[1+\rho_r(\mathfrak{s})]f_{\mathfrak{st}} = (\psi_r^{\mathcal{O}})^2 f_{\mathfrak{st}}.\end{aligned}$$

The cases when $i_r \rightarrow i_{r+1}$ and $i_r \leftrightarrows i_{r+1}$ are similar. □

Set $\mathcal{B}_r^{\mathcal{O}} = \psi_r^{\mathcal{O}}\psi_{r+1}^{\mathcal{O}}\psi_r^{\mathcal{O}} - \psi_{r+1}^{\mathcal{O}}\psi_r^{\mathcal{O}}\psi_{r+1}^{\mathcal{O}}$, for $1 \le r < n-1$.

4.29. **Proposition.** *Suppose that $1 \le r < n$ and $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$, with $\mathfrak{s} \in \operatorname{Std}(\mathbf{i})$ for $\mathbf{i} \in I^n$. Then*

$$\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = \begin{cases} (y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} + y_{r+2}^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\langle 1-\rho_r(\mathbf{i})\rangle}) f_{\mathfrak{s}\mathfrak{t}}, & \text{if } i_{r+2} = i_r \rightleftarrows i_{r+1},\\ -t^{1+\rho_r(\mathbf{i})} f_{\mathfrak{s}\mathfrak{t}}, & \text{if } i_{r+2} = i_r \to i_{r+1},\\ f_{\mathfrak{s}\mathfrak{t}}, & \text{if } i_{r+2} = i_r \leftarrow i_{r+1},\\ 0, & \text{otherwise.} \end{cases}$$

Proof. We mimic the proof of the braid relations from Lemma 3.12.

Define (not necessarily standard) tableaux $\mathfrak{u}_1 = \mathfrak{s}(r, r+1)$, $\mathfrak{u}_2 = \mathfrak{s}(r+1, r+2)$, $\mathfrak{u}_{12} = \mathfrak{u}_1(r+1, r+2)$, $\mathfrak{u}_{21} = \mathfrak{u}_2(r, r+1)$ and $\mathfrak{u}_{121} = \mathfrak{u}_{12}(r, r+1) = \mathfrak{u}_{212}$. To ease notation set $i = i_r$, $j = i_{r+1}$ and $k = i_{r+2}$. The relationship between these tableaux, and their residues $\{\operatorname{res}_s(\mathfrak{u}) \mid r \le s \le r+2\} = \{i, j, k\}$, is illustrated in the following diagram.

$$\begin{tikzcd}
 & \mathfrak{s} \sim (i,j,k) \arrow[dl, "s_r"'] \arrow[dr, "s_{r+1}"] & \\
\mathfrak{u}_1 \sim (j,i,k) \arrow[d, "s_{r+1}"'] & & \mathfrak{u}_2 \sim (i,k,j) \arrow[d, "s_r"] \\
\mathfrak{u}_{12} \sim (j,k,i) \arrow[dr, "s_r"'] & & \mathfrak{u}_{21} \sim (k,i,j) \arrow[dl, "s_{r+1}"] \\
 & \mathfrak{u}_{121} = \mathfrak{u}_{212} \sim (k,j,i) &
\end{tikzcd}$$

Note that if any tableau $\mathfrak{u} \in \{\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_{12}, \mathfrak{u}_{21}, \mathfrak{u}_{121}\}$ is not standard then, by definition, $f_{\mathfrak{u}\mathfrak{t}} = 0$ so this term can be ignored in all of the calculations below.

We need to compute $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}}$. To start with, observe that by Lemma 4.23, the coefficient of $f_{\mathfrak{u}_{121}\mathfrak{t}}$ in $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}}$ is equal to

$$\beta_r(\mathfrak{s})\beta_{r+1}(\mathfrak{u}_1)\beta_r(\mathfrak{u}_{12}) - \beta_{r+1}(\mathfrak{s})\beta_r(\mathfrak{u}_2)\beta_{r+1}(\mathfrak{u}_{21}).$$

By definition, the scalars $[\rho_r(\mathfrak{s})]$ and $[1-\rho_r(\mathfrak{s})]$ are determined by the positions of r and $r+1$ in $\mathfrak{s}$, so it is easy to see that

$$\begin{aligned} \rho_r(\mathfrak{s}) &= \rho_{r+1}(\mathfrak{u}_{21}), & \rho_r(\mathfrak{u}_1) &= \rho_{r+1}(\mathfrak{u}_{121}), & \rho_r(\mathfrak{u}_2) &= \rho_{r+1}(\mathfrak{u}_1),\\ \rho_r(\mathfrak{u}_{12}) &= \rho_{r+1}(\mathfrak{s}), & \rho_r(\mathfrak{u}_{21}) &= \rho_{r+1}(\mathfrak{u}_{12}), & \rho_r(\mathfrak{u}_{121}) &= \rho_{r+1}(\mathfrak{u}_2). \end{aligned} \tag{4.30}$$

Observe that $\alpha_r(\mathfrak{s})\alpha_{r+1}(\mathfrak{u}_1)\alpha_r(\mathfrak{u}_{12}) = \alpha_{r+1}(\mathfrak{s})\alpha_r(\mathfrak{u}_2)\alpha_{r+1}(u_{21})$ by Definition 3.9(a). Keeping track of the exponent of t, (4.21) and (4.30) now imply that $\beta_r(\mathfrak{s})\beta_{r+1}(\mathfrak{u}_1)\beta_r(\mathfrak{u}_{12}) = \beta_{r+1}(\mathfrak{s})\beta_r(\mathfrak{u}_2)\beta_{r+1}(\mathfrak{u}_{21})$. Note that Definition 3.9(a) is crucial here. Therefore, the coefficient of $f_{\mathfrak{u}_{121}\mathfrak{t}}$ in $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}}$ is zero for any choice of i, j and k. As the coefficient of $f_{\mathfrak{u}_{121}\mathfrak{t}}$ in $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}}$ is always zero we will omit $f_{\mathfrak{u}_{121}\mathfrak{t}}$ from most of the calculations that follow.

There are five cases to consider.

Case 1. *i, j and k are pairwise distinct.*
By Lemma 4.23 and the last paragraph,

$$\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = \big(\beta_r(\mathfrak{s})\beta_{r+1}(\mathfrak{u}_1)\beta_r(\mathfrak{u}_{12}) - \beta_{r+1}(\mathfrak{s})\beta_r(\mathfrak{u}_2)\beta_{r+1}(\mathfrak{u}_{21})\big) f_{\mathfrak{u}_{121}\mathfrak{t}} = 0,$$

as required by the statement of the proposition.

Case 2. *$i = j \ne k$.*
In this case, using Lemma 4.23,

$$\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = \beta_{r+1}(\mathfrak{s})\beta_r(\mathfrak{u}_2)\Big(-\frac{t^{\hat{\imath} - c_{r+1}(\mathfrak{s})}}{[\rho_r(\mathfrak{s})]} + \frac{t^{\hat{\imath} - c_{r+2}(\mathfrak{u}_{21})}}{[\rho_{r+1}(\mathfrak{u}_{21})]}\Big) f_{\mathfrak{u}_{21}\mathfrak{t}}.$$

Now $\rho_r(\mathfrak{s}) = \rho_{r+1}(\mathfrak{u}_{21})$ and $c_{r+1}(\mathfrak{s}) = c_{r+2}(\mathfrak{u}_{21})$, as in (4.30). Hence, $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = 0$ when $i = j \neq k$.

Case 3. $i \neq j = k$.
This is almost identical to Case 2, so we leave the details to the reader.

Case 4. $i = k \neq j$.
Typographically, it is convenient to set $c = c_r(\mathfrak{s})$, $c' = c_{r+1}(\mathfrak{s})$ and $c'' = c_{r+2}(\mathfrak{s})$. According to the statement of the proposition, this is the only case where $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} \neq 0$. Using Lemma 4.23, we see that

$$\begin{aligned}\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} &= \Big(-\beta_r(\mathfrak{s}) \frac{t^{\hat{\imath}-c_{r+2}(\mathfrak{u}_1)}}{[\rho_{r+1}(\mathfrak{u}_1)]} \beta_r(\mathfrak{u}_1) + \beta_{r+1}(\mathfrak{s}) \frac{t^{\hat{\imath}-c_{r+1}(\mathfrak{u}_2)}}{[\rho_r(\mathfrak{u}_2)]} \beta_{r+1}(\mathfrak{u}_2) \Big) f_{\mathfrak{s}\mathfrak{t}}. \\ &= \frac{t^{\hat{\imath}-c''}}{[c-c'']} \big(-\beta_r(\mathfrak{s})\beta_r(\mathfrak{u}_1) + \beta_{r+1}(\mathfrak{s})\beta_{r+1}(\mathfrak{u}_2) \big) f_{\mathfrak{s}\mathfrak{t}}.\end{aligned}$$

Expanding the last equation using Lemma 4.26 shows that

$$\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = \begin{cases} -\dfrac{t^{c}[1-\rho_r(\mathfrak{s})][1+\rho_r(\mathfrak{s})] - t^{c''}[1-\rho_{r+1}(\mathfrak{s})][1+\rho_{r+1}(\mathfrak{s})]}{t^{c''-c'+\hat{\jmath}}[c-c'']} f_{\mathfrak{s}\mathfrak{t}}, & \text{if } i \leftrightarrows j, \\ -\dfrac{[1+\rho_r(\mathfrak{s})] - [1-\rho_{r+1}(\mathfrak{s})]}{t^{c''-\hat{\imath}-c'+\hat{\jmath}}[c-c'']} f_{\mathfrak{s}\mathfrak{t}}, & \text{if } i \rightarrow j, \\ -\dfrac{t^{c}[1-\rho_r(\mathfrak{s})] - t^{c''}[1+\rho_{r+1}(\mathfrak{s})]}{t^{c''}[c-c'']} f_{\mathfrak{s}\mathfrak{t}}, & \text{if } i \leftarrow j, \\ 0, & \text{otherwise.} \end{cases}$$

(Note that, by assumption, the case $i = j$ does not arise.) If $i \leftrightarrows j$ then a straightforward calculation shows that in this case

$$\begin{aligned}\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} &= -\Big([c'-\hat{\jmath}+2] + [c'-\hat{\jmath}] - [c+1-\hat{\jmath}] - [c''+1-\hat{\jmath}] \Big) f_{\mathfrak{s}\mathfrak{t}} \\ &= -\big(y_{r+1}^{\langle 1+\rho_r(\mathbf{i})\rangle} + y_{r+1}^{\langle 1-\rho_r(\mathbf{i})\rangle} - y_{r}^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+2}^{\langle 1+\rho_r(\mathbf{i})\rangle} \big) f_{\mathfrak{s}\mathfrak{t}},\end{aligned}$$

where the last equality uses Lemma 4.25 and the observation that, because $e = 2$, we have $\{1 \pm \rho_r(\mathbf{i})\} = \{0, 2\}$ and $\{\hat{\imath}, \hat{\jmath}\} = \{0, 1\}$. A similar, but easier, calculation shows that if $i \rightarrow j$ then $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = -t^{1+\hat{\imath}-\hat{\jmath}} f_{\mathfrak{s}\mathfrak{t}} = -t^{1+\rho_r(\mathbf{i})} f_{\mathfrak{s}\mathfrak{t}}$ and if $i \leftarrow j$ then $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = f_{\mathfrak{s}\mathfrak{t}}$. If $i \neq j$ and $i \not- j$ then we have already seen that $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = 0$, so this completes the proof of Case 4.

Case 5. $i = j = k$.
We continue to use the notation for c, c', c'' from Case 4. By Lemma 4.23 (compare with the proof of Lemma 3.12), $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}}$ is equal to

$$\begin{aligned} -\Big(\tfrac{t^{3\hat{\imath}-2c'-c''}}{[\rho_r(\mathfrak{s})]^2[\rho_{r+1}(\mathfrak{s})]} - \tfrac{t^{3\hat{\imath}-c'-2c''}}{[\rho_{r+1}(\mathfrak{s})]^2[\rho_r(\mathfrak{s})]} + \tfrac{t^{\hat{\imath}-c''}\beta_r(\mathfrak{s})\beta_r(\mathfrak{u}_1)}{[\rho_{r+1}(\mathfrak{u}_1)]} - \tfrac{t^{\hat{\imath}-c''}\beta_{r+1}(\mathfrak{s})\beta_{r+1}(\mathfrak{u}_2)}{[\rho_r(\mathfrak{u}_2)]} \Big) f_{\mathfrak{s}\mathfrak{t}} \\ + t^{2\hat{\imath}}\beta_r(\mathfrak{s}) \Big(\tfrac{t^{-c''-c}}{[\rho_{r+1}(\mathfrak{u}_1)][\rho_r(\mathfrak{u}_1)]} + \tfrac{t^{-c'-c''}}{[\rho_r(\mathfrak{s})][\rho_{r+1}(\mathfrak{s})]} - \tfrac{t^{-2c''}}{[\rho_{r+1}(\mathfrak{s})][\rho_{r+1}(\mathfrak{u}_1)]} \Big) f_{\mathfrak{u}_1\mathfrak{t}} \\ + t^{2\hat{\imath}}\beta_{r+1}(\mathfrak{s}) \Big(\tfrac{t^{-c'-c''}}{[\rho_r(\mathfrak{s})][\rho_r(\mathfrak{u}_2)]} - \tfrac{t^{-c''-c'}}{[\rho_r(\mathfrak{u}_2)][\rho_{r+1}(\mathfrak{u}_2)]} - \tfrac{t^{-c''-c'}}{[\rho_{r+1}(\mathfrak{s})][\rho_r(\mathfrak{s})]} \Big) f_{\mathfrak{u}_2\mathfrak{t}} \\ - t^{\hat{\imath}-c''}\beta_r(\mathfrak{s})\beta_{r+1}(\mathfrak{u}_1) \Big(\tfrac{1}{[\rho_r(\mathfrak{u}_{12})]} - \tfrac{1}{[\rho_{r+1}(\mathfrak{s})]} \Big) f_{\mathfrak{u}_{12}\mathfrak{t}} \\ - t^{\hat{\imath}-c'}\beta_{r+1}(\mathfrak{s})\beta_r(\mathfrak{u}_2) \Big(\tfrac{1}{[\rho_r(\mathfrak{s})]} - \tfrac{1}{[\rho_{r+1}(\mathfrak{u}_{21})]} \Big) f_{\mathfrak{u}_{21}\mathfrak{t}}. \end{aligned}$$

Using (4.30) it is easy to see that the coefficients of $f_{\mathfrak{u}_{12}\mathfrak{t}}$ and$f_{\mathfrak{u}_{21}\mathfrak{t}}$ are both zero. On the other hand, if $t \neq 1$ then the coefficient of $t^{2\hat{\imath}}\beta_r(\mathfrak{s}) f_{\mathfrak{u}_1\mathfrak{t}}$ in $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}}$ is

$$\frac{t-1}{(t^{c'}-t^{c})(t^{c}-t^{c''})} + \frac{t-1}{(t^{c}-t^{c'})(t^{c'}-t^{c''})} - \frac{t-1}{(t^{c'}-t^{c''})(t^{c}-t^{c''})} = 0.$$

The case when $t = 1$ now follows by specialisation. Similarly, the coefficient of $f_{\mathfrak{u}_2\mathfrak{t}}$ in $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}}$ is also zero. Finally, using Lemma 4.26 and (4.30), the coefficient of $f_{\mathfrak{s}\mathfrak{t}}$ in $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}}$ is zero as the four terms above, which give the coefficient of $f_{\mathfrak{s}\mathfrak{t}}$ in the displayed equation, cancel out in pairs. Hence, $\mathcal{B}_r^{\mathcal{O}} f_{\mathfrak{st}} = 0$ when $i = j = k$, as required.

This completes the proof. □

4.4. **A deformation of the quiver Hecke algebra.** Using the results of the last two sections we now describe $\mathcal{H}_n^\Lambda(\mathcal{O})$ by generators and relations using the '$\mathcal{O}$-KLR generators' of $\mathcal{H}_n^\Lambda(\mathcal{O})$.

Let $R_n(\mathcal{O})$ be the abstract algebra defined by the generators and relations in the statement of Theorem A. We abuse notation and use the same symbols for the generators of $R_n(\mathcal{O})$ and the corresponding elements in $\mathcal{H}_n^\Lambda(\mathcal{O})$ that we defined in Section 4.3. The previous section shows that there is a surjection $R_n(\mathcal{O}) \twoheadrightarrow \mathcal{H}_n^\Lambda(\mathcal{O})$. We want to prove that this map is an isomorphism.

The next lemma, which is modelled on [6, Lemma 2.1], will be used to show that $R_n(\mathcal{O})$ is finitely generated as an $\mathcal{O}$-module.

4.31. **Lemma.** *Suppose that $1 \le r \le n$ and $\mathbf{i} \in I^n$. Then there exists a multiset $X_r(\mathbf{i}) \subseteq e\mathbb{Z}$ such that*

$$\prod_{c \in X_r(\mathbf{i})} (y_r^{\mathcal{O}} - [c]) f_{\mathbf{i}}^{\mathcal{O}} = 0.$$

Proof. We argue by induction on r. If $r = 1$ then the relations in $R_n(\mathcal{O})$ say that we can take $X_1(\mathbf{i})$ to be the multiset with elements $\kappa_l - \hat{\imath}_1$, where $\kappa_l \equiv i_1 \pmod e$ for $1 \le l \le \ell$. By induction we assume that we have proved the result for $y_r^{\mathcal{O}}$ and use this to prove the result for $y_{r+1}^{\mathcal{O}}$. There are three cases to consider. Throughout, let $\mathbf{j} = \mathfrak{s}_r \cdot \mathbf{i}$.

Case 1. $i_{r+1} \not\leftrightarrow i_r$.
Set $X_{r+1}(\mathbf{i}) = X_r(\mathbf{j})$. Then, using the relations in $R_n(\mathcal{O})$,

$$\begin{aligned}\prod_{c \in X_{r+1}(\mathbf{i})} (y_{r+1}^{\mathcal{O}} - [c]) f_{\mathbf{i}}^{\mathcal{O}} &= \prod_{c \in X_{r+1}(\mathbf{i})} (y_{r+1}^{\mathcal{O}} - [c]) (\psi_r^{\mathcal{O}})^2 f_{\mathbf{i}}^{\mathcal{O}} \\ &= \psi_r^{\mathcal{O}} \prod_{c \in X_r(\mathbf{j})} (y_r^{\mathcal{O}} - [c]) f_{\mathbf{j}}^{\mathcal{O}} \psi_r^{\mathcal{O}} = 0,\end{aligned}$$

where the last equality follows by induction.

Case 2. $i_r \to i_{r+1}$ *or* $i_r \leftarrow i_{r+1}$.
We consider only the case when $i_r \to i_{r+1}$. The case when $i_r \leftarrow i_{r+1}$ is similar. Using quadratic relation for $\psi_r^{\mathcal{O}}$ in $R_n(\mathcal{O})$,

$$(y_{r+1}^{\mathcal{O}} - [c]) f_{\mathbf{i}}^{\mathcal{O}} = t^{1+\rho_i(\mathbf{i})} (y_r^{\mathcal{O}} - [c - 1 - \rho_i(\mathbf{i})]) f_{\mathbf{i}}^{\mathcal{O}} - (\psi_r^{\mathcal{O}})^2 f_{\mathbf{i}}^{\mathcal{O}}.$$

Let $X_{r+1}(\mathbf{i})$ be the disjoint union $X_r(\mathbf{j}) \sqcup X_{r+1}^+(\mathbf{i})$, where $X_{r+1}^+(\mathbf{i})$ is the multiset $\{\, c + 1 + \rho_r(\mathbf{i}) \mid c \in X_r(\mathbf{i}) \,\}$. If $d = c+1+\rho_r(\mathbf{i}) \in X_{r+1}^+(\mathbf{i})$ then, by the last displayed equation,

$$\begin{aligned}\prod_{c' \in X_{r+1}(\mathbf{i})} (y_{r+1}^{\mathcal{O}} - [c']) f_{\mathbf{i}}^{\mathcal{O}} &= \prod_{c' \in X_{r+1}(\mathbf{i}) \setminus \{d\}} (y_{r+1}^{\mathcal{O}} - [c']) \cdot \big(t^{1+\rho_r(\mathbf{i})} (y_r^{\mathcal{O}} - [c]) - (\psi_r^{\mathcal{O}})^2\big) f_{\mathbf{i}}^{\mathcal{O}} \\ &= t^{1+\rho_r(\mathbf{i})} (y_r^{\mathcal{O}} - [c]) \prod_{c' \in X_{r+1}(\mathbf{i}) \setminus \{d\}} (y_{r+1}^{\mathcal{O}} - [c']) f_{\mathbf{i}}^{\mathcal{O}} \\ &\quad - \psi_r^{\mathcal{O}} \prod_{c' \in X_{r+1}(\mathbf{i}) \setminus \{d\}} (y_r^{\mathcal{O}} - [c']) f_{\mathbf{j}}^{\mathcal{O}} \psi_r^{\mathcal{O}}.\end{aligned}$$

The second summand is zero by induction because $X_r(\mathbf{j})$ is contained in $X_{r+1}(\mathbf{i})$. Therefore, arguing this way for every $d \in X_{r+1}^+(\mathbf{i})$, there exists $N \in \mathbb{Z}$ such that

$$\prod_{c' \in X_{r+1}(\mathbf{i})} (y_{r+1}^{\mathcal{O}} - [c']) f_{\mathbf{i}}^{\mathcal{O}} = t^{Ne} \prod_{c' \in X_r(\mathbf{j})} (y_{r+1}^{\mathcal{O}} - [c']) \cdot \prod_{c \in X_r(\mathbf{i})} (y_r^{\mathcal{O}} - [c]) f_{\mathbf{i}}^{\mathcal{O}} = 0,$$

where we again use induction for the last equality. This completes the proof of the inductive step when $i_r \to i_{r+1}$.

Case 3. $i_r \leftrightarrows i_{r+1}$.
This is similar to Case 2 but slightly more involved. Define $X_{r+1}(\mathbf{i}) = X_r(\mathbf{j}) \sqcup X^+_{r+1}(\mathbf{i}) \sqcup X^-_{r+1}(\mathbf{i})$, where $X^\pm_{r+1} = \{\, c \pm 1 + \rho_r(\mathbf{i}) \mid c \in X_r(\mathbf{i}) \,\}$. If $c \in X_r(\mathbf{i})$ set

$$c^+ := c + 1 + \rho_r(\mathbf{i}), \qquad c^- := c - 1 + \rho_r(\mathbf{i}).$$

Using the equality

$$(\psi_r^{\mathcal{O}})^2 f_{\mathbf{i}}^{\mathcal{O}} = (t^{1+\rho_r(\mathbf{i})} y_r^{\mathcal{O}} + [1+\rho_r(\mathbf{i})] - y_{r+1}^{\mathcal{O}})(t^{1-\rho_r(\mathbf{i})} y_{r+1}^{\mathcal{O}} + [1-\rho_r(\mathbf{i})] - y_r^{\mathcal{O}}) f_{\mathbf{i}}^{\mathcal{O}},$$

we see that

$$(\psi_r^{\mathcal{O}})^2 f_{\mathbf{i}}^{\mathcal{O}} = -t^{1-\rho_r(\mathbf{i})}(y_{r+1}^{\mathcal{O}} - [c^+])(y_{r+1}^{\mathcal{O}} - [c^-]) f_{\mathbf{i}}^{\mathcal{O}} + (y_r^{\mathcal{O}} - [c]) f_{\mathbf{i}}^{\mathcal{O}} F_r(y,c),$$

where $F_r(y,c)$ is a polynomial in $y_r^{\mathcal{O}}$ and $y_{r+1}^{\mathcal{O}}$ with coefficients in $\mathbb{Z}[t,t^{-1}]$. Hence,

$$\begin{aligned}
&\prod_{c' \in X_{r+1}(\mathbf{i})} (y_{r+1}^{\mathcal{O}} - [c']) f_{\mathbf{i}}^{\mathcal{O}} \\
&= \prod_{c' \in X_{r+1}(\mathbf{i}) \setminus \{c^+, c^-\}} (y_{r+1}^{\mathcal{O}} - [c']) \cdot t^{\rho_r(\mathbf{i})-1}\big((y_r^{\mathcal{O}} - [c]) f_{\mathbf{i}}^{\mathcal{O}} F_r(y,c) - (\psi_r^{\mathcal{O}})^2 f_{\mathbf{i}}^{\mathcal{O}}\big) \\
&= t^{\rho_r(\mathbf{i})-1}(y_r^{\mathcal{O}} - [c]) \prod_{c' \in X_{r+1}(\mathbf{i}) \setminus \{c^+, c^-\}} (y_{r+1}^{\mathcal{O}} - [c']) f_{\mathbf{i}}^{\mathcal{O}} F_r(y,c) \\
&\qquad - t^{\rho_r(\mathbf{i})-1} \psi_r^{\mathcal{O}} \prod_{c' \in X_{r+1}(\mathbf{i}) \setminus \{c^+, c^-\}} (y_r^{\mathcal{O}} - [c']) f_{\mathbf{j}}^{\mathcal{O}} \psi_r^{\mathcal{O}}.
\end{aligned}$$

The second summand is zero by induction because $X_r(\mathbf{j})$ is contained in $X_{r+1}(\mathbf{i})$. Therefore, arguing this way for every $c \in X_r(\mathbf{i})$, there exists $N \in \mathbb{Z}$ such that

$$\prod_{c' \in X_{r+1}(\mathbf{i})} (y_{r+1}^{\mathcal{O}} - [c']) f_{\mathbf{i}}^{\mathcal{O}} = t^{Ne} \prod_{c' \in X_r(\mathbf{j})} (y_{r+1}^{\mathcal{O}} - [c']) \cdot \prod_{c \in X_r(\mathbf{i})} (y_r^{\mathcal{O}} - [c]) f_{\mathbf{i}}^{\mathcal{O}} F_r(y) = 0,$$

where $F_r(y)$ is a polynomial in $y_1^{\mathcal{O}}, \cdots, y_n^{\mathcal{O}}$ with coefficients in $\mathbb{Z}[t,t^{-1}]$ and we again use induction for the last equality. This completes the proof of the inductive step when $i_r \leftrightarrows i_{r+1}$.

Case 4. $i_{r+1} = i_r$.
Let $\phi_r = \psi_r^{\mathcal{O}}(y_r^{\mathcal{O}} - y_{r+1}^{\mathcal{O}}) f_{\mathbf{i}}^{\mathcal{O}}$. Then $\phi_r \psi_r^{\mathcal{O}} = -2\psi_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}$, so that $(1+\phi_r)^2 f_{\mathbf{i}}^{\mathcal{O}} = f_{\mathbf{i}}^{\mathcal{O}}$. Moreover, an easy albeit uninspiring calculation reveals that

$$(1+\phi_r) y_r^{\mathcal{O}} (1+\phi_r) f_{\mathbf{i}}^{\mathcal{O}} = (y_r^{\mathcal{O}} + \phi_r y_r^{\mathcal{O}} + y_r^{\mathcal{O}} \phi_r + \phi_r y_r^{\mathcal{O}} \phi_r) f_{\mathbf{i}}^{\mathcal{O}} = y_{r+1}^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}.$$

Therefore, setting $X_{r+1}(\mathbf{i}) = X_r(\mathbf{i})$,

$$\prod_{c \in X_{r+1}(\mathbf{i})} (y_{r+1}^{\mathcal{O}} - [c]) f_{\mathbf{i}}^{\mathcal{O}} = (1+\phi_r) \prod_{c \in X_r(\mathbf{i})} (y_r^{\mathcal{O}} - [c]) f_{\mathbf{i}}^{\mathcal{O}} (1+\phi_r) = 0,$$

where the last equality follows by induction. This completes the proof. □

Suppose that $(\mathcal{O},t)$ is an idempotent subring of $\mathscr{K}$. So far we have not used the assumption that $[de] \in J(\mathcal{O})$, for $d \in \mathbb{Z}$. This comes into play in the next theorem, which is Theorem A from the introduction.

4.32. **Theorem.** *Suppose that $(\mathcal{O},t)$ is an e-idempotent subring of $\mathscr{K}$. Then $R_n(\mathcal{O}) \cong \mathcal{H}_n^\Lambda(\mathcal{O})$ as $\mathcal{O}$-algebras.*

Proof. By the results in the last two sections, the elements given in Definition 4.14 satisfy all of the relations of the corresponding generators of $R_n(\mathcal{O})$. Hence, by Proposition 4.16, there is a surjective $\mathcal{O}$-algebra homomorphism $\theta{:}R_n(\mathcal{O}) \twoheadrightarrow \mathcal{H}_n^\Lambda(\mathcal{O})$, which maps the generators of $R_n(\mathcal{O})$ to the corresponding elements of $\mathcal{H}_n^\Lambda(\mathcal{O})$.

If $w \in \mathfrak{S}_n$ then set $\psi_w^\mathcal{O} = \psi_{r_1}^\mathcal{O} \dots \psi_{r_k}^\mathcal{O}$, where $w = s_{r_1} \dots s_{r_k}$ is a reduced expression for w. In general, $\psi_w^\mathcal{O}$ will depend upon the choice of reduced expression, however, using the relations in $R_n(\mathcal{O})$ it follows that every element in $R_n(\mathcal{O})$ can be written as a linear combination of elements of the form $f(y)\psi_w e(\mathbf{i})$, where $f(y) \in \mathcal{O}[y_1^\mathcal{O}, \dots, y_n^\mathcal{O}]$, $w \in \mathfrak{S}_n$ and $\mathbf{i} \in I^n$. Therefore, $R_n(\mathcal{O})$ is finitely generated as an $\mathcal{O}$-module by Lemma 4.31.

Now suppose that $\mathfrak{m}$ is a maximal ideal of $\mathcal{O}$ and let $K = \mathcal{O}/\mathfrak{m} \cong \mathcal{O}_\mathfrak{m}/\mathfrak{m}\mathcal{O}_\mathfrak{m}$ and $\zeta = t + \mathfrak{m}$. Then $1 + \zeta + \dots + \zeta^{e-1} = 0$ in K, since $[e] \in J(\mathcal{O}) \subseteq \mathfrak{m}$. Note also that $1 + \zeta + \dots + \zeta^{k-1} \neq 0$ if $k \notin e\mathbb{Z}$ since $\mathcal{O}$ is an e-idempotent subring. Consequently, $y_r^{\langle de\rangle} \otimes 1_K = y_r^\mathcal{O} \otimes 1_K$, for all $d \in \mathbb{Z}$. It is easy to see that all of the shifts $1 \pm \rho_r(\mathbf{i})$ appearing in the statement of theorem are equal to either 0 or to e. Therefore,upon base change to K the relations of $R_n(\mathcal{O}_\mathfrak{m}) \otimes_{\mathcal{O}_\mathfrak{m}} K$ coincide with the relations of the quiver Hecke algebra $\mathcal{R}_n^\Lambda(K)$, see Definition 2.10 and Theorem 2.14. Consequently, $R_n(\mathcal{O}_\mathfrak{m}) \otimes_{\mathcal{O}_\mathfrak{m}} K \cong \mathcal{R}_n^\Lambda(K)$, so that $\dim R_n(\mathcal{O}_\mathfrak{m}) \otimes_{\mathcal{O}_\mathfrak{m}} K = \dim \mathcal{H}_n^\Lambda(K)$ by [7, Theorem 4.20].

By the last paragraph, if $K = \mathcal{O}/\mathfrak{m}$, for any maximal ideal $\mathfrak{m}$ of $\mathcal{O}$, then $\dim R_n(\mathcal{O}_\mathfrak{m}) \otimes_{\mathcal{O}_\mathfrak{m}} K = \dim \mathcal{H}_n^\Lambda(K) = \ell^n n!$. Moreover, by the second paragraph of the proof, $R_n(\mathcal{O}_\mathfrak{m})$ is a finitely generated $\mathcal{O}_\mathfrak{m}$-algebra. Therefore, Nakayama's lemma applies and it implies that $R_n(\mathcal{O}_\mathfrak{m})$ is a free $\mathcal{O}_\mathfrak{m}$-module of rank $\ell^n n!$. Hence, the map $\theta_\mathfrak{m} : R_n(\mathcal{O}_\mathfrak{m}) \xrightarrow{\sim} \mathcal{H}_n^\Lambda(\mathcal{O}_\mathfrak{m})$ is an isomorphism of $\mathcal{O}_\mathfrak{m}$-algebras. It follows that θ is an isomorphism of $\mathcal{O}$-algebras, as required. □

The proof of Theorem 4.32 gives the following.

4.33. **Corollary.** *Suppose that $K = \mathcal{O}/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal of $\mathcal{O}$. Then*

$$\mathcal{R}_n^\Lambda(K) \cong R_n(\mathcal{O}) \otimes_\mathcal{O} K \cong \mathcal{H}_n^\Lambda(K).$$

4.34. *Remarks.* (a) The proof of Theorem 4.32 uses [7, Theorem 4.20] to bound the rank of $R_n(\mathcal{O})$. The proof of [7, Theorem 4.20] does not depend on Brundan-Kleshchev's isomorphism Theorem 2.14 ([6, Theorem 1.1]). Instead, [7, Theorem 4.20] depends on the Ariki-Brundan-Kleshchev Categorification Theorem [7, Theorem 4.18]. Consequently, Theorem 4.32 gives a new proof of Brundan and Kleshchev's Isomorphism Theorem 2.14. It should be possible to prove Theorem 4.32 directly, without appealing to [7, Theorem 4.18], by adapting the arguments of [6, Theorem 3.3].

(b) In proving Theorem 2.14, Brundan and Kleshchev [6] construct a family of isomorphisms $\mathcal{R}_n^\Lambda \xrightarrow{\sim} \mathcal{H}_n^\Lambda(\mathscr{K})$ that depend on a choice of polynomials $Q_r(\mathbf{i})$ that can be varied subject to certain constraints. In our setting this amounts to choosing certain 'scalars' $q_r(\mathbf{i})$, which are rational functions in L_r and L_{r+1}, such that $q_r(\mathbf{i})f_\mathbf{i}^\mathcal{O} \in \mathcal{H}_n^\Lambda(\mathcal{O})$ and then defining

$$\psi_r^\mathcal{O} f_\mathbf{i}^\mathcal{O} = \begin{cases} (T_r+1)\frac{t^{\hat{\imath}_r}}{M_r} f_\mathbf{i}^\mathcal{O}, & \text{if } i_r = i_{r+1},\\ (T_rL_r - L_rT_r)q_r(\mathbf{i})f_\mathbf{i}^\mathcal{O}, & \text{otherwise,}\end{cases}$$

such that the corresponding β-coefficients still satisfy the constraints of Lemma 4.26 and Definition 3.9(a). To make this more precise, as in Lemma 4.23 write

$$\psi_r^\mathcal{O} f_{\mathfrak{s}\mathfrak{t}} = \beta_r'(\mathfrak{s}) f_{\mathfrak{u}\mathfrak{t}} - \delta_{i_r i_{r+1}} \frac{t^{\hat{\imath}_{r+1} - c_{r+1}(\mathfrak{s})}}{[\rho_r(\mathfrak{s})]},$$

where $\mathfrak{u} = \mathfrak{s}(r, r+1)$, $\beta'_r(\mathfrak{s}) \in \mathscr{K}$, $\mathfrak{s} \in \operatorname{Std}(\mathbf{i})$, $\mathbf{i} \in I^n$ and $1 \le r < n$. (Explicitly, $\beta'_r(\mathfrak{s}) = t^{c_{r+1}(\mathfrak{s})}[\rho_r(\mathfrak{s})]q_r(\mathfrak{s})$ by Lemma 4.11, where $q_r(\mathfrak{s}) \in \mathscr{K}$ is the scalar such that $q_r(\mathbf{i})f_{\mathfrak{s}\mathfrak{t}} = q_r(\mathfrak{s})f_{\mathfrak{s}\mathfrak{t}}$.) Then we require that the scalars $\beta'_r(\mathfrak{s})$ satisfy Lemma 4.26 and the "braid relation" of Definition 3.9(a). If the $q_r(\mathbf{i})$ are chosen so that these two identities are satisfied then it is easy to see that argument used to prove Theorem 4.32 applies, virtually without change, using these more general elements. The key point is that Lemma 4.26 still holds. The corresponding identities in Brundan and Kleshchev's work are [6, (3.28), (3.29), (4.34) and (4.35)].

We end this section by using Theorem 4.32 to give an upper bound for the nilpotency index of the KLR generators $y_1, \dots, y_n$. If $1 \le r \le n$ and $\mathbf{i} \in I^n$ set

$$\mathscr{D}_r(\mathbf{i}) = \{\, c_r(\mathfrak{t}) - \hat{\imath}_r \mid \mathfrak{t} \in \operatorname{Std}(\mathbf{i}) \,\}$$

and define $d_r(\mathbf{i}) = \#\mathscr{D}_r(\mathbf{i})$. For example, $\mathscr{D}_1(\mathbf{i}) \subseteq \{\kappa_1 - \hat{\imath}_1, \dots, \kappa_\ell - \hat{\imath}_1\}$ so that $d_1(\mathbf{i}) = (\Lambda, \alpha_{i_1})$.

Two nodes $\gamma = (l, r, c)$ and $\gamma' = (l', r', c')$ are on the same **diagonal** if they have the same content. That is, γ and γ' are on the same diagonal if and only if $l = l'$ and $c - r = c' - r'$. The set of diagonals is indexed by pairs (l, d), with $1 \le l \le \ell$ and $d \in \mathbb{Z}$, and where the corresponding diagonal is the set of nodes $\mathbb{D}_{l,d} = \{\, (l, r, c) \mid \kappa_l + c - r = d \,\}$. Hence, $d_r(\mathbf{i}) = \#\mathscr{D}_r(\mathbf{i})$ counts the number of different diagonals that r appears on in $\operatorname{Std}(\mathbf{i})$. More precisely, we have:

4.35. **Lemma.** *Suppose that $1 \le r \le n$ and $\mathbf{i} \in I^n$. Then*

$$d_r(\mathbf{i}) = \#\{\, (l, d) \mid d \equiv i_r \pmod e \text{ and } \mathfrak{t}^{-1}(r) \in \mathbb{D}_{l,d} \text{ for some } \mathfrak{t} \in \operatorname{Std}(\mathbf{i}) \,\}.$$

That is, $d_r(\mathbf{i})$ is equal to the number of distinct diagonals that r appears on for some tableau $\mathfrak{t} \in \operatorname{Std}(\mathbf{i})$.

The next result is a stronger version of Lemma 4.31. We do not know how to prove this result using only the relations in $R_n(\mathcal{O})$.

4.36. **Proposition.** *Suppose that $1 \le r \le n$ and $\mathbf{i} \in I^n$. Then*

$$\prod_{c \in \mathscr{D}_r(\mathbf{i})} (y_r^{\mathcal{O}} - [c]) f_{\mathbf{i}}^{\mathcal{O}} = 0.$$

Proof. By Lemma 4.4 and Lemma 4.23,

$$\begin{aligned}\prod_{c \in \mathscr{D}_r(\mathbf{i})} (y_r^{\mathcal{O}} - [c]) f_{\mathbf{i}}^{\mathcal{O}} &= \sum_{\mathfrak{t} \in \operatorname{Std}(\mathbf{i})} \prod_{c \in \mathscr{D}_r(\mathbf{i})} (y_r^{\mathcal{O}} - [c]) \frac{1}{\gamma_{\mathfrak{t}}} f_{\mathfrak{t}\mathfrak{t}} \\ &= \sum_{\mathfrak{t} \in \operatorname{Std}(\mathbf{i})} \frac{1}{\gamma_{\mathfrak{t}}} \prod_{c \in \mathscr{D}_r(\mathbf{i})} ([c_r(\mathfrak{t}) - \hat{\imath}_r] - [c]) f_{\mathfrak{t}\mathfrak{t}} = 0,\end{aligned}$$

where the last equality follows because $c_r(\mathfrak{t}) - \hat{\imath}_r \in \mathscr{D}_r(\mathbf{i})$, for all $\mathfrak{t} \in \operatorname{Std}(\mathbf{i})$. □

Even though Proposition 4.36 is very easy to prove within our framework, it gives strong information about the nilpotency index of $y_r e(\mathbf{i})$, for $\mathbf{i} \in I^n$ and $1 \le e \le n$. By Proposition 4.36, and Corollary 4.33, we have the following.

4.37. **Corollary.** *Suppose that $\mathbf{i} \in I^n$ and $1 \le r \le n$. Then $y_r^{d_r(\mathbf{i})} e(\mathbf{i}) = 0$ in $\mathcal{R}_n^\Lambda$.*

When $e = 0$ Brundan and Kleshchev [6, Conjecture 2.3] conjectured that $y_r^\ell = 0$, for $1 \le r \le n$. Hoffnung and Lauda proved this conjecture as the main result of their paper [13]. Using Corollary 4.37 we obtain a quick proof of this result and, at the same time, a generalization of it to include the cases when $e > n$.

4.38. **Corollary.** *Suppose that $e = 0$ or $e > n$. Then $y_r^\ell = 0$, for $1 \le r \le n$.*

Proof. If $e = 0$ then we may assume that $e \gg 0$ by Corollary 2.15, so the results of this section apply. Hence, we may assume that $e > n$.

To prove the corollary it is enough to show that $y_r^\ell e(\mathbf{i}) = 0$, whenever $\mathbf{i} = \operatorname{res}(\mathfrak{t})$ for some standard tableau $\mathfrak{t} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$. By Corollary 4.37, this will follow if we show that each component contains at most one diagonal with content congruent to i_r upon which r can appear in any standard tableau $\mathfrak{s}$ with $\operatorname{res}(\mathfrak{s}) = \mathbf{i}$. Suppose by way of contradiction that there exists a standard tableau $\mathfrak{s}$, with $\operatorname{res}(\mathfrak{s}) = \mathbf{i}$, and such that r appears in the same component of $\mathfrak{s}$ and $\mathfrak{t}$ but on different diagonals. Then the axial distance between the nodes $\mathfrak{s}^{-1}(r)$ and $\mathfrak{t}^{-1}(r)$ is at least e, so every residue in I must appear in any connected path between these two nodes. As $\operatorname{res}(\mathfrak{s}) = \mathbf{i} = \operatorname{res}(\mathfrak{t})$ it follows that $\{i_1, \dots, i_r\} = I$. This is a contradiction, however, because $|I| = e > n \geq r$. $\square$

5. Integral bases for $\mathcal{H}_n^\Lambda(\mathcal{O})$

Now that we have proved Theorem A, we begin to use the machinery of seminormal forms to study the cyclotomic quiver Hecke algebras $\mathcal{R}_n^\Lambda$. In this chapter we reconstruct the 'natural' homogeneous bases for the cyclotomic Hecke algebras $\mathcal{H}_n^\Lambda(K)$ and their Specht modules over a field.

5.1. **The ψ-basis.** Theorem 4.32 links the KLR grading on $\mathcal{H}_n^\Lambda \cong \mathcal{R}_n^\Lambda$ with the semisimple representation theory of $\mathcal{H}_n^\Lambda(\mathscr{K})$. We next want to try and understand the graded Specht modules of $\mathcal{H}_n^\Lambda$ [8, 14, 23] in terms of the seminormal form. We start by lifting the homogeneous basis $\{\psi_{\mathfrak{s}\mathfrak{t}}\}$ of $\mathcal{H}_n^\Lambda$ to $\mathcal{H}_n^\Lambda(\mathcal{O})$. This turns out to be easier than the approach taken in [14]. Throughout this section, $\mathcal{O}$ is an e-idempotent subring of $\mathscr{K}$.

By Theorem 4.32, there is a unique anti-isomorphism $\diamond$ of $\mathcal{H}_n^\Lambda(\mathcal{O})$ such that

$$(\psi_r^{\mathcal{O}})^\diamond = \psi_r^{\mathcal{O}}, \quad (y_s^{\mathcal{O}})^\diamond = y_s^{\mathcal{O}} \quad \text{and} \quad (f_{\mathbf{i}}^{\mathcal{O}})^\diamond = f_{\mathbf{i}}^{\mathcal{O}},$$

for $1 \leq r < n$, $1 \leq s \leq n$ and $\mathbf{i} \in I^n$. Lemma 4.23 shows that, in general, the automorphisms $*$ and $\diamond$ do not coincide.

Recall from Definition 3.7 that a $\diamond$-seminormal basis of $\mathcal{H}_n^\Lambda(\mathscr{K})$ is a basis $\{f_{\mathfrak{s}\mathfrak{t}}\}$ of two-sided eigenvalues for $\mathscr{L}$ such that $f_{\mathfrak{s}\mathfrak{t}} = f_{\mathfrak{t}\mathfrak{s}}^\diamond$, for all $(\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$. We define a **$\diamond$-seminormal coefficient system** to be a set of scalars $\{\beta_r(\mathfrak{t})\}$ that satisfy the identity in Lemma 4.26 and the "braid relations" of Definition 3.9(a) (with α replaced by β) as well as the relation Definition 3.9(b) (with α replaced by β). The reader may check that the $\diamond$-seminormal coefficients correspond to the more general setup considered in Remark 4.34(b).

The main difference between a $*$-seminormal basis and a $\diamond$-seminormal basis is that $T_r f_{\mathfrak{s}\mathfrak{t}} = (f_{\mathfrak{t}\mathfrak{s}} T_r)^*$ for a $*$-seminormal basis whereas $\psi_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = (f_{\mathfrak{t}\mathfrak{s}} \psi_r^{\mathcal{O}})^\diamond$ for a $\diamond$-seminormal basis.

5.1. **Lemma.** *Suppose that $\{f_{\mathfrak{s}\mathfrak{t}}\}$ is a $\diamond$-seminormal basis of $\mathcal{H}_n^\Lambda(\mathscr{K})$. Then there exists a unique $\diamond$-seminormal coefficient system $\{\beta_r(\mathfrak{t})\}$ such that if $1 \leq r < n$ and $(\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ then*

$$f_{\mathfrak{s}\mathfrak{t}} \psi_r^{\mathcal{O}} = \beta_r(\mathfrak{t}) f_{\mathfrak{s}\mathfrak{v}} - \delta_{i_r i_{r+1}} \frac{t^{\hat{\imath}_{r+1} - c_{r+1}(\mathfrak{t})}}{[\rho_r(\mathfrak{t})]} f_{\mathfrak{s}\mathfrak{t}},$$

where $\mathfrak{v} = \mathfrak{t}(r, r+1)$ and $\mathfrak{t} \in \operatorname{Std}(\mathbf{i})$, for $\mathbf{i} \in I^n$. Conversely, as in Theorem 3.13, a $\diamond$-seminormal coefficient system, together with a choice of scalars $\{\, \gamma_{\mathfrak{t}^\lambda} \mid \boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \,\}$, determines a unique $\diamond$-seminormal basis.

Proof. By (4.21), a set of scalars $\{\beta_r(\mathfrak{t})\}$ is a $\diamond$-seminormal coefficient system if and only if $\{\alpha_r(\mathfrak{t})\}$ is a $*$-seminormal coefficient system, where

$$\alpha_r(\mathfrak{t}) = \begin{cases} \beta_r(\mathfrak{t})t^{c_r(\mathfrak{t})-\hat{\imath}_r}[1-\rho_r(\mathfrak{t})], & \text{if } i_r = i_{r+1},\\ \dfrac{\beta_r(\mathfrak{t})t^{\hat{\imath}_r - c_{r+1}(\mathfrak{t})}}{[\rho_r(\mathfrak{t})]}, & \text{if } i_r = i_{r+1}+1,\\ \dfrac{\beta_r(\mathfrak{t})[1-\rho_r(\mathfrak{t})]}{[\rho_r(\mathfrak{t})]}, & \text{otherwise.} \end{cases}$$

Therefore, as seminormal coefficient systems are determined by the action of the corresponding generators of $\mathcal{H}_n^\Lambda$ on its right regular representation, the result follows from Theorem 3.13 and Lemma 4.23. □

Henceforth, we will work with $\diamond$-seminormal bases. Lemma 5.1 also describes the left action of $\psi_r^{\mathcal{O}}$ on the $\diamond$-seminormal basis because $\psi_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = (f_{\mathfrak{t}\mathfrak{s}}\psi_r^{\mathcal{O}})^\diamond$.

Exactly as in Theorem 3.13, if $\{f_{\mathfrak{s}\mathfrak{t}}\}$ is a $\diamond$-seminormal basis then there exists scalars $\gamma_\mathfrak{t} \in \mathscr{K}$ such that $f_{\mathfrak{s}\mathfrak{t}} f_{\mathfrak{u}\mathfrak{v}} = \delta_{\mathfrak{u}\mathfrak{t}}\gamma_\mathfrak{t} f_{\mathfrak{s}\mathfrak{v}}$, for $(\mathfrak{s},\mathfrak{t}),(\mathfrak{u},\mathfrak{v}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$. Repeating the argument of Corollary 3.16, these scalars satisfy the following recurrence relation.

5.2. **Corollary.** *Suppose that* $\mathfrak{t} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$ *and that* $\mathfrak{v} = \mathfrak{t}(r,r+1)$ *is standard, where* $1 \le r < n$. *Then* $\beta_r(\mathfrak{v})\gamma_\mathfrak{t} = \beta_r(\mathfrak{t})\gamma_\mathfrak{v}$.

Motivated by [14], we now define a new basis of $\mathcal{H}_n^\Lambda(\mathcal{O})$ that is cellular with respect to the anti-involution $\diamond$. Fix $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$ and let $\mathbf{i}^{\boldsymbol{\lambda}} = (i_1^{\boldsymbol{\lambda}},\dots,i_n^{\boldsymbol{\lambda}})$, so that $i_r^{\boldsymbol{\lambda}} = \operatorname{res}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(r)$ for $1 \le r \le n$. Following [14, Definition 4.7], define

$$\mathscr{A}_{\boldsymbol{\lambda}}(r) = \left\{ \alpha \;\middle|\; \begin{array}{c}\alpha \text{ is an addable } i_r^{\boldsymbol{\lambda}}\text{-node of the multipartition}\\ \operatorname{Shape}(\mathfrak{t}^{\boldsymbol{\lambda}}_{\downarrow r}) \text{ that is } below\ (\mathfrak{t}^{\boldsymbol{\lambda}})^{-1}(r)\end{array}\right\},$$

for $1 \le r \le n$.

Up until now we have worked with an arbitrary seminormal basis of $\mathcal{H}_n^\Lambda(\mathscr{K})$. In order to define a 'nice' basis of $\mathcal{H}_n^\Lambda(\mathcal{O})$ that is compatible with Theorem 4.32 we now fix the choice of γ-coefficients by requiring that

$$\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} = \prod_{r=1}^{n} \prod_{\alpha \in \mathscr{A}_{\boldsymbol{\lambda}}(r)} [\mathrm{c}_r(\mathfrak{t}^{\boldsymbol{\lambda}}) - c_\alpha], \tag{5.3}$$

for all $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$. Together with a choice of seminormal coefficient system, this determines $\gamma_\mathfrak{t}$ for all $\mathfrak{t} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$ by Corollary 5.2. By definition, $\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}}$ is typically a non-invertible element of $\mathcal{O}$. Nonetheless, if $\mathbf{i} \in I^n$ then $f_\mathbf{i}^{\mathcal{O}} = \sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})} \frac{1}{\gamma_\mathfrak{s}} f_{\mathfrak{s}\mathfrak{s}}$ belongs to $\mathcal{H}_n^\Lambda(\mathcal{O})$ by Lemma 4.4.

We also fix a choice of seminormal coefficient system by requiring that $\beta_r(\mathfrak{s}) = 1$ whenever $\mathfrak{s} \rhd \mathfrak{t} = \mathfrak{s}(r,r+1)$, for $\mathfrak{s} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$ and $1 \le r < n$. More precisely, if $\mathbf{i} \in I$ and $\mathfrak{s} \in \operatorname{Std}(\mathbf{i})$ then we define

$$\beta_r(\mathfrak{s}) = \begin{cases} 1, & \text{if } \mathfrak{s} \rhd \mathfrak{t} \text{ or } i_r \not\sim i_{r+1},\\ -\frac{t^{2\hat{\imath}_r - 2c_{r+1}(\mathfrak{s})}}{[\rho_r(\mathfrak{s})]^2}, & \text{if } \mathfrak{t} \rhd \mathfrak{s} \text{ and } i_r = i_{r+1},\\ t^{c_r(\mathfrak{s})+c_{r+1}(\mathfrak{s})-\hat{\imath}_r-\hat{\imath}_{r+1}}[1-\rho_r(\mathfrak{s})][1+\rho_r(s)], & \text{if } \mathfrak{t} \rhd \mathfrak{s} \text{ and } i_r \leftrightarrows i_{r+1},\\ t^{c_r(\mathfrak{s})-\hat{\imath}_r}[1-\rho_r(\mathfrak{s})], & \text{if } \mathfrak{t} \rhd \mathfrak{s} \text{ and } i_r \leftarrow i_{r+1},\\ t^{c_{r+1}(\mathfrak{s})-\hat{\imath}_{r+1}}[1+\rho_r(s)], & \text{if } \mathfrak{t} \rhd \mathfrak{s} \text{ and } i_r \to i_{r+1}. \end{cases} \tag{5.4}$$

where $\mathfrak{s} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$ and $\mathfrak{t} = \mathfrak{s}(r,r+1)$ is standard, for $1 \le r < n$. The reader is invited to check that this defines a $\diamond$-seminormal coefficient system. As the definition of $\psi_r^{\mathcal{O}}$ is independent of the choice of seminormal coefficient system this choice is not strictly necessary for what follows but it simplifies many of the formulas.

By Lemma 5.1, this choice of $\diamond$-seminormal coefficient system and γ-coefficients determines a unique $\diamond$-seminormal basis $\{f_{\mathfrak{s}\mathfrak{t}}\}$ of $\mathcal{H}_n^\Lambda(\mathscr{K})$. We will use this basis to define new homogeneous basis of $\mathcal{H}_n^\Lambda$. The first step is to define

$$\begin{aligned} y_{\mathcal{O}}^{\boldsymbol{\lambda}} f_{\mathbf{i}^\lambda}^{\mathcal{O}} &= \prod_{r=1}^{n} \prod_{\alpha\in\mathscr{A}_{\boldsymbol{\lambda}}(r)} t^{-c_r(\mathfrak{t}^\lambda)}(L_r - [c_\alpha]) f_{\mathbf{i}^\lambda}^{\mathcal{O}} \\ &= \prod_{r=1}^{n} \prod_{\alpha\in\mathscr{A}_{\boldsymbol{\lambda}}(r)} t^{\hat{\imath}_r^{\lambda} - c_r(\mathfrak{t}^\lambda)}(y_r^{\mathcal{O}} - [c_\alpha - \hat{\imath}_r^{\lambda}]) f_{\mathbf{i}^\lambda}^{\mathcal{O}}, \end{aligned}$$

where the second equation follows by rewriting $L_k f_{\mathbf{i}}^{\mathcal{O}}$ in terms of $y_k f_{\mathbf{i}}^{\mathcal{O}}$ as in the proof of Proposition 4.16. In particular, these equations show that $y_{\mathcal{O}}^{\boldsymbol{\lambda}} f_{\mathbf{i}^\lambda}^{\mathcal{O}} \otimes_{\mathcal{O}} 1_K$ is a monomial in $y_1, \dots, y_n$ and, further, that it is (up to a sign) equal to the element $y^{\boldsymbol{\lambda}}$ defined in [14, Definition 4.15].

The next result is a essentially a translation of [14, Lemma 4.13] into the current setting for the special case of the tableau $\mathfrak{t}^{\boldsymbol{\lambda}}$.

5.5. **Lemma.** *Suppose that $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$. Then there exist scalars $a_{\mathfrak{s}} \in \mathscr{K}$ such that*

$$y_{\mathcal{O}}^{\boldsymbol{\lambda}} f_{\mathbf{i}^\lambda}^{\mathcal{O}} = f_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} + \sum_{\mathfrak{s}\blacktriangleright\mathfrak{t}^\lambda} a_{\mathfrak{s}} f_{\mathfrak{s}\mathfrak{s}}.$$

In particular, $y_{\mathcal{O}}^{\boldsymbol{\lambda}} f_{\mathbf{i}^\lambda}^{\mathcal{O}}$ is a non-zero element of $\mathcal{H}_n^\Lambda(\mathcal{O})$.

Proof. By Lemma 4.4, $f_{\mathbf{i}^\lambda}^{\mathcal{O}} = \sum_{\mathfrak{s}} \frac{1}{\gamma_{\mathfrak{s}}} f_{\mathfrak{s}\mathfrak{s}}$, so that $y_{\mathcal{O}}^{\boldsymbol{\lambda}} f_{\mathbf{i}^\lambda}^{\mathcal{O}} = \sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i}^\lambda)} a_{\mathfrak{s}} f_{\mathfrak{s}\mathfrak{s}}$, for some $a_{\mathfrak{s}} \in \mathscr{K}$, by (3.14). It remains to show that $a_{\mathfrak{t}^\lambda} = 1$ and that $a_{\mathfrak{s}} \neq 0$ only if $\mathfrak{s} \trianglerighteq \mathfrak{t}^{\boldsymbol{\lambda}}$. Using (3.14), and recalling the definition of $\gamma_{\mathfrak{t}^\lambda}$ from (5.3),

$$\frac{1}{\gamma_{\mathfrak{t}^\lambda}} y_{\mathcal{O}}^{\boldsymbol{\lambda}} f_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} = \frac{1}{\gamma_{\mathfrak{t}^\lambda}} \prod_{r=1}^{n} \prod_{\alpha\in\mathscr{A}_{\boldsymbol{\lambda}}(r)} t^{-c_r(\mathfrak{t}^\lambda)}([c_r(\mathfrak{t}^{\boldsymbol{\lambda}})] - [c_\alpha]) \cdot f_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} = f_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}.$$

To complete the proof we claim that there exist scalars $a_{\mathfrak{s}}(k) \in \mathscr{K}$, $1 \le k \le n$, such that

$$\prod_{r=1}^{k} \prod_{\alpha\in\mathscr{A}_{\boldsymbol{\lambda}}(r)} t^{-c_r(\mathfrak{t}^\lambda)}(L_r - [c_\alpha]) f_{\mathbf{i}^\lambda}^{\mathcal{O}} = \sum_{\substack{\mathfrak{s}\in\operatorname{Std}(\mathbf{i}^\lambda) \\ \mathfrak{s}_{\downarrow k} \trianglerighteq \mathfrak{t}^\lambda_{\downarrow k}}} a_{\mathfrak{s}}(k) f_{\mathfrak{s}\mathfrak{s}}$$

where $a_{\mathfrak{t}^\lambda}(k) = 1$. We prove this by induction on k. If $k = 1$ then the result is immediate from (3.14). Suppose that $k > 1$. By induction, it is enough to show that

$$(L_k - [c_\alpha]) f_{\mathfrak{s}\mathfrak{s}} = ([c_\alpha] - [c_k(\mathfrak{s})]) f_{\mathfrak{s}\mathfrak{s}} = 0$$

whenever $\mathfrak{s}_{\downarrow(k-1)} \trianglerighteq \mathfrak{t}^{\boldsymbol{\lambda}}_{\downarrow(k-1)}$ and $\mathfrak{s}_{\downarrow k} \not\trianglerighteq \mathfrak{t}^{\boldsymbol{\lambda}}_{\downarrow k}$, for $\mathfrak{s} \in \operatorname{Std}(\mathbf{i}^{\boldsymbol{\lambda}})$. Fix such a tableau $\mathfrak{s}$. Since $\mathfrak{s}_{\downarrow(k-1)} \trianglerighteq \mathfrak{t}^{\boldsymbol{\lambda}}_{\downarrow(k-1)}$ we must have $(\mathfrak{s}_{\downarrow k})^{(l)} = \emptyset$ whenever $l > \operatorname{comp}_{\mathfrak{t}^\lambda}(k)$, so the node $\alpha = \mathfrak{s}^{-1}(k)$ must be below $(\mathfrak{t}^{\boldsymbol{\lambda}})^{-1}(k)$. Therefore, $\alpha \in \mathscr{A}_{\boldsymbol{\lambda}}(k)$, and $c_k(\mathfrak{s}) = \mathrm{c}_\alpha$ for this α, and forcing $a_{\mathfrak{s}}(k) = 0$ as claimed. This completes the proof. □

For each $w \in \mathfrak{S}_n$ we now fix a reduced expression $w = s_{r_1} \dots s_{r_k}$ for w, with $1 \le r_j < n$ for $1 \le j \le k$, and define $\psi_w^{\mathcal{O}} = \psi_{r_1}^{\mathcal{O}} \dots \psi_{r_k}^{\mathcal{O}}$. By Theorem 4.32 the elements $\psi_r^{\mathcal{O}}$ do not satisfy the braid relations so, in general, $\psi_w^{\mathcal{O}}$ will depend upon this (fixed) choice of reduced expression.

5.6. **Definition.** *Suppose that $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$. Define*

$$\psi_{\mathfrak{s}\mathfrak{t}}^{\mathcal{O}} = (\psi_{d(\mathfrak{s})}^{\mathcal{O}})^\diamond y_{\mathcal{O}}^{\boldsymbol{\lambda}} f_{\mathbf{i}^\lambda}^{\mathcal{O}} \psi_{d(\mathfrak{t})}^{\mathcal{O}},$$

for $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$.

We can now lift the graded cellular basis of [14, Definitions 5.1] to $\mathcal{H}_n^\Lambda(\mathcal{O})$.

5.7. **Theorem.** *Suppose that $\mathcal{O}$ is an idempotent subring. Then*

$$\{\, \psi^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\mu}) \text{ for } \boldsymbol{\mu} \in \mathcal{P}^{\Lambda}_{n} \,\}$$

is a cellular basis of $\mathcal{H}^{\Lambda}_{n}(\mathcal{O})$ with respect to the anti-involution $\diamond$.

Proof. In view of (3.14) and Lemma 4.23, Lemma 5.5 implies that

$$\psi^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} = f_{\mathfrak{s}\mathfrak{t}} + \sum_{(\mathfrak{u},\mathfrak{v})\blacktriangleright(\mathfrak{s},\mathfrak{t})} a_{\mathfrak{u}\mathfrak{v}} f_{\mathfrak{u}\mathfrak{v}}, \tag{5.8}$$

for some $a_{\mathfrak{u}\mathfrak{v}} \in \mathscr{K}$. Therefore, $\{\, \psi^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} \mid (\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}^{\Lambda}_{n}) \,\}$ is a basis of $\mathcal{H}^{\Lambda}_{n}(\mathscr{K})$. In fact, these elements are a basis for $\mathcal{H}^{\Lambda}_{n}(\mathcal{O})$ because if $h \in \mathcal{H}^{\Lambda}_{n}(\mathcal{O})$ then we can write $h = \sum r_{\mathfrak{u}\mathfrak{v}} f_{\mathfrak{u}\mathfrak{v}}$, for some $r_{\mathfrak{u}\mathfrak{v}} \in \mathscr{K}$. Pick $(\mathfrak{s},\mathfrak{t})$ to be minimal with respect to dominance such that $r_{\mathfrak{s}\mathfrak{t}} \neq 0$. Then $r_{\mathfrak{s}\mathfrak{t}} \in \mathcal{O}$ because $h \in \mathcal{H}^{\Lambda}_{n}(\mathcal{O})$. Consequently, $h - r_{\mathfrak{s}\mathfrak{t}}\psi^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} \in \mathcal{H}^{\Lambda}_{n}(\mathcal{O})$ so, by continuing in this way, we can write h as a linear combination of the ψ-basis.

It remains to show that the ψ-basis is cellular with respect to the anti-involution $\diamond$. By definition, if $\boldsymbol{\lambda} \in \mathcal{P}^{\Lambda}_{n}$ then $y^{\boldsymbol{\lambda}}_{\mathcal{O}}$ and $f^{\mathcal{O}}_{\mathbf{i}^{\boldsymbol{\lambda}}}$ commute and they are fixed by the automorphism $\diamond$. Therefore, $(\psi^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}})^{\diamond} = \psi^{\mathcal{O}}_{\mathfrak{t}\mathfrak{s}}$, for all $\mathfrak{s},\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. By Lemma 5.1, the $\diamond$-seminormal basis $\{f_{\mathfrak{s}\mathfrak{t}}\}$ is a cellular basis with cellular anti-involution $\diamond$. It remains to verify (GC$_2$) from Definition 2.4. As in Theorem 3.13, the seminormal basis $\{f_{\mathfrak{u}\mathfrak{v}}\}$ is cellular. Therefore, if $(\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\boldsymbol{\lambda})$ and $h \in \mathcal{H}^{\Lambda}_{n}(\mathcal{O})$ then, using (5.8) twice,

$$\begin{aligned}
\psi^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} h = (\psi^{\mathcal{O}}_{d(\mathfrak{s})})^{\diamond}\psi^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} &\equiv (\psi^{\mathcal{O}}_{d(\mathfrak{s})})^{\diamond}\Big(f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} + \sum_{\mathfrak{v}\rhd\mathfrak{t}} a_{\mathfrak{v}} f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{v}}\Big)h \equiv (\psi^{\mathcal{O}}_{d(\mathfrak{s})})^{\diamond} \sum_{\mathfrak{v}\in\operatorname{Std}(\boldsymbol{\lambda})} a'_{\mathfrak{v}} f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{v}} \\
&\equiv (\psi^{\mathcal{O}}_{d(\mathfrak{s})})^{\diamond} \sum_{\mathfrak{v}\in\operatorname{Std}(\boldsymbol{\lambda}} b_{\mathfrak{v}}\psi^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{v}} \quad \equiv \sum_{\mathfrak{v}\in\operatorname{Std}(\boldsymbol{\lambda}} b_{\mathfrak{v}}\psi^{\mathcal{O}}_{\mathfrak{s}\mathfrak{v}} \quad \pmod{\mathcal{H}^{\rhd\boldsymbol{\lambda}}_{n}},
\end{aligned}$$

where $a_{\mathfrak{v}}, a'_{\mathfrak{v}} \in \mathscr{K}$ and $b_{\mathfrak{v}} \in \mathcal{O}$ with the scalars $b_{\mathfrak{v}}$ being independent of $\mathfrak{s}$. Hence, (GC$_2$) holds, completing the proof. $\square$

If $K = \mathcal{O}/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of $\mathcal{O}$ then $\mathcal{H}^{\Lambda}_{n}(K) \cong \mathcal{H}^{\Lambda}_{n}(\mathcal{O}) \otimes_{\mathcal{O}} K$. Set $\psi_{\mathfrak{s}\mathfrak{t}} = \psi^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} \otimes 1_K$.

5.9. **Corollary** ([14, Theorem 5.8])**.** *Suppose that $K = \mathcal{O}/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of $\mathcal{O}$. Then $\{\, \psi_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\mu})$ for $\boldsymbol{\mu} \in \mathcal{P}^{\Lambda}_{n} \,\}$ is a graded cellular basis of $\mathcal{H}^{\Lambda}_{n}(K)$ with $\deg\psi_{\mathfrak{s}\mathfrak{t}} = \deg\mathfrak{s} + \deg\mathfrak{t}$, for $(\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}^{\Lambda}_{n})$.*

5.2. **Graded Specht modules and Gram determinants.** By Theorem 5.7, $\{\psi^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}}\}$ is a cellular basis of $\mathcal{H}^{\Lambda}_{n}(\mathcal{O})$ so we can use it to define Specht modules for $\mathcal{H}^{\Lambda}_{n}(\mathcal{O})$ that specialise to the graded Specht modules in characteristic zero and in positive characteristic.

5.10. **Definition.** *Suppose that $\boldsymbol{\lambda} \in \mathcal{P}^{\Lambda}_{n}$. The Specht module $S^{\boldsymbol{\lambda}}(\mathcal{O})$ is the right $\mathcal{H}^{\Lambda}_{n}(\mathcal{O})$-module with basis $\{\, \psi^{\mathcal{O}}_{\mathfrak{t}} \mid \mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \,\}$, where $\psi^{\mathcal{O}}_{\mathfrak{t}} = \psi^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} + \mathcal{H}^{\rhd\boldsymbol{\lambda}}_{n}(\mathcal{O})$.*

By Theorem 5.7 and [14, Corollary 5.10], ignoring the grading, $S^{\boldsymbol{\lambda}}(\mathcal{O}) \otimes_{\mathcal{O}} K$ can be identified with the graded Specht module $S^{\boldsymbol{\lambda}}$ of $\mathcal{H}^{\Lambda}_{n}$ defined by Brundan, Kleshchev and Wang [8]. The action of $\mathcal{H}^{\Lambda}_{n}(\mathscr{K})$ on a graded Specht module is completely determined by the relations for these modules that are given in [23]. In contrast, in view of (5.8) and Theorem 4.32, the action of $\mathcal{H}^{\Lambda}_{n}(\mathcal{O})$ on the Specht module $S^{\boldsymbol{\lambda}}(\mathcal{O})$ is completely determined by the (choice of) seminormal form.

We now turn to computing the determinant of the Gram matrix

$$\mathcal{G}^{\boldsymbol{\lambda}} = \big(\langle\psi^{\mathcal{O}}_{\mathfrak{s}}, \psi^{\mathcal{O}}_{\mathfrak{t}}\rangle\big)_{\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})}.$$

A priori, it is unclear how the bilinear form on $S^{\boldsymbol{\lambda}}(\mathcal{O})$ is related to the usual (ungraded) bilinear from on the Specht module that is defined using the Murphy basis that we considered in Theorem 3.21. The main problem in relating these two bilinear forms is that the cellular algebra anti-involutions $*$ and $\diamond$, which are used to define these bilinear forms, are different.

Note that the cellular algebra anti-involutions $*$ and $\diamond$ on $\mathcal{H}_n^\Lambda(\mathcal{O})$ naturally extend to anti-involutions on the algebra $\mathcal{H}_n^\Lambda(\mathscr{K})$. The key point to understanding the graded bilinear form is the following.

5.11. **Lemma.** *Suppose that* $\mathfrak{t}\in\operatorname{Std}(\mathcal{P}_n^\Lambda)$. *Then* $(F_\mathfrak{t})^\diamond = F_\mathfrak{t}$.

Proof. By definition, $F_\mathfrak{t}$ is a linear combination of products of Jucys-Murphy elements, so it can also be written as a polynomial, with coefficients in $\mathscr{K}$, in $y_r^\mathcal{O}$, $f_\mathbf{i}^\mathcal{O}$, for $1\le r\le n$ and $\mathbf{i}\in I^n$. As $(y_r^\mathcal{O})^\diamond = y_r^\mathcal{O}$, $(f_\mathbf{i}^\mathcal{O})^\diamond = f_\mathbf{i}^\mathcal{O}$, for $1\le r\le n$ and $\mathbf{i}\in I^n$, the result follows. □

Recall that if $\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})$ then $\psi_\mathfrak{t}^\mathcal{O} = \psi_{\mathfrak{t}^\lambda\mathfrak{t}}^\mathcal{O} + \mathcal{H}_n^{\rhd\boldsymbol{\lambda}}$ is a basis element of the Specht module $S^{\boldsymbol{\lambda}}(\mathcal{O})$. In order to compute $\det\mathcal{G}^{\boldsymbol{\lambda}}$, set $f_\mathfrak{t} = \psi_\mathfrak{t}^\mathcal{O}F_\mathfrak{t}$, for $\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})$. Recall that $S^{\boldsymbol{\lambda}}(\mathscr{K}) = S^{\boldsymbol{\lambda}}(\mathcal{O})\otimes_\mathcal{O}\mathscr{K}$.

5.12. **Lemma.** *Suppose that* $\boldsymbol{\lambda}\in\mathcal{P}_n^\Lambda$. *Then* $\{\, f_\mathfrak{t} \mid \mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})\,\}$ *is a basis of* $S^{\boldsymbol{\lambda}}(\mathscr{K})$. *Moreover,* $\det\mathcal{G}^{\boldsymbol{\lambda}} = \det\big(\langle f_\mathfrak{s}, f_\mathfrak{t}\rangle\big) = \prod_{\mathfrak{s}\in\operatorname{Std}(\boldsymbol{\lambda})}\gamma_\mathfrak{s}$.

Proof. By definition, $f_\mathfrak{t} = f_{\mathfrak{t}^\lambda\mathfrak{t}} + (\mathcal{H}_n^\Lambda(\mathscr{K}))^{\rhd\boldsymbol{\lambda}}$. Therefore, $f_\mathfrak{t}\in S^{\boldsymbol{\lambda}}(\mathscr{K})$ and $f_\mathfrak{t} = \psi_\mathfrak{t}^\mathcal{O} + \sum_{\mathfrak{v}\rhd\mathfrak{t}} r_{\mathfrak{t}\mathfrak{v}}\psi_\mathfrak{v}^\mathcal{O}$ by (5.8), for some scalars $r_{\mathfrak{t}\mathfrak{v}}\in\mathscr{K}$. Set $r_{\mathfrak{t}\mathfrak{t}}=1$ and $U=\big(r_{\mathfrak{t}\mathfrak{v}}\big)$. Then $\{\, f_\mathfrak{t} \mid \mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})\,\}$ is a $\mathscr{K}$-basis of $S^{\boldsymbol{\lambda}}(\mathscr{K})$ and $\mathcal{G}^{\boldsymbol{\lambda}} = (U^{-1})^{tr}\big(\langle f_\mathfrak{s}, f_\mathfrak{t}\rangle\big)U^{-1}$ Taking determinants shows that $\det\mathcal{G}^{\boldsymbol{\lambda}} = \deg\big(\langle f_\mathfrak{s}, f_\mathfrak{t}\rangle\big)$ since U is unitriangular. To complete the proof observe that $\langle f_\mathfrak{s}, f_\mathfrak{t}\rangle f_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} \equiv f_{\mathfrak{t}^\lambda\mathfrak{s}}f_{\mathfrak{t}\mathfrak{t}^\lambda} = \delta_{\mathfrak{s}\mathfrak{t}}\gamma_\mathfrak{s}f_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} \pmod{\mathcal{H}_n^{\rhd\boldsymbol{\lambda}}}$, where we are implicitly using Lemma 5.11. The result follows. □

Lemma 5.12 is subtly different from (3.18) because, in spite of our notation, the $\gamma_\mathfrak{t}$'s appearing in the two formulas satisfy different recurrence relations. It is not hard to show that the quotient of $\gamma_\mathfrak{t}$, as defined in this section, by the $\gamma_\mathfrak{t}$ defined in Section 3.3 in a unit in $\mathcal{O}$, for all $\mathfrak{t}\in\operatorname{Std}(\mathcal{P}_n^\Lambda)$.

5.13. **Lemma.** *Suppose that* $\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})$, *for* $\boldsymbol{\lambda}\in\mathcal{P}_n^\Lambda$. *Then* $\gamma_\mathfrak{t} = u_\mathfrak{t}\Phi_e(t)^{\deg_e(\mathfrak{t})}$, *for some unit* $u_\mathfrak{t}\in\mathcal{O}^\times$.

Proof. We argue by induction on the dominance order on $\operatorname{Std}(\boldsymbol{\lambda})$. If $\mathfrak{t}=\mathfrak{t}^{\boldsymbol{\lambda}}$ then (5.3) ensures that $\gamma_{\mathfrak{t}^\lambda} = u_{\mathfrak{t}^\lambda}\Phi_e(t)^{\deg_e(\mathfrak{t}^{\boldsymbol{\lambda}})}$, for some unit $u_{\mathfrak{t}^\lambda}\in\mathcal{O}$. Now suppose that $\mathfrak{t}^{\boldsymbol{\lambda}}\rhd\mathfrak{t}$. Then there exists a standard tableau $\mathfrak{s}\in\operatorname{Std}(\boldsymbol{\lambda})$ such that $\mathfrak{s}\rhd\mathfrak{t}$ and $\mathfrak{t}=\mathfrak{s}(r,r+1)$, where $1\le r<n$. Arguing exactly as in Corollary 5.2 shows that $\beta_r(\mathfrak{s})\gamma_\mathfrak{t} = \beta_r(\mathfrak{t})\gamma_\mathfrak{s}$. Therefore, $\gamma_\mathfrak{t} = \frac{\beta_r(\mathfrak{t})}{\beta_r(\mathfrak{s})}\gamma_\mathfrak{s} = \beta_r(\mathfrak{t})\gamma_\mathfrak{s}$. Hence, the lemma follows by induction exactly as in the proof of Theorem 3.21. □

5.14. *Remark.* Looking at the definition of a $\diamond$-seminormal coefficient system shows that the quantities $\frac{\beta_r(\mathfrak{t})}{\beta_r(\mathfrak{s})}$, which are used in the proof of Lemma 5.13, are independent of the choice of $\diamond$-seminormal coefficient system. This shows that the choice of $\diamond$-seminormal coefficient system made in (5.4) really is only for convenience.

By general nonsense, the determinants of $\mathcal{G}^{\boldsymbol{\lambda}}$ and $\underline{\mathcal{G}}^{\boldsymbol{\lambda}}$ differ by a scalar in $\mathscr{K}$. The last two results readily imply the next theorem, the real content of which is that this scalar is a unit in $\mathcal{O}$.

5.15. **Theorem.** *Suppose that* $\boldsymbol{\lambda}\in\mathcal{P}_n^\Lambda$. *Then* $\det\mathcal{G}^{\boldsymbol{\lambda}} = u\Phi_e(t)^{\deg_e(\boldsymbol{\lambda})}$, *for some unit* $u\in\mathcal{O}^\times$. *Consequently,* $\det\mathcal{G}^{\boldsymbol{\lambda}} = u'\det\underline{\mathcal{G}}^{\boldsymbol{\lambda}}$, *for some unit* $u'\in\mathcal{O}^\times$.

If $\mathbf{i}\in I^n$ and $\boldsymbol{\lambda}\in\mathcal{P}_n^\Lambda$ let $\operatorname{Std}_\mathbf{i}(\boldsymbol{\lambda}) = \{\,\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda}) \mid \operatorname{res}(\mathfrak{t})=\mathbf{i}\,\}$.

The Specht module $S^{\boldsymbol{\lambda}}$ over $\mathcal{O}$ decomposes as a direct sum of generalised eigenspaces as an $\mathscr{L}(\mathcal{O})$-module: $S^{\boldsymbol{\lambda}} = \bigoplus_{\mathbf{i}\in I^n} S^{\boldsymbol{\lambda}}_\mathbf{i}$, where $S^{\boldsymbol{\lambda}}_\mathbf{i} = S^{\boldsymbol{\lambda}}f_\mathbf{i}^\mathcal{O}$. The weight space $S^{\boldsymbol{\lambda}}_\mathbf{i}$

has basis $\{\,\psi^{\mathcal{O}}_{\mathfrak{t}} \mid \mathfrak{t}\in\operatorname{Std}_{\mathbf{i}}(\boldsymbol{\lambda})\,\}$ and the bilinear linear form $\langle\ ,\ \rangle$ on $S^{\boldsymbol{\lambda}}$ respects the weight space decomposition of $S^{\boldsymbol{\lambda}}$. Set

$$\deg_{e,\mathbf{i}}(\boldsymbol{\lambda}) = \sum_{\mathfrak{t}\in\operatorname{Std}_{\mathbf{i}}(\boldsymbol{\lambda})} \deg\mathfrak{t}.$$

and let $\mathcal{G}^{\boldsymbol{\lambda}}_{\mathbf{i}}$ be restriction of the Gram matrix of $S^{\boldsymbol{\lambda}}$ to $S^{\boldsymbol{\lambda}}_{\mathbf{i}}$, for $\mathbf{i}\in I^n$. Then we have the following refinement of Theorem 5.15 (and Theorem 3.21).

5.16. **Corollary.** *Suppose that $\boldsymbol{\lambda}\in\mathcal{P}^{\Lambda}_n$ and $\mathbf{i}\in I^n$. Then $\deg\mathcal{G}^{\boldsymbol{\lambda}}_{\mathbf{i}} = u_{\mathbf{i}}\Phi_e(t)^{\deg_{e,\mathbf{i}}(\boldsymbol{\lambda})}$, for some unit $u_{\mathbf{i}}\in\mathcal{O}^{\times}$. Moreover, $\deg_{e,\mathbf{i}}(\boldsymbol{\lambda})\ge 0$.*

6. A distinguished homogeneous basis for $\mathcal{H}^{\Lambda}_n$

The ψ-basis of $\mathcal{H}^{\Lambda}_n(\mathcal{O})$, the homogeneous bases of $\mathcal{H}^{\Lambda}_n$ constructed in [14], and the homogeneous basis of the graded Specht modules given by Brundan, Kleshchev and Wang [8], are all indexed by pairs of standard tableaux. Unfortunately, unlike in the ungraded case, these basis elements depend upon choices of reduced expressions for the permutations corresponding to these tableaux. In this section we construct new bases for these modules that depend only on the corresponding tableaux.

6.1. **A new basis of $\mathcal{H}^{\Lambda}_n(\mathcal{O})$.** To construct our new basis for $\mathcal{H}^{\Lambda}_n$ we need to work over a complete discrete valuation ring. We start by setting up the necessary machinery.

Recall that the algebra $\mathcal{H}^{\Lambda}_n$ is defined over the field K with parameter ξ and that $e>1$ is minimal such that $[e]_\xi=0$. Let x be an indeterminate over K and let $\mathcal{O}=K[x]_{(x)}$ and $t=x+\xi$. Then $(\mathcal{O},t)$ is an idempotent subring by Example 4.2(b) and $K(x)$ is the field of fractions of $\mathcal{O}$. Note that $\mathcal{O}$ is a local ring with maximal ideal $\mathfrak{m}=x\mathcal{O}$.

Let $\widehat{\mathcal{O}}$ be the $\mathfrak{m}$-adic completion of $\mathcal{O}$. Then $\widehat{\mathcal{O}}$ is a complete discrete valuation ring with field of fractions $K((x))$ Let $\widehat{\mathscr{K}}=K((x))$ be the $\mathfrak{m}$-adic completion of $K(x)$. Then $\widehat{\mathcal{O}}$ is an idempotent subring of $\widehat{\mathscr{K}}$.

Define a valuation on $\widehat{\mathscr{K}}^{\times}$ by setting $\nu_x(a)=n$ if $a=ux^n$, where $n\in\mathbb{Z}$ and $u\in\widehat{\mathcal{O}}^{\times}$ is a unit in $\widehat{\mathcal{O}}$. We need to work with a complete discrete valuation ring because of the following fundamental but elementary fact that is proved, for example, as [35, Proposition II.5].

6.1. **Lemma.** *Suppose that $a\in\widehat{\mathscr{K}}$. Then a can be written uniquely as a convergent series*

$$a=\sum_{n\in\mathbb{Z}} a_n x^n, \qquad \text{with } a_n\in K,$$

such that if $a\neq 0$ then $a_n\neq 0$ only if $n\ge\nu_x(a)$. Moreover, $a\in\widehat{\mathcal{O}}$ if and only if $a_n=0$ for all $n<0$.

In particular, $x^{-1}K[x^{-1}]\cap\widehat{\mathcal{O}}=0$, where we embed $x^{-1}K[x^{-1}]$ into $\widehat{\mathscr{K}}$ in the obvious way.

6.2. **Theorem.** *Suppose that $(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathcal{P}^{\Lambda}_n)$. There exists a unique element $B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}}\in\mathcal{H}^{\Lambda}_n(\widehat{\mathcal{O}})$ such that*

$$B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} = f_{\mathfrak{s}\mathfrak{t}} + \sum_{\substack{(\mathfrak{u},\mathfrak{v})\in\operatorname{Std}^2(\mathcal{P}^{\Lambda}_n)\\ (\mathfrak{u},\mathfrak{v})\blacktriangleright(\mathfrak{s},\mathfrak{t})}} p^{\mathfrak{s}\mathfrak{t}}_{\mathfrak{u}\mathfrak{v}}(x^{-1}) f_{\mathfrak{u}\mathfrak{v}},$$

where $p^{\mathfrak{s}\mathfrak{t}}_{\mathfrak{u}\mathfrak{v}}(x)\in xK[x]$. Moreover, $\{\,B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} \mid (\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathcal{P}^{\Lambda}_n)\,\}$ is a cellular basis of $\mathcal{H}^{\Lambda}_n(\widehat{\mathcal{O}})$.

Proof. The existence of an element $B^{\mathcal{O}}_{\mathfrak{st}}$ with the required properties follows directly from (5.8) and Lemma 6.1 using Gaussian elimination. (See the proof of Proposition 6.4, below, which proves a stronger result in characteristic zero.) To prove uniqueness of the element $B^{\mathcal{O}}_{\mathfrak{st}}$, suppose, by way of contradiction, that there exist two elements $B^{\mathcal{O}}_{\mathfrak{st}}$ and $B'_{\mathfrak{st}}$ in $\mathcal{H}^\Lambda_n(\widehat{\mathcal{O}})$ with the required properties. Then $B^{\mathcal{O}}_{\mathfrak{st}} - B'_{\mathfrak{st}} = \sum r_{\mathfrak{uv}} f_{\mathfrak{uv}} \in \mathcal{H}^\Lambda_n(\widehat{\mathcal{O}})$ and, by assumption, $r_{\mathfrak{uv}} \in x^{-1}K[x^{-1}]$ with $r_{\mathfrak{uv}} \neq 0$ only if $(\mathfrak{u},\mathfrak{v}) \blacktriangleright (\mathfrak{s},\mathfrak{t})$. Pick $(\mathfrak{a},\mathfrak{b})$ minimal with respect to dominance such that $r_{\mathfrak{ab}} \neq 0$. Then, by Theorem 5.7, if we write $B^{\mathcal{O}}_{\mathfrak{st}} - B'_{\mathfrak{st}}$ as a linear combination of ψ-basis elements then $\psi^{\widehat{\mathcal{O}}}_{\mathfrak{ab}}$ appears with coefficient $r_{\mathfrak{ab}}$. Therefore, $r_{\mathfrak{ab}} \in x^{-1}K[x^{-1}] \cap \widehat{\mathcal{O}} = 0$, a contradiction. Hence, $B^{\mathcal{O}}_{\mathfrak{st}} = B'_{\mathfrak{st}}$ as claimed.

By (5.8), the transition matrix between the B-basis and the ψ-basis is unitriangular, so $\{B^{\mathcal{O}}_{\mathfrak{st}}\}$ is a basis of $\mathcal{H}^\Lambda_n(\widehat{\mathcal{O}})$. To show that the B-basis is cellular we need to check properties (GC$_1$)–(GC$_3$) from Definition 2.4. We have already verified (GC$_1$) Moreover, (GC$_3$) holds because $(B^{\mathcal{O}}_{\mathfrak{st}})^\diamond = B^{\mathcal{O}}_{\mathfrak{ts}}$ by the uniqueness of $B^{\mathcal{O}}_{\mathfrak{ts}}$ since $\{f_{\mathfrak{uv}}\}$ is $\diamond$-seminormal basis. It remains to prove (GC$_2$), which we do in three steps.

Step 1. We claim that if $h \in \mathcal{H}^\Lambda_n(\widehat{\mathcal{O}})$ and $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ then

$$B^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} h \equiv \sum_{\mathfrak{v}\in\operatorname{Std}(\boldsymbol{\lambda})} b_{\mathfrak{v}} B^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{v}} \pmod{\mathcal{H}^{\rhd\boldsymbol{\lambda}}_n},$$

for some scalars $b_{\mathfrak{v}} \in \widehat{\mathcal{O}}$ that depend only on $\mathfrak{t}$, $\mathfrak{v}$ and h (and not on $\mathfrak{t}^{\boldsymbol{\lambda}}$).

To see this first note that $\psi^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} = f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} + \sum_{\mathfrak{v}\blacktriangleright\mathfrak{t}} a_{\mathfrak{v}} f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{v}}$ by (5.8), for some $a_{\mathfrak{v}} \in K(x)$. Therefore, it follows by induction on the dominance order that if $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ then

$$B^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} = f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} + \sum_{\mathfrak{v}\blacktriangleright\mathfrak{t}} p_{\mathfrak{tv}} f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{v}} \pmod{\mathcal{H}^{\rhd\boldsymbol{\lambda}}_n},$$

for some $p_{\mathfrak{tv}} \in x^{-1}K[x^{-1}]$. As the seminormal basis is cellular, and the transition matrix between the seminormal basis and the B-basis is unitriangular, our claim now follows.

Step 2. As the Specht module $S^{\boldsymbol{\lambda}}$ is cyclic there exists an element $D^{\mathcal{O}}_{\mathfrak{t}} \in \mathcal{H}^\Lambda_n(\widehat{\mathcal{O}})$ such that $B^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} \equiv B^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} D^{\mathcal{O}}_{\mathfrak{t}} \pmod{\mathcal{H}^{\rhd\boldsymbol{\lambda}}_n}$. We claim that

$$B^{\mathcal{O}}_{\mathfrak{st}} \equiv (D^{\mathcal{O}}_{s})^\diamond B^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} D^{\mathcal{O}}_{\mathfrak{t}} \pmod{\mathcal{H}^{\rhd\boldsymbol{\lambda}}_n},$$

for all $\mathfrak{s},\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$.

To prove this claim, embed $\mathcal{H}^\Lambda_n(\widehat{\mathcal{O}})$ in $\mathcal{H}^\Lambda_n(\widehat{\mathscr{K}})$. Note that $f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} f_{\mathfrak{uv}} = 0$ if $\mathfrak{u} \neq \mathfrak{t}^{\boldsymbol{\lambda}}$, so we may assume that $D^{\mathcal{O}}_{\mathfrak{t}} \equiv \sum_{\mathfrak{v}} q_{\mathfrak{tv}} f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{v}} \pmod{\mathcal{H}^{\rhd\boldsymbol{\lambda}}_n}$, for some $q_{\mathfrak{tv}} \in \widehat{\mathscr{K}}$. Then

$$B^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}} \equiv B^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} D^{\mathcal{O}}_{\mathfrak{t}} = \sum_{\mathfrak{v}\in\operatorname{Std}(\boldsymbol{\lambda})} \gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}} q_{\mathfrak{tv}} f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{v}} \pmod{\mathcal{H}^{\rhd\boldsymbol{\lambda}}_n}.$$

Therefore, $q_{\mathfrak{tv}} = \frac{1}{\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}}} p_{\mathfrak{tv}}$, where $p_{\mathfrak{tv}} \in \delta_{\mathfrak{tv}} + x^{-1}K[x^{-1}]$ is as in Step 1. In particular, $q_{\mathfrak{tt}} = \frac{1}{\gamma_{\mathfrak{t}^{\boldsymbol{\lambda}}}}$ and $q_{\mathfrak{tv}} \neq 0$ only if $\mathfrak{v} \trianglerighteq \mathfrak{t}$. Consequently,

$$\begin{aligned}(D^{\mathcal{O}}_{s})^\diamond B^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} D^{\mathcal{O}}_{\mathfrak{t}} &\equiv \sum_{\substack{(\mathfrak{u},\mathfrak{v}) \trianglerighteq (\mathfrak{s},\mathfrak{t})\\ \mathfrak{u},\mathfrak{v}\in\operatorname{Std}(\boldsymbol{\lambda})}} q_{\mathfrak{su}} q_{\mathfrak{tv}} f_{\mathfrak{ut}^{\boldsymbol{\lambda}}} f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} f_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{v}} = \sum_{(\mathfrak{u},\mathfrak{v}) \trianglerighteq (\mathfrak{s},\mathfrak{t})} \gamma^2_{\mathfrak{t}^{\boldsymbol{\lambda}}} q_{\mathfrak{su}} q_{\mathfrak{tv}} f_{\mathfrak{uv}}\\ &= f_{\mathfrak{st}} + \sum_{(\mathfrak{u},\mathfrak{v})\blacktriangleright(\mathfrak{s},\mathfrak{t})} p_{\mathfrak{su}} p_{\mathfrak{tv}} f_{\mathfrak{uv}} \pmod{\mathcal{H}^{\rhd\boldsymbol{\lambda}}_n}.\end{aligned}$$

By construction, $(D^{\mathcal{O}}_{s})^\diamond B^{\mathcal{O}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} D^{\mathcal{O}}_{\mathfrak{t}} \in \mathcal{H}^\Lambda_n(\widehat{\mathcal{O}})$. Consequently, our claim now follows using the uniqueness property of $B^{\mathcal{O}}_{\mathfrak{st}}$ since $p_{\mathfrak{su}} p_{\mathfrak{tv}} \in x^{-1}K[x^{-1}]$ when $\mathfrak{s} \neq \mathfrak{u}$ or $\mathfrak{t} \neq \mathfrak{v}$.

Step 3. We can now verify (GC$_2$). If $h\in\mathcal{H}_n^\Lambda(\widehat{\mathcal{O}})$ then, using steps 1 and 2,

$$B^{\mathcal{O}}_{\mathfrak{st}}h\equiv(D^{\mathcal{O}}_s)^\diamond B^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}}h\equiv\sum_{\mathfrak{v}\in\operatorname{Std}(\boldsymbol{\lambda})}b_{\mathfrak{v}}(D^{\mathcal{O}}_s)^\diamond B^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{v}}\equiv\sum_{\mathfrak{v}\in\operatorname{Std}(\boldsymbol{\lambda})}b_{\mathfrak{v}}B^{\mathcal{O}}_{\mathfrak{sv}}\quad(\operatorname{mod}\mathcal{H}_n^{\rhd\boldsymbol{\lambda}}),$$

where $b_{\mathfrak{v}}$ depends only on $\mathfrak{t}$, $\mathfrak{v}$ and h and not on $\mathfrak{s}$. Hence, the B-basis satisfies all of the cellular basis axioms and the theorem is proved. □

By Theorem 6.2, if $(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ then $B^{\mathcal{O}}_{\mathfrak{st}}\in\mathcal{H}_n^\Lambda(\widehat{\mathcal{O}})$, however, our notation suggests that $B^{\mathcal{O}}_{\mathfrak{st}}\in\mathcal{H}_n^\Lambda(\mathcal{O})$, where $\mathcal{O}=K[x]_{(x)}$. The next result justifies our notation and shows that we can always work over the ring $\mathcal{O}$.

6.3. **Corollary.** *Let $\mathcal{O}=K[x]_{(x)}$. Then $\{\,B^{\mathcal{O}}_{\mathfrak{st}}\mid(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathcal{P}_n^\Lambda)\,\}$ is a graded cellular basis of $\mathcal{H}_n^\Lambda(\mathcal{O})$.*

Proof. Fix $(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathcal{P}_n^\Lambda)$. Then it is enough to prove that $B^{\mathcal{O}}_{\mathfrak{st}}\in\mathcal{H}_n^\Lambda(\mathcal{O})$. First note that by construction the $\diamond$-seminormal basis is defined over the rational function field $K(x)$, so $B^{\mathcal{O}}_{\mathfrak{st}}$ is defined over the ring $R=K(x)\cap\widehat{\mathcal{O}}$ since if $(\mathfrak{u},\mathfrak{v})\in\operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ then $p^{\mathfrak{st}}_{\mathfrak{uv}}(x^{-1})\in K[x^{-1}]\subset K(x)$ by Theorem 6.2. Every element of $K(x)$ can be written in the form $f(x)/g(x)$, for $f(x),g(x)\in K[x]$ with $\gcd(f,g)=1$. Expanding f/g into a power series, as in Lemma 6.1, it is not difficult to see that if $f/g\in\widehat{\mathcal{O}}$ then $g(0)\neq0$. Therefore, $R\subseteq\mathcal{O}$ so that $B^{\mathcal{O}}_{\mathfrak{st}}$ is defined over $\mathcal{O}$ as claimed. □

By similar arguments, $D^{\mathcal{O}}_{\mathfrak{t}}\in\mathcal{H}_n^\Lambda(\mathcal{O})$, for all $\mathfrak{t}\in\operatorname{Std}(\mathcal{P}_n^\Lambda)$.

If K is a field of characteristic zero then we can determine the degree of the polynomials $p^{\mathfrak{st}}_{\mathfrak{uv}}\neq0$, for $(\mathfrak{u},\mathfrak{v})\trianglerighteq(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathcal{P}_n^\Lambda)$.

6.4. **Proposition.** *Suppose that K is a field of characteristic zero. Suppose that $(\mathfrak{u},\mathfrak{v})\blacktriangleright(\mathfrak{s},\mathfrak{t})$ for $(\mathfrak{s},\mathfrak{t}),(\mathfrak{u},\mathfrak{v})\in\operatorname{Std}^2(\mathcal{P}_n^\Lambda)$. Then $p^{\mathfrak{st}}_{\mathfrak{uv}}(x)\in xK[x]$ and*

$$\deg p^{\mathfrak{st}}_{\mathfrak{uv}}(x)\le\tfrac12(\deg\mathfrak{u}-\deg\mathfrak{s}+\deg\mathfrak{v}-\deg\mathfrak{t}).$$

In particular, $p^{\mathfrak{st}}_{\mathfrak{uv}}(x)\neq0$ only if $\deg\mathfrak{u}+\deg\mathfrak{v}>\deg\mathfrak{s}+\deg\mathfrak{t}$.

Proof. We argue by induction on the dominance orders on $\mathcal{P}_n^\Lambda$ and $\operatorname{Std}(\mathcal{P}_n^\Lambda)$. Note that $\deg p(x)=d$ if and only if $\nu_x\big(p(x^{-1})\big)=-d$. For convenience, throughout the proof given two tableaux $\mathfrak{s},\mathfrak{u}\in\operatorname{Std}^2(\mathcal{P}_n^\Lambda)$ set $\deg(\mathfrak{s},\mathfrak{u})=\deg\mathfrak{s}-\deg\mathfrak{u}$. Therefore, the proposition is equivalent to the claim that $\nu_x\big(p^{\mathfrak{st}}_{\mathfrak{uv}}(x^{-1})\big)\ge\frac12\big(\deg(\mathfrak{s},\mathfrak{u})+\deg(\mathfrak{t},\mathfrak{v})\big)$.

Suppose first that $\boldsymbol{\lambda}=(n|0|\dots|0)$. Then $\mathfrak{s}=\mathfrak{t}^{\boldsymbol{\lambda}}=\mathfrak{t}$ and $\psi^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}=f_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$ so there is nothing to prove. Hence, we may assume that $\boldsymbol{\lambda}\neq(n|0|\dots|0)$ and that the proposition holds for all more dominant shapes.

Next, consider the case when $\mathfrak{s}=\mathfrak{t}^{\boldsymbol{\lambda}}=\mathfrak{t}$. By the proof of Lemma 5.5, if $\mathfrak{s}\in\operatorname{Std}(\mathbf{i}^{\boldsymbol{\lambda}})$ and $\mathfrak{s}\trianglerighteq\mathfrak{t}^{\boldsymbol{\lambda}}$ then $y^{\boldsymbol{\lambda}}_{\mathcal{O}}f_{\mathfrak{ss}}=u'_{\mathfrak{s}}\gamma_{\mathfrak{t}^\lambda}f_{\mathfrak{ss}}$ for some unit $u'_{\mathfrak{s}}\in\mathcal{O}^\times$. Therefore, by Lemma 5.13, there exist units $u_{\mathfrak{s}}\in\mathcal{O}^\times$ so that in $\mathcal{H}_n^\Lambda(K(x))$

$$\psi^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}=\sum_{\mathfrak{s}\,\trianglerighteq\,\mathfrak{t}^\lambda}\frac{u'_{\mathfrak{s}}\gamma_{\mathfrak{t}^\lambda}}{\gamma_{\mathfrak{s}}}f_{\mathfrak{ss}}=f_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}+\sum_{\mathfrak{s}\blacktriangleright\mathfrak{t}^\lambda}u_{\mathfrak{s}}\Phi_e(t)^{\deg(\mathfrak{t}^\lambda,\mathfrak{s})}f_{\mathfrak{ss}}.$$

Since $t=x+\xi$, the constant term of $\Phi_e(t)$ is $\Phi_e(\xi)=0$, so x divides $\Phi_e(t)$ and $\nu_x(\Phi_e(t)^{\deg(\mathfrak{t}^\lambda,\mathfrak{s})})=\deg(\mathfrak{t}^{\boldsymbol{\lambda}},\mathfrak{s})$ since the coefficient of x in $\Phi_e(t)$ is non-zero. (If K is field of positive characteristic this may not be true.) Expanding each unit $u_{\mathfrak{s}}$ into a power series, as in Lemma 6.1, the coefficient of $f_{\mathfrak{ss}}$ can be written as $b_{\mathfrak{s}}+c_{\mathfrak{s}}$ where $b_{\mathfrak{s}}\in x^{-1}K[x^{-1}]$ and $c_{\mathfrak{s}}\in\mathcal{O}$. In particular, if $b_{\mathfrak{s}}\neq0$ and $c_{\mathfrak{s}}\neq0$ then $\nu_x(c_{\mathfrak{s}})\ge0>\nu_x(b_{\mathfrak{s}})$ and $\nu_x(c_{\mathfrak{s}})>\nu_x(b_{\mathfrak{s}})\ge\deg(\mathfrak{t}^{\boldsymbol{\lambda}},\mathfrak{s})$. Pick $\mathfrak{t}$ minimal with respect to dominance such that $c_{\mathfrak{t}}\neq0$. Note that $\nu_x(c_{\mathfrak{t}})\ge\deg(\mathfrak{t}^{\boldsymbol{\lambda}},\mathfrak{t})$, with equality only if $b_{\mathfrak{t}}=0$. Using induction, replace $\psi^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$ with the element $A_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}=\psi^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}-c_{\mathfrak{t}}B^{\mathcal{O}}_{\mathfrak{tt}}$. By construction

$A_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} \in \mathcal{H}_n^\Lambda(\mathcal{O})$ and, by (5.8), the coefficient of $f_{\mathfrak{t}\mathfrak{t}}$ in $A_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$ is $b_\mathfrak{t} \in x^{-1}K[x^{-1}]$. If $(\mathfrak{u},\mathfrak{v}) \blacktriangleright (\mathfrak{t},\mathfrak{t})$ then, $f_{\mathfrak{u}\mathfrak{v}}$ appears in $B^{\mathcal{O}}_{\mathfrak{t}\mathfrak{t}}$ with coefficient $p^{\mathfrak{t}\mathfrak{t}}_{\mathfrak{u}\mathfrak{v}}(x^{-1})$ and, by induction, $\nu_x\big(p^{\mathfrak{t}\mathfrak{t}}_{\mathfrak{u}\mathfrak{v}}(x^{-1})\big) \ge \frac12\big(\deg(\mathfrak{t},\mathfrak{u})+\deg(\mathfrak{t},\mathfrak{v})\big)$. Therefore,

$$\begin{aligned}\nu_x\big(c_\mathfrak{t} p^{\mathfrak{t}\mathfrak{t}}_{\mathfrak{u}\mathfrak{v}}(x^{-1})\big) &= \nu_x(c_\mathfrak{t}) + \nu_x\big(p^{\mathfrak{t}\mathfrak{t}}_{\mathfrak{u}\mathfrak{v}}(x^{-1})\big) \ge \deg(\mathfrak{t}^\lambda,\mathfrak{t}) + \tfrac12(\deg(\mathfrak{t},\mathfrak{u})+\deg(\mathfrak{t},\mathfrak{v}))\\ &= \tfrac12\big(\deg(\mathfrak{t}^\lambda,\mathfrak{u})+\deg(\mathfrak{t}^\lambda,\mathfrak{v})\big).\end{aligned}$$

It follows that if $f_{\mathfrak{u}\mathfrak{v}}$ appears in $A_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$ with non-zero coefficient $a_{\mathfrak{u}\mathfrak{v}}$ then $\nu_x(a_{\mathfrak{u}\mathfrak{v}}) \ge \frac12\big(\deg(\mathfrak{t}^\lambda,\mathfrak{u})+\deg(\mathfrak{t}^\lambda,\mathfrak{v})\big)$. If $A_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$ now has the required properties then we can set $B_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} = A_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$. Otherwise, let $(\mathfrak{s},\mathfrak{t})$ be a pair of tableau that is minimal with respect to dominance such that the coefficient of $f_{\mathfrak{s}\mathfrak{t}}$ in $A_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$ is of the form $b_{\mathfrak{s}\mathfrak{t}}+c_{\mathfrak{s}\mathfrak{t}}$ with $c_{\mathfrak{s}\mathfrak{t}} \ne 0$, $\nu_x(c_{\mathfrak{s}\mathfrak{t}}) \ge 0$, $b_{\mathfrak{s}\mathfrak{t}} \in x^{-1}K[x^{-1}]$ and $\nu_x(b_{\mathfrak{s}\mathfrak{t}}) \ge \frac12\big(\deg(\mathfrak{t}^\lambda,\mathfrak{s})+\deg(\mathfrak{t}^\lambda,\mathfrak{t})\big)$. Replacing $A_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$ with $A_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} - c_{\mathfrak{s}\mathfrak{t}}B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}}$ and continuing in this way we will, in a finite number of steps, construct an element $B'_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$ with all of the required properties. By the uniqueness statement in Theorem 6.2, $B^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} = B'_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$ so this proves the proposition for the polynomials $p^{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}_{\mathfrak{u}\mathfrak{v}}(x^{-1})$.

Finally, suppose that $(\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\boldsymbol{\lambda})$ with $(\mathfrak{t}^\lambda,\mathfrak{t}^\lambda) \rhd (\mathfrak{s},\mathfrak{t})$. Without loss of generality, suppose that $\mathfrak{s} = \mathfrak{a}(r,r+1)$ where $\mathfrak{a} \in \operatorname{Std}(\mathbf{i})$, for $\mathbf{i} \in I^n$, and $\mathfrak{a} \rhd \mathfrak{s}$. Using Lemma 4.23,

$$\begin{aligned}\psi^{\mathcal{O}}_r B^{\mathcal{O}}_{\mathfrak{a}\mathfrak{t}} &= \sum_{(\mathfrak{u},\mathfrak{v}) \,\underline{\blacktriangleright}\, (\mathfrak{a},\mathfrak{t})} p^{\mathfrak{a}\mathfrak{t}}_{\mathfrak{u}\mathfrak{v}}(x^{-1})\psi^{\mathcal{O}}_r f_{\mathfrak{u}\mathfrak{v}}\\ &= \sum_{(\mathfrak{u},\mathfrak{v}) \,\underline{\blacktriangleright}\, (\mathfrak{a},\mathfrak{t})} p^{\mathfrak{a}\mathfrak{t}}_{\mathfrak{u}\mathfrak{v}}(x^{-1})\Big(\beta_r(\mathfrak{u})f_{\mathfrak{u}(r,r+1),\mathfrak{v}} - \delta_{i_r i_{r+1}}\frac{t^{\hat{\imath}_{r+1}-c_{r+1}(\mathfrak{u})}}{[\rho_r(\mathfrak{u})]}f_{\mathfrak{u}\mathfrak{v}}\Big).\end{aligned}$$

By induction, $\nu_x(p^{\mathfrak{a}\mathfrak{t}}_{\mathfrak{u}\mathfrak{v}}) \ge \frac12(\deg(\mathfrak{a},\mathfrak{u})+\deg(\mathfrak{t},\mathfrak{v}))$. Therefore, using Lemma 5.13 (as in the proof of Theorem 3.21), it follows that if $c_{\mathfrak{u}\mathfrak{v}} \ne 0$ is the coefficient of $f_{\mathfrak{u}\mathfrak{v}}$ in the last equation then $\nu_x(c_{\mathfrak{u}\mathfrak{v}}) \ge \frac12\big(\deg(\mathfrak{s},\mathfrak{u})+\deg(\mathfrak{t},\mathfrak{v})\big)$. Hence, the proposition follows by repeating the argument of the last paragraph. □

6.2. **A distinguished homogeneous basis of $\mathcal{H}_n^\Lambda(K)$.** This section uses Theorem 6.2 to construct a new graded cellular basis of $\mathcal{H}_n^\Lambda(K)$. The existence of such a basis is not automatically guaranteed by Theorem 6.2 because the elements $B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} \otimes 1_K$, for $(\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$, are not necessarily homogeneous.

The isomorphisms $K \cong \mathcal{O}/x\mathcal{O} \cong \widehat{\mathcal{O}}/x\widehat{\mathcal{O}}$ extend to K-algebra isomorphisms

$$\mathcal{H}_n^\Lambda(K) \cong \mathcal{H}_n^\Lambda(\mathcal{O}) \otimes_{\mathcal{O}} K \cong \mathcal{H}_n^\Lambda(\widehat{\mathcal{O}}) \otimes_{\widehat{\mathcal{O}}} 1_K.$$

We identify these three K-algebras.

6.5. **Lemma.** *Suppose that $(\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda)$. Then*

$$B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} \otimes 1_K = \psi_{\mathfrak{s}\mathfrak{t}} + \sum_{(\mathfrak{u},\mathfrak{v})\blacktriangleright(\mathfrak{s},\mathfrak{t})} a_{\mathfrak{u}\mathfrak{v}}\psi_{\mathfrak{u}\mathfrak{v}},$$

for some $a_{\mathfrak{u}\mathfrak{b}} \in K$. In particular, the homogeneous component of $B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} \otimes 1_K$ of degree $\deg\mathfrak{s}+\deg\mathfrak{t}$ is non-zero.

Proof. This is immediate from Theorem 6.2, (5.8) and Corollary 5.9. □

Recall from Step 2 in the proof of Theorem 6.2 that for each $\mathfrak{v} \in \operatorname{Std}(\boldsymbol{\lambda})$ there exists an element $D^{\mathcal{O}}_\mathfrak{v} \in \mathcal{H}_n^\Lambda(\mathcal{O})$ such that $B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} \equiv (D^{\mathcal{O}}_\mathfrak{s})^\diamond B_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} D^{\mathcal{O}}_\mathfrak{t} \pmod{\mathcal{H}_n^{\rhd\boldsymbol{\lambda}}}$.

6.6. **Definition.** *Suppose that $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$.*

a) *If $\mathfrak{v} \in \operatorname{Std}(\boldsymbol{\lambda})$ let $D_\mathfrak{v}$ be the homogeneous component of $D^{\mathcal{O}}_\mathfrak{v} \otimes 1_K$ of degree $\deg\mathfrak{v} - \deg\mathfrak{t}^\lambda$.*

b) *Define* $B_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$ *to be the homogeneous component of* $B^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}\otimes 1_K$ *of degree* $2\deg\mathfrak{t}^\lambda$. *More generally, if* $\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})$ *define* $B_{\mathfrak{s}\mathfrak{t}} = D^\diamond_{\mathfrak{s}} B_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} D_{\mathfrak{t}}$.

By Theorem 6.2, $(B^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda})^\diamond = B^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$ which implies that $B^\diamond_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} = B_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$. Consequently, if $\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})$ then $B^\diamond_{\mathfrak{s}\mathfrak{t}} = B_{\mathfrak{t}\mathfrak{s}}$. If $B_{\mathfrak{s}\mathfrak{t}}\neq 0$ then, by construction, $B_{\mathfrak{s}\mathfrak{t}}$ is homogeneous of degree $\deg\mathfrak{s}+\deg\mathfrak{t}$. Unfortunately, it is not clear from the definitions that $B_{\mathfrak{s}\mathfrak{t}}$ is non-zero.

6.7. **Proposition.** *Suppose that* $(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathcal{P}^\Lambda_n)$. *Then*

$$B_{\mathfrak{s}\mathfrak{t}} \equiv \psi_{\mathfrak{s}\mathfrak{t}} + \sum_{(\mathfrak{u},\mathfrak{v})\blacktriangleright(\mathfrak{s},\mathfrak{t})} b_{\mathfrak{u}\mathfrak{v}}\psi_{\mathfrak{u}\mathfrak{v}} \qquad (\mathit{mod}\ \mathcal{H}^{\rhd\boldsymbol{\lambda}}_n),$$

for some $b_{\mathfrak{u}\mathfrak{v}}\in K$. *In particular,* $B_{\mathfrak{s}\mathfrak{t}}\neq 0$.

Proof. Fix $\boldsymbol{\lambda}\in\mathcal{P}^\Lambda_n$ and suppose that $\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\boldsymbol{\lambda})$. If $\mathfrak{s}=\mathfrak{t}=\mathfrak{t}^{\boldsymbol{\lambda}}$ then $B_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$ is the homogeneous component of $B^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}\otimes 1_K$ of degree $2\deg\mathfrak{t}^{\boldsymbol{\lambda}}$, so the result is just Lemma 6.5 in this case. Now consider the case when $\mathfrak{s}=\mathfrak{t}^{\boldsymbol{\lambda}}$ and $\mathfrak{t}$ is an arbitrary standard $\boldsymbol{\lambda}$-tableau. Then, since $B^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}\equiv\psi^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}\ (\mathrm{mod}\ \mathcal{H}^{\rhd\boldsymbol{\lambda}}_n)$,

$$B^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}}\otimes 1_K \equiv (\psi^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}\otimes 1_K)(D^{\mathcal{O}}_{\mathfrak{t}}\otimes 1_K)\ (\mathrm{mod}\ \mathcal{H}^{\rhd\boldsymbol{\lambda}}_n).$$

Looking at the homogeneous component of degree $\deg\mathfrak{t}^{\boldsymbol{\lambda}}+\deg\mathfrak{t}$ shows that

$$B_{\mathfrak{t}^\lambda\mathfrak{t}} = B_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}D_{\mathfrak{t}} \equiv \psi_{\mathfrak{t}^\lambda\mathfrak{t}} + \sum_{\mathfrak{v}\blacktriangleright\mathfrak{t}} a_{\mathfrak{t}^\lambda\mathfrak{v}}\psi_{\mathfrak{t}^\lambda\mathfrak{v}} \qquad (\mathrm{mod}\ \mathcal{H}^{\rhd\boldsymbol{\lambda}}_n),$$

by Lemma 6.5. Set $b_{\mathfrak{t}^\lambda\mathfrak{v}} = a_{\mathfrak{t}^\lambda\mathfrak{v}}$ with $b_{\mathfrak{t}^\lambda\mathfrak{t}}=1$. Similarly,

$$D^\diamond_{\mathfrak{s}}\psi_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} \equiv D^\diamond_{\mathfrak{s}}B_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} = B_{\mathfrak{s}\mathfrak{t}^\lambda} \equiv \sum_{\mathfrak{u}\,\underline{\blacktriangleright}\,\mathfrak{s}} b_{\mathfrak{u}\mathfrak{t}^\lambda}\psi_{\mathfrak{u}\mathfrak{t}^\lambda} \qquad (\mathrm{mod}\ \mathcal{H}^{\rhd\boldsymbol{\lambda}}_n),$$

where $b_{\mathfrak{u}\mathfrak{t}^\lambda} = a_{\mathfrak{t}^\lambda\mathfrak{u}}$ with $b_{\mathfrak{s}\mathfrak{t}^\lambda}=1$. By Corollary 5.9, $\{\psi_{\mathfrak{u}\mathfrak{v}}\}$ is a graded cellular basis of $\mathcal{H}^\Lambda_n(K)$ so, working modulo $\mathcal{H}^{\rhd\boldsymbol{\lambda}}_n$,

$$\begin{aligned} B_{\mathfrak{s}\mathfrak{t}} = D^\diamond_{\mathfrak{s}}B_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}D_{\mathfrak{t}} &\equiv \sum_{\mathfrak{v}\,\underline{\blacktriangleright}\,\mathfrak{t}} b_{\mathfrak{t}^\lambda\mathfrak{v}}D^\diamond_{\mathfrak{s}}\psi_{\mathfrak{t}^\lambda\mathfrak{v}} \equiv \sum_{\mathfrak{v}\,\underline{\blacktriangleright}\,\mathfrak{t}}\sum_{\mathfrak{u}\,\underline{\blacktriangleright}\,\mathfrak{s}} b_{\mathfrak{t}^\lambda\mathfrak{v}}b_{\mathfrak{u}\mathfrak{t}^\lambda}\psi_{\mathfrak{u}\mathfrak{v}} \\ &= \psi_{\mathfrak{s}\mathfrak{t}} + \sum_{(\mathfrak{u},\mathfrak{v})\blacktriangleright(\mathfrak{s},\mathfrak{t})} b_{\mathfrak{u}\mathfrak{t}^\lambda}b_{\mathfrak{t}^\lambda\mathfrak{v}}\psi_{\mathfrak{u}\mathfrak{v}} \qquad (\mathrm{mod}\ \mathcal{H}^{\rhd\boldsymbol{\lambda}}_n). \end{aligned}$$

Setting $b_{\mathfrak{u}\mathfrak{v}} = \mathfrak{b}_{\mathfrak{s}\mathfrak{t}^\lambda}b_{\mathfrak{t}^\lambda\mathfrak{v}}$ completes the proof. □

Combining these results gives us a new graded cellular basis of $\mathcal{H}^\Lambda_n$.

6.8. **Theorem.** *Suppose that* K *is a field. Then* $\{\,B_{\mathfrak{s}\mathfrak{t}}\mid(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathcal{P}^\Lambda_n)\,\}$ *is a graded cellular basis of* $\mathcal{H}^\Lambda_n(K)$ *with cellular algebra automorphism* $\diamond$.

Proof. By Proposition 6.7 and Corollary 5.9, $\{\,B_{\mathfrak{s}\mathfrak{t}}\mid(\mathfrak{s},\mathfrak{t}\in\operatorname{Std}^2(\mathcal{P}^\Lambda_n)\,\}$ is a basis of $\mathcal{H}^\Lambda_n(K)$. By definition, if $(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathcal{P}^\Lambda_n)$ then $B_{\mathfrak{s}\mathfrak{t}}$ is homogeneous of degree $\deg\mathfrak{s}+\deg\mathfrak{t}$ and $B^\diamond_{\mathfrak{s}\mathfrak{t}} = B_{\mathfrak{t}\mathfrak{s}}$. Therefore, the basis $\{B_{\mathfrak{s}\mathfrak{t}}\}$ satisfies (GC_1), (GC_3) and (GC_d) from Definition 2.4. Finally, since $B_{\mathfrak{s}\mathfrak{t}}\equiv D^\diamond_{\mathfrak{s}}B_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}D_{\mathfrak{t}}\ (\mathrm{mod}\ \mathcal{H}^{\rhd\boldsymbol{\lambda}}_n)$, ($\mathrm{GC}_2$) follows by repeating the argument from Step 3 in the proof of Theorem 6.2. □

The graded cellular basis $\{\,B_{\mathfrak{s}\mathfrak{t}}\mid(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathcal{P}^\Lambda_n)\,\}$ of $\mathcal{H}^\Lambda_n(K)$ is distinguished in the sense that, unlike $\psi_{\mathfrak{s}\mathfrak{t}}$, the element $B_{\mathfrak{s}\mathfrak{t}}$ depends only on $(\mathfrak{s},\mathfrak{t})\in\operatorname{Std}^2(\mathcal{P}^\Lambda_n)$ and not on a choice of reduced expressions for the permutations $d(\mathfrak{s})$ and $d(\mathfrak{t})$.

6.9. **Example** We give an example to show what B-basis elements look like. Suppose that K is a field of characteristic zero, that $e>2$, and let $\Lambda = 2\Lambda_0+\Lambda_1$. Fix a multicharge $\boldsymbol{\kappa}=(\kappa_1,\kappa_2,\kappa_3)$ such that $\boldsymbol{\kappa}\equiv(0,1,0)\ (\mathrm{mod}\ e)$ and $\boldsymbol{\kappa}$ satisfies (3.2).

We use the notation of the last two sections, so we work over the rings $(\mathscr{K}, \widehat{\mathcal{O}}, K)$ and $t = x+\xi \in \mathcal{O}$.

Let $\boldsymbol{\lambda} = (1|1|1)$ and set $\mathfrak{t} = \left(\boxed{3}\middle|\boxed{2}\middle|\boxed{1}\right)$. The permutation $d(\mathfrak{t})$ has two reduced expressions: $s_1s_2s_1$ and $s_2s_1s_2$. Let $\psi_{\mathfrak{t}^\lambda\mathfrak{t}}$ and $\hat{\psi}_{\mathfrak{t}^\lambda\mathfrak{t}}$, respectively, be the ψ-basis elements corresponding to these two reduced expressions. By Definition 2.10, $\psi_1\psi_2\psi_2 e(0,1,0) = \psi_2\psi_1\psi_2 e(0,1,0) - e(0,1,0)$, so $\psi_{\mathfrak{t}^\lambda\mathfrak{t}} = \hat{\psi}_{\mathfrak{t}^\lambda\mathfrak{t}} - \psi_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} \neq \hat{\psi}_{\mathfrak{t}^\lambda\mathfrak{t}}$, where $0 \neq \psi_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} = y_1 e(0,1,0)$. The set of standard tableau with residue sequence $(0,1,0)$ and which are dominant than or equal to $\mathfrak{t}$ is $\{\mathfrak{t}^\nu, \mathfrak{t}^\nu, \mathfrak{t}^\lambda, \mathfrak{t}\}$, where $\boldsymbol{\mu} = (2|-|1)$ and $\boldsymbol{\nu} = (1|1^2|-)$. Of these tableaux, only $\mathfrak{t}$ and $\mathfrak{t}^\lambda$ have degree 1, so it follows that $B_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} = \psi_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$ and $B_{\mathfrak{t}^\lambda\mathfrak{t}} = \psi_{\mathfrak{t}^\lambda\mathfrak{t}} + c\psi_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$, for some $c \in K$. To compute c it is enough to work with the seminormal basis $\{\, f_{\mathfrak{s}} \mid \mathfrak{s} \in \operatorname{Std}(\boldsymbol{\lambda}) \,\}$ of the Specht module $S^{\boldsymbol{\lambda}}$ over $\mathcal{O}$. Using Lemma 5.1 and (5.4),

$$\psi_{\mathfrak{t}}^{\mathcal{O}} = f_{\mathfrak{t}^\lambda}\psi_1^{\mathcal{O}}\psi_2^{\mathcal{O}}\psi_1^{\mathcal{O}} = f_{\mathfrak{t}} - \frac{t^{\kappa_2-1-\kappa_3}[1+\kappa_1-\kappa_2]_t}{[\kappa_1-\kappa_3]_t} f_{\mathfrak{t}^\lambda}.$$

Since $\boldsymbol{\kappa} \equiv (0,1,0)$ we can write $1+\kappa_1-\kappa_2 = ae$ and $\kappa_1-\kappa_3 = be$, for some $a, b \in \mathbb{Z}$. Moreover, $a, b \neq 0$ by (3.2). It is straightforward to check that when $x = 0$ the coefficient of $f_{\mathfrak{t}^\lambda}$ above is equal to $-\frac{a}{b}$, so this coefficient is invertible in $\widehat{\mathcal{O}}$. Hence, $B^{\mathcal{O}}_{\mathfrak{t}^\lambda\mathfrak{t}} = f_{\mathfrak{t}^\lambda\mathfrak{t}} + c_1 f_{\mathfrak{t}^\mu\mathfrak{t}^\mu} + c_2 f_{\mathfrak{t}^\nu\mathfrak{t}^\nu} \in \mathcal{H}_n^\Lambda(\mathcal{O})$, for some $c_1, c_2 \in x^{-1}K[x^{-1}]$. Since $\deg \mathfrak{t}^\mu = \deg \mathfrak{t}^\nu = 2 > 1$, we conclude that $B_{\mathfrak{t}^\lambda\mathfrak{t}} = \psi_{\mathfrak{t}^\lambda\mathfrak{t}} + \frac{a}{b}\psi_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}$. $\diamond$

We have not yet proved Theorem B from the introduction because it is not clear that $B_{\mathfrak{s}\mathfrak{t}}$ is the homogeneous component of $B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} \otimes 1_K$ of degree $\deg \mathfrak{s} + \deg \mathfrak{t}$. In fact, there is no reason why this should be true.

As in Theorem B, suppose that K is a field of characteristic zero. Using Proposition 6.4, it follows by induction on the dominance ordering that the homogeneous components of $B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} \otimes 1_K$ have degree greater than or equal to $\deg \mathfrak{s} + \deg \mathfrak{t}$. Moreover, if $B'_{\mathfrak{s}\mathfrak{t}}$ is the homogeneous component of $B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} \otimes 1_K$ of degree $\deg \mathfrak{s} + \deg \mathfrak{t}$ then

$$B'_{\mathfrak{s}\mathfrak{t}} \equiv B_{\mathfrak{s}\mathfrak{t}} \pmod{\mathcal{H}_n^{\rhd\boldsymbol{\lambda}}}$$

by the proof of Theorem 6.2 (specifically the definition of $D^{\mathcal{O}}_{\mathfrak{s}}$ and $D^{\mathcal{O}}_{\mathfrak{t}}$). In particular, $B'_{\mathfrak{s}\mathfrak{t}} \neq 0$. As $B'_{\mathfrak{s}\mathfrak{t}}$ is the minimial homogeneous component of $B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} \otimes 1_K$, Theorem 6.2 readily implies that $\{\, B'_{\mathfrak{s}\mathfrak{t}} \mid (\mathfrak{s},\mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda) \,\}$ is a graded cellular basis of $\mathcal{H}_n^\Lambda$. Hence, all of the claims in Theorem B now follow.

If K is a field of positive characteristic it is not clear if $B^{\mathcal{O}}_{\mathfrak{s}\mathfrak{t}} \otimes 1_K$ has homogeneous components of degree less than $\deg \mathfrak{s} + \deg \mathfrak{t}$. It is precisely for this reason that we need the elements $D_{\mathfrak{s}}$ and $D_{\mathfrak{t}}$ in Definition 6.6.

Appendix A. Seminormal forms for the linear quiver

In this appendix we show how the results in this paper work when $e = 0$ so that $\xi \in K$ is either not a root of unity or $\xi = 1$ and K is a field of characteristic zero. In order to define a modular system we have to leave the case where the cyclotomic parameters $Q_1, \dots, Q_\ell$ are *integral*, that is, when $Q_l = [\kappa_l]$ for $1 \le l \le \ell$. This causes quite a few notational inconveniences, but otherwise the story is much the same as for the case when $e > 0$. We do not develop the full theory of "0-idempotent subrings" here. Rather, we show just one way of proving the results in this paper when $e = 0$.

Fix a field K and $0 \neq \xi \in K$ of quantum characteristic e. That is, either $\xi = 1$ and K is a field of characteristic zero or $\xi^d \neq 1$ for $d \in \mathbb{Z}$. The multicharge $\boldsymbol{\kappa} \in \mathbb{Z}^\ell$ is arbitrary.

Let $\mathcal{O} = \mathbb{Z}[x,\xi]_{(x)}$ be the localisation of $\mathbb{Z}[x,\xi]$ at the principal ideal generated by x. Let $\mathscr{K} = \mathbb{Q}(x,\xi)$ be the field of fractions of $\mathcal{O}$. Define $\mathcal{H}_n^\Lambda(\mathcal{O})$ to be the cyclotomic Hecke algebra of type A with Hecke parameter $t = \xi$, a unit in $\mathcal{O}$, and cyclotomic parameters

$$Q_l = x^l + [\kappa_l], \qquad \text{for } 1 \le l \le \ell,$$

where, as before, $[k] = [k]_t$ for $k \in \mathbb{Z}$. Then $\mathcal{H}_n^\Lambda(\mathscr{K}) = \mathcal{H}_n^\Lambda(\mathcal{O}) \otimes_{\mathcal{O}} \mathscr{K}$ is split semisimple in view of Ariki's semisimplicity condition [1]. Moreover, by definition, $\mathcal{H}_n^\Lambda(K) \cong \mathcal{H}_n^\Lambda(\mathcal{O}) \otimes_{\mathcal{O}} K$, where we consider K as an $\mathcal{O}$-module by setting x act on K as multiplication by zero.

Define a new **content** function for $\mathcal{H}_n^\Lambda(\mathcal{O})$ by setting

$$C_\gamma = t^{c-r}x^l + [\kappa_l + c - r],$$

for a node $\gamma = (l, r, c)$. We will also need the previous definition of contents below. If $\mathfrak{t} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$ is a tableau and $1 \le k \le n$ then set $C_k(\mathfrak{t}) = C_\gamma$, where γ is the unique node such that $\mathfrak{t}(\gamma) = k$.

As in Section 2.5, let $\{\, m_{\mathfrak{s}\mathfrak{t}} \mid (\mathfrak{s}, \mathfrak{t}) \in \operatorname{Std}^2(\mathcal{P}_n^\Lambda) \,\}$ be the Murphy basis of $\mathcal{H}_n^\Lambda(\mathcal{O})$. Then the analogue of Lemma 2.9 is that if $1 \le r \le n$ then

$$m_{\mathfrak{s}\mathfrak{t}} L_r = C_r(\mathfrak{t}) m_{\mathfrak{s}\mathfrak{t}} + \sum_{(\mathfrak{u},\mathfrak{v}) \rhd (\mathfrak{s},\mathfrak{t})} r_{\mathfrak{u}\mathfrak{v}} m_{\mathfrak{u}\mathfrak{v}},$$

for some $r_{\mathfrak{u}\mathfrak{v}} \in \mathcal{O}$. As in Section 3.1 define a $*$-seminormal basis of $\mathcal{H}_n^\Lambda(\mathscr{K})$ to be a basis $\{f_{\mathfrak{s}\mathfrak{t}}\}$ of simultaneous two-sided eigenvectors for $L_1, \dots, L_n$ such that $f_{\mathfrak{s}\mathfrak{t}}^* = f_{\mathfrak{t}\mathfrak{s}}$.

Define a **seminormal coefficient system** for $\mathcal{H}_n^\Lambda(\mathcal{O})$ to be a set of scalars $\boldsymbol{\alpha} = \{\alpha_r(\mathfrak{s})\}$ satisfying Definition 3.9(a), Definition 3.9(b) and such that if $\mathfrak{s} \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$ and $\mathfrak{u} = \mathfrak{s}(r, r+1) \in \operatorname{Std}(\mathcal{P}_n^\Lambda)$ then

$$\text{(A1)} \qquad \alpha_r(\mathfrak{s})\alpha_r(\mathfrak{u}) = \frac{(1 - C_r(\mathfrak{s}) + tC_r(\mathfrak{u}))(1 + tC_r(\mathfrak{s}) - C_r(\mathfrak{u}))}{P_r(\mathfrak{s})P_r(\mathfrak{u})},$$

where $P_r(\mathfrak{s}) = C_r(\mathfrak{u}) - C_r(\mathfrak{s})$, and where $\alpha_r(\mathfrak{s}) = 0$ if $\mathfrak{u} \notin \operatorname{Std}(\mathcal{P}_n^\Lambda)$.

As in Theorem 3.13, each seminormal basis of $\mathcal{H}_n^\Lambda(\mathscr{K})$ is determined by a seminormal coefficient system $\boldsymbol{\alpha} = \{\alpha_r(\mathfrak{s})\}$, such that

$$T_r f_{\mathfrak{s}\mathfrak{t}} = \alpha_r(\mathfrak{s}) f_{\mathfrak{u}\mathfrak{t}} + \frac{1 + (t-1)C_{r+1}(\mathfrak{s})}{P_r(\mathfrak{s})} f_{\mathfrak{s}\mathfrak{t}}, \qquad \text{where } \mathfrak{u} = \mathfrak{s}(r, r+1),$$

together with a set of scalars $\{\, \gamma_{\mathfrak{t}^\lambda} \mid \boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda \,\}$. Notice that $I = \mathbb{Z}$, since $e = 0$, so if $\mathbf{i} \in I^n$ then $\mathfrak{t} \in \operatorname{Std}(\mathbf{i})$ if and only if $c_r(\mathfrak{t}) = i_r$ and, in turn, this is equivalent to the constant term of $C_r(\mathfrak{t})$ being equal to $[i_r]$, for $1 \le r \le n$. Arguing as in Lemma 4.4,

$$f_{\mathbf{i}}^{\mathcal{O}} = \sum_{\mathfrak{t} \in \operatorname{Std}(\mathbf{i})} \frac{1}{\gamma_{\mathfrak{t}}} f_{\mathfrak{t}\mathfrak{t}} \in \mathcal{H}_n^\Lambda(\mathcal{O}).$$

With these definitions in place all of the arguments in Chapter 4 go through with only minor changes. In particular, if $1 \le r \le n$ and $\mathbf{i} \in I^n$ then Definition 4.14 should be replaced by

$$\psi_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} (T_r + 1)\frac{1}{M_r} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r = i_{r+1} \\ (T_r L_r - L_r T_r) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r = i_{r+1} + 1, \\ (T_r L_r - L_r T_r)\frac{1}{M_r} f_{\mathbf{i}}^{\mathcal{O}}, & \text{otherwise,} \end{cases}$$

and $y_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \big(L_r - C_r(\mathfrak{t})\big) f_{\mathbf{i}}^{\mathcal{O}}$ where, as before, $M_r = 1 - L_r + tL_{r+1}$. With these new definitions, if $\mathfrak{s} \in \operatorname{Std}(\mathbf{i})$, for $\mathbf{i} \in I^m$, and $1 \le r \le n$ then Lemma 4.23 becomes

$$\psi_r^{\mathcal{O}} f_{\mathfrak{s}\mathfrak{t}} = B_r(\mathfrak{s}) f_{\mathfrak{s}\mathfrak{t}} + \frac{\delta_{i_r i_{r+1}}}{P_r(\mathfrak{s})} f_{\mathfrak{u}\mathfrak{t}},$$

where $\mathfrak{u}=\mathfrak{s}(r,r+1)$ and

$$B_r(\mathfrak{s})=\begin{cases}\frac{\alpha_r(\mathfrak{s})}{1-C_r(\mathfrak{s})+tC_{r+1}(\mathfrak{s})}, & \text{if } i_r=i_{r+1},\\ \alpha_r(\mathfrak{s})P_r(\mathfrak{s}), & \text{if } i_r=i_{r+1}+1,\\ \frac{\alpha_r(\mathfrak{s})P_r(\mathfrak{s})}{1-C_r(\mathfrak{s})+tC_{r+1}(\mathfrak{s})}, & \text{otherwise.}\end{cases}$$

Observe that if $\mathfrak{u}=\mathfrak{s}(r,r+1)$ is a standard tableau then, using (A1), the definitions imply that

$$B_r(\mathfrak{s})B_r(\mathfrak{u})=\begin{cases}\frac{1}{P_r(\mathfrak{s})P_r(\mathfrak{u})}, & \text{if } i_r=i_{r+1},\\ (1-C_r(\mathfrak{s})+tC_r(\mathfrak{u}))(1+tC_r(\mathfrak{s})-C_r(\mathfrak{u})), & \text{if } i_r\leftrightarrows i_{r+1},\\ (1+tC_r(\mathfrak{s})-C_r(\mathfrak{u})), & \text{if } i_r\rightarrow i_{r+1},\\ (1-C_r(\mathfrak{s})+tC_r(\mathfrak{u})), & \text{if } i_r\leftarrow i_{r+1},\\ 1, & \text{otherwise.}\end{cases}$$

Comparing this with Lemma 4.26, it is now easy to see that analogues of Proposition 4.28 and Proposition 4.29 both hold in this situation. Hence, repeating the arguments of Section 4.4, a suitable modification of Theorem A also holds. Similarly, the construction of the bases in Chapter 5 and Chapter 6 now goes though largely without change.

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