

Weighted Sets of Probabilities and Minimax Weighted Expected Regret: New Approaches for Representing Uncertainty and Making Decisions *

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Abstract

We consider a setting where an agent’s uncertainty is represented by a set of probability measures, rather than a single measure. Measure-by-measure updating of such a set of measures upon acquiring new information is well-known to suffer from problems; agents are not always able to learn appropriately. To deal with these problems, we propose using *weighted sets of probabilities*: a representation where each measure is associated with a *weight*, which denotes its significance. We describe a natural approach to updating in such a situation and a natural approach to determining the weights. We then show how this representation can be used in decision-making, by modifying a standard approach to decision making—minimizing expected regret—to obtain *minimax weighted expected regret* (MWER). We provide an axiomatization that characterizes preferences induced by MWER both in the static and dynamic case.

1 Introduction

Agents must constantly make decisions; these decisions are typically made in a setting with uncertainty. For decisions based on the outcome of the toss of a fair coin, the uncertainty can be well characterized by probability. However, what is the probability of you getting cancer if you eat fries at every meal? What if you have salads instead? Even experts would not agree on a single probability.

Representing uncertainty by a single probability measure and making decisions by maximizing expected utility leads to further problems. Consider the following stylized problem, which serves as a running example in this paper.

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	1 broken	10 broken
<i>cont</i>	10,000	-10,000
<i>back</i>	0	0
<i>check</i>	5,001	-4,999

Table 1: Payoffs for the robot delivery problem. Acts are in the leftmost column. The remaining two columns describe the outcome for the two sets of states that matter.

The baker’s delivery robot, T-800, is delivering 1,000 cupcakes from the bakery to a banquet. Along the way, T-800 takes a tumble down a flight of stairs and breaks some of the cupcakes. The robot’s map indicates that this flight of stairs must be either ten feet or fifteen feet high. For simplicity, assume that a fall of ten feet results in one broken cupcake, while a fall of fifteen feet results in ten broken cupcakes.

T-800’s choices and their consequences are summarized in Table 1. Decision theorists typically model decision problems with states, acts, and outcomes: the world is in one of many possible states, and the decision maker chooses an *act*, a function mapping states to outcomes. A natural state space in this problem is $\{good, broken\}^{1000}$, where each state is a possible state of the cupcakes. However, all that matters about the state is the number of broken cakes, so we can further restrict to states with either one or ten broken cakes.

T-800 can choose among three acts: *cont*: continue the delivery attempt; *back*: go back for new cupcakes; or *check*: open the container and count the number of broken cupcakes, and then decide to continue or go back, depending on the number of broken cakes. The client will tolerate one broken cupcake, but not ten broken cupcakes. Therefore, if T-800 chooses *cont*, it obtains a utility of 10,000 if there is only one broken cake, but a utility of $-10,000$ if there are ten broken cakes. If T-800 chooses to go *back*, then it gets a utility of 0. Finally, checking the cupcakes costs 4,999 units of utility but is reliable, so if T-800 chooses *check*, it ends up with a utility of 5,001 if there is one broken cake, and a utility of $-4,999$ if there are ten broken cakes.

If we try to maximize expected utility, we must assume some probability over states. What measure should be used? There are two hypotheses that T-800 entertains: (1) the stairs are ten feet high and (2) the stairs are fifteen feet high. Each of these places a different probability on states. If the stairs are ten feet high, we can take all of the 1,000 states where there is exactly one broken cake to be equally probable, and take the remaining states to have probability 0; if the stairs are fifteen feet high, we can take all of the $C(1000, 10)$ states where there are exactly ten broken cakes to be equally probable, and take the remaining states to have probability 0. One way to model T-800’s uncertainty about the height of the stairs is to take each hypothesis to be equally likely. However, not having any idea about which hypothesis holds is very different from believing that all hypotheses are equally likely. It is easy to check that taking each hypothesis to be equally likely makes *check* the act that maximizes

utility, but taking the probability that the stairs are fifteen feet high to be .51 makes *back* the act that maximizes expected utility, and taking the probability that the stairs are ten feet high to be .51 makes *cont* the act that maximizes expected utility. What makes any of these choices the “right” choice?

It is easy to construct many other examples where a single probability measure does not capture uncertainty, and does not result in what seem to be reasonable decisions, when combined with expected utility maximization. A natural alternative, which has often been considered in the literature, is to represent the agent’s uncertainty by a *set* of probability measures. For example, in the delivery problem, the agent’s beliefs could be represented by two probability measures, Pr_1 and Pr_{10} , one for each hypothesis. Thus, Pr_1 assigns uniform probability to all states with exactly one broken cake, and Pr_{10} assigns uniform probability to all states with exactly ten broken cakes.

But this representation also has problems. Consider the delivery example again. Why should T-800 be sure that there is exactly either one broken cake or ten broken cakes? Of course, we can replace these two hypotheses by hypotheses that say that the probability of a cake being broken is either .001 or .01, but this doesn’t solve the problem. Why should the agent be sure that the probability is either exactly .001 or exactly .01? Couldn’t it also be .0999? Representing uncertainty by a set of measures still places a sharp boundary on what measures are considered possible and impossible.

A second problem involves updating beliefs. How should beliefs be updated if they are represented by a set of probability measures? The standard approach for updating a single measure is by conditioning. The natural extension of conditioning to sets of measure is measure-by-measure updating: conditioning each measure on the information (and also removing measures that give the information probability 0).

However, measure-by-measure updating can produce some rather counterintuitive outcomes. In the delivery example, suppose that a passer-by tells T-800 the information *E*: the first 100 cupcakes are good. Assuming that the passer-by told the truth, intuition tells us that there is now more reason to believe that there is only one broken cupcake.

However, $\text{Pr}_1 \mid E$ places uniform probability on all states where the first 100 cakes are good, and there is exactly one broken cake among the last 900. Similarly, $\text{Pr}_{10} \mid E$ places uniform probability on all states where the first 100 cakes are good, and there are exactly ten broken cakes among the last 900. $\text{Pr}_1 \mid E$ still places probability 1 on there being one broken cake, just like Pr_1 , $\text{Pr}_{10} \mid E$ still places probability 1 on there being ten broken cakes. There is no way to capture the fact that T-800 now views the hypothesis Pr_{10} as less likely, even if the passer-by had said instead that the first 990 cakes are all good!

Of course, both of these problems would be alleviated if we placed a probability on hypotheses, but, as we have already observed, this leads to other problems. In this paper, we propose an intermediate approach: representing uncertainty using *weighted sets of probabilities*. That is, each probability measure is associated with a weight. These weights can be viewed as probabilities; indeed, if the set of probabilities is finite, we can normalize them so that they

are effectively probabilities. Moreover, in one important setting, we update them in the same way that we would update probabilities, using likelihood (see below). On the other hand, these weights do not act like probabilities if the set of probabilities is infinite. For example, if we had a countable set of hypotheses, we could assign them all weight 1 (so that, intuitively, they are all viewed as equally likely), but there is no uniform measure on a countable set.

More importantly, when it comes to decision making, we use the weights quite differently from how we would use second-order probabilities on probabilities. Second-order probabilities would let us define a probability on events (by taking expectation) and maximize expected utility, in the usual way. Using the weights, we instead define a novel decision rule, *minimax weighted expected regret* (MWER), that has some rather nice properties, which we believe will make it widely applicable in practice. If all the weights are 1, then MWER is just the standard *minimax expected regret* (MER) rule (described below). If the set of probabilities is a singleton, then MWER agrees with (subjective) expected utility maximization (SEU). More interestingly perhaps, if the weighted set of measures converges to a single measure (which will happen in one important special case, discussed below), MWER converges to SEU. Thus, the weights give us a smooth, natural way of interpolating between MER and SEU.

In summary, weighted sets of probabilities allow us to represent ambiguity (uncertainty about the correct probability distribution). Real individuals are sensitive to this ambiguity when making decisions, and the MWER decision rule takes this into account. Updating the weighted sets of probabilities using likelihood allows the initial ambiguity to be resolved as more information about the true distribution is obtained.

We now briefly explain MWER, by first discussing MER. MER is a probabilistic variant of the minimax regret decision rule proposed by Niehans [13] and Savage [17]. Most likely, at some point, we’ve second-guessed ourselves and thought “had I known this, I would have done that instead”. That is, in hindsight, we regret not choosing the act that turned out to be optimal for the realized state, called the *ex post* optimal act. The *regret* of an act a in a state s is the difference (in utility) between the *ex post* optimal act in s and a . Of course, typically one does not know the true state at the time of decision. Therefore the regret of an act is the worst-case regret, taken over all states. The *minimax regret* rule orders acts by their regret.

The definition of regret applies if there is no probability on states. If an agent’s uncertainty is represented by a single probability measure, then we can compute the *expected regret* of an act a : just multiply the regret of an act a at a state s by the probability of s , and then sum. It is well known that the order on acts induced by minimizing expected regret is identical to that induced by maximizing expected utility (see [8] for a proof). If an agent’s uncertainty is represented by a set \mathcal{P} of probabilities, then we can compute the expected regret of an act a with respect to each probability measure $\text{Pr} \in \mathcal{P}$, and then take the worst-case expected regret. The MER (Minimax Expected Regret) rule orders acts according to their worst-case expected regret, preferring the act that minimizes the worst-case regret. If the set of measures is the set of *all* probability

measures on states, then it is not hard to show that MER induces the same order on acts as (probability-free) minimax regret. Thus, MER generalizes both minimax regret (if \mathcal{P} consists of all measures) and expected utility maximization (if \mathcal{P} consists of a single measure).

MWER further generalizes MER. If we start with a *weighted* set of measures, then we can compute the weighted expected regret for each one (just multiply the expected regret with respect to \Pr by the weight of \Pr) and compare acts by their worst-case weighted expected regret.

Sarver [16] also proves a representation theorem that involves putting a multiplicative weight on a regret quantity. However, his representation is fundamentally different from MWER. In his representation, regret is a factor only when comparing two *sets* of acts; the ranking of individual acts is given by expected utility maximization. By way of contrast, we do not compare sets of acts.

It is standard in decision theory to axiomatize a decision rule by means of a representation theorem. For example, Savage [18] showed that if an agent's preferences \succeq satisfied several axioms, such as completeness and transitivity, then the agent is behaving as if she is maximizing expected utility with respect to some utility function and probabilistic belief.

If uncertainty is represented by a set of probability measures, then we can generalize expected utility maximization to *maximin expected utility (MMEU)*. MMEU compares acts by their worst-case expected utility, taken over all measures. MMEU has been axiomatized by Gilboa and Schmeidler [7]. MER was axiomatized by Hayashi [8] and Stoye [20]. We provide an axiomatization of MWER. We make use of ideas introduced by Stoye [20] in his axiomatization of MER, but the extension seems quite nontrivial.

We also consider a dynamic setting, where beliefs are updated by new information. If observations are generated according to a probability measure that is stable over time, then, as we suggested above, there is a natural way of updating the weights given observations, using ideas of likelihood. The idea is straightforward. After receiving some information E , we update each probability $\Pr \in \mathcal{P}$ to $\Pr \mid E$, and take its weight to be $\alpha_{\Pr} = \Pr(E) / \sup_{\Pr' \in \mathcal{P}} \Pr'(E)$. If more than one $\Pr \in \mathcal{P}$ gets updated to the same $\Pr \mid E$, the sup of all such weights is used. Thus, the weight of \Pr after observing E is modified by taking into account the likelihood of observing E assuming that \Pr is the true probability. We refer to this method of updating weights as *likelihood updating*.

If observations are generated by a stable measure (e.g., we observe the outcomes of repeated flips of a biased coin) then, as the agent makes more and more observations, the weighted set of probabilities of the agent will, almost surely, look more and more like a single measure. The weight of the measures in \mathcal{P} closest to the measure generating the observations converges to 1, and the weight of all other measures converges to 0. This would not be the case if uncertainty were represented by a set of probability measures and we did measure-by-measure updating, as is standard. As we mentioned above, this means that MWER converges to SEU.

We provide an axiomatization for dynamic MWER with likelihood updat-

ing. We remark that a dynamic version of MMEU with measure-by-measure updating has been axiomatized by Jaffray [10], Pires [14], and Siniscalchi [19].

Likelihood updating is somewhat similar in spirit to an updating method implicitly proposed by Epstein and Schneider [5]. They also represented uncertainty by using (unweighted) sets of probability measures. They choose a threshold α with $0 < \alpha < 1$, update by conditioning, and eliminate all measures whose relative likelihood does not exceed the threshold. This approach also has the property that, over time, all that is left in \mathcal{P} are the measures closest to the measure generating the observations; all other measures are eliminated. However, it has the drawback that it introduces a new, somewhat arbitrary, parameter α .

Chateauneuf and Faro [2] also consider weighted sets of probabilities (they model the weights using what they call *confidence functions*), although they impose more constraints on the weights than we do. They then define and provide a representation of a generalization of MMEU using weighted sets of probabilities that parallels our generalization of MER. Chateauneuf and Faro do not discuss the dynamic situation; specifically, they do not consider how weights should be updated in the light of new information.

The rest of this paper is organized as follows. Section 2 introduces the weighted sets of probabilities representation, and Section 3 introduces the MWER decision rule. Axiomatic characterizations of static and dynamic MWER are provided in Sections 4 and 5, respectively. We conclude in Section 7.

2 Weighted Sets of Probabilities

A set \mathcal{P}^+ of *weighted probability measures* on a set S consists of pairs (\Pr, α_{\Pr}) , where $\alpha_{\Pr} \in [0, 1]$ and \Pr is a probability measure on S .¹ Let $\mathcal{P} = \{\Pr : \exists \alpha(\Pr, \alpha) \in \mathcal{P}^+\}$. We assume that, for each $\Pr \in \mathcal{P}$, there is exactly one α such that $(\Pr, \alpha) \in \mathcal{P}^+$. We denote this number by α_{\Pr} , and view it as the *weight of* \Pr . We further assume for convenience that weights have been normalized so that there is at least one measure $\Pr \in \mathcal{P}$ such that $\alpha_{\Pr} = 1$.² We remark that, just as we do, Chateauneuf and Faro [2] take weights to be in the interval $[0, 1]$. They impose additional requirements on the weights. For example, they require that the weight of a convex combination of two probability measures is at least as high as the weight of each one. This does not seem reasonable in our applications. For example, an agent may know that one of two measures is generating his observations, and give them both weight 1, while giving all other distributions weight 0.

¹In this paper, for ease of exposition, we take the state space S to be finite, and assume that all sets are measurable. We can easily generalize to arbitrary measure spaces.

²While we could take weights to be probabilities, and normalize them so that they sum to 1, if \mathcal{P} is finite, this runs into difficulties if we have an infinite number of measures in \mathcal{P} . For example, if we are tossing a coin, and \mathcal{P} includes all probabilities on heads from $1/3$ to $2/3$, using a uniform probability, we would be forced to assign each individual probability measure a weight of 0, which would not work well in the definition of MWER.

As we observed in the introduction, one way of updating weighted sets of probabilities is by using likelihood updating. We use $\mathcal{P}^+ \mid E$ to denote the result of applying likelihood updating to \mathcal{P}^+ . Define $\overline{\mathcal{P}}^+(E) = \sup\{\alpha_{\text{Pr}} \text{Pr}(E) : \text{Pr} \in \mathcal{P}\}$; if $\overline{\mathcal{P}}^+(E) > 0$, set $\alpha_{\text{Pr},E} = \sup\{\text{Pr}' \in \mathcal{P} : \text{Pr}'|E = \text{Pr}|E\} \frac{\alpha_{\text{Pr}} \text{Pr}(E)}{\overline{\mathcal{P}}^+(E)}$. Note that given a measure $\text{Pr} \in \mathcal{P}$, there may be several distinct measures Pr' in \mathcal{P} such that $\text{Pr}' \mid E = \text{Pr} \mid E$. Thus, we take the weight of $\text{Pr} \mid E$ to be the sup of the possible candidate values of $\alpha_{\text{Pr},E}$. By dividing by $\overline{\mathcal{P}}^+(E)$, we guarantee that $\alpha_{\text{Pr},E} \in [0, 1]$, and that there is some measure Pr such that $\alpha_{\text{Pr},E} = 1$, as long as there is some pair $(\alpha_{\text{Pr}}, \text{Pr}) \in \mathcal{P}$ such that $\alpha_{\text{Pr}} \text{Pr}(E) = \overline{\mathcal{P}}^+(E)$. If $\overline{\mathcal{P}}^+(E) > 0$, we take $\mathcal{P}^+ \mid E$ to be

$$\{(\text{Pr} \mid E, \alpha_{\text{Pr},E}) : \text{Pr} \in \mathcal{P}\}.$$

If $\overline{\mathcal{P}}^+(E) = 0$, then $\mathcal{P}^+ \mid E$ is undefined.

In computing $\mathcal{P}^+ \mid E$, we update not just the probability measures in \mathcal{P} , but also their weights. The new weight combines the old weight with the likelihood. Clearly, if all measures in \mathcal{P} assign the same probability to the event E , then likelihood updating and measure-by-measure updating coincide. This is not surprising, since such an observation E does not give us information about the relative likelihood of measures. We stress that using likelihood updating is appropriate only if the measure generating the observations is assumed to be stable. For example, if observations of heads and tails are generated by coin tosses, and a coin of possibly different bias is tossed in each round, then likelihood updating would not be appropriate.

It is well known that, when conditioning on a single probability measure, the order that information is acquired is irrelevant; the same observation easily extends to sets of probability measures. As we now show, it can be further extended to weighted sets of probability measures.

Proposition 1. *Likelihood updating is consistent in the sense that for all $E_1, E_2 \subseteq S$, $(\mathcal{P}^+ \mid E_1) \mid E_2 = (\mathcal{P}^+ \mid E_2) \mid E_1 = \mathcal{P}^+ \mid (E_1 \cap E_2)$, provided that $\mathcal{P}^+ \mid (E_1 \cap E_2)$ is defined.*

Proof. By standard results, $(\text{Pr} \mid E_1) \mid E_2 = (\text{Pr} \mid E_2) \mid E_1 = \text{Pr} \mid (E_1 \cap E_2)$. Since the weight of the measure $\text{Pr} \mid E_1$ is proportional to $\alpha_{\text{Pr}} \text{Pr}(E_1)$, the weight of $(\text{Pr} \mid E_1) \mid E_2$ is proportional to $\alpha_{\text{Pr}} \text{Pr}(E_1) \text{Pr}(E_2 \mid E_1) = \alpha_{\text{Pr}} \text{Pr}(E_1 \cap E_2)$. Likewise, the weight of $(\text{Pr} \mid E_2) \mid E_1$ is proportional to $\alpha_{\text{Pr}} \text{Pr}(E_2) \text{Pr}(E_1 \mid E_2) = \alpha_{\text{Pr}} \text{Pr}(E_1 \cap E_2)$. Since, in all these cases, the sup of the weights is normalized to 1, the weights of corresponding measures in $\mathcal{P}^+ \mid (E_1 \cap E_2)$, $(\mathcal{P}^+ \mid E_1) \mid E_2$ and $(\mathcal{P}^+ \mid E_2) \mid E_1$ must be equal. \square

3 MWER

We now define MWER formally. Given a set S of states and a set X of outcomes, an *act* f (over S and X) is a function mapping S to X . For simplicity in this paper, we take S to be finite. Associated with each outcome $x \in X$ is a utility:

$u(x)$ is the utility of outcome x . We call a tuple (S, X, u) a *(non-probabilistic) decision problem*. To define regret, we need to assume that we are also given a set $M \subseteq X^S$ of feasible acts, called the *menu*. The reason for the menu is that, as is well known (and we will demonstrate by example shortly), regret can depend on the menu. Moreover, we assume that every menu M has utilities bounded from above. That is, we assume that for all menus M , $\sup_{g \in M} u(g(s))$ is finite. This ensures that the regret of each act is well defined.³ For a menu M and act $f \in M$, the regret of f with respect to M and decision problem (S, X, u) in state s is

$$\text{reg}_M(f, s) = \left(\sup_{g \in M} u(g(s)) \right) - u(f(s)).$$

That is, the regret of f in state s (relative to menu M) is the difference between $u(f(s))$ and the highest utility possible in state s (among all the acts in M). The regret of f with respect to M and decision problem (S, X, u) is the worst-case regret over all states:

$$\max_{s \in S} \text{reg}_M(f, s).$$

We denote this as $\text{reg}_M^{(S, X, u)}(f)$, and usually omit the superscript (S, X, u) if it is clear from context. If there is a probability measure \Pr over the states, then we can consider the *probabilistic decision problem* (S, X, u, \Pr) . The *expected regret* of f with respect to M is

$$\text{reg}_{M, \Pr}(f) = \sum_{s \in S} \Pr(s) \text{reg}_M(f, s).$$

If there is a set \mathcal{P} of probability measures over the states, then we consider the \mathcal{P} -decision problem (S, X, u, \mathcal{P}) . The maximum expected regret of $f \in M$ with respect to M and (S, X, u, \mathcal{P}) is

$$\text{reg}_{M, \mathcal{P}}(f) = \sup_{\Pr \in \mathcal{P}} \left(\sum_{s \in S} \Pr(s) \text{reg}_M(f, s) \right).$$

Finally, if beliefs are modeled by weighted probabilities \mathcal{P}^+ , then we consider the \mathcal{P}^+ -decision problem (S, X, u, \mathcal{P}^+) . The maximum weighted expected regret of $f \in M$ with respect to M and (S, X, u, \mathcal{P}^+) is

$$\text{reg}_{M, \mathcal{P}^+}(f) = \sup_{\Pr \in \mathcal{P}} \left(\alpha_{\Pr} \sum_{s \in S} \Pr(s) \text{reg}_M(f, s) \right).$$

The MER decision rule is thus defined for all $f, g \in X^S$ as

$$f \succeq_{M, \mathcal{P}}^{S, X, u} g \text{ iff } \text{reg}_{M, \mathcal{P}}^{(S, X, u)}(f) \leq \text{reg}_{M, \mathcal{P}}^{(S, X, u)}(g).$$

³ Stoye [21] assumes that, for each menu M , there is a finite set A_M of acts such that M consists of all the convex combinations of the acts in A_M . Our assumption is clearly much weaker than Stoye's.

	1 broken cake		10 broken cakes	
	Payoff	Regret	Payoff	Regret
<i>cont</i>	10,000	0	-10,000	10,000
<i>back</i>	0	10,000	0	0
<i>check</i>	5,001	4,999	-4,999	4,999

Table 2: Payoffs and regrets for delivery example.

	1 broken cake		10 broken cakes	
	Payoff	Regret	Payoff	Regret
<i>cont</i>	10,000	10,000	-10,000	10,000
<i>back</i>	0	20,000	0	0
<i>check</i>	5,001	14,999	-4,999	4,999
<i>new</i>	20,000	0	-20,000	20,000

Table 3: Payoffs and regrets for the delivery problem with a new choice added.

That is, f is preferred to g if the maximum expected regret of f is less than that of g . We can similarly define $\succeq_{M,reg}$, $\succeq_{M,Pr}^{S,X,u}$, and $\succeq_{M,\mathcal{P}^+}^{S,X,u}$ by replacing $reg_{M,\mathcal{P}}^{(S,X,u)}$ by $reg_M^{(S,X,u)}$, $reg_{M,Pr}^{(S,X,u)}$, and $reg_{M,\mathcal{P}^+}^{(S,X,u)}$, respectively. Again, we usually omit the superscript (S, X, u) and subscript Pr or \mathcal{P}^+ , and just write \succeq_M , if it is clear from context.

To see how these definitions work, consider the delivery example from the introduction. There are 1,000 states with one broken cake, and $C(1000, 10)$ states with ten broken cakes. The regret of each action in a state depends only on the number of broken cakes, and is given in Table 2. It is easy to see that the action that minimizes regret is *check*, with *cont* and *back* having equal regret. If we represent uncertainty using the two probability measures Pr_1 and Pr_{10} , the expected regret of each of the acts with respect to Pr_1 (resp., Pr_{10}) is just its regret with respect to states with one (resp. ten) broken cakes. Thus, the action that minimizes maximum expected regret is again *check*.

As we said above, the ranking of acts based on MER or MWER can change if the menu of possible choices changes. For example, suppose that we introduce a new choice in the delivery problem, whose gains and losses are twice those of *cont*, resulting in the payoffs and regrets described in Table 3. In this new setting, *cont* has a lower maximum expected regret (10,000) than *check* (14,999), so MER prefers *cont* over *check*. Thus, the introduction of a new choice can affect the relative order of acts according to MER (and MWER), even though other acts are preferred to the new choice. By way of contrast, the decision rules MMEU and SEU are *menu-independent*; the relative order of acts according to MMEU and SEU is not affected by the addition of new acts.

We next consider a dynamic situation, where the agent acquires information. Specifically, in the context of the delivery problem, suppose that T-800 learns E —the first 100 items are good. Initially, suppose that T-800

has no reason to believe that one hypothesis is more likely than the other, so assigns both hypotheses weight 1. Note that $P_1(E) = 0.9$ and $\Pr_{10}(E) = C(900, 10)/C(1000, 10) \approx 0.35$. Thus,

$$\mathcal{P}^+ \mid E = \{(\Pr_1 \mid E, 1), (\Pr_{10} \mid E, C(900, 10)/(0.9C(1000, 10)))\}.$$

We can also see from this example that MWER interpolates between MER and expected utility maximization. Suppose that a passer-by tells T-800 that the first N cupcakes are good. If $N = 0$, MWER with initial weights 1 is the same as MER. On the other hand, if $N \geq 991$, then the likelihood of \Pr_{10} is 0, and the only measure that has effect is \Pr_1 , which means minimizing maximum weighted expected regret is just maximizing expected utility with respect to \Pr_1 . If $0 < N < 991$, then the likelihoods (hence weights) of \Pr_1 and \Pr_{10} are 1 and $\frac{C(1000-N, 10)}{C(1000, 10)} \times \frac{1000}{1000-N} < ((999 - N)/999)^9$. Thus, as N increases, the weight of \Pr_{10} goes to 0, while the weight of \Pr_1 stays at 1.

4 An axiomatic characterization of MWER

We now provide a representation theorem for MWER. That is, we provide a collection of properties (i.e., axioms) that hold of MWER such that a preference order on acts that satisfies these properties can be viewed as arising from MWER. To get such an axiomatic characterization, we restrict to what is known in the literature as the *Anscombe-Aumann* (AA) framework [1], where outcomes are restricted to lotteries. This framework is standard in the decision theory literature; axiomatic characterizations of SEU [1], MMEU [7], and MER [8, 20] have already been obtained in the AA framework. We draw on these results to obtain our axiomatization.

Given a set Y (which we view as consisting of *prizes*), a *lottery* over Y is just a probability with finite support on Y . Let $\Delta(Y)$ consist of all finite probabilities over Y . In the AA framework, the set of outcomes has the form $\Delta(Y)$. So now acts are functions from S to $\Delta(Y)$. (Such acts are sometimes called *Anscombe-Aumann acts*.) We can think of a lottery as modeling objective uncertainty, while a probability on states models subjective uncertainty; thus, in the AA framework we have both objective and subjective uncertainty. The technical advantage of considering such a set of outcomes is that we can consider convex combinations of acts. If f and g are acts, define the act $\alpha f + (1 - \alpha)g$ to be the act that maps a state s to the lottery $\alpha f(s) + (1 - \alpha)g(s)$.

In this setting, we assume that there is a utility function U on prizes in Y . The utility of a lottery l is just the expected utility of the prizes obtained, that is,

$$u(l) = \sum_{\{y \in Y: l(y) > 0\}} l(y)U(y).$$

This makes sense since $l(y)$ is the probability of getting prize y if lottery l is played. The expected utility of an act f with respect to a probability \Pr is then

just $u(f) = \sum_{s \in S} \Pr(s)u(f(s))$, as usual. We also assume that there are at least two prizes y_1 and y_2 in Y , with different utilities $U(y_1)$ and $U(y_2)$.

Given a set Y of prizes, a utility U on prizes, a state space S , and a set \mathcal{P}^+ of weighted probabilities on S , we can define a family $\succeq_{M, \mathcal{P}^+}^{S, \Delta(Y), u}$ of preference orders on Anscombe-Aumann acts determined by weighted regret, one per menu M , as discussed above, where u is the utility function on lotteries determined by U . For ease of exposition, we usually write $\succeq_{M, \mathcal{P}^+}^{S, Y, U}$ rather than $\succeq_{M, \mathcal{P}^+}^{S, \Delta(Y), u}$.

We state the axioms in a way that lets us clearly distinguish the axioms for SEU, MMEU, MER, and MWER. The axioms are universally quantified over acts f, g , and h , menus M and M' , and $p \in (0, 1)$. We assume that $f, g \in M$ when we write $f \succeq_M g$.⁴ We use l^* to denote a constant act that maps all states to l .

Axiom 1. (*Transitivity*) $f \succeq_M g \succeq_M h \Rightarrow f \succeq_M h$.

Axiom 2. (*Completeness*) $f \succeq_M g$ or $g \succeq_M f$.

Axiom 3. (*Nontriviality*) $f \succ_M g$ for some acts f and g and menu M .

Axiom 4. (*Monotonicity*) If $(f(s))^* \succeq_{\{(f(s))^*, (g(s))^*\}} (g(s))^*$ for all $s \in S$, then $f \succeq_M g$.

Axiom 5. (*Mixture Continuity*) If $f \succ_M g \succ_M h$, then there exist $q, r \in (0, 1)$ such that

$$qf + (1 - q)h \succ_{M \cup \{qf + (1 - q)h\}} g \succ_{M \cup \{rf + (1 - r)h\}} rf + (1 - r)h.$$

Menu-independent versions of Axioms 1–5 are standard. Clearly (menu-independent versions of) Axioms 1, 2, 4, and 5 hold for MMEU, MER, and SEU; Axiom 3 is assumed in all the standard axiomatizations, and is used to get a unique representation.

Axiom 6. (*Ambiguity Aversion*)

$$f \sim_M g \Rightarrow pf + (1 - p)g \succeq_{M \cup \{pf + (1 - p)g\}} g.$$

Ambiguity Aversion says that the decision maker weakly prefers to hedge her bets. It also holds for MMEU, MER, and SEU, and is assumed in the axiomatizations for MMEU and MER. It is not assumed for the axiomatization of SEU, since it follows from the Independence axiom, discussed next. Independence also holds for MWER, provided that we are careful about the menus involved. Given a menu M and an act h , let $pM + (1 - p)h$ be the menu $\{pf + (1 - p)h : p \in M\}$.

⁴Stoye [21] assumed that menus were convex, so that if $f, g \in M$, then so is $pf + (1 - p)g$. We do not make this assumption, although our results would still hold if we did (with the axioms slightly modified to ensure that menus are convex). While it may seem reasonable to think that, if f and g are feasible for an agent, then so is $pf + (1 - p)g$, this is not always the case. For example, it may be difficult for the agent to randomize, or it may be infeasible for the agent to randomize with probability p for some choices of p (e.g., for p irrational).

Axiom 7. (*Independence*)

$$f \succeq_M g \text{ iff } pf + (1-p)h \succeq_{pM+(1-p)h} pg + (1-p)h.$$

Independence holds in a strong sense for SEU, since we can ignore the menus. The menu-independent version of Independence is easily seen to imply Ambiguity Aversion. Independence does not hold for MMEU.

Although we have menu independence for SEU and MMEU, we do not have it for MER or MWER. The following two axioms are weakened versions of menu independence that do hold for MER and MWER.

Axiom 8. (*Menu independence for constant acts*) If l^* and $(l')^*$ are constant acts, then $l^* \succeq_M (l')^*$ iff $l^* \succeq_{M'} (l')^*$.

In light of this axiom, when comparing constant acts, we omit the menu.

An act h is *never strictly optimal relative to M* if, for all states $s \in S$, there is some $f \in M$ such that $(f(s))^* \succeq (h(s))^*$.

Axiom 9. (*Independence of Never Strictly Optimal Alternatives (INA)*) If every act in M' is *never strictly optimal relative to M* , then $f \succeq_M g$ iff $f \succeq_{M \cup M'} g$.

Axiom 10. (*Boundedness of menus*) For every menu M , there exists a lottery $\bar{l} \in \Delta(Y)$ such that for all $f \in M$ and $s \in S$, $(f(s))^* \preceq \bar{l}^*$.

The boundedness axiom enforces the assumption that we made earlier that every menu has utilities that are bounded from above. Recall that this assumption is necessary for regret to be finite.

We now present our representation theorem for MWER. Roughly, the representation theorem states that a family of preferences satisfies Axioms 1–10 if and only if it has a MWER representation with respect to some utility function and weighted probabilities. In the representation theorem for SEU [1], not only is the utility function unique (up to affine transformations, so that we can replace U by $aU + b$, where $a > 0$ and b are constants), but the probability is unique as well. Similarly, in the MMEU representation theorem of Gilboa and Schmeidler [7], the utility function is unique, and the set of probabilities is also unique, as long as one assume that the set is convex and closed.

To get uniqueness in the representation theorem for MWER, we need to consider a different representation of weighted probabilities. Define a *sub-probability measure* \mathbf{p} on S to be like a probability measure (i.e., a function mapping measurable subsets of S to $[0, 1]$ such that $\mathbf{p}(T \cup T') = \mathbf{p}(T) + \mathbf{p}(T')$ for disjoint sets T and T'), without the requirement that $\mathbf{p} = 1$. We can identify a weighted probability distribution (\Pr, α) with the sub-probability measure $\alpha \Pr$. (Note that given a sub-probability measure \mathbf{p} , there is a unique pair (α, \Pr) such that $\mathbf{P} = \alpha \Pr$: we simply take $\alpha = \mathbf{p}(S)$ and $\Pr = \mathbf{p}/\alpha$.) A set C of sub-probability measures is *downward-closed* if, whenever $\mathbf{p} \in C$ and $\mathbf{q} \leq \mathbf{p}$, then $\mathbf{q} \in C$. We get a unique set of sub-probability measures in our representation theorem if we restrict to sets that are convex, downward-closed, closed, and contain at least one (proper) probability measure. (The latter requirement corresponds to

having $\alpha_{Pr} = 1$ for some $Pr \in \mathcal{P}^+$.) For convenience, we will call a set *regular* if it is convex, downward-closed, and closed.

We identify each set of weighted probabilities \mathcal{P}^+ with the set of sub-probability measures

$$C(\mathcal{P}^+) = \{\alpha Pr : (Pr, \alpha_{Pr}) \in \mathcal{P}^+, 0 \leq \alpha \leq \alpha_{Pr}\}.$$

Note that if $(\alpha, Pr) \in \mathcal{P}^+$, then $C(\mathcal{P}^+)$ includes all the sub-probability measures between the all-zero measure and $\alpha_{Pr} Pr$.

We need to restrict to closed and convex sets of sub-probability measures to get uniqueness in the representation of MWER for much the same reason that we need to restrict to closed and convex sets to get uniqueness in the representation of MMEU. To see why convexity is needed, consider the delivery example and the expected regrets in Table 2, and the distribution $a Pr_1 + (1-a) Pr_{10}$, for some $a \in (0, 1)$. The weighted expected regret of any act with respect to $a Pr_1 + (1-a) Pr_{10}$ is bounded above by the maximum weighted expected regret of that act with respect to Pr_1 and Pr_{10} . Therefore, adding $a Pr_1 + (1-a) Pr_{10}$ to \mathcal{P}^+ for some weight $a \in (0, 1)$ does not change the resulting family of preferences. Similarly, we need to restrict to closed sets for uniqueness, since if we start with a set C of sub-probability measures that is not closed, taking the closure of C would result in the same family of preferences.

While convexity is easy to define for a set of sub-probability measures, there seems to be no natural notion of convexity for a set \mathcal{P}^+ of weighted probabilities. Moreover, the requirement that \mathcal{P}^+ is closed is different from the requirement that $C(\mathcal{P}^+)$ is closed. The latter requirement seems more reasonable. For example, fix a probability measure Pr , and let $\mathcal{P}^+ = \{(1, Pr)\} \cup \{(0, Pr') : Pr' \neq Pr\}$. Thus, \mathcal{P}^+ essentially consists of a single probability measure, namely Pr , with weight 1; all the weighted probability measures $(0, Pr')$ have no impact. This represents the uncertainty of an agent who is sure that that Pr is true probability. Clearly \mathcal{P}^+ is not closed, since we can find a sequence Pr_n such that $(0, Pr_n) \rightarrow (0, Pr)$, although $(0, Pr) \notin \mathcal{P}^+$. But $C(\mathcal{P}^+)$ is closed.

Restricting to closed, convex sets of sub-probability measures does not suffice to get uniqueness; we also need to require downward-closedness. This is so because if \mathbf{p} is in C , then adding any $\mathbf{q} \leq \mathbf{p}$ to the set leaves all regrets unchanged. Finally, the presence of a proper probability measure is also required, since for any $a \in (0, 1]$, scaling each element in the set C by a leaves the family of preferences unchanged.

In summary, if we consider arbitrary sets of sub-probability measures, then the set of sub-probability measures that represent a given family of MWER preferences would be unique if we required the set to be regular and contain a probability measure.

Theorem 1. *For all Y , U , S , and \mathcal{P}^+ , the family of preference orders $\succeq_{M, \mathcal{P}^+}^{S, Y, U}$ satisfies Axioms 1–10. Conversely, if a family of preference orders \succeq_M on the acts in $\Delta(Y)^S$ satisfies Axioms 1–10, then there exist a utility U on Y and a weighted set \mathcal{P}^+ of probabilities on S such that $C(\mathcal{P}^+)$ is regular and $\succeq_M = \succeq_{M, \mathcal{P}^+}^{S, Y, U}$. Moreover, U is unique up to affine transformations, and $C(\mathcal{P}^+)$*

is unique in the sense that if \mathcal{Q}^+ represents \succeq_M , and $C(\mathcal{Q}^+)$ is regular, then $C(\mathcal{Q}^+) = C(\mathcal{P}^+)$.

Showing that $\succeq_{M, \mathcal{P}^+}^{S, Y, U}$ satisfies Axioms 1–10 is fairly straightforward; we leave details to the reader. The proof of the converse is quite nontrivial, although it follows the lines of the proof of other representation theorems. We provide an outline of the proof here; details can be found in the appendix.

Using standard techniques, we can show that the axioms guarantee the existence of a utility function U on prizes that can be extended to lotteries in the obvious way, so that $l^* \succeq (l')^*$ iff $U(l) \geq U(l')$. We then use techniques of Stoye [21] to show that it suffices to get a representation theorem for a single menu, rather than all menus: the menu consisting of all acts f such that $U(f(s)) \leq 0$ for all states $s \in S$. This allows us to use techniques in the spirit of those used by Gilboa and Schmeidler [6] to represent (unweighted) MMEU. However, there are technical difficulties that arise from the fact that we do not have a key axiom that is satisfied by MMEU: C-independence (discussed below). The heart of the proof involves dealing with the lack of C-independence; as we said, the details can be found in the appendix.

It is instructive to compare Theorem 1 to other representation results in the literature. Anscombe and Aumann [1] showed that the menu-independent versions of axioms 1–5 and 7 characterize SEU. The presence of Axiom 7 (menu-independent Independence) greatly simplifies things. Gilboa and Schmeidler [7] showed that axioms 1–6 together with one more axiom that they call *Certainty-independence* characterizes MMEU. Certainty-independence, or C-independence for short, is a weakening of independence (which, as we observed, does not hold for MMEU), where the act h is required to be a constant act. Since MMEU is menu-independent, we state it in a menu-independent way.

Axiom 11. (*C-Independence*) If h is a constant act, then $f \succeq g$ iff $pf + (1 - p)h \succeq pg + (1 - p)h$.

As we observed, in general, we have Ambiguity Aversion (Axiom 6) for regret. *Betweenness* [3] is a stronger notion than ambiguity aversion, which states that if an agent is indifferent between two acts, then he must also be indifferent among all convex combinations of these acts. While betweenness does not hold for regret, Stoye [20] gives a weaker version that does hold. A menu M has *state-independent outcome distributions* if the set $L(s) = \{y \in \Delta(Y) : \exists f \in M, f(s) = y\}$ is the same for all states s .

Axiom 12. If h is a constant act, and M has state-independent outcome distributions, then

$$h \sim_M f \Rightarrow pf + (1 - p)h \sim_{M \cup \{pf + (1 - p)h\}} f.$$

The assumption that the menu has state-independent outcome distributions is critical in Axiom 12.

Stoye [20] shows that Axioms 1–9 together with Axiom 12 characterize MER.⁵ Non-probabilistic regret (which we denote REG) can be viewed as a

⁵Stoye actually worked with choice correspondences; see Section 7.

	1 broken cake		10 broken cakes	
	Payoff	Regret	Payoff	Regret
<i>cont</i>	10,000	0	-10,000	10,000
$\frac{1}{2}cont + \frac{1}{2}back$	5,000	5,000	-5,000	5,000
<i>back</i>	0	10,000	0	0
<i>check</i>	5,001	4,999	-4,999	4,999

Table 4: Payoffs and regrets for the delivery problem, with *cont* mixed with the constant act *back*.

	1 broken cake		10 broken cakes	
	Payoff	Regret	Payoff	Regret
<i>cont</i>	10,000	0	-10,000	20,000
$\frac{1}{2}cont + \frac{1}{2}back$	5,000	5,000	-5,000	15,000
<i>back</i>	0	10,000	0	10,000
<i>check1</i>	-5,000	15,000	5,000	5,000
<i>check2</i>	-10,000	20,000	10,000	0

Table 5: Payoffs and regrets for the delivery problem, with state-independent outcome distributions.

special case of MER, where \mathcal{P} consists of all distributions. This means that it satisfies all the axioms that MER satisfies. As Stoye [21] shows, REG is characterized by Axioms 1–9 and one additional axiom, which he calls Symmetry. We omit the details here.

The assumption that the menu has state-independent outcome distributions is critical in Axiom 12. For example, suppose that we change the payoffs in the delivery problem so that *cont* has the same maximum expected regret as *back* (10,000). However, as seen in Table 4, $\frac{1}{2}cont + \frac{1}{2}back$ has lower maximum expected regret (5,000) than *cont* (10,000), showing that the variant of Axiom 12 without the state-independent outcome distribution requirement does not hold.

Although Axiom 12 is sound for unweighted minimax expected regret, it is no longer sound once we add weights. For example, suppose that we modified the delivery problem so that all states we care about have the same outcome distributions, as required by Axiom 12. Then the payoffs and regrets will be those shown in Table 5. Suppose that the weights on Pr_1 and Pr_{10} are 1 and 0.5, respectively. Then *cont* has the same maximum weighted expected regret as *back* (10,000). However, $\frac{1}{2}cont + \frac{1}{2}back$ has lower maximum weighted expected regret (7,500) than *cont*, showing that Axiom 12 with weighted probabilities does not hold.

Table 6 describes the relationship between the axioms characterizing the decision rules.

	SEU	REG	MER	MWER	MMEU
Ax. 1-6,8-10	✓	✓	✓	✓	✓
Ind	✓	✓	✓	✓	
C-Ind	✓				✓
Ax. 12	✓	✓	✓		
Symmetry	✓	✓			

Table 6: Characterizing axioms for several decision rules.

5 Characterizing MWER with Likelihood Updating

We next consider a more dynamic setting, where agents learn information. For simplicity, we assume that the information is always a subset E of the state space. If the agent is representing her uncertainty using a set \mathcal{P}^+ of weighted probability measures, then we would expect her to update \mathcal{P}^+ to some new set \mathcal{Q}^+ of weighted probability measures, and then apply MWER with uncertainty represented by \mathcal{Q}^+ . In this section, we characterize what happens in the special case that the agent uses likelihood updating, so that $\mathcal{Q}^+ = (\mathcal{P}^+ \mid E)$.

For this characterization, we assume that the agent has a family of preference orders $\succeq_{E,M}$ indexed not just by the menu M , but by the information E . Each preference order $\succeq_{E,M}$ satisfies Axioms 1–10, since the agent makes decisions after learning E using MWER. Somewhat surprisingly, all we need is one extra axiom for the characterization; we call this axiom MDC, for ‘menu-dependent dynamic consistency’.

To explain the axiom, we need some notation. As usual, we take fEh to be the act that agrees with f on E and with h off of E ; that is

$$fEh(s) = \begin{cases} f(s) & \text{if } s \in E \\ h(s) & \text{if } s \notin E. \end{cases}$$

In the delivery example, the act *check* can be thought of as $(cont)E(back)$, where E is the set of states where there is only one broken cake.

Roughly speaking, MDC says that you prefer f to g once you learn E if and only if, for any act h , you also prefer fEh to gEh before you learn anything. This seems reasonable, since learning that the true state was in E is conceptually similar to knowing that none of your choices matter off of E .

To state MDC formally, we need to be careful about the menus involved. Let $MEh = \{fEh : f \in M\}$. We can identify unconditional preferences with preferences conditional on S ; that is, we identify \succeq_M with $\succeq_{S,M}$. We also need to restrict the sets E to which MDC applies. Recall that conditioning using likelihood updating is undefined for an event such that $\overline{\mathcal{P}}^+(E) = 0$. That is, $\alpha_{Pr} \Pr(E) = 0$ for all $Pr \in \mathcal{P}$. As is commonly done, we capture the idea that conditioning on E is possible using the notion of a *non-null* event.

Definition 1. An event E is null if, for all $f, g \in \Delta(Y)^S$ and menus M with

$fEg, g \in M$, we have $fEg \sim_M g$.

MDC. For all non-null events E , $f \succeq_{E,M} g$ iff $fEh \succeq_{MEh} gEh$ for some $h \in M$.⁶

The key feature of MDC is that it allows us to reduce all the conditional preference orders $\succeq_{E,M}$ to the unconditional order \succeq_M , to which we can apply Theorem 1.

Theorem 2. For all Y, U, S , and \mathcal{P}^+ , the family of preference orders $\succeq_{\mathcal{P}^+|E,M}^{S,Y,U}$ for events E such that $\overline{\mathcal{P}}^+(E) > 0$ satisfies Axioms 1–10 and MDC. Conversely, if a family of preference orders $\succeq_{E,M}$ on the acts in $\Delta(Y)^S$ satisfies Axioms 1–10 and MDC, then there exists a utility U on Y and a weighted set \mathcal{P}^+ of probabilities on S such that $C(\mathcal{P}^+)$ is regular, and for all non-null E , $\succeq_{E,M} = \succeq_{\mathcal{P}^+|E,M}^{S,Y,U}$. Moreover, U is unique up to affine transformations, and $C(\mathcal{P}^+)$ is unique in the sense that if \mathcal{Q}^+ represents $\succeq_{E,M}$, and $C(\mathcal{Q}^+)$ is regular, then $C(\mathcal{Q}^+) = C(\mathcal{P}^+)$.

Proof. Since $\succeq_M = \succeq_{S,M}$ satisfies Axioms 1–10, there must exist a weighted set \mathcal{P}^+ of probabilities on S and a utility function U such that $f \succeq_M g$ iff $f \succeq_{M,\mathcal{P}^+}^{S,Y,U} g$. We now show that if E is non-null, then $\overline{\mathcal{P}}^+(E) > 0$, and $f \succeq_{E,M} g$ iff $f \succeq_{M,\mathcal{P}^+|E}^{(S,X,u)} g$.

For the first part, it clearly is equivalent to show that if $\overline{\mathcal{P}}^+(E) = 0$, then E is null. So suppose that $\overline{\mathcal{P}}^+(E) = 0$. Then $\alpha_{\text{Pr}} \Pr(E) = 0$ for all $\text{Pr} \in \mathcal{P}$. This means that $\alpha_{\text{Pr}} \Pr(s) = 0$ for all $\text{Pr} \in \mathcal{P}$ and $s \in E$. Thus, for all acts f and g ,

$$\begin{aligned} & \text{reg}_{M,\mathcal{P}^+}(fEg) \\ &= \sup_{\text{Pr} \in \mathcal{P}} (\alpha_{\text{Pr}} \sum_{s \in S} \Pr(s) \text{reg}_M(fEg, s)) \\ &= \sup_{\text{Pr} \in \mathcal{P}} (\alpha_{\text{Pr}} (\sum_{s \in E} \Pr(s) \text{reg}_M(f, s)) \\ & \quad + \sum_{s \in E^c} \Pr(s) \text{reg}_M(g, s)) \\ &= \sup_{\text{Pr} \in \mathcal{P}} (\alpha_{\text{Pr}} \sum_{s \in S} \Pr(s) \text{reg}_M(g, s)) \\ &= \text{reg}_{M,\mathcal{P}^+}(g). \end{aligned}$$

Thus, $fEg \sim_M g$ for all acts f, g and menus M containing fEg and g , which means that E is null.

For the second part, we first show that if $\overline{\mathcal{P}}^+(E) > 0$, then for all $f, h \in M$, we have that

$$\text{reg}_{MEh,\mathcal{P}^+}(fEh) = \overline{\mathcal{P}}^+(E) \text{reg}_{M,\mathcal{P}^+|E}(f).$$

⁶Although we do not need this fact, it is worth noting that the MWER decision rule has the property that $fEh \succeq_{MEh} gEh$ for some act h iff $fEh \succeq_{MEh} gEh$ for all acts h . Thus, this property follows from Axioms 1–10.

We proceed as follows:

$$\begin{aligned}
& reg_{MEh, \mathcal{P}^+}(fEh) \\
&= \sup_{Pr \in \mathcal{P}} (\alpha_{Pr} \sum_{s \in S} Pr(s) reg_{MEh}(fEH, s)) \\
&= \sup_{Pr \in \mathcal{P}} (\alpha_{Pr} Pr(E) \sum_{s \in E} Pr(s | E) reg_M(f, s) \\
&\quad + \alpha_{Pr} \sum_{s \in E^c} Pr(s) reg_{\{h\}}(h, s)) \\
&= \sup_{Pr \in \mathcal{P}} (\alpha_{Pr} Pr(E) \sum_{s \in E} Pr(s|E) reg_M(s, f)) \\
&= \sup_{Pr \in \mathcal{P}} \left(\overline{\mathcal{P}}^+(E) \alpha_{Pr, E} \sum_{s \in E} Pr(s|E) reg_M(f, s) \right) \\
&\quad [\text{since } \alpha_{Pr, E} = \sup_{\{Pr' \in \mathcal{P} : Pr'|E = Pr|E\}} \frac{\alpha_{Pr'} Pr'(E)}{\overline{\mathcal{P}}^+(E)}] \\
&= \overline{\mathcal{P}}^+(E) \cdot reg_{M, \mathcal{P}^+|E}(f).
\end{aligned}$$

Thus, for all $h \in M$,

$$\begin{aligned}
& reg_{MEh, \mathcal{P}^+}(fEh) \leq reg_{MEh, \mathcal{P}^+}(gEh) \\
&\text{iff } \overline{\mathcal{P}}^+(E) \cdot reg_{M, \mathcal{P}^+|E}(f) \leq \overline{\mathcal{P}}^+(E) \cdot reg_{M, \mathcal{P}^+|E}(g) \\
&\text{iff } reg_{M, \mathcal{P}^+|E}(f) \leq reg_{M, \mathcal{P}^+|E}(g).
\end{aligned}$$

It follows that the order induced by \mathcal{P}^+ satisfies MDC.

Moreover, if 1–10 and MDC hold, then for a weighted set \mathcal{P}^+ that represents \succeq_M , we have

$$\begin{aligned}
& f \succeq_{E, M} g \\
&\text{iff } \text{for some } h \in M, fEh \succeq_{MEh} gEh \\
&\text{iff } reg_{M, \mathcal{P}^+|E}(f) \leq reg_{M, \mathcal{P}^+|E}(g),
\end{aligned}$$

as desired.

Finally, the uniqueness of $C(\mathcal{P}^+)$ follows from Theorem 1, which says that the family $\succeq_{S, M}$ of preferences is already sufficient to guarantee the uniqueness of $C(\mathcal{P}^+)$. \square

Analogues of MDC have appeared in the literature before in the context of updating preference orders. In particular, Epstein and Schneider [4] discuss a menu-independent version of MDC, although they do not characterize updating in their framework. Sinischalchi [19] also uses an analogue of MDC in his axiomatization of measure-by-measure updating of MMEU. Like us, he starts with an axiomatization for unconditional preferences, and adds an axiom called *constant-act dynamic consistency* (CDC), somewhat analogous to MDC, to extend the axiomatization of MMEU to deal with conditional preferences.

6 Dynamic Inconsistency

There is an important issue when one attempts to apply MWER with likelihood updating to dynamic decision problems. If you want to execute a plan, at every step you'll need to stick with that plan and execute the corresponding part of the plan. However, after following the initial steps of an ex-ante optimal plan,

a MWER agent may no longer wish to adhere to the plan. In such a situation, the agent is said to have *dynamically inconsistent* preferences. Dynamic inconsistency is well known to hold for regret. Indeed, as Epstein and Le Breton [4] show, dynamic inconsistency arises for any non-Bayesian approach to decision making (i.e., any approach that does not involve maximizing expected utility) that satisfies certain minimal assumptions. Not surprisingly, it arises for MWER as well. In the rest of this section, we discuss the problem and some standard approaches to dealing with it, and illustrate some subtleties that arise in dealing with it in the context of MWER.

To understand the problem in the context of regret, consider the two-stage decision problem of having dinner, represented as a decision tree in Figure 1. Solid circles denote decision points, and empty circles denote points where nature reveals information to the agent. The decision tree also includes information about what states are considered possible at each node. The set of states considered possible at the root is always the entire state space, and nature’s actions at each nature decision point partitions the set of possible states.

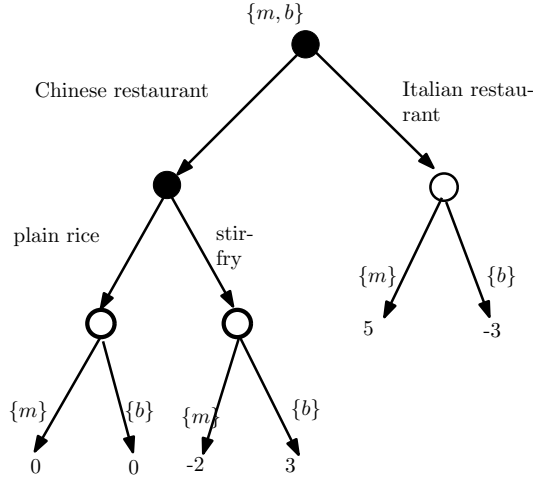


Figure 1: Dynamic inconsistency example.

First, you have to choose between a Chinese restaurant and an Italian restaurant. Once you have arrived at a particular restaurant, you cannot change your mind and go to another; so in the second stage you must order something from the menu at the chosen restaurant. Your utility is a combination of how much you enjoy the food, and whether you get an allergic reaction. Initially, you know that there are two possible states: you must be either allergic to MSG (state m) or to basil (state b), but not both. Assume that all Italian foods will have traces of basil, and Chinese stir-fry has MSG but plain rice does not. However, you do not enjoy eating plain rice, so the utility of ordering rice is 0.

Suppose that you make decisions using the minimax regret decision rule, viewing a plan (i.e., a strategy) as an act. A straightforward computation

shows that, *ex ante*, going to the Chinese restaurant and ordering plain rice has the lowest regret (5). However, if you go to the Chinese restaurant, the choice of going to the Italian restaurant is now irrelevant. If we now compute regret with respect to the menu of the two remaining choices, then the regret of ordering stir-fry is lower (2) than that of ordering rice (3). You thus end up ordering the stir-fry. The plan of going to the Chinese restaurant and ordering plain rice cannot be carried out.

More generally, dynamic consistency requires that the plan considered optimal *ex ante* continues to be considered optimal at any later time. As we said earlier, Epstein and Le Breton [4] show that dynamic inconsistency will arise for essentially all non-Bayesian decision rules. A standard approach for dealing with this lack of dynamic consistency in the literature is to consider ‘sophisticated’ agents, who are aware of the potential for dynamic inconsistency, and thus use backward induction to determine the feasible plans. In the restaurant example, a sophisticated agent believes correctly that she will prefer stir-fry over rice, once she is at the Chinese restaurant. Therefore, she no longer considers going to the Chinese restaurant and ordering plain rice a viable plan. The only feasible options are going to the Italian restaurant, or having stir-fry at the Chinese restaurant.

A subtlety arises when trying to apply backward induction to menu-dependent decision rules: which menu do we use when comparing the viable plans? For example, in the restaurant example, do we use the menu consisting of all three plans, or the menu consisting of just the viable plans. It turns out not to matter in this example—with respect to both menus, going to the Italian restaurant minimizes regret. However, in general, the choice of menu can matter. Hayashi [9] uses the menu of viable plans in computing for minimax expected regret agents in optimal stopping problems, but it seems to us that both choices (and perhaps others) can be justified.

A second subtlety that arises when considering sophisticated agents: What choice do they make when they are indifferent between two plans? Siniscalchi [19] axiomatizes *consistent planning* (with menu-independent preferences over plans), which augments backward induction with a tie-breaking assumption. This tie-breaking assumption in consistent planning allows an agent to commit to a plan as long as each stage of the plan is considered to be one of the best at each local decision node.

In order to axiomatize consistent planning, Siniscalchi must assume that the agent has preferences that are more general than preferences over plans. Rather, the agent must be assumed to have preferences over *decision trees* (such as that in Figure 1). Plans are the special case of decision trees with no branching at decision nodes; we can identify a decision tree with a set of plans (essentially, the branches in the decision tree). Sophistication is captured by an axiom that says, roughly, that the agent is indifferent between a decision tree and the same tree with a non-optimal (based on backward-induction) plan removed. Preferences over decision trees are similar in spirit to preferences over menus [11].

If we try to apply Siniscalchi’s approach to regret, we encounter further difficulties. In a menu-independent setting, we can compare two decision trees

by comparing the best plans in each decision tree (if we identify a decision tree with a set of plans). But once menus become relevant, we must decide what menu to use when making this comparison. It is not clear which menu to choose. What we really have here are menus over menus; it is not even clear how to apply regret in this setting. Defining and axiomatizing consistent planning in a regret-based setting remains an open problem.

7 Conclusion

We proposed an alternative belief representation using *weighted sets of probabilities*, and described a natural approach to updating in such a situation and a natural approach to determining the weights. We also showed how weighted sets of probabilities can be combined with regret to obtain a decision rule, MWER, and provided an axiomatization that characterizes static and dynamic preferences induced by MWER.

We have considered preferences indexed by menus here. Stoye [21] used a different framework: *choice functions*. A choice function maps every finite set M of acts to a subset M' of M . Intuitively, the set M' consists of the ‘best’ acts in M . Thus, a choice function gives less information than a preference order; it gives only the top elements of the preference order. The motivation for working with choice functions is that an agent can reveal his most preferred acts by choosing them when the menu is offered. In a menu-independent setting, the agent can reveal his whole preference order; to decide if $f \succ g$, it suffices to present the agent with a choice among $\{f, g\}$. However, with regret-based choices, the menu matters; the agent’s most preferred choice(s) when presented with $\{f, g\}$ might no longer be the most preferred choice(s) when presented with a larger menu. Thus, a whole preference order is arguably not meaningful with regret-based choices. Stoye [21] provides a representation theorem for MER where the axioms are described in terms of choice functions. The axioms that we have attributed to Stoye are actually the menu-based analogue of his axioms. We believe that it should be possible to provide a characterization of MWER using choice functions, although we have not yet proved this.

Finally, we briefly considered the issue of dynamic consistency and consistent planning. As we showed, making this precise in the context of regret involves a number of subtleties. We hope to return to this issue in future work.

A Proof of Theorem 1

We show here that if a family of menu-dependent preferences \succeq_M satisfies axioms 1-10, then \succeq_M can be represented as minimizing expected regret with respect to a set of weighted probabilities and a utility function. Since the proof is somewhat lengthy and complicated, we split it into several steps, each in a separate subsection.

A.1 Simplifying the Problem

Our proof starts in much the same way as the proof by Stoye [21] of a representation theorem for regret. Lemma 1 guarantees the existence of a utility function U on prizes that can be extended to lotteries in the obvious way, so that $l^* \succeq (l')^*$ iff $U(l) \geq U(l')$. In other words, preferences over all constant acts are represented by the maximization of U on the corresponding lotteries that the constant acts map to. Lemma 1 is a consequence of standard results. Our menus are arbitrary sets of acts, as opposed to convex hulls of a finite number of acts in [21]; Lemma 3 shows that Stoye's technique can be adapted to work when menus are arbitrary sets of acts. Finally, following Stoye [21], we reduce the proof of existence of a minimax weighted regret representation for the family \succeq_M to the proof of existence of a minimax weighted regret representation for a single menu-independent preference ordering \succeq (Lemma 4).

Lemma 1. *If Axioms 1-3, 5, 7, and 8 hold, then there exists a nonconstant function $U : X \rightarrow \mathbb{R}$, unique up to positive affine transformations, such that for all constant acts l^* and $(l')^*$ and menus M ,*

$$l^* \succeq_M (l')^* \Leftrightarrow \sum_{\{y: l^*(y) > 0\}} l(y)U(y) \geq \sum_{\{y: l'(y) > 0\}} l'(y)U(y).$$

Proof. By menu independence for constant acts, the family of preferences \succeq_M all agree when restricted to constant acts. The lemma then follows from standard results (see, e.g., [12]), since menu-independence for constant acts, combined with independence, gives the standard independence (substitution) axiom from expected utility theory. \square

As is commonly done, given U , we define $u(l) = \sum_{\{y: l(y) > 0\}} l(y)U(y)$. Thus, $u(l)$ is the expected utility of lottery l . We extend u to constant acts by taking $u(l^*) = u(l)$. Thus, Lemma 1 says that, for all menus M , $l^* \succeq (l')^*$ iff $u(l^*) \geq u(l')$. If c is the utility of some lottery, let l_c^* be a constant lottery that $u(l_c^*) = c$. The following is now immediate. We state it as a lemma so that we can refer to it later.

Lemma 2. *$u(l_c^*) \geq u(l_{c'}^*)$ iff $l_c^* \succeq l_{c'}^*$; similarly, $u(l_c^*) = u(l_{c'}^*)$ iff $l_c^* \sim l_{c'}^*$, and $u(l_c^*) > u(l_{c'}^*)$ iff $l_c^* \succ l_{c'}^*$.*

The key step in showing that we can reduce to a single menu is to show that, roughly speaking, for each menu, there exists a menu-dependent function g_M such that $u(g_M(s)) = -\sup_{f \in M} u(f(s))$. Stoye [21] proved a similar result, but he assumed that all menus were obtained by taking the convex hull of a finite set of acts. Because we allow arbitrary bounded menus, this result is not quite true for us. For example, suppose that the range of u is $(-1, \infty]$. Then there may be a menu M such that $\sup_{f \in M} u(f(s)) = 5$, so $-\sup_{f \in M} u(f(s)) = -5$. But there is no act g such that $u(g(s)) = -5$, since u is bounded below by -1 . The following weakening of this result suffices for our purpose.

Lemma 3. *There exists a utility function U such that for every menu M , there exists $\epsilon \in (0, 1]$ and constant act l^* such that for all $f, g \in M$, $f \succeq_M g \Leftrightarrow t(f) \succeq_{t(M)} t(g)$, where t has the form $t(f) = \epsilon f + (1 - \epsilon)l^*$ and $t(M) = \{t(f) : f \in M\}$. Moreover, there exists an act $g_{t(M)}$ such that $u(g_{t(M)}(s)) = -\sup_{f \in t(M)} u(f(s))$ for all $s \in S$.*

Proof. The nontriviality and monotonicity axioms imply there must exist prizes x and y such that $U(x) > U(y)$. We consider four cases.

Case 1: The range of U is bounded above and below. Then we can rescale so that the range of U is $[-1, 1]$. Thus, there must be prizes x and y such that $U(x) = 1$ and $U(y) = -1$. For all $c \in [-1, 1]$, there must be a prize x' that is a convex combination of x and y such that $u(x') = c$, so we can clearly define a function g_M such that, for all $s \in S$, we have $u(g_M(s)) = -\sup_{f \in M} u(f(s))$. Furthermore, we know that such a g_M exists because it can be formed as an act which maps each state to an appropriate lottery over the prizes x and y . More generally, we know that an act with a certain utility profile exists if its utility for each state is within the range of U . This fact will be used in the other cases as well.

Thus, in this case we can take t to be the identity (i.e., $\epsilon = 1$).

Case 2: The range of U is $(-\infty, \infty)$. Again, for all $c \in (-\infty, \infty)$, there must exist a prize x such that $u(x) = c$. Since menus are assumed to be bounded above, we can again define the required function g and take $\epsilon = 1$.

Case 3: The range of U is bounded above and unbounded below. Then we can assume without loss of generality that the range is $(-\infty, 1]$, and for all c in the range, there is a prize x such that $u(x) = c$. For all menus M , $\epsilon > 0$, and acts $f, g \in M$, by Independence, we have that

$$f \succeq_M g \Leftrightarrow \epsilon f + (1 - \epsilon)l_1^* \succeq_{\epsilon M + (1 - \epsilon)l_1^*} \epsilon g + (1 - \epsilon)l_1^*.$$

There exists an $\epsilon > 0$ such that for all $s \in S$,

$$1 \geq \sup_{f \in M} \epsilon u(f(s)) + (1 - \epsilon) \geq -1.$$

Let $t(f) = \epsilon f + (1 - \epsilon)l_1^*$. Clearly there exists an act $g_{t(M)}$ such that $u(g_{t(M)}(s)) = -\sup_{f \in t(M)} u(f(s))$ for all $s \in S$.

Case 4: The range of U is bounded below and unbounded above. By the upper-boundedness axiom, every menu has an upper bound on its utility range. Therefore, for every menu M , $\epsilon > 0$, and all acts f and g in M , by Independence,

$$f \succeq_M g \Leftrightarrow \epsilon f + (1 - \epsilon)l_{-1}^* \succeq_{\epsilon M + (1 - \epsilon)l_{-1}^*} \epsilon g + (1 - \epsilon)l_{-1}^*.$$

There exists $\epsilon > 0$ such that for all $s \in S$,

$$\sup_{f \in M} \epsilon u(f(s)) + (1 - \epsilon)u(l_{-1}^*(s)) \leq 1.$$

Let $t(f) = \epsilon f + (1 - \epsilon)l_{-1}^*$. Again, it is easy to see that $g_{t(M)}$ exists. \square

In light of Lemma 3, we henceforth assume that the utility function u derived from U is such that its range is either $(-\infty, \infty)$, $[-1, 1]$, $(-\infty, 1]$, or $[-1, \infty)$. In any case, its *range always includes* $[-1, 1]$.

Before proving the key lemma, we establish some useful notation for acts and utility acts. Given a utility act b , let f_b , the act corresponding to b , be the act such that $f_b(s) = l_{b(s)}$, if such an act exists. Conversely, let b_f , the utility act corresponding to the act f , be defined by taking $b_f(s) = u(f(s))$. Note that monotonicity implies that if $f_b = g_b$, then $f \sim_M g$ for all menus M . That is, only utility acts matter. If c is a real, we take c^* to be the constant utility act such that $c^*(s) = c$ for all $s \in S$.

Lemma 4. *Let M^* be the menu consisting of all acts f such that $(-1)^* \leq b_f \leq 0^*$. Then (U, \mathcal{P}^+) represents \succeq_{M^*} (i.e., $\succeq_{M^*} = \succeq_{M^*, \mathcal{P}^+}^{S, X, U}$) iff (U, \mathcal{P}^+) represents \succeq_M for all menus M .*

Proof. Our arguments are similar in spirit to those of Stoye [21].

By Lemma 3, there exists t such that $t(f) = \epsilon f + (1 - \epsilon)h$ for a constant function h such that

$$f \succeq_M g \text{ iff } t(f) \succeq_{t(M)} t(g);$$

moreover, for this choice of t , the act $g_{t(M)}$ defined in Lemma 3 exists.

By Independence,

$$t(f) \succeq_{t(M)} t(g) \text{ iff } \frac{1}{2}t(f) + \frac{1}{2}g_{t(M)} \succeq_{\frac{1}{2}t(M) + \frac{1}{2}g_{t(M)}} \frac{1}{2}t(g) + \frac{1}{2}g_{t(M)}.$$

Let M^* be the menu that contains all acts with utilities in $[-1, 0]$. By INA, we know that for all acts f and g , and menus M for which g_M is defined, we have

$$f \succeq_M g \text{ iff } \frac{1}{2}f + \frac{1}{2}g_M \succeq_{M^*} \frac{1}{2}g + \frac{1}{2}g_M.$$

This is because acts of the form $\frac{1}{2}f + \frac{1}{2}g_M$ are never strictly optimal with respect to the menu $\frac{1}{2}M + \frac{1}{2}g_M$. At every state there must be some act in $\frac{1}{2}M + \frac{1}{2}g_M$ that has utility 0 (namely, the mixture that involves the act $\arg\max_{f \in M} u(f(s))$).

Thus,

$$f \succeq_M g \text{ iff } \frac{1}{2}t(f) + \frac{1}{2}g_{t(M)} \succeq_{M^*} \frac{1}{2}t(g) + \frac{1}{2}g_{t(M)}.$$

Since the MWER representation also satisfies Independence and INA, we know that for all menus M , and acts f and g in M ,

$$f \succeq_{M, \mathcal{P}^+}^{S, X, U} g \Leftrightarrow t(f) \succeq_{t(M), \mathcal{P}^+}^{S, X, U} t(g) \Leftrightarrow \frac{1}{2}t(f) + \frac{1}{2}g_{t(M)} \succeq_{M^*, \mathcal{P}^+}^{S, X, U} \frac{1}{2}t(g) + \frac{1}{2}g_{t(M)}.$$

Therefore, to show that \succeq_M has a MWER representation with respect to (U, \mathcal{P}^+) , it suffices to show that \succeq_{M^*} has a MWER representation with respect to (U, \mathcal{P}^+) . \square

In the sequel, we drop the menu subscript when we refer to the family of preferences, and just write \succeq (to denote \succeq_{M^*}); by Lemma 4, it suffices to consider \succeq_{M^*} .

A.2 Defining a functional on utility acts

As we said, Stoye [20] also started his proof of a representation theorem for MER by reducing to a single preference order \succeq_{M^*} . He then noted that, the expected regret of an act f with respect to a probability \Pr and menu M^* is just the negative of the expected utility of f . Thus, the worst-case expected regret of f with respect to a set \mathcal{P} of probability measures is the negative of the worst-case expected utility of f with respect to \mathcal{P} . Thus, it sufficed for Stoye to show that \succeq_{M^*} had an MMEU representation, which he did by showing that \succeq_{M^*} satisfied Gilboa and Schmeidler's [6] axioms for MMEU, and then appealing to their representation theorem.

This argument does not quite work for us, because now \succeq_{M^*} does not satisfy the C-independence axiom. (This is because our preference order \succeq_{M^*} is based on *weighted* regret, not regret.) However, we can get a representation theorem for weighted regret by using some of the techniques used by Gilboa and Schmeidler to get a representation theorem for MMEU, appropriately modified to deal with lack of C-independence. Specifically, like Gilboa and Schmeidler, we define a functional I on utility acts such that the preference order on utility acts is determined by their value according to I (see Lemma 6). Using I , we can then determine the weight of each probability in $\Delta(S)$, and prove the desired representation theorem.

Recall that u represents \succeq on constant acts, and that only utility acts matter to \succeq . The space of all utility acts is the Banach space \mathcal{B} of real-valued functions on S . Let \mathcal{B}^- be the set of nonpositive functions in \mathcal{B} , where the function b is nonpositive if $b(s) \leq 0$ for all $s \in S$.

We now define a functional I on utility acts in \mathcal{B}^- such that for all f, g with $b_f, b_g \in \mathcal{B}^-$, we have $I(b_f) \geq I(b_g)$ iff $f \succeq g$. Let

$$R_f = \{\alpha' : l_{\alpha'}^* \succeq f\}.$$

If $0^* \geq b \geq (-1)^*$, then f_b exists, and we define

$$I(b) = \inf(R_{f_b}).$$

For the remaining $b \in \mathcal{B}^-$, we extend I by homogeneity. Let $\|b\| = |\min_{s \in S} b(s)|$. Note that if $b \in \mathcal{B}^-$, then $0^* \geq b/\|b\| \geq (-1)^*$, so we define

$$I(b) = \|b\|I(b/\|b\|).$$

Lemma 5. *If $b_f \in \mathcal{B}^-$, then $f \sim l_{I(b_f)}^*$.*

Proof. Suppose that $b_f \in \mathcal{B}^-$ and, by way of contradiction, that $l_{I(b_f)}^* \prec f$. If $f \sim l_0^*$, then it must be the case that $I(b_f) = 0$, since $I(b_f) \leq 0$ by definition of \inf , and $f \sim l_0^* \succ l_\epsilon^*$ for all $\epsilon < 0$ by Lemma 2, so $I(b_f) > \epsilon$ for all $\epsilon < 0$. Therefore, $f \sim l_{I(b_f)}^*$. Otherwise, since $b_f \in \mathcal{B}^-$, by monotonicity, we must have $l_0^* \succ f$, and thus $l_0^* \succ f \succ l_{I(b_f)}^*$. By mixture continuity, there is some $q \in (0, 1)$ such that $q \cdot l_0^* + (1 - q) \cdot l_{I(b_f)}^* \sim l_{(1-q)I(b_f)} \prec f$, contradicting the fact that $I(b)$ is the greatest lower bound of R_f .

If, on the other hand, $l_{I(b_f)}^* \succ f$, then $l_{I(b_f)}^* \succ f \succeq l_{\underline{c}}^*$ for some $\underline{c} \in \mathbb{R}$. If $f \sim l_{\underline{c}}^*$ then it must be the case that $I(b_f) = \underline{c}$. $I(b_f) \leq \underline{c}$ since $l_{\underline{c}}^* \succeq l_{\underline{c}}^*$, and $I(b_f) \geq \underline{c}$ since for all $c' < \underline{c}$, $l_{c'}^* \prec f \sim l_{\underline{c}}^*$.

Otherwise, $l_{I(b_f)}^* \succ f \succ l_{\underline{c}}^*$, and by mixture continuity, there is some $q \in (0, 1)$ such that $q \cdot l_{I(b_f)}^* + (1-q)l_{\underline{c}}^* \succ f$. Since $qI(b_f) + (1-q)\underline{c} < I(b_f)$, this contradicts the fact that $I(b_f)$ is a lower bound of R_f . Therefore, it must be the case that $l_{I(b_f)}^* \sim f$. \square

We can now show that I has the required property.

Lemma 6. *For all acts f, g such that $b_f, b_g \in \mathcal{B}^-$, $f \succeq g$ iff $I(b_f) \geq I(b_g)$.*

Proof. Suppose that $b_f, b_g \in \mathcal{B}^-$. By Lemma 5, $l_{I(b_f)}^* \sim f$ and $g \sim l_{I(b_g)}^*$. Thus, $f \succeq g$ iff $l_{I(b_f)}^* \succeq l_{I(b_g)}^*$, and by Lemma 2, $l_{I(b_f)}^* \succeq l_{I(b_g)}^*$ iff $I(b_f) \geq I(b_g)$. \square

In order to invoke a standard separation result for Banach spaces, we extend the definition of I to the Banach space \mathcal{B} . We extend I to \mathcal{B} by taking $I(b) = I(b^-)$ for $b \in \mathcal{B} - \mathcal{B}^-$, where for all $b \in \mathcal{B}$, b^- is defined as

$$b^-(s) = \begin{cases} b(s), & \text{if } b(s) \leq 0, \\ 0, & \text{if } b(s) > 0. \end{cases}$$

Clearly $b^- \in \mathcal{B}^-$ and $b = b^-$ if $b \in \mathcal{B}^-$.

We show that the axioms guarantee that I has a number of standard properties. Since we have artificially extended I to \mathcal{B} , our arguments require more cases than those in [6]. (We remark that such an “artificial” extension seem unavoidable in our setting.) Moreover, we must work harder to get the result that we want. We need different arguments from that for MMEU [6], since the preference order induced by MMEU satisfies C-independence, while our preference order does not.

Lemma 7. (a) *If $c \leq 0$, then $I(c^*) = c$.*

(b) *I satisfies positive homogeneity: if $b \in \mathcal{B}$ and $c > 0$, then $I(cb) = cI(b)$.*

(c) *I is monotonic: if $b, b' \in \mathcal{B}$ and $b \geq b'$, then $I(b) \geq I(b')$.*

(d) *I is continuous: if $b, b_1, b_2, \dots \in \mathcal{B}$, and $b_n \rightarrow b$, then $I(b_n) \rightarrow I(b)$.*

(e) *I is superadditive: if $b, b' \in \mathcal{B}$, then $I(b + b') \geq I(b) + I(b')$.*

Proof. For part (a), If c is in the range of u , then it is immediate from the definition of I and Lemma 2 that $I(c^*) = c$. If c is not in the range of u , then since $[-1, 0]$ is a subset of the range of u , we must have $c < -1$, and by definition of I , we have $I(c^*) = |c|I(c^*/|c|) = c$.

For part (b), first suppose that $\|b\| \leq 1$ and $b \in \mathcal{B}^-$ (i.e., $0^* \geq b \geq (-1)^*$). Then there exists an act f such that $b_f = b$. By Lemma 5, $f \sim l_{I(b)}^*$. We now need to consider the case that $c \leq 1$ and $c > 1$ separately. If $c \leq 1$, by

Independence, $cf_b + (1-c)l_0^* \sim cl_{I(b)}^* + (1-c)l_0^*$. By Lemma 6, $I(b_{cf_b+(1-c)l_0^*}) = I(b_{cl_{I(b)}^*+(1-c)l_0^*})$. It is easy to check that $b_{cf_b+(1-c)l_0^*} = cb$, and $b_{cl_{I(b)}^*} + (1-c)l_0^* = cI(b)^*$. Thus, $I(cb) = I(cI(b)^*)$. By part (a), $I(cI(b)^*) = cI(b)$. Thus, $I(cb) = cI(b)$, as desired.

If $c > 1$, there are two subcases. If $\|cb\| \leq 1$, since $1/c < 1$, by what we have just shown $I(b) = I(\frac{1}{c}(cb)) = \frac{1}{c}I(cb)$. Crossmultiplying, we have that $I(cb) = cI(b)$, as desired. And if $\|cb\| > 1$, by definition, $I(cb) = \|cb\|I(bc/\|cb\|) = c\|b\|I(b/\|b\|)$ (since $bc/\|cb\| = b/\|b\|$). Since $\|b\| \leq 1$, by what we have shown $I(b) = I(\|b\|I(b/\|b\|)) = \|b\|I(b/\|b\|)$, so $I(b/\|b\|) = \frac{1}{\|b\|}I(b)$. Again, it follows that $I(cb) = cI(b)$.

Now suppose that $\|b\| > 1$. Then $I(b) = \|b\|I(b/\|b\|)$. Again, we have two subcases. If $\|cb\| > 1$, then

$$I(cb) = \|cb\|I(cb/\|cb\|) = c\|b\|I(b/\|b\|) = cI(b).$$

And if $\|cb\| \leq 1$, by what we have shown for the case $\|b\| \leq 1$,

$$I(b) = I(\frac{1}{c}(cb)) = \frac{1}{c}I(cb),$$

so again $I(cb) = cI(b)$.

For part (c), first note that if $b, b' \in \mathcal{B}^-$. If $\|b\| \leq 1$ and $\|b'\| \leq -1$, then the acts f_b and $f_{b'}$ exist. Moreover, since $b \geq b'$, we must have $(f_b(s))^* \succeq (f_{b'}(s))^*(s)$ for all states $s \in S$. Thus, by Monotocity, $f_b \succeq f_{b'}$. If either $\|b\| > 1$ or $\|b'\| > 1$, let $n = \max(\|b\|, \|b'\|)$. Then $\|b/n\| \leq 1$ and $\|b'/n\| \leq 1$. Thus, $I(b/n) \geq I(b'/n)$, by what we have just shown. By part (b), $I(b) \geq I(b')$. Finally, if either $b \in \mathcal{B} - \mathcal{B}^-$ or $b' \in \mathcal{B} - \mathcal{B}^-$, note that if $b \geq b'$, then $b^- \geq (b')^-$. By definition, $I(b) = I(b^-)$ and $I(b') = I(b')^-$; moreover, $b^-, (b')^- \in \mathcal{B}^-$. Thus, by the argument above, $I(b) \geq I(b^-)$.

For part (d), note that if $b_n \rightarrow b$, then for all k , there exists n_k such that $b_n - (1/k)^* \leq b_n \leq b_n + (1/k)^*$ for all $n \geq n_k$. Moreover, by the monotonicity of I (part (c)), we have that $I(b - (1/k)^*) \leq I(b_n) \leq I(b + (1/k)^*)$. Thus, it suffices to show that $I(b - (1/k)^*) \rightarrow I(b)$ and that $I(b + (1/k)^*) \rightarrow I(b)$.

To show that $I(b - (1/k)^*) \rightarrow I(b)$, we must show that for all $\epsilon > 0$, there exists k such that $I(b - (1/k)^*) \geq I(b) - \epsilon$. By positive homogeneity (part (b)), we can assume without loss of generality that $\|b - (1/2)^*\| \leq 1$ and that $\|b\| \leq 1$. Fix $\epsilon > 0$. If $I(b - (1/2)^*) \geq I(b) - \epsilon$, then we are done. If not, then $I(b) > I(b) - \epsilon > I(b - (1/2)^*)$. Since $\|b\| \leq 1$ and $\|b - (1/2)^*\| \leq 1$, f_b and $f_{b-(1/2)^*}$ exist. Moreover, by Lemma 6, $f_b \succ f_{(I(b)-\epsilon)^*} \succ f_{b-(1/2)^*}$. By mixture continuity, for some $p \in (0, 1)$, we have $pf_b + (1-p)f_{b-(1/2)^*} \succ f_{(I(b)-\epsilon)^*}$. It is easy to check that $b_{pf_b+(1-p)f_{b-(1/2)^*}} = b - (1-p)(1/2)^*$. Thus, by Lemma 6, $f_{b-(1-p)(1/2)^*} \succeq f_{(I(b)-\epsilon)^*}$, and $I(b - (1-p)(1/2)^*) > I(b) - \epsilon$. Choose k such that $1/k < (1-p)(1/2)$. Then $I(b - (1/k)^*) \geq I(b - (1-p)(1/2)^*) > I(b) - \epsilon$, as desired.

The argument that $I(b + (1/k)^*) \rightarrow I(b)$ is similar and left to the reader.

For part (e), first suppose that $b, b' \in \mathcal{B}^-$. If $\|b\|, \|b'\| \leq 1$, and $I(b), I(b') \neq 0$, consider $\frac{b}{I(b)}$ and $\frac{b'}{I(b')}$. Since $I(\frac{b}{I(b)}) = I(\frac{b'}{I(b')}) = 1$, it follows from Lemma 5

that $f_{\frac{b}{I(b)}} \sim f_{\frac{b'}{I(b')}}$. By Ambiguity Aversion, for all $p \in (0, 1]$, $pf_{\frac{b}{I(b)}} + (1 - p)f_{\frac{b'}{I(b')}} \succeq f_{\frac{b}{I(b)}}$. Thus, $I(\frac{I(b)}{I(b)+I(b')} \frac{b}{I(b)} + \frac{I(b')}{I(b)+I(b')} \frac{b'}{I(b')}) \geq I(\frac{b}{I(b)}) = I(\frac{b'}{I(b')}) = 1$. Hence, $I(b + b') \geq I(b) + I(b')$.

If $b, b^- \in \mathcal{B}^-$ and either $\|b\| > 1$ or $\|b'\| > 1$, and both $I(b) \neq 0$ and $I(b') \neq 0$, then the result easily follows by positive homogeneity (property (b)).

If $b, b^- \in \mathcal{B}^-$ and either $I(b) = 0$ or $I(b') = 0$, let $b_n = b - \frac{1}{n}^*$ and $b'_n = b' - \frac{1}{n}^*$. Clearly $\|b_n\| > 0$, $\|b'_n\| > 0$, $b_n \rightarrow b$, and $b'_n \rightarrow b'$. By our argument above, $I(b_n + b'_n) \geq I(b_n) + I(b'_n)$ for all $n \geq 1$. The result now follows from continuity.

Finally, if either $b \in \mathcal{B} - \mathcal{B}^-$ or $b' \in \mathcal{B} - \mathcal{B}^-$, observe that

$$(b + b')^-(s) \begin{cases} = b^-(s) + b'^-(s), & \text{if } b(s) \leq 0, b'(s) \leq 0 \\ = b^-(s) + b'^-(s), & \text{if } b(s) \geq 0, b'(s) \geq 0 \\ \geq b^-(s) + b'^-(s), & \text{if } b(s) > 0, b'(s) \leq 0 \\ \geq b^-(s) + b'^-(s), & \text{if } b(s) \leq 0, b'(s) > 0. \end{cases}$$

Therefore, $(b + b')^- \geq b^- + b'^-$. Thus, $I(b + b') = I((b + b')^-) \geq I(b^- + b'^-)$ by the monotonicity of I , and $I(b^- + b'^-) \geq I(b^-) + I(b'^-)$ by superadditivity of I on \mathcal{B}^- . Therefore, $I(b + b') \geq I(b) + I(b')$. \square

A.3 Defining the weights

In this section, we use I to define a weight α_{Pr} for each probability $\text{Pr} \in \Delta(S)$. The heart of the proof involves showing that the resulting set \mathcal{P}^+ so determined gives us the desired representation.

Given a set \mathcal{P}^+ of weighted probability measures, for $b \in \mathcal{B}^-$, define

$$NWREG(b) = \inf_{\text{Pr} \in \mathcal{P}} \alpha_{\text{Pr}} \left(\sum_{s \in S} b(s) \text{Pr}(s) \right).$$

Note that $NWREG$ is the negative of the weighted regret when the menu is \mathcal{B}^- . Define

$$NREG(b) = \inf_{\text{Pr} \in \mathcal{P}} \sum_{s \in S} b(s) \text{Pr}(s).$$

and

$$NREG_{\text{Pr}}(b) = \sum_{s \in S} b(s) \text{Pr}(s) = E_{\text{Pr}} b.$$

For each probability $\text{Pr} \in \Delta(S)$, define

$$\alpha_{\text{Pr}} = \sup \{ \alpha \in \mathbb{R} : \alpha NREG_{\text{Pr}}(b) \geq I(b) \text{ for all } b \in \mathcal{B}^- \}. \quad (1)$$

Note that $\alpha_{\text{Pr}} \geq 0$ for all distributions $\text{Pr} \in \Delta(S)$, since $0 \geq I(b)$ for $b \in \mathcal{B}^-$ (by monotonicity); and $\alpha_{\text{Pr}} \leq 1$, since $NREG_{\text{Pr}}((-1)^*) = I((-1)^*) = -1$ for all distributions Pr . Thus, $\alpha_{\text{Pr}} \in [0, 1]$. Moreover, it is immediate from the definition of α_{Pr} that $\alpha_{\text{Pr}} NREG_{\text{Pr}}(b) \geq I(b)$ for all $b \in \mathcal{B}^-$. The next lemma shows that there exists a probability Pr where we have equality.

Lemma 8. (a) For some distribution Pr , we have $\alpha_{\text{Pr}} = 1$.

(b) For all $b \in \mathcal{B}^-$, there exists Pr such that $\alpha_{\text{Pr}} \text{NREG}_{\text{Pr}}(b) = I(b)$.

Proof. The proofs of both part (a) and (b) use a standard separation result: If U is an open convex subset of \mathcal{B} , and $b \notin U$, then there is a linear functional λ that separates U from b , that is, $\lambda(b') > \lambda(b)$ for all $b' \in U$. We proceed as follows

For part (a), we must show that for some Pr , for all $b \in \mathcal{B}^-$, $\text{NREG}_{\text{Pr}}(b) \geq I(b)$. Since $\text{NREG}_{\text{Pr}}(b) = E_{\text{Pr}}b$, it suffices to show that $E_{\text{Pr}}(b) \geq I(b)$ for all $b \in \mathcal{B}^-$.

Let $U = \{b' \in \mathcal{B} : I(b') > -1\}$. U is open (by continuity of I), and convex (by positive homogeneity and superadditivity of I), and $(-1)^* \notin U$. Thus, there exists a linear functional λ such that $\lambda(b') > \lambda((-1)^*)$ for $b' \in U$.

We want to show that λ is a positive linear functional, that is, that $\lambda(b) \geq 0$ if $b \geq 0^*$. Since $0^* \in U$, and $\lambda(0^*) = 0$, it follows that $\lambda((-1)^*) < 0$. Since λ is linear, we can assume without loss of generality that $\lambda((-1)^*) = -1$. Thus, for all $b' \in \mathcal{B}^-$, $I(b') > -1$ implies $\lambda(b') > -1$. (The fact that $I(cb') = I(0^*)$ follows from the definition of I on elements in $\mathcal{B} - \mathcal{B}^-$.) Suppose that $c > 0$ and $b' \geq 0^*$. From the definition of I , it follows that $I(cb') = I(0^*) = 0 > -1$. So $c\lambda(b') = \lambda(cb') > -1$, so $\lambda(b') > -1/c$. Since this is true for all $c > 0$, it must be the case that $\lambda(b') \geq 0$. Thus, λ is a positive functional.

Define the probability distribution Pr on S by taking $\text{Pr}(s) = \lambda(1_s)$. To see that Pr is indeed a probability distribution, note that since $1_s \geq 0$ and λ is positive, we must have $\lambda(1_s) \geq 0$. Moreover, $\sum_{s \in S} \text{Pr}(s) = \lambda(1^*) = 1$. In addition, for all $b' \in \mathcal{B}$, we have

$$\lambda(b') = \sum_{s \in S} \lambda(1_s) b'(s) = \sum_{s \in S} \text{Pr}(s) b'(s) = E_{\text{Pr}}(b').$$

Next note that, for $b \in \mathcal{B}^-$,

$$\text{for all } c < 0, \text{ if } I(b) > c, \text{ then } \lambda(b) > c. \quad (2)$$

For if $I(b) > c$, then $I(b/|c|) > -1$ by positive homogeneity, so $\lambda(b/|c|) > -1$ and $\lambda(b) > c$. The result now follows. For if $b \in \mathcal{B}^-$, then $I(b) \leq I(0^*) = 0$ by monotonicity. Thus, if $c < I(b)$, then $c < 0$, so, by (2), $\lambda(b) > c$. Since $\lambda(b) > c$ whenever $I(b) > c$, it follows that $E_{\text{Pr}}(b) = \lambda(b) \geq I(b)$, as desired.

The proof of part (b) is similar to that of part (a). We want to show that, given $b \in \mathcal{B}^-$, there exists Pr such that $\alpha_{\text{Pr}} \text{NREG}_{\text{Pr}}(b) = I(b)$. First suppose that $\|b\| \leq 1$. If $I(b) = 0$, then there must exist some s such that $b(s) = 0$, for otherwise there exists $c < 0$ such that $b \leq c^*$, so $I(b) \leq c$. If $b(s) = 0$, let Pr_s be such that $\text{Pr}_s(s) = 1$. Then $\text{NREG}_{\text{Pr}_s}(b) = 0$, so (b) holds in this case.

If $\|b\| \leq 1$ and $I(b) < 0$, let $U = \{b' : I(b') > I(b)\}$. Again, U is open and convex, and $b \notin U$, so there exists a linear functional λ such that $\lambda(b') > \lambda(b)$ for $b' \in U$. Since $0^* \in U$ and $\lambda(0^*) = 0$, we must have $\lambda(b) < 0$. Since $(-1)^* \leq b$, $(-1)^*$ is not in U , and therefore we also have $\lambda((-1)^*) < 0$. Thus, we can

assume without loss of generality that $\lambda((-1)^*) = -1$, and hence $\lambda((1)^*) = 1$. The same argument as above shows that λ is positive: for all $c > 0$ and $b' \geq 0^*$, $I(cb') = 0$ as before. Since $I(b) < 0$, it follows that $I(cb') > I(b)$, so $cb' \in U$ and $\lambda(cb') > \lambda(b) \geq \lambda((-1)^*) = -1$. Thus, as before, for all $c > 0$, $b' \geq 0^*$, $\lambda(b') > \frac{-1}{c}$, so λ is a positive functional.

Therefore, λ determines a probability distribution \Pr such that, for all $b' \in \mathcal{B}^-$, we have $\lambda(b') = E_{\Pr}(b')$. This, of course, will turn out to be the desired distribution. To show this, we need to show that $\alpha_{\Pr} = I(b)/NREG_{\Pr}(b)$. Clearly $\alpha_{\Pr} \leq I(b)/NREG_{\Pr}(b)$, since if $\alpha > I(b)/NREG_{\Pr}(b)$, then $\alpha NREG_{\Pr}(b) < I(b)$ (since $NREG_{\Pr}(b) = \lambda(b) < 0$). To show that $\alpha_{\Pr} \geq I(b)/NREG_{\Pr}(b)$, we must show that $(I(b)/NREG_{\Pr}(b))NREG_{\Pr}(b') \geq I(b')$ for all $b' \in \mathcal{B}^-$. Equivalently, we must show that $I(b)\lambda(b')/\lambda(b) \geq I(b')$ for all $b' \in \mathcal{B}^-$.

Essentially the same argument used to prove (2) also shows

$$\text{for all } c > 0, \text{ if } I(b') > cI(b), \text{ then } \lambda(b') > c\lambda(b).$$

In particular, if $I(b') > cI(b)$, then by positive homogeneity, $\frac{I(b')}{c} > I(b)$, so $\frac{b'}{c} \in U$, and $\lambda(\frac{b'}{c}) > \lambda(b)$ and hence $\lambda(b') > c\lambda(b)$.

Thus, if $I(b')/(-I(b)) > c$ and $c < 0$, then $I(b') > -cI(b)$, and hence $\lambda(b')/(-\lambda(b)) > c$. It follows that $\lambda(b')/(-\lambda(b)) \geq I(b')/(-I(b))$ for all $b' \in \mathcal{B}^-$. Thus, $I(b)\lambda(b')/\lambda(b) \geq I(b')$ for all $b' \in \mathcal{B}^-$, as required.

Finally, if $\|b\| > 1$, let $b' = b/\|b\|$. By the argument above, there exists a probability measure \Pr such that $\alpha_{\Pr}NREG_{\Pr}(b/\|b\|) = I(b/\|b\|)$. Since $NREG_{\Pr}(b/\|b\|) = NREG_{\Pr}(b)/\|b\|$, and $I(b/\|b\|) = I(b)/\|b\|$, we must have that $\alpha_{\Pr}NREG_{\Pr}(b) = I(b)$. \square

We can now complete the proof of Theorem 1. By Lemma 8 and the definition of α_{\Pr} , for all $b \in \mathcal{B}^-$,

$$\begin{aligned} I(b) &= \inf_{\Pr \in \Delta(S)} \alpha_{\Pr} NREG(b) \\ &= \inf_{\Pr \in \Delta(S)} \left(\alpha_{\Pr} \sum_{s \in S} b(s) \Pr(s) \right) \\ &= \sup_{\Pr \in \mathcal{P}} \left(-\alpha_{\Pr} \sum_{s \in S} b(s) \Pr(s) \right). \end{aligned} \tag{3}$$

Recall that, by Lemma 6, for all acts f, g such that $b_f, b_g \in \mathcal{B}^-$, $f \succeq g$ iff $I(b_f) \geq I(b_g)$. Thus, $f \succeq g$ iff

$$\sup_{\Pr \in \Delta(S)} \left(-\alpha_{\Pr} \sum_{s \in S} u(f(s)) \Pr(s) \right) \leq \sup_{\Pr \in \Delta(S)} \left(-\alpha_{\Pr} \sum_{s \in S} u(g(s)) \Pr(s) \right).$$

Note that, for $f \in M^* = \mathcal{B}^-$, we have $reg_{M^*, \Pr}(f) = \sup(-u(f(s)) \Pr(s))$, since 0^* dominates all acts in M^* . Thus, $\succeq = \succeq_{M^*, \mathcal{P}^+}^{S, Y, U}$, where $\mathcal{P}^+ = \{(\Pr, \alpha_{\Pr}) : \Pr \in$

$\Delta(S)\}$. By Lemma 4, this means (U, \mathcal{P}^+) represents \succeq_M for all menus M , as required.

We have already observed that U is unique up to affine transformations, so it remains to show that \mathcal{P}^+ is maximal. This follows from the definition of α_{Pr} . If $\succeq_M = \succeq_{M, (\mathcal{P}')^+}^{S, Y, U}$, and $(\alpha', Pr) \in (\mathcal{P}')^+$, then we claim that $\alpha' \in \{\alpha \in \mathbb{R} : \alpha NREG_{Pr}(b) \geq I(b) \text{ for all } b \in \mathcal{B}^-\}$. If not, there would be some $b \in \mathcal{B}^-$ with $\|b\| \leq \frac{1}{2}$, such that $\alpha' NREG_{Pr}(b) < I(b)$, which, by the definition of $\prec_{M^*, (\mathcal{P}')^+}^{S, Y, U}$, means that $l_{-1}^* \prec_{M^*, (\mathcal{P}')^+}^{S, Y, U} f_b \prec_{M^*, (\mathcal{P}')^+}^{S, Y, U} l_{I(b)}^*$. Recall that $I(b_f) = \inf\{\gamma : l_\gamma^* \succeq_{M^*} f\}$. Moreover, since $\prec_{M^*, (\mathcal{P}')^+}^{S, Y, U}$ satisfies the Mixture Continuity, there exists some $p \in (0, 1)$ such that $f_b \prec_{M^*, (\mathcal{P}')^+}^{S, Y, U} pl_{-1}^* + (1-p)l_{I(b)}^* \prec_{M^*, (\mathcal{P}')^+}^{S, Y, U} \prec_{M^*, (\mathcal{P}')^+}^{S, Y, U} l_{I(b)}^*$. This contradicts the definition of $I(b)$. Therefore, $\alpha' \in \{\alpha \in \mathbb{R} : \alpha NREG_{Pr}(b) \geq I(b) \text{ for all } b \in \mathcal{B}^-\}$, and hence $\alpha' \leq \alpha_{Pr}$.

A.4 Uniqueness of Representation

In the preceding sections, we have shown that if a family of menu-dependent preferences \succeq_M satisfies axioms 1 – 10, then \succeq_M can be represented as minimizing weighted expected regret with respect to a canonical set \mathcal{P}^+ of weighted probabilities and a utility function. We now want to show uniqueness.

In this section, we show that the canonical set of weighted probabilities we constructed, when viewed as a set of subnormal probability measures, is regular and includes at least one proper probability measure. Moreover, this set of sub-probability measures is the only regular set that induces a family of preferences \succeq_M that satisfies axioms 1 – 10. Our uniqueness result is analogous to the uniqueness results of Gilboa and Schmeidler [7], who show that the convex, closed, and non-empty set of probability measures in their representation theorem for MMEU is unique.

By Lemma 4, it suffices to consider the preference relation \succeq_{M^*} . The argument is based on two lemmas: the first lemma says that the canonical set of sub-probability measures is regular; and the second lemma says that a set of sub-probability measures representing \succeq_{M^*} that is regular and contains at least one proper probability measure is unique. The proof of this second lemma, like the proof of uniqueness in Gilboa and Schmeidler [7], uses a separating hyperplane theorem to show the existence of acts on which two different representations must ‘disagree’. However, a slightly different argument is required in our case, since our acts in M^* must have utilities corresponding to nonpositive vectors in $\mathbb{R}^{|S|}$.

Lemma 9. *Let \mathcal{P}^+ be the canonical set of weighted probability measures representing \succeq_{M^*} . The set $C(\mathcal{P}^+)$ of sub-probability measures is regular.*

Proof. It is useful to note that, by definition, $\mathbf{p} \in C(\mathcal{P}^+)$ if and only if

$$E_{\mathbf{p}}(b) \geq I(b) \text{ for all } b \in \mathcal{B}^-$$

(where expectation with respect to a subnormal probability measure is defined in the obvious way).

Recall that a set is regular if it is convex, closed, and downward-closed. We first show that $C(\mathcal{P}^+)$ is downward-closed. Suppose that $\mathbf{p} \in C(\mathcal{P}^+)$ and $\mathbf{q} \leq \mathbf{p}$ (i.e., $\mathbf{q}(s) \leq \alpha \Pr(s)$ for all $s \in S$). Since $\mathbf{p} \in C(\mathcal{P}^+)$, $E_{\mathbf{p}}(b) \geq I(b)$ for all $b \in \mathcal{B}^-$. Since $\mathbf{q} \leq \mathbf{p}$ and, if $b \in cB^-$, we have $b \leq 0^*$, it follows that $E_{\mathbf{q}}(b) \geq E_{\mathbf{p}}(b) \geq I(b)$ for all $b \in \mathcal{B}^-$, and thus $\mathbf{q} \in C(\mathcal{P}^+)$.

To see that $C(\mathcal{P}^+)$ is closed, let $\mathbf{p} = \lim_{n \rightarrow \infty} \mathbf{p}_n$, where each $\mathbf{p}_n \in C(\mathcal{P}^+)$. Since $\mathbf{p}_n \in C(\mathcal{P}^+)$ it must be the case that $E_{\mathbf{p}_n}(b) \geq I(b)$ for all $b \in \mathcal{B}^-$. By the continuity of expectation, it follows that $E_{\mathbf{p}}(b) \geq I(b)$ for all $b \in \mathcal{B}^-$. Thus, $\mathbf{p} \in C(\mathcal{P}^+)$.

To show that $C(\mathcal{P}^+)$ is convex, suppose that $\mathbf{p}, \mathbf{q} \in C(\mathcal{P}^+)$. Then $E_{\mathbf{p}}(b) \geq I(b)$ and $E_{\mathbf{q}}(b) \geq I(b)$ for all $b \in \mathcal{B}^-$. It easily follows that for all $a \in (0, 1)$, $E_{a\mathbf{p} + (1-a)\mathbf{q}}(b) \geq I(b)$ for all $b \in \mathcal{B}^-$. Thus, $a\mathbf{p} + (1-a)\mathbf{q} \in C(\mathcal{P}^+)$. \square

Lemma 10. *A set of sub-probability measures representing \succeq_{M^*} that is regular, and has at least one proper probability measure is unique.*

Proof. Suppose for contradiction that there exists two regular sets of subnormal probability distributions, C_1 and C_2 , that represent \succeq_{M^*} and have at least one proper probability measure.

First, without loss of generality, let $\mathbf{q} \in C_2 \setminus C_1$. We actually look at an extension of C_1 that is downward-closed in each component to $-\infty$. Let $\overline{C}_1 = \{\mathbf{p} \in \mathbb{R}^{|S|} : \mathbf{p} \leq \mathbf{p}'\}$. Note an element \mathbf{p} of \overline{C}_1 may not be subnormal probability measures; we do not require that $\mathbf{p}(s) \geq 0$ for all $s \in S$. Since \overline{C}_1 and $\{\mathbf{q}\}$ are closed, convex, and disjoint, and $\{\mathbf{q}\}$ is compact, the separating hyperplane theorem [15] says that there exists $\theta \in \mathbb{R}^{|S|}$ and $c \in \mathbb{R}$ such that

$$\theta \cdot \mathbf{p} > c \text{ for all } \mathbf{p} \in \overline{C}_1, \text{ and } \theta \cdot \mathbf{q} < c. \quad (4)$$

By scaling c appropriately, we can assume that $|\theta(s)| \leq 1$ for all $s \in S$. Now we argue that it must be the case that $\theta(s) \leq 0$ for all $s \in S$ (so that θ corresponds to the utility profile of some act in M^*). Suppose that $\theta(s') > 0$ for some $s' \in S$. By (4), $\theta \cdot \mathbf{p} > c$ for all $\mathbf{p} \in \overline{C}_1$. However, consider $\mathbf{p}^* \in \overline{C}_1$ defined by

$$\mathbf{p}^*(s) = \begin{cases} 0, & \text{if } s \neq s' \\ \frac{-|c|}{\theta(s')}, & \text{if } s = s'. \end{cases}$$

Clearly, $\theta \cdot \mathbf{p}^* \leq c$, contradicting (4). Thus it must be the case that $\theta(s) \leq 0$ for all $s \in S$.

Consider the θ given by the separating hyperplane theorem, and let f be an act such that $u \circ f = \theta$. By continuity, $f \sim_{M^*} l_d^*$ for some constant act l_d^* . Since C_1 and C_2 both represent \succeq_{M^*} , and C_1 and C_2 both contain a proper probability measure,

$$\min_{\mathbf{p} \in C_1} \mathbf{p} \cdot (u \circ f) = \min_{\mathbf{p} \in C_1} \mathbf{p} \cdot (u \circ l_d^*) = d = \min_{\mathbf{p} \in C_2} \mathbf{p} \cdot (u \circ f).$$

However, by (4),

$$\min_{\mathbf{p} \in C_1} \mathbf{p} \cdot (u \circ f) > c > \min_{\mathbf{p} \in C_2} \mathbf{p} \cdot (u \circ f),$$

which is a contradiction. □

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