Applied Toric Geometry
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Contents

Chapter 1

Monomial maps and toric ideals

Toric varieties often arise in applications as the image of a (Laurent) monomial map in affine or projective space. Such embedded toric varieties are the topic of this chapter. We study their dimension and defining equations, i.e., their ideals. More intrinsic descriptions via semigroups, cones and fans are left for later chapters.

1.1 First examples

In this chapter, a toric variety is encoded by an integer matrix of size $d \times n$:

$$A = (a_1 \; a_2 \; \cdots \; a_n) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{d1} & a_{d2} & \cdots & a_{dn} \end{pmatrix} \in \mathbb{Z}^{d\times n}.$$

The columns are the exponent vectors $a_1, \ldots, a_n \in \mathbb{Z}^d$ of a (Laurent) monomial map $\phi_A : (\mathbb{C}^*)^d \to (\mathbb{C}^*)^n$. That is, the monomial map ϕ_A associated to A is

$$\phi_A(t) = (t^{a_1}, \ldots, t^{a_n}). \tag{1.1.1}$$

Here $t = (t_1, \ldots, t_d)$ is a d-tuple of nonzero complex numbers and t^{a_i} is short for $t_1^{a_{1i}} \cdots t_d^{a_{di}}$. The domain of ϕ_A is the *algebraic torus* $(\mathbb{C}^*)^d = (\mathbb{C} \setminus \{0\})^d \subset \mathbb{C}^d$. Note that, when A has some negative entries, this cannot be extended to the affine space $\mathbb{C}^d$.

We associate an affine variety $Y_A \subseteq \mathbb{C}^n$ to A by taking the Zariski closure of $\operatorname{im} \phi_A$:

$$Y_A = \overline{\operatorname{im} \phi_A} = \overline{\{\phi_A(t) : t \in (\mathbb{C}^*)^d\}} \subseteq \mathbb{C}^n.$$

Our ad hoc definition of an *affine toric variety* uses this construction: for now, an affine toric variety is a variety which arises as Y_A for some matrix $A \in \mathbb{Z}^{d\times n}$. A more general definition is given in Section 4.3. The construction of Y_A provides first evidence for the claim that *toric varieties are represented by combinatorial data*, the data here being integer matrices. Many familiar affine varieties are toric. Here are some examples.

Example 1.1.1 (affine spaces and tori)**.** The variety Y_A for the identity matrix $A = \mathrm{id}_d$ is $\mathbb{C}^d$. Appending a column with entries -1, we obtain the $d \times (d+1)$-matrix

$$A = \begin{pmatrix} 1 & & & -1 \\ & \ddots & & \vdots \\ & & 1 & -1 \end{pmatrix}.$$

Now Y_A is parametrized by $(t_1, \ldots, t_d) \mapsto (t_1, \ldots, t_d, (t_1 \cdots t_d)^{-1}) \in \mathbb{C}^{d+1}$. Its equation is $Y_A = \{x_1 \cdots x_d x_{d+1} = 1\} \subset \mathbb{C}^{d+1}$. We will see that $Y_A \simeq (\mathbb{C}^*)^d$ (Exercise 1.2.1). ◇

Example 1.1.2 (moment curves)**.** The *moment curve* of degree δ arises from

$$A = \begin{pmatrix} 1 & 2 & \cdots & \delta \end{pmatrix} \quad \in \mathbb{Z}^{1\times\delta}.$$

It is embedded in $\mathbb{C}^\delta$ via the parametrization $\phi_A(t) = (t, t^2, \ldots, t^\delta)$. For $\delta = 2$, the moment curve is the parabola $\{x^2 - y = 0\}$ in the affine plane $\mathbb{C}^2$. For $\delta = 3$, it is the twisted cubic from [18, p. 8], defined by $Y_{(1\ 2\ 3)} = \{x^2 - y = x^3 - z = 0\} \subset \mathbb{C}^3$. ◇

Exercise 1.1.3. Show that the moment curve of degree n from Example 1.1.2 is defined by the $n-1$ equations $x_1^k - x_k = 0$ for $k = 2, \ldots, n$.

Exercise 1.1.4. Show that $(1, \ldots, 1) \in \mathbb{R}^n$ is contained in Y_A for any $A \in \mathbb{Z}^{d\times n}$.

Example 1.1.5 (the cuspidal cubic)**.** Coordinate projections of Y_A are obtained by deleting columns of A. For instance, the *cuspidal cubic* $\{z^2 - y^3 = 0\} \subset \mathbb{C}^2$ is obtained as a projection of the twisted cubic $Y_{(1\ 2\ 3)} \subset \mathbb{C}^3$ from Example 1.1.2 onto the (y, z)-plane, see Figure 1.1. It is the toric curve $Y_{A'}$ for $A' = (2\ \ 3)$. ◇

Example 1.1.6 (toric surfaces)**.** Consider the 2×3-matrices

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad A_3 = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

These correspond to toric surfaces Y_{A_1}, Y_{A_2} and Y_{A_3} in $\mathbb{C}^3$, with defining equation $x - yz = 0, x^2 - yz = 0$ and $x^3 - yz = 0$ respectively. The parametrization of A_1 is $\phi_{A_1}(t_1, t_2) = (t_1 t_2, t_1, t_2)$, and $x - yz$ vanishes on the image. The degree of Y_{A_1} and Y_{A_2} is 2, while Y_{A_3} has degree 3. Some real points of these surfaces are shown in Figure 1.2. Note that Y_{A_1} is smooth, while Y_{A_2} and Y_{A_3} have a singular point at the origin. ◇

Example 1.1.7 (rank-one matrices)**.** Consider the matrix $A \in \mathbb{Z}^{5\times 6}$ given by

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}. \tag{1.1.2}$$

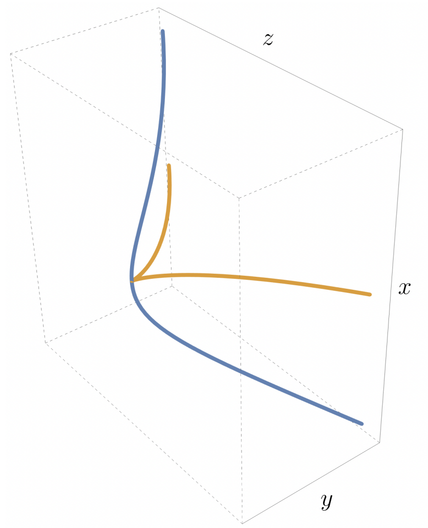

Figure 1.1: The twisted cubic (blue) and its projection onto the (y, z)-plane: the cuspical cubic (orange).

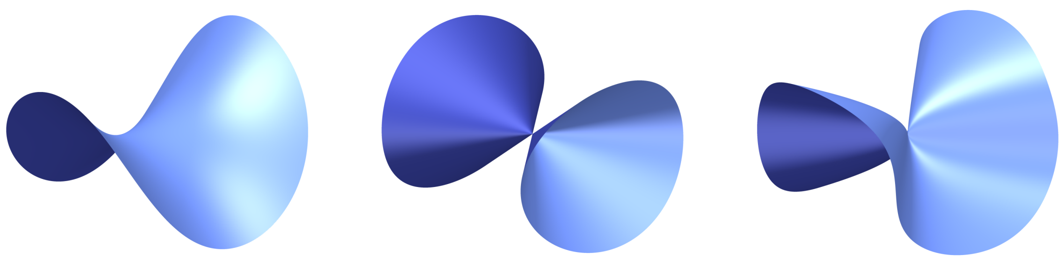

Figure 1.2: Three toric surfaces in three-space.

The corresponding monomial map ϕ_A parametrizes rank-one 2×3-matrices:

$$\phi_A(t_1, \dots, t_5) \;=\; \begin{pmatrix} t_1t_3 & t_1t_4 & t_1t_5 \\ t_2t_3 & t_2t_4 & t_2t_5 \end{pmatrix} \;=\; \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \cdot \begin{pmatrix} t_3 & t_4 & t_5 \end{pmatrix} \;\in \mathbb{C}^{2\times 3} = \mathbb{C}^6.$$

Analogously, one easily finds the matrix $A \in \mathbb{Z}^{(d_1+d_2)\times d_1 d_2}$ for rank-one $d_1 \times d_2$-matrices. These matrices and their toric varieties appear in *optimal transport* in Chapter 6. $\diamond$

We now switch to the projective toric variety defined by A. We replace (1.1.1) by

$$\Phi_A : (\mathbb{C}^*)^d \longrightarrow \mathbb{P}^{n-1}, \quad \text{where} \quad \Phi_A(t) = (t^{a_1} : \dots : t^{a_n}). \tag{1.1.3}$$

We associate a projective variety $X_A \subset \mathbb{P}^{n-1}$ to A by taking the Zariski closure of $\operatorname{im} \Phi_A$:

$$X_A \;=\; \overline{\operatorname{im} \Phi_A} \;=\; \overline{\{\Phi_A(t) \,:\, t \in (\mathbb{C}^*)^d\}} \;\subset \mathbb{P}^{n-1}. \tag{1.1.4}$$

We here define a *projective toric variety* as a variety arising as X_A for some matrix $A \in \mathbb{Z}^{d\times n}$. We revisit some previously seen matrices from the projective point of view.

Example 1.1.8 (projective space is toric)**.** The variety X_A for $A = \mathrm{id}_d$ is $\mathbb{P}^{d-1}$. $\diamond$

Exercise 1.1.9. Show that the projective toric variety $X_A \subset \mathbb{P}^\delta$ of the matrix $A = (0\ \ 1\ \ 2\ \ \cdots\ \ \delta) \in \mathbb{Z}^{1\times(\delta+1)}$ is the *rational normal curve* of degree δ. This is the curve in $\mathbb{P}^\delta$ defined by

$$\operatorname{rank}\begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_{\delta-1} \\ x_1 & x_2 & x_3 & \cdots & x_\delta \end{pmatrix} \le 1.$$

Show that its affine chart with $x_0 \neq 0$ is the degree δ moment curve from Example 1.1.2.

Example 1.1.10 (Example 1.1.6 in $\mathbb{P}^2$)**.** The matrices A_1, A_3 from Example 1.1.6 give the projective toric variety $X_{A_1} = X_{A_3} = \mathbb{P}^2$. Indeed, the maps Φ_{A_1} and Φ_{A_3} are dominant. The matrix A_2 leads to a smooth toric curve $X_{A_2} = \{x_0^2 - x_1x_2 = 0\} \subset \mathbb{P}^2$. $\diamond$

Example 1.1.11 (Segre embedding)**.** The threefold $X_A \subset \mathbb{P}^5$ corresponding to the matrix A from (1.1.2) is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ into $\mathbb{P}^5$. It is described by

$$\operatorname{rank}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \end{pmatrix} \le 1.$$

That is, the defining equations are the 2×2 minors of this matrix. $\diamond$

Exercise 1.1.12. A projective variety $X \subset \mathbb{P}^m$ is called *unirational* if it equals the closure of the image of a parametrization map $\mathbb{C}^d \dashrightarrow \mathbb{P}^m$ whose $m+1$ coordinates are given by rational functions. Show that every unirational projective variety is a linear projection of a projective toric variety.

1.2 Tori and monomial maps

The d-dimensional algebraic torus is $(\mathbb{C}^*)^d$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Clearly, $(\mathbb{C}^*)^d$ is a Zariski open subset of the affine space $\mathbb{C}^d$. However, it can be embedded as a closed subvariety of $\mathbb{C}^{d+1}$. By the next exercise, the algebraic torus $(\mathbb{C}^*)^d$ is an affine algebraic variety with coordinate ring $\mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$. This is the ring of *Laurent polynomials* in $t = (t_1, \ldots, t_d)$. For more on coordinate rings of affine varieties, see [18, Chapter 5, §4].

Exercise 1.2.1. Find a bijective map $(\mathbb{C}^*)^d \to V = V_{\mathbb{C}^{d+1}}(x_1 \cdots x_d y - 1) \subset \mathbb{C}^{d+1}$. Show that the coordinate ring $\mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_d, y]/\langle x_1 \cdots x_d y - 1\rangle$ of the affine variety V is isomorphic to $\mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ as a $\mathbb{C}$-algebra.

The affine variety $(\mathbb{C}^*)^d$ is also a group under component-wise multiplication:

$$t \cdot t' = (t_1, \ldots, t_d) \cdot (t_1', \ldots, t_d') = (t_1t_1', \ldots, t_dt_d') \in (\mathbb{C}^*)^d \quad \text{for } t, t' \in (\mathbb{C}^*)^d.$$

An affine variety V with a group operation $V \times V \to V$ which is a morphism is called an *algebraic group*. The algebraic torus has this property, and its algebraic group structure is crucial in this book. Below, all our tori will be algebraic groups isomorphic to $(\mathbb{C}^*)^d$.

Definition 1.2.2 (Torus). *A torus T of dimension d is an algebraic variety isomorphic to $(\mathbb{C}^*)^d$. Note that T is an algebraic group with the operation induced by $T \simeq (\mathbb{C}^*)^d$.*

Remark 1.2.3. The *topological d-torus* is the subgroup of $(\mathbb{C}^*)^d$ consisting of points whose coordinates have modulus 1. The case $d = 2$ gives a product of two circles, which is the familiar 'doughnut' picture. We emphasize that *in this book, all tori are algebraic.*

By definition, the coordinate ring $\mathbb{C}[T]$ of a d-dimensional torus T is isomorphic to the Laurent polynomial ring $\mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$. It consists of all morphisms $T \to \mathbb{C}$. *Characters* are the non-vanishing morphisms on T which respect the group structure.

Definition 1.2.4 (Character). *A character of a torus T is a morphism of varieties $T \to \mathbb{C}^*$ which is also a group homomorphism.*

Proposition 1.2.5. *The characters of $(\mathbb{C}^*)^d$ are the morphisms $(\mathbb{C}^*)^d \to \mathbb{C}^*$ given by*

$$(t_1, \ldots, t_d) \longmapsto t^m = t_1^{m_1} \cdots t_d^{m_d} \qquad \textit{for } m = (m_1, \ldots, m_d) \in \mathbb{Z}^d.$$

Proof. We claim that the only morphisms $(\mathbb{C}^*)^d \to \mathbb{C}^*$ are those given by $t \mapsto \lambda t^m$, for $m \in \mathbb{Z}^d$ and $\lambda \in \mathbb{C}^*$. It suffices to show that if $f \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ has more than one term, then $V_{(\mathbb{C}^*)^d}(f) \neq \emptyset$. This is easy to see for univariate Laurent polynomials ($d = 1$). For $d > 1$, suppose that f has more than one term and pick generic integers $u_1, \ldots, u_d$. The univariate Laurent polynomial $g(s) = f(s^{u_1}, \ldots, s^{u_d}) \in \mathbb{C}[s^{\pm 1}]$ has more than one term, and a nonzero root $s \in V_{\mathbb{C}^*}(g)$ corresponds to a root $(s^{u_1}, \ldots, s^{u_d}) \in (\mathbb{C}^*)^d$ of f. Among all morphisms $t \mapsto \lambda t^m$, group homomorphisms must have $\lambda = 1$. $\square$

By Proposition 1.2.5, the characters of a torus T form a lattice, i.e., a free abelian group of finite rank. This is called the *character lattice* of T, and it is conventionally denoted by $M = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{C}^*) \simeq \mathbb{Z}^d$, where $d = \dim T$.

Corollary 1.2.6. *The morphisms $(\mathbb{C}^*)^d \to (\mathbb{C}^*)^n$ which are group homomorphisms are precisely the maps ϕ_A from* (1.1.1)*, for $A \in \mathbb{Z}^{d \times n}$.*

We now study the fibers of the map ϕ_A. These are the sets of pre-images

$$\phi_A^{-1}(x) = \{t \in (\mathbb{C}^*)^d \,:\, \phi(t) = x\}, \quad x \in \operatorname{im} \phi_A.$$

Our main tool to study $\phi_A^{-1}(x)$ is the *Smith normal form* of A. We recall the definition.

Definition 1.2.7 (Smith normal form). *A* Smith normal form *(SNF) of a matrix $A \in \mathbb{Z}^{d \times n}$ is a diagonal matrix $S \in \mathbb{Z}^{d \times n}$ such that there exist matrices $P \in \mathbb{Z}^{d \times d}$ and $Q \in \mathbb{Z}^{n \times n}$ with determinant ± 1 for which $P\,A\,Q = S$, and the diagonal entries $(s_1, \ldots, s_d)$ of S, called* invariant factors*, satisfy $s_i | s_{i+1}$. The invariant factors are unique up to signs.*

Requiring that all invariant factors are nonnegative allows us to speak of *the* Smith normal form of A. The rank of the matrix A, viewed as a matrix over $\mathbb{Q}$, is denoted by $\mathrm{rank}(A)$. By elementary linear algebra, we have that $\mathrm{rank}(A)$ equals the number of nonzero invariant factors.

Example 1.2.8. The Smith normal form of A is computed using `Oscar.jl` as follows:

```
A = [1 1 1 0 0 0; 0 0 0 1 1 1; 1 0 0 1 0 0; 0 1 0 0 1 0; 0 0 1 0 0 1];
A = matrix_space(ZZ,size(A)...)(A) # bring A into Oscar format
S, P, Q = snf_with_transform(A)
```

This uses A from Example 1.1.7. The invariant factors are obtained via `diagonal(S)`. $\diamond$

The columns $a_1, \ldots, a_n \in \mathbb{Z}^d$ of A generate a sublattice

$$\mathbb{Z}A \,=\, \{c_1\, a_1 + \cdots + c_n\, a_n \,:\, c_i \in \mathbb{Z}\} \subset \mathbb{Z}^d.$$

This is the image of the $\mathbb{Z}$-linear map $A : \mathbb{Z}^n \to \mathbb{Z}^d$. The rank of $\mathbb{Z}A$ equals $\operatorname{rank}(A)$. If $\operatorname{rank}(A) = d$, then the quotient group $\mathbb{Z}^d/\mathbb{Z}A$ is finite. Its order is called the *lattice index* of $\mathbb{Z}A$ in $\mathbb{Z}^d$, and we denote this number by $[\mathbb{Z}^d : \mathbb{Z}A]$. It is revealed by the SNF.

Exercise 1.2.9. Show that, if $P \in \mathbb{Z}^{d\times d}$ is such that $\det(P) = \pm 1$, then P is invertible over the integers, i.e., $P^{-1} \in \mathbb{Z}^{d\times d}$. Use this to show that, if $\operatorname{rank}(A) = d$, then the lattice index of $\mathbb{Z}A$ in $\mathbb{Z}^d$ is given by $[\mathbb{Z}^d : \mathbb{Z}A] = |s_1 \cdot s_2 \cdot \cdots \cdot s_d|$.

Example 1.2.10. For an example with $d = 2$ and $[\mathbb{Z}^2 : \mathbb{Z}A] = 2$, see Figure 1.4 (left). $\diamond$

We establish two more facts before stating our main result about fibers.

Exercise 1.2.11. Show that, if A factors as $A = A_1A_2$, then $\phi_A = \phi_{A_2} \circ \phi_{A_1}$.

Lemma 1.2.12. *If $P \in \mathbb{Z}^{d\times d}$ is such that $\det(P) = \pm 1$, then $\phi_P : (\mathbb{C}^*)^d \to (\mathbb{C}^*)^d$ is an isomorphism of algebraic groups. That is, it is an isomorphism of algebraic varieties which is also a group homomorphism.*

Proof. Since $\det(P) = \pm 1$, we have $P^{-1} \in \mathbb{Z}^{d\times d}$ (Exercise 1.2.9). By Exercise 1.2.11, $\phi_P \circ \phi_{P^{-1}}$ is the identity on $(\mathbb{C}^*)^d$, and so is $\phi_{P^{-1}} \circ \phi_P$. $\square$

Proposition 1.2.13. *Let $A \in \mathbb{Z}^{d\times n}$ and let $PAQ = S$ be its Smith normal form. Write*

$$S \;=\; \begin{pmatrix} s_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & s_r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \in \mathbb{N}^{d\times n},$$

where $r = \operatorname{rank}(A) \leq d$. The fiber $\phi_A^{-1}(x)$ for $x \in \operatorname{im}\phi_A$ is isomorphic to $\ker\phi_A = \{t \in (\mathbb{C}^*)^d \,:\, \phi_A(t) = (1, \ldots, 1) \in (\mathbb{C}^*)^n\}$ as an algebraic variety. Moreover, we have

$$\ker\phi_A \;\simeq\; G_{s_1} \times \cdots \times G_{s_r} \times (\mathbb{C}^*)^{d-r}, \tag{1.2.1}$$

where $G_{s_i} \subset \mathbb{C}^*$ is the multiplicative subgroup of s_i-th roots of unity.

Proof. The fiber $\phi_A^{-1}(x)$ for $x = \phi_A(t)$ consists of all $t' \in (\mathbb{C}^*)^d$ such that $\phi_A(t) = \phi_A(t')$. Notice that $\phi_A(t) = \phi_A(t') \iff \phi_A(t)^{-1} \cdot \phi_A(t') = \mathbf{1} \iff \phi_A(t^{-1} \cdot t') = \mathbf{1}$. Here $\mathbf{1}$ is the neutral element $(1, \dots, 1)$ in $(\mathbb{C}^*)^n$. We conclude that $\phi_A^{-1}(\phi_A(t)) = t \cdot \ker \phi_A \simeq \ker \phi_A$. To describe the kernel of ϕ_A, we use the Smith normal form. By Exercise 1.2.11, we have $\phi_A = \phi_{Q^{-1}} \circ \phi_S \circ \phi_{P^{-1}}$. By Lemma 1.2.12 and the fact that $\phi_Q(\mathbf{1}) = \mathbf{1}$, $\phi_A(t) = \mathbf{1}$ if and only if $\phi_S \circ \phi_{P^{-1}}(t) = \mathbf{1}$. Setting $\tau = \phi_{P^{-1}}(t)$, we see that $\phi_S(\tau) = \mathbf{1}$ is equivalent to

$$(\tau^{s_1}, \dots, \tau^{s_r}, 1, \dots, 1) \;=\; (1, \dots, 1).$$

Hence τ belongs to the product in (1.2.1), and

$$\ker \phi_A = \phi_P(G_{s_1} \times \cdots \times G_{s_r} \times (\mathbb{C}^*)^{d-r}). \tag{1.2.2}$$

The proposition follows from the fact that ϕ_P is an isomorphism by Lemma 1.2.12. $\square$

Corollary 1.2.14. *Each fiber $\phi_A^{-1}(x)$ of ϕ_A has dimension $d - \mathrm{rank}(A)$. If $\mathrm{rank}(A) = d$, the map ϕ_A is $[\mathbb{Z}^d : \mathbb{Z}A]$-to-one, where $[\mathbb{Z}^d : \mathbb{Z}A] = |s_1 \cdot s_2 \cdot \cdots \cdot s_d|$.*

Example 1.2.15. For $A = (2) \in \mathbb{Z}^{1\times 1}$, we have $\phi_A(t) = t^2$, $\mathrm{im}\,\phi_A = \mathbb{C}^*$ and $\phi_A^{-1}(x) = \{\pm\sqrt{x}\}$. Hence, the fibers contain two points, which agrees with the lattice index $[\mathbb{Z} : \mathbb{Z}A] = [\mathbb{Z} : 2\mathbb{Z}] = 2$. For $A = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \in \mathbb{Z}^{2\times 1}$, the fiber $\phi^{-1}(x)$ is the hyperbola $\{t \in (\mathbb{C}^*)^2 : t_1 t_2 = x\}$ of dimension $d - \mathrm{rank}(A) = 2 - 1 = 1$. $\diamond$

Corollary 1.2.16. *The map ϕ_A is one-to-one if and only if $\mathbb{Z}A = \mathbb{Z}^d$.*

In cases where ϕ_A is not one-to-one, the Smith normal form can be used to compute an alternative matrix $\bar{A}$ for which $\phi_{\bar{A}}$ is one-to-one and has the same image as ϕ_A. A parametrization which is one-to-one is called *identifiable*, which is a desirable property.

Proposition 1.2.17. *Let A, P, Q, S be as in Proposition 1.2.13. Let $\bar{A} \in \mathbb{Z}^{r\times n}$ be the matrix consisting of the first $r = \mathrm{rank}(A)$ rows of Q^{-1}. The map $\phi_{\bar{A}} : (\mathbb{C}^*)^r \to (\mathbb{C}^*)^n$ is one-to-one, and $\mathrm{im}\,\phi_{\bar{A}} = \mathrm{im}\,\phi_A$.*

Proof. We use the factorization $\phi_A = \phi_{Q^{-1}} \circ \phi_S \circ \phi_{P^{-1}}$ (Exercise 1.2.11). Every point in the image of ϕ_S has coordinates $(y_1, \dots, y_r, 1, \dots, 1)$ for some $y \in (\mathbb{C}^*)^r$, and it is easy to verify that $\phi_S \circ \phi_P^{-1}$ is onto $(\mathbb{C}^*)^r \times \{(1, \dots, 1)\}$. Therefore $\mathrm{im}\,\phi_A = \phi_Q^{-1}((\mathbb{C}^*)^r \times \{(1, \dots, 1)\}) = \mathrm{im}\,\phi_{\bar{A}}$. To show that $\phi_{\bar{A}}$ is one-to-one, by Corollary 1.2.16 it suffices to prove that $\mathbb{Z}\bar{A} = \mathbb{Z}^r$. For this, notice that for any $w \in \mathbb{Z}^r$, we have $w = \bar{A}Q\left(\begin{smallmatrix}w\\ \mathbf{0}\end{smallmatrix}\right)$. $\square$

Exercise 1.2.18. Show that the map $\phi_{\bar{A}} : (\mathbb{C}^*)^r \to \mathrm{im}\,\phi_{\bar{A}}$ from Proposition 1.2.17 is an isomorphism, whose inverse is given by the restriction of the map $\phi_{Q_{1:r}} : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^r$ to $\mathrm{im}\,\phi_{\bar{A}}$. Here $Q_{1:r}$ is the matrix consisting of the first r columns of Q.

Exercise 1.2.19. Compute $[\mathbb{Z}^2 : \mathbb{Z}A]$ for $A = \left(\begin{smallmatrix}3 & 1 & 2\\ 0 & 4 & -4\end{smallmatrix}\right)$, possibly using computer algebra.

Suppose that A has rank d. We outline an algorithm for computing the fiber $\phi_A^{-1}(1)$ of the neutral element, i.e., we compute $\ker \phi_A$. The reader is encouraged to check the details, and to implement this algorithm using any computer algebra system. We need to solve the system of equations $t^{a_1} = t^{a_2} = \cdots = t^{a_n} = 1$. Let $P = (p_1 \ p_2 \ \cdots \ p_d) \in \mathbb{Z}^{d \times d}$, $Q = (q_1 \ q_2 \ \cdots \ q_n) \in \mathbb{Z}^{n \times n}$ and $S = \operatorname{diag}(s_1, \ldots, s_d) \in \mathbb{Z}_{\geq 0}^{d \times n}$ be the matrices from the Smith normal form decomposition $P\,A\,Q = S$. First, apply the change of coordinates $t = (u^{p_1}, \ldots, u^{p_d}) = \phi_P(u)$ to our equations, where $u = (u_1, \ldots, u_d)$ are new variables. Since P has determinant ± 1, this change of coordinates is invertible by Lemma 1.2.12. We obtain the following equivalent system of equations:

$$(u^{Pa_1},\, u^{Pa_2},\, \ldots,\, u^{Pa_n}) \;=\; (1, 1, \ldots, 1). \tag{1.2.3}$$

Now, we apply another invertible change of coordinates $v \mapsto \phi_Q(v)$ to the left- and righthand side of (1.2.3). The result is $u_1^{s_1} = u_2^{s_2} = \cdots = u_d^{s_d} = 1$, which is easy to solve. The number of solutions, i.e., the number of points in the fiber, is $s_1 \cdot s_2 \cdots s_d$. This is the product of the invariant factors, which equals the lattice index of $\mathbb{Z}A$ in $\mathbb{Z}^d$. The method explained here can easily be adapted to compute any fiber $\phi_A^{-1}(x)$ for $x \in \operatorname{im} \phi_A$. This is summarized in Algorithm 1.

Algorithm 1 Computing the fiber $\phi_A^{-1}(x)$ for $A \in \mathbb{Z}^{d \times n}$ and $x \in (\mathbb{C}^*)^n$

```
1: P, S, Q ← Smith normal form of A
2: r ← rank(A)
3: if φ_Q(x)_{r+1} = ··· = φ_Q(x)_n = 1 then                     ▷ x ∈ im φ_A
4:     U ← {(u_1, ..., u_r) : u_i^{s_i} = φ_Q(x)_i, i = 1, ..., r}
5:     return φ_P(U × (C*)^{d−r})
6: else                                                          ▷ x ∉ im φ_A
7:     return ∅
8: end if
```

By definition, the image of ϕ_A is a dense subset of the affine toric variety Y_A, see Section 1.1. The next proposition describes the tangent spaces of $\operatorname{im} \phi_A$ in terms of A.

Proposition 1.2.20. *The tangent space of Y_A at $\phi_A(t)$ is spanned by the rows of*

$$A \cdot \begin{pmatrix} t^{a_1} & 0 & \cdots & 0 \\ 0 & t^{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t^{a_n} \end{pmatrix} = \begin{pmatrix} a_{11}t^{a_1} & a_{12}t^{a_2} & \cdots & a_{1n}t^{a_n} \\ a_{21}t^{a_1} & a_{22}t^{a_2} & \cdots & a_{2n}t^{a_n} \\ \vdots & \vdots & & \vdots \\ a_{d1}t^{a_1} & a_{d2}t^{a_2} & \cdots & a_{dn}t^{a_n} \end{pmatrix}.$$

Proof. The matrix in the lemma is the transpose of the Jacobian matrix of the map ϕ_A, multiplied by the $d \times d$ diagonal matrix with the entries of t on the diagonal. Hence, its row span equals the column span of the Jacobian matrix of ϕ_A. □

Proposition 1.2.21. *The image of* ϕ_A *in* $(\mathbb{C}^*)^n$ *is a subtorus of dimension* $r = \mathrm{rank}(A)$. *It is a closed subvariety of* $(\mathbb{C}^*)^n$*, defined by the following binomial equations. Let* $B = (b_1 \ \cdots \ b_{n-r}) \in \mathbb{Z}^{n\times(n-r)}$ *be a matrix whose columns form a* $\mathbb{Z}$*-basis for* $\ker A$*. We have*

$$\mathrm{im}\,\phi_A \;=\; \{x \in (\mathbb{C}^*)^n \,:\, x^{b_1} = \cdots = x^{b_{n-r}} = 1\}. \tag{1.2.4}$$

Proof. The fact that $\mathrm{im}\,\phi_A$ is isomorphic to a torus follows from Proposition 1.2.17 and Exercise 1.2.18. The statement about the dimension also follows from Proposition 1.2.20. The equality (1.2.4) will follow from the fact that $\mathrm{im}\,\phi_A = \ker\phi_B$. If $PAQ = S$ is the SNF of A, we can choose B to be the submartix of Q consisting of its last $n-r$ columns. The SNF of B is then given by $\left(\begin{smallmatrix} A' \\ \bar{A}\end{smallmatrix}\right) B = \left(\begin{smallmatrix} \mathrm{id}_{(n-r)} \\ \mathbf{0}\end{smallmatrix}\right)$, where A' consists of the last $n-r$ rows of Q^{-1}, and $\bar{A}$ consists of its first r rows. It now follows from (1.2.2) that $\ker\phi_B = \mathrm{im}\,\phi_{\bar{A}} = \mathrm{im}\,\phi_A$, where the last equality is Proposition 1.2.17. □

Exercise 1.2.22. Find an example where $n \geq 2$ and the image of ϕ_A is closed in $\mathbb{C}^n$.

Since the subtorus $\mathrm{im}\,\phi_A \subset (\mathbb{C}^*)^n$ is defined by algebraic equations, its dimension equals that of its closure in $\mathbb{C}^n$. Here is a direct consequence.

Corollary 1.2.23. *The affine toric variety* Y_A *has dimension* $r = \mathrm{rank}(A)$.

Remark 1.2.24. Proposition 1.2.21 implies that the image of ϕ_A only depends on the row span of A over $\mathbb{Q}$. Indeed, if A and $\bar{A}$ have the same row span over $\mathbb{Q}$, then their kernels over $\mathbb{Z}$ agree. This is the case for A and $\bar{A}$ from Proposition 1.2.17.

1.3 Toric ideals

The toric varieties in this chapter come with a parametrization map ϕ_A. This section studies a different representation: which polynomials vanish on a toric variety?

Definition 1.3.1 (toric ideal)**.** *An ideal of the polynomial ring* $\mathbb{C}[x_1, \ldots, x_n]$ *is* toric *if it is the vanishing ideal* $I(Y_A)$ *of an affine toric variety* Y_A *for some matrix* $A \in \mathbb{Z}^{d\times n}$.

We saw several examples in Section 1.1. Note that, in these examples, the toric ideal is generated by binomials. This is true in general. Our first theorem provides an infinite set of binomial generators in terms of the matrix A.

Theorem 1.3.2. *The toric ideal* $I_A = I(Y_A) \subset \mathbb{C}[x_1, \ldots, x_n]$ *is generated by the set*

$$\mathcal{B}_A \;=\; \{x^u - x^v \,:\, u, v \in \mathbb{N}^n,\ A(u-v) = 0\}.$$

Proof. By definition of Y_A, a polynomial $f(x_1, \ldots, x_n)$ belongs to I_A if and only if

$$f(t^{a_1}, \ldots, t^{a_n}) \;=\; 0.$$

The inclusion $\langle \mathcal{B}_A \rangle \subset I_A$ is clear. For the opposite inclusion, we follow the proof of [65, Lemma 4.1]. It suffices to show that each $f \in I_A$ is a (finite) $\mathbb{C}$-linear combination of elements in $\mathcal{B}_A$. We proceed by contradiction. Suppose there exists $f \in I_A$ such that f is not a linear combination of $\mathcal{B}_A$. We fix a monomial order $\prec$ on $\mathbb{C}[x_1, \ldots, x_n]$, and pick such an f whose $\prec$-leading term $c_u\, x^u$ is minimal. We write $f = c_u\, x^u + \sum_{x^v \prec x^u} c_v x^v$, so that $f(t^{a_1}, \ldots, t^{a_n}) = c_u\, t^{Au} + \sum_{x^v \prec x^u} c_v t^{Av} = 0$. The term $c_u\, t^{Au}$ must cancel, so there must exist a monomial $x^v \prec x^u$ for which $Au = Av$. The leading monomial of the polynomial $g = f - c_u(x^u - x^v)$ is strictly smaller than that of f, but it is no linear combination of $\mathcal{B}_A$ and it belongs to I_A. This is the desired contradiction. □

By Hilbert's basis theorem, some finite subset of $\mathcal{B}_A$ generates I_A.

Exercise 1.3.3. Show that, for any monomial order, the reduced Gröbner basis of I_A consists of a finite subset of $\mathcal{B}_A$. Hint: apply Buchberger's algorithm to a finite set of binomials in $\mathcal{B}_A$ generating I_A [65, Corollary 4.4].

Exercise 1.3.4. Suppose that $A' \in \mathbb{Z}^{r \times n}$ is an integer matrix whose row span over $\mathbb{Q}$ equals that of $A \in \mathbb{Z}^{d \times n}$. Use Theorem 1.3.2 to show that $Y_A = Y_{A'}$. In particular, $Y_A = Y_{\bar{A}}$, with $\bar{A}$ as in Proposition 1.2.17.

Exercise 1.3.5. Let $d = 2, n = 3$ and $A = \left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{smallmatrix}\right)$. What is the dimension of Y_A? What are its defining equations? Same questions for $A = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix}\right)$.

Exercise 1.3.6. Let $B = (b_1 \ \cdots \ b_r) \in \mathbb{Z}^{n \times (n-r)}$ be a matrix whose columns are a $\mathbb{Z}$-basis for $\ker A$. Write each of these vectors as $b_i = u_i - v_i$, where $u_i, v_i \in \mathbb{N}^n$ have disjoint support. Use Proposition 1.2.21 to show that I_A is the ideal saturation

$$\langle x^{u_1} - x^{v_1}, \ldots, x^{u_r} - x^{v_r} \rangle : \langle x_1 x_2 \cdots x_n \rangle^\infty.$$

Exercise 1.3.7. For each column $a_i \in \mathbb{Z}^d$ of A, write $a_i = a_i^+ - a_i^-$, where $a_i^+, a_i^- \in \mathbb{N}^d$. Show that I_A is the elimination ideal $J \cap \mathbb{C}[x_1, \ldots, x_n]$, where

$$J \;=\; \langle t_0 t_1 \cdots t_d - 1,\; t^{a_1^-} x - t^{a_1^+},\; \ldots,\; t^{a_n^-} x - t^{a_n^+} \rangle \;\subset\; \mathbb{C}[t_0, t_1, \ldots, t_d, x_1, \ldots, x_n].$$

Example 1.3.8. We illustrate how to compute I_A in `Oscar.jl`. Consider the matrix

$$A = \begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 & 1 \end{pmatrix} \tag{1.3.1}$$

from [52, Example 8.9]. The associated toric variety Y_A has dimension 3 (Corollary 1.2.23) and is embedded in $\mathbb{C}^7$. To compute the toric ideal $I_A = I(Y_A)$, we use the function `toric_ideal`. An example of in- and output is shown in Figure 1.3. According to `Oscar` conventions, `toric_ideal` requires the *transpose* of A as an input. The result, as in [52, Example 8.9], is an ideal generated by 9 binomials. Using the kernel matrix

$$B^\top = \begin{pmatrix} -1 & 1 & -1 & 0 & 0 & 0 & 1 \\ -2 & 2 & -1 & 0 & 0 & 1 & 0 \\ -2 & 3 & -2 & 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & 1 & 0 & 0 & 0 \end{pmatrix},$$

```
In [56]: A = [2 2 1 0 0 1 1; 1 0 0 1 2 2 1; 0 1 2 2 1 0 1]
         I = toric_ideal(transpose(A))

Out[56]: ideal(x4*x6 - x5*x7, x3*x6 - x7^2, -x1*x7 + x2*x6, x3*x5 - x4*x7, x2*x5 - x7^2,
         x1*x5 - x6*x7, x2*x4 - x3*x7, x1*x4 - x7^2, x1*x3 - x2*x7)
```

Figure 1.3: Computing toric ideals in `Oscar.jl`.

computed using `Oscar.jl`, the ideal $\langle x^{u_1} - x^{v_1}, \ldots, x^{u_r} - x^{v_r}\rangle$ from Exercise 1.3.6 is

$$\langle x_1x_3 - x_2x_7,\, x_1^2x_3 - x_2^2x_6,\, x_1^2x_3^2 - x_2^3x_5,\, x_1x_3^2 - x_2^2x_4\rangle.$$

It defines a 5-dimensional variety in $\mathbb{C}^7$. The nine associated primes are

$$\begin{gathered}\langle x_1, x_2\rangle, \quad \langle x_2, x_3\rangle, \quad \langle x_2, x_3, x_7\rangle, \quad \langle x_1, x_2, x_7\rangle, \quad \langle x_1, x_2, x_3\rangle, \\ \langle x_1, x_2, x_6, x_7\rangle, \quad \langle x_1, x_2, x_3, x_7\rangle, \quad \langle x_2, x_3, x_4, x_7\rangle \quad \text{and} \quad I_A.\end{gathered}$$

These depend on the choice of B. After saturating by $\langle x_1x_2\cdots x_7\rangle$, only I_A is left. $\diamond$

Theorem 1.3.9. *An ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$ is prime and generated by binomials if and only if I is toric.*

Proof. If I is toric, it is prime since it is the vanishing ideal of an irreducible variety. It is generated by binomials by Theorem 1.3.2. For the other implication, suppose $I = \langle x^{u_i} - x^{v_i}, i = 1, \ldots, r\rangle$ is prime. Let $b_i = u_i - v_i$ and consider the matrix $B \in \mathbb{Z}^{n\times r}$ whose columns are b_i. Let $A \in \mathbb{Z}^{d\times n}$ be its left kernel matrix (over $\mathbb{Z}$). That is, $A \cdot B = 0$ and $\operatorname{rank}(A) = \operatorname{rank}(\operatorname{coker} B) = d$. By construction, we have $I \subset I_A = I(Y_A)$. We show that this inclusion is in fact an equality. Notice that $(1, \ldots, 1) \in V(I)$. The Jacobian matrix of $x^{u_i} - x^{v_i}, i = 1, \ldots, r$ at $(1, \ldots, 1)$ is $B^\top$. Therefore, $V(I)$ has dimension at most $\dim(\ker B) = d$. It also has dimension at least d, because it contains Y_A (Corollary 1.2.23). Since I is prime, $V(I) = Y_A$ and, by the Nullstellensatz, $I = I_A$. $\square$

Exercise 1.3.10. A *circuit* of the matrix A is a binomial $x^u - x^v$ so that $u - v \in \ker A$, the entries of $u - v$ are pairwise coprime, and the *support* of the vector $u - v$ is minimal with respect to inclusion. Here the support of a vector is the index set of its nonzero entries. Geometrically, a circuit is the defining equation of a codimension-one coordinate projection of Y_A. The set of all circuits of A is denoted by $\mathcal{C}_A$. It was proved in [26] that the variety of the binomial ideal $\langle \mathcal{C}_A\rangle$ generated by all circuits defines the toric variety Y_A set-theoretically. That is, $\sqrt{\langle \mathcal{C}_A\rangle} = I_A$. Verify this statement for the circuit ideal $\langle \mathcal{C}_A\rangle$ of the matrix $A = \left(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3\end{smallmatrix}\right)$, and check that it is not prime.

Our discussion on toric ideals was limited to affine toric varieties so far. The vanishing ideal $I(X_A)$ of the projective toric variety X_A is toric too, in the sense of Definition 1.3.1.

Theorem 1.3.11. *The vanishing ideal $I(X_A)$ of the projective toric variety X_A associated to $A \in \mathbb{Z}^{d\times n}$ is the toric ideal $I_{\hat{A}}$, where*

$$\hat{A} = \begin{pmatrix} A \\ \mathbf{1}^\top \end{pmatrix} \quad \in \mathbb{Z}^{(d+1)\times n}. \tag{1.3.2}$$

Here, $\mathbf{1} = (1,\dots,1)^\top$ is the all-ones vector of length n.

Proof. By Theorem 1.3.2 and the fact that the last row of $\hat{A}$ is $\mathbf{1}^\top$, the ideal $I_{\hat{A}}$ is homogeneous. It is also prime: it is the vanishing ideal of $Y_{\hat{A}}$. Finally, observe that $Y_{\hat{A}}$ is the affine cone over X_A. This follows immediately from the definition of $\phi_{\hat{A}}$:

$$\phi_{\hat{A}}(t_1,\dots,t_d,u) = (u\,t^{a_1},\dots,u\,t^{a_n}).$$

We used parameters $(t_1,\dots,t_d,u)$ to emphasize the special role of the last row of A. □

Example 1.3.12 ($d=2, n=4$)**.** The toric surface $Y_A \subset \mathbb{C}^4$ corresponding to the matrix A from Exercise 1.3.10 is the affine cone over the twisted cubic curve X_A. ⋄

Exercise 1.3.13. Use toric ideals to solve Exercise 1.1.4: Y_A contains $(1,\dots,1) \in \mathbb{R}^n$. Similarly, each projective toric variety X_A contains $(1:\dots:1) \in \mathbb{RP}^{n-1}$.

Proposition 1.3.14. *Let $T \in \mathbb{Q}^{d\times d}$ and $m \in \mathbb{Q}^d$ be such that* $\det(T) \neq 0$ *and the entries of $T\cdot A + m = \begin{pmatrix} T\cdot a_1 + m & \cdots & T\cdot a_n + m\end{pmatrix} \in \mathbb{Q}^{d\times n}$ are integers. We have $X_A = X_{T\cdot A+m}$. That is, affine transformations of A do not change its projective toric variety.*

Proof. Let $\mathcal{A} = T\cdot A + m \in \mathbb{Z}^{d\times n}$ and observe that

$$\begin{pmatrix} T & m \\ \mathbf{0}^\top & 1 \end{pmatrix} \cdot \hat{A} = \begin{pmatrix} T\cdot A + m \\ \mathbf{1}^\top \end{pmatrix} = \hat{\mathcal{A}}.$$

Hence $\hat{A}$ and $\hat{\mathcal{A}}$ have the same row span over $\mathbb{Q}$, so by Exercise 1.3.4 and Theorem 1.3.11 we have $I(X_A) = I_{\hat{A}} = I_{\hat{\mathcal{A}}} = I(X_{\mathcal{A}})$. The proposition now follows from the projective Nullstellensatz. □

Theorem 1.3.11 together with Corollary 1.2.23 gives a formula for $\dim X_A$ in terms of A. This is commonly expressed in terms of a lattice associated to A.

Definition 1.3.15 (affine lattice)**.** *The* lattice affinely generated by $A \in \mathbb{Z}^{d\times n}$ *is*

$$\mathbb{Z}'A = \left\{ \sum_{i=1}^n c_i a_i \;:\; c_i \in \mathbb{Z},\; \sum_{i=1}^n c_i = 0 \right\} \subset \mathbb{Z}^d.$$

Example 1.3.16. The lattices $\mathbb{Z}A$ and $\mathbb{Z}'A$ may coincide. This happens for instance for $A = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. In Figure 1.4, we show $\mathbb{Z}A$ and $\mathbb{Z}'A$ for $A = \left(\begin{smallmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{smallmatrix}\right)$. Notice that $\mathbb{Z}A$ has rank 2 and lattice index 2 inside $\mathbb{Z}^2$. The affine lattice $\mathbb{Z}'A$ has rank 1, the dimension of the smallest affine-linear space containing A (hence the name). The lattice $\mathbb{Z}'A$ is contained in the unique translate of the affine line containing A passing through 0. ⋄

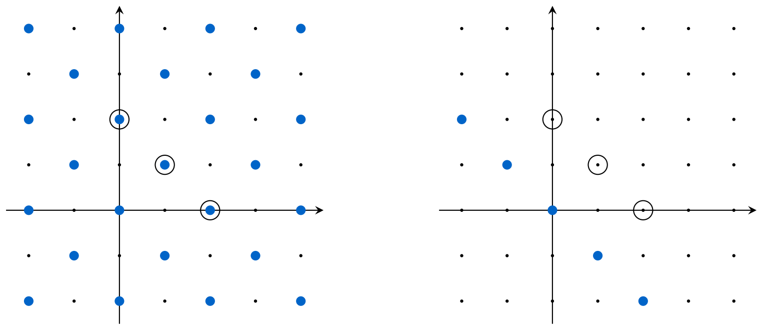

Figure 1.4: Left: the lattice $\mathbb{Z}A$ generated by $A = \left(\begin{smallmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{smallmatrix}\right)$ (blue) inside the ambient lattice $\mathbb{Z}^2$ (in black). Right: the affine lattice $\mathbb{Z}'A$.

Exercise 1.3.17. Write $A - a_i = (a_1 - a_i \ \cdots \ a_{i-1} - a_i \ \ a_{i+1} - a_i \cdots a_n - a_i)$ for the $d \times (n-1)$-matrix obtained by subtracting a_i from each column of A, and deleting the i-th vector $a_i - a_i = 0$. Show that $\mathbb{Z}'A = \mathbb{Z}(A - a_i)$, for $i = 1, \ldots, n$.

Corollary 1.3.18. *The dimension of* X_A *is given by* $\dim X_A = \operatorname{rank}(\hat{A}) - 1 = \operatorname{rank}(\mathbb{Z}'A)$.

Proof. The equality $\dim X_A = \operatorname{rank}(\hat{A}) - 1$ is a direct consequence of the fact that $Y_{\hat{A}}$ is the affine cone over X_A, and has dimension $\operatorname{rank}(\hat{A})$ (Corollary 1.2.23). The equality $\operatorname{rank}(\hat{A}) - 1 = \operatorname{rank}(\mathbb{Z}'A)$ follows from the fact that $\operatorname{rank}(\hat{A})$ is the rank of

$$\hat{A} \cdot \begin{pmatrix} 1 & -1 & -1 & \cdots & -1 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 - a_1 & a_3 - a_1 & \cdots & a_n - a_1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \qquad \square$$

Exercise 1.3.19. Show that the toric ideal I_A is homogeneous if and only if $\mathbf{1}^\top$ lies in the row span of A over $\mathbb{Q}$. Hint: in that case, there exists a vector $w \in \mathbb{Z}^d$ such that $w^\top A = k \cdot \mathbf{1}^\top$, for some $k \in \mathbb{N} \setminus \{0\}$. Show that, for any $u \in \mathbb{C}^*$, we then have

$$u^k \cdot \phi_A(t) \ = \ \phi_A(\lambda_w(u) \cdot t),$$

where $\lambda_w : \mathbb{C}^* \to (\mathbb{C}^*)^d$ is the group homomorphism $u \mapsto (u^{w_1}, \ldots, u^{w_d})$, $u^k \cdot \phi_A(t)$ is multiplication in the algebraic group $(\mathbb{C}^*)^n$, and $\lambda_w(u) \cdot t$ is multiplication in $(\mathbb{C}^*)^d$.

Remark 1.3.20. Let $T \simeq (\mathbb{C}^*)^d$ be a torus. A morphism $\lambda_w : \mathbb{C}^* \to T$ of algebraic groups, like that in Exercise 1.3.19, is called a *co-character* or *one-parameter subgroup* of T. We will encounter these again in Section 4.4.

Exercise 1.3.21. Show that the projective toric variety $X_{A'} \subset \mathbb{P}^6$ with

$$A' \ = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 & 1 \end{pmatrix}$$

has dimension 2, and it equals the projective toric variety X_A with A as in (1.3.1).

Further reading

The embedded toric varieties Y_A and X_A are discussed, for instance, in [19, §1.1 and §2.1] and [33, §5.1]. For more on binomial and toric ideals, see [26, 65]. We saw in Exercise 1.3.7 that the toric ideal I_A can be computed via variable elimination. Faster algorithms for computing toric ideals are found in [9] and [65, Chapter 12]. In some situations, a prime ideal is *secretly toric*, in the sense that a linear change of coordinates makes it toric. For recent progress on exposing such hidden toric structures, see [39, 41, 51].

Chapter 2

Cones and affine toric varieties

This chapter describes the coordinate ring of an affine toric variety Y_A in terms of semigroups related to A. This description of Y_A is intrinsic, in the sense that it does not depend on the embedding of Y_A in affine space. Important geometric properties of a toric variety, like normality and smoothness, can be translated into properties of semigroups. Normal affine toric varieties arise from semigroups of polyhedral cones.

2.1 Semigroup algebras

The character lattice of a torus $T \simeq (\mathbb{C}^*)^d$ is $M \simeq \mathbb{Z}^d$, see Section 1.2. An element $m \in M$ represents a group homomorphism $(\mathbb{C}^*)^d \to \mathbb{C}^*$, given by the Laurent monomial t^m. For concreteness, we simply set $T = (\mathbb{C}^*)^d$ and $M = \mathbb{Z}^d$. Naming our lattice M helps to avoid confusion in later parts of the book, where it will be important to distinguish M from its dual lattice $N = M^\vee \simeq \mathbb{Z}^d$. The columns of $A \in \mathbb{Z}^{d\times n}$ are elements of M.

The coordinate ring of a toric variety is encoded by the following algebraic structure.

Definition 2.1.1. *An* affine semigroup *in $M = \mathbb{Z}^d$ is a subset $\mathsf{S} \subseteq M$ of the form*

$$\mathbb{N}\mathscr{A} = \left\{ \sum_{a\in\mathscr{A}} c_a\, a \;:\; c_a \in \mathbb{N} \right\},$$

where $\mathscr{A} \subseteq \mathsf{S}$ is finite. We say that S is generated by *$\mathscr{A}$.*

Notice that, for any affine semigroup S, we have $0 \in \mathsf{S}$ and S is closed under addition. The affine semigroups in this chapter are generated by the columns of the matrix $A \in \mathbb{Z}^{d\times n}$. To make this explicit, we denote such a semigroup by $\mathbb{N}A \subseteq \mathbb{Z}A \subseteq M$.

Example 2.1.2. The semigroup $\mathsf{S} = \mathbb{N}A$ generated by the columns of $A = \mathrm{id}_d$ is $\mathbb{N}^d \subset \mathbb{Z}A = \mathbb{Z}^d$. The entire lattice $M = \mathbb{Z}^d$ is the semigroup of $A = (\mathrm{id}_d \;\; -\mathrm{id}_d)$. $\diamond$

Exercise 2.1.3. Show that the semigroup $\mathsf{S} = \mathbb{N}A$ for $A = (2 \;\; 3)$ equals $\mathbb{N} \setminus \{1\} \subset M$.

Linear combinations of characters (i.e., Laurent monomials) are regular functions on T. Semigroups of M lead to important subrings of $\mathbb{C}[T] \simeq \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$.

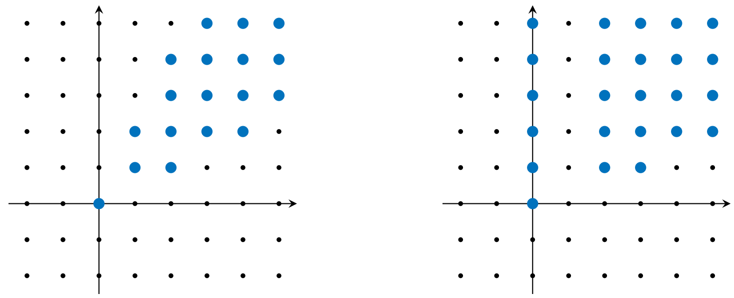

Figure 2.1: Two affine semigroups in $\mathbb{Z}^2$.

Definition 2.1.4 (Semigroup algebra)**.** *Let $M = \mathbb{Z}^d$ be the character lattice of $T = (\mathbb{C}^*)^d$. The* semigroup algebra *associated to an affine semigroup $\mathsf{S} \subseteq M$ is the $\mathbb{C}$-algebra*

$$\mathbb{C}[\mathsf{S}] = \left\{ \sum_{m\in\mathsf{S}} c_m t^m \;:\; c_m \in \mathbb{C}, \textit{ finitely many } c_m \textit{ are nonzero} \right\} \quad \subseteq \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}].$$

Notice that $\mathbb{C}[\mathsf{S}]$ is closed under addition and multiplication, and multiplication is induced by the group operation in M: $t^m t^{m'} = t^{m+m'}$. If $\mathsf{S} = \mathbb{N}A$ is the semigroup of $M = \mathbb{Z}^d$ generated by the columns $a_1, \ldots, a_n$ of A, then $\mathbb{C}[\mathbb{N}A]$ is generated as a $\mathbb{C}$-algebra by the Laurent monomials t^{a_i}, i.e., $\mathbb{C}[\mathbb{N}A] = \mathbb{C}[t^{a_1}, \ldots, t^{a_n}]$. For that reason, our semigroup algebras are also called *monomial subalgebras* of $\mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$. Monomial algebras form a rich subject in their own right, see for instance [74].

Example 2.1.5. The ring $\mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ is the semigroup algebra $\mathbb{C}[M] = \mathbb{C}[\mathbb{Z}^d]$. ◇

Example 2.1.6. It is instructive to represent semigroups and their algebras pictorially. The columns of the matrix $A = \left(\begin{smallmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{smallmatrix}\right)$ generate an infinite subset of $M = \mathbb{Z}^2$, shown as blue dots in the left part of Figure 2.1. Elements of the monomial algebra $\mathbb{C}[\mathbb{N}A]$ are polynomials $\sum_{m\in\mathbb{N}A} c_m t^m$ whose exponents correspond to blue dots. A different semigroup for $A = \left(\begin{smallmatrix} 0 & 2 & 3 \\ 1 & 1 & 1 \end{smallmatrix}\right)$ is shown in the right part of Figure 2.1. ◇

Our interest in semigroups algebras comes from the fact that they are coordinate rings of toric varieties. Here is the precise statement.

Proposition 2.1.7. *Let $\mathsf{S} = \mathbb{N}A \subset M$ be an affine semigroup. The semigroup algebra $\mathbb{C}[\mathsf{S}]$ is a finitely generated $\mathbb{C}$-algebra. Moreover, it is an integral domain. The corresponding affine variety, denoted by* $\operatorname{Specm}(\mathbb{C}[\mathsf{S}])$*, is isomorphic to the affine toric variety Y_A. That is, the coordinate ring of Y_A is isomorphic to $\mathbb{C}[\mathsf{S}]$.*

Proposition 2.1.7 uses the correspondence between isomorphism classes of finitely generated, nilpotent-free $\mathbb{C}$-algebras and isomorphism classes of affine varieties over $\mathbb{C}$, see for instance [61, Section 2.5]. That correspondence sends an affine variety V to its coordinate ring $\mathbb{C}[V]$, and a $\mathbb{C}$-algebra R to its *maximal spectrum* $\operatorname{Specm}(R)$. This restricts to a correspondence between finitely generated integral domains and irreducible affine varieties. We briefly recall how to compute the variety Specm(R). Suppose R is generated as a $\mathbb{C}$-algebra by $f_1, \ldots, f_n$: $R = \mathbb{C}[f_1, \ldots, f_n]$. Let $\varphi : \mathbb{C}[x_1, \ldots, x_n] \to R$ be the map that sends $x_i \mapsto f_i$. The kernel of φ is an ideal I. Since R is nilpotent-free, that ideal is radical. The variety Specm(R) is isomorphic to $V(I) \subseteq \mathbb{C}^n$.

Proof of Proposition 2.1.7. By definition, a semigroup algebra is finitely generated. We have $\mathbb{C}[\mathsf{S}] = \mathbb{C}[t^{a_1}, \ldots, t^{a_n}]$, where $a_1, \ldots, a_n \in M$ generate S. Note that $\mathbb{C}[\mathsf{S}] \subseteq \mathbb{C}[M] \simeq \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ is a subring of an integral domain. It is therefore an integral domain itself. For the final statement, let $\mathsf{S} = \mathbb{N}A$ and let $\phi_A^* : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[M]$ be the pullback map of ϕ_A, given by $f \mapsto f \circ \phi_A$. The isomorphism $\mathbb{C}[\mathsf{S}] \simeq \mathbb{C}[x_1, \ldots, x_n]/I(Y_A) = \mathbb{C}[Y_A]$ is seen from the short exact sequence

$$0 \to I(Y_A) \hookrightarrow \mathbb{C}[x_1, \ldots, x_n] \xrightarrow{\phi_A^*} \mathbb{C}[\mathsf{S}] \to 0. \qquad \square$$

Corollary 2.1.8. *If the two semigroups $\mathbb{N}A \subseteq M$ and $\mathbb{N}A' \subseteq M'$ are isomorphic, then the affine toric varieties Y_A and $Y_{A'}$ are isomorphic as well.*

Corollary 2.1.8 is an important consequence of Proposition 2.1.7. The semigroup S represents the toric variety Specm$(\mathbb{C}[\mathsf{S}])$ *independently of its embedding.* Choosing two different sets of generators for S leads to two different embeddings $Y_A \simeq Y_{A'}$.

Example 2.1.9. The matrix $A = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{smallmatrix}\right)$ appeared as A_1 in Example 1.1.6. The corresponding toric surface is $Y_A = \{x - yz = 0\}$. The columns of A generate the semigroup $\mathbb{N}A = \mathbb{N}^2$, which is the same as $\mathbb{N}\,\mathrm{id}_2$. The identity matrix id_2 corresponds to the isomorphic toric variety $Y_{\mathrm{id}_2} = \mathbb{C}^2$ (Example 1.1.1). The isomorphism $Y_A \simeq \mathbb{C}^2$ is simply the projection $(x, y, z) \mapsto (y, z)$, mirroring the fact that id_2 is a submatrix of A. $\diamond$

Proposition 2.1.7 says that finitely generated monomial algebras are the coordinate rings of affine toric varieties. We now state a projective version. The coordinate ring $\mathbb{C}[X]$ of a projective variety $X \subset \mathbb{P}^{n-1}$ is the graded ring $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]/I(\hat{X})$, where $\hat{X} \subset \mathbb{C}^n$ is the affine cone over X. Here the ideal $I(\hat{X})$ is homogeneous, and the grading on $\mathbb{C}[X]$ is induced by the standard grading on $\mathbb{C}[x_1, \ldots, x_n]$.

Proposition 2.1.10. *The coordinate ring $\mathbb{C}[X_A]$ of the projective toric variety $X_A \subset \mathbb{P}^{n-1}$ is isomorphic to the semigroup algebra $\mathbb{C}[\mathbb{N}\hat{A}]$, where $\hat{A}$ is as in Theorem 1.3.11. This is a graded isomorphism, where the grading on $\mathbb{C}[\mathbb{N}\hat{A}]$ is given by*

$$\mathbb{C}[\mathbb{N}\hat{A}] = \bigoplus_{k=0}^{\infty} \mathbb{C}[\mathbb{N}\hat{A}]_k = \bigoplus_{k=0}^{\infty} \bigoplus_{\substack{m \in \mathbb{N}\hat{A} \\ m_{d+1} = k}} \mathbb{C} \cdot t^m.$$

In other words, the degree of a monomial t^m in $\mathbb{C}[\mathbb{N}\hat{A}]$ is the last coordinate of m.

Proof. This follows from Proposition 2.1.7, using $Y_{\hat{A}} = \hat{X}_A$ and the fact that

$$0 \to I(Y_{\hat{A}}) \hookrightarrow \mathbb{C}[x_1, \ldots, x_n] \xrightarrow{\phi_{\hat{A}}^*} \mathbb{C}[\mathbb{N}\hat{A}] \to 0.$$

is an exact sequence of graded rings (see Exercise 2.1.11). $\square$

Exercise 2.1.11. Show that $\phi_{\hat{A}}^*(\mathbb{C}[x_1, \ldots, x_n]_k) = \mathbb{C}[\mathbb{N}\hat{A}]_k$.

Exercise 2.1.12. Consider the semigroup depicted in Figure 2.1, right. Write down three generators as the columns of $\hat{A} \in \mathbb{Z}^{2\times 3}$. Identify the corresponding semigroup algebra with the graded coordinate ring $\mathbb{C}[x, y, z]/\langle y^3 - xz^2\rangle$ of the cuspidal cubic curve X_A. Can you read the first values of the Hilbert function of X_A from the figure?

2.2 Rational polyhedral cones

In Example 2.1.6, we plotted two different affine semigroups in the plane $\mathbb{R}^2 \supset \mathbb{Z}^2 = M$. The blue points in Figure 2.2 lie inside the *cone* in $\mathbb{R}^2$ obtained by taking nonnegative linear combinations of the columns of A. In the next section, we go in the other direction: a cone σ gives an affine semigroup S_σ, and S_σ gives an affine toric variety $\mathrm{Specm}(\mathbb{C}[\mathsf{S}_\sigma])$. The aim of this section is to introduce the cones we will need, and their properties.

If $M \simeq \mathbb{Z}^d$ is an abstract lattice, its affine semigroups $\mathsf{S} \subset M$ consist of lattice points inside cones of the vector space $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R} \simeq \mathbb{R}^d$. The dual vector space $N_\mathbb{R} = (M_\mathbb{R})^\vee$ is obtained in a similar way from the dual lattice $N = \mathrm{Hom}_\mathbb{Z}(M, \mathbb{Z})$: $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R} \simeq \mathbb{R}^d$. For concreteness, one can replace $N_\mathbb{R}$ and $M_\mathbb{R}$ by $\mathbb{R}^d$, bearing in mind that they are each other's dual. It is conventional in toric geometry to start from a cone σ in $\mathbb{N}_\mathbb{R}$.

Definition 2.2.1 (Rational convex polyhedral cone)**.** *A* rational convex polyhedral cone *in $N_\mathbb{R}$ is a set of the form*

$$\sigma = \mathrm{Cone}(S) = \left\{ \sum_{u \in S} \lambda_u \, u \;:\; \lambda_u \in \mathbb{R}_{\geq 0} \right\},$$

for some finite set $S \subset N$. The set S is called a set of (cone) generators *of σ.*

More generally, a convex cone is a set of the form $\mathrm{Cone}(S)$ for $S \subset N_\mathbb{R}$. Requiring S to be finite makes the cone polyhedral, and requiring $S \subset N$ makes it rational. The cones related to toric varieties are rational, convex and polyhedral. In what follows, we will omit the adjectives 'polyhedral' and 'convex' and simply write 'cone' in the statements that hold for any convex polyhedral cone. When we write 'rational cone', we mean 'rational convex polyhedral cone'. Given a cone $\sigma \subset N_\mathbb{R}$, its *dual cone* $\sigma^\vee$ is

$$\sigma^\vee = \{m \in M_\mathbb{R} \;:\; \langle u, m\rangle \geq 0 \text{ for all } u \in \sigma\}.$$

Here $\langle \cdot, \cdot \rangle : N_\mathbb{R} \times M_\mathbb{R} \to \mathbb{R}$ is the natural pairing between our dual vector spaces: under the isomorphisms $N_\mathbb{R} \simeq \mathbb{R}^d$ and $M_\mathbb{R} \simeq \mathbb{R}^d$, it is the usual dot product of vectors. We have $(\sigma^\vee)^\vee = \sigma$, and if σ is rational then so is $\sigma^\vee$. For a point $m \in M_\mathbb{R}$, we set

$$H_m = \{u \in N_\mathbb{R} \;:\; \langle u, m\rangle = 0\} \quad \text{and} \quad H_m^+ = \{u \in N_\mathbb{R} \;:\; \langle u, m\rangle \geq 0\}.$$

Definition 2.2.2 (Face of a cone)**.** *A* face τ *of a cone σ is a subset of the form $\tau = \sigma \cap H_m$ for $m \in \sigma^\vee$.*

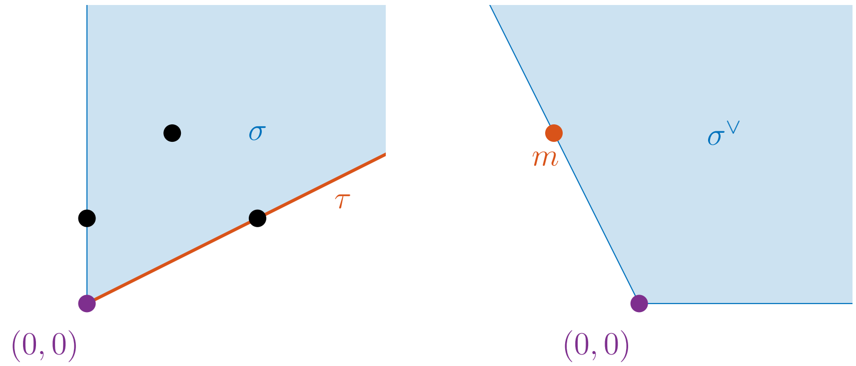

Figure 2.2: A rational cone σ and its dual.

Faces of (rational) cones are again (rational) cones. Note that σ is considered to be face of itself, as $0 \in \sigma^\vee$. If τ is a face of σ, we write $\tau \preceq \sigma$ and $\tau \prec \sigma$ if $\tau \neq \sigma$.

Example 2.2.3. Let $N = \mathbb{Z}^2$. The cone $\sigma = \mathrm{Cone}((0,1),(1,2),(2,1))$ and its dual are shown in Figure 2.2. The point $m = (-1,2) \in \sigma^\vee$ gives the face $\tau = \sigma \cap H_m = \mathrm{Cone}((2,1))$. ◇

A useful way of representing cones uses a finite set of linear inequalities.

Proposition 2.2.4. *The set* $\{m_i\}_{i=1}^s \subset M$ *generates* $\sigma^\vee$ *if and only if* $\sigma = \bigcap_{i=1}^s H_{m_i}^+$.

In the following, we summarize a list of properties of cones and their faces. For proofs, we refer to [19, §1.2] and references therein.

Proposition 2.2.5. *Let* σ, τ, τ' *be cones. Then we have*

1. $\tau \preceq \sigma$ *and* $\tau' \preceq \sigma$ *implies* $\tau \cap \tau' \preceq \sigma$,
2. $\tau \preceq \sigma$ *and* $\tau' \preceq \tau$ *implies* $\tau' \preceq \sigma$,
3. *if* $\tau \preceq \sigma$ *and* $v, w \in \sigma$, *then* $v + w \in \tau$ *implies* $v \in \tau$ *and* $w \in \tau$,
4. *faces of* σ *and faces of* $\sigma^\vee$ *are in bijective, inclusion reversing correspondence.*

Example 2.2.6. The bijection in item 4 of Proposition 2.2.5 works as follows. Let $\tau \preceq \sigma^\vee$ be a face and let $m \in \tau$ be a point in the relative interior of τ, i.e., m is on none of the faces of τ except for τ itself. Then $H_m \cap \sigma$ is the corresponding face of σ. ◇

The dimension $\dim \sigma$ of a cone $\sigma \subset N_\mathbb{R}$ is the dimension of the smallest linear subspace of $N_\mathbb{R}$ containing σ. Here are some further properties of cones we will need.

Definition 2.2.7 (Strong convexity). *A cone* $\sigma \subset N_\mathbb{R} \simeq \mathbb{R}^d$ *is called* pointed *or* strongly convex *if* $\sigma \cap (-\sigma) = \{0\}$. *This is equivalent to* $\{0\} \preceq \sigma$, *and to* $\dim \sigma^\vee = d$.

The one-dimensional faces $\rho \preceq \sigma$ are called the *rays* of σ. If σ is strongly convex, for each ray ρ there is a unique lattice point u_ρ such that $\rho \cap N = \mathbb{N} \cdot u_\rho$.

Definition 2.2.8. *The set of primitive ray generators* $\{u_\rho \mid \rho \preceq \sigma, \dim \rho = 1\}$ *generates* σ. *It is called the set of* minimal generators *of* σ.

Definition 2.2.9 (Smoothness)**.** *A strongly convex rational cone* $\sigma \subset N_{\mathbb{R}}$ *is* smooth *if its minimal generators are a subset of a* $\mathbb{Z}$*-basis of* N.

Definition 2.2.10 (Simplicial)**.** *A strongly convex cone* σ *is* simplicial *if its minimal generators are linearly independent over* $\mathbb{R}$.

Example 2.2.11. All two-dimensional cones are simplicial. The cone σ from Example 2.2.3 is not smooth. Its minimal generators $(0,1)$ and $(2,1)$ span a sublattice of $\mathbb{Z}^2$. The three-dimensional cone over $\{(1,0,0),(0,1,0),(1,0,1),(0,1,1)\}$ has 4 minimal generators. Hence it is not simplicial. ◇

Here is how to work with rational convex polyhedral cones in `Oscar.jl`. We define the cone σ from Example 2.2.3 and its dual as follows:

```
σ = positive_hull([0 1;1 2;2 1]); σ_dual = polarize(σ);
```

The dimension of σ and its facet inequalities are obtained via

```
dim_σ = dim(σ); facets_σ = facets(σ);
```

This shows that σ has dimension two and it is given by $x_1 \geq 0$ and $x_1 - 2x_2 \leq 0$. The package offers several commands for checking properties of cones:

```
is_simplicial(σ),is_smooth(σ),is_fulldimensional(σ),is_pointed(σ); # 1,0,1,1
```

Computations involving polyhedral geometry in `Oscar.jl`, and thus many computations related to toric varieties, rely on the software `Polymake` [32]. ◇

2.3 Affine toric varieties from cones

Let σ be a rational cone (that is, a rational convex polyhedral cone) in $N_{\mathbb{R}}$. We associate an affine toric variety to σ, by considering the semigroup

$$\mathsf{S}_\sigma = \sigma^\vee \cap M \subset M.$$

This only makes sense if we can show that S_σ is finitely generated.

Lemma 2.3.1 (Gordan's lemma)**.** *The semigroup* S_σ *is finitely generated.*

Proof. We sketch the proof given in [19, Proposition 1.2.17]. Since the dual cone $\sigma^\vee$ is rational and polyhedral, it is generated by finitely many elements of M. Denote this finite set of cone generators by $T \subset M$. A lattice point in $\sigma^\vee$ is a nonnegative combination of T. This can be rewritten as the sum of a non-negative *integer* combination of the elements in T and a lattice point in the bounded region $K = \{\sum_{t\in T} c_t t \,:\, 0 \leq c_t < 1\}$. Since $K \cap M$ is finite, $T \cup (K \cap M)$ is a finite set of generators for $\sigma^\vee \cap M$. □

Gordan's lemma entails that S_σ is an affine semigroup, in the sense of Definition 2.1.1. That is, $\mathsf{S}_\sigma \simeq \mathbb{N}A$ for some matrix $A \simeq \mathbb{Z}^{d\times n}$. Together with Proposition 2.1.7, Gordan's lemma justifies the following definition.

Definition 2.3.2 (Toric variety of a cone)**.** *Let $\sigma \subset N_\mathbb{R}$ be a rational cone with associated semigroup $\mathsf{S}_\sigma = \sigma^\vee \cap M$. The affine toric variety associated to σ is $\mathcal{Y}_\sigma = \mathrm{Specm}(\mathbb{C}[\mathsf{S}_\sigma])$.*

Throughout the book, we use calligraphic font to denote toric varieties with no specified embedding, such as $\mathcal{Y}_\sigma$ in Definition 2.3.2. This is in constrast with Y_A, which is embedded in the affine space $\mathbb{C}^n$ whose coordinates are indexed by the columns of A.

Proposition 2.3.3. *Let $\mathcal{Y}_\sigma = \mathrm{Specm}(\mathbb{C}[\mathsf{S}_\sigma])$ be the affine toric variety of $\sigma \subset N_\mathbb{R}$. We have that $\dim \mathcal{Y}_\sigma = \dim \sigma^\vee$. In particular, $\dim \mathcal{Y}_\sigma = d$ if and only if σ is pointed.*

Proof. Consider the lattice $\mathbb{Z}\mathsf{S}_\sigma$ of finite integer combinations of elements in S_σ. Using Corollary 1.2.23, the first statement follows from the fact that the rank of the lattice $\mathbb{Z}\mathsf{S}_\sigma$ equals the dimension of the cone $\sigma^\vee$. The second statement follows from Definition 2.2.7: $\dim \sigma^\vee = d$ if and only if σ is pointed. $\square$

Example 2.3.4. Let $\sigma = \mathrm{Cone}(e_1, \ldots, e_r) \subset \mathbb{R}^d$, where $r \le d$ and e_i is the i-th standard basis vector. One checks that $\sigma^\vee = \mathrm{Cone}(e_1', \ldots, e_r', \pm e_{r+1}', \ldots, \pm e_d')$, where e_i' is the dual basis vector of $(\mathbb{R}^d)^\vee$. The associated affine toric variety $\mathcal{Y}_\sigma$ is

$$\mathrm{Specm}(\mathbb{C}[x_1, \ldots, x_r, x_{r+1}^{\pm 1}, \ldots, x_d^{\pm 1}]) \simeq \mathbb{C}^r \times (\mathbb{C}^*)^{d-r}.$$

By changing bases in the lattice $\mathbb{Z}^n$, we see that the affine toric variety corresponding to any smooth cone σ is the product of an affine space with a torus. $\diamond$

Example 2.3.5. Let $\sigma = \mathrm{Cone}(e_1, -e_1, e_2) \subset \mathbb{R}^2$ be the upper half plane. This cone is not pointed, and the dual cone $\sigma^\vee = \mathrm{Cone}(e_2')$ has dimension 1. We have $\mathcal{Y}_\sigma \simeq \mathbb{C}$. $\diamond$

Different sets of generators for the semigroup S_σ lead to different embeddings of the same toric variety, see Corollary 2.1.8. When σ is full-dimensional, there is a unique *minimal* set of semigroup generators (with respect to inclusion), which thus leads to an embedding of $\mathcal{Y}_\sigma$ in an ambient affine space of minimal dimension. These minimal generators are expressed in terms of *irreducible elements*. An element $m \in \mathsf{S}_\sigma$ is irreducible if $m = m' + m''$ for $m', m'' \in \mathsf{S}_\sigma$, implies that $m' = 0$ or $m'' = 0$.

Proposition 2.3.6. *Let $\sigma \subset N_\mathbb{R}$ be full dimensional, and let $\mathscr{H} \subset \mathsf{S}_\sigma$ be the set of all irreducible elements of S_σ. The set $\mathscr{H}$ is the unique minimal set of generators for S_σ with respect to inclusion.*

Proof. Any set of generators $\mathscr{A}$ for S_σ contains $\mathscr{H}$ by definition. Indeed, if $m \in \mathscr{H} \setminus \mathscr{A}$, then $m \notin \mathbb{N}\mathscr{A}$ by irreducibility and $\mathbb{N}\mathscr{A} \subsetneq \mathsf{S}_\sigma$. By Gordan's lemma (Lemma 2.3.1), it follows from this observation that $\mathscr{H}$ is finite. It remains to show that every $m \in \mathsf{S}_\sigma$ can be written as a finite sum of irreducible elements. We follow the argument in the proof of [19, Prop. 1.2.23]. Since $\sigma^\vee$ is pointed, there is $u \in N$ such that $\langle u, m\rangle \ge 0$ for all

$m \in \mathsf{S}_\sigma$ and $\langle u, m\rangle = 0$ if and only if $m = 0$. Suppose m is reducible. The decomposition $m = m' + m''$ gives $\langle u, m'\rangle + \langle u, m''\rangle = \langle u, m\rangle$, so that both terms on the lefthand side are non-negative but strictly smaller than $\langle u, m\rangle$. This shows that after finitely many such decompositions, we arrive at irreducible elements. □

The set $\mathscr{H}$ from Proposition 2.3.6 is called *Hilbert basis* of S_σ, or of $\sigma^\vee$. A consequence of this proposition is that $\mathcal{Y}_\sigma$ can be embedded via a monomial map in an affine space of dimension $|\mathscr{H}|$. If $\mathsf{S}_\sigma \subset \mathbb{Z}^d$, this is the monomial map ϕ_H as in (1.1.1), where $H \in \mathbb{Z}^{d\times|\mathscr{H}|}$ is a matrix whose columns are $\mathscr{H}$. We will see in Proposition 2.4.9 that $\mathcal{Y}_\sigma$ cannot be embedded in an affine space of dimension smaller than $|\mathscr{H}|$.

Exercise 2.3.7. Let $\sigma = \text{Cone}(e_2) \subset \mathbb{R}^2$. Show that the irreducible elements of S_σ do not generate S_σ as a semigroup.

Exercise 2.3.8. Show that the Hilbert basis $\mathscr{H}$ of S_σ, for a full-dimensional cone $\sigma \subset N_\mathbb{R}$, contains the minimal generators of $\sigma^\vee$.

We will now use `Oscar.jl` to compute the embedding of $\mathcal{Y}_\sigma$, where σ is the cone from Example 2.2.3. We generated this cone and its dual in Example 2.2. The first step is to compute the Hilbert basis of $\sigma^\vee$:

```
H = hilbert_basis(σ_dual)
```

This gives $H = \left(\begin{smallmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \end{smallmatrix}\right)$. From it, we can compute the toric ideal $I(Y_H)$ via

```
I = toric_ideal(H)
```

It is generated by $x_1x_2 - x_3^2$, which is the equation for $\mathcal{Y}_\sigma \simeq Y_H$ embedded in $\mathbb{C}^3$. ⋄

2.4 Normality and smoothness

Recall that an integral domain R is *normal* if it is integrally closed in its field of fractions K. This means that if $x \in K$ satisfies a monic polynomial relation $x^d + r_{d-1}x^{d-1} + \cdots + r_1x + r_0 = 0$ with coefficients $r_i \in R$, then we must have $x \in R$. An irreducible affine variety V is called *normal* if its coordinate ring $\mathbb{C}[V]$ is normal. Normality of toric varieties will be important in our discussion on divisors in Chapter 9. It turns out that normal affine toric varieties are exactly those affine toric varieties corresponding to cones.

Definition 2.4.1 (Saturated semigroup)**.** *An affine semigroup* $\mathsf{S} \subset M \simeq \mathbb{Z}^d$ *is* saturated *in* M *if* $km \in \mathsf{S}$ *for* $k \in \mathbb{N}_{>0}, m \in M$ *implies that* $m \in \mathsf{S}$.

Theorem 2.4.2. *Let* $\mathcal{Y} = \text{Specm}(\mathbb{C}[\mathsf{S}])$ *be the affine toric variety of an affine semigroup* $\mathsf{S} \subset M$ *such that* $\mathbb{Z}\mathsf{S} = M$*. The following statements are equivalent:*

1. $\mathcal{Y}$ *is normal,*

2. S *is saturated in* M*,*

3. $\mathsf{S} = \mathsf{S}_\sigma$ *for some cone* $\sigma \subset N_\mathbb{R}$*.*

Proof. We can choose $A \in \mathbb{Z}^{d\times n}$ such that $\mathbb{Z}A = \mathbb{Z}^d = M$, $\mathbb{N}A \simeq \mathsf{S}$ and thus $Y_A \simeq \mathcal{Y}$. We prove the implications $1 \Rightarrow 2 \Rightarrow 3$ and point to [19, Theorem 1.3.5] for the remaining step.
$1 \Rightarrow 2$) Suppose that Y_A is normal, i.e., $\mathbb{C}[\mathbb{N}A]$ is integrally closed in its field of fractions. Suppose $km \in \mathbb{N}A$ with $k \in \mathbb{N}_{>0}$ and $m \in \mathbb{Z}^d$. Since $\mathbb{Z}A = \mathbb{Z}^d$, the monomial t^m belongs to the fraction field of $\mathbb{C}[\mathbb{N}A]$, and it satisfies the monic relation $(t^m)^k - t^{km} = 0$ with coefficients in $\mathbb{C}[\mathbb{N}A]$. We conclude that $t^m \in \mathbb{C}[\mathbb{N}A]$, and thus $m \in \mathbb{N}A$.
$2 \Rightarrow 3$) Let $\sigma^\vee \subset M_\mathbb{R} = \mathbb{R}^d$ be the cone generated by columns of A. The inclusion $\mathbb{N}A \subseteq \sigma^\vee \cap M = \sigma^\vee \cap \mathbb{Z}^d$ always holds. The claim is that if $\mathbb{N}A$ is saturated, then this inclusion is an equality. The columns of A contain a set of minimal cone generators for $\sigma^\vee$. Any $m \in \sigma^\vee \cap M$ can be written as a $\mathbb{Q}$-linear combination of these minimal generators with nonnegative coefficients. By clearing denominators, we see that $km \in \mathbb{N}A$ for some positive integer k. Since $\mathbb{N}A$ is saturated, we conclude $m \in \mathbb{N}A$, which shows that $\sigma^\vee \cap M \subseteq \mathbb{N}A$. □

Example 2.4.3. The affine toric curve Y_A with $A = (2 \;\; 3) \in \mathbb{Z}^{1\times 2}$ is the cuspidal cubic. It is a standard example of a non-normal variety. The semigroup $\mathbb{N}A = \{0, 2, 3, 4, \ldots\}$ is strictly contained in $\mathrm{Cone}(A) \cap \mathbb{Z} = \mathbb{N}$. ◇

Corollary 2.4.4. *The affine toric variety* Y_A *is normal if and only if the affine semigroup* $\mathbb{N}A$ *is saturated in* $\mathbb{Z}A$*.*

The next example illustrates why the words "in $\mathbb{Z}A$" of Corollary 2.4.4 are important.

Example 2.4.5. The affine toric variety Y_A with $A = (2\ 4)$ is the parabola $\{y - x^2 = 0\}$. This affine toric variety is smooth, and therefore normal. Since $\mathbb{N}A = 2\mathbb{N} \subset \mathbb{Z}$, this seemingly contradicts the implication $1 \Rightarrow 3$ from Theorem 2.4.2. Here we see that it is important to work in the right lattice. The semigroup $\mathbb{N}A$ is saturated in the lattice it generates, i.e., in $\mathbb{Z}A = 2\mathbb{Z} \subset \mathbb{Z}$. We have $\mathbb{N}A = \sigma^\vee \cap 2\mathbb{Z}$. By Exercise 1.3.4, we may replace A by $(1 \;\; 2)$, for which $\mathbb{Z}A = \mathbb{Z}$ and the proof of Theorem 2.4.2 applies. ◇

Next, we discuss which affine toric varieties are smooth. Smoothness implies normality, so by Theorem 2.4.2, the only candidates are toric varieties coming from cones.

Theorem 2.4.6. *The affine toric variety* $\mathcal{Y}_\sigma = \mathrm{Specm}(\mathbb{C}[\mathsf{S}_\sigma])$ *is smooth if and only if* σ *is a smooth rational cone (Definition 2.2.9).*

The following Lemmas will be useful for the proof.

Lemma 2.4.7. *Let* $\mathsf{S} \subset M$ *be an affine semigroup such that* $\mathrm{Cone}(\mathsf{S}) \subset M_\mathbb{R}$ *is pointed. The ideal* $\mathfrak{m} = \langle t^m \,:\, m \in \mathsf{S} \setminus \{0\}\rangle \subset \mathbb{C}[\mathsf{S}]$ *is maximal. We have* $\mathfrak{m}^2 = \bigoplus_{m \text{ reducible}} \mathbb{C} \cdot t^m$*, where the sum is over all* $m \in \mathsf{S}$ *for which we have* $m = m' + m''$ *for nonzero* $m', m'' \in \mathsf{S}$*.*

Proof. Note that $\mathfrak{m} \subsetneq \mathbb{C}[\mathsf{S}]$ is a proper ideal because Cone(S) is pointed. Maximality of $\mathfrak{m}$ follows from the fact that the quotient $\mathsf{S}/\mathfrak{m} = \mathbb{C}$ is a field. The second statement is left as an easy exercise for the reader. □

Lemma 2.4.8. *Let $\sigma \subset N_{\mathbb{R}}$ be a rational cone. The group $M/\mathbb{Z}\mathsf{S}_\sigma$ is torsion free. In particular, if $\operatorname{rank} \mathbb{Z}\mathsf{S}_\sigma = d$, then $\mathbb{Z}\mathsf{S}_\sigma = M$.*

Proof. This is proved in [19, Proposition 1.2.18]. We encourage the reader to show this as an exercise. □

Proof of Theorem 2.4.6. If σ is smooth, then Specm($\mathbb{C}[\mathsf{S}_\sigma]$) is smooth by Example 2.3.4. Conversely, suppose $\mathcal{Y}_\sigma$ is smooth and $\dim \sigma = \dim \mathcal{Y}_\sigma = d$. Then in particular the maximal ideal $\mathfrak{m} = \langle t^m \ : \ m \in \mathsf{S}_\sigma \setminus \{0\} \rangle$ represents a smooth point p of $\mathcal{Y}_\sigma$. Therefore, the tangent space at p has dimension d: $\dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2 = d$. Observe that, by Lemma 2.4.7,

$$\mathfrak{m} = \bigoplus_{m \in \mathsf{S}_\sigma \setminus \{0\}} \mathbb{C} \cdot t^m = \bigoplus_{m \text{ irreducible}} \mathbb{C} \cdot t^m \oplus \bigoplus_{m \text{ reducible}} \mathbb{C} \cdot t^m = \bigoplus_{m \in \mathscr{H}} \mathbb{C} \cdot t^m \oplus \mathfrak{m}^2,$$

where $\mathscr{H}$ is the Hilbert basis of S_σ (Proposition 2.3.6). Hence $|\mathscr{H}| = d$, which implies that $\sigma^\vee \cap M$ is generated by d elements. The Hilbert basis contains the minimal generators (Exercise 2.3.8), and σ has dimension d by assumption, so that σ has d rays. Moreover, $\mathbb{Z}\mathsf{S}_\sigma = \mathbb{Z}\mathscr{H} = M$ (Lemma 2.4.8), which means that $\mathscr{H}$ is a $\mathbb{Z}$-basis for M. We conclude that $\sigma^\vee$ is smooth, and smoothness is preserved by duality (exercise). For the case $\dim \sigma < d$, one observes that $\mathcal{Y}_\sigma \simeq \mathcal{Y}_{\sigma'} \times (\mathbb{C}^*)^{d-\dim(\sigma)}$, where $\dim(\sigma') = \dim(\mathcal{Y}_{\sigma'})$. Then the above argument shows smoothness of $(\sigma')^\vee \times \mathbb{R}^{d-\dim(\sigma)}$, and hence of σ'. We refer to [19, Theorem 1.3.12] for more details. □

Proposition 2.4.9. *The cardinality $|\mathscr{H}|$ of the Hilbert basis for S_σ is the smallest integer n for which $\mathcal{Y}_\sigma$ can be embedded in $\mathbb{C}^n$.*

Proof. The proof of Theorem 2.4.6 implies that in general, when σ is a strongly convex rational cone of dimension d, the tangent space at the point defined by the ideal $\mathfrak{m}$ has dimension $|\mathscr{H}|$. For any affine embedding of $\mathcal{Y}_\sigma$, the tangent space is at most the dimension of the ambient affine space. □

Consider the rational cone

$$\sigma = \text{Cone}((1,2,3),(2,1,3),(1,3,2),(3,1,2),(2,3,1),(3,2,1)) \subset \mathbb{R}^3.$$

This is the cone over the two-dimensional *permutohedron*. Since $\sigma \subset \mathbb{R}^3_{\geq 0}$, σ is strongly convex and hence $\dim \mathcal{Y}_\sigma = 3$ (Proposition 2.3.3). We check this in Julia by constructing an affine normal toric variety from σ:

```
Y_σ = affine_normal_toric_variety(σ); dim(Y_σ)
```

For any embedding $\mathcal{Y}_\sigma \simeq Y_A \subset \mathbb{C}^s$, s is at least 15, since this is the number of elements in the Hilbert basis of $\sigma^\vee$, computed as in Example 2.2. The function `toric_ideal` can be used on the object returned by `affine_normal_toric_variety`:

```
I = toric_ideal(Y_σ)
```

This computes 77 binomial generators for $I(Y_A)$. ◇

Exercise 2.4.10. Let $A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 & 6 & 4 \\ 0 & 0 & 1 & -1 & 17 & 3 & 19 \end{pmatrix}$. Use `Oscar.jl` to compute the toric ideal I_A. Costruct the `affine_normal_toric_variety` $\mathcal{Y}_\sigma \simeq Y_A$ for $\sigma^\vee = \mathrm{Cone}(A)$. Compute a Hilbert basis for $\sigma^\vee$ and the corresponding embedding Y_H of $\mathcal{Y}_\sigma$. Can you find the isomorphism $Y_A \to Y_H$?

Further reading

Affine toric varieties from cones are the point of departure in [30]. See also [19, §1.2 and §1.3]. For normalization of affine toric varieties, see [33, Chapter 5, §2B] and [19, Proposition 1.3.8].

Chapter 3

Polytopes and projective toric varieties

In Chapter 2 we have explored first connections between affine toric varieties and polyhedral cones. In this chapter, we focus on the projective setting. We have seen in Proposition 2.1.10 that the coordinate ring of the affine cone $Y_{\hat{A}}$ over the projective toric variety X_A is the semigroup algebra $\mathbb{C}[\mathbb{N}\hat{A}]$. The polyhedral cone $\mathrm{Cone}(\mathbb{N}\hat{A})$ over this semigroup is a cone over a bounded polyhedron, i.e., a polytope. This chapter explains some aspects of how that polytope encodes information about X_A, and how to construct normal toric varieties from polytopes. We start with an introduction to polytopes.

3.1 Convex lattice polytopes

In this section, we introduce some definitions and basic facts related to convex lattice polytopes. A standard reference on this topic is [75], to which we refer for more details. Our polytopes, like our dual cones $\sigma^\vee$, live in the real vector space $M_\mathbb{R} = \mathbb{R}^d$.

Definition 3.1.1 (Polytope). *A (convex)* polytope $\mathcal{P}$ *in* $M_\mathbb{R}$ *is the convex hull of a finite set of points* $\mathscr{A} = \{a_1, \ldots, a_n\}$ *in* $M_\mathbb{R}$*:*

$$\mathcal{P} = \mathrm{Conv}(\mathscr{A}) = \left\{ \sum_{i=1}^n c_i a_i \in M_\mathbb{R} \,:\, c_i \in \mathbb{R}, \sum_{i=1}^n c_i = 1, c_i \geq 0 \right\} \subset M_\mathbb{R}.$$

If $\mathscr{A} \subset M \subset M_\mathbb{R}$*, then* $\mathcal{P}$ *is called a* lattice polytope.

When $M = \mathbb{Z}^d$ and $A \in \mathbb{Z}^{d \times n}$ is a matrix whose columns are the points in $\mathscr{A}$, we write $\mathrm{Conv}(A) = \mathrm{Conv}(\mathscr{A})$ for the corresponding lattice polytope.
The dimension $\dim \mathcal{P}$ of a polytope $\mathcal{P} \subset M_\mathbb{R}$ is defined as the dimension of the smallest affine subspace of $M_\mathbb{R}$ containing $\mathcal{P}$. A polytope in $M_\mathbb{R}$ is said to be *full-dimensional* if $\dim \mathcal{P} = d$. A point $u \in N_\mathbb{R} \setminus \{0\}$ and a scalar $b \in \mathbb{R}$ give

$$H_{u,b} = \{m \in M_\mathbb{R} \,:\, \langle u, m\rangle + b = 0\} \quad \text{and} \quad H^+_{u,b} = \{m \in M_\mathbb{R} \,:\, \langle u, m\rangle + b \geq 0\}.$$

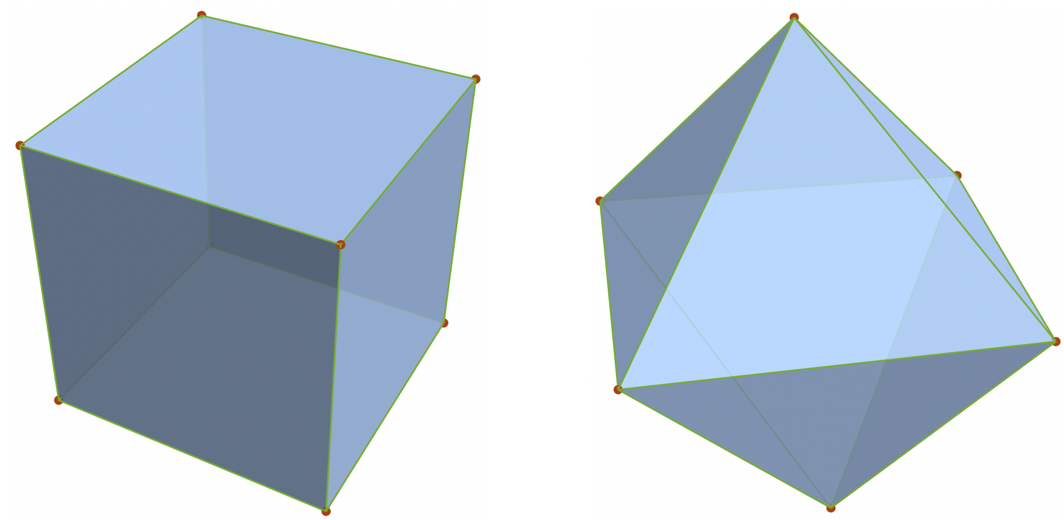

Figure 3.1: Left: the three-dimensional cube. Right: the octahedron.

Definition 3.1.2 (Faces of a polytope)**.** *Take $u \in N_{\mathbb{R}} \setminus \{0\}$, $a \in \mathbb{R}$ and let $\mathcal{P} \subset M_{\mathbb{R}}$ be a convex polytope. The set $H_{u,b} \cap \mathcal{P}$ is a* face *of $\mathcal{P}$ if $\mathcal{P} \subset H_{u,b}^{+}$ and $b = -\min_{m \in \mathcal{P}} \langle u, m \rangle$. We say that $\mathcal{P}$ is a face of $\mathcal{P}$ by convention. We write $Q \preceq \mathcal{P}$ if Q is a face of $\mathcal{P}$.*

An affine hyperplane $H_{u,b}$ for which $H_{u,b} \cap \mathcal{P}$ is a face of $\mathcal{P}$ is called a *supporting hyperplane*. A face Q of a polytope is again a polytope. The *codimension* of a face $Q \preceq \mathcal{P}$ is $\dim \mathcal{P} - \dim Q$. A face of codimension 1 in $\mathcal{P}$ is called a *facet*. A face of dimension 1 is an *edge*, and a face of dimension 0 is a *vertex*.

Example 3.1.3. A *d-simplex* is a polytope of dimension d with the minimal amount of vertices. For instance, a 1-simplex is a line segment, a 2-simplex is a triangle and a 3-simplex is a tetrahedron. ◇

Example 3.1.4. The columns of the following matrix are the vertices of the cube $[0,1]^3$:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

The polytope $\mathrm{Conv}(A)$ is shown in the left part of Figure 3.1. It has eight vertices, twelve edges and six facets. This polytope is called *simple*, because each vertex is contained in dimension-many facets. A non-simple polytope is obtained from the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

It represents a three-dimensional polytope $\mathcal{P} = \mathrm{Conv}(A)$ in $\mathbb{R}^4$: each column is contained in the three-dimensional affine hyperplane $\{m \in \mathbb{R}^4 \ : \ m_1 + m_2 + m_3 + m_4 = 2\}$.

Projecting $\mathcal{P}$ onto its last three coordinates ($m \mapsto (m_2, m_3, m_4)$) we obtain the three-dimensional polytope shown in the right part of Figure 3.1. This is called the *octahedron*. The octahedron has six vertices, twelve edges and eight facets. It is an example of a *simplicial* polytope. Simplicial polytopes are polytopes whose faces are simplices. $\diamond$

Exercise 3.1.5. Show that the only lattice points contained in the three-cube and the octahedron from Example 3.1.4 are the vertices. That is, in both cases, $\text{int}(\mathcal{P}) \cap \mathbb{Z}^3 = \emptyset$. Polytopes with this property are called *hollow*.

Exercise 3.1.6. Draw the polytope $\mathcal{P} = \text{Conv}(A) \subset \mathbb{R}^3$ corresponding to the matrix

$$A = \begin{pmatrix} 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and conclude that $\mathcal{P}$ is a quadrilateral pyramid. Show that it is neither simple nor simplicial (see Example 3.1.4 for definitions).

One way of representing a polytope $\mathcal{P}$ in terms of faces uses vertices:

$$\mathcal{P} = \text{Conv}(v \in M_{\mathbb{R}} \,:\, v \text{ is a vertex of } \mathcal{P}).$$

This is called a *vertex representation* or *V-representation*. Alternatively, any polytope can be expressed as the intersection of finitely many closed half-spaces $H^+_{u,b}$ associated to supporting hyperplanes. That is, any polytope $\mathcal{P} \subset M_{\mathbb{R}}$ can be written as

$$\mathcal{P} = H^+_{u_1,b_1} \cap \cdots \cap H^+_{u_k,b_k} = \{m \in M_{\mathbb{R}} \,:\, \langle u_i, m\rangle + b_i \geq 0, i = 1, \ldots, k\} \tag{3.1.1}$$

for some $u_1, \ldots, u_k \in N_{\mathbb{R}}$, $b_1, \ldots, b_k \in \mathbb{R}$. The representation in (3.1.1) is called a *half-space representation* or *H-representation* of the polytope $\mathcal{P}$. There exist infinitely many different H-representations for any polytope. However, if $\mathcal{P}$ is full-dimensional, there exists an *essentially unique*, *minimal* H-representation of $\mathcal{P}$, in the sense that it consists of a minimal number k of inequalities where the inequalities are uniquely defined up to multiplication with a nonzero scalar. Suppose that $\mathcal{P}$ is full-dimensional. For a supporting hyperplane $H_{u,b}$ corresponding to a facet Q of $\mathcal{P}$, the vector u is uniquely determined up to a nonzero scalar factor. For every facet Q, let u_Q, b_Q be such that $\mathcal{P} \subset H^+_{u_Q,b_Q}, H_{u_Q,b_Q} \cap \mathcal{P} = Q$. The minimal H-representation of $\mathcal{P}$ is given by

$$\mathcal{P} = \bigcap_{Q \text{ facet of } \mathcal{P}} H^+_{u_Q,b_Q} \tag{3.1.2}$$

If $\mathcal{P}$ is a full-dimensional lattice polytope, then for any facet $Q \subset \mathcal{P}$, u_Q can be chosen in a unique way as the generator of the rank-one sublattice

$$\{u \in N \,:\, \langle u, m\rangle = 0 \text{ for all } m \in Q\},$$

for which $\mathcal{P} \in H^+_{u_Q,b_Q}$. This vector u_Q is called the *primitive, inward pointing facet normal* of Q. It is the inward pointing integer vector perpendicular to Q of the smallest length. Below, by 'the facet normal' associated to Q we mean the primitive, inward pointing facet normal.

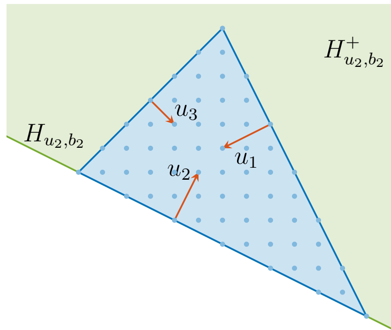

Figure 3.2: Illustration of a lattice polytope of dimension 2 and its primitive inward pointing facet normals.

Example 3.1.7. Figure 3.2 shows a full-dimensional polytope in $\mathbb{R}^2$ (a 2-dimensional polytope is also called a *polygon*) together with its interior lattice points and primitive inward pointing facet normals. The minimal H-representation is given by

$$u_1 = (-2,-1), \quad u_2 = (1,2) \quad \text{and} \quad u_3 = (1,-1).$$

The supporting hyperplane H_{u_2,b_2} is also shown in Figure 3.2, and its corresponding half-space $H^+_{u_2,b_2}$ (shaded in green) contains the polytope. We note that, strictly speaking, the orange arrows do not belong in the same picture: they live in the dual plane $(\mathbb{R}^2)^\vee$. ◇

Exercise 3.1.8. Compute a minimal H-representation for the three-cube and the octahedron from Example 3.1.4.

For any polytope $\mathcal{P} \subset M_\mathbb{R}$ and any $\lambda \in \mathbb{R}, \lambda \geq 0$, we define the polytope $\lambda \cdot \mathcal{P}$ as $\lambda \cdot \mathcal{P} = \{\lambda p \,:\, p \in \mathcal{P}\}$. This is called a *dilation* of the polytope $\mathcal{P}$. A famous result by Ehrhart counts the lattice points contained in integer dilations of a lattice polytope [25].

Theorem 3.1.9. *Let $\mathcal{P} \subset \mathbb{R}^n$ be a convex lattice polytope of dimension d. The function $E_\mathcal{P} : \mathbb{N} \to \mathbb{N}$ given by $k \mapsto |k \cdot \mathcal{P} \cap \mathbb{Z}^n|$ is a polynomial with leading term* $\mathrm{Vol}(\mathcal{P})\, k^d$.

The quantity $\mathrm{Vol}(\mathcal{P})$ in Theorem 3.1.9 is the Euclidean volume $\int_\mathcal{P} \mathrm{d}x_1 \cdots \mathrm{d}x_d$. The polynomial $E_\mathcal{P}$ is called the *Ehrhart polynomial* of the polytope $\mathcal{P}$.

Example 3.1.10. The Ehrhart polynomial of a polytope $\mathcal{P}$ can be computed using `Oscar.jl`. We illustrate this and some more functionalities for the polytope $\mathcal{P} = \mathrm{Conv}((0,0),(1,0),(0,1),(2,1),(1,2))$ in Figure 3.3. ◇

Exercise 3.1.11. Compute an H-representation of the polygon $\mathcal{P} = \mathrm{Conv}(A')$, with A' as in Exercise 1.3.21. How many vertices and edges does this polygon have? What is its Ehrhart polynomial?

Exercise 3.1.12. Compute $E_\mathcal{P}$ for the cube $\mathcal{P} = [0,1]^3$ from Example 3.1.4.

In [84]:
```
P = convex_hull([0 0; 1 0; 0 1; 2 1; 1 2])
```
Out[84]: A polyhedron in ambient dimension 2

In [85]:
```
dim(P), length(vertices(P)), length(facets(P))
```
Out[85]: (2, 5, 5)

In [86]:
```
E_P = ehrhart_polynomial(P)
```
Out[86]: $5//2x^2 + 5//2x + 1$

Figure 3.3: The polygon $\mathcal{P}$ has dimension 2. It has 5 vertices and 5 edges, and its Ehrhart polynomial is $\frac{5}{2}x^2 + \frac{5}{2}x + 1$.

3.2 Affine charts of a projective toric variety

Let $U_i = \mathbb{P}^{n-1} \setminus V(x_i)$ for $i = 1, \ldots, n$ be the standard affine charts of $\mathbb{P}^{n-1}$. Recall that

$$\psi_i : (x_1 : \cdots : x_n) \longmapsto \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i} \right). \tag{3.2.1}$$

is an isomorphism $U_i \simeq \mathbb{C}^{n-1}$. The intersection $\bigcap_{i=1}^n U_i$ of all these charts is isomorphic to $(\mathbb{C}^*)^{n-1}$ under any map ψ_i, and it contains $\operatorname{im} \Phi_A$ for any integer matrix $A \in \mathbb{Z}^{d \times n}$. For the definition of Φ_A, see (1.1.3). The composition of Φ_A with ψ_i is

$$\phi_{A-a_i} : (\mathbb{C}^*)^d \to (\mathbb{C}^*)^{n-1}, \quad \phi_{A-a_i}(t) = (t^{a_1-a_i}, \ldots, t^{a_{i-1}-a_i}, t^{a_{i+1}-a_i}, \ldots, t^{a_n-a_i}).$$

This leads to a description of the standard affine covering of $X_A = \bigcup_{i=1}^n (X_A \cap U_i)$.

Proposition 3.2.1. *The affine chart $X_A \cap U_i$ of X_A is isomorphic to the affine toric variety Y_{A-a_i} where $A - a_i = \begin{pmatrix} a_1 - a_i & \cdots & a_{i-1} - a_i & a_{i+1} - a_i & \cdots & a_n - a_i \end{pmatrix} \in \mathbb{Z}^{d \times (n-1)}$.*

Proof. The affine variety $\psi_i(X_A \cap U_i) \subset \mathbb{C}^{n-1}$ is closed, irreducible and contains $\operatorname{im} \phi_{A-a_i}$. Therefore, it has dimension $\operatorname{rank} \mathbb{Z}'A = \operatorname{rank} \mathbb{Z}(A - a_i)$ (Exercise 1.3.17), see Corollary 1.3.18. We conclude that $\psi_i(X_A \cap U_i)$ equals the closure of $\operatorname{im} \phi_{A-a_i}$, which is Y_{A-a_i}. $\square$

Exercise 3.2.2. Show that the projective closure of the affine toric variety $Y_A \subset \mathbb{C}^n$ is given by the projective toric variety $X_{0 \cup A}$, where $0 \cup A \in \mathbb{Z}^{d \times (n+1)}$ is the matrix obtained by prepending a column of zeros to A.

We have seen in Exercise 1.3.4 that one can always compute a one-to-one parametrization of an affine toric variety Y_A via the Smith normal form. Here is the analogous statement for the projective toric variety X_A.

Proposition 3.2.3. *Let $\mathcal{A} = 0 \cup (A - a_1)$ be the matrix $(0 \; a_2 - a_1 \; \cdots \; a_n - a_1) \in \mathbb{Z}^{d \times n}$ and let $P\mathcal{A}Q = S$ be its Smith normal form (Definition 1.2.7). Let $r = \operatorname{rank}(\mathcal{A})$ and let $\overline{\mathcal{A}} \in \mathbb{Z}^{r \times n}$ be the matrix consisting of the first r rows of Q^{-1}. We have $X_A = X_{\mathcal{A}} = X_{\bar{\mathcal{A}}}$ and the parametrization $\Phi_{\bar{\mathcal{A}}} : (\mathbb{C}^*)^r \to \operatorname{im} \Phi_{\bar{\mathcal{A}}}$ of X_A is one-to-one.*

Proof. The equalities $X_A = X_{\mathcal{A}} = X_{\bar{A}}$ follow from Proposition 1.3.14. We have seen above that $\psi_1 \circ \Phi_{\bar{A}} = \phi_{\bar{A}-\bar{a}_1}$, where $\bar{a}_1 = 0$ is the first column of $\bar{\mathcal{A}}$. Since $\phi_{\bar{\mathcal{A}}} : (\mathbb{C}^*)^r \to (\mathbb{C}^*)^n$ is one-to-one (Proposition 1.2.17), so is $\phi_{\bar{A}-\bar{a}_1} : (\mathbb{C}^*)^r \to (\mathbb{C}^*)^{n-1}$. Clearly, ψ_1 is one-to-one. This proves the proposition. □

Any column of A can be used to construct the matrices $\mathcal{A}$ and $\bar{A}$ in Proposition 3.2.3.

The affine covering $X_A = \bigcup_{i=1}^n (X_A \cap U_i)$ is often redundant, meaning that several varieties in this intersection can be omitted. To identify the minimal affine pieces of X_A, we use the polytope $\mathrm{Conv}(A) \subset \mathbb{R}^d$; the convex hull of the columns of A.

Proposition 3.2.4. *Let* $\mathcal{V} = \{i \,:\, a_i \in A$ *is a vertex of* $\mathrm{Conv}(A)\}$*. We have*

$$X_A = \bigcup_{i \in \mathcal{V}} (X_A \cap U_i).$$

This statement will be implied by our very explicit description of $X_A \backslash \mathrm{im}\, \Phi_A$ in Chapter 4. Here, we give a proof using semigroup algebras following [19, Proposition 2.1.9].

For an affine variety V with coordinate ring $\mathbb{C}[V]$ we write V_f for the affine variety $\{p \in V \,:\, f(p) \neq 0\} = \mathrm{Specm}(\mathbb{C}[V]_f)$, $f \in \mathbb{C}[V]$. Here $\mathbb{C}[V]_f$ is the localization of $\mathbb{C}[V]$ at the set $\{f^\ell \,:\, \ell \in \mathbb{N}\}$. The inclusions

$$X_A \cap U_i \,\supset\, X_A \cap U_i \cap U_j \,\subset\, X_A \cap U_j$$

can be described in a coordinate-free way using the semigroup algebras $\mathbb{C}[\mathsf{S}_i]$, with $\mathsf{S}_i = \mathbb{N}(A - a_i)$, as follows. For the inclusion $X_A \cap U_i \supset X_A \cap U_i \cap U_j$, note that

$$X_A \cap U_i \cap U_j = \{x \in X_A \cap U_i \,:\, x_j/x_i \neq 0\} = (X_A \cap U_i)_{x_j/x_i} \simeq \mathrm{Specm}(\mathbb{C}[\mathsf{S}_i]_{t^{a_j - a_i}}).$$

Hence $X_A \cap U_i \supset X_A \cap U_i \cap U_j$ is given by the inclusion of $\mathbb{C}[\mathsf{S}_i]$ in its localization $\mathbb{C}[\mathsf{S}_i]_{t^{a_j-a_i}}$. Arguing analogously for $X_A \cap U_j$, we obtain

$$\mathbb{C}[\mathsf{S}_i] \subset \mathbb{C}[\mathsf{S}_i]_{t^{a_j-a_i}} \simeq \mathbb{C}[\mathsf{S}_j]_{t^{a_i-a_j}} \supset \mathbb{C}[\mathsf{S}_j].$$

Proof of Proposition 3.2.4. We show that $X_A \cap U_j \subset U_{i^*}$ for some $i^* \in \mathcal{V}$. Let $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$. Since $a_j \in P \cap M_{\mathbb{Q}}$, there exist $r_i \in \mathbb{Q}_{\geq 0}, i \in \mathcal{V}$ such that $a_j = \sum_{i \in \mathcal{V}} r_i a_i$ and $\sum_{i \in \mathcal{V}} r_i = 1$. Setting $r_i = k_i/k$ with $k, k_i \in \mathbb{N}$ and clearing denominators gives

$$k a_j = \sum_{i \in \mathcal{V}} k_i a_i, \quad \text{with} \quad \sum_{i \in \mathcal{V}} k_i = k.$$

This gives $\sum_{i \in \mathcal{V}} k_i (a_j - a_i) = 0$. We now choose i^* such that $k_{i^*} > 0$ and rewrite

$$a_j - a_{i^*} = \sum_{i \in \mathcal{V} \backslash \{i^*\}} k_i (a_i - a_j) + (k_{i^*} - 1)(a_{i^*} - a_j),$$

which shows that $a_j - a_{i^*} \in \mathsf{S}_j$. Therefore $t^{a_{i^*} - a_j}$ is invertible in $\mathbb{C}[\mathsf{S}_j]$, and $\mathbb{C}[\mathsf{S}_j]_{t^{a_{i^*}-a_j}} = \mathbb{C}[\mathsf{S}_j]$. The statement follows from Proposition 3.2.1 and

$$X_A \cap U_{i^*} \cap U_j \simeq \mathrm{Specm}(\mathbb{C}[\mathsf{S}_j]_{t^{a_{i^*}-a_j}}) = \mathrm{Specm}(\mathbb{C}[\mathsf{S}_j]) \simeq X_A \cap U_j.$$ □

Remark 3.2.5. It follows from the proof of Proposition 3.2.4 that $X_A \cap U_j \subset U_{i^*}$ for every $i^* \in \mathcal{V}$ such that the vertex a_{i^*} lies on the smallest face of $\mathrm{conv}(A)$ containing a_j. Indeed, for such a vertex, there exists a convex combination $r_i = k_i/k$ with $k_{i^*} > 0$.

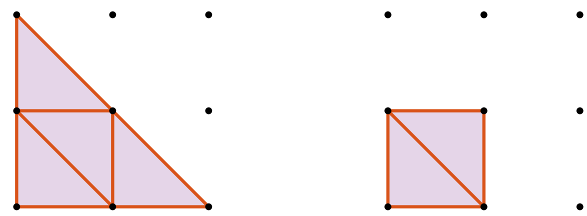

Figure 3.4: The degree of a projective toric surface is the area of a polygon.

3.3 The degree of a projective toric variety

The degree of the projective toric variety $X_A \subset \mathbb{P}^{n-1}$ is the number of intersection points of X_A with a generic linear space of dimension $n-1-\dim X_A$. This invariant can be described in terms of the volume of the polytope $\mathrm{Conv}(A)$. For brevity, we write $\mathcal{P} = \mathrm{Conv}(A) \subset \mathbb{R}^d$, and its volume is $\mathrm{Vol}(\mathcal{P}) = \mathrm{Vol}(A) = \int_{\mathcal{P}} \mathrm{d}x_1 \cdots \mathrm{d}x_d$. This volume is nonzero if and only if $\dim \mathcal{P} = d$, which is equivalent to $\mathrm{rank}\, \mathbb{Z}'A = d$. Here $\mathbb{Z}'A$ is the lattice affinely generated by A from Definition 1.3.15. The rank condition $\mathrm{rank}\, \mathbb{Z}'A = d$ is, in turn, equivalent to the parametrization Φ_A being finite-to-one, see Propositions 1.2.13 and 3.2.1. Here is this section's main result, due to Kushnirenko [43].

Theorem 3.3.1 (Kushnirenko's theorem). *If $A \in \mathbb{Z}^{d\times n}$ is such that $\mathbb{Z}'A = \mathbb{Z}^d$, then the degree of $X_A \subset \mathbb{P}^{n-1}$ is the normalized volume $d!\,\mathrm{Vol}(A)$ of the lattice polytope $\mathcal{P} =$* $\mathrm{Conv}(A)$*. More generally, if* $\mathrm{rank}\, \mathbb{Z}'A = d$*, we have* $\deg X_A = [\mathbb{Z}^d : \mathbb{Z}'A]^{-1} \cdot d!\,\mathrm{Vol}(A)$.

Remark 3.3.2. The term *normalized volume* in Theorem 3.3.1 indicates that the measure is normalized such that a standard simplex $\Delta_d = \mathrm{Conv}(0, e_1, \ldots, e_d)$ in the lattice $\mathbb{Z}^d$ has volume one: $d!\,\mathrm{Vol}(\Delta_d) = 1$. The lattice index $[\mathbb{Z}^d : \mathbb{Z}A]$ is as in Exercise 1.2.9.

Example 3.3.3. We consider two toric surfaces $X_{A_1} \subset \mathbb{P}^5, X_{A_2} \subset \mathbb{P}^3$ corresponding to

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

These are familiar surfaces from classical algebraic geometry: X_{A_1} is the second Veronese embedding $\nu_2(\mathbb{P}^2)$ of $\mathbb{P}^2$, and $X_{A_2} \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is the Segre quadric. Both matrices generate a lattice of index one in $\mathbb{Z}^2$. That is, $\mathbb{Z}A_1 = \mathbb{Z}A_2 = \mathbb{Z}^2$. The polygons $\mathcal{P}_1 = \mathrm{Conv}(A_1)$ and $\mathcal{P}_2 = \mathrm{Conv}(A_2)$ are shown in Figure 3.4. The degree of X_{A_1} is $d!\,\mathrm{Vol}(A_1) = 2!\cdot 2 = 4$, and that of X_{A_2} is $d!\,\mathrm{Vol}(A_2) = 2! \cdot 1 = 2$. The reader is encouraged to verify these numbers. The normalized volume $d!\,\mathrm{Vol}(A_i)$ counts how many standard simplices fit into $\mathcal{P}_i$. This is illustrated by the triangulations in Figure 3.4. $\diamond$

Notice that, by Proposition 3.2.3, the assumption that $\mathrm{rank}\, \mathbb{Z}'A = d$ or even $\mathbb{Z}'A = \mathbb{Z}^d$ is not restrictive: one can replace the matrix A by a different matrix of the same rank so that these assumptions are satisfied. The final statement of Theorem 3.3.1 will follow from the claim for the case $\mathbb{Z}'A = \mathbb{Z}^d$ via Proposition 3.2.3 and the following lemma.

Lemma 3.3.4. *Let* $A, \mathcal{A}$ *and* $\bar{\mathcal{A}}$ *be as in Proposition 3.2.3. If* $\operatorname{rank} \mathbb{Z}'A = d$, *then we have* $\operatorname{Vol}(A) = \operatorname{Vol}(\mathcal{A}) = [\mathbb{Z}^d : \mathbb{Z}'A] \cdot \operatorname{Vol}(\bar{\mathcal{A}})$.

Proof. The equality $\operatorname{Vol}(A) = \operatorname{Vol}(\mathcal{A})$ follows from the pact that the polytope $\operatorname{Conv}(\mathcal{A})$ is a translation of $\operatorname{Conv}(A)$, which preserves the volume. To see that $\operatorname{Vol}(\mathcal{A}) = [\mathbb{Z}^d : \mathbb{Z}'A] \cdot \operatorname{Vol}(\bar{\mathcal{A}})$, note that the SNF formula $P\mathcal{A}Q = S$ implies that $\mathcal{A} = P^{-1} \cdot S_{1:d} \cdot \bar{\mathcal{A}}$, where $S_{1:d}$ is the square diagonal matrix consisting of the first d columns of S. Hence, $P^{-1} \cdot S_{1:d}$ is a linear coordinate change that takes $\operatorname{Conv}(\bar{\mathcal{A}})$ to $\operatorname{Conv}(\mathcal{A})$. The volume is multiplied by the absolute value of its determinant, which, by Exercise 1.2.9, equals $\det S_{1:d} = [\mathbb{Z}^d : \mathbb{Z}\mathcal{A}] = [\mathbb{Z}^d : \mathbb{Z}'A]$. ◻

We present a proof of Theorem 3.3.1 following [64, Section 2.2] and [65, Theorem 4.16]. It uses the following classical result by Hilbert, see [34, Chapter I, Theorems 7.5 and 7.7]. For a projective variety $X \subset \mathbb{P}^{n-1}$ with coordinate ring $\mathbb{C}[X]$, let $\mathbb{C}[X]_k$ be its k-th graded piece. Here the grading of $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]/I(X)$ is the one induced by the standard grading on the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$: $\mathbb{C}[X]_k = \mathbb{C}[x_1, \ldots, x_n]_k / I(X)_k$.

Theorem 3.3.5. *Let* $X \subset \mathbb{P}^{n-1}$ *be an irreducible projective variety of dimension* d *with coordinate ring* $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]/I(X)$. *For* $k \gg 0$, *the* Hilbert function

$$\operatorname{HF}_X : \mathbb{Z} \to \mathbb{N}, \quad k \mapsto \dim_{\mathbb{C}} \mathbb{C}[X]_k$$

is given by a polynomial $\operatorname{HP}_X(k)$, *called the* Hilbert polynomial *of* X, *with leading term*

$$\frac{\deg(X)}{d!} k^d.$$

We have seen in Proposition 2.1.10 that the coordinate ring $\mathbb{C}[X_A]$ of X_A is the semigroup algebra $\mathbb{C}[\mathbb{N}\hat{A}] = \mathbb{C}[t^{a_1}u, \ldots, t^{a_n}u] \subset \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}, u]$, with grading

$$\mathbb{C}[\mathbb{N}\hat{A}] = \bigoplus_{k \in \mathbb{Z}} \mathbb{C}[\mathbb{N}\hat{A}]_k, \quad \mathbb{C}[\mathbb{N}\hat{A}]_k = \bigoplus_{m \in kA} \mathbb{C} \cdot t^m u^k.$$

Here $kA = \{a_{i_1} + \cdots + a_{i_k} \, : \, 1 \leq i_1 \leq \cdots \leq i_k \leq n\}$ and $\mathbb{C}[\mathbb{N}\hat{A}]_k = 0$ for $k < 0$.

Example 3.3.6. We use the running example from [64, Section 2.2]. Let $A = (0\ 2\ 3) \in \mathbb{Z}^{1 \times 3}$, such that $\hat{A} = \left(\begin{smallmatrix} 0 & 2 & 3 \\ 1 & 1 & 1 \end{smallmatrix}\right)$. The semigroup $\mathbb{N}\hat{A}$ is shown in Figure 2.1 (right). The number of monomials in $\in \mathbb{C}[\mathbb{N}\hat{A}]_k$ is $1, 3, 6, 9, 12, \ldots$ for $k = 0, 1, 2, 3, 4, \ldots$. These are obtained by counting the blue dots at vertical level k. Since the graded pieces are spanned by the corresponding monomials, counting these dots gives the Hilbert function HF_{X_A} of $X_A \subset \mathbb{P}^2$. The Hilbert polynomial is $\operatorname{HP}_{X_A} = 3k$, and it agrees with HF_{X_A} for $k \geq 1$. The vanishing ideal $I(X_A) = I(Y_{\hat{A}})$ is $\langle xz^2 - y^3 \rangle$. ⋄

The previous example illustrates the following corollary of Proposition 2.1.10.

Corollary 3.3.7. *The Hilbert function of* X_A *is given by* $\operatorname{HF}_{X_A}(k) = |kA|$, *i.e., the number of lattice points obtained as the sum of* k *not necessarily distinct columns of* A.

Counting lattice points "at level k" in the affine semigroup $\mathbb{N}\hat{A}$ is reminiscent of counting integer points in the k-th dilation of the convex polytope $\mathrm{Conv}(A)$. Recall that this is what is counted by the Ehrhart polynomial (Theorem 3.1.9). In particular, note that we have the obvious inequality $\mathrm{HF}_{X_A}(k) \leq E_{\mathcal{P}}(k)$, for $k \geq 0$ and $\mathcal{P} = \mathrm{Conv}(A)$. If equality holds, we are done, since combining Theorems 3.3.5 and 3.1.9 would give $\deg(X_A) = d!\mathrm{Vol}(\mathcal{P})$. In general, the inequality may be strict for all k, see Example 3.3.6. However, the Hilbert and Ehrhart polynomials do have the same leading term, which is our strategy for proving Theorem 3.3.1.

Let $\sigma^\vee = \mathrm{Cone}(\hat{A}) \subset \mathbb{R}^{d+1}$. This gives a saturated semigroup $\mathsf{S}_\sigma = \sigma^\vee \cap \mathbb{Z}^{d+1}$.

Lemma 3.3.8. *If $\mathbb{Z}'A = \mathbb{Z}^d$, then there exists $m \in \mathbb{N}\hat{A}$ such that $m + \mathsf{S}_\sigma \subset \mathbb{N}\hat{A}$.*

Proof. Let $\mathcal{T} = \{\sum_{i=1}^n \lambda_i(a_i,1) \in \mathbb{Z}^{d+1} \,:\, \lambda_i \in [0,1) \cap \mathbb{Q}\}$. For a geometric interpretation, see Example 3.3.9. For each $b = (b_1, \ldots, b_d, b_{d+1}) \in \mathcal{T}$, there exist integers $c_i(b) \in \mathbb{Z}, i = 1, \ldots, n$ such that $b = \sum_{i=1}^n c_i(b)(a_i,1)$. Indeed, because of the assumption $\mathbb{Z}'A = \mathbb{Z}^d$ we can write $b - b_{d+1}(a_1,1) = \sum_{i=1}^n c_i(a_i,1)$ for some $c_i \in \mathbb{Z}$ satisfying $\sum_{i=1}^n c_i = 0$. The $c_i(b)$ are obtained from $b = b_{d+1}(a_1,1) + \sum_{i=1}^n c_i(a_i,1)$. Fix $\nu \in \mathbb{N}$ such that $-\nu \leq c_i(b)$ for all $b \in \mathcal{T}$ and $i = 1, \ldots, n$. We define $m = \nu \sum_{i=1}^n (a_i,1)$ and show that $m + \mathsf{S}_\sigma \subset \mathbb{N}\hat{A}$.

If $m' \in m + \mathsf{S}_\sigma$, there exist $\alpha_i \in \mathbb{Q}_{\geq 0}$ such that $m' - m = \sum_{i=1}^n \alpha_i\,(a_i,1)$. Set $\alpha_i = \lambda_i + \gamma_i$ where $\lambda_i \in [0,1) \cap \mathbb{Q}$ and $\gamma_i \in \mathbb{N}$. Then $m' - m = \sum_{i=1}^n (\lambda_i + \gamma_i)(a_i,1)$ and

$$m' = m + \sum_{i=1}^n \lambda_i(a_i,1) + \sum_{i=1}^n \gamma_i(a_i,1) = \sum_{i=1}^n (\nu + c_i(b))(a_i,1) + \sum_{i=1}^n \gamma_i(a_i,1),$$

where $b = \sum_{i=1}^n \lambda_i(a_i,1) \in \mathcal{T}$, and both sums are in $\mathbb{N}\hat{A}$ by construction. $\square$

Proof of Theorem 3.3.1. Suppose that $\mathbb{Z}'A = \mathbb{Z}^d$. By Lemma 3.3.8, there exist $\tilde{m} \in \mathbb{Z}^d$, $e \in \mathbb{N}$ such that $m = (\tilde{m}, e) \in \mathbb{N}\hat{A}$ satisfies $m + \mathsf{S}_\sigma \subset \mathbb{N}\hat{A}$. Therefore, for $k \geq e$ we have

$$|(k-e)\cdot\mathcal{P} \cap \mathbb{Z}^d| \leq |kA| \leq |k\cdot\mathcal{P} \cap \mathbb{Z}^d|.$$

In terms of Ehrhart polynomials and Hilbert functions, by Corollary 3.3.7 this reads

$$E_{\mathcal{P}}(k-e) \leq \mathrm{HF}_{X_A}(k) \leq E_{\mathcal{P}}(k) \tag{3.3.1}$$

and for $k \gg 0$ we may replace $\mathrm{HF}_{X_A}(k)$ by $\mathrm{HP}_{X_A}(k)$. This implies that $\mathrm{HP}_{X_A}(k)$ and $E_{\mathcal{P}}(k)$ have the same leading term, which implies Theorem 3.3.1 when $\mathbb{Z}'A = \mathbb{Z}^d$ by Theorems 3.3.5 and 3.1.9. The case $\mathrm{rank}\,\mathbb{Z}'A = d$ follows from Lemma 3.3.4. $\square$

Example 3.3.9. The set $\mathcal{T}$ from the proof of Lemma 3.3.8 contains the origin in $\mathbb{Z}^{d+1}$, and the interior lattice points of a *zonotope* obtained by taking the Minkowski sum (see Definition 12.0.1) of the line segments connecting 0 with $(a_i,1)$. This is shown in Figure 3.5 for the matrix in Example 3.3.6. Blue dots represent points in $\mathbb{N}\hat{A}$, as in Figure 2.1. The lattice points in $\mathcal{T}$ are encircled in orange. We chose the following linear combinations $b = \sum_{i=1}^n c_i(b)(a_i,1)$:

$$(1,1) \;=\; 2(a_1,1) - (a_3,1), \quad (3,2) \;=\; (a_1,1) + (a_3,1), \quad (4,2) \;=\; 2(a_2,1).$$

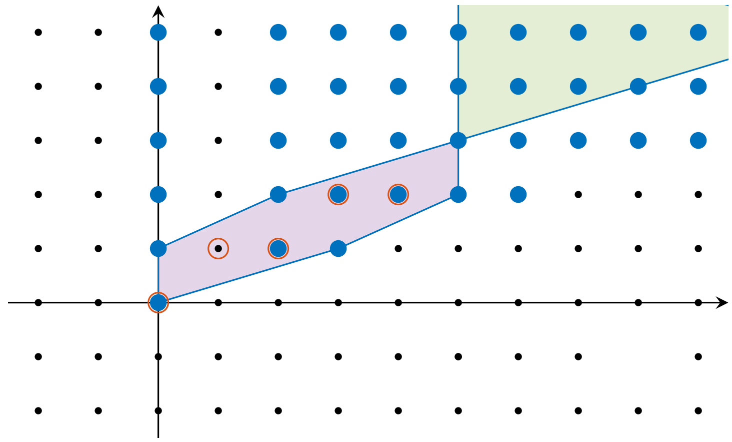

Figure 3.5: Illustration of the proof of Kushnirenko's theorem for $A = (0\ 2\ 3)$.

This gives $\nu = 1$, $m = (a_1, 1) + (a_2, 1) + (a_3, 1) = (5, 3)$ and $e = 3$. The green cone with apex in $(5, 3)$ is $m + \sigma^\vee$, and its lattice points are $m + \mathsf{S}_\sigma \subset \mathbb{N}\hat{A}$. At level $k = 3$, the chain of inequalities (3.3.1) reads $1 \leq 9 \leq 10$. For $k = 4$, we have $4 \leq 12 \leq 13$. $\diamond$

Example 3.3.10. Let $A = \left(\begin{smallmatrix} 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{smallmatrix}\right)$. The polytope $P = \mathrm{Conv}(A)$ was computed in `Oscar.jl` in Example 3.1.10. It is the pentagon shown in Figure 3.6 (blue and green). By Kushnirenko's theorem, the surface $X_A \subset \mathbb{P}^5$ has degree 5. This can be computed in `Oscar.jl` using the command `2*volume(P)`. We compute the `toric_ideal` associated

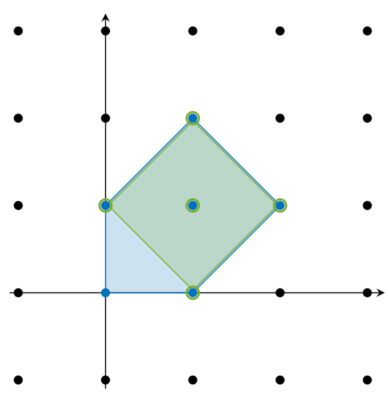

Figure 3.6: The polygons $\mathrm{Conv}(A)$ and $\mathrm{Conv}(A')$ from Example 3.3.10.

to $\hat{A}$ using `Oscar.jl`. The output is the ideal of $\mathbb{C}[x_1, x_2, x_3, x_4, x_5]$ generated by

$$x_3x_4 - x_2x_5,\ x_2x_3^2 - x_1^2x_5,\ x_2^2x_3 - x_1^2x_4,\ x_1^2x_4^2 - x_2^3x_5.$$

These are the defining equations of $X_A \subset \mathbb{P}^4$.

We remove the column $(0, 0)$ from A to obtain A' (Figure 3.6, green). Notice that the inclusion $\mathbb{Z}'A' \subset \mathbb{Z}^2$ is now strict. The degree of $X_{A'} \subset \mathbb{P}^4$ is 2, while the normalized volume of $P' = \mathrm{Conv}(A')$ in $\mathbb{R}^2$ is 4. The factor 2 is the lattice index of $\mathbb{Z}'A'$ in $\mathbb{Z}^2$. $\diamond$

Remark 3.3.11. We close the section with the observation that the dimension of X_A is also encoded by $\mathcal{P} = \mathrm{Conv}(A)$. By Corollary 1.3.18 and the fact that $\mathbb{Z}'A = \dim \mathcal{P}$, we have $\dim X_A = \dim \mathcal{P}$. This holds with no assumptions on A.

3.4 (Projective) normality and smoothness

This section discusses the properties of normality, projective normality and smoothness for projective toric varieties. This is phrased in terms of properties of polytopes in Section 3.5. The definition of a *normal affine variety* was given in Section 2.4.

Definition 3.4.1 (Normal)**.** *An irreducible variety X with affine covering $X = \bigcup_k U_k$ is* normal *if each of the affine varieties $U_k \subseteq X$ is normal.*

Definition 3.4.1 is independent of the choice of affine covering. We point out that Definition 3.4.1 applies when X is not projective (this is relevant in later chapters). In what follows, we write $\mathcal{V}(\mathcal{P})$ for the set of vertices of a polytope $\mathcal{P}$.

Proposition 3.4.2. *The projective toric variety $X_A \subset \mathbb{P}^{n-1}$ is normal if and only if each of the affine semigroups $\mathbb{N}(A - a_i)$, with $a_i \in \mathcal{V}(\mathrm{Conv}(A))$, is saturated in $\mathbb{Z}'A$.*

Using Theorem 2.4.2, the proof of Proposition 3.4.2 is easy and left as an exercise.

Exercise 3.4.3. Let X_{A_1}, X_{A_2} and X_{A_3} be the projective toric varieties defined by

$$A_1 = \begin{pmatrix} 0 & 2 & 3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 1 & 3 & 4 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Show that X_{A_3} is normal, while X_{A_1} and X_{A_2} are not.

A stronger condition than normality is *projective normality.*

Definition 3.4.4 (Projective normality)**.** *An irreducible projective variety $X \subset \mathbb{P}^{n-1}$ is* projectively normal *if its affine cone $\hat{X} \subset \mathbb{C}^n$ is a normal affine variety. Equivalently, X is projectively normal if the coordinate ring $\mathbb{C}[X]$ is a normal domain.*

For the definition of a normal domain, see Section 2.4. Any projectively normal projective variety is normal. The projectively normal toric varieties X_A are precisely those for which the Hilbert function agrees with the Ehrhart polynomial of $\mathrm{Conv}(A)$, see Corollary 3.3.7 and the discussion following it.

Proposition 3.4.5. *Suppose $\mathbb{Z}'A = \mathbb{Z}^d$. The projective toric variety $X_A \subset \mathbb{P}^{n-1}$ is* projectively normal *if and only if the following equivalent conditions are satisfied:*

1. $E_{\mathrm{Conv}(A)}(k) = \mathrm{HF}_{X_A}(k)$ *for* $k \in \mathbb{N}$,

2. $kA = \{a_{i_1} + \cdots + a_{i_k} \,:\, 1 \le i_1 \le \cdots \le i_k \le n\} = (k \cdot \mathrm{Conv}(A)) \cap \mathbb{Z}^d$ *for* $k \in \mathbb{N}$,

3. *the semigroup* $\mathbb{N}\hat{A}$ *with* $\hat{A}$ *as in Theorem 1.3.11 is saturated in* $\mathbb{Z}^{d+1}$.

Proof. The assumption $\mathbb{Z}'A = \mathbb{Z}^d$ ensures that $\mathbb{Z}\hat{A} = \mathbb{Z}^{d+1}$. We have seen this in the proof of Lemma 3.3.8. The equivalence of X_A being projectively normal and the semigroup $\mathbb{N}\hat{A} \subseteq \mathbb{Z}^{d+1}$ being saturated follows from Theorem 2.4.2 and the fact that $Y_{\hat{A}}$ is the affine cone over X_A. For 2 $\Leftrightarrow$ 3, notice that $\mathbb{N}\hat{A}$ is saturated in $\mathbb{Z}^{d+1}$ if and only if it equals $\mathrm{Cone}(\hat{A}) \cap \mathbb{Z}^{d+1}$. The elements of $\mathbb{N}\hat{A}$ with last coordinate equal to k are precisely the k-element sums of $\hat{A}$, which are in one-to-one correspondence with elements of kA by dropping the last coordinate. The elements of $\mathrm{Cone}(\hat{A}) \cap \mathbb{Z}^{d+1}$ with last coordinate equal to k are the points in $((k \cdot \mathrm{Conv}(A)) \cap \mathbb{Z}^d) \times \{k\}$. Finally, 1 $\Leftrightarrow$ 2 follows from Corollary 3.3.7 and the obvious inclusion $kA \subset (k \cdot \mathrm{Conv}(A)) \cap \mathbb{Z}^d$. □

We will see in Example 3.4.3 that projective normality depends on the embedding of a projective variety. On the other hand, normality does not: it is a local notion [34, Chapter I, Exercise 3.17]. Another local notion is smoothness.

Proposition 3.4.6. *Suppose* $\mathbb{Z}'A = \mathbb{Z}^d$ *and* $X_A \subset \mathbb{P}^{n-1}$ *is normal. Then* X_A *is smooth if and only if each of the cones* $\mathrm{Cone}(A - a_i)$, *with* $a_i \in \mathcal{V}(\mathrm{Conv}(A))$, *is smooth.*

Proof. The cone $\mathrm{Cone}(A - a_i)$ is smooth if and only if Y_{A-a_i} is smooth (Theorem 2.4.6). A projective variety is smooth if and only if it is covered by smooth affine varieties. □

Remark 3.4.7. It is conjectured in [65, Conjecture 13.19] that if X_A is projectively normal and smooth, then $I_{\hat{A}}$ is generated by quadratic binomials.

3.5 The projective toric variety of a polytope

In Section 2.3 we constructed an affine toric variety $\mathcal{Y}_\sigma$ from a rational polyhedral cone $\sigma \subset N_\mathbb{R}$. This is the normal toric variety whose coordinate ring is the semigroup algebra $\mathbb{C}[\mathsf{S}_\sigma]$. Here, we construct a projective toric variety $\mathcal{X}_\mathcal{P}$ from a lattice polytope $\mathcal{P}$.

A natural candidate for the toric variety of the polytope $\mathcal{P} \subset M_\mathbb{R} \simeq \mathbb{R}^d$ is $X_{\mathcal{P}\cap M}$. This is the projective toric variety embedded via the monomial map whose exponents are the lattice points contained in $\mathcal{P}$. This is not the conventional definition, for a number of good reasons. The first is *normality*. Like in the affine case (Theorem 2.4.2), we would like the toric variety $\mathcal{X}_\mathcal{P}$ of $\mathcal{P}$ to be normal. Theorems 2.4.2 and 3.2.4 imply that normality of $X_{\mathcal{P}\cap M}$ translates into the following condition on $\mathcal{P}$.

Definition 3.5.1 (Very ample)**.** *Let* $\mathcal{P} \subset M_\mathbb{R}$ *be a lattice polytope and let* $\mathcal{P}\cap M$ *be the set of its lattice points. We say that* $\mathcal{P}$ *is* very ample *if the semigroup* $\mathsf{S}_v = \mathbb{N}(\mathcal{P} \cap M - v) = \mathbb{N}\{m - v \,:\, m \in \mathcal{P} \cap M\}$ *is saturated in* M *for each vertex* $v \in \mathcal{V}(\mathcal{P})$.

Proposition 3.5.2. *Let* $\mathcal{P} \subset M_\mathbb{R}$ *be a lattice polytope such that* $\mathbb{Z}'(\mathcal{P} \cap M) = M$. *Then* $\mathcal{P}$ *is very ample if and only if the projective toric variety* $X_{\mathcal{P}\cap M} \subset \mathbb{P}^{|\mathcal{P}\cap M|-1}$ *is normal.*

Exercise 3.5.3. Prove Proposition 3.5.2.

Exercise 3.5.4. The assumption $\mathbb{Z}'(\mathcal{P} \cap M) = M$ in Proposition 3.5.2 is necessary. Verify that the simplex in $\mathbb{R}^3 \supset \mathbb{Z}^3 = M$ given by the convex hull of

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

is not very ample, yet $X_A = \mathbb{P}^3$ is smooth, and hence normal. After replacing $\mathbb{Z}^3$ by its sublattice $\mathbb{Z}'A$, Proposition 3.5.2 applies.

Exercise 3.5.5. Show that every 1-dimensional polytope is very ample.

The following statement gives an easy way to obtain a very ample polytope from any lattice polytope $\mathcal{P}$ [19, Corollary 2.2.19].

Theorem 3.5.6. *Let $\mathcal{P} \subset M_{\mathbb{R}} \simeq \mathbb{R}^d$ be a d-dimensional lattice polytope. We have that the k-dilation $k \cdot \mathcal{P} \subset \mathbb{R}^d$ is very ample for all $k \geq d-1$.*

Exercise 3.5.7. Use `Oscar.jl` to verify that the 3-dimensional simplex $\mathcal{P}$ given as the convex hull of the columns of A_3 from Exercise 3.4.3 is not very ample, but $2 \cdot \mathcal{P}$ is. Show that $X_{\mathcal{P} \cap M} \not\simeq X_{(2 \cdot \mathcal{P}) \cap M}$.

Definition 3.5.8 (Projective toric variety of a polytope)**.** *Let $\mathcal{P} \subset M_{\mathbb{R}} \simeq \mathbb{R}^d$ be a lattice polytope of dimension d. Let $k \in \mathbb{N}$ be such that the k-dilation $k \cdot \mathcal{P}$ is very ample. The variety $\mathcal{X}_{\mathcal{P}}$ is isomorphic to the projective toric variety $X_{(k \cdot \mathcal{P}) \cap M} \subset \mathbb{P}^{|(k \cdot \mathcal{P}) \cap M| - 1}$.*

Exercise 3.5.9. Show that the projective toric variety of the standard simplex $\mathcal{P} = \operatorname{Conv}(0, e_1, \ldots, e_d)$ is $\mathbb{P}^d$.

Exercise 3.5.10. Show that the projective toric variety of a square $\mathcal{P} = [0,1]^2 \subset \mathbb{R}^2$ is the product of two projective lines: $\mathcal{X}_{\mathcal{P}} = \mathbb{P}^1 \times \mathbb{P}^1$.

The variety $\mathcal{X}_{\mathcal{P}}$ from Definition 3.5.8 does not come with a fixed embedding. In fact, for the definition to make sense, we must show that all projective toric varieties $X_{(k \cdot \mathcal{P}) \cap M}$ for $k \cdot \mathcal{P}$ very ample are different embeddings of the same (abstract) projective variety. First, we show that vertices of very ample dilations give rise to the same semigroups.

Lemma 3.5.11. *Let $\mathcal{P} \subset M_{\mathbb{R}} \simeq \mathbb{R}^d$ be a lattice polytope of dimension d. Let $k \in \mathbb{N}$ be such that $k \cdot \mathcal{P}$ is very ample. For every vertex $v \in \mathcal{V}(\mathcal{P})$ of $\mathcal{P}$, we have an equality of semigroups*

$$\operatorname{Cone}(\mathcal{P} \cap M - v) \cap M \;=\; \mathbb{N}((k \cdot \mathcal{P}) \cap M - kv).$$

Proof. Notice that kv is a vertex of $k \cdot \mathcal{P}$ and $\operatorname{Cone}(\mathcal{P} \cap M - v) = \operatorname{Cone}((k \cdot \mathcal{P}) \cap M - kv)$. Since $k \cdot \mathcal{P}$ is very ample, the saturated semigroup $\operatorname{Cone}((k \cdot \mathcal{P}) \cap M - kv) \cap M$ is generated by $(k \cdot \mathcal{P}) \cap M - kv$. The lemma follows. $\square$

Lemma 3.5.11 implies that for k, ℓ such that $k \cdot \mathcal{P}$ and $\ell \cdot \mathcal{P}$ are very ample, we have

$$\mathbb{N}((k \cdot \mathcal{P}) \cap M - kv) \;=\; \mathbb{N}((\ell \cdot \mathcal{P}) \cap M - \ell v).$$

This means that $X_{k \cdot \mathcal{P} \cap M}$ and $X_{\ell \cdot \mathcal{P} \cap M}$ have isomorphic affine open charts. In what follows, we make these isomorphisms explicit and show that they agree on overlaps.

Let us write $[(k \cdot \mathcal{P}) \cap M - kv]$ for the matrix whose columns are the lattice points in $(k \cdot \mathcal{P}) \cap M - kv$, and similarly for ℓ. The sizes of these matrices are $d \times E_{\mathcal{P}}(k)$ and $d \times E_{\mathcal{P}}(\ell)$ respectively, where $E_{\mathcal{P}}$ is the Ehrhart polynomial from Theorem 3.1.9. By Lemma 3.5.11, for every vertex $v \in \mathcal{P}$ there exist matrices $A^v_{k \to \ell} \in \mathbb{N}^{E_{\mathcal{P}}(k) \times E_{\mathcal{P}}(\ell)}$ and $A^v_{\ell \to k} \in \mathbb{N}^{E_{\mathcal{P}}(\ell) \times E_{\mathcal{P}}(k)}$ with nonnegative integer entries such that

$$\begin{aligned} [(k \cdot \mathcal{P}) \cap M - kv] \cdot A^v_{k \to \ell} \;&=\; [(\ell \cdot \mathcal{P}) \cap M - \ell v], \\ [(\ell \cdot \mathcal{P}) \cap M - \ell v] \cdot A^v_{\ell \to k} \;&=\; [(k \cdot \mathcal{P}) \cap M - kv]. \end{aligned} \tag{3.5.1}$$

Each lattice point in $(k \cdot \mathcal{P}) \cap M$ corresponds to a coordinate of the projective space $\mathbb{P}^{E_{\mathcal{P}}(k)-1}$ in which $X_{(k \cdot \mathcal{P}) \cap M}$ is embedded. For each vertex $v \in \mathcal{V}(P)$ and each $k \in \mathbb{N}$, let $U_{kv} \subset \mathbb{P}^{E_{\mathcal{P}}(k)-1}$ be the standard affine chart on which the coordinate x_{kv} corresponding to the vertex kv of $k \cdot \mathcal{P}$ is non-zero. We define the monomial maps

$$\varphi^v_{k \to \ell} : U_{kv} \to U_{\ell v}, \quad \varphi^v_{\ell \to k} : U_{\ell v} \to U_{kv}$$

as follows: the map $\varphi^v_{k \to \ell}$ first sends the point with projective coordinates $(x_m)_{m \in k \cdot \mathcal{P}}$ to $(\frac{x_m}{x_{kv}})_{m \in k \cdot \mathcal{P}}$, and then it applies the monomial map whose exponents are the columns of $A^v_{k \to \ell}$. Notice that this map is well-defined and regular on U_{kv}, as $x_{kv} \neq 0$ and $A^v_{k \to \ell}$ has entries in $\mathbb{N}$. Similarly, $\varphi^v_{\ell \to k}$ sends $(\frac{y_m}{y_{\ell v}})_{m \in \ell \cdot \mathcal{P}}$ through the monomial map of $A^v_{\ell \to k}$.

Example 3.5.12. Let $\mathcal{P} = \mathrm{Conv}((0,0),(1,0),(0,1))$ be the standard simplex in $\mathbb{R}^2$. By Theorem 3.5.6, $k \cdot \mathcal{P}$ is very ample for every $k \in \mathbb{N}$. Let $k = 1$ and $\ell = 2$ and $v = (1,0)$. The matrices of lattice points are given by

$$[(k \cdot \mathcal{P}) \cap M - kv] \;=\; \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad [(\ell \cdot \mathcal{P}) \cap M - \ell v] \;=\; \begin{pmatrix} -2 & -1 & -2 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}.$$

Their columns generate the same saturated semigroup, whose Hilbert basis consists of the nonzero columns of $[(k \cdot \mathcal{P}) \cap M - kv]$. The transition matrices are

$$A^v_{k \to \ell} \;=\; \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}, \quad A^v_{\ell \to k} \;=\; \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^{\top}.$$

The map $\varphi^v_{k \to \ell}$ is the composition of $(x_1 : x_2 : x_3) \mapsto (\frac{x_1}{x_2} : 1 : \frac{x_3}{x_2})$ with the monomial map of $A^v_{k \to \ell}$, and $\varphi^v_{\ell \to k}$ composes $(y_1 : \cdots : y_6) \mapsto (\frac{y_1}{y_4} : \cdots \frac{y_6}{y_4})$ with the map of $A^v_{\ell \to k}$:

$$\begin{aligned} \varphi^v_{k \to \ell} &: (x_1 : x_2 : x_3) \longmapsto \left(\left(\frac{x_1}{x_2}\right)^2 : \frac{x_1}{x_2} : \left(\frac{x_1}{x_2}\right)\left(\frac{x_3}{x_2}\right) : 1 : \frac{x_3}{x_2} : \left(\frac{x_3}{x_2}\right)^2 \right), \\ \varphi^v_{\ell \to k} &: (y_1 : y_2 : y_3 : y_4 : y_5 : y_6) \mapsto \left(\frac{y_2}{y_4} : 1 : \frac{y_5}{y_4} \right). \end{aligned}$$

$\diamond$

For any $k, \ell \in \mathbb{N}$ such that $k \cdot \mathcal{P}$ and $\ell \cdot \mathcal{P}$ are very ample, let $\varphi_{k \to \ell} : \mathbb{P}^{E_{\mathcal{P}}(k)-1} \to \mathbb{P}^{E_{\mathcal{P}}(\ell)-1}$ be the map which sends $x \in X_{(k \cdot \mathcal{P}) \cap M}$ to $\varphi^v_{k \to \ell}(x)$ for any $v \in \mathcal{V}(P)$ such that $x \in U_{kv}$. It turns out that this map is well-defined on $X_{(k \cdot \mathcal{P}) \cap M}$, and it establishes the isomorphism $X_{(k \cdot \mathcal{P}) \cap M} \to X_{(\ell \cdot \mathcal{P}) \cap M}$ we need for Definition 3.5.8 to make sense.

Lemma 3.5.13. *Let $k, \ell \in \mathbb{N}$ be such that $k \cdot \mathcal{P}$ and $\ell \cdot \mathcal{P}$ are very ample. Let $v, w \in \mathcal{V}(\mathcal{P})$ be two vertices of $\mathcal{P}$ such that $x \in X_{(k \cdot \mathcal{P}) \cap M} \cap U_{kv} \cap U_{kw}$. We have $\phi^v_{k \to \ell}(x) = \phi^w_{k \to \ell}(x)$.*

Proof. The matrices $A^v_{k \to \ell}$ and $A^w_{k \to \ell}$ defined as in Equation (3.5.1) satisfy

$$[(k \cdot \mathcal{P}) \cap M - kv] \cdot A^v_{k \to \ell} \, = \, [(k \cdot \mathcal{P}) \cap M - kw] \cdot A^w_{k \to \ell} \, + \, \ell \, (w - v).$$

Here "$+\ell \, (w - v)$", similar to "$-kv$", means that we add this vector to each column. Adding the same integer vector to each column of a matrix $A \in \mathbb{Z}^{d \times n}$ does not change the image of its monomial map $\Phi_A : (\mathbb{C}^*)^d \to \mathbb{P}^{n-1}$. Products of matrices correspond to compositions of monomial maps. With these observations, a slight adaptation of Exercise 1.2.11 shows that $\varphi^v_{k \to \ell} \circ \Phi_{(k \cdot \mathcal{P}) \cap M} = \varphi^w_{k \to \ell} \circ \Phi_{(k \cdot \mathcal{P}) \cap M}$. We conclude that $\varphi^v_{k \to \ell}$ and $\varphi^w_{k \to \ell}$ agree on the image of $\Phi_{(k \cdot \mathcal{P}) \cap M}$, and hence on its closure in $U_{kv} \cap U_{kw}$. $\square$

Proposition 3.5.14. *Let $k, \ell \in \mathbb{N}$ be such that $k \cdot \mathcal{P}$ and $\ell \cdot \mathcal{P}$ are very ample. The restriction of $\phi_{k \to \ell}$ to $X_{(k \cdot \mathcal{P}) \cap M}$ is an isomorphism between $X_{(k \cdot \mathcal{P}) \cap M}$ and $X_{(\ell \cdot \mathcal{P}) \cap M}$. Its inverse is the restriction of $\varphi_{\ell \to k}$ to $X_{(\ell \cdot \mathcal{P}) \cap M}$.*

Proof. Using Exercise 1.2.11, one sees that Equation (3.5.1) implies that $\varphi_{k \to \ell} \circ \Phi_{(k \cdot \mathcal{P}) \cap M} = \Phi_{(\ell \cdot \mathcal{P}) \cap M}$. This means that the image of $(\varphi^v_{k \to \ell})_{|X_{(k \cdot \mathcal{P}) \cap M} \cap U_{kv}}$ is contained in $X_{(\ell \cdot \mathcal{P}) \cap M}$ for any vertex $v \in \mathcal{V}(\mathcal{P})$. Hence, by Lemma 3.5.13, $\varphi_{k \to \ell} : X_{(k \cdot \mathcal{P}) \cap M} \to X_{(\ell \cdot \mathcal{P}) \cap M}$ is a morphism of projective varieties. By (3.5.1), we also have

$$[(k \cdot \mathcal{P}) \cap M - kv] \cdot A^v_{k \to \ell} \cdot A^v_{\ell \to k} \, = \, [(k \cdot \mathcal{P}) \cap M - kv],$$

which implies that $\varphi^v_{\ell \to k} \circ \varphi^v_{k \to \ell}$ is the identity map on the image of $\Phi_{(k \cdot \mathcal{P}) \cap M}$. Therefore, it is the identity map on U_{kv}. Hence, $\varphi_{\ell \to k} \circ \varphi_{k \to \ell}$ is the identity on $X_{(k \cdot P) \cap M}$. We encourage the reader to check the details. $\square$

Exercise 3.5.15. Show that the morphism $\varphi_{k \to \ell}$ for the varieties in Example 3.5.12 is the 2-uple Veronese embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$.

We have shown that the lattice points of any two very ample dilations of a given polytope $\mathcal{P}$ give rise to isomorphic projective toric varieties. A stronger result is true: if two polytopes $\mathcal{P}, \mathcal{Q} \subset M_{\mathbb{R}}$ have the same *normal fan* (see Chapter 7), then their toric varieties are the same: $\mathcal{X}_{\mathcal{P}} \simeq \mathcal{X}_{\mathcal{Q}}$. More concretely, if $\mathcal{P}$ and $\mathcal{Q}$ have the same normal fan, then for any very ample dilations $k \cdot \mathcal{P}$ and $\ell \cdot \mathcal{Q}$, we have $X_{(k \cdot \mathcal{P}) \cap M} \simeq X_{(\ell \cdot \mathcal{Q}) \cap M}$. Our proof of Proposition 3.5.14 can easily be adapted to construct this isomorphism.

Exercise 3.5.16. We have shown that the projective toric variety $\mathcal{X}_{\mathcal{P}}$ can be embedded into projective space using any very ample dilation of $\mathcal{P}$. In this exercise, you will see that there are other options. Let $\mathcal{P}$ be the triangle from Example 3.5.12 and consider

$$A \, = \, \begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 2 & 3 \end{pmatrix}.$$

Show that $X_A \simeq \mathcal{X}_{\mathcal{P}} \simeq \mathbb{P}^2$, and A cannot be written as $\mathcal{Q} \cap \mathbb{Z}^2$ for any polytope $\mathcal{Q}$.

We conclude by explaining what projective normality means in terms of polytopes.

Definition 3.5.17 (Normal polytope)**.** *A lattice polytope $\mathcal{P} \subset M_{\mathbb{R}} \simeq \mathbb{R}^d$ is* normal *if for every $k \in \mathbb{N}$, we have*

$$k(\mathcal{P} \cap M) = \left\{ \sum_{i=1}^{k} m_i \,:\, m_i \in \mathcal{P} \cap M \right\} = (k \cdot \mathcal{P}) \cap M.$$

Here is an easy consequence of Proposition 3.4.5.

Corollary 3.5.18. *Let $\mathcal{P} \subset M_{\mathbb{R}} \simeq \mathbb{R}^d$ be a d-dimensional lattice polytope. If $\mathcal{P}$ is normal, then the projective toric variety $X_{\mathcal{P}\cap M}$ is projectively normal.*

Exercise 3.5.19. Prove Corollary 3.5.18.

Any normal lattice polytope is very ample [19, Proposition 2.2.18], while the converse is not true [44]. The following theorem shows that the projective toric variety of any polytope admits a projectively normal embedding [19, Theorem 2.2.12].

Theorem 3.5.20. *Let $\mathcal{P} \subset M_{\mathbb{R}} \simeq \mathbb{R}^d$ be a d-dimensional lattice polytope. We have that the k-dilation $k \cdot \mathcal{P} \subset \mathbb{R}^d$ is normal for all $k \geq d-1$.*

Exercise 3.5.21. Verify using `Oscar.jl` that the 5-dimensional polytope $\mathcal{P}$ from [19, Example 2.2.20] is not normal. The polytope $\mathcal{P} \subset \mathbb{R}^6$ equals $\mathrm{Conv}(A)$ for

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Find the smallest k for which $k \cdot \mathcal{P}$ is normal. Show that $X_{\mathcal{P}\cap M} \simeq X_{(k\cdot\mathcal{P})\cap M}$ are isomorphic as projective varieties, while $X_{(k\cdot\mathcal{P})\cap M}$ is projectively normal and $X_{\mathcal{P}\cap M}$ is not.

Exercise 3.5.22. Let $\mathcal{P}_1 \subset (M_1)_{\mathbb{R}} \simeq \mathbb{R}^{d_1}$ and $\mathcal{P}_2 \subset (M_2)_{\mathbb{R}} \simeq \mathbb{R}^{d_2}$ be lattice polytopes. The product polytope $\mathcal{P}_1 \times \mathcal{P}_2 \subset (M_1)_{\mathbb{R}} \times (M_2)_{\mathbb{R}}$ is $\{(m_1, m_2) \in (M_1)_{\mathbb{R}} \times (M_2)_{\mathbb{R}} : m_1 \in \mathcal{P}_1, m_2 \in \mathcal{P}_2\}$. Show that $\mathcal{X}_{\mathcal{P}_1 \times \mathcal{P}_2} \simeq \mathcal{X}_{\mathcal{P}_1} \times \mathcal{X}_{\mathcal{P}_2}$.

Further reading

A nice introduction to Ehrhart theory is found, for instance, in [53, Chapter 12]. For more on Remark 3.4.7, see [54]. The polytope $\mathrm{Conv}(A)$ is an example of a *Newton-Okounkov body*. Roughly speaking, these are convex bodies associated to a semigroup whose volume measures the degree of a projective variety. See [42] for more information.

Chapter 4

Toric boundaries

Embedded toric varieties are obtained as the closure of the image of a monomial map. This chapter investigates precisely what is added when taking the closure. The points in $Y \setminus \operatorname{im}\phi_A$ constitute the *boundary* of Y_A. This chapter explains how the boundary is naturally stratified according to which coordinates of $x \in Y_A \setminus \operatorname{im}\phi_A$ are zero. The strata correspond to faces of the polyhedral cone $\operatorname{Cone}(A)$. Similarly, in the projective setting, the boundary $X_A \setminus \operatorname{im}\Phi_A$ breaks up into strata indexed by the lower dimensional faces of the polytope $\operatorname{Conv}(A)$. We also explain how these strata are orbits of a group action, namely, the action of the torus $\operatorname{im}\phi_A$, resp. $\operatorname{im}\Phi_A$, on Y_A, resp. X_A.

4.1 The boundary of an affine toric variety

Lemma 4.1.1. *Let $A \in \mathbb{Z}^{d\times n}$ and let $Y_A \subset \mathbb{C}^n$ be the corresponding affine toric variety, parametrized by ϕ_A from* (1.1.1)*. We have* $\operatorname{im}\phi_A = Y_A \cap (\mathbb{C}^*)^n$.

Proof. The inclusion $\operatorname{im}\phi_A \subset Y_A \cap (\mathbb{C}^*)^n$ is clear. To show the reverse inclusion, it suffices to observe that $x \in Y_A \cap (\mathbb{C}^*)^n$ satisfies all equations in (1.2.4). □

Lemma 4.1.1 implies that the boundary $Y \setminus \operatorname{im}\phi_A$ consists of the points in Y_A with at least one zero coordinate. For $x \in \mathbb{C}^n$, we define the *support* of x as

$$\operatorname{supp}(x) \;=\; \{a_i \in A \,:\, x_i \neq 0\}.$$

Here we abuse notation slightly by writing A for both our usual matrix and the set of its columns: $a_i \in A$ reads 'a_i is a column of A'. The support of x is the set of columns of A indexing nonzero coordinates of x. For example, when $x \in (\mathbb{C}^*)^n$, we have $\operatorname{supp}(x) = A$.

Proposition 4.1.2. *Let $x \in Y_A \subset \mathbb{C}^n$. We have* $\operatorname{supp}(x) = \tau \cap A$ *for some face τ of the cone* $\operatorname{Cone}(A) = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \,:\, \lambda_i \in \mathbb{R}_{\geq 0}\}$.

Proof. Let $\tau \preceq \operatorname{Cone}(A)$ be the smallest face containing $\operatorname{supp}(x)$. We need to show $\operatorname{supp}(x) = \tau \cap A$. Let $\{a_{i_1}, \ldots, a_{i_q}\} \subset A$, with $\dim\tau \leq q \leq n$, be a set of minimal generators for τ. By this we mean that $\tau = \{\lambda_1 a_{i_1} + \cdots + \lambda_q a_{i_q} \,:\, \lambda_i \in \mathbb{R}_{\geq 0}\}$ and no

a_{i_j} can be omitted. In particular, $a_{i_1}, \dots, a_{i_q}$ generate the rays of τ but they are not necessarily the primitive ray generators from Definition 2.2.8.

The point $a_* = \sum_{a_i \in \mathrm{supp}(x)} a_i$ is an integer point inside relint(Cone(supp(x))) $\subset$ relint(τ). Therefore, there are positive integers $c_*, c_{i_1}, \dots, c_{i_q} \in \mathbb{R}_{>0}$ such that $c_* a_* = c_{i_1} a_{i_1} + \cdots + c_{i_q} a_{i_q}$. By Theorem 1.3.2, this implies

$$\prod_{a_i \in \mathrm{supp}(x)} x_i^{c_*} - \prod_{j=1}^{q} x_{i_j}^{c_{i_j}} \in I_A.$$

This binomial vanishes at our point x, which means $x_{i_j} \neq 0$, and hence $a_{i_j} \in \mathrm{supp}(x)$ for $j = 1, \dots, q$. We apply a similar argument for any $a_k \in \tau \cap A$: there are nonnegative integers $c_{i_1}, \dots, c_{i_q}$ and a positive integer $c_k > 0$ such that $c_k a_k = c_{i_1} a_{i_1} + \cdots + c_{i_q} a_{i_q}$. This means $x_k^{c_k} = \prod_{j=1}^{q} x_{i_j}^{c_{i_j}} \neq 0$, and hence $a_k \in \mathrm{supp}(x)$. This implies $\mathrm{supp}(x) = \tau \cap A$. $\square$

Corollary 4.1.3. *The affine toric variety Y_A is a disjoint union of* open strata

$$Y_A = \bigsqcup_{\tau \preceq \mathrm{Cone}(A)} Y_{A,\tau}^{\circ}, \tag{4.1.1}$$

where the open stratum $Y_{A,\tau}^{\circ}$ is given by $Y_{A,\tau}^{\circ} = \{x \in Y_A \,:\, \mathrm{supp}(x) = \tau \cap A\}$.

Notice that, by Lemma 4.1.1, the open stratum $Y_{A,\mathrm{Cone}(A)}^{\circ}$ of the full cone $\tau = \mathrm{Cone}(A)$ is the image of the map ϕ_A. Consequently, the toric boundary $Y_A \setminus \mathrm{im}\, \phi_A$ is

$$Y_A \setminus \mathrm{im}\, \phi_A = \bigsqcup_{\tau \prec \mathrm{Cone}(A)} Y_{A,\tau}^{\circ}.$$

Example 4.1.4. The matrix $A = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{smallmatrix}\right)$ gives rise to the smooth toric surface $Y_A = \{z - xy = 0\} \subset \mathbb{C}^3$, isomorphic to $\mathbb{C}^2$ (see Example 2.1.9). The cone Cone(A) is the nonnegative quadrant in $\mathbb{R}^2$. Equation (4.1.1) decomposes Y_A into four pieces, one for each face of that cone, see Figure 4.1. For instance, the green ray $\mathbb{R}_{\geq 0} \cdot (1,0)$ intersects A in the point a_2. The corresponding open stratum consists of points whose support is a_2, i.e., points (x, y, z) with $x = z = 0$, and $y \neq 0$. This is the punctured y-axis, shown in green in the right part of the figure. There is no point on Y_A for which $z = 0$ and $x, y \neq 0$, because a_1, a_2 do not form the intersection $\tau \cap A$ for some face $\tau \preceq \mathrm{Cone}(A)$. One also sees this from the defining equation $z = xy$. $\diamond$

The subset of columns $\tau \cap A$ defines an affine toric variety embedded in an affine space of dimension $|\tau \cap A| \leq n$. Such toric varieties are coordinate projections of Y_A.

Proposition 4.1.5. *For a face $\tau \preceq$ Cone(A), let $\tau \cap A = \{a_{i_1}, \dots, a_{i_\ell}\}$ and consider the coordinate projection $\pi_\tau : \mathbb{C}^n \to \mathbb{C}^\ell$ which drops all coordinates not indexed by $\tau \cap A$: $\pi_\tau(x) = (x_{i_1}, \dots, x_{i_\ell})$. Let $Y_{A,\tau} = \{x \in Y_A,\, \mathrm{supp}(x) \subseteq \tau \cap A\}$. We have*

$$\pi_\tau(Y_A) = \pi_\tau(Y_{A,\tau}) = Y_{\tau \cap A}.$$

Moreover, the map $(\pi_\tau)_{|Y_{A,\tau}} : Y_{A,\tau} \to Y_{\tau \cap A}$ is an isomorphism and $\pi_\tau(Y_{A,\tau}^{\circ}) = \mathrm{im}\, \phi_{\tau \cap A}$.

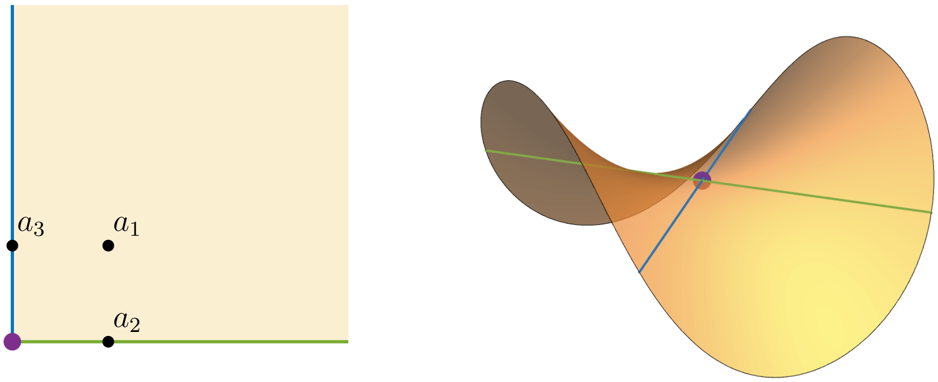

Figure 4.1: Faces of $\mathrm{Cone}(A)$ correspond to open strata of Y_A.

Proof. We first show that $\pi_\tau(Y_A) \subset Y_{\tau\cap A}$. We define the support of an exponent vector $u \in \mathbb{N}^n$ to be $\mathrm{supp}(u) = \{a_i \in A \,:\, u_i \neq 0\}$. We also set $\pi_\tau(u) = (u_{i_1}, \ldots, u_{i_\ell})$. Let u, v be such that $\mathrm{supp}(u) \subset \tau \cap A$, $\mathrm{supp}(v) \subset \tau \cap A$ and $A(u - v) = 0$. Clearly, for the matrix $\tau \cap A = (a_{i_1} \ \cdots \ a_{i_\ell})$, we have $(\tau \cap A)(\pi_\tau(u) - \pi_\tau(v)) = 0$. For $x \in Y_A$, we have $x^u - x^v = \pi_\tau(x)^{\pi_\tau(u)} - \pi_\tau(x)^{\pi_\tau(v)} = 0$, and hence $\pi_\tau(x) \in Y_{\tau\cap A}$.

Now, we show that π_τ maps $Y_{A,\tau}$ surjectively to $Y_{\tau\cap A}$. Let $\tilde{x} \in Y_{\tau\cap A}$. Clearly, $\tilde{x}$ is the image of $x \in \mathbb{C}^n$ under π_τ, where

$$x_i = \begin{cases} \tilde{x}_j & a_i = a_{i_j} \in \tau \cap A, \ \text{for some } j, \\ 0 & \text{otherwise} \end{cases}.$$

We claim that $x \in Y_{A,\tau} \subset Y_A$. It is clear that $\mathrm{supp}(x) \subset \tau \cap A$, so we need only show that $x \in Y_A$. For this, by Theorem 1.3.2, it suffices to check that $x^u - x^v = 0$ for all exponent vectors u, v such that $A(u - v) = 0$. If $\mathrm{supp}(u) \subset \tau \cap A$ and $\mathrm{supp}(v) \subset \tau \cap A$, this follows clearly from $\tilde{x} \in Y_{\tau\cap A}$. If $u_i > 0$ for some $a_i \notin \tau$, then $x^u = 0$. Moreover, $Au \notin \tau$ (Exercise 4.1.6). Since $Au = Av$, we must have $\mathrm{supp}(v) \not\subset \tau$: otherwise we would have $Av \in \tau$. Therefore $x^v = 0$, so that $x^u - x^v = 0$.

The map described above which sends $\tilde{x}$ to x is the inverse of the map $(\pi_\tau)_{|Y_{A,\tau}}$ from the proposition. It sends the torus $\mathrm{im}\,\phi_{\tau\cap A}$ to the open stratum $Y^\circ_{A,\tau}$. □

Exercise 4.1.6. Let $\mathrm{Cone}(A) \subset \mathbb{R}^d$ be the cone generated by the columns of A. Let $\tau \preceq \mathrm{Cone}(A)$ be a face, generated by the subset of columns $\tau\cap A$. Assume that $a_1 \notin \tau\cap A$. Show that a vector $u_1 a_1 + \cdots + u_n a_n$ with $u_1 > 0$, $u_2, \ldots, u_n \geq 0$ does not lie on τ.

The varieties $Y_{A,\tau}$ from Proposition 4.1.5 are called *closed strata*: they are obtained as the closure of the open strata from Corollary 4.1.3: $Y_{A,\tau} = \overline{Y^\circ_{A,\tau}}$.

Corollary 4.1.7. *The decomposition* (4.1.1) *is a stratification of* Y_A *into tori:* $Y^\circ_{A,\tau} \simeq \mathrm{im}\,\phi_{\tau\cap A}$ *is a torus of dimension* $\dim(\tau)$*. The closed stratum* $Y_{A,\tau}$ *satisfies*

$$Y_{A,\tau} = \overline{Y^\circ_{A,\tau}} = \bigsqcup_{\tau' \preceq \tau} Y^\circ_{A,\tau'},$$

where the disjoint union ranges over all faces τ' *of* $\tau \preceq \mathrm{Cone}(A)$*.*

Example 4.1.8. The open strata of the surface Y_A from Example 4.1.4 are

$$\begin{aligned} Y^\circ_{A,\mathrm{Cone}(A)} &= \mathrm{im}\ \phi_A, & Y^\circ_{A,\mathbb{R}_{\geq 0}\cdot(1,0)} &= \{0\} \times \mathrm{im}\ \phi_{\binom{1}{0}} \times \{0\}, \\ Y^\circ_{A,\mathbb{R}_{\geq 0}\cdot(0,1)} &= \{(0,0)\} \times \mathrm{im}\ \phi_{\binom{0}{1}}, & Y^\circ_{A,\{(0,0)\}} &= \{(0,0,0)\}. \end{aligned}$$

We have $Y_{A,\mathbb{R}_{\geq 0}\cdot(1,0)} = \overline{Y^\circ_{A,\mathbb{R}_{\geq 0}\cdot(1,0)}} = Y^\circ_{A,\mathbb{R}_{\geq 0}\cdot(1,0)} \sqcup Y^\circ_{A,\mathbb{R}_{\geq 0}\cdot(0,0)}$. $\diamond$

Exercise 4.1.9. If two matrices $A \in \mathbb{Z}^{d\times n}$ and $A' \in \mathbb{Z}^{d\times n'}$ generate the same semigroup, then $Y_A \simeq Y_{A'}$. Show that, in this case, $\mathrm{Cone}(A) = \mathrm{Cone}(A')$ and the isomorphism preserves each of the open strata: $Y^\circ_{A,\tau} \simeq Y^\circ_{A',\tau}$, $\tau \preceq \mathrm{Cone}(A)$.

4.2 The boundary of a projective toric variety

Projective toric varieties are stratified into tori in a similar manner. We extend our notation to the projective setting. The support of a point $x \in \mathbb{P}^{n-1}$ is

$$\mathrm{supp}(x) = \{a_i \in A \,:\, x_i \neq 0\}.$$

Let $Q \subset \mathrm{Conv}(A)$ be a face of the polytope obtained by taking the convex hull of A. We define subsets of X_A according to a prescribed support:

$$X^\circ_{A,Q} = \{x \in X_A \,:\, \mathrm{supp}(x) = Q \cap A\}, \quad X_{A,Q} = \{x \in X_A \,:\, \mathrm{supp}(x) \subseteq Q \cap A\}.$$

Here $Q \cap A$ consists of the columns of A which lie on Q.

Lemma 4.2.1. *Let $X_A \subset \mathbb{P}^{n-1}$ be the projective toric variety corresponding to $A \in \mathbb{Z}^{d\times n}$, parametrized by Φ_A from* (1.1.3). *We have* $\mathrm{im}\,\Phi_A = X_A \cap \{x \in \mathbb{P}^{n-1} \,:\, x_1 \cdots x_n \neq 0\}$.

Proof. Recall that the affine cone over X_A is the affine toric variety $Y_{\hat{A}}$. That is, X_A is the image of $Y_{\hat{A}} \setminus \{0\}$ under the map $\pi : \mathbb{C}^n \setminus \{0\} \to \mathbb{P}^{n-1}$ given by $\pi(x_1, \ldots, x_n) = (x_1 : \cdots : x_n)$. By Lemma 4.1.1, we have $\mathrm{im}\,\phi_{\hat{A}} = Y_{\hat{A}} \cap (\mathbb{C}^*)^n$. As a consequence,

$$\pi(\mathrm{im}\,\phi_A) = \mathrm{im}\,\Phi_A = X_A \cap \{x \in \mathbb{P}^{n-1} \,:\, x_1 \cdots x_n \neq 0\}. \qquad \square$$

For each face $Q \preceq \mathrm{Conv}(A)$, the submatrix $Q \cap A$ consisting of columns which lie on Q defines a projective toric variety $X_{Q\cap A} \subset \mathbb{P}^{|Q\cap A|-1}$.

Theorem 4.2.2. *Let $X_A \subset \mathbb{P}^{n-1}$ be the projective toric variety of $A \in \mathbb{Z}^{d\times n}$.*

1. *We have $X_A = \bigsqcup_{Q \preceq \mathrm{Conv}(A)} X^\circ_{A,Q}$ and $X_{A,Q} = \bigsqcup_{Q' \preceq Q} X^\circ_{A,Q}$.*
2. *For a face $Q \preceq \mathrm{Conv}(A)$, let $Q \cap A = \{a_{i_1}, \ldots, a_{i_\ell}\}$. The map*

$$\pi_Q : X_{A,Q} \longrightarrow X_{Q\cap A}, \quad x \longmapsto (x_{i_1} : \cdots : x_{i_\ell})$$

is a well defined isomorphism, and $\pi_Q(X^\circ_{A,Q}) = X^\circ_{Q\cap A,Q} = \mathrm{im}\,\Phi_{Q\cap A}$.

Proof. Faces $Q \subset \mathrm{Conv}(A)$ are in one-to-one correspondence with positive dimensional faces of the pointed cone $\mathrm{Cone}(\hat{A}) \subset \mathbb{R}^{d+1}$. For $Q \preceq \mathrm{Conv}(A)$, let $\tau_Q \preceq \mathrm{Cone}(\hat{A})$ be the corresponding face. By Corollary 4.1.3, we have

$$Y_{\hat{A}} = \bigsqcup_{Q \preceq \mathrm{Conv}(A)} Y^{\circ}_{\hat{A},\tau_Q} \sqcup \{0\}.$$

Indeed, the faces of $\mathrm{Conv}(A)$ account for all faces of dimension > 0, and the origin $\{0\}$ is the stratum $Y^{\circ}_{\hat{A},\{0\}}$ of the zero-dimensional face. The map $\pi : \mathbb{C}^n \setminus \{0\} \to \mathbb{P}^{n-1}$ given by $\pi(x_1, \dots, x_n) = (x_1 : \cdots : x_n)$ is defined on all strata except $\{0\}$. We have

$$X_A = \pi(Y_{\hat{A}} \setminus \{0\}) = \bigsqcup_{Q \preceq \mathrm{Conv}(A)} \pi(Y^{\circ}_{\hat{A},\tau_Q}) = \bigsqcup_{Q \preceq \mathrm{Conv}(A)} X^{\circ}_{A,Q}.$$

The stratification $X_{A,Q} = \bigsqcup_{Q' \preceq Q} X^{\circ}_{A,Q'}$ follows analogously from Corollary 4.1.7. This concludes the proof of the first statement.

The second statement, follows from the following commutative diagram

$$\begin{tikzcd}
Y_{\hat{A},\tau_Q} \setminus \{0\} \arrow[r, "\pi_{\tau_Q}"] \arrow[d, "\pi"] & Y_{\tau_Q \cap \hat{A}} \setminus \{0\} \arrow[d, "\tilde{\pi}"] \\
X_{A,Q} \arrow[r, "\pi_Q"] & X_{Q \cap A}
\end{tikzcd}$$

which uses the map π_{τ_Q} from Proposition 4.1.5, and $\tilde{\pi} : \mathbb{C}^{\ell} \setminus \{0\} \to \mathbb{P}^{\ell-1}$. □

Exercise 4.2.3. Consider the matrix $A = \left(\begin{smallmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{smallmatrix}\right)$. The corresponding projective toric variety X_A is the projective closure of the surface from Example 4.1.4, see Exercise 3.2.2. It is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$. List all strata $X^{\circ}_{A,Q}$ by describing them in coordinates. Match four of these strata with those listed in Example 4.1.8.

Exercise 4.2.4. In this exercise you work with the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

- Compute the toric ideal of the toric threefold $X_A \subset \mathbb{P}^5$ associated to A.
- Show that X_A equals the toric variety corresponding to the 5×6 matrix in Exercise 1.1.7.
- Draw the polytope $\mathrm{Conv}(A)$ and enumerate its faces. Conclude that X_A is the projective toric variety of a triangular prism.
- What is $\deg(X_A)$? Confirm this using the ideal I_A you computed.

- Parametrize the open strata of $X_A = \bigsqcup_{Q \subset \mathrm{Conv}(A)} X^\circ_{A,Q}$.
- Show that $X_A \simeq \mathbb{P}^1 \times \mathbb{P}^2$, and describe the above strata in the bihomogeneous coordinates $(x_0 : x_1), (y_0 : y_1 : y_2)$. See Exercise 3.5.22.

Exercise 4.2.5. Show that the topological Euler characteristic of X_A equals the number of vertices of $\mathrm{Conv}(A)$. Hint: use Theorem 4.2.2 and additivity of the Euler characteristic to show that $\chi(X_A) = \sum_{Q \preceq \mathrm{Conv}(A)} \chi(X^\circ_{A,Q})$. Then use $\chi(X^\circ_{A,Q}) \simeq (\mathbb{C}^*)^{\dim Q}$.

4.3 Torus orbits

The open strata $Y^\circ_{A,\tau}$ and $X^\circ_{A,Q}$ are orbits of a natural group action on the respective toric varieties. We recall some terminology. Let G be an algebraic group and let X be a variety. An algebraic group action of G on X is a morphism $G \times X \to X$, denoted by $(g, x) \mapsto g \bullet x$, satisfying the following axioms:

1. $e \bullet x = x$ for all $x \in X$ and for $e \in G$ the identity element,
2. $g \bullet (h \bullet x) = (g \cdot h) \bullet x$ for all $x \in X$ and all $g, h \in G$. Here $g \cdot h$ denotes the group operation in G.

An algebraic group always acts on itself: the group operation morphism $G \times G \to G$ satisfies axioms 1 and 2. Let $A \in \mathbb{Z}^{d \times n}$ be such that $\mathbb{Z}A = \mathbb{Z}^d$. Recall that this is not restrictive by Proposition 1.2.17. Consider the morphism

$$(\mathbb{C}^*)^d \times Y_A \longrightarrow Y_A, \quad (t, x) \longmapsto \phi_A(t) \cdot x = (t^{a_1} x_1, \ldots, t^{a_n} x_n). \tag{4.3.1}$$

Here $\phi_A(t) \cdot x$ denotes the group operation in $(\mathbb{C}^*)^n$, i.e., entrywise multiplication. We have seen in Proposition 1.2.21 that $\mathrm{im}\, \phi_A$ is a subtorus of $(\mathbb{C}^*)^n$. In particular, it is a group itself with coordinate-wise multiplication: $\phi_A(t_1) \cdot \phi_A(t_2) = \phi_A(t_1 \cdot t_2)$.

Proposition 4.3.1. *The morphism* (4.3.1) *is an algebraic group action of* $(\mathbb{C}^*)^d \simeq \mathrm{im}\, \phi_A$ *on* Y_A *which extends the action of* $\mathrm{im}\, \phi_A$ *on itself.*

Proof. For $t \in (\mathbb{C}^*)^d$ and $x \in Y_A$, define $t \bullet x = \phi_A(t) \cdot x$. We first confirm that the image of (4.3.1) lies in Y_A. By Theorem 1.3.2, it suffices to check that binomials $(t \bullet x)^u - (t \bullet x)^v$ for $A(u - v) = 0$ vanish for all t. We compute

$$(t \bullet x)^u - (t \bullet x)^v \;=\; t^{Au} x^u - t^{Av} x^v \;=\; 0.$$

The last equality uses $Au = Av$ and $x^u = x^v$ since $x \in Y_A$. It is straightforward to verify that the axioms 1 and 2 are satisfied. The action of $(\mathbb{C}^*)^d \simeq \mathrm{im}\, \phi_A$ on itself is given by

$$\phi_A(t_1) \cdot \phi_A(t_2) \;=\; \phi_A(t_1 \cdot t_2) \;=\; t_1 \bullet \phi_A(t_2).$$

This is extended by (4.3.1), in the sense that the following diagram commutes:

$$\begin{tikzcd}
(\mathbb{C}^*)^d \times \operatorname{im}\phi_A \arrow[r] \arrow[d, hook] & \operatorname{im}\phi_A \arrow[d, hook] \\
(\mathbb{C}^*)^d \times Y_A \arrow[r] & Y_A
\end{tikzcd}. \qquad \square$$

The *orbit* of $x \in X$ under the group action $G \times X \to X$ is $O_x = \{g \bullet x \,:\, g \in G\}$.

Theorem 4.3.2. *Let $A \in \mathbb{Z}^{d\times n}$ be such that $\mathbb{Z}A = \mathbb{Z}^d$ and let Y_A be the corresponding affine toric variety. The stratification $Y_A = \bigsqcup_{\tau \preceq \mathrm{Cone}(A)} Y^\circ_{A,\tau}$ from* (4.1.1) *decomposes Y_A into $(\mathbb{C}^*)^d$-orbits, where the action of $(\mathbb{C}^*)^d$ is that from* (4.3.1)*.*

Proof. For any $x \in Y_A$, let τ be the unique face of $\mathrm{Cone}(A)$ such that $x \in Y^\circ_{A,\tau}$. We need to show that $Y^\circ_{A,\tau} = O_x$. Notice that, by Proposition 4.1.5, there exists $u \in (\mathbb{C}^*)^d$ such that $\phi_{\tau\cap A}(u) = \pi_\tau(x)$. The theorem follows from the following chain of equivalences:

$$\begin{aligned}
x' \in O_x &\iff x' = t \bullet x \quad \text{ for some } t \in (\mathbb{C}^*)^d \\
&\iff \mathrm{supp}(x') = \tau \cap A \text{ and } \pi_\tau(x') = \phi_{\tau\cap A}(t)\cdot\phi_{\tau\cap A}(u) \\
&\iff x' \in Y^\circ_{A,\tau}.
\end{aligned}$$

The first equivalence is the definition of an orbit. The second equivalence follows from the fact that $x' = t \bullet x$ implies $\mathrm{supp}(x') = \mathrm{supp}(x)$, and from the definition (4.3.1). For the third equivalence, the arrow $\Rightarrow$ is clear. For $\Leftarrow$, note that $x' \in Y^\circ_{A,\tau} \Rightarrow \pi_\tau(x') = \phi_{\tau\cap A}(t')$ for some $t' \in (\mathbb{C}^*)^d$. Now write $t' = t \cdot u$, where $t = t' \cdot u^{-1}$. $\square$

In the literature, the association of a torus orbit to each face of $\mathrm{Cone}(A)$ is called the *orbit-cone correspondence*. In fact, one conventionally uses faces of the dual cone instead. Recall from point 4 in Proposition 2.2.5 that there is a bijective, inclusion reversing correspondence between the faces of a cone $\sigma \subset N_\mathbb{R}$ and those of its dual cone $\sigma^\vee \subset M_\mathbb{R}$. This bijection sends $\tau \preceq \sigma$ to $\tilde{\tau} = \{m \in \sigma^\vee \,:\, \langle u, m\rangle = 0, \text{ for all } u \in \tau\} \preceq \sigma^\vee$.

Definition 4.3.3. *The* orbit-cone correspondence *for the toric variety Y_A is a bijection between faces of the dual cone* $\mathrm{Cone}(A)^\vee$ *and $(\mathbb{C}^*)^d$-orbits of Y_A, given by $\tau \mapsto Y^\circ_{A,\tilde{\tau}}$.*

Exercise 4.3.4. Show that the orbit-cone correspondence sends a k-dimensional cone $\tau \preceq \mathrm{Cone}(A)^\vee$ to a $(d-k)$-dimensional torus orbit.

Exercise 4.3.5. Let τ and σ be faces of $\mathrm{Cone}(A)^\vee$. Show that τ is a face of σ if and only if the torus orbit corresponding to σ is contained in the closure of the torus orbit corresponding to τ. That is, $\tau \preceq \sigma$ if and only if $Y^\circ_{A,\tilde{\sigma}} \subseteq \overline{Y^\circ_{A,\tilde{\tau}}}$. Hint: use Corollary 4.1.7.

Exercise 4.3.6. Let $A = \mathrm{id}_d$, so that $Y_A = \mathbb{C}^d$. The cone $\mathrm{Cone}(A) = \mathbb{R}^d_{\geq 0}$ is self-dual. Show that the $(\mathbb{C}^*)^d$-orbit corresponding to the face $\mathrm{Cone}(e_{i_1}, \dots, e_{i_\ell}) \preceq \mathrm{Cone}(A)^\vee$ is $\{x \in \mathbb{C}^d \,:\, x_j = 0 \Leftrightarrow j \in \{i_1, \dots, i_\ell\}\}$.

Similar statements hold in the projective case, and the proofs are completely analogous. Here we assume that A is such that $\Phi_A : (\mathbb{C}^*)^d \to \mathbb{P}^{n-1}$ is one-to-one. Again, this is not restrictive, see Proposition 3.2.3. The action of $(\mathbb{C}^*)^d$ on X_A is similar to (4.3.1):

$$(\mathbb{C}^*)^d \times X_A \longrightarrow X_A, \quad (t, x) \longmapsto (t^{a_1}x_1 : \ldots : t^{a_n}x_n). \tag{4.3.2}$$

Proposition 4.3.7. *The morphism* (4.3.2) *is an algebraic group action of* $(\mathbb{C}^*)^d \simeq \operatorname{im}\Phi_A$ *on* X_A *which extends the action of* $\operatorname{im}\Phi_A$ *on itself.*

Theorem 4.3.8. *The orbits of* (4.3.2) *are the open strata* $X^\circ_{A,Q}$ *from 4.2.2. That is, the disjoint union* $X_A = \bigsqcup_{Q \preceq \mathrm{Conv}(A)} X^\circ_{A,Q}$ *decomposes* X_A *into* $(\mathbb{C}^*)^d$*-orbits.*

Exercise 4.3.9. Prove Proposition 4.3.7 and Theorem 4.3.8.

There is an orbit-cone correspondence like that in Definition 4.3.3 for projective toric varieties as well. For this, we need the terminology of *normal fans.* We defer the projective orbit-cone correspondence to Definition 7.3.11. Theorem 4.3.8 establishes an *orbit-face correspondence* for X_A, which sends a k-dimensional face $Q \preceq \mathrm{Conv}(A)$ to the k-dimensional $(\mathbb{C}^*)^d$-orbit $X^\circ_{A,Q}$.

Exercise 4.3.10. The projective plane $\mathbb{P}^2$ is a projective toric variety X_A corresponding to the matrix $A = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Describe its open strata in the $(\mathbb{C}^*)^2$-orbit stratification, and write the standard affine charts as unions of those. More generally, which strata constitute the affine chart of X_A corresponding to a vertex $v \in \mathcal{V}(\mathrm{Conv}(A))$?

Exercise 4.3.11. Show that the Grassmannian $\mathrm{Gr}(k, n)$ of $(k-1)$-planes in $\mathbb{P}^{n-1}$ has an action of $(\mathbb{C}^*)^n$ given by $t \bullet p = p \cdot \mathrm{diag}(t)$. Here $p \in \mathbb{C}^{k \times n}$ is a matrix representative of a point $[p]$ on $\mathrm{Gr}(k, n)$ and $\mathrm{diag}(t)$ is a diagonal $n \times n$-matrix whose diagonal entries are the coordinates of t. In other words, t acts by scaling the columns of p.

Consider the orbit O_p of a generic point $[p] \in \mathbb{P}^{\binom{n}{k}-1}$ in the Plücker embedding of $\mathrm{Gr}(k, n)$ under this action. Show that the closure of O_p is isomorphic to the projective toric variety X_A, where the columns of A are the vertices of the *hypersimplex*

$$\mathcal{P}(k, n) \;=\; \mathrm{Conv}\left(\sum_{i \in I} e_i \;:\; I \subset [n], |I| = k\right).$$

What is the dimension of such an orbit? Is X_A normal/smooth? Hint: you encountered the hypersimplex $\mathcal{P}(2, 4)$ in Example 3.1.4.

So far, we have worked with the ad hoc definition of a toric variety given in Section 1.1: a(n affine/projective) toric variety is the image of a monomial map. Propositions 4.3.1 and 4.3.7 establish the fact that our toric varieties Y_A, X_A fall under the following, more formal definition, see [33, Chapter 5, Definition 1.4] or [19, Definition 3.1.1].

Definition 4.3.12. *A* toric variety $\mathcal{X}$ *is an irreducible algebraic variety containing a torus* $T \simeq (\mathbb{C}^*)^d$ *as a dense open subset, such that the action of* T *on itself extends to an algebraic action* $T \times \mathcal{X} \to \mathcal{X}$*.*

In practice, toric varieties often arise as the image of a monomial map. Sometimes, this is only true after a linear change of coordinates [39]. All toric varieties we have seen so far are either affine or projective. Definition 4.3.12 uses the term *algebraic variety* in a more general sense, so that it applies to our varieties in Chapter 7.

4.4 Co-characters and distinguished points

In this section we construct the points on the boundary of a toric variety as limits of points in the torus. We use *one-parameter subgroups* for this. These are also called *co-characters*. The reason for both terminologies is clear from the definition.

Definition 4.4.1 (Co-character). *A* co-character *of a torus T is a morphism of varieties $\mathbb{C}^* \to T$ which is also a group homomorphism.*

The co-characters are in one-to-one correspondence with the lattice $N \simeq \mathbb{Z}^d$ first introduced in Section 3.2. A point $u \in \mathbb{Z}^d$ gives a co-character $\lambda_u(t) = (t^{u_1}, \dots, t^{u_d})$, where $t \in \mathbb{C}^*$. This parametrizes a one-dimensional subtorus of $(\mathbb{C}^*)^d$. We are interested in its image in an affine toric variety $Y_A \supset (\mathbb{C}^*)^d$. We keep assuming that $\mathbb{Z}A = \mathbb{Z}^d$.

Proposition 4.4.2. *The limit $\lim_{t\to 0} \phi_A(\lambda_u(t))$ exists in Y_A if and only if $u \in \mathrm{Cone}(A)^\vee$.*

Proof. One checks that $\phi_A(\lambda_u(t)) = (t^{\langle u,a_1\rangle}, \dots, t^{\langle u,a_n\rangle})$. This converges for $t \to 0$ in $\mathbb{C}^n$, and hence in Y_A, if and only if $\langle u, a_i\rangle \geq 0, i = 1, \dots, n$. This is equivalent to $u \in \mathrm{Cone}(A)^\vee$ because the rays of $\mathrm{Cone}(A)$ are among the a_i. □

The set of limit points of one-parameter subgroups is finite, and there is one for each face of the cone $\mathrm{Cone}(A)^\vee$. That is, there is one limit point for each stratum in Corollary 4.1.3, and for each torus orbit in Theorem 4.3.2. For each face $\tau \preceq \mathrm{Cone}(A)^\vee$, let $\tilde{\tau}$ be the associated face of $\mathrm{Cone}(A)$ (defined above Definition 4.3.3). Let $\gamma_\tau \in \mathbb{C}^n$ be the point whose i-th coordinate is 1 if $a_i \in \tilde{\tau}$, and 0 otherwise.

Exercise 4.4.3. Show that $\gamma_\tau \in Y_A$ for each face $\tau \preceq \mathrm{Cone}(A)^\vee$.

Proposition 4.4.4. *If $u \in \mathrm{relint}(\tau)$ for $\tau \in \mathrm{Cone}(A)^\vee$, then $\lim_{t\to 0} \phi_A(\lambda_u(t)) = \gamma_\tau$.*

Proof. Since $u \in \mathrm{Cone}(A)^\vee$, $\langle u, a_i\rangle \geq 0$ for $i = 1, \dots, n$. By the assumption that $u \in \mathrm{relint}(\tau)$, we have $\langle u, a_i\rangle = 0$ if and only if $a_i \in \tilde{\tau}$ (the reader should check this carefully). From this and $\phi_A(\lambda_u(t)) = (t^{\langle u,a_1\rangle}, \dots, t^{\langle u,a_n\rangle})$, the claim follows. □

The point γ_τ is called the *distinguished point* associated to τ. The distinguished points are orbit representatives for the $(\mathbb{C}^*)^d$ action, in the sense that for any point $p \in Y_A$, there is a distinguished point γ_τ with the same $(\mathbb{C}^*)^d$ orbit: $O_p = O_{\gamma_\tau}$. For this, one uses the unique $\tau \preceq \mathrm{Cone}(A)^\vee$ such that $p \in Y^\circ_{A,\tilde{\tau}}$.

Exercise 4.4.5. The matrix A from Example 4.1.4 defines an affine toric surface Y_A with four distinguished points. Let $\sigma = \{0\}$ so that γ_σ represents the dense torus orbit. Let $u = (1, 0) \in N = \mathbb{Z}^2$ and check that $\lim_{t\to 0} \lambda_u(t) \bullet \gamma_\sigma = \gamma_\tau$ for some $\tau \preceq \sigma$.

Exercise 4.4.6. The affine toric surface of the matrix $A = \left(\begin{smallmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ is $\mathbb{C} \times \mathbb{C}^*$ (Example 2.3.4). Identify the two distinguished points associated to the two faces of $\mathrm{Cone}(A)^\vee$.

The distinguished points of an affine toric variety can be characterized in a coordinate-free way in terms of the semigroup algebra $\mathbb{C}[\mathsf{S}]$.

Lemma 4.4.7. *Let $\mathsf{S} \subset M = \mathbb{Z}^d$ be an affine semigroup and let $\sigma = \mathrm{Cone}(\mathsf{S})^\vee \subset N_\mathbb{R}$. Fix $\tau \preceq \sigma$ and consider the $\mathbb{C}$-algebra homomorphism $\psi_\tau : \mathbb{C}[\mathsf{S}] \to \mathbb{C}$ defined by*

$$t^m \mapsto \begin{cases} 1 & m \in \tilde{\tau} \\ 0 & m \notin \tilde{\tau} \end{cases}.$$

The kernel $\ker \psi_\tau \subset \mathbb{C}[\mathsf{S}]$ *is a maximal ideal in* $\mathbb{C}[\mathsf{S}]$.

Proof. It is left to the reader to check that ψ_τ is well-defined, and its kernel is an ideal. By the short exact sequence $0 \to \ker \psi_\tau \hookrightarrow \mathbb{C}[\mathsf{S}] \to \mathbb{C} \to 0$, the quotient $\mathbb{C}[\mathsf{S}]/\ker \psi_\tau$ is a field. Hence, the ideal $\ker \psi_\tau$ is maximal. □

Definition 4.4.8. *The* distinguished point γ_τ *of* $\mathrm{Specm}(\mathbb{C}[\mathsf{S}])$ *corresponding to* $\tau \preceq \mathrm{Cone}(\mathsf{S})^\vee$ *is the maximal ideal* $\ker \psi_\tau \subset \mathbb{C}[\mathsf{S}]$ *from Lemma 4.4.7.*

Exercise 4.4.9. Check that our two definitions of γ_τ agree. More precisely, if $\mathsf{S} = \mathbb{N}A$, then the isomorphism $\mathrm{Specm}(\mathbb{C}[\mathsf{S}]) \simeq Y_A$ (Proposition 2.1.7) identifies $\ker \psi_\tau$ with γ_τ from Exercise 4.4.3.

Limits of one-parameter subgroups also index all strata in a projective toric variety X_A, or equivalently, all faces of the polytope $\mathrm{Conv}(A)$. For each face $Q \preceq \mathrm{Conv}(A)$, let

$$\sigma_Q = \{u \in N_\mathbb{R} : \min_{m \in \mathcal{P}} \langle u, m \rangle \text{ is attained for all } m \in Q\}.$$

We will see that these cones form the *normal fan* of $\mathrm{Conv}(A)$ (see Equation (7.3.1)). Let $\gamma_Q \in \mathbb{P}^{n-1}$ be the point represented by the following homogeneous coordinates: the i-th coordinate is 1 if $a_i \in Q$, and 0 otherwise.

Exercise 4.4.10. Show that $\gamma_Q \in X_A$ for each face $Q \preceq \mathrm{Conv}(A)$.

Proposition 4.4.11. *Fix $u \in N$ and let Q be the smallest face of* $\mathrm{Conv}(A)$ *such that $u \in \sigma_Q$. We have $\lim_{t \to 0} \Phi_A(\lambda_u(t)) = \gamma_Q \in X_A$.*

The proof of Proposition 4.4.11 is left to the reader. A first step is to write $\Phi_A(\lambda_u(t))$ in coordinates, and proceed as in Proposition 4.4.4.

Exercise 4.4.12. The projective toric surface X_A with A as in Exercise 4.2.3 has nine distinguished points. Write these points in coordinates on $\mathbb{P}^3 \supset X_A$ and in homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^1 \simeq X_A$.

4.5 Toric singularities

In this section, we describe the singular locus of Y_A. More precisely, we present a combinatorial formula for the multiplicity $\mathrm{mult}_x Y_A$ of Y_A at a point $x \in Y_A$. That multiplicity is one if and only if x is a smooth point of Y_A. It will follow from the main result, Theorem 4.5.7, that the singular locus $\mathrm{Sing}(Y_A)$ is a union of torus orbits. *We assume throughout that* $\mathrm{Cone}(A)$ *is a pointed cone of dimension* d. Since multiplicity is defined locally, the discussion also applies to projective toric varieties X_A: one computes $\mathrm{mult}_x X_A$ by passing to an affine chart $X_A \cap U_i \simeq Y_{A-a_i}$ containing x, see Section 3.2.

We start by recalling the definition of multiplicity. Let $Y \subset \mathbb{C}^n$ be an irreducible d-dimensional affine variety and let $x \in Y$ be one of its points. Consider a generic affine-linear subspace L of dimension $n-d$ which contains x. After slightly perturbing L we obtain an $(n-d)$-dimensional affine-linear space $\tilde{L}$ which intersects Y in $\mathrm{mult}_x Y$-many points in a small neighborhood of x. A more formal definition is purely algebraic. Let $\mathfrak{m}_x \subset \mathbb{C}[Y]$ be the maximal ideal corresponding to x. For large $k \gg 0$, we have

$$\dim_{\mathbb{C}} \mathfrak{m}_x^k/\mathfrak{m}_x^{k+1} \;=\; \frac{\mathrm{mult}_x Y}{(d-1)!}\, k^{d-1} \;+\; \text{lower order terms.} \tag{4.5.1}$$

We will express the multiplicity $\mathrm{mult}_x Y_A$ as a product of two combinatorially defined quantities. The first is a lattice index. Let $\sigma^\vee = \mathrm{Cone}(A) \subset M_{\mathbb{R}} = \mathbb{R}^d$ be the cone generated by the columns of $A \in \mathbb{Z}^{d\times n}$ and let $\tau \preceq \sigma^\vee$ be a face. The linear span of τ is denoted by $\mathbb{R}\tau$. This is a subvector space of $M_{\mathbb{R}}$ of dimension $\dim \tau$.

Definition 4.5.1. *The lattice index of A with respect to $\tau \preceq \sigma^\vee$ is*

$$i(\tau, A) \;=\; [\mathbb{Z}A \cap \mathbb{R}\tau : \mathbb{Z}(\tau \cap A)].$$

For $\tau = \{0\}$, we set $i(\{0\}, A) = 1$.

In words, the quantity $i(\tau, A)$ is the index of the lattice generated by $\tau \cap A = \{a_i : a_i \in \tau\}$ inside the (possibly bigger) lattice obtained by intersecting the lattice generated by A with the linear span of τ. For $\tau = \sigma^\vee$, we have $i(\sigma^\vee, A) = 1$.

Example 4.5.2. *Whitney's umbrella* is the toric surface $Y_A \subset \mathbb{C}^3$ coming from $A = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \end{smallmatrix}\right)$. Figure 4.2 shows this surface and highlights its strata (i.e., its torus orbits). The surface is defined by $y^2 - x^2 z = 0$, hence it contains the axis $y = z = 0$, shown in green, and $x = y = 0$, shown in orange. The latter is the closed stratum $Y_{A,\tau}$ corresponding to the orange cone τ in the left part of the figure. We have $\mathbb{Z}A = \mathbb{Z}^2$ and $i(\tau, A) = 2$. One checks that A has lattice index 1 with respect to all other faces of $\sigma^\vee$. ◇

The second quantity we need in our formula is the *subdiagram volume*.

Definition 4.5.3. *The* subdiagram volume *of A is defined as*

$$\mathrm{SDV}(A) \;=\; \frac{d!\,\mathrm{Vol}(\mathrm{Cone}(A) \setminus \mathrm{Conv}(\mathbb{N}A \setminus \{0\}))}{[\mathbb{Z}^d : \mathbb{Z}A]}\,.$$

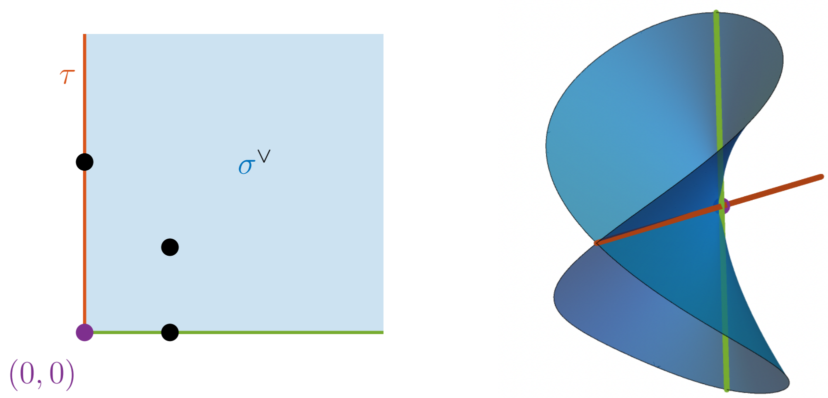

Figure 4.2: Whitney's umbrella (right) is the toric surface of $A = \left(\begin{smallmatrix}1&1&0\\0&1&2\end{smallmatrix}\right)$.

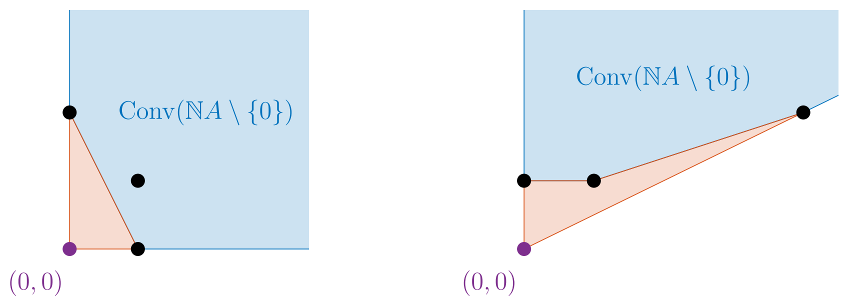

Figure 4.3: The subdiagram volumes (orange) of $\left(\begin{smallmatrix}1&1&0\\0&1&2\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&1&4\\1&1&2\end{smallmatrix}\right)$.

Example 4.5.4. The subdiagram volume is the normalized volume of a possibly non-convex polyhedron in $M_\mathbb{R}$. For the matrices $A_1 = \left(\begin{smallmatrix}1&1&0\\0&1&2\end{smallmatrix}\right)$ (see Example 4.5.2) and $A_2 = \left(\begin{smallmatrix}0&1&4\\1&1&2\end{smallmatrix}\right)$, the subdiagram volumes are $\mathrm{SDV}(A_1) = 2$ and $\mathrm{SDV}(A_2) = 3$. These are the areas of the orange regions in Figure 4.3, multiplied by 2!. $\diamond$

Exercise 4.5.5. Show that $\mathrm{SDV}(A) = 1$ if and only if $Y_A \simeq \mathbb{C}^d$. Hint: it suffices to show that $\mathrm{SDV}(A) = 1$ is equivalent to $\mathbb{N}A \simeq \mathbb{N}^d$.

Here is the geometric interpretation of the subdiagram volume $\mathrm{SDV}(A)$.

Proposition 4.5.6. *Let $A \in \mathbb{Z}^{d\times n}$ be such that $\sigma^\vee = \mathrm{Cone}(A)$ is pointed of dimension d. Let $0 \in Y_A$ be the torus invariant point $Y^\circ_{A,\{0\}}$. We have $\mathrm{mult}_0 Y_A = \mathrm{SDV}(A)$.*

Proof. We sketch the proof and refer to [33, Chapter 5, Theorem 3.14] for more details. We will assume without loss of generality that $\mathbb{Z}A = \mathbb{Z}^d$. The ideal $\mathfrak{m}_0 \subset \mathbb{C}[\mathbb{N}A]$ is generated by $\{t^m : m \in \mathbb{N}A \setminus \{0\}\}$. Adopting notation from [33], let us write

$$K_+ = \mathrm{Conv}(\mathbb{N}A \setminus \{0\}) \quad \text{and} \quad K_- = \overline{\mathrm{Cone}(A) \setminus \mathrm{Conv}(\mathbb{N}A \setminus \{0\})},$$

where we took the Euclidean closure for K_-. The ideal $\mathfrak{m}_0^k$ is generated by a subset of the monomials in $k \cdot K_+$. For $k \gg 0$, the dimension of the quotient vector space $\mathfrak{m}_0^k/\mathfrak{m}_0^{k+1}$

grows asymptotically like the number of lattice points in

$$k \cdot K_+ \setminus (k+1) \cdot K_+ \;=\; (k+1) \cdot K_- \setminus k \cdot K_-.$$

Recall that $k \cdot K_-$ is a (possibly non-convex) polyhedron, and its number of lattice points grows like a degree d polynomial with leading term $\mathrm{Vol}(K_-)k^d$ (Theorem 3.1.9). Writing "l.o.t" for *lower order terms*, we arrive at

$$\dim_{\mathbb{C}} \mathfrak{m}_0^k/\mathfrak{m}_0^{k+1} \;=\; \mathrm{Vol}(K_-)(k+1)^d - \mathrm{Vol}(K_-)(k)^d + \text{l.o.t.} \;=\; d\,\mathrm{Vol}(K_-)k^{d-1} + \text{l.o.t.}\ .$$

Using (4.5.1), we find that $\mathrm{mult}_0 Y_A = d \cdot (d-1)! \cdot \mathrm{Vol}(K_-) = \mathrm{SDV}(A)$. □

For each face $\tau \preceq \sigma^\vee$, consider the quotient lattice

$$\mathbb{Z}A/(\mathbb{Z}A \cap \mathbb{R}\tau) \;\simeq\; \mathbb{Z}^{d-\dim\tau}.$$

Let $A/\tau \in \mathbb{Z}^{(d-\dim\tau)\times n}$ be the matrix whose columns are the images of $a_1, \ldots, a_n$ in this quotient. To compute this matrix, one constructs a left kernel matrix $B \in \mathbb{Z}^{(d-\dim\tau)\times d}$ of $\tau \cap A$ and applies B from the left to each column of A.

Theorem 4.5.7. *Let $A \in \mathbb{Z}^{d\times n}$ be such that $\sigma^\vee = \mathrm{Cone}(A)$ is pointed of dimension d. For any point $x \in Y_A$, let $\tau \preceq \sigma^\vee$ be such that $x \in Y^\circ_{A,\tau}$. The multiplicity of Y_A at x is*

$$\mathrm{mult}_x\, Y_A \;=\; i(\tau, A) \cdot \mathrm{SDV}(A/\tau),$$

where $i(\tau, A)$ and $\mathrm{SDV}(A)$ are as in Definitions 4.5.1 and 4.5.3.

The proof of Theorem 4.5.7 given in [33, Theorems 3.1 and 3.16] argues that locally at x, Y_A looks like a product of $(\mathbb{C}^*)^{\dim\tau}$ and a union of $i(\tau, A)$ branches isomorphic to $Y_{A/\tau}$, each of which has multiplicity $\mathrm{SDV}(A)$ at x (Proposition 4.5.6).

The number $i(\tau, A) \cdot \mathrm{SVD}(A/\tau)$ is the multiplicity of Y_A at a generic point of the closed subvariety $Y_{A,\tau}$, For that reason, it is called the *multiplicity of Y_A along $Y_{A,\tau}$*.

Remark 4.5.8. Notice in particular that Y_A has multiplicity one at each point of its dense orbit $Y_A \cap (\mathbb{C}^*)^n$. That is, the points $Y_A \cap (\mathbb{C}^*)^n$ are smooth in Y_A. This is no surprise, as $Y_A \cap (\mathbb{C}^*)^n \simeq (\mathbb{C}^*)^d$ as varieties (Lemma 4.1.1).

Example 4.5.9. We return to Whitney's umbrella from Example 4.5.2. A left kernel matrix of $\tau \cap A = \left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right)$ is $B = \begin{pmatrix} 1 & 0 \end{pmatrix}$. Applying B from the left to A gives $A/\tau = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$, which has subdiagram volume 1. Hence, for any point x on the orbit $Y^\circ_{A,\tau}$ (shown in orange in Figure 4.2), we have $\mathrm{mult}_x Y_A \;=\; i(\tau, A) \cdot \mathrm{SDV}(A) \;=\; 2 \cdot 1 \;=\; 2.$ ⋄

Exercise 4.5.10. Let $A = \left(\begin{smallmatrix} 0 & 1 & 4 \\ 1 & 1 & 2 \end{smallmatrix}\right)$ be the matrix used in the right part of Figure 4.3. Determine the multiplicity of Y_A along each of its boundary strata using Theorem 4.5.7.

Exercise 4.5.11. Use Theorem 4.5.7 to determine the multiplicity of the cuspidal cubic plane curve $Y_{(2\ 3)}$ at its cusp $x = (0, 0)$.

Exercise 4.5.12. The matrix $A = \left(\begin{smallmatrix} 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{smallmatrix}\right)$ defines a toric surface $X_A \subset \mathbb{P}^3$. Show that X_A has three isolated singular points, and these have multiplicity 2. Hint: recall from Proposition 3.2.4 that X_A is covered by the affine charts Y_{A-a_1}, Y_{A-a_3} and Y_{A-a_4}.

Exercise 4.5.13. To illustrate that the results in this section can be computed effectively in a computer algebra system, we end with `Oscar` code [55]. The functions `get_face_lattice_index`, `get_SDV` and `get_multiplicity` implement Definitions 4.5.1, 4.5.3 and Theorem 4.5.7. We also include some auxiliary functions. The reader is encouraged to try out these functions to solve previous exercises and verify their correctness.

```
using Oscar
# Here is an example input for get_multiplicity
A = [1 1 0; 0 1 2] # Whitney's umbrella
τ = positive_hull(A[:,3]) # The ray τ from Figure 4.2
# Compute a matrix whose columns form a basis of the lattice generated by
# the columns of A
function get_lattice_basis(A)
    S,P,Q = snf_with_transform(A) # see Definition 1.2.7
    return inv(P)*S[:,1:rank(A)]
end
# Compute a basis for the sublattice of A obtained by intersecting A
# with the linear space generated by the columns of L.
function sublattice_in_linspace(A,L)
    N = nullspace(transpose(L))[2]
    V = nullspace(transpose(N)*A)[2]
    return get_lattice_basis(A*V)
end
# Compute the lattice index i(τ,A) of A with respect to a face τ of Cone(A)
function get_face_lattice_index(τ,A)
    inds = findall(i->A[:,i] in τ, 1:size(A,2))
    L = A[:,inds]
    Rτ = sublattice_in_linspace(A,L)
    newA = solve(Rτ,A[:,inds]; side = :right)
    return get_lattice_index(newA)
end
# Return the index of the lattice generated by the columns of A in
function get_lattice_index(A)
    return prod(diagonal(snf(A))[1:size(A,1)])
end
# Compute the subdiagram volume of A
function SDV(A)
    A = A[:,findall(i->sum(abs.(A[:,i]))!=0,1:size(A,2))]
    d, n = size(A)
    Awith0 = zero(matrix_space(ZZ,d,n+1)); Awith0[:,2:end] = A
    V1 = volume(convex_hull(transpose(Awith0)))
    V2 = volume(convex_hull(transpose(A)))
```

```
    vol = V1 - V2
    return factorial(d)*vol/get_lattice_index(A)
end
# Compute the multiplicity of Y_A along τ
function get_multiplicity(τ,A)
    A = matrix_space(ZZ,size(A)...)(A) # turn A into Oscar format
    A = hcat(matrix_space(ZZ,size(A,1),1)(zeros(Int,size(A,1))),A) # add 0
    inds = findall(i->A[:,i] in τ, 1:size(A,2))
    B = nullspace(transpose(A[:,inds]))[2]
    Amodτ = transpose(B)*A
    get_face_lattice_index(τ,A)*SDV(Amodτ)
end
```

For completeness, we also include Julia code for computing the multiplicity of the projective toric variety X_A along the orbit $X^\circ_{A,Q}$ corresponding to a face $Q \preceq \mathrm{Conv}(A)$. For each face Q, we compute a vertex $a_i \in \mathrm{Conv}(A)$ such that $a_i \in Q$. The orbit $X^\circ_{A,Q}$ is contained in the affine open subset $U_i \simeq Y_{A-a_i}$. We apply the affine function `get_multiplicity` using the shifted version $A - a_i$ of our matrix and $\tau = \mathrm{Cone}(Q - a_i)$.

```
# Compute the multiplicity of X_A along Q
function get_multiplicity_proj(A,Q)
    if dim(Q) == 0
        At = matrix_space(ZZ,size(A)...)(A .- lattice_points(Q)[1])
        return get_multiplicity([zeros(Int,size(A,1))], At )
    else
        Qinds = findall(p -> A[:,p] in Q, 1:size(A,2))
        vtcs = vertices(convex_hull(transpose(A)))
        # find a vertex contained in Q
        vtx = vtcs[findfirst(v->v in Q, vtcs)]
        # work on corresponding affine chart by shifting A
        At = matrix_space(ZZ,size(A)...)(A .- vtx)
        Qt = A[:,Qinds] .- vtx
        τQ = positive_hull([Qt[:,j] for j = 1:size(Qt,2)])
        return get_multiplicity(τQ, At)
    end
end
```

The code assumes that $\mathrm{Conv}(A)$ has dimension d. This is not restrictive, see Proposition 3.2.3. Here is how to compute the multiplicity of X_A along each face for the matrix $A = \left(\begin{smallmatrix} 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{smallmatrix}\right)$ from Example 4.5.12.

```
A = [0 1 1 2; 0 1 2 1]
P = convex_hull(transpose(A))
for i = 0:dim(P)
```

```
    println("faces of dimension $i:")
    for Q in faces(P,i) println(get_multiplicity_proj(A,Q)) end
end
```

Further reading

We will discuss the orbit-cone correspondence for projective toric varieties and more general toric varieties in Chapter 7. For more on the local structure and singularities of embedded toric varieties, see [33, Chapter 5, §3].

Chapter 5

Positive toric geometry

In many applications, the points of a (projective) toric variety Y_A (X_A) with positive or non-negative coordinates are of particular interest. The real points of the affine (resp. projective) toric variety Y_A (resp. X_A) are the points

$$Y_A(\mathbb{R}) = Y_A \cap \mathbb{R}^n, \quad X_A(\mathbb{R}) = X_A \cap \mathbb{RP}^{n-1}.$$

By Theorem 1.3.2, points in $Y_A(\mathbb{R})$ are real coordinate vectors $(x_1, \ldots, x_n) \in \mathbb{R}^n$ satisfying $x^u - x^v = 0$, for all $u - v \in \ker A$. Similarly, $X_A(\mathbb{R}) = \{(x_1 : \cdots : x_n) \in \mathbb{RP}^{n-1} : x^u - x^v = 0, \text{ for all } u - v \in \ker \hat{A}\}$. Among those are the non-negative and positive points. Section 5.1 describes these points in terms of the parametrizations ϕ_A and Φ_A. Section 5.2 is about an algebraic version of the moment map from symplectic geometry. That map is a homeomorphism between the nonnegative points of Y_A (resp. X_A) and the cone $\mathrm{Cone}(A)$ (resp. the polytope $\mathrm{Conv}(A)$).

5.1 Non-negative points of a toric variety

We start with positive points. We use the notation $\mathbb{R}^n_{>0}, \mathbb{R}^n_{\geq 0}$ for the positive, resp. non-negative orthant in $\mathbb{R}^n$. We write $\mathbb{P}^{n-1}_{>0}$ for points in projective space which can be represented by positive homogeneous coordinates:

$$\mathbb{P}^{n-1}_{>0} = \{(x_1 : \cdots : x_n) \in \mathbb{RP}^{n-1} : x_i > 0, \, i = 1, \ldots, n\}.$$

Similarly, $\mathbb{P}^{n-1}_{\geq 0}$ consists of points with non-negative homogeneous coordinates. Throughout the section, $A \in \mathbb{Z}^{d \times n}$, $Y_A \subset \mathbb{C}^n$, $X_A \subset \mathbb{P}^{n-1}$ are as defined in Chapter 1.

Definition 5.1.1. *The* positive affine toric manifold $(Y_A)_{>0} \subset Y_A(\mathbb{R})$ *is given by* $(Y_A)_{>0} = Y_A \cap \mathbb{R}^n_{>0}$. *We say that* $(Y_A)_{>0}$ *is the* positive part *of* Y_A. *Similarly, the* positive projective toric manifold $(X_A)_{>0} \subset X_A(\mathbb{R})$, *also called the* positive part *of* X_A, *is* $X_A \cap \mathbb{P}^{n-1}_{>0}$.

The semi-algebraic sets $(Y_A)_{>0}$ and $(X_A)_{>0}$ are sometimes called "positive toric varieties". The name "variety" in this context is a slight abuse of terminology.

Example 5.1.2. The positive part of affine space $Y_{\mathrm{id}_d} = \mathbb{C}^d$ is $\mathbb{R}^d_{>0}$. Similarly, we have $((\mathbb{C}^*)^d)_{>0} = \mathbb{R}^d_{>0}$, and $(X_{\mathrm{id}_d})_{>0} = (\mathbb{P}^{d-1})_{>0} = \mathbb{P}^{d-1}_{>0}$. $\diamond$

The monomial map $\phi_A : (\mathbb{C}^*)^d \to \mathbb{C}^n$ from (1.1.1) restricts to $\mathbb{R}^d_{>0} \subset (\mathbb{C}^*)^d$, and the image of that restriction is clearly contained in $(Y_A)_{>0}$. Similarly, $\Phi_A(\mathbb{R}^d_{>0})$ gives positive points on X_A. It turns out the converse statement is true as well.

Proposition 5.1.3. *The following equalities hold:*

$$(Y_A)_{>0} \;=\; \phi_A(\mathbb{R}^d_{>0}), \quad (X_A)_{>0} \;=\; \Phi_A(\mathbb{R}^d_{>0}).$$

Proof. The inclusion $\phi_A(\mathbb{R}^d_{>0}) \subseteq (Y_A)_{>0}$ is easy. To show the opposite inclusion, notice that $(Y_A)_{>0} \subseteq Y_A \cap (\mathbb{C}^*)^n = \operatorname{im} \phi_A$. The last equality is Lemma 4.1.1. Hence, a point $x \in (Y_A)_{>0}$ can be written as

$$x \;=\; (t^{a_1}, \ldots, t^{a_n}) \;=\; (|t^{a_1}|, \ldots, |t^{a_n}|) \;=\; (|t|^{a_1}, \ldots, |t|^{a_n}) \quad \text{for some } t \in (\mathbb{C}^*)^d.$$

Here $|\cdot|$ takes the absolute value, and $|t| = (|t_1|, \ldots, |t_n|) \in \mathbb{R}^d_{>0}$. This shows that $x \in \phi_A(\mathbb{R}^d_{>0})$, and concludes the proof for Y_A. The proof for X_A is analogous. $\square$

Another way to view positive toric manifolds is as the image of a linear space under the exponential map. Let $\exp : \mathbb{R}^n \to \mathbb{R}^n_{>0}$ be given by $\exp(x) = (e^{x_1}, \ldots, e^{x_n})$ and $\log : \mathbb{R}^d_{>0} \to \mathbb{R}^d$ is the coordinate-wise natural logarithm: $\log(t) = (\log(t_1), \ldots, \log(t_d))$. Here $e \approx 2.71828$ is Euler's number and $x \mapsto e^x$ is the inverse of the logarithm: $e^{\log x} = x$. We write $\mathrm{Row}_{\mathbb{R}}(A)$ for the row span of A, viewed as an $\mathbb{R}$-vector space.

Proposition 5.1.4. *We have* $(Y_A)_{>0} = \exp(\mathrm{Row}_{\mathbb{R}}(A))$.

Proof. By Proposition 5.1.3, every point $x \in (Y_A)_{>0}$ can be written as $x = \phi_A(t)$ for some $t \in \mathbb{R}^d_{>0}$. Rewriting the equality $x = \phi_A(t)$, we find

$$x \;=\; (t^{a_1}, \ldots, t^{a_n}) \;=\; \exp(\log(t)^\top A) \;\in\; \exp(\mathrm{Row}_{\mathbb{R}}(A)).$$

If $x \in \exp(\mathrm{Row}_{\mathbb{R}}(A))$, then $x = \exp(w^\top A) = \phi_A(e^{w_1}, \ldots, e^{w_d})$ for some $w \in \mathbb{R}^d$. $\square$

By Proposition 5.1.4, any linear space $\Lambda \subset \mathbb{R}^n$ defined over $\mathbb{Q}$ exponentiates to a positive toric manifold. Conversely, the logarithm of every positive toric manifold $(Y_A)_{>0}$ is a linear space, defined over $\mathbb{Q}$. The latter is the reason why, in statistics, positive toric manifolds go by the name of *log-linear models*, see Chapter 14.

Definition 5.1.5. *The* nonnegative affine toric manifold $(Y_A)_{\geq 0} \subset Y_A(\mathbb{R})$ *is* $Y_A \cap \mathbb{R}^n_{\geq 0}$. *We say that* $(Y_A)_{\geq 0}$ *is the* non-negative part *of* Y_A. *Similarly, the* non-negative projective toric manifold $(X_A)_{\geq 0}$, *also called the* non-negative part *of* X_A, *is* $(X_A)_{\geq 0} \cap \mathbb{P}^{n-1}_{\geq 0}$.

We should point out that $(Y_A)_{\geq 0}$ and $(X_A)_{\geq 0}$ may not be smooth manifolds. In fact, if $\mathrm{Cone}(A)$ is simplicial, resp. $\mathrm{Conv}(A)$ is simple, they are manifolds with corners, as we will see in Section 5.2.

Example 5.1.6. The nonnegative part of affine space $\mathbb{C}^d$ is the nonnegative orthant in $\mathbb{R}^d$: $(\mathbb{C}^d)_{\geq 0} = \mathbb{R}^d_{\geq 0}$. The nonnegative part of $(\mathbb{C}^*)^d$ equals its positive part: $((\mathbb{C}^*)^d)_{\geq 0} = \mathbb{R}^d_{>0}$. The nonnegative part of projective space is $(\mathbb{P}^{d-1})_{\geq 0} = \mathbb{P}^{d-1}_{\geq 0}$. ◇

Proposition 5.1.7. *The following equalities hold:*

$$(Y_A)_{\geq 0} \;=\; \overline{(Y_A)_{>0}}, \quad (X_A)_{\geq 0} \;=\; \overline{(X_A)_{>0}},$$

where the closures are the euclidean closures in $\mathbb{R}^n$ and $\mathbb{RP}^{n-1}$ respectively.

Proof. In general, the closure of an intersection is contained in the intersection of the closures. We have $\overline{(Y_A)_{>0}} = \overline{Y_A \cap \mathbb{R}^n_{>0}} \subseteq \overline{Y_A} \cap \overline{\mathbb{R}^n_{>0}} = Y_A \cap \mathbb{R}^n_{\geq 0}$, which proves one inclusion for $(Y_A)_{\geq 0}$. To show the other inclusion, let $x \in Y_A \cap \mathbb{R}^n_{\geq 0}$. Let $\operatorname{supp}(x) = \{a_i \in A : x_i \neq 0\}$ as in Chapter 4. By Proposition 4.1.5, we have

1. $\operatorname{supp}(x) = \tau \cap A$, for some face $\tau \preceq \operatorname{Cone}(A)$,
2. there is $t \in (\mathbb{C}^*)^d$ such that

$$x_i = \begin{cases} t^{a_i} & a_i \in \operatorname{supp}(x), \\ 0 & \text{otherwise} \end{cases}.$$

Since $x_i > 0$ for $a_i \in \operatorname{supp}(x)$, we may replace t by $|t|$, and assume $t \in \mathbb{R}^d_{>0}$. Let $w \in (\mathbb{R}^d)^\vee$ be such that $\langle w, a \rangle = 0$ for all $a \in \tau$, and $\langle w, a \rangle > 0$ for all $a \in A \setminus \tau$ (see Exercise 5.1.8). We have that

$$\lim_{u \to 0} \phi_A(u^{w_1} t_1, \ldots, u^{w_d} t_d) \;=\; \lim_{u \to 0} (t^{a_1} u^{\langle w, a_1 \rangle}, \ldots, t^{a_n} u^{\langle w, a_n \rangle}) \;=\; x.$$

Here u is a positive parameter. This shows that $x \in \overline{(Y_A)_{>0}}$. The proof for the nonnegative projective toric variety $(X_A)_{\geq 0}$ is analogous, and left as an exercise. □

Exercise 5.1.8. Let $\sigma \subset \mathbb{R}^d$ be a cone and let $\tau \preceq \sigma$ be one of its faces. Let

$$\tau^\perp = \{m \in (\mathbb{R}^d)^\vee \,:\, \langle u, m \rangle = 0 \text{ for all } u \in \tau\}.$$

Show that $\sigma^\vee \cap \tau^\perp$ is a face of $\sigma^\vee$, and that any point u in the relative interior of that face satisfies $\langle u, m \rangle > 0$ for $m \in \sigma \setminus \tau$. Start by verifying the statement in Figure 2.2.

Example 5.1.9. Let A_3 be the 2×3 matrix from Example 1.1.6. The intersection of the toric surface $Y_{A_3} = \{x^3 - yz = 0\} \subset \mathbb{C}^3$ with $\mathbb{R}^3_{\geq 0}$ is its nonnegative part. This is the blue surface in Figure 5.1, obtained by intersecting Y_{A_3} with $\mathbb{R}^3_{\geq 0}$. ◇

We can associate a positive and non-negative part to each of the closed strata $Y_{A,\tau}$ from Proposition 4.1.5:

$$(Y_{A,\tau})_{>0} \;=\; Y^\circ_{A,\tau} \cap \mathbb{R}^n_{\geq 0}, \quad (Y_{A,\tau})_{\geq 0} \;=\; Y_{A,\tau} \cap \mathbb{R}^n_{\geq 0}.$$

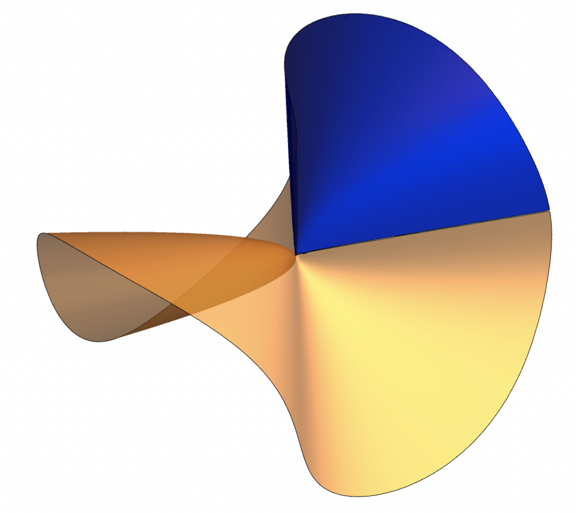

Figure 5.1: The nonnegative part $(Y_{A_3})_{\geq 0}$ of the surface $Y_{A_3} = \{x^3 - yz = 0\}$.

Proposition 5.1.10. *The absolute value map $\mathbb{R}^n \to \mathbb{R}^n_{\geq 0}$ given by $x \mapsto (|x_1|, \ldots, |x_n|)$ restricts to a retraction $r_A : Y_A \to (Y_A)_{\geq 0}$. For any face $\tau \preceq \operatorname{Conv}(A)$, the fiber of r_A over a point $\tilde{x} \in (Y_{A,\tau})_{>0}$ is isomorphic to $(S^1)^{\dim \tau} = \{(t_1, \ldots, t_{\dim \tau}) \in (\mathbb{C}^*)^{\dim \tau} : |t_i| = 1\}$.*

Proof. If $r_A(x) = \tilde{x}$, then $x \in Y^\circ_{A,\tau}$ and by Proposition 4.1.5, $Y^\circ_{A,\tau} \simeq \operatorname{im} \phi_{A,\tau}$. By Proposition 1.2.17, we may replace $\tau \cap A$ by a matrix $\overline{\tau \cap A}$ so that

$$\phi_{\overline{\tau \cap A}} : (\mathbb{C}^*)^{\dim \tau} \longrightarrow \mathbb{C}^{|\tau \cap A|}$$

is injective and $\operatorname{im} \phi_{\overline{\tau \cap A}} = \operatorname{im} \phi_{\tau \cap A}$. We write the columns of $\overline{\tau \cap A}$ as $\tilde{a}_i$, where i runs over the indices for which $a_i \in \tau \cap A$. Combining injectivity of $\phi_{\overline{\tau \cap A}}$ and Proposition 5.1.3, we observe that there is a unique $t \in \mathbb{R}^{\dim \tau}_{>0}$ such that

$$\tilde{x}_i = \begin{cases} t^{\tilde{a}_i} & a_i \in \tau \cap A, \\ 0 & \text{otherwise} \end{cases}.$$

The fiber $r_A^{-1}(\tilde{x})$ is clearly contained in $Y^\circ_{A,\tau}$, and it is given by

$$r_A^{-1}(\tilde{x}) = \{x \in Y^\circ_{A,\tau} : x_i = \tilde{t}^{\tilde{a}_i} \text{ for } a_i \in \tau \cap A \text{ and } |\tilde{t}| = t\}.$$

Since $\phi_{\overline{\tau \cap A}}$ is injective, this is isomorphic to $\{\tilde{t} \in (\mathbb{C}^*)^{\dim \tau} : |\tilde{t}| = t\} \simeq (S^1)^{\dim \tau}$. □

Again, the same proof idea applies for the projective version of Proposition 5.1.10. The statement uses the positive and non-negative parts of the strata in Theorem 4.2.2:

$$(X_{A,Q})_{>0} = X^\circ_{A,Q} \cap \mathbb{P}^{n-1}_{\geq 0}, \quad (X_{A,Q})_{\geq 0} = X_{A,Q} \cap \mathbb{P}^{n-1}_{\geq 0}.$$

Proposition 5.1.11. *The absolute value map $\mathbb{P}^{n-1} \to \mathbb{P}^{n-1}_{\geq 0}$ given by $x \mapsto (|x_1| : \ldots : |x_n|)$ restricts to a retraction $R_A : X_A \to (X_A)_{\geq 0}$. The fiber of R_A over a point $\tilde{x} \in (X_{A,Q})_{>0}$ is isomorphic to $(S^1)^{\dim Q}$.*

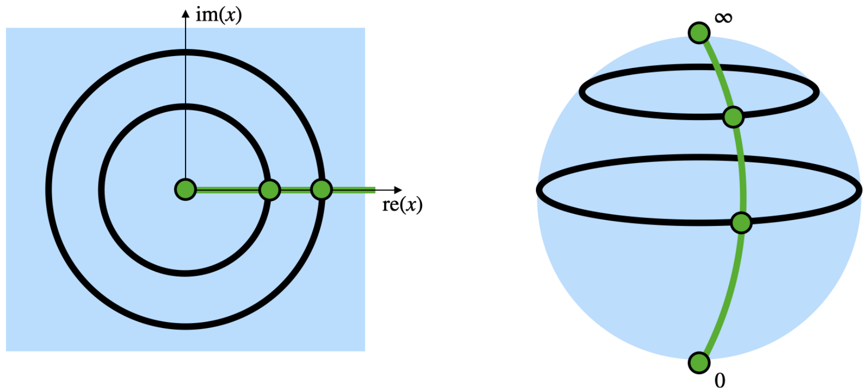

Figure 5.2: Left: $\mathbb{C}$ retracts to the nonnegative axis $\mathbb{R}_{\geq 0}$. Fibers of positive points are circles, while the fiber over 0 is 0. Right: $\mathbb{P}^1$, shown as a Riemann sphere, retracts to a line segment $(\mathbb{P}^1)_{\geq 0}$ connecting 0 and ∞ (in green). Again, fibers over interior points are illustrated as black circles.

Example 5.1.12. For $A = (1) \in \mathbb{Z}^{1\times 1}$, the retraction $r_A : \mathbb{C} \to \mathbb{R}$ is given by $x \mapsto |x|$. The cone $\mathrm{Cone}(A) = \mathbb{R}_{\geq 0}$ has two faces: $\tau_0 = \{0\}$ and $\tau_1 = \mathrm{Cone}(A)$. We have $(Y_{A,\tau_0})_{>0} = \{0\}$, whose fiber $r_A^{-1}(0)$ consists of a single point $\{0\} \simeq (S^1)^0$. For τ_1, we have $(Y_{A,\tau_1})_{>0} = \mathbb{R}_{>0}$, and $r_A^{-1}(\tilde{x})$ for $\tilde{x} \in \mathbb{R}_{>0}$ is the circle $\{x \in \mathbb{C} : |x| = \tilde{x}\} \simeq (S^1)^1$.

Adding the point at infinity corresponds to considering $A = (0\ 1) \in \mathbb{Z}^{1\times 2}$ and $X_A = \mathbb{P}^1$. The polytope $\mathrm{Conv}(A)$ has three faces, $Q_0 = \{0\}, Q_1 = [0,1], Q_\infty = \{1\}$. The positive parts of the corresponding strata are

$$(X_{A,Q_0})_{>0} = \{(1:0)\}, \quad (X_{A,Q_1})_{>0} = \{(1:x) \,:\, x \in \mathbb{R}_{>0}\}, \quad (X_{A,Q_\infty})_{>0} = \{(0:1)\}.$$

The retraction R_A is given by $R_A(x_0 : x_1) = (|x_0| : |x_1|)$. The fibers over $(X_{A,Q_0})_{>0}$ and $(X_{A,Q_\infty})_{>0}$ consist of a point, while those over $(X_{A,Q_1})_{>0}$ are circles. See Figure 5.2. $\diamond$

5.2 The algebraic moment map

We now identify the non-negative part $(Y_A)_{\geq 0}$ of the affine toric variety Y_A with the cone $\mathrm{Cone}(A)$. The map that realizes this identification is the *algebraic moment map.*

Definition 5.2.1. *The* algebraic moment map $\mu_{A,w} : Y_A \to \mathbb{R}^d$ *with weights* $w \in \mathbb{R}^n_{>0}$ *is*

$$\mu_{A,w}(x) \;=\; \sum_{i=1}^{n} w_i \cdot |x_i| \cdot a_i.$$

Note that, when restricted to the nonnegative part $(Y_A)_{\geq 0}$, $\mu_{A,w}$ agrees with the linear map $\mathbb{R}^n \to \mathbb{R}^d$ given by the matrix $A \cdot \mathrm{diag}(w)$. Here is the theorem that justifies our claim that $(Y_A)_{\geq 0} \simeq \mathrm{Cone}(A)$.

Theorem 5.2.2. *For any $w \in \mathbb{R}^n_{>0}$, the restriction of the affine algebraic moment map $\mu_{A,w}$ to the nonnegative affine toric variety $(Y_A)_{\geq 0}$ is a homeomorphism onto* $\mathrm{Cone}(A)$.

Example 5.2.3. The image of the blue surface in Figure 5.1 under the linear projection $A_3 : \mathbb{R}^3 \to \mathbb{R}^2$ is the cone $\mathrm{Cone}(A_3)$ shown in blue in Figure 5.3. $\diamond$

To prove Theorem 5.2.2, we follow the approach in [30, Chapter 4]. It is a consequence of Theorem 5.2.4 below. By Proposition 4.1.2, we have $(Y_A)_{\geq 0} = \bigsqcup_{\tau \preceq \mathrm{Cone}(A)} (Y_{A,\tau})_{>0}$. This is a disjoint union over all faces of $\mathrm{Cone}(A)$, including $\mathrm{Cone}(A)$ itself, for which $(Y_{A,\mathrm{Cone}(A)})_{>0} = (Y_A)_{>0}$.

Theorem 5.2.4. *For each face τ of* $\mathrm{Cone}(A)$, *the restriction of the algebraic moment map $\mu_{A,w}$ to $(Y_{A,\tau})_{>0}$ is a real analytic isomorphism onto the relative interior of τ.*

Proof. Our proof strategy is to show that every map except $\mu_{A,w}$ in the diagram

$$\begin{tikzcd}
\mathbb{R}^{\dim\tau}_{>0} \arrow[r, "\phi_{\overline{\tau\cap A}}"] & (Y_{\tau\cap A})_{>0} \arrow[r, "\simeq"] & (Y_{A,\tau})_{>0} \arrow[d, "\mu_{A,w}"] \\
\mathbb{R}^{\dim\tau} \arrow[u, "\exp"] \arrow[r, "F"] & \mathrm{int}(\mathrm{Cone}(\overline{\tau\cap A})) \arrow[r, "\simeq"] & \mathrm{relint}(\tau)
\end{tikzcd}$$

is a real analytic isomorphism. As a consequence, $\mu_{A,w}$ is a real analytic isomorphism as well. We explain each of the maps. By Proposition 4.1.5, $(Y_{A,\tau})_{>0}$ is isomorphic to the positive part of the affine toric variety $Y_{\tau\cap A} \subset \mathbb{C}^{|\tau\cap A|}$, parametrized by the monomial map $\phi_{\tau\cap A}$. We replace $\tau \cap A$ by $\overline{\tau\cap A}$ so that $\phi_{\overline{\tau\cap A}}$ is one-to-one. The restriction of $\phi_{\overline{\tau\cap A}}$ to $\mathbb{R}^{\dim\tau}_{>0}$ is a real analytic isomorphism onto $(Y_{\tau\cap A})_{>0}$. The map exp is given by $\exp(y_1, \ldots, y_q) = (e^{y_1}, \ldots, e^{y_{\dim\tau}})$, and the identification between $\mathrm{Cone}(\overline{\tau\cap A})$ and τ is induced by a coordinate change $\mathbb{R}^q \simeq \mathrm{span}_{\mathbb{R}}(\tau)$. Finally, the map F is defined such that it makes the diagram commute. Let $\overline{\tau\cap A} = (\tilde{a}_{i_1} \ \cdots \ \tilde{a}_{i_\ell})$

$$F : \mathbb{R}^{\dim\tau} \longrightarrow \mathrm{relint}(\mathrm{Cone}(\overline{\tau\cap A})), \quad y \longmapsto \sum_{j=1}^{\ell} w_{i_j} \cdot e^{\langle y, \tilde{a}_{i_j}\rangle} \cdot \tilde{a}_{i_j},$$

We have now reduced the problem to showing that F is a real analytic isomorphism. This is part of Theorem 5.2.6 below. $\square$

Example 5.2.5. We consider again the matrix A_3 from Example 1.1.6. The cone $C = \mathrm{Cone}(A_3)$ is the image of $(Y_{A_3})_{\geq 0}$ under the moment map. The proof of Theorem 5.2.4 parametrizes this cone in two more ways. First, the map $\mathbb{R}^2_{>0} \to \mathrm{int}(C)$, using weights 1, is given by

$$(t_1, t_2) \longmapsto (t_1t_2 + t_1t_2^2 + 2t_1^2t_2,\ t_1t_2 + 2t_1t_2^2 + t_1^2t_2).$$

This is obtained by composing ϕ_{A_3} with the moment map. The meshed orange area in the left part of Figure 5.3 is the image of $(0,1)^2$ under this map. In lighter shades of

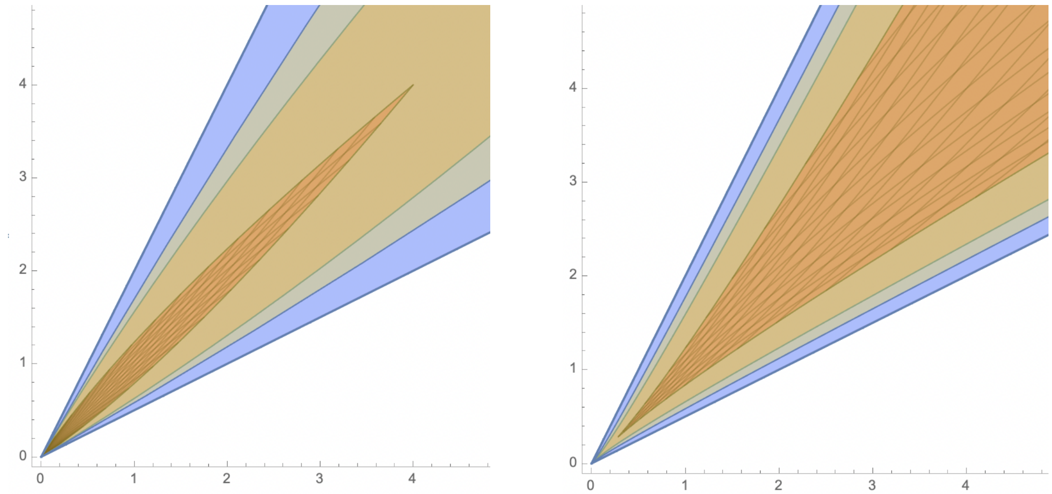

Figure 5.3: The cone $\mathrm{Cone}(A_3)$ is parametrized by $\mathbb{R}^2_{>0}$ and by $\mathbb{R}^2$.

orange, the images of $(0,2)^2$ and $(0,3)^2$ are shown. When $\alpha \to \infty$, the image of $(0,\alpha)^2$ fills the interior. The other parametrization of $\mathrm{int}(C)$ precomposes this map with the exponential:

$$(y_1, y_2) \longmapsto (e^{y_1+y_2} + e^{y_1+2y_2} + 2e^{2y_1+y_2},\; e^{y_1+y_2} + 2e^{y_1+2y_2} + e^{2y_1+y_2}).$$

This is the map $\mathbb{R}^2 \to \mathrm{int}(C)$ from the proof of Theorem 5.2.4. The images of $(-\alpha,\alpha)^2$ for $\alpha = 1, 2, 3$ are shown in the right part of Figure 5.3. $\diamond$

Theorem 5.2.6. *Let $F : \mathbb{R}^q \to \mathbb{R}^q$ be a map of the form $F(y) = \sum_{j=1}^r w_j \cdot e^{\langle y, u_j\rangle} \cdot u_j$, where $u_1, \ldots, u_r \in \mathbb{R}^q$ span $\mathbb{R}^q$ and $w_k > 0$. We have*

(A_q) *F is a real analytic isomorphism onto $\mathrm{int}(C)$, where $C = \mathrm{Cone}(u_1, \ldots, u_r)$.*

$(B_{q,m})$ *For any linear surjection $\pi : \mathbb{R}^q \to \mathbb{R}^m$, $\pi \circ F$ is onto $\mathrm{int}(\pi(C))$, and the fiber of $\pi \circ F$ over any point of $\mathrm{int}(\pi(C))$ is a connected manifold isomorphic to $\mathbb{R}^{q-m}$.*

Proof. The proof uses a clever induction argument. The plan is to show (A_1), then $(A_m) \Rightarrow (B_{q,m})$ for $q \geq m$ and finally $(B_{q,q-1}) \Rightarrow (A_q)$. The theorem then follows from $(A_1) \Rightarrow (B_{q,1}) \Rightarrow (A_2) \Rightarrow (B_{q,2}) \Rightarrow (A_3) \Rightarrow \ldots$.

First, we show (A_1). The derivative of $F(y) = \sum_{j=1}^r w_j e^{y u_j} u_j$ is $\sum_{j=1}^r w_j u_j^2 e^{y u_j} > 0$. This shows that F is injective. To show surjectivity onto $\mathrm{int}(C)$, consider three cases:

1. all u_j are negative, $C = \mathbb{R}_{\leq 0}$, $\lim_{y\to-\infty} F(y) = -\infty$, $\lim_{y\to\infty} F(y) = 0$,
2. all u_j are positive, $C = \mathbb{R}_{\geq 0}$, $\lim_{y\to-\infty} F(y) = 0$, $\lim_{y\to\infty} F(y) = \infty$,
3. mixed signs, $C = \mathbb{R}$, $\lim_{y\to-\infty} F(y) = -\infty$, $\lim_{y\to\infty} F(y) = \infty$.

This proves (A_1). Before continuing the induction argument, we show two claims:

(i) F is one-to-one and

(ii) the Jacobian matrix $\left(\frac{\partial F_i}{\partial y_k}\right)_{1\le i,k\le n}$ is positive definite for all $y \in \mathbb{R}^q$.

For (i), it suffices to show that the restriction of F to any line is one-to-one. This is clearly necessary, and it is sufficient because if $F(y) = F(y')$, then F is not injective on the line connecting y and y'. Fix any line $L \subset \mathbb{R}^q$. After a change of coordinates, we may assume that L is given by fixing the last $q-1$ coordinates: $L = \{y_2 = y_2^*, \dots, y_q = y_q^*\}$. Let $\tilde{w}_j = w_j e^{y_2^* u_{j,2}+\cdots+y_q^* u_{j,q}}$, where $u_{j,k}$ is the k-th coordinate of u_j. The restriction $F_{|L}$ is

$$y_1 \longmapsto \sum_{j=1}^{r} \tilde{w}_j \cdot e^{y_1 u_{j,1}} \cdot u_j.$$

The first coordinate of this function is one-to-one by (A_1), so $F_{|L}$ is one-to-one.

To show (ii), we compute $\frac{\partial F_i}{\partial y_k} = \sum_{j=1}^r w_j \cdot u_{j,k} \cdot e^{\langle y,u_j\rangle} u_{j,i}$. Note that this matrix is symmetric. It represents a positive definite quadratic form given by

$$v = (v_1,\dots,v_q)^t \longmapsto v^t \cdot \left(\frac{\partial F_i}{\partial y_k}\right)_{1\le i,k\le n} \cdot v \;=\; \sum_j w_j \cdot e^{\langle y,u_j\rangle} \langle v,u_j\rangle^2.$$

The next step is to show $(A_m) \Rightarrow (B_{q,m})$ for all $m \le q$. After changing coordinates $\pi : \mathbb{R}^q \to \mathbb{R}^m$ is the projection $(y_1,\dots,y_q) \mapsto (y_1,\dots,y_m)$ onto the first m coordinates. We write $\bar{\pi} : (y_1,\dots,y_q) \mapsto (y_{m+1},\dots,y_q)$ for the complementary projection. To simplify notation, let us write $\underline{y} = \pi(y) \in \mathbb{R}^m$, and $\bar{y} = \bar{\pi}(y) \in \mathbb{R}^{q-m}$. Notice that

$$F(y) \;=\; F(\underline{y},\bar{y}) \;=\; \sum_{j=1}^{r} w_j \cdot e^{\langle \underline{y},\underline{u}_j\rangle} \cdot e^{\langle \bar{y},\bar{u}_j\rangle} \cdot u_j.$$

When we fix the last $q-m$ coordinates of y, that is, we fix $\bar{y} = \bar{\pi}(y)$, we see that the map $F_{\bar{y}} : \underline{y} \mapsto \pi(F(\underline{y},\bar{y}))$ is a real analytic isomorphism $\mathbb{R}^m \to \mathrm{int}(\pi(C))$ by (A_m). The positive weights are $\tilde{w}_j = w_j \cdot e^{\langle \bar{y},\bar{u}_j\rangle}$. This proves, in particular, that $\pi \circ F$ is onto $\mathrm{int}(\pi(C))$. To show that fibres are isomorphic to $\mathbb{R}^{q-m}$, for each $p \in \mathrm{int}(\pi(C))$, define

$$G_p : \mathbb{R}^{q-m} \longrightarrow (\pi\circ F)^{-1}(p), \quad \bar{y} \longmapsto (F_{\bar{y}}^{-1}(p), \bar{y}).$$

By the above discussion, this is one-to-one and onto. To show that it is an isomorphism of manifolds, we use the implicit function theorem. Using coordinates $(\underline{z},\bar{z})$ on the image, we see that the graph of our map is given by $\mathcal{G}(\bar{y},\underline{z},\bar{z}) = 0$, where $\mathcal{G} : \mathbb{R}^{(q-m)+q} \to \mathbb{R}^q$ is

$$\mathcal{G}(\bar{y},\underline{z},\bar{z}) = ((\pi\circ F)(\underline{z},\bar{z}) - p,\; \bar{z} - \bar{y}).$$

Indeed, we have $\mathcal{G}(\bar{y}, G_p(\bar{y})) = 0$. The derivatives with respect to the variables $\underline{z},\bar{z}$ give a $q\times q$ Jacobian matrix of $\mathcal{G}$, whose first m rows are the first m rows of the Jacobian matrix from (ii). The last $q-m$ rows consist of a $(q-m)\times m$ block of zeros, and a $(q-m) \times (q-m)$ identity matrix. By our computations in the proof of (ii), the Jacobian is invertible for all values of $(\bar{y}, \underline{z}, \bar{z})$. Hence G_p is analytic, and establishes $\mathbb{R}^{q-m} \simeq (\pi \circ F)^{-1}(p)$ as manifolds.

The last step is to show $(B_{q,q-1}) \Rightarrow (A_q)$. By (i)-(ii), F is one-to-one, and it is a local isomorphism. We need to show that $(B_{q,q-1})$ implies that F is onto. We first prove that

$$\operatorname{im} F \text{ contains a point arbitrarily close to any point on each ray of } C. \tag{5.2.1}$$

Suppose that u_1 spans a ray. Let $J \subset [r] = \{1, \ldots, r\}$ be defined as $J = \{j \in [r] : u_j = s_j u_1 \text{ for some } s_j \in \mathbb{R}\}$: it indexes the vectors u_j which lie on the same line through 0 as u_1. There is $v \in \mathbb{R}^q$ such that $\langle v, u_j \rangle = 0$ for $j \in J$ and $\langle v, u_j \rangle < 0$ for $j \notin J$. We have

$$\lim_{\lambda \to \infty} F(\lambda v + v') \;=\; \lim_{\lambda \to \infty} \sum_{j=1}^{r} w_j \cdot e^{\langle \lambda v + v', u_j \rangle} \cdot u_j \;=\; \left(\sum_{j \in J} w_j e^{\langle v', u_1 \rangle s_j} s_j \right) \cdot u_1 .$$

Here the s_j are the scaling factors appearing in the definition of J, and $v' \in \mathbb{R}^q$ is arbitrary. We now apply (A_1) to the expression between parentheses on the right hand side. If all s_j are positive, we can choose v' to approach any point on the ray $\mathbb{R}_{>0} \cdot u_1$. If at least one of the s_j is negative, than $\mathbb{R} \cdot u_1$ belongs to the lineality space of C, and v' can be chosen to approach any point on this line. This establishes the claim (5.2.1).

Knowing (5.2.1), to show that F is onto, it suffices to show that $\operatorname{im} F \subset \mathbb{R}^q$ is convex. Equivalently, the intersection of $\operatorname{im} F$ with any line $L \subset \mathbb{R}^q$ is either connected or empty. Any such line L is the fiber $\pi^{-1}(p)$ of a linear projection $\pi : \mathbb{R}^q \to \mathbb{R}^{q-1}$. We have

$$F(\mathbb{R}^q) \cap \pi^{-1}(p) \;=\; F((\pi \circ F)^{-1}(p)).$$

By $(B_{q,q-1})$, $(\pi \circ F)^{-1}(p)$ is connected or empty. Since F is continuous, the same holds for $F((\pi \circ F)^{-1}(p))$. We have proved $(B_{q,q-1}) \Rightarrow (A_q)$, which concludes the proof. □

We now turn to projective toric varieties. We start with a projective moment map.

Definition 5.2.7. *The* algebraic moment map $\bar{\mu}_{A,w} : X_A \to \mathbb{R}^d$ *with weights* $w \in \mathbb{R}^n_{>0}$ *is*

$$\bar{\mu}_{A,w}(x) \;=\; \frac{1}{|w_1 \cdot x_1| + \cdots + |w_n \cdot x_n|} \sum_{i=1}^{n} w_i \cdot |x_i| \cdot a_i .$$

Notice that $\bar{\mu}_{A,w}$ is well defined on $\mathbb{P}^{n-1}$. Ignoring absolute values, the map $\bar{\mu}_{A,w}$ is given by the matrix $\hat{A} \cdot \operatorname{diag}(w)$, viewed as a map between projective spaces: $\hat{A} \cdot \operatorname{diag}(w) : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^d$, followed by the dehomogenization $\mathbb{P}^d \dashrightarrow \mathbb{R}^d$ which divides by the last coordinate.

Theorem 5.2.8. *For any positive weights w, the restriction of the algebraic moment map $\bar{\mu}_{A,w}$ to the nonnegative projective toric variety $(X_A)_{\geq 0}$ is a homeomorphism onto* $\operatorname{Conv}(A)$.

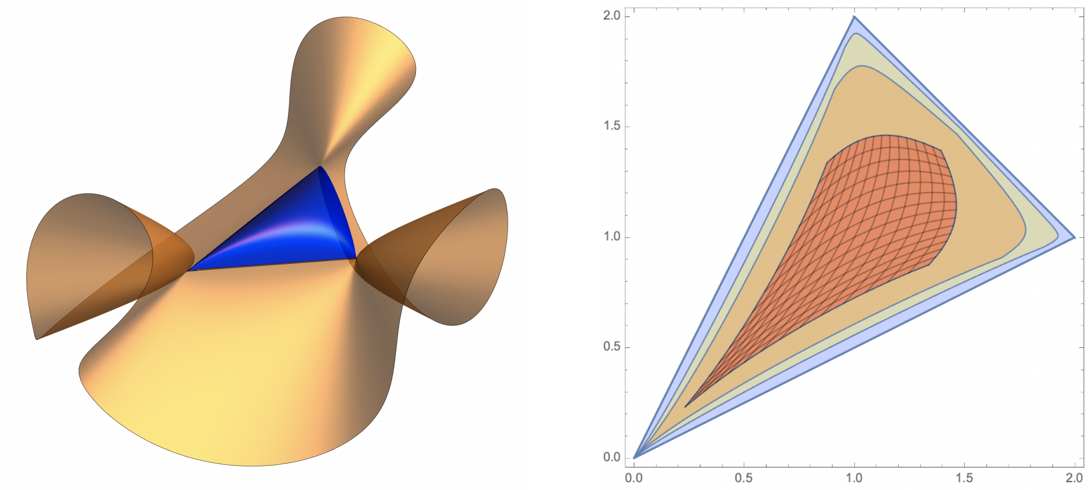

Figure 5.4: The image of the nonnegative toric variety $(X_A)_{\geq 0}$ in Example 5.2.9 is a triangle.

Example 5.2.9. The surface $Y_{A_3} = \{x^3 - yz = 0\}$ is an affine open subset of $X_A = \{x^3 - yzw = 0\} \subset \mathbb{P}^3$. The matrix A is given by

$$A = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix}.$$

The corresponding polygon $\mathrm{conv}(A_3)$ is the triangle in the right part of Figure 5.4. The surface X_A (orange) and its nonnegative part $(X_A)_{\geq 0}$ (blue) are plotted in Figure 5.4, using $x + y + z + w = 1$. The surface has three singular points, one for each vertex of the triangle. Each of these singular points has multiplicity 2 (Exercise 4.5.12). The nonnegative part is homeomorphic to our triangle, as predicted by Theorem 5.2.6. $\diamond$

Like in the affine case, Theorem 5.2.8 follows from the following statement:

Theorem 5.2.10. *For each face Q of* $\mathrm{Conv}(A)$*, the restriction of the algebraic moment map $\bar{\mu}_{A,w}$ to $(X_{A,Q})_{>0}$ is a real analytic isomorphism onto the relative interior of Q.*

Here $X_{A,Q} = \{x \in X_A \,:\, \mathrm{supp}(x) \subset Q\}$. After suitable coordinate changes, we are left with the following analog of Theorem 5.2.6:

Theorem 5.2.11. *For positive weights w_j and $u_j \in \mathbb{R}^q$, let $\overline{F} : \mathbb{R}^q \to \mathbb{R}^q$ be the map*

$$\overline{F}(y) = \frac{1}{\sum_{j=1}^{r} w_j \cdot e^{\langle y, u_j \rangle}} \cdot \sum_{j=1}^{r} w_j \cdot e^{\langle y, u_j \rangle} \cdot u_j.$$

If $P = \mathrm{Conv}(u_1, \ldots, u_r)$ has dimension q, then $\overline{F}$ is a real analytic isomorphism onto $\mathrm{int}(P)$.

Example 5.2.12. In Example 5.2.9, the two-dimensional face $Q = \mathrm{Conv}(A)$ leads to the following map $\mathbb{R}^2 \to \mathbb{R}^2$ (we take all weights w_j to be 1):

$$(y_1, y_2) \longmapsto \left(\frac{e^{y_1+y_2} + e^{y_1+2y_2} + 2e^{2y_1+y_2}}{1 + e^{y_1+y_2} + e^{y_1+2y_2} + e^{2y_1+y_2}}, \frac{e^{y_1+y_2} + 2e^{y_1+2y_2} + e^{2y_1+y_2}}{1 + e^{y_1+y_2} + e^{y_1+2y_2} + e^{2y_1+y_2}} \right).$$

Theorem 5.2.11 claims this is a real analytic isomorphism of the plane onto the interior of the triangle $\mathrm{Conv}(A)$. The image of $[-\alpha, \alpha]^2$ for $\alpha = 1, 2, 3$ is shown in the right part of Figure 5.4 in different shades of orange. When $\alpha \to \infty$, the image fills the blue triangle. $\diamond$

Proof of Theorem 5.2.11. Let $U = (u_1 \ \cdots \ u_r) \in \mathbb{Z}^{q \times r}$ be the matrix whose columns are u_j, and $\hat{U}$ is the $(q+1) \times r$-matrix with an extra row of ones. The assumption $\dim P = q$ implies $\mathrm{rank}(\hat{U}) = q + 1$. Theorem 5.2.6 says that $F : \mathbb{R}^{q+1} \to \mathrm{int}(\mathrm{Cone}(\hat{U}))$, with

$$F(y, y_{q+1}) = \sum_{j=1}^{r} w_j \cdot e^{\langle y, u_j \rangle} \cdot e^{y_{q+1}} \cdot \hat{u}_j,$$

is a real analytic isomorphism. Here $(y, y_{q+1}) = (y_1, \ldots, y_q, y_{q+1})$ are coordinates on $\mathbb{R}^{q+1}$ and $\hat{u}_j = (u_j, 1) \in \mathbb{R}^{q+1}$. We identify $\mathrm{int}(P) \simeq \mathrm{int}(\mathrm{Cone}(\hat{U})) \cap \{\text{first coordinate equal to } 1\}$. Its preimage under F is $y_{q+1} = -\log(\sum_{j=1}^{r} w_j \cdot e^{\langle y, u_j \rangle})$. The restriction of F to this preimage is precisely $\bar{F}$. More precisely, we have $F(-\log(\sum_{j=1}^{r} w_j \cdot e^{\langle y, u_j \rangle}), y) = (1, \overline{F}(y))$. $\square$

Further reading

For more on real and positive toric varieties and their applications in geometric modelling, see Sottile's tutorial [63]. Further connections with algebraic statistics are explained in [16]. Our proof of Theorem 5.2.2 follows [30, Chapter 4]. A different proof is given in [19, Theorem 12.2.5]. For a symplectic view on toric varieties and moment maps, see [20] and references therein.

Chapter 6

Toric geometry of linear programming

Linear programming means minimizing a linear function subject to linear equality and inequality constraints. More precisely, we aim to solve the optimization problem

$$\text{Minimize } \langle v, y\rangle \quad \text{subject to} \quad B_1\, y \;=\; c_1 \;\text{ and }\; B_2\, y + c_2 \geq 0. \tag{6.0.1}$$

Here $v \in \mathbb{R}^m$ is a *cost vector* and the equality/inequality constraints are given by matrices B_i with m columns and real vectors c_i. The objective function $\langle v, x\rangle$ is the linear function $v_1\, y_1 + \cdots + v_m\, y_m$. A *feasible point* is a point $y \in \mathbb{R}^m$ satisfying the equations $B_1\, y = c_1$ and the (entry-wise) inequalities $B_2\, y + c_2 \geq 0$. The set of all feasible points is a polyhedron, called the *feasible region*. You will show in Exercise 6.0.1 that any linear program can equivalently be written in the form

$$\text{Minimize } \langle w, x\rangle \quad \text{subject to} \quad A\, x \;=\; b \;\text{ and }\; x \in \mathbb{R}^n_{\geq 0}. \tag{6.0.2}$$

A linear program of the form (6.0.2) is said to be in *standard form*. The polyhedral feasible region is represented as a linear section $A\, x = b$ of the positive orthant $\mathbb{R}^n_{\geq 0}$.

Exercise 6.0.1. Show that the linear program (6.0.1) can be written in the standard form (6.0.2). Hint: first eliminate the equality constraints $B_1\, y = c_1$ by parametrizing the affine solution space: $y = y_0 + M\, z$, where M is a kernel matrix of B_1, y_0 is any solution to $B_1\, y = c_1$, and z is a vector of new unknowns. Next, check that the change of coordinates $x = B_2 M\, z + B_2\, y_0 + c_2$ makes the problem take the form (6.0.2).

We will make the assumption that the linear space $A\, x = 0$ is defined over $\mathbb{Q}$. That is, A has integer entries. Moreover, we assume that the entries of A are nonnegative, and no column of A is filled with zeros. This ensures that the feasible region $P = \{x \in \mathbb{R}^n : A\, x = b \text{ and } x \in \mathbb{R}^n_{\geq 0}\}$ is a polytope, i.e., the feasible polyhedron is bounded. These assumptions are satisfied, e.g., in optimal transport problems. Here is an example.

Exercise 6.0.2. Suppose a company needs to distribute 117 units of a product stored in two places S_1, S_2 among three clients T_1, T_2, T_3. The cost of transporting one unit from S_k to T_ℓ is the entry $w_{k,\ell}$ of

$$w = \begin{pmatrix} 3 & 5 & 7 \\ 11 & 13 & 17 \end{pmatrix}.$$

The number of units stored at S_1 is 66, so S_2 holds 51 units. The number of units desired at T_1, T_2, T_3 is $36, 26, 55$ respectively. The goal of the company is to come up with a transportation plan $x = (x_{k,\ell}) \in \mathbb{N}^{2\times 3}$, i.e., the plan is to transport $x_{k,\ell}$ units from S_k to T_ℓ, so that the total cost $\langle w, x\rangle$ is minimized. Write this optimal transport problem as a linear program in standard form. You saw the A-matrix in Example 1.1.7.

We can consider the same problem over the positive real numbers (e.g., replace units by kilograms). The feasible region is a convex polytope in $\mathbb{R}^n_{\geq 0}$. What is its dimension? How many vertices does it have?

This chapter discusses two ways in which toric ideals/varieties enter in solving (6.0.1). In Section 6.1, we discuss *integer programming*. This means we optimize over $x \in \mathbb{R}^n_{\geq 0} \cap \mathbb{Z}^n$. In Section 6.2, we consider an interior point method using *entropic regularization*.

6.1 Integer programming

This section deals with the following discrete version of (6.0.2):

$$\text{Minimize } \langle w, x\rangle \quad \text{subject to} \quad A\,x = b \ \text{ and } \ x \in \mathbb{N}^n, \tag{6.1.1}$$

where $w \in \mathbb{R}^n$, $A \in \mathbb{N}^{d\times n}$ and $b \in \mathbb{N}^d$. Our aim is to show that this can be solved by computing normal forms modulo the toric ideal I_A from Definition 1.3.1.

Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be an ideal and $\prec$ a monomial order. We write $\mathrm{LT}_\prec(f)$ for the leading term of a polynomial f with respect to $\prec$, and $\mathrm{in}_\prec(I)$ for the $\prec$-initial ideal of I. For each $f \in \mathbb{C}[x_1, \ldots, x_n]$, there exists a unique polynomial $\mathrm{NF}^I_\prec(f)$, called the $\prec$*-normal form* of f mod I, such that

$$f - \mathrm{NF}^I_\prec(f) \in I \quad \text{and} \quad \mathrm{LT}_\prec(\mathrm{NF}^I_\prec(f)) \notin \mathrm{in}_\prec(I). \tag{6.1.2}$$

The normal form $\mathrm{NF}^I_\prec(f)$ is computed as the remainder upon euclidean division of f by a $\prec$-Gröbner basis for I, see [18, Chapter 2, §6]. Here is a special property of normal forms of monomials in the case where $I = I_A$ is a toric ideal.

Lemma 6.1.1. *Let $I_A \subset \mathbb{C}[x_1, \ldots, x_n]$ be a toric ideal. For any monomial order $\prec$ and any monomial $x^u \in \mathbb{C}[x_1, \ldots, x_n]$, the normal form $\mathrm{NF}^{I_A}_\prec(x^u)$ is a monomial x^v, and the exponent vector v is such that $A(u - v) = 0$.*

Proof. Every reduced Gröbner basis of I_A consists of binomials of the form $x^u - x^v$ (Exercise 1.3.3). It follows that Euclidean division of a monomial by this Gröbner basis will give a monomial remainder. If $\mathrm{NF}^{I_A}_\prec(x^u) = x^v$, then by definition $x^u - x^v \in I_A$, which implies $A(u - v) = 0$. $\square$

We use the weight vector $w \in \mathbb{R}^n$ in (6.1.1) to define an order $\prec_w$ on monomials of $\mathbb{C}[x_1, \ldots, x_n]$ as follows. For two monomials $x^u, x^v \in \mathbb{C}[x_1, \ldots, x_n]$, we have

$$x^v \prec_w x^u \quad \Longleftrightarrow \quad \langle w, v\rangle < \langle w, u\rangle \quad \text{or} \quad \langle w, v\rangle = \langle w, u\rangle \quad \text{and} \quad x^v \prec x^u,$$

where $\prec$ is some fixed monomial order.

Remark 6.1.2. The order $\prec_w$ might not be a monomial order in the usual sense, as w is allowed to have negative entries. However, any weight w represents a monomial order for I_A because of homogeneity, see [65, Proposition 1.12 and page 43].

Theorem 6.1.3. *Let A, b, w be the data defining an integer program* (6.1.1). *If there exists a tuple $u \in \mathbb{N}^n$ satisfying $A\,u = b$, then the exponent v of the normal form $x^v = \mathrm{NF}^{I_A}_{\prec_w}(x^u)$ gives an optimal solution to* (6.1.1).

Proof. By Lemma 6.1.1, the normal form $\mathrm{NF}^{I_A}_{\prec_w}(x^u)$ is indeed a monomial x^v and we have $A\,v = b$. We need to show that $\mathrm{NF}^{I_A}_{\prec_w}(x^u) = x^v$ implies that $\langle w, v\rangle$ is minimal among $\{\langle w, u\rangle \,:\, u \in \mathbb{N}^n \text{ and } A\,u = b\}$. Suppose that there exists $v' \in \mathbb{N}^n$ such that $A\,v' = b$ and $\langle w, v'\rangle < \langle w, v\rangle$. Then $A(v - v') = 0$, hence $x^v - x^{v'} \in I_A$ (Theorem 1.3.2). But then $\mathrm{LT}_{\prec_w}(x^v - x^{v'}) = x^v \in \mathrm{in}_{\prec_w}(I)$. This contradicts the fact that (the leading term of) a normal form never lies in the initial ideal, see (6.1.2). $\square$

Notice that if no vector $u \in \mathbb{N}^n$ satisfying $A\,u = b$ exists, then (6.1.1) is infeasible. Theorem 6.1.3 provides an algorithm for solving the integer program (6.1.1), once we establish how to find $u \in \mathbb{N}^n$ such that $A\,u = b$. This can again be done using normal forms. We work in the ring $\mathbb{C}[t_0, t_1, \ldots, t_d, x_1, \ldots, x_n]$ and its ideal J from Exercise 1.3.7. We fix a monomial order $\prec$ satisfying $x_i \prec t_j$ for all i, j. It is easy to check that, if the integer program (6.1.1) is feasible, then $\mathrm{NF}^J_{\prec}(t^b) = x^u$, where $A\,u = b$.

Example 6.1.4. We return to the linear program from Exercise 6.0.2. We start by computing a feasible transport plan $u \in \mathbb{N}^{2\times 3}$. The ideal $J \subset \mathbb{C}[t_0, \ldots, t_5, x_{11}, \ldots, x_{23}]$ is

$$J \;=\; \langle t_0t_1t_2t_3t_4t_5 - 1,\, x_{11} - t_1t_3,\, x_{12} - t_1t_4,\, x_{13} - t_1t_5,\, x_{21} - t_2t_3,\, x_{22} - t_2t_4,\, x_{23} - t_2t_5\rangle.$$

This uses the matrix A from (1.1.2). In exercise 6.0.2, you found the corresponding righthand side vector $b = (66, 51, 36, 26, 55)^\top \in \mathbb{N}^5$. Using the lexicographic ordering $\prec$ with $x_{11} \prec x_{12} \prec x_{13} \prec x_{21} \prec x_{22} \prec x_{23} \prec t_1 \prec \cdots \prec t_5 \prec t_0$, we compute that

$$\mathrm{NF}^J_{\prec}(t_1^{66}t_2^{51}t_3^{36}t_4^{26}t_5^{55}) \;=\; x_{11}^{36}x_{12}^{26}x_{13}^{4}x_{23}^{51}.$$

This gives $u = \left(\begin{smallmatrix} 36 & 26 & 4 \\ 0 & 0 & 51 \end{smallmatrix}\right)$. One checks that $A \cdot (36, 26, 4, 0, 0, 51)^\top = b$, so u is indeed feasible. Here is the code for computing u in `Oscar.jl`.

```
A = [1 1 1 0 0 0; 0 0 0 1 1 1; 1 0 0 1 0 0; 0 1 0 0 1 0; 0 0 1 0 0 1]
w = [3; 5; 7; 11; 13; 17]; b = [66; 51; 36; 26; 55]
R, vrs = polynomial_ring(QQ,[["x_$i" for i = 1:6];["t_$j" for j = 1:6]])
x = vrs[1:6]; t = vrs[7:end]
J = ideal([prod(t)-1;[x[i]-prod(t[1:end-1].^(A[:,i])) for i = 1:6]])
NF = normal_form(prod(t[1:end-1].^b),J)
u = [36;26;4;0;0;51] # NF is a monomial, from which we read the exponents u
```

Next, by Theorem 6.1.3, we must compute $\mathrm{NF}^{I_A}_{\prec_w}(x^u)$. Here $\prec_w$ is the weight ordering given by w, refined by any monomial order on $\mathbb{C}[x_{11}, \ldots, x_{23}]$. Choosing the lexicographic ordering with $x_{11} \prec x_{12} \prec x_{13} \prec x_{21} \prec x_{22} \prec x_{23}$, we find

$$\mathrm{NF}^{I_A}_{\prec_{w,\mathrm{lex}}}(x^u) \;=\; x_{12}^{11} x_{13}^{55} x_{21}^{36} x_{22}^{15}.$$

This can be reproduced using the following snippet of `Oscar.jl` code:

```
IA = toric_ideal(transpose(A))
S = base_ring(IA); x = gens(S)
wo = weight_ordering(w,lex(S)) # Create weight ordering refined by lex
with_ordering(S,wo) do
    normal_form(prod(x.^u),IA)
end
```

The corresponding transport plan $v = \left(\begin{smallmatrix} 0 & 11 & 55 \\ 36 & 15 & 0 \end{smallmatrix}\right)$ has minimal cost 1031. Replacing the lexicographic ordering by the degree reverse lexicographic ordering, we obtain the alternative optimal transport plan $v' = \left(\begin{smallmatrix} 11 & 0 & 55 \\ 25 & 26 & 0 \end{smallmatrix}\right)$, with the same cost of 1031. The feasible polytope $\{x \in \mathbb{R}^6_{\geq 0} \,:\, A\,x = b\}$ is a two dimensional lattice polytope with 933 lattice points. The pairing $\langle w, \cdot \rangle$ is minimized on an edge, connecting the two vertices v, v' mentioned above and containing 12 optimal solutions in total. ◇

6.2 Entropic regularization

We switch back to linear programming over the real numbers. The program is (6.0.2):

$$\text{Minimize } \langle w, x \rangle \quad \text{subject to} \quad A\,x \;=\; b \;\text{ and }\; x \in \mathbb{R}^n_{\geq 0}, \tag{6.2.1}$$

where $A \in \mathbb{N}^{d \times n}$ has nonnegative integer entries and each column has at least one nonzero entry. This ensures that the feasible polyhedron $\mathcal{P}_b = \{x \in \mathbb{R}^n_{\geq 0} \,:\, A\,x = b\}$ is a bounded polytope. For $\mathcal{P}_b$ to be nonempty, we must have $b \in \mathrm{Cone}(A) \subset \mathbb{R}^d$, where $\mathrm{Cone}(A)$ is the polyhedral cone generated by the columns of A, see Section 3.2. We choose to index the feasible polytope $\mathcal{P}_b$ by the righthand side vector b, because later in this section we will be interested in the problem (6.2.1) for varying $b \in \mathrm{Cone}(A)$.

Exercise 6.2.1. Show that the set of optimizers x^* of (6.2.1) (i.e., the values of x^* for which $\langle w, x \rangle$ attains its minimal value on $\mathcal{P}_b$) is a face of $\mathcal{P}_b$. More precisely, it is the unique face of $\mathcal{P}_b$ so that w belongs to the relative interior of the corresponding cone in the *normal fan* of $\mathcal{P}_b$. We will study normal fans in Chapter **??**.

Example 6.2.2. Figure 6.1 illustrates Exercise 6.2.2. The feasible polytope $\mathcal{P}_b$ is the pentagon shown in green in the plane $\{Ax = b\}$. The projection of a weight vector w_1 onto the dual plane is shown in red. The inner product with w_1 is minimized on the lower left vertex, colored in red. The same vertex is optimal for all other choices of w

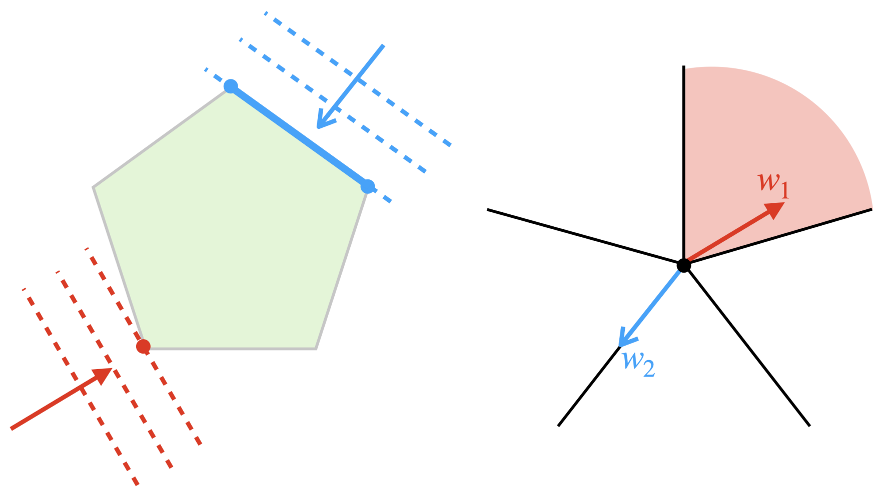

Figure 6.1: The optimizers of a linear program lie on a face of $\mathcal{P}_b$.

in the interior of the red cone shown in the right part of the figure. The weight vector w_2 exposes an edge of $\mathcal{P}_b$, highlighted in blue. There are infinitely many optimizers for this choice of weights. This happens only if w aligns with one of the five black rays in the right part of the figure. $\diamond$

Important practical methods for solving linear programs over $\mathbb{R}$ are the so-called *interior point methods*. A general reference is [57]. To apply these methods, we will assume that the vector b lies in the relative interior of the rank(A)-dimensional cone Cone(A). As a consequence, $\mathcal{P}_b$ is $(n - \mathrm{rank}(A))$-dimensional and has an "interior" in the affine linear space $\{A\,x = b\}$. The idea of interior point methods is to introduce a positive parameter $\varepsilon \in \mathbb{R}_{\geq 0}$ and perturb the objective function $\langle w, x\rangle$ into a convex function on $\mathcal{P}_b$. This is called *regularization*. Concretely, we replace (6.2.1) by

$$\text{Minimize } \langle w, x\rangle + \varepsilon \cdot \sum_{i=1}^{n} H(x_i) \quad \text{subject to} \quad A\,x = b \;\text{ and }\; x \in \mathbb{R}^n_{\geq 0}, \tag{6.2.2}$$

where $H : \mathbb{R}^n_{>0} \to \mathbb{R}$ is strictly convex and differentiable, and $\lim_{t\to 0^+} H'(t) = +\infty$. Recall that *strictly convex* means $H(\alpha t_1 + (1-\alpha)t_2) > \alpha H(t_1) + (1-\alpha)H(t_2)$ for any $0 < \alpha < 1$ and $0 < t_1 < t_2$. These requirements for H make sure that for all $\varepsilon > 0$, the optimization problem (6.2.2) has a unique minimizer in the relative interior of $\mathcal{P}_b$. That minimizer depends on ε and on b, which we will emphasize by denoting it as $x_b^*(\varepsilon)$. If the optimizer x^* is a vertex of $\mathcal{P}_b$, which happens for generic w (see Example 6.2.2), then $\lim_{\varepsilon\to 0^+} x_b^*(\varepsilon) = x^*$. Interior point methods first compute $x_b^*(\varepsilon)$ for a positive value of ε, possibly for $\varepsilon = +\infty$, and then track $x_b^*(\varepsilon)$ to the optimal vertex by letting $\varepsilon \to 0$. The first step can be done by any iterative method for convex optimization, such as gradient descent. Tracking for $\varepsilon \to 0$ is a numerical continuation procedure.

Proposition 6.2.3. *With the above assumptions on H and A, $b \in \mathrm{relint}(\mathrm{Cone}(A))$ and $\varepsilon > 0$, the optimizer $x_b^*(\varepsilon)$ of (6.2.2) is the unique point $x \in \mathrm{relint}(\mathcal{P}_b)$ satisfying*

$$(H'(x_1), \ldots, H'(x_n)) \ \in \ \mathrm{Row}_{\mathbb{R}}(A) - w/\varepsilon. \tag{6.2.3}$$

Here, $\mathrm{Row}_{\mathbb{R}}(A) - w/\varepsilon$ *denotes the affine linear space* $\{A^\top \lambda - w/\varepsilon \in \mathbb{R}^n \,:\, \lambda \in \mathbb{R}^d\}$.

Proof. By assumption, the optimizer lies in the relative interior of $\mathcal{P}_b$, and hence in $\mathbb{R}^n_{>0}$. Therefore, it satisfies the first order optimality conditions obtained from the method of Lagrange multipliers. The Lagrangian function of (6.2.2) is

$$\mathcal{L} \ = \ \langle w, x \rangle + \varepsilon \cdot \sum_{i=1}^n H(x_i) - \lambda^\top (Ax - b).$$

Setting the partial derivatives of $\mathcal{L}$ with respect to x to zero gives (6.2.3). By convexity, precisely one point in $\mathrm{relint}(\mathcal{P}_b)$ satisfies this condition. That point is $x_b^*(\varepsilon)$. □

So far, our general discussion on interior point methods does not involve any toric varieties. The following choice of H brings us into the realm of toric geometry:

$$H(t) \ = \ t \cdot \log(t) - t. \tag{6.2.4}$$

Regularizing as in (6.2.2) using this choice of H is *entropic regularization*:

$$\text{Minimize } \langle w, x \rangle + \varepsilon \cdot \sum_{i=1}^n (x_i \log(x_i) - x_i) \quad \text{subject to} \quad A\,x \ = \ b \ \text{ and } \ x \in \mathbb{R}^n_{\geq 0}. \tag{6.2.5}$$

Exercise 6.2.4. Verify that the entropy function H from (6.2.4) satisfies our assumptions: H is strictly convex, differentiable, and $\lim_{t \to 0^+} H'(t) = +\infty$. Show that the derivative of H is the logarithm: $H'(t) = \log(t)$ for $t \in \mathbb{R}_{>0}$.

A reason for using the entropy function (6.2.4) in interior point methods is the fact that there exists an algorithm for solving (6.2.5) efficiently for positive ε. This is the topic of the next section. Here, we explain the toric geometry of (6.2.5).

When $\varepsilon \to +\infty$, the linear objective $\langle w, x \rangle$ is disregarded and (6.2.5) simply minimizes the entropy $\sum_{i=1}^n (x_i \log(x_i) - x_i)$ on $\mathcal{P}_b$. The following statement makes the desired connection to toric varieties.

Proposition 6.2.5. *Consider the entropically regularized linear program given by (6.2.5), with optimizer $x_b^*(\varepsilon)$. For any $b \in \mathrm{relint}(\mathrm{Cone}(A))$, the limit $\lim_{\varepsilon \to +\infty} x_b^*(\varepsilon)$ is the unique intersection point of $\mathcal{P}_b$ with the positive part $(Y_A)_{>0}$ of the affine toric variety Y_A.*

Proof. Let us write $x_b^*(\infty) = \lim_{\varepsilon \to +\infty} x_b^*(\varepsilon)$. Using Proposition 6.2.3 and the fact that $H'(t) = \log(t)$ (Exercise 6.2.4), we find that the coordinates of $x_b^*(\infty)$ satisfy

$$(H'(x_1), \ldots, H'(x_n)) \ = \ (\log(x_1), \ldots, \log(x_n)) \ \in \ \mathrm{Row}_{\mathbb{R}}(A).$$

This is equivalent to $x_b^*(\varepsilon) \in \exp(\mathrm{Row}_{\mathbb{R}}(A))$. Now apply Proposition 5.1.4. □

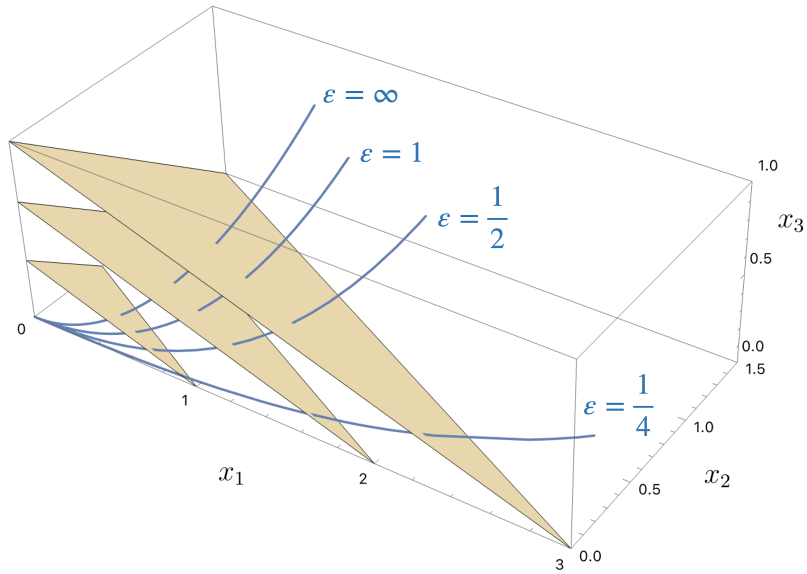

Figure 6.2: The twisted cubic (blue, $\varepsilon = \infty$) intersects the triangle $\mathcal{P}_b = \{x \in \mathbb{R}^3_{>0} : x_1 + 2x_2 + 3x_3 = b\}$ (shown in orange for $b = 1, 2, 3$) in its Birch point. When $\varepsilon \to 0$, the intersection of the scaled twisted cubic $\exp(-w/\varepsilon) \star Y_A$ with $\mathcal{P}_b$ approaches the optimal vertex of the linear program (6.2.1).

We name the distinguished interior point $x_b^*(\infty)$ after Birch, because of its relation to Birch's theorem in algebraic statistics, see Theorem 14.0.1.

Definition 6.2.6 (Birch point). *For fixed* $b \in \operatorname{relint}(\operatorname{Cone}(A))$, *the point* $\{x_b^*(\infty)\} = \mathcal{P}_b \cap (Y_A)_{>0}$ *is called the* Birch point *of* $\mathcal{P}_b$.

Corollary 6.2.7. *The map* $\operatorname{relint}(\operatorname{Cone}(A)) \to (Y_A)_{>0}$ *which sends b to the Birch point* $x_b^*(\infty)$ *of* $\mathcal{P}_b$ *is a real analytic isomorphism.*

Proof. It suffices to observe that $b \mapsto x_b^*(\infty)$ is the inverse of the moment map $\mu_{A,w}$ from Theorem 5.2.4, where $w = (1, \ldots, 1)$ is the all-ones vector. $\square$

In other words, the positive toric variety $(Y_A)_{>0}$ parametrizes all minimizers of the entropy function $\sum_{i=1}^n x_i \log(x_i) - x_i$ on $\mathcal{P}_b$, as b varies over $\operatorname{int}(\operatorname{Cone}(A))$.

Example 6.2.8. The matrix $A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ gives rise to the twisted cubic curve $Y_A \subset \mathbb{C}^3$ (Example 1.1.2). Its positive part $(Y_A)_{>0}$ consists of the Birch points of all triangles $\mathcal{P}_b = \{x \in \mathbb{R}^3_{>0} : x_1 + 2x_2 + 3x_3 = b\}, b \in \mathbb{R}_{\geq 0}$. This is illustrated in Figure 6.2. The map $\mathbb{R}_{>0} \to (Y_A)_{>0}$ which sends b to the Birch point $x_b^*(\infty) \in \mathcal{P}_b$ is a real analytic isomorphism between the positive real line and the blue curve labeled "$\varepsilon = \infty$". $\diamond$

Exercise 6.2.9. We revisit Exercise 6.0.2. The feasible polytope $\mathcal{P}_b$ for optimal transport problems is called the *transportation polytope*. Show that the Birch point of $\mathcal{P}_b$ is the unique rank-one 2×3 matrix contained in $\mathcal{P}_b$.

In order to make similar statements for finite ε, we must consider *scaled toric varieties*. For a vector $z \in (\mathbb{C}^*)^n$ and a toric variety Y_A, we define

$$z \star Y_A \,=\, \{z \star x \,=\, (z_1x_1, \dots, z_nx_n) \,:\, x \in Y_A\}. \tag{6.2.6}$$

Clearly, $z \star Y_A$ is isomorphic to Y_A for all $z \in (\mathbb{C}^*)^n$. Its ideal is generated by binomials obtained from $x^u - x^v \in I_A$ by adapting the coefficients: $z^{-u}x^u - z^{-v}x^v \in I(z \star Y_A)$. If the scaling vector z is positive, i.e., $z \in \mathbb{R}^n_{>0}$, then the positive part $(z \star Y_A)_{>0}$ is

$$(z \star Y_A)_{>0} \,=\, \big\{(z_1x_1, \dots, z_nx_n) \,:\, x \in (Y_A)_{>0}\big\} \,=\, (z \star Y_A) \cap \mathbb{R}^n_{>0}.$$

Proposition 6.2.10. *Consider the entropically regularized linear program given by* (6.2.5)*, with optimizer $x_b^*(\varepsilon)$. For any $b \in \operatorname{relint}(\operatorname{Cone}(A))$ and $\varepsilon > 0$, $x_b^*(\varepsilon)$ is the unique intersection point of $\mathcal{P}_b$ with the positive scaled toric manifold* $(\exp(-w/\varepsilon) \star Y_A)_{>0}$.

Proof. The proof is identical to that of Proposition 6.2.5, with the observation that $\exp(\operatorname{Row}_{\mathbb{R}}(A) - w/\varepsilon) = (\exp(-w/\varepsilon) \star Y_A)_{>0}$. □

Corollary 6.2.11. *The map* $\operatorname{relint}(\operatorname{Cone}(A)) \to (\exp(-w/\varepsilon) \star Y_A)_{>0}$ *which sends b to $x_b^*(\varepsilon) \in \mathcal{P}_b$ is a real analytic isomorphism.*

Proof. the map $b \mapsto x_b^*(\varepsilon)$ is equivalently given by $b \mapsto z \star \mu_{A,z}^{-1}(b)$, where $z = \exp(-w/\varepsilon)$ and $\mu_{A,z}$ is the weighted moment map from Definition (5.2.1). To see this, it suffices to observe that $z \star \mu_{A,z}^{-1}(b) \in \mathcal{P}_b$. Clearly, $z \star \mu_{A,z}^{-1}(b) \in \mathbb{R}^n_{>0}$, and since $\mu_{A,z}(x) = A(z \star x)$ for any $x \in (Y_A)_{>0}$, we have $A(z \star \mu_{A,z}^{-1}(b)) = \mu_{A,z}(\mu_{A,z}^{-1}(b)) = b$. Hence, the map in the corollary is the composition of the real analytic isomorphism $\mu_{A,z}^{-1}$ with $x \mapsto z \star x$. □

Example 6.2.12. Let $A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ and $w = (-2, -3, -5)$. For any $b \in \mathbb{R}_{>0}$, the cost function $\langle w, x \rangle$ is minimized on the vertex $(b, 0, 0)$ of $\mathcal{P}_b$. When $\varepsilon \to 0$, the minimizer of (6.2.5) moves from the Birch point to the optimal vertex. For finite ε, the minimizer is obtained as the unique intersection point of $\mathcal{P}_b$ with the curve $t \mapsto (e^{2/\varepsilon}t, e^{3/\varepsilon}t^2, e^{5/\varepsilon}t^3)$. The situation is illustrated in Figure 6.2 for several values of ε and $b = 1, 2, 3$. ◇

6.3 Iterative proportional scaling

This section discusses how to compute $x_b^*(\varepsilon)$, i.e., the optimizer of (6.2.5), in practice. From an algebraic point of view, $x_b^*(\varepsilon)$ is the unique positive solution to the system of polynomial equations given by

$$x \,\in\, \exp(-w/\varepsilon) \star Y_A \quad \text{and} \quad A\,x \,=\, b. \tag{6.3.1}$$

The left set of equations is given by binomials, and the rest of the equations are linear. Alternatively, we may implement the condition $x \in \exp(-w/\varepsilon) \star Y_A$ by replacing $x = (e^{-w_1/\varepsilon}t^{a_1}, \dots, e^{-w_1/\varepsilon}t^{a_n})$ in $A\,x = b$. This gives d equations in the coordinates of $t \in (\mathbb{C}^*)^d$, and Proposition 5.1.3 ensures that it suffices to compute a solution in $\mathbb{R}^d_{>0}$.

Unfortunately, solving (6.3.1) using algebraic methods often requires to compute all complex solutions first, and then select the unique positive solution among them. This has the obvious disadvantage that many solutions need to be discarded. This section explains an alternative, iterative method which computes *only* the positive solution.

We present a toric view on the method of *iterative proportional scaling*, originally introduced as *generalized iterative scaling* in [22]. We make the assumption that the columns of A all sum to the same positive number c. That is, the row vector $\mathbf{1}_d^\top = (1,\dots,1) \in \mathbb{Q}^d$ satisfies $\mathbf{1}_d^\top A = c \cdot \mathbf{1}_n^\top$. You will show in the next exercise that it actually suffices that $\mathbf{1}_n^\top$ lies in the row span of A. By Exercise (1.3.19), this translates into the condition that the toric ideal I_A is homogeneous. In optimal transport (Exercise 6.0.2), our assumption is satisfied.

Exercise 6.3.1. Suppose that the all-ones row vector of length n, denoted $\mathbf{1}_n^\top = (1,\dots,1)$, lies in the row span of A. This is the case in [22], where x represents a discrete probability distribution, so that one of the linear conditions is $\sum_{i=1}^n x_i = 1$. Let $v^\top = \mathbf{1}_d^\top A$ and $c = \max_i v_i$. We have $c > 0$ because A has no zero columns. Show that appending the row vector $c \cdot \mathbf{1}^\top - v^\top$ to A results in a nonnegative integer matrix $\tilde{A}$ of rank equal to $\operatorname{rank}(A)$ and with constant column sum c. Show that one can find $\tilde{b} \in \operatorname{relint}(\operatorname{Cone}(\tilde{A}))$ so that the affine linear spaces $\{Ax = b\}$ and $\{\tilde{A}x = \tilde{b}\}$ are equal. Conclude that, if $\mathbf{1}_n^\top \in \operatorname{Row}_{\mathbb{R}}(A)$, then one can replace A, b by $\tilde{A}, \tilde{b}$ in (6.2.5) to obtain an equivalent optimization problem which satisfies the assumption $\mathbf{1}_d^\top A = c \cdot \mathbf{1}_n^\top$.

For $\varepsilon > 0$ and $A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \in \mathbb{N}^{d\times n}$ satisfying $\mathbf{1}_d^\top A = c \cdot \mathbf{1}_n^\top$, and $w \in \mathbb{R}^n, b \in \operatorname{int}(\operatorname{Cone}(A))$ as in (6.2.5), we consider the following sequence of points in $\mathbb{R}^n$:

$$x^{(0)} = \exp(-w/\varepsilon), \quad x_i^{(k+1)} = x_i^{(k)} \cdot \left(\frac{b^{a_i}}{(A\,x^{(k)})^{a_i}} \right)^{\frac{1}{c}}, \quad i = 1,\dots,n, \; k \in \mathbb{N}. \qquad (6.3.2)$$

Here $x_i^{(k)}$ is the i-th coordinate of $x^{(k)}$, the k-th point in the sequence.

Theorem 6.3.2. *The solution to* (6.2.5) *with* $b \in \operatorname{relint}(\operatorname{Cone}(A))$ *is the unique limit point of the sequence* (6.3.2)*.*

To prove Theorem 6.3.2, we need a few lemmas.

Lemma 6.3.3. *For all* $k \in \mathbb{N}$*, the point* $x^{(k)}$ *in* (6.3.2) *lies in* $(\exp(-w/\varepsilon) \star Y_A)_{>0}$.

Proof. The proof is by induction on k. The base case is $k = 0$: $z \in (z \star Y_A)_{>0}$ for any $z \in \mathbb{R}^n_{>0}$ by Exercise 1.1.4. A simple calculation shows that

$$\log(x^{(k+1)})^\top = \log(x^{(k)})^\top + \frac{1}{c}\,(b^\top - (A\,x^{(k)})^\top) \cdot A.$$

By the induction hypothesis $x^{(k)} \in (\exp(-w/\varepsilon) \star Y_A)_{>0}$, we have $\log(x^{(k)}) = -w/\varepsilon + w'$ for some $w' \in \operatorname{Row}_{\mathbb{R}}(A)$. Therefore, $\log(x^{(k+1)}) = -w/\varepsilon + w''$ for some $w'' \in \operatorname{Row}_{\mathbb{R}}(A)$. Applying exp proves that $x^{(k+1)} \in (\exp(-w/\varepsilon) \star Y_A)_{>0}$. $\square$

We will also use the following standard fact from statistics [22, Lemma 1]:

Lemma 6.3.4. *Fix $p, q \in \mathbb{R}^n_{>0}$ with $\sum_{i=1}^n p_i = 1$ and $\sum_{i=1}^n q_i \leq 1$. The* Kullback-Leibler divergence *$K[p,q] = \sum_{i=1}^n p_i \log(p_i/q_i)$ satisfies $K[p,q] \geq 0$, and $K[p,q] = 0$ if and only if $p = q$.*

Also the following particular instance of *Jensen's Inequality* will be useful. Concavity of $\log(\,\cdot\,)$ implies that, for all $\alpha_i, y_i > 0$, we have

$$\frac{\sum_{i=1}^n \alpha_i \log(y_i)}{\sum_{i=1}^n \alpha_i} \leq \log\left(\frac{\sum_{i=1}^n \alpha_i y_i}{\sum_{i=1}^n \alpha_i}\right), \qquad \text{and thus} \qquad \prod_{i=1}^n y_i^{\frac{\alpha_i}{\sum_{i=1}^n \alpha_i}} \leq \frac{\sum_{i=1}^n \alpha_i y_i}{\sum_{i=1}^n \alpha_i}. \tag{6.3.3}$$

Proof of Theorem 6.3.2. Fix $p \in \mathbb{R}^n_{>0}$ such that $A\,p = b$. Such a p exists by the assumption that $b \in \operatorname{relint}(\operatorname{Cone}(A))$. Let $|p| = \sum_{i=1}^n p_i > 0$ be the coordinate sum of p. In the first step of the proof we show that the Kullback-Leibler divergence $K[\frac{p}{|p|}, \frac{x^{(k)}}{|p|}]$ is weakly decreasing for $x^{(k)}$ as in (6.3.2) and $k \geq 1$. We observe that $x_i^{(k)} > 0$ for all k, i. To apply Lemma 6.3.4, we need to show that $\sum_{i=1}^n x_i^{(k)} \leq |p|$ for $k \geq 1$. We have

$$\begin{aligned}
\sum_{i=1}^n x_i^{(k)} \;=\; \sum_{i=1}^n x_i^{(k-1)} \prod_{j=1}^d \left(\frac{b_j}{(A\,x^{(k-1)})_j}\right)^{\frac{a_{ji}}{c}} &\leq \sum_{i=1}^n x_i^{(k-1)} \left(\sum_{j=1}^d \frac{a_{ji}}{c} \frac{b_j}{(A\,x^{(k-1)})_j}\right) \\
&= \frac{1}{c} \sum_{j=1}^d b_j \left(\frac{\sum_{i=1}^n a_{ji} x_i^{(k-1)}}{(A\,x^{(k-1)})_j}\right) \\
&= \frac{\sum_{j=1}^d b_j}{c} \;=\; |p|.
\end{aligned}$$

Here the inequality is (6.3.3). The last equality follows from $\mathbf{1}_d^\top A\,p = c \cdot |p| = \mathbf{1}_d^\top b$.

The inequality above implies $\sum_{j=1}^d (A\,x^{(k)})_j \leq c \cdot |p|$. Hence, by Lemma 6.3.4,

$$K\left[\frac{b}{c \cdot |p|}, \frac{A\,x^{(k)}}{c \cdot |p|}\right] \;=\; \sum_{j=1}^d \frac{b_j}{c \cdot |p|} \log \frac{b_j}{(A\,x^{(k)})_j} \;\geq\; 0.$$

The sequence $K[\frac{p}{|p|}, \frac{x^{(k)}}{|p|}]$ is bounded below by 0, and it is indeed non-increasing because

$$\begin{aligned}
K\left[\frac{p}{|p|}, \frac{x^{(k+1)}}{|p|}\right] &= \sum_{i=1}^n \frac{p_i}{|p|} \log \frac{p_i}{x_i^{(k+1)}} \\
&= \sum_{i=1}^n \frac{p_i}{|p|} \log \frac{p_i}{x_i^{(k)}} - \sum_{i=1}^n \frac{p_i}{|p|} \log \left(\frac{b^{a_i}}{(A\,x^{(k)})^{a_i}}\right)^{\frac{1}{c}} \\
&= K\left[\frac{p}{|p|}, \frac{x^{(k)}}{|p|}\right] - \sum_{i=1}^n \frac{p_i}{|p|} \sum_{j=1}^d \frac{a_{ji}}{c} \log \frac{b_j}{(A\,x^{(k)})_j}
\end{aligned}$$

$$
\begin{aligned}
&= K\left[\frac{p}{|p|}, \frac{x^{(k)}}{|p|}\right] - \sum_{j=1}^{d} \frac{b_j}{c \cdot |p|} \log \frac{b_j}{(A\,x^{(k)})_j} \\
&= K\left[\frac{p}{|p|}, \frac{x^{(k)}}{|p|}\right] - K\left[\frac{b}{c \cdot |p|}, \frac{Ax^{(k)}}{c \cdot |p|}\right].
\end{aligned}
$$

Hence, the sequence $K\left[\frac{p}{|p|}, \frac{x^{(k+1)}}{|p|}\right]$ has a limit, and thus $K\left[\frac{b}{c\cdot|p|}, \frac{Ax^{(k)}}{c\cdot|p|}\right] \to 0$. This implies, by Lemma 6.3.4, that $Ax^{(k)} \to b$. Hence, any limit point of (6.3.2) satisfies $A\left(\lim_{k\to\infty} x^{(k)}\right) = b$ and $\lim_{k\to\infty} x^{(k)} \in (\exp(-w/\varepsilon) \star Y_A)_{>0}$ by Lemma 6.3.3. The unique point satisfying these conditions is the optimizer $x_b^*(\varepsilon)$ of the convex optimization problem (6.2.5). Hence, the sequence $x^{(k)}$ has a unique limit point, which is $x_b^*(\varepsilon)$. $\square$

Example 6.3.5. It is straightforward to implement iterative proportional scaling in Julia. The following code snippet solves the optimal transport problem from Exercise 6.0.2 over the real numbers. The matrix $A \in \mathbb{Z}^{5\times 6}$ in Example 1.1.7 has rank 4, so we may remove its last row and the last entry of $b = (66, 51, 36, 26, 55)$ to obtain an equivalent program. We do this to illustrate the "preprocessing" step explained in Example 6.3.1, which adds these entries back in. The matrix A in line 18 has column sum $c = 2$.

```
1  using LinearAlgebra
2  using Oscar
3  
4  A = [1 1 1 0 0 0; 0 0 0 1 1 1; 1 0 0 1 0 0; 0 1 0 0 1 0]
5  d,n = size(A)
6  w = [3,5,7,11,13,17]
7  b = [66,51,36,26]
8  ε = 1 # regularization parameter
9  
10 # preprocessing
11 vt = ones(Int,1,d)*A
12 newrow = maximum(vt)*ones(Int,1,n)-vt
13 if norm(newrow) !=0
14     Anew = [A;newrow]
15     A_oscar = matrix_space(QQ,d,n)(A)
16     newrow_oscar = matrix_space(QQ,1,n)(newrow)
17     bnew = [b; Rational{Int64}((solve(A_oscar,newrow_oscar)*b)[1])]
18     A = Anew; b = bnew
19 end
20 
21 c = sum(A[:,1]) # constant column sum
22 x = exp.(-w/ε) # 0-th point in the iteration
23 tol = 1e-8 # stop iterating when the relative difference err < tol
24 err = Inf # initialize the error to be infinite
25 maxiter = 1000 # maximal number of iterations
26 iter = 0 # current iteration
```

```
27
28 while err > tol && iter < maxiter
29     xold = x
30     x = x.*([prod((b./(A*x)).^(A[:,i])) for i = 1:n]).^(1/c)
31     err = norm(x-xold)/norm(x)
32     iter += 1
33     if mod(iter,5) == 0
34         println("iter = $iter, error = $err, coordinate sum = $(sum(x))")
35     end
36 end
```

The print statement in line 34 produces the following output:

```
iter = 5, error = 0.13712059288564354, coordinate sum = 113.73920337599048
iter = 10, error = 0.018245722010035933, coordinate sum = 116.89504522056544
iter = 15, error = 0.0034277601860021366, coordinate sum = 116.9958906897446
iter = 20, error = 0.0006812383508837578, coordinate sum = 116.99983491442295
iter = 25, error = 0.00013664291004961645, coordinate sum = 116.9999933368572
iter = 30, error = 2.745578037568848e-5, coordinate sum = 116.9999997308164
iter = 35, error = 5.518624823362526e-6, coordinate sum = 116.99999998912328
iter = 40, error = 1.1093230937616797e-6, coordinate sum = 116.9999999995605
iter = 45, error = 2.2299302500178605e-7, coordinate sum = 116.99999999998225
iter = 50, error = 4.4825559278522627e-8, coordinate sum = 116.99999999999929
iter = 55, error = 9.010738803226357e-9, coordinate sum = 116.99999999999997
```

Here the error in iteration step `iter` $= k$ is measured as $\|x^{(k)} - x^{(k-1)}\|_2/\|x^{(k)}\|_2$. The coordinate sum $\sum_{i=1}^6 x_i^{(k)}$ is bounded by $c^{-1} \cdot \sum_{i=1}^5 b_i = 117$ in each step. After 55 iterations, the approximate optimizer is

$$x_b^*(1) \approx (12.7243, 9.18974, 44.086, 23.2758, 16.8103, 10.914). \qquad \diamond$$

Once $x_b^*(\varepsilon)$ is computed for finite $\varepsilon > 0$, we can use numerical homotopy continuation to track $x_b^*(\varepsilon)$ as a solution of (6.3.1) when ε moves to 0. Hence, the algebraic formulation (6.3.1) may be useful in practice after all, in the second step of our interior point method.

Further reading

The material in Section 6.1 can be found with more details in [65, Chapter 4]. The relation between entropic regularization of linear programs and toric varieties is described in [66]. That paper also contains a description of the path described by $x_b^*(\varepsilon)$ for $\varepsilon \in \mathbb{R}_{>0}$ inside the feasible polytope. This is called the *entropic path*, and its Zariski closure is an algebraic curve when the cost vector w has rational entries. More information on iterative proportional scaling can be found in the original paper [22]. For a modern treatment, a discussion on the convergence rate and more references, see [60]. The story of Sections 6.2 and 6.3 extends to semidefinite programming, where toric varieties have a non-commutative analog called *Gibbs manifolds* [56].

Chapter 7

Fans and gluing toric varieties

Affine varieties are the building blocks for projective varieties, in the sense that each projective variety $X \subset \mathbb{P}^{n-1}$ is a union of affine varieties $Y_i \simeq X \cap U_i, i = 1, \ldots, n$, where $U_i = \mathbb{P}^{n-1} \setminus V(x_i)$ is as in Section 3.2. Going the other way, one can reconstruct X by considering the disjoint union of $Y_i \simeq X \cap U_i \hookrightarrow X$ and identifying points which have the same image under $Y_i \hookrightarrow X$. This is called *gluing* and can be applied more generally to construct varieties which are neither affine nor projective from a collection of affine charts and gluing morphisms. We recall this in Section 7.1. In toric geometry, we glue together normal affine toric varieties $\mathcal{Y}_\sigma$ coming from cones, see Section 2.3. They are glued along distinguished open subsets coming from faces of σ, as defined in Section 7.2. The gluing morphisms are encoded by a combinatorial object called a *polyhedral fan*, see Section 7.3. Once these ingredients are introduced, we present the gluing construction of an abstract toric variety $\mathcal{X}_\Sigma$ from a fan Σ in Section 7.4.

7.1 Gluing varieties

Consider a set $\{Y_i\}_{i \in \mathscr{J}}$ of affine varieties for some index set $\mathscr{J}$. For $i, j \in \mathscr{J}$, let $Y_{ij} \subset Y_i$ be a dense Zariski open subset and suppose that there exist morphisms $\{\phi_{ij}\}_{i,j \in \mathscr{J}}$ such that for all $i, j, k \in \mathscr{J}$,

1. $\phi_{ij} : Y_{ij} \to Y_{ji}$ and $\phi_{ji} : Y_{ji} \to Y_{ij}$ satisfy $\phi_{ij} \circ \phi_{ji} = \mathrm{id}_{Y_{ji}}, \phi_{ji} \circ \phi_{ij} = \mathrm{id}_{Y_{ij}}$,
2. $\phi_{ij}(Y_{ij} \cap Y_{ik}) = Y_{ji} \cap Y_{jk}$,
3. $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $Y_{ik} \cap Y_{ij}$.

The disjoint union $\bigsqcup_{i \in \mathscr{J}} Y_i$ is the set

$$\hat{X} = \bigsqcup_{i \in \mathscr{J}} Y_i = \{(x, Y_i) \mid i \in \mathscr{J}, x \in Y_i\}.$$

It is a topological space with the disjoint union topology, which is such that the open subsets of $\hat{X}$ are disjoint unions of open subsets in the Y_i. We define an equivalence relation $\sim$ on $\hat{X}$ by setting $(x, Y_i) \sim (y, Y_j)$ if $x \in Y_{ij}$, $y \in Y_{ji}$ and $\phi_{ij}(x) = y$. The first condition on the ϕ_{ij} makes $\sim$ reflexive and symmetric, the second and third conditions make it transitive. We consider the quotient space $X = \hat{X}/\sim$ with its quotient topology, called the *Zariski topology* on X. In this topological space,

$$U_i = \{[(x, Y_i)] \mid x \in Y_i\} \subset X$$

are open subsets isomorphic to Y_i (here we write $[\cdot]$ for an equivalence class in the quotient). The space X is called an *abstract variety*. The affine varieties $\{Y_i\}_{i \in \mathscr{J}}$ and the isomorphisms $\{\phi_{ij}\}_{i,j \in \mathscr{J}}$ are called the *gluing data* for the construction of X.

Example 7.1.1. The projective line $\mathbb{P}^1$ is covered by $\mathbb{P}^1 = U_x \cup U_y$ where

$$U_x = \{(x : y) \in \mathbb{P}^1 \mid x \neq 0\}, \quad U_y = \{(x : y) \in \mathbb{P}^1 \mid y \neq 0\}.$$

Consider the isomorphisms

$$h_x : U_x \to \mathbb{C}_t \quad \text{and} \quad h_y : U_y \to \mathbb{C}_u,$$

where $\mathbb{C}_t$ is $\mathbb{C}$ with coordinate t and analogously for u, given by $h_x(x : y) = y/x$ and $h_y(x : y) = x/y$. For a point $(x : y) \in U_x \cap U_y$, we have $h_x(x : y) = h_y(x : y)^{-1}$. Let

$$\mathbb{C}_{tu} = \mathbb{C}_t^* = \mathbb{C}_t \setminus \{0\}, \quad \mathbb{C}_{ut} = \mathbb{C}_u^* = \mathbb{C}_u \setminus \{0\}$$

and $\phi_{tu} : \mathbb{C}_{tu} \to \mathbb{C}_{ut}$ given by $\phi_{tu}(t) = t^{-1}$, $\phi_{ut} = \phi_{tu}^{-1}$. This gives a commutative diagram

$$\begin{tikzcd}
U_x \cap U_y \arrow[r, "h_x"] \arrow[d, "h_y"] & \mathbb{C}_{tu} \arrow[dl, shift left, "\phi_{tu}"] \\
\mathbb{C}_{ut} \arrow[ur, shift left, "\phi_{ut}"] &
\end{tikzcd}$$

The projective line $\mathbb{P}^1$ is a gluing of two copies of $\mathbb{C}$ with gluing data $\{\mathbb{C}_t, \mathbb{C}_u\}$ and $\{\phi_{tu}, \phi_{ut}\}$. The two affine lines $\mathbb{C}_t$ and $\mathbb{C}_u$ are glued together along the open subsets $\mathbb{C}_t^*$ and $\mathbb{C}_u^*$, which give the open subset $U_x \cap U_y \subset \mathbb{P}^1$. The missing points $\mathbb{P}^1 \setminus (U_x \cap U_y) = \{(1 : 0), (0 : 1)\}$ correspond to the origins in $\mathbb{C}_t$ and $\mathbb{C}_u$. If we consider $\mathbb{P}^1$ as the compactification of $\mathbb{C}_t \subset \mathbb{P}^1$, the *point at infinity* $\mathbb{P}^1 \setminus \mathbb{C}_t$ corresponds to the origin in $\mathbb{C}_u$. This is illustrated in Figure 7.1. $\diamond$

Example 7.1.2. We replace the isomorphisms in the glueing data from Example 7.1.1 by $\phi_{tu}(t) = t$ and $\phi_{ut}(u) = u$. This way, we obtain an abstract variety X that, like $\mathbb{P}^1$, is a union of $\mathbb{C}^*$ and two points. This is illustrated in Figure 7.2. However, this variety is not *separated*, meaning that the *classical* topology on X is not Hausdorff. Toric varieties glued from fans as in Section 7.4 are always separated, see [19, Theorem 3.1.5]. $\diamond$

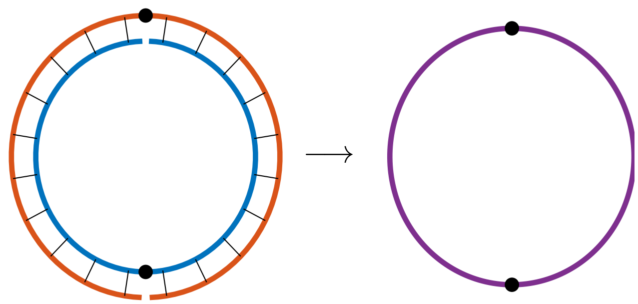

Figure 7.1: Illustration of the construction of $\mathbb{P}^1$ as the gluing of two affine lines. The affine lines are represented as circles with a missing point ('at infinity'). The origin in each line is indicated with a black dot and the gluing isomorphism is illustrated by black line segments.

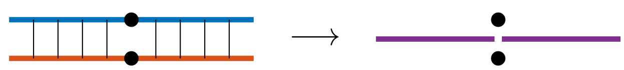

Figure 7.2: A non-separated variety obtained from gluing two affine lines.

Example 7.1.3. We glue $\mathbb{P}^2$ from three copies of $\mathbb{C}^2$. Consider the isomorphisms

$$h_x : U_x \to \mathbb{C}^2_t,\ h_y : U_y \to \mathbb{C}^2_u,\ \text{and } h_z : U_z \to \mathbb{C}^2_v$$

where $\mathbb{C}^2_t$ is the affine plane with coordinates t_1, t_2 (analogously for u, v) and

$$h_x(x:y:z) = (y/x, z/x),\ h_y(x:y:z) = (x/y, z/y),\ h_z(x:y:z) = (x/z, y/z).$$

The gluing morphisms $\phi_{tv} = \phi_{vt}^{-1}$ come from identifying the images of points in $U_x \cap U_z$ under h_x and h_z, e.g. on $\mathbb{C}^2_{tv} = \mathbb{C}^2_t \setminus V(t_2)$

$$\phi_{tv}(t_1, t_2) = (t_2^{-1}, t_1 t_2^{-1}) \quad \text{comes from} \quad \left(\frac{x}{z}, \frac{y}{z}\right) = \left(\left(\frac{z}{x}\right)^{-1}, \left(\frac{y}{x}\right)\left(\frac{z}{x}\right)^{-1}\right). \quad \diamond$$

Example 7.1.4. More generally, the projective space $\mathbb{P}^{n-1}$ is glued from n copies of $\mathbb{C}^{n-1}$. Let $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n$ be coordinates on $Y_i = \mathbb{C}^n$. For $i \neq j$, we set $Y_{ij} = \{x \in Y_i : y_j \neq 0\}$. The morphism ϕ_{ij} is such that

$$\phi_{ij}(x) = \left(\frac{y_1}{y_j}, \ldots, \frac{y_{j-1}}{y_j}, \frac{y_{j+1}}{y_j}, \ldots, \frac{y_n}{y_j}\right), \quad \text{where} \quad y_i = 1. \tag{7.1.1}$$

The resulting abstract variety X is $\mathbb{P}^n$, and $U_i \simeq Y_i$ is the isomorphism from (3.2.1). $\diamond$

Example 7.1.5. Any projective variety $X \subset \mathbb{P}^{n-1}$ can be glued from the affine varieties $\mathbb{C}^n \supset Y_i \simeq X \cap U_i$ by restricting the maps (7.1.1), defined on an open subset of $\mathbb{C}^{n-1}$, to the intersection of that open subset with Y_i. $\diamond$

Exercise 7.1.6. Glue $\mathbb{P}^1 \times \mathbb{P}^1$ from four copies of $\mathbb{C}^2$.

7.2 Faces and affine open subsets

The first step in specializing the gluing construction in Section 7.1 to toric varieties is identifying the affine open subsets of the normal affine toric variety $\mathcal{Y}_\sigma$ along which we will glue. In the rest of the section, $\sigma \subset N_\mathbb{R}$ is a pointed cone. Therefore $\mathcal{Y}_\sigma$ has dimension d (Proposition 2.3.3). A cone σ defines an algebra $\mathbb{C}[\mathsf{S}_\sigma] = \mathbb{C}[\sigma^\vee \cap M]$ and an inclusion of algebras $\mathbb{C}[\mathsf{S}_\sigma] \subset \mathbb{C}[\mathsf{S}_\tau] \subset \mathbb{C}[M]$ corresponds to an opposite inclusion of varieties $(\mathbb{C}^*)^d \subset \mathcal{Y}_\tau \subset \mathcal{Y}_\sigma$. The relevant affine open subsets of $\mathcal{Y}_\sigma$ come from faces of σ.

Proposition 7.2.1. *Let $\tau \preceq \sigma$ be a face, given by $\tau = \sigma \cap H_m$ with $m \in \sigma^\vee \cap M$. That is, $\tau = \{u \in \sigma \,:\, \langle u, m\rangle = 0\}$. We have*

$$\tau^\vee \cap M \;=\; \sigma^\vee \cap M + \mathbb{Z}\cdot(-m) \;=\; \{m' + c\cdot m \,:\, m \in \sigma^\vee \cap M, c \in \mathbb{Z}\}.$$

Proof. Since $\tau \preceq \sigma$, we have $\sigma^\vee \cap M \subseteq \tau^\vee \cap M$. The inclusion $\sigma^\vee \cap M + \mathbb{Z}\cdot(-m) \subseteq \tau^\vee \cap M$ is then clear from $\langle u, m\rangle = 0$ for all $u \in \tau$. For the other inclusion, pick $m'' \in \tau^\vee \cap M$ and let $u_1, \ldots, u_\ell$ be generators of σ, with $u_i \in N$. Let $c = \max_i |\langle u_i, m''\rangle|$. We claim that $m'' + c\cdot m \in \sigma^\vee \cap M$, which implies $m'' = m' - c\cdot m$ for some $m' \in \sigma^\vee \cap M$. Indeed,

$$\langle u_i, m'' + c\cdot m\rangle \;=\; \langle u_i, m''\rangle + c\cdot\langle u_i, m\rangle \;\geq\; 0.$$

To see this, note that if $\langle u_i, m\rangle = 0$, then $u_i \in \tau$ and thus $\langle u_i, m''\rangle \geq 0$. Otherwise, if $\langle u_i, m\rangle > 0$, then $c \cdot \langle u_i, m\rangle \geq |\langle u_i, m''\rangle|$. □

Example 7.2.2. Proposition 7.2.1 is illustrated in Figure 7.3 for a two-dimensional cone σ and one of its faces τ. Every lattice point in $\tau^\vee$, shaded in orange, can be obtained by adding an integer multiple of m to a lattice point in $\sigma^\vee$, shaded in blue. ⋄

In the next corollary we use the following standard notation. Let f be an element of the coordinate ring R of an affine variety $\mathcal{Y}$. The localization $R_f = R[f^{-1}]$ of R at $\{f^k, k \in \mathbb{N}\}$ is the coordinate ring of the affine open subset $\mathcal{Y}_f = \{p \in \mathcal{Y} \,:\, f(p) \neq 0\}$.

Corollary 7.2.3. *Let σ, τ, m be as in Proposition 7.2.1. We have $\mathbb{C}[\mathsf{S}_\tau] \simeq \mathbb{C}[\mathsf{S}_\sigma]_{t^m}$. The inclusion $\mathsf{S}_\sigma \subset \mathsf{S}_\tau$ induces $\mathbb{C}[\mathsf{S}_\sigma] \subset \mathbb{C}[\mathsf{S}_\tau]$ and $\mathcal{Y}_\tau \simeq (\mathcal{Y}_\sigma)_{t^m} \subset \mathcal{Y}_\sigma$.*

Thus, we have now associated an affine open subset $\mathcal{Y}_\tau \simeq (\mathcal{Y}_\sigma)_{t^m} \subset \mathcal{Y}_\sigma$ to every face $\tau \preceq \sigma$. Of course, every such affine open subset has dimension d. This is fundamentally different from the orbit-cone correspondence (Definition 4.3.3), which associates to a k-dimensional face $\tau \preceq \sigma$ a $(d-k)$-dimensional torus orbit $O(\tau) \subset \mathcal{Y}_\sigma$. Here $O(\tau)$ is the orbit O_{γ_τ} of the distinguished point $\gamma_\tau \in \mathcal{Y}_\sigma$, see the discussion following Proposition 4.4.4.

Example 7.2.4. For concreteness, we work out the inclusions in Corollary 7.2.3 for Example 7.2.2 in coordinates. We embed $\mathcal{Y}_\sigma \simeq Y_{A_\sigma}$ and $\mathcal{Y}_\tau \simeq Y_{A_\tau}$ in $\mathbb{C}^3$ via the matrices

$$A_\sigma \;=\; \begin{pmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \quad A_\tau \;=\; \begin{pmatrix} -1 & 0 & 1 \\ 2 & 1 & -2 \end{pmatrix}.$$

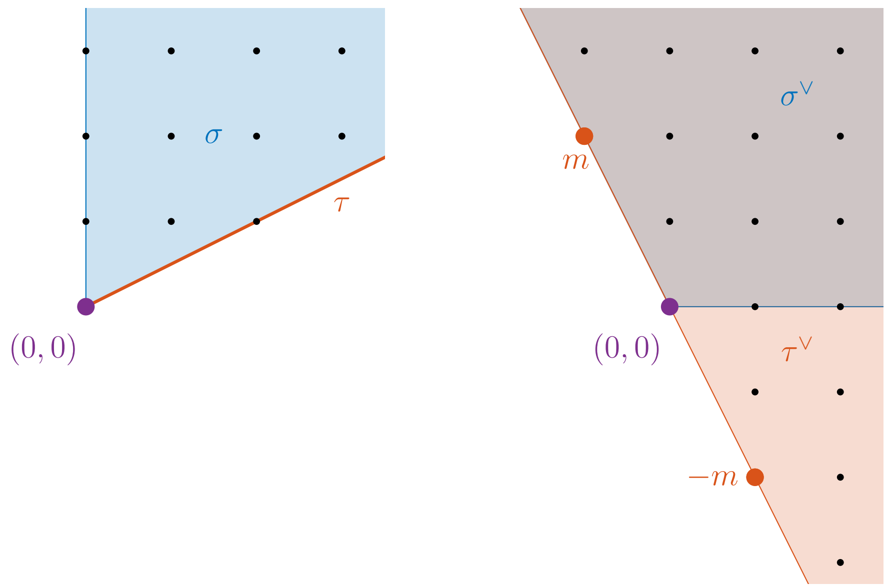

Figure 7.3: An illustration of Proposition 7.2.1.

The columns of these matrices generate the semigroups S_σ and S_τ respectively. The semigroup algebras and their representations corresponding to these matrices are

$$\begin{aligned}\mathbb{C}[\mathsf{S}_\sigma] &= \mathbb{C}[\mathbb{N}A_\sigma] = \mathbb{C}[t_1^{-1}t_2^2, t_2, t_1] \simeq \mathbb{C}[x,y,z]/\langle xz - y^2\rangle,\\ \mathbb{C}[\mathsf{S}_\tau] &= \mathbb{C}[\mathbb{N}A_\tau] = \mathbb{C}[t_1^{-1}t_2^2, t_2, t_1t_2^{-2}] \simeq \mathbb{C}[u,v,w]/\langle uw - 1\rangle.\end{aligned}$$

By Corollary 7.2.3, the algebra $\mathbb{C}[\mathsf{S}_\tau]$ also equals the localization $\mathbb{C}[t_1^{-1}t_2^2, t_2, t_1]_{t_1^{-1}t_2^2}$. The inclusion $\mathbb{C}[\mathsf{S}_\sigma] \subset \mathbb{C}[\mathsf{S}_\tau]$ is given by $x \mapsto u$, $y \mapsto v$ and $z \mapsto v^2w$. This corresponds to an injective map $Y_{A_\tau} \to Y_{A_\sigma}$ given by $(u,v,w) \mapsto (u,v,v^2w)$ whose image is $(Y_{A_\sigma})_x$. ⋄

Lemma 7.2.5. *Let $\tau, \tau' \preceq \sigma$ be faces of σ such that $\tau = \sigma \cap H_m$ and $\tau' = \sigma \cap H_{m'}$ for $m, m' \in \sigma^\vee \cap M$. The intersection of the corresponding affine open subsets $(\mathcal{Y}_\sigma)_{t^m}, (\mathcal{Y}_\sigma)_{t^{m'}} \subset \mathcal{Y}_\sigma$ is isomorphic to $\mathcal{Y}_{\tau\cap\tau'}$. That is,*

$$(\mathcal{Y}_\sigma)_{t^m} \cap (\mathcal{Y}_\sigma)_{t^{m'}} = (\mathcal{Y}_\sigma)_{t^{m+m'}} \simeq \mathcal{Y}_{\tau\cap\tau'}.$$

Proof. The equality $(\mathcal{Y}_\sigma)_{t^m} \cap (\mathcal{Y}_\sigma)_{t^{m'}} = (\mathcal{Y}_\sigma)_{t^{m+m'}}$ is by definition: t^m and $t^{m'}$ do not vanish at $p \in \mathcal{Y}_\sigma$ if and only if the product $t^m \cdot t^{m'} = t^{m+m'}$ does not vanish. The isomorphism $(\mathcal{Y}_\sigma)_{t^{m+m'}} \simeq \mathcal{Y}_{\tau\cap\tau'}$ is Corollary 7.2.3, once we show that $\sigma \cap H_{m+m'} = \tau \cap \tau'$. The inclusion $\supset$ is clear. Suppose $u \in \sigma \cap H_{m+m'}$. That is, $\langle u, m+m'\rangle = \langle u,m\rangle + \langle u,m'\rangle = 0$. This implies $\langle u,m\rangle = \langle u,m'\rangle = 0$, because $m, m' \in \sigma^\vee$. □

Exercise 7.2.6. Show that the intersection of the affine open subsets of $\mathcal{Y}_\sigma$ corresponding to all faces of σ is isomorphic to $(\mathbb{C}^*)^d$. Hence, it equals the dense torus of $\mathcal{Y}_\sigma$.

It is convenient to introduce the following notation. For a face $\tau \preceq \sigma$ of a pointed cone $\sigma \subset N_\mathbb{R}$, we write $\mathcal{Y}_\sigma^\tau \subset \mathcal{Y}_\sigma$ for the affine open subset of the affine toric variety $\mathcal{Y}_\sigma$

corresponding to τ (see Corollary 7.2.3). That is, $\mathcal{Y}_\tau \simeq \mathcal{Y}_\sigma^\tau = (\mathcal{Y}_\sigma)_{t^m}$ where $m \in \sigma^\vee \cap M$ is such that $\tau = \sigma \cap H_m$. This makes the notation independent of the choice of m, and helps to distinguish between $\mathcal{Y}_\tau$, an abstract affine variety, and $\mathcal{Y}_\sigma^\tau$, an open subset of $\mathcal{Y}_\sigma$.

Exercise 7.2.7. Show that the affine open subset $\mathcal{Y}_\sigma^\tau$ is the union of torus orbits $\bigsqcup_{\tau' \preceq \tau} O(\tau')$, where $O(\tau)$ is the image of τ under the orbit-cone correspondence from Definition 4.3.3.

Suppose two pointed cones $\sigma_1, \sigma_2 \subset N_\mathbb{R}$ intersect in a cone $\tau = \sigma_1 \cap \sigma_2$ which is a face of each. Then Corollary 7.2.3 shows that $\mathcal{Y}_\tau$ is naturally viewed as an affine open subset of both $\mathcal{Y}_{\sigma_1}$ and $\mathcal{Y}_{\sigma_2}$. We will glue $\mathcal{Y}_{\sigma_1}$ and $\mathcal{Y}_{\sigma_2}$ along such common open subsets, which motivates us to look at collections of cones which intersect in common faces.

7.3 Fans

Definition 7.3.1 (Fan)**.** *A* pointed polyhedral fan *(or simply* fan*) in a real vector space is a finite collection Σ of pointed rational polyhedral cones such that*

- *if $\sigma \in \Sigma$ and $\tau \preceq \sigma$ is a face, then $\tau \in \Sigma$,*
- *if $\tau = \sigma \cap \sigma'$ for $\sigma, \sigma' \in \Sigma$, then $\tau \preceq \sigma$ and $\tau \preceq \sigma'$.*

Example 7.3.2. The set Σ consisting of a pointed cone $\sigma \subset N_\mathbb{R}$ and all its faces is a fan in $N_\mathbb{R}$. The set $\{\tau^\vee : \tau \preceq \sigma\}$ does not form a fan in $M_\mathbb{R}$. For instance, the intersection of $\sigma^\vee$ and $\tau^\vee$ in Figure 7.3 is not a face of $\tau^\vee$. $\diamond$

Example 7.3.3. Figure 7.4 shows eight fans in $\mathbb{R}^2$. The two-dimensional cones are shaded in blue, one-dimensional cones are in orange, and the zero-dimensional cone $\{(0,0)\}$ is represented by a purple dot. The fan in Figure 7.4(a) illustrates Example 7.3.2: it consists of $\mathbb{R}^2_{\geq 0}$ and all its faces. $\diamond$

Exercise 7.3.4. Let Σ be a fan and $\tau \in \Sigma$ one of its cones. Show that $\Sigma' = \Sigma \setminus \{\sigma \in \Sigma : \tau \preceq \sigma\}$ is a fan. That is, we can obtain a new fan Σ' from Σ by removing one of its cones τ and all other cones of which τ is a face. In Figure 7.4, the fan (d) is obtained from fan (a) by removing one cone. Similarly, fan (c) is fan (f) with six cones removed.

Fans can be obtained from polytopes. Let $\mathcal{P} \subset M_\mathbb{R}$ be a full-dimensional polytope in $M_\mathbb{R} \simeq \mathbb{R}^d$. For each of its vertices $v \in \mathcal{V}(\mathcal{P})$, consider the cone $\sigma_v^\vee = \mathrm{Cone}(\mathcal{P} - v)$. The dual cones $\sigma_v \subset N_\mathbb{R}, v \in \mathcal{V}(\mathcal{P})$ fit together nicely. Together with all their faces, they form a fan called the *normal fan* of $\mathcal{P}$.

Example 7.3.5. The fan in Figure 7.4(e) is the normal fan of a triangle in $\mathbb{R}^2$. Each of the two-dimensional cones corresponds to a vertex. This is illustrated in Figure 7.5. $\diamond$

There is a bijective correspondence between cones in the normal fan of $\mathcal{P}$ and faces of $\mathcal{P}$. This is made explicit in the following, more formal definition.

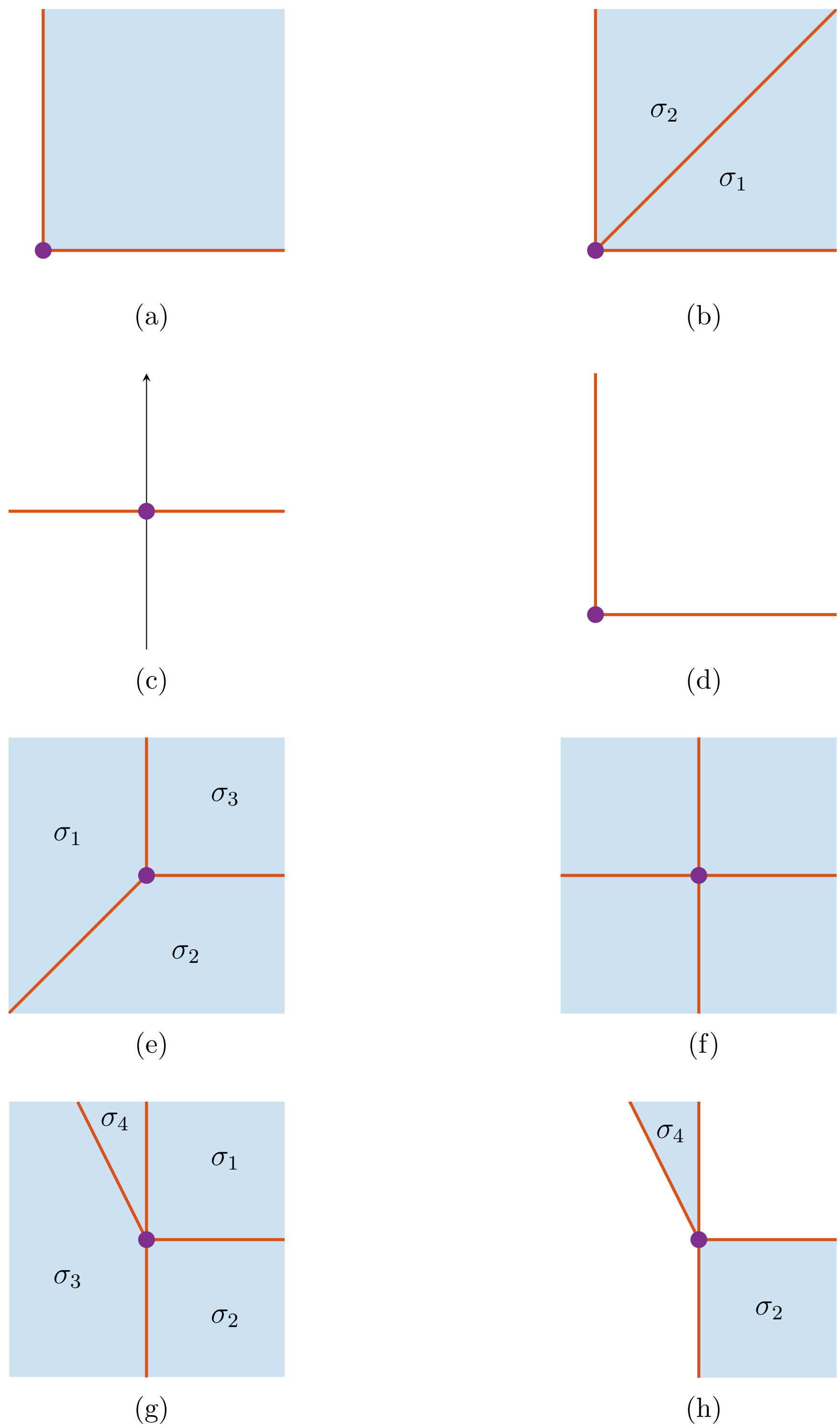

Figure 7.4: Eight fans in $\mathbb{R}^2$.

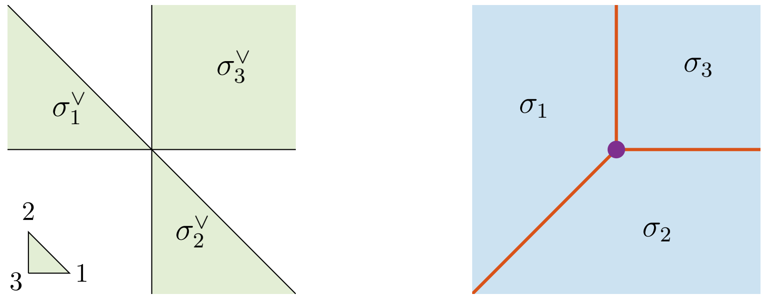

Figure 7.5: A fan from a triangle.

Definition 7.3.6 (Normal fan). *Let $P \subset M_{\mathbb{R}}$ be full dimensional and let*

$$\mathcal{P} = \bigcap_{Q \text{ facet of } \mathcal{P}} H^+_{u_Q, a_Q}$$

be the minimal H-representation of $\mathcal{P}$ (as introduced in (3.1.2)*). The* normal fan *of $\mathcal{P}$ is*

$$\Sigma_{\mathcal{P}} = \{\sigma_Q \ : \ Q \preceq \mathcal{P}\}, \quad \textit{where } \sigma_Q = \operatorname{Cone}(\{u_{Q'} \ : \ Q \textit{ is a face of the facet } Q'\}).$$

Here we set $\sigma_{\mathcal{P}} = \operatorname{Cone}(\emptyset) = \{0\}$ by convention.

Example 7.3.7. The fan in Figure 7.4(f) is the normal fan $\Sigma_{\mathcal{P}}$ of a square $\mathcal{P} \subset \mathbb{R}^2$, see Figure 7.6. Faces of $\mathcal{P}$ correspond to cones of $\Sigma_{\mathcal{P}}$. The square has four vertices, four edges, and one two-dimensional face. Its normal fan has four two-dimensional cones, four rays and one zero-dimensional cone. ◇

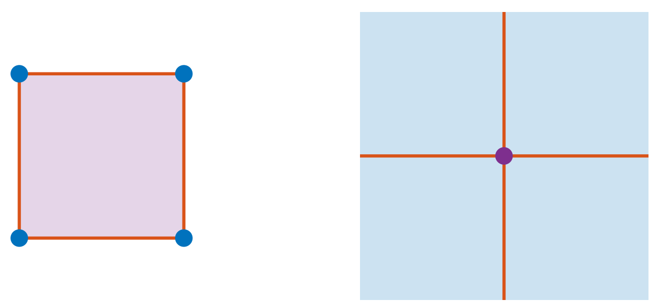

Figure 7.6: The normal fan of a square.

The fact that $\Sigma_{\mathcal{P}}$ is a fan is [19, Theorem 2.3.2]. Here is an alternative definition of the normal fan, which works when $\mathcal{P}$ is not full-dimensional.

Exercise 7.3.8. Let $\mathcal{P} \subset M_{\mathbb{R}} \simeq \mathbb{R}^d$ be a convex polytope. For each face $Q \preceq P$, we define a cone $\sigma_Q \subset N_{\mathbb{R}}$ as follows:

$$\sigma_Q \ = \ \{u \in N_{\mathbb{R}} \ : \ \min_{m \in \mathcal{P}} \langle u, m \rangle \text{ is attained for every } m \in Q \ \}. \tag{7.3.1}$$

The set of cones $\Sigma_{\mathcal{P}} = \{\sigma_Q \ : \ Q \preceq \mathcal{P}\}$ is the *normal fan* of $\mathcal{P}$.

1. The *lineality space* of a fan Σ in $N_{\mathbb{R}}$ is the largest linear subspace of $N_{\mathbb{R}}$ contained in each cone of Σ. Show that if $\dim \mathcal{P} = d - k$, then the lineality space of $\Sigma_{\mathcal{P}}$ has dimension k.

2. Show that the definition of σ_Q above agrees with the definition in terms of inward pointing facet normals in the case where $\dim \mathcal{P} = d$.

Notice that the definition in Exercise 7.3.8 is more directly related to our observations in Example 6.2.2 and Figure 6.1: the face $Q \preceq \mathcal{P}$ minimizes the cost function $\langle w, x\rangle$ for any $w \in \sigma_Q$. Our definition of the normal fan of $\mathcal{P}$ is sometimes called the *inner normal fan*, because we use inward instead of outward pointing facet normals in Definition 7.3.6 and, equivalently, min instead of max in (7.3.1). This is the convention in [19] and [30] as well. The standard reference [75] in polyhedral geometry uses the *outer normal fan* instead, see [75, Example 7.3]. Here are some properties of normal fans.

Proposition 7.3.9. *Let $\mathcal{P} \subset M_{\mathbb{R}} \simeq \mathbb{R}^d$ be a full-dimensional lattice polytope. For each face $Q \preceq \mathcal{P}$, let σ_Q be as in Definition 7.3.6. We have*

1. *cones and faces have complimentary dimension:* $\dim Q + \dim \sigma_Q = d$ *for all* $Q \preceq \mathcal{P}$,

2. *the normal fan $\Sigma_{\mathcal{P}}$ is* complete, *meaning that its cones cover $N_{\mathbb{R}}$:*

$$N_{\mathbb{R}} = \bigcup_{v \in \mathcal{V}(\mathcal{P})} \sigma_v = \bigcup_{Q \preceq \mathcal{P}} \sigma_Q,$$

3. *the normal fan is invariant under dilations and translations of $\mathcal{P}$: for any $m \in M_{\mathbb{R}}$ and $k \in \mathbb{N} \setminus \{0\}$, we have $\Sigma_{k \cdot \mathcal{P} + m} = \Sigma_{\mathcal{P}}$.*

For a proof of Proposition 7.3.9, see [19, Propositions 2.3.8 and 2.3.9]. Point 2 says that any normal fan is complete, but the converse is not true: not every complete fan comes from a polytope, see [30, page 71] or [75, Example 7.5].

Example 7.3.10. Normal fans can be computed in `Oscar.jl`. We consider the pentagon $\mathcal{P}$ from Example 3.1.10 and execute `normal_fan(P)`. The result is Figure 7.7. ◇

Let X_A be a projective toric variety. Assume that $\mathrm{Conv}(A)$ is a d-dimensional polytope in $M_{\mathbb{R}} = \mathbb{R}^d$ and $\mathbb{Z}A = \mathbb{Z}^d$. The normal fan $\Sigma_{\mathcal{P}}$ assigns a cone to each face of $\mathcal{P}$. In turn, each face Q of $\mathcal{P}$ corresponds to a $(\mathbb{C}^*)^d$-orbit $X^\circ_{A,Q} \subset X_A$ (Theorem 4.3.8). This allows us to define the orbit-cone correspondence for X_A.

Definition 7.3.11. *The* orbit-cone correspondence *for the toric variety X_A is a bijection between cones in the normal fan $\Sigma_{\mathcal{P}}$ and $(\mathbb{C}^*)^d$-orbits of X_A, given by $\sigma_Q \mapsto X^\circ_{A,Q}$.*

Exercise 7.3.12. Let Q, Q' be faces of P. Show that the orbit closures $X_{A,Q} = \overline{X^\circ_{A,Q}}$ and $X_{A,Q'} = \overline{X^\circ_{A,Q'}}$ intersect in X_A if and only if $Q \cap Q' \neq \emptyset$. Moreover, this is equivalent to $\sigma_Q \cap \sigma_{Q'} = \{0\}$, and the dimension of the intersection is $\dim(Q \cap Q')$. Hint: use Theorem 4.2.2.

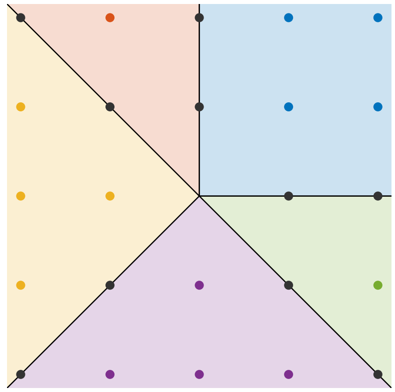

Figure 7.7: The normal fan from Example 7.3.10.

7.4 Gluing data from fans

Let $\tau \preceq \sigma$ be a face of σ. Recall from Section 7.2 that the coordinate ring of the affine open subset $\mathcal{Y}_\sigma^\tau \simeq (\mathcal{Y}_\sigma)_{t^m}$ is $\mathbb{C}[\mathsf{S}_\tau] \simeq \mathbb{C}[\mathsf{S}_\sigma]_{t^m}$, where $m \in \sigma^\vee \cap M$ is such that $\tau = \sigma \cap H_m$. Below we write $\mathbb{C}[\mathsf{S}_\sigma]_\tau = \mathbb{C}[\mathsf{S}_\sigma]_{t^m}$, because the precise choice of m satisfying these conditions does not matter. Corollary 7.2.3 provides an isomorphism $\mathbb{C}[\mathsf{S}_\sigma]_\tau \simeq \mathbb{C}[\mathsf{S}_\tau]$.

We are now ready to define an abstract variety from a fan. The gluing data from Section 7.1 are the following. We consider a set $\{Y_\sigma\}_{\sigma\in\Sigma}$ of affine varieties, whose index set $\mathscr{J} = \Sigma$ is a fan Σ in $N_\mathbb{R}$. For $\sigma \in \Sigma$, the variety Y_σ is the d-dimensional normal affine toric variety $\mathcal{Y}_\sigma$. By Definition 7.3.1, the intersection $\tau = \sigma \cap \sigma'$ of two cones $\sigma, \sigma' \in \Sigma$ is a face of both. The dense open subset $Y_{\sigma\sigma'} \subset Y_\sigma = \mathcal{Y}_\sigma$ is the open subset $\mathcal{Y}_\sigma^\tau$ corresponding to that face. The isomorphism $\phi_{\sigma\sigma'} : Y_{\sigma\sigma'} \to Y_{\sigma'\sigma}$ is (Corollary 7.2.3)

$$\phi_{\sigma\sigma'} \;:\; Y_{\sigma\sigma'} \;\simeq\; \mathcal{Y}_\tau \;\simeq\; Y_{\sigma'\sigma}. \tag{7.4.1}$$

Theorem 7.4.1. *Let Σ be a fan in $N_\mathbb{R}$. The data $\{Y_\sigma\}_{\sigma\in\Sigma}$ and $\{\phi_{\sigma\sigma'}\}_{\sigma,\sigma'\in\Sigma}$ defined above are the gluing data for an abstract algebraic variety. We denote this variety by $\mathcal{X}_\Sigma$.*

Proof. We need to show that these data satisfy the axioms 1-3 listed at the beginning of Section 7.1. Point 1 is clear: $\phi_{\sigma\sigma'}$ is a composition of isomorphisms, and $\phi_{\sigma'\sigma} = (\phi_{\sigma\sigma'})^{-1}$. For point 2, consider $\sigma_0, \sigma_1, \sigma_2 \in \Sigma$. We need to show that $\phi_{\sigma_0\sigma_1}(Y_{\sigma_0\sigma_1} \cap Y_{\sigma_0\sigma_2}) = Y_{\sigma_1\sigma_0} \cap Y_{\sigma_1\sigma_2}$. By Lemma 7.2.5, the intersection $Y_{\sigma_0\sigma_1} \cap Y_{\sigma_0\sigma_2}$ is the affine open subset $\mathcal{Y}_{\sigma_0}^{\sigma_0\cap\sigma_1\cap\sigma_2} \subseteq \mathcal{Y}_{\sigma_0}$. We have the following two dual commutative diagrams:

$$\begin{tikzcd}
\mathbb{C}[\mathsf{S}_{\sigma_0}]_{\sigma_0\cap\sigma_1\cap\sigma_2} \arrow[ddr, bend right, "\simeq"'] & & \mathbb{C}[\mathsf{S}_{\sigma_1}]_{\sigma_0\cap\sigma_1} \arrow[ll, hook', "\phi^*"'] \arrow[dl, "\simeq"] \arrow[dd, hook] \\
& \mathbb{C}[\mathsf{S}_{\sigma_0\cap\sigma_1}] \arrow[ul, hook] \arrow[d, hook] & \\
& \mathbb{C}[\mathsf{S}_{\sigma_0\cap\sigma_1\cap\sigma_2}] & \mathbb{C}[\mathsf{S}_{\sigma_1}]_{\sigma_0\cap\sigma_1\cap\sigma_2} \arrow[l, hook', "\simeq"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{Y}_{\sigma_0}^{\sigma_0\cap\sigma_1\cap\sigma_2} \arrow[rr, hook, "\phi"] \arrow[dr, hook] & & \mathcal{Y}_{\sigma_1}^{\sigma_0\cap\sigma_1} \\
& \mathcal{Y}_{\sigma_0\cap\sigma_1} \arrow[ur, "\simeq"] & \\
& \mathcal{Y}_{\sigma_0\cap\sigma_1\cap\sigma_2} \arrow[uul, bend left, "\simeq"] \arrow[u, hook] \arrow[r, "\simeq"] & \mathcal{Y}_{\sigma_1}^{\sigma_0\cap\sigma_1\cap\sigma_2} \arrow[uu, hook]
\end{tikzcd}$$

On the left, the isomorphisms are as in Corollary 7.2.3, and ϕ^* is the pullback of $\phi = (\phi_{\sigma_0\sigma_1})_{|\mathcal{Y}_{\sigma_0}^{\sigma_0\cap\sigma_1\cap\sigma_2}}$. The other maps are defined to make the left diagram commute. The diagram on the right is obtained by taking Specm. The right diagram shows that the image of ϕ is $\mathcal{Y}_{\sigma_1}^{\sigma_0\cap\sigma_1\cap\sigma_2}$, which establishes $\phi_{\sigma_0\sigma_1}(Y_{\sigma_0\sigma_1} \cap Y_{\sigma_0\sigma_2}) = Y_{\sigma_1\sigma_0} \cap Y_{\sigma_1\sigma_2}$.

It remains to show that $\phi_{\sigma_1\sigma_2} \circ \phi_{\sigma_0\sigma_1} = \phi_{\sigma_0\sigma_2}$ on $Y_{\sigma_0\sigma_1} \cap Y_{\sigma_0\sigma_2}$. The diagrams above show that the restriction of $\phi_{\sigma_0\sigma_1}$ to $Y_{\sigma_0\sigma_1} \cap Y_{\sigma_0\sigma_2} = \mathcal{Y}_{\sigma_0}^{\sigma_0\cap\sigma_1\cap\sigma_2}$ is the isomorphism $\mathcal{Y}_{\sigma_0}^{\sigma_0\cap\sigma_1\cap\sigma_2} \to \mathcal{Y}_{\sigma_1}^{\sigma_0\cap\sigma_1\cap\sigma_2}$ coming from the map of rings (bottom right $\to$ top left)

$$\mathbb{C}[\mathsf{S}_{\sigma_1}]_{\sigma_0\cap\sigma_1\cap\sigma_2} \simeq \mathbb{C}[\mathsf{S}_{\sigma_0\cap\sigma_1\cap\sigma_2}] \simeq \mathbb{C}[\mathsf{S}_{\sigma_0}]_{\sigma_0\cap\sigma_1\cap\sigma_2}.$$

The desired relation $\phi_{\sigma_1\sigma_2} \circ \phi_{\sigma_0\sigma_1} = \phi_{\sigma_0\sigma_2}$ is now seen from dualizing the following diagram

$$\begin{tikzcd}
\mathbb{C}[\mathsf{S}_{\sigma_0}]_{\sigma_0\cap\sigma_1\cap\sigma_2} & & \mathbb{C}[\mathsf{S}_{\sigma_1}]_{\sigma_0\cap\sigma_1\cap\sigma_2} \arrow[ll, "\phi^*_{\sigma_0\sigma_1}"'] \\
& \mathbb{C}[\mathsf{S}_{\sigma_0\cap\sigma_1\cap\sigma_2}] \arrow[ul, dash, "\simeq"'] \arrow[ur, dash, "\simeq"] & \\
& \mathbb{C}[\mathsf{S}_{\sigma_2}]_{\sigma_0\cap\sigma_1\cap\sigma_2} \arrow[u, dash, "\simeq"'] \arrow[uul, bend left, "\phi^*_{\sigma_0\sigma_2}"] \arrow[uur, bend right, "\phi^*_{\sigma_1\sigma_2}"'] &
\end{tikzcd}$$

□

Example 7.4.2. Let $N = \mathbb{Z}$ and consider the fan Σ in $N_{\mathbb{R}} = \mathbb{R}$ consisting of

$$\sigma_0 = \mathbb{R}_{\geq 0}, \quad \sigma_1 = \mathbb{R}_{\leq 0} \quad \text{and} \quad \tau = \{0\}.$$

This is the only complete fan in $\mathbb{R}$. The gluing data are

$$\{\mathcal{Y}_{\sigma_0}, \mathcal{Y}_{\sigma_1}, \mathcal{Y}_{\tau}\} \quad \text{and} \quad \{\phi_{\sigma_0\sigma_1}, \phi_{\sigma_1\sigma_0}, \phi_{\sigma_0\tau}, \phi_{\tau\sigma_0}, \phi_{\sigma_1\tau}, \phi_{\tau\sigma_1}\}.$$

By Example 2.3.4, we have $\mathcal{Y}_{\sigma_0} \simeq \mathcal{Y}_{\sigma_1} \simeq \mathbb{C}$ and $\mathcal{Y}_{\tau} \simeq \mathbb{C}^*$. The coordinate rings are

$$\mathbb{C}[\mathcal{Y}_{\sigma_0}] = \mathbb{C}[\mathsf{S}_{\sigma_0}] = \mathbb{C}[t],\ \mathbb{C}[\mathcal{Y}_{\sigma_1}] = \mathbb{C}[\mathsf{S}_{\sigma_1}] = \mathbb{C}[t^{-1}] \quad \text{and} \quad \mathbb{C}[\mathcal{Y}_{\tau}] = \mathbb{C}[M] = \mathbb{C}[t, t^{-1}].$$

With these choices of generators, $\mathcal{Y}_{\sigma_0}$ is $\mathbb{C}$ with coordinate t, $\mathcal{Y}_{\sigma_1}$ is $\mathbb{C}$ with coordinate $u = t^{-1}$ and $\mathcal{Y}_{\tau} = \{(t,u) \in \mathbb{C}^2 : tu = 1\}$. The isomorphism $\phi_{\sigma_0\sigma_1}$ is defined on $\mathcal{Y}_{\sigma_0}^{\sigma_0\cap\sigma_1} = \mathcal{Y}_{\sigma_0}^{\tau} \simeq \mathbb{C}^*$. It is given by the composition in (7.4.1), which is

$$t \longmapsto (t, t^{-1}) \longmapsto t^{-1}.$$

Its inverse $\phi_{\sigma_1\sigma_0}$ is $u \mapsto u^{-1}$. Make sure that you see where the exponent -1 comes from. Similarly $\phi_{\sigma_0\tau}(t) = (t, t^{-1})$ and $\phi_{\sigma_1\tau}(u) = (u^{-1}, u)$. The inverses of these last two maps simply embed $\mathbb{C}^*$ into $\mathbb{C}$. Our three affine varieties give three affine open subsets $U_{\sigma_0}, U_{\sigma_1}, U_{\tau}$ of $\mathcal{X}_{\Sigma}$ as in Section 7.1. These are such that $U_{\tau} = U_{\sigma_0} \cap U_{\sigma_1} \simeq \mathbb{C}^*$. Hence, $\mathcal{Y}_{\tau}$ is redundant as an affine building block of $\mathcal{X}_{\Sigma}$, and the same abstract variety is obtained from $\{\mathcal{Y}_{\sigma_0}, \mathcal{Y}_{\sigma_1}\}$ with maps $\{\phi_{\sigma_0\sigma_1}, \phi_{\sigma_1\sigma_0}\}$. We recognize that these are precisely the gluing data for the projective line $\mathbb{P}^1$ as in Example 7.1.1. That is, $\mathcal{X}_{\Sigma} \simeq \mathbb{P}^1$. ◇

Remark 7.4.3. Example 7.4.2 shows that the gluing data of a fan can be redundant. If $\sigma, \tau \in \Sigma$ are such that $\tau \preceq \sigma$, then the gluing map $\phi_{\tau\sigma} : \mathcal{Y}_\tau \hookrightarrow \mathcal{Y}_\sigma^\tau$ simply embeds $\mathcal{Y}_\tau$ into $\mathcal{Y}_\sigma$, so that $U_\tau \subseteq U_\sigma \subseteq \mathcal{X}_\sigma$. Therefore, $\mathcal{X}_\Sigma$ can be glued from the affine varieties $\mathcal{Y}_\sigma$ where σ runs over all cones in Σ which are maximal with respect to inclusion.

Example 7.4.4. We consider the fan Σ shown in Figure 7.4(e). By Remark 7.4.3, $\mathcal{X}_\Sigma$ is glued from $\{\mathcal{Y}_{\sigma_1}, \mathcal{Y}_{\sigma_2}, \mathcal{Y}_{\sigma_3}\}$ using the maps $\phi_{\sigma_1\sigma_2}, \phi_{\sigma_1\sigma_3}, \phi_{\sigma_2\sigma_3}$ and their inverses (here the cones σ_i are as indicated in the figure). The coordinate rings are

$$\mathbb{C}[\mathcal{Y}_{\sigma_1}] = \mathbb{C}[t_1^{-1}, t_1^{-1}t_2], \quad \mathbb{C}[\mathcal{Y}_{\sigma_2}] = \mathbb{C}[t_2^{-1}, t_1t_2^{-1}], \quad \mathbb{C}[\mathcal{Y}_{\sigma_3}] = \mathbb{C}[t_1, t_2].$$

We investigate how $\mathcal{Y}_{\sigma_2}$ and $\mathcal{Y}_{\sigma_3}$ are glued. Both varieties are isomorphic to $\mathbb{C}^2$. Compatibly with our choice of generators for their coordinate rings, we use coordinates t_1, t_2 on $\mathcal{Y}_{\sigma_3} \simeq \mathbb{C}^2$ and $u_1 = t_2^{-1}, u_2 = t_1t_2^{-1}$ on $\mathcal{Y}_{\sigma_2} \simeq \mathbb{C}^2$. The morphism $\phi_{\sigma_3\sigma_2} : \mathcal{Y}_{\sigma_3}^{\sigma_2\cap\sigma_3} \to \mathcal{Y}_{\sigma_2}^{\sigma_2\cap\sigma_3}$ is induced by the ring isomorphism

$$\mathbb{C}[t_2^{-1}, t_1t_2^{-1}]_{t_2^{-1}} \longrightarrow \mathbb{C}[t_1, t_2]_{t_2}.$$

It is given by $\mathbb{C} \times \mathbb{C}^* \to \mathbb{C}^* \times \mathbb{C} : (t_1, t_2) \mapsto (t_2^{-1}, t_1t_2^{-1})$. Its inverse sends (u_1, u_2) to $(u_1^{-1}u_2, u_1^{-1})$. This is precisely the map we found in Example 7.1.3. In fact, the reader should check that $\mathcal{X}_\Sigma \simeq \mathbb{P}^2$, and $U_{\sigma_i} = U_i, i = 1, \ldots, 3$ are standard affine open charts. $\diamond$

Exercise 7.4.5. Show that, if Σ is a fan consisting of a pointed cone σ and all its faces, then $\mathcal{X}_\Sigma$ is the normal affine toric variety of σ, i.e., $\mathcal{X}_\Sigma \simeq \mathcal{Y}_\sigma$.

Example 7.4.6. We show that the fan in Figure 7.4(b) gives rise to the variety $\mathrm{Bl}_0\mathbb{C}^2$, the blow-up of $\mathbb{C}^2$ at the origin. By Remark 7.4.3, it suffices to glue the surfaces $\mathcal{Y}_{\sigma_1}$ and $\mathcal{Y}_{\sigma_2}$, see Figure 7.4(b). We choose the following generators for their coordinate rings:

$$\begin{aligned} \mathbb{C}[\mathcal{Y}_{\sigma_1}] &= \mathbb{C}[t_1, t_2, t_1t_2^{-1}] \simeq \mathbb{C}[x_1, x_2, x_3]/\langle x_1 - x_2x_3\rangle, \\ \mathbb{C}[\mathcal{Y}_{\sigma_2}] &= \mathbb{C}[t_1, t_2, t_1^{-1}t_2] \simeq \mathbb{C}[y_1, y_2, y_3]/\langle y_1y_3 - y_2\rangle. \end{aligned}$$

The exponents generate the semigroups of the cones seen in Figure 7.8. The cones σ_1, σ_2 intersect in the ray τ. The corresponding affine toric variety has coordinate ring $\mathbb{C}[\mathcal{Y}_\tau] = \mathbb{C}[t_1, t_2, t_1t_2^{-1}, t_1^{-1}t_2]$, isomorphic to a localization of both rings above. The gluing morphism $\phi_{\sigma_1\sigma_2}$ is defined where the coordinate $t_1t_2^{-1}$ is nonzero. It is given by

$$(t_1, t_2, t_1t_2^{-1}) \longmapsto (t_1, t_2, t_1t_2^{-1}, (t_1t_2^{-1})^{-1}) \longmapsto (t_1, t_2, (t_1t_2^{-1})^{-1}),$$

or in the x-coordinates on the surface $\{x_1 - x_2x_3 = 0, x_3 \neq 0\} \simeq (\mathcal{Y}_{\sigma_1})_{t_1t_2^{-1}}$:

$$\{x_1 - x_2x_3\} \ni (x_1, x_2, x_3) \longmapsto (x_1, x_2, x_3, x_3^{-1}) \longmapsto (x_1, x_2, x_3^{-1}) \in \{y_1y_3 - y_2 = 0\}.$$

This gluing identifies our surfaces with the charts $x_0 \neq 0$ and $x_1 \neq 0$ of

$$\{((x, y), (x_0 : x_1)) \in \mathbb{C}^2 \times \mathbb{P}^1 : x_0x - x_1y = 0\} \simeq \mathrm{Bl}_0\mathbb{C}^2. \qquad \diamond$$

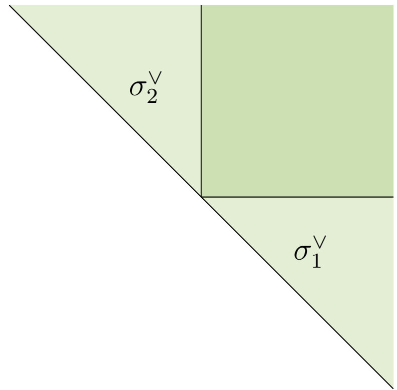

Figure 7.8: Dual cones for the fan in Figure 7.4(b).

Exercise 7.4.7. Let Σ be the fan in Figure 7.4(c). Show that $\mathcal{X}_\Sigma \simeq \mathbb{C} \times \mathbb{C}^*$.

Theorem 7.4.8. *Let $\mathcal{P} \subset M_\mathbb{R} \simeq \mathbb{R}^d$ be a d-dimensional convex lattice polytope with normal fan $\Sigma_\mathcal{P}$. The abstract variety $\mathcal{X}_{\Sigma_\mathcal{P}}$ obtained from the gluing* (7.4.1) *is isomorphic to the projective toric variety $\mathcal{X}_\mathcal{P}$ from Definition 3.5.8.*

Proof. Let $k \in \mathbb{N}$ be such that $k \cdot \mathcal{P}$ is very ample. By Definition 3.5.8, $\mathcal{X}_\mathcal{P}$ is the projective toric variety X_A, where A is a matrix whose columns $a_1, \ldots, a_n \in M = \mathbb{Z}^d$ are the lattice points of $k \cdot \mathcal{P}$. Since $\mathcal{P}$ is a lattice polytope, the vertices $\mathcal{V}(k \cdot \mathcal{P})$ are among these columns. By Proposition 3.2.4 we have $X_A = \bigcup_i (X_A \cap U_i)$, where $U_i = \mathbb{P}^{n-1} \setminus V(x_i)$ and i runs over all indices in $\{1, \ldots, n\}$ for which a_i is a vertex. By Proposition 3.2.1, the affine variety $X_A \cap U_i$ is isomorphic to Y_{A-a_i}. By Example 7.1.5, these affine pieces are glued to X_A by restricting the maps (7.1.1) to $Y_{A-a_i} \subset \mathbb{C}^n$.

On the other hand $\mathcal{X}_{\Sigma_\mathcal{P}}$ is glued from $\{\mathcal{Y}_\sigma\}_{\sigma \in \Sigma_\mathcal{P}}$. By Remark 7.4.3, it suffices to use the affine varieties corresponding to maximal cones in $\Sigma_\mathcal{P}$. These are the cones $\sigma_v \in \Sigma_\mathcal{P}$ corresponding to the vertices of $\mathcal{P}$. For a vertex $v \in \mathcal{V}(\mathcal{P})$ let $a_i = k \cdot v$ be the corresponding column of A. Very ampleness of $k \cdot \mathcal{P}$ implies that Y_{A-a_i} is isomorphic to $\mathcal{Y}_{\sigma_v}$ (Definition 3.5.1). Hence, it suffices to show that the varieties $\mathcal{Y}_{\sigma_v}$ are glued together to form $\mathcal{X}_{\Sigma_\mathcal{P}}$ in the same way that Y_{A-a_i} are glued together to form $X_A \simeq \mathcal{X}_\mathcal{P}$. This is seen as follows. Let $a_i = k \cdot v_i, a_j = k \cdot v_j$ be two vertices of $k \cdot \mathcal{P}$. The gluing morphism $\phi_{\sigma_{v_i}\sigma_{v_j}}$ from (7.4.1) is induced by the following isomorphisms of semigroup algebras

$$\mathbb{C}[\mathbb{N}(A-a_j)]_{t^{a_i-a_j}} \simeq \mathbb{C}[\mathbb{N}(A-a_i) + \mathbb{N}(A-a_j)] \simeq \mathbb{C}[\mathbb{N}(A-a_i)]_{t^{a_j-a_i}}.$$

Writing this out in coordinates, we see that $\phi_{\sigma_{v_i}\sigma_{v_j}}$ is indeed the restriction of (7.1.1) to $(Y_{A-a_i})_{t^{a_j-a_i}}$. We encourage the reader to check the details. $\square$

Example 7.4.9. The fan Σ in Figure 7.4(e) is the normal fan of a the standard simplex $\mathrm{Conv}(0, e_1, e_2) \subset \mathbb{R}^2$. By Theorem 7.4.8 and Exercise 3.5.9, we indeed have $\mathcal{X}_\Sigma \simeq \mathbb{P}^2$. $\diamond$

Exercise 7.4.10. Use Exercise 3.5.10 to show that the fan Σ in Figure 7.4(f) gives $\mathcal{X}_\Sigma \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Compare your gluing in Exercise 7.1.6 with that of this section.

Example 7.4.11. Let Σ be the fan consisting of a pointed cone σ and all its faces. We have $X_\Sigma \simeq \mathcal{Y}_\sigma$. In Exercise 7.3.4, we saw that we can obtain a different fan Σ' by removing a cone τ from Σ, along with all cones it is a face of. By Exercise 7.2.7, the result is $\mathcal{X}_{\Sigma'} = \mathcal{Y}_\sigma \setminus \overline{O(\tau)}$, where $\overline{\cdot}$ is the closure in $\mathcal{Y}_\sigma$ and $O(\tau)$ is the image of τ under the orbit-cone correspondence from Definition 4.3.3. For instance, the fan Σ' in Figure 7.4(d) is obtained by removing the two-dimensional cone from Figure 7.4(a), the fan of $\mathbb{C}^2$. This gives $\mathcal{X}_{\Sigma'} \simeq \mathbb{C}^2 \setminus \{0\}$, which is neither affine nor projective. $\diamond$

Exercise 7.4.12. This is the projective version of Example 7.4.11. Let $\Sigma_\mathcal{P}$ be the normal fan of $\mathcal{P}$ and let Σ' be obtained by removing a cone τ from $\Sigma_\mathcal{P}$ along with all other cones it is a face of. Show that $\mathcal{X}_{\Sigma'}$ is obtained from the projective toric variety $\mathcal{X}_\mathcal{P}$ by removing the closure of the $(\mathbb{C}^*)^d$-orbit $O(\tau)$, i.e., the image of τ under the orbit-cone correspondence 7.3.11. Show that the surface $\mathcal{X}_{\Sigma'}$ with Σ' the fan in Figure 7.4(h) is obtained by removing two points from a Hirzebruch surface. Hint: the latter is the projective toric surface obtained from the fan in Figure 7.4(g).

The abstract varieties $\mathcal{X}_\Sigma$ obtained in this section are also called *toric*. The torus $(\mathbb{C}^*)^d$ is a dense open subset of $\mathcal{X}_\Sigma$ via $\mathcal{X}_\Sigma \supset U_{\{0\}} \simeq \mathcal{Y}_{\{0\}}$. That is, it is the affine open subset associated to the cone $\{0\} \in \Sigma$.

To discuss the action of $(\mathbb{C}^*)^d$, let us write $O_\sigma(\tau)$ for the orbit corresponding to τ in $\mathcal{Y}_\sigma$, with $\tau \preceq \sigma$ (see Definition 4.3.3). The gluing morphisms $\phi_{\sigma\sigma'}$ identify each torus orbit $O_\sigma(\tau') \subset \mathcal{Y}_\sigma$, for $\tau' \preceq \tau = \sigma \cap \sigma'$, with the torus orbit $O_{\sigma'}(\tau')$ of τ' in $\mathcal{Y}_{\sigma'}$. Hence, the action of $(\mathbb{C}^*)^d$ on itself extends to an algebraic action on $\mathcal{X}_\Sigma$. The orbits of this action are in bijection with the cones of Σ.

Definition 7.4.13. *The* orbit-cone correspondence *for the toric variety $\mathcal{X}_\Sigma$ is a bijection between cones of Σ and $(\mathbb{C}^*)^d$-orbits of $\mathcal{X}_\Sigma$, given by $\sigma \mapsto O_\sigma(\sigma) \subset \mathcal{Y}_\sigma \simeq U_\sigma \subset \mathcal{X}_\Sigma$.*

If Σ is the normal fan of a very ample polytope $\mathcal{P}$, then Definition 7.4.13 agrees with Definition 7.3.11 after identifying $\mathcal{X}_\Sigma$ and $\mathcal{X}_\mathcal{P}$ as in Theorem 7.4.8.

Example 7.4.14. Consider the fan Σ in $\mathbb{R}^2$ whose rays have generators $u_1 = (1,2), u_2 = (1,0), u_3 = (-3,-2), u_4 = (0,1)$. The surface $\mathcal{X}_\Sigma$ can be embedded in $\mathbb{P}^{88}$ via the 89 lattice points of the polygon $P = \mathrm{Conv}((0,15),(0,1),(2,0),(10,0))$ with normal fan Σ.

```
P = convex_hull([0 15; 0 1; 2 0; 10 0]); Σ = normal_fan(P);
X_Σ = NormalToricVariety(Σ)
```

It is covered by 4 normal affine toric surfaces $\mathcal{Y}_{\sigma_{12}}, \mathcal{Y}_{\sigma_{23}}, \mathcal{Y}_{\sigma_{34}}, \mathcal{Y}_{\sigma_{14}}$ where $\sigma_{ij} = \mathrm{Cone}(u_i, u_j)$. Out of these, only $\mathcal{Y}_{\sigma_{14}}$ is smooth. Here is how to check this in `Oscar.jl`.

```
cover = affine_open_covering(X_Σ);
[issmooth(Y) for Y in cover]
```

This returns a boolean vector `[0;0;1;0]`. We embed $\mathcal{Y}_{\sigma_{12}}$ and $\mathcal{Y}_{\sigma_{23}}$ in affine space and determine $\phi_{\sigma_{12},\sigma_{23}}$ in coordinates. With `Oscar.jl` we compute the `hilbert_basis` of the two cones $\sigma_{12}^\vee, \sigma_{23}^\vee$. These provide the embeddings

$$\mathcal{Y}_{\sigma_{12}} \simeq Y_{12} = \{xz - y^2 = 0\} \subset \mathbb{C}^3, \qquad \mathcal{Y}_{\sigma_{23}} \simeq Y_{23} = \{uw - v^2 = 0\} \subset \mathbb{C}^3.$$

The coordinates x, y, z correspond to the blue marked lattice points in Figure 7.9, ordered from left to right. Similarly, u, v, w correspond to the green dots. The overlap $\mathcal{Y}_{\sigma_{12}} \cap \mathcal{Y}_{\sigma_{23}} \subset \mathcal{X}_\Sigma$ is given by points on Y_{12} with $x \neq 0$, and points on Y_{23} with $u \neq 0$. On these open sets, we have

$$\phi_{\sigma_{12},\sigma_{23}}(x,y,z) = \left(\frac{1}{x}, \frac{y}{x^2}, \frac{y^2}{x^3}\right) = \left(\frac{1}{x}, \frac{y}{x^2}, \frac{z}{x^2}\right).$$

This map can be obtained from the cones $\sigma_{12}^\vee, \sigma_{23}^\vee$, writing the green lattice points in Figure 7.9 as $\mathbb{Z}$-linear combinations of the blue ones. $\diamond$

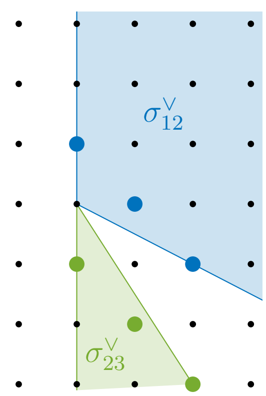

Figure 7.9: The Hilbert bases of $\sigma_{12}^\vee$ and $\sigma_{23}^\vee$ give the isomorphism $\phi_{\sigma_{12},\sigma_{23}}$.

Further reading

Abstract normal toric varieties coming from cones and fans are the starting point in Fulton's book [30]. They are also treated in [19, Section 3.1], and the orbit-cone correspondence for normal toric varieties is the topic of [19, Section 3.2]. Normality will be a key assumption in our study of divisors in Chapter 9. Toric varieties arising in applications typically come with a monomial parametrization, or with a polytope. In the latter case, the abstract toric variety comes from the normal fan $\Sigma_{\mathcal{P}}$ of that polytope (Theorem 7.4.8). The "fan perspective" is still useful in such cases, as we sill see later.

Chapter 8

Maps between toric varieties

This chapter discusses *toric morphisms* between toric varieties. Roughly speaking, these are morphisms which interact nicely with the structures we have identified in previous chapters. For instance, they are induced by semigroup homomorphisms, or by linear maps which are *compatible* with polyhedral cones/fans. As we will see, toric morphisms are also equivariant with respect to the torus action. We start with affine toric varieties in Section 8.1, and proceed with abstract normal toric varieties in Section 8.2.

8.1 Toric morphisms between affine toric varieties

In this section, our maps go between affine toric varieties. We start by fixing and recalling some notation. Consider two affine semigroups $\mathsf{S}_1 \subseteq M_1 \simeq \mathbb{Z}^{d_1}$ and $\mathsf{S}_2 \subseteq M_2 \simeq \mathbb{Z}^{d_2}$. Picking bases for M_1, M_2, the corresponding tori are $(\mathbb{C}^*)^{d_1}$ and $(\mathbb{C}^*)^{d_2}$. By Proposition 2.1.7, each of these semigroups defines an affine toric variety $\mathcal{Y}_i$ with coordinate ring $\mathbb{C}[\mathsf{S}_i]$, $i = 1, 2$. The semigroup ring $\mathbb{C}[\mathsf{S}_i]$ is a subring of a Laurent polynomial ring:

$$\mathbb{C}[\mathsf{S}_1] \subset \mathbb{C}[M_1] = \mathbb{C}[t_1^{\pm 1}, \ldots, t_{d_1}^{\pm 1}], \quad \mathbb{C}[\mathsf{S}_2] \subset \mathbb{C}[M_2] = \mathbb{C}[s_1^{\pm 1}, \ldots, s_{d_2}^{\pm 1}].$$

Choosing generators of S_i amounts to writing $\mathsf{S}_i = \mathbb{N}A_i$ for some matrix $A_i \in \mathbb{Z}^{d_i \times n_i}$, and this results in two embedded affine toric varieties $Y_{A_1} \subset \mathbb{C}^{n_1}$ and $Y_{A_2} \subset \mathbb{C}^{n_2}$. Switching lattices if necessary, we may assume that $\mathbb{Z}A_i = \mathbb{Z}^{d_i}$ (Proposition 1.2.17), so that the dense torus $T_i \subseteq Y_{A_i} \simeq \mathcal{Y}_i$ is isomorphic to $(\mathbb{C}^*)^{d_i}$ via ϕ_{A_i}. Equivalently, $\mathbb{Z}\mathsf{S}_i = M_i$.

Definition 8.1.1. *A morphism $\phi : \mathcal{Y}_1 \to \mathcal{Y}_2$ of affine toric varieties is* toric *if $\phi(T_1) \subseteq T_2$ and $\phi_{|T_1} : T_1 \to T_2$ is a group homomorphism.*

Our first example is a map from a surface to a curve.

Example 8.1.2. Let $A_1 = \left(\begin{smallmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{smallmatrix}\right)$ and $A_2 = \begin{pmatrix} 2 & 3 \end{pmatrix}$. The corresponding affine toric varieties are the cone $\mathcal{Y}_1 \simeq Y_{A_1} = \{(x, y, z) \in \mathbb{C}^3 : xz = y^2\}$ and the cuspidal cubic $\mathcal{Y}_2 \simeq Y_{A_2} = \{(v, w) \in \mathbb{C}^2 : v^3 = w^2\}$. We define a morphism $\phi : Y_{A_1} \to Y_{A_2}$ given by $\phi(x, y, z) = (x^2, x^3)$. This morphism is toric: it restricts nicely to the dense torus of Y_{A_1}, i.e., to the image $\operatorname{im}\phi_{A_1} = T_1$ of the monomial parametrization (Proposition 1.2.21 and Lemma 4.1.1). For $(x,y,z),(x',y',z') \in T_1$, we have

$$\phi(xx', yy', zz') \;=\; \phi((x,y,z)\cdot(x',y',z')) = \phi(x,y,z)\cdot\phi(x',y',z').$$

Notice also that ϕ is *equivariant* with respect to the action of $(\mathbb{C}^*)^2 \simeq T_1$ on Y_{A_1}: for $t \in (\mathbb{C}^*)^2$, we have $\phi(t\cdot(x,y,z)) = \phi(t_1^{-1}t_2x, t_2y, t_1t_2z) = \phi(\phi_{A_1}(t))\cdot\phi(x,y,z)$. ◇

Exercise 8.1.3. Check that the following three maps are other examples of toric morphisms $\phi : Y_{A_1} :\to Y_{A_2}$ with $Y_{A_i} \simeq \mathcal{Y}_i$ as in Example (8.1.2):

$$\phi(x,y,z) \;=\; (z^2, z^3), \quad \phi(x,y,z) \;=\; (xz, y^3), \quad \phi(x,y,z) \;=\; (xy^2z^3, y^6z^3). \qquad (8.1.1)$$

A toric morphism in the opposite direction $\phi : Y_{A_2} \to Y_{A_1}$ is $\phi(v,w) = (v,v,v)$. Check that $\phi : Y_{A_1} \to Y_{A_2}$ given by $(x,y,z) \mapsto ((x-1)^2, (x-1)^3)$ is *not* a toric morphism.

At the level of algebras, a morphism $\phi : \mathcal{Y}_1 \to \mathcal{Y}_2$ is given by its pullback $\phi^* : \mathbb{C}[\mathsf{S}_2] \to \mathbb{C}[\mathsf{S}_1]$. For reasons that will become clear later, we are interested in maps $\mathbb{C}[\mathsf{S}_2] \to \mathbb{C}[\mathsf{S}_1]$ which are induced by a semigroup homomorphism $\mathsf{S}_2 \to \mathsf{S}_1$.

Lemma 8.1.4. *A semigroup homomorphism $\varphi : \mathsf{S}_2 \to \mathsf{S}_1$ induces a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[\mathsf{S}_2] \to \mathbb{C}[\mathsf{S}_1]$ given by $s^m \mapsto t^{\varphi(m)}$.*

Exercise 8.1.5. Prove Lemma 8.1.4.

Example 8.1.6. Let A_1, A_2 be as in Example 8.1.2 and consider the semigroup homomorphism $\varphi : \mathbb{N}A_2 \to \mathbb{N}A_1$ given by $2 \mapsto (-2,2), 3 \mapsto (-3,3)$. This induces the $\mathbb{C}$-algebra homomorphism $\mathbb{C}[s^2,s^3] \to \mathbb{C}[t_1^{-1}t_2, t_2, t_1t_2]$ given by $s^2 \mapsto t_1^{-2}t_2^2, s^3 \mapsto t_1^{-3}t_2^3$. ◇

Maps obtained as in Lemma 8.1.4 lead to toric morphisms $\phi : \mathcal{Y}_1 \to \mathcal{Y}_2$.

Proposition 8.1.7. *Let $\phi : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism. The following are equivalent:*

1. *$\phi^* : \mathbb{C}[\mathsf{S}_2] \to \mathbb{C}[\mathsf{S}_1]$ is induced by a semigroup homomorphism, as in Lemma 8.1.4.*

2. *The morphism ϕ is toric.*

Morphisms satisfying these two equivalent conditions are called *toric morphisms.*

Exercise 8.1.8. Find the semigroup homomorphisms which induce the toric morphisms from Exercise 8.1.3.

We need an easy lemma to prove Proposition 8.1.7. Below, we write $F \in \mathbb{Z}^{d_2\times d_1}$ for an integer matrix, representing a group homomorphism $N_1 \to N_2$ between the dual lattices.

Lemma 8.1.9. *Assume that $\mathbb{Z}\mathsf{S}_i = M_i, i = 1,2$. A semigroup homomorphism $\varphi : \mathsf{S}_2 \to \mathsf{S}_1$ extends uniquely to a group homomorphism $F^\top : M_2 \to M_1$.*

Proof. For any $m_2 \in M_2$, let $m_2 = A_2 v \in M_2$ for some $v \in \mathbb{Z}^{n_2}$. We can find such a v because of our assumption $\mathbb{Z}\mathsf{S}_2 = M_2$. Write $v = v^+ - v^-$, where $v^+, v^- \in \mathbb{N}^{d_2}$ are the smallest nonnegative integer vectors (entrywise) for which this identity holds. We define $F^\top m_2 = \varphi(A_2 v^+) - \varphi(A_2 v^-) \in M_1$. We need to show that this is independent of the choice of v. If $A_2 v = A_2 w$, then $A_2(v^+ + w^-) = A_2(v^- + w^+)$. Hence, we have

$$\varphi(A_2v^+) - \varphi(A_2v^-) - \varphi(A_2w^+) + \varphi(A_2w^-) \;=\; \varphi(A_2(v^+ + w^-)) - \varphi(A_2(v^- + w^+)) \;=\; 0.$$

To group the terms in the first "=", we used that φ is a semigroup homomorphism. □

Example 8.1.10. The semigroup homomorphism φ in Example 8.1.6 extends to a group homomorphism $F^\top : \mathbb{Z} \to \mathbb{Z}^2$, where F is the matrix $\begin{pmatrix} -1 & 1 \end{pmatrix}$. ⋄

We will write $u_1^\top, \ldots, u_{d_2}^\top$ for the rows of the matrix $F \in \mathbb{Z}^{d_2 \times d_1}$. The transposed matrix $F^\top : M_2 \to M_1$ induces a monomial map like ϕ_A from Chapter 1:

$$\phi_{F^\top} : (\mathbb{C}^*)^{d_1} \longrightarrow (\mathbb{C}^*)^{d_2}, \quad t \longmapsto (t^{u_1}, \ldots, t^{u_{d_2}}). \tag{8.1.2}$$

Proof of Proposition 8.1.7. We start with the implication 1 ⇒ 2. Consider the following diagrams of semigroups, algebras and affine varieties:

$$\begin{tikzcd} \mathsf{S}_2 \arrow[r, "\varphi"] \arrow[d, hook] & \mathsf{S}_1 \arrow[d, hook] \\ M_2 \arrow[r, "F^\top"] & M_1 \end{tikzcd} \longrightarrow \begin{tikzcd} \mathbb{C}[\mathsf{S}_2] \arrow[r, "\phi^*"] \arrow[d, hook] & \mathbb{C}[\mathsf{S}_1] \arrow[d, hook] \\ \mathbb{C}[M_2] \arrow[r, "\phi^*_{F^\top}"] & \mathbb{C}[M_1] \end{tikzcd} \xrightarrow{\text{Specm}} \begin{tikzcd} \mathcal{Y}_2 & \mathcal{Y}_1 \arrow[l, "\phi"'] \\ (\mathbb{C}^*)^{d_2} \arrow[u, hook] & (\mathbb{C}^*)^{d_1} \arrow[l, "\phi_{F^\top}"] \arrow[u, hook] \end{tikzcd}$$

The map $\varphi : \mathsf{S}_2 \to \mathsf{S}_1$ on the left induces $\phi^* : \mathbb{C}[\mathsf{S}_2] \to \mathbb{C}[\mathsf{S}_1]$ in the middle following Lemma 8.1.4. It extends to $F^\top : M_2 \to M_1$ by Lemma 8.1.9. The bottom horizontal map in the middle diagram sends $s^m \mapsto t^{F^\top m}$, so its restriction to $\mathbb{C}[\mathsf{S}_2]$ is ϕ^*. Taking Specm we see that ϕ restricts to the group homomorphism (8.1.2). For 2 ⇒ 1, we can use the same diagrams, but start on the right. If ϕ restricts to a group homomorphism $T_1 \to T_2$, then in coordinates this restriction is $\phi_{F^\top}$ for some matrix $F^\top$ (Proposition 1.2.17). Passing to coordinate rings, $\phi^*_{F^\top}$ is $s^m \mapsto t^{F^\top m}$, hence $F^\top m_2 \in \mathsf{S}_1$ for any $m_2 \in M_2$. The restriction of $F^\top$ to S_2 gives the desired semigroup homomorphism φ. □

Example 8.1.11. For Example 8.1.2, the map $\phi_{F^\top} : (\mathbb{C}^*)^2 \to \mathbb{C}^*, (t_1, t_2) \mapsto t_1^{-1} t_2$ makes

$$\begin{tikzcd} Y_{A_2} & Y_{A_1} \arrow[l, "\phi"'] \\ (\mathbb{C}^*)^{d_2} \arrow[u, hook, "\phi_{A_2}"] & (\mathbb{C}^*)^{d_1} \arrow[l, "\phi_{F^\top}"] \arrow[u, hook, "\phi_{A_1}"] \end{tikzcd}$$

commute. This is the rightmost diagram in the proof of Proposition 8.1.7. ⋄

Proposition 8.1.12. *A toric morphism $\phi : \mathcal{Y}_1 \to \mathcal{Y}_2$ is* equivariant *with respect to the torus action, meaning that $\phi(t \cdot p) = \phi(t) \cdot \phi(p)$ for $t \in T_1$, $p \in \mathcal{Y}_1$.*

Proof. The statement comes down to the fact that the diagram

$$\begin{array}{ccc} T_1 \times \mathcal{Y}_1 & \longrightarrow & \mathcal{Y}_1 \\ \Big\downarrow{\scriptstyle \phi_{|T_1} \times \phi} & & \Big\downarrow{\scriptstyle \phi} \\ T_2 \times \mathcal{Y}_2 & \longrightarrow & \mathcal{Y}_2 \end{array} \text{, which restricts to} \quad \begin{array}{ccc} T_1 \times T_1 & \longrightarrow & T_1 \\ \Big\downarrow{\scriptstyle \phi_{|T_1} \times \phi_{|T_1}} & & \Big\downarrow{\scriptstyle \phi_{|T_1}} \\ T_2 \times T_2 & \longrightarrow & T_2 \end{array},$$

commutes. The commutation of the right diagram follows from the fact that $\phi_{|T_1}$ is a group homomorphism. The left diagram gives two maps $T_1 \times \mathcal{Y}_1 \to \mathcal{Y}_2$. They agree on the dense open subset $T_1 \times T_1$, hence they agree on $T_1 \times \mathcal{Y}_1$. □

If $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are normal affine toric varieties, then $\mathcal{Y}_i = \mathcal{Y}_{\sigma_i}$ for two pointed cones $\sigma_i \subset (N_i)_\mathbb{R} = \mathbb{R}^{d_i}$, $i = 1, 2$, see Section 2.3. The semigroups S_i are saturated in $M_i = \mathbb{Z}^{d_i}$ and equal to $\mathsf{S}_i = \sigma_i^\vee \cap M_i$.

Proposition 8.1.13. *Let $F \in \mathbb{Z}^{d_2 \times d_1}$ be an integer matrix and let $F_\mathbb{R} : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ be the $\mathbb{R}$-linear map represented by F. The monomial map $\phi_{F^\top} : (\mathbb{C}^*)^{d_1} \to (\mathbb{C}^*)^{d_2}$ extends to a toric morphism $\phi : \mathcal{Y}_{\sigma_1} \to \mathcal{Y}_{\sigma_2}$ if and only if $F_\mathbb{R}(\sigma_1) \subseteq \sigma_2$.*

Proof. By Proposition 8.1.7, $\phi_{F^\top}$ is the restriction of a toric morphism if and only if $F^\top : \mathbb{Z}^{d_2} \to \mathbb{Z}^{d_1}$ restricts to a semigroup homomorphism $\mathsf{S}_2 \to \mathsf{S}_1$. Since $\mathsf{S}_i = \sigma_i^\vee \cap M_i$, the proposition follows from the following chain of equivalences:

$$\begin{aligned} & F^\top m_2 \in \sigma_1^\vee \cap M_1 \quad \text{for all } m_2 \in \sigma_2^\vee \cap M_2 \\ \iff & \langle u_1, F^\top m_2 \rangle \geq 0 \quad \text{for all } u_1 \in \sigma_1,\ m_2 \in \sigma_2 \\ \iff & \langle F u_1, m_2 \rangle \geq 0 \quad \text{for all } u_1 \in \sigma_1,\ m_2 \in \sigma_2 \\ \iff & F u_1 \in (\sigma_2^\vee)^\vee = \sigma_2 \quad \text{for all } u_1 \in \sigma_1. \end{aligned}$$

□

Example 8.1.14. If $N_1 = N_2 = N = \mathbb{Z}^d$ and $\tau \preceq \sigma$, with τ and σ two pointed cones, then $F = \mathrm{id}_N$ represents the identity map on $(\mathbb{C}^*)^d$. This extends to the inclusion map $\mathcal{Y}_\tau \subseteq \mathcal{Y}_\sigma$ of the affine open subset $\mathcal{Y}_\tau$ of $\mathcal{Y}_\sigma$, see Corollary 7.2.3. That inclusion morphism is toric, since $F_\mathbb{R}(\tau) \subseteq \sigma$. ◇

8.2 Toric morphisms between normal toric varieties

Proposition 8.1.13 relates toric morphisms between normal affine toric varieties to linear maps between cones. This section generalizes that result: we construct toric morphisms between abstract normal toric varieties from linear maps of fans. Throughout the section, Σ_i is a fan in $(N_i)_\mathbb{R} = \mathbb{R}^{d_i}$ for $i = 1, 2$. The corresponding toric varieties are $\mathcal{X}_{\Sigma_1}$ and $\mathcal{X}_{\sigma_2}$. They have dimension d_1 and d_2 respectively, and their dense tori are $T_i \simeq (\mathbb{C}^*)^{d_i}$, $i = 1, 2$. We generalize Definition 8.1.1 as follows.

Definition 8.2.1. *A morphism $\phi : \mathcal{X}_{\Sigma_1} \to \mathcal{X}_{\Sigma_2}$ is* toric *if $\phi(T_1) \subseteq T_2$ and $\phi_{|T_1} : T_1 \to T_2$ is a group homomorphism.*

Here is a first easy observation.

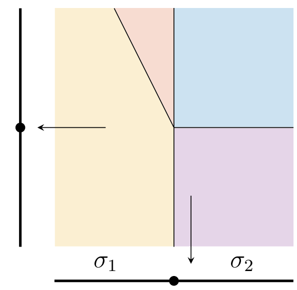

Figure 8.1: A compatible and a non-compatible projection of the fan of a Hirzebruch surface.

Lemma 8.2.2. *Toric morphisms are equivariant: $\phi(t \cdot p) = \phi(t) \cdot \phi(p)$.*

Proof. The proof is identical to that of Proposition 8.1.12. $\square$

We now identify toric morphisms from integer matrices. It turns out that the right way to generalize the condition $F_{\mathbb{R}}(\sigma_1) \subseteq \sigma_2$ from Proposition 8.1.13 is the following.

Definition 8.2.3. *A group homomorphism $N_1 \to N_2$ given by $F \in \mathbb{Z}^{d_1 \times d_2}$ is* compatible *with Σ_1 and Σ_2 if for each cone $\sigma_1 \in \Sigma_1$, there exists $\sigma_2 \in \Sigma_2$ such that $F_{\mathbb{R}}(\sigma_1) \subseteq \sigma_2$.*

Example 8.2.4. Consider the fan Σ_1 in $\mathbb{R}^2$ shown in Figure 8.1 and the complete fan Σ_2 in $\mathbb{R}$. The variety $\mathcal{X}_{\Sigma_1}$ is known as the *Hirzebruch surface* $\mathscr{H}_2$, and $\mathcal{X}_{\Sigma_2}$ is $\mathbb{P}^1$ by Example 7.4.2. The coordinate projection $F(u_1, u_2) = u_1$ is compatible with Σ_1 and Σ_2. The orange and the yellow cone of Σ_1, as well as all their faces, are mapped into σ_1 by $F_{\mathbb{R}}$. The blue and purple cones are mapped into σ_2. The other coordinate projection $F(u_1, u_2) = u_2$ is not compatible with Σ_1, Σ_2, as the image of the yellow cone is not contained in any of the cones of Σ_2. $\diamond$

Theorem 8.2.5. *Let Σ_i be a fan in $(N_i)_{\mathbb{R}}, i = 1, 2$ and let $\mathcal{X}_{\Sigma_1}, \mathcal{X}_{\Sigma_2}$ be the corresponding toric varieties with dense tori T_1 and T_2 respectively.*

1. *If $F : N_1 \to N_2$ is a $\mathbb{Z}$-linear map, compatible with Σ_1, Σ_2, then there is a toric morphism $\phi : \mathcal{X}_{\Sigma_1} \to \mathcal{X}_{\Sigma_2}$ satisfying $\phi_{|T_1} = \phi_{F^\top} : T_1 \to T_2$.*

2. *If $\phi : \mathcal{X}_{\Sigma_1} \to \mathcal{X}_{\Sigma_2}$ is a toric morphism, then ϕ induces a $\mathbb{Z}$-linear map $F : N_1 \to N_2$, compatible with Σ_1, Σ_2, such that $\phi_{|T_1} = \phi_{F^\top} : T_1 \to T_2$.*

Proof. 1. To show point 1, we argue that $\phi_{F^\top}$ extends to a toric morphism on each of the affine charts $\{U_{\sigma_1}\}_{\sigma_1 \in \Sigma_1}$, and these morphisms agree on overlaps. Here, the affine open subsets $U_{\sigma_1} \simeq \mathcal{Y}_{\sigma_1}$ are those used in the gluing construction from Section 7.4. Fix any cone $\sigma_1 \in \Sigma_1$. By compatibility, there is a cone $\sigma_2 \in \Sigma_2$ such that $F_{\mathbb{R}}(\sigma_1) \subseteq \sigma_2$. By Proposition 8.1.13, $\phi_{F^\top}$ extends to a morphism $\phi_{\sigma_1} : U_{\sigma_1} \to U_{\sigma_2} \subseteq \mathcal{X}_{\Sigma_2}$. The morphism ϕ_{σ_1} is induced by the map of semigroups $\varphi : m \mapsto F^\top m$, which is the restriction of $F^\top : M_2 \to M_1$ to the semigroup $\mathsf{S}_2 = \sigma_2^\vee \cap M_2$. We now see that the maps $\{\phi_{\sigma_1}\}_{\sigma_1 \in \Sigma_1}$

agree on overlaps. They glue to a morphism $\phi : \mathcal{X}_{\Sigma_1} \to \mathcal{X}_{\Sigma_2}$. That morphism is toric since $\phi_{\{0\}} : T_1 \to T_2$ is the group homomorphism $\phi_{F^\top}$.

2. By definition, $\phi_{|T_1} : T_1 \to T_2$ is a group homomorphism. It corresponds to an integer matrix $F \in \mathbb{Z}^{d_1 \times d_2}$ by Corollary 1.2.6. Since ϕ is equivariant, it sends the orbit $O(\sigma_1)$ into an orbit $O(\sigma_2)$, where $\sigma_i \in \Sigma_i$. By Proposition 8.1.13, to show that $F_\mathbb{R}(\sigma_1) \subseteq \sigma_2$, it suffices to show that $\phi(U_{\sigma_1}) \subseteq U_{\sigma_2}$. By Exercise 7.2.7, we have $U_{\sigma_i} = \bigsqcup_{\tau_i \preceq \sigma_i} O(\tau_i), i = 1, 2$. We need to show that for each $\tau_1 \preceq \sigma_1$, there is a face $\tau_2 \preceq \sigma_2$ such that $\phi(O(\tau_1)) \subseteq O(\tau_2)$. Let τ_2 be such that $\phi(O(\tau_1)) \subseteq O(\tau_2)$. By Exercise 4.3.5, $O(\sigma_1) \subseteq \overline{O(\tau_1)}$. Since ϕ is continuous, we have $\phi(\overline{O(\tau_1)}) \subset \overline{O(\tau_2)}$. We conclude that $\phi(O(\sigma_1)) \subseteq O(\sigma_2) \cap \overline{O(\tau_2)}$. We know that $\overline{O(\tau_2)}$ is a union of torus orbits (see Corollary 4.1.7 and Theorem 4.3.2), hence we must have $O(\sigma_2) \subset \overline{O(\tau_2)}$, which implies $\tau_2 \preceq \sigma_2$. $\square$

Example 8.2.6. Let $N_i = \mathbb{Z}^2, i = 1, 2$ and $F : u \mapsto \ell u$ for $\ell \in \mathbb{Z}_{>0}$. This is compatible with $\Sigma = \Sigma_1 = \Sigma_2$, where Σ is the fan in Figure 7.5. The matrix F is $\left(\begin{smallmatrix} \ell & 0 \\ 0 & \ell \end{smallmatrix}\right)$, so that $\phi_{|(\mathbb{C}^*)^2}$ is $(t_1, t_2) \mapsto (t_1^\ell, t_2^\ell)$. Globally, ϕ is given by $\phi((x_0 : x_1 : x_2)) = (x_0^\ell : x_1^\ell : x_2^\ell)$. $\diamond$

Exercise 8.2.7. Let $\phi : \mathcal{X}_{\Sigma_1} \to \mathbb{P}^1$ be the toric morphism corresonding to the compatible coordinate projection from Example 8.2.4. Embed $\mathcal{X}_{\Sigma_1}$ in projective space and compute the map ϕ in homogeneous coordinates.

Proposition 8.2.8. *Let $F : N_1 \to N_2$ be a $\mathbb{Z}$-linear map, compatible with two fans Σ_1, Σ_2 in $(N_1)_\mathbb{R}$ and $(N_2)_\mathbb{R}$ respectively. Let $\phi : \mathcal{X}_{\Sigma_1} \to \mathcal{X}_{\Sigma_2}$ be the corresponding toric morphism. For each cone $\sigma_1 \in \Sigma_1$, let $\sigma_2 \in \Sigma_2$ be the smallest cone such that $F_\mathbb{R}(\sigma_1) \subseteq \sigma_2$. We have*

$$\phi(O(\sigma_1)) \subseteq O(\sigma_2) \quad \text{and} \quad \phi(U_{\sigma_1}) \subseteq U_{\sigma_2}.$$

Moreover, if F has rank $d_1 = \mathrm{rank}(N_1)$, then $\phi(O(\sigma_1)) = O(\sigma_2)$.

Proof. The fact that $\phi(U_{\sigma_1}) \subseteq U_{\sigma_2}$ follows directly from the proof of Theorem 8.2.5. The first claim, that $\phi(O(\sigma_1)) \subseteq O(\sigma_2)$, is seen as follows. By Proposition 4.4.4, we have

$$\phi(\gamma_{\sigma_1}) = \phi\left(\lim_{t\to 0} \lambda_u(t)\right) = \lim_{t \to 0} \phi(\lambda_u(t)) = \lim_{t\to 0} \lambda_{Fu}(t) = \gamma_{\sigma_2}.$$

Here γ_{σ_2} is the distinguished point of $O(\sigma_2)$, and we used the fact that Fu lies in the relative interior of σ_2. The proposition now follows from equivariance. $\square$

Notice that $\phi(U_{\sigma_1}) \subseteq U_{\sigma_2}$ holds under the weaker assumption that $F_\mathbb{R}(\sigma_1) \subseteq \sigma_2$ (σ_2 need not be minimal with respect to this property).

Example 8.2.9. The fan Σ_1 in Figure 7.4(b) corresponds to the toric surface $\mathcal{X}_{\Sigma_1} = \mathrm{Bl}_0\mathbb{C}^2$ (Exercise 7.4.6). The identity matrix $F = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ gives a compatible map for Σ_1 and Σ_2, where Σ_2 is the positive quadrant and all its faces (Figure 7.4(a)). The origin in $\mathbb{C}^2 = \mathcal{X}_{\Sigma_2}$ is the torus fixed point corresponding to the unique 2-dimensional cone $\mathbb{R}^2_{\geq 0} \in \Sigma_2$. This is the smallest cone of Σ_2 containing $F_\mathbb{R}(\sigma_1)$, $F_\mathbb{R}(\sigma_1 \cap \sigma_2)$ and $F_\mathbb{R}(\sigma_2)$ (the labeling of cones is as in Figure 7.4(b)). Hence, by Proposition 8.2.8, the union of orbits $O(\sigma_1) \sqcup O(\sigma_1 \cap \sigma_2) \sqcup O(\sigma_2)$ gets sent to the origin. This union is the exceptional divisor in $\mathrm{Bl}_0\mathbb{C}^2$, and the toric morphism ϕ is the blow-down morphism. $\diamond$

Further Reading

Toric morphisms will be useful in our discussion on the Cox ring of a toric variety in Chapter 10. Section 3.3 of the book by Cox, Little and Schenck [19] contains many examples and applications of toric morphisms.

Chapter 9

Divisors on a toric variety

Divisors on an algebraic variety are formal sums of irreducible codimension-one subvarieties. They conveniently encode (possibly reducible and/or non-reduced) hypersurfaces. The theory works best for normal varieties. We start with some general background before presenting the basics of divisors on normal toric varieties.

9.1 Background on divisors

Let X be an irreducible variety. A *prime divisor* on X is an irreducible subvariety $D \subset X$ of codimension one. Let $\mathrm{Div}(X)$ be the free abelian group generated by the prime divisors on X. A *Weil divisor* is an element $E = \sum_{D \subset X} a_D D$ of $\mathrm{Div}(X)$, where the sum is over all prime divisors D on X. Only finitely many integer coefficients a_D are allowed to be nonzero. A Weil divisor E is *effective* if all coefficients are nonnegative. In this case we write $E \geq 0$. The *support* of E is the subvariety $\mathrm{Supp}(E) = \bigcup_{a_D \neq 0} D \subset X$. Prime divisors are Weil divisors with only one non-zero coefficient, and that coefficient is one. Some Weil divisors represent the loci of zeros and poles of a rational function on X. We explain how this works in the case where X is a normal variety.

We start with the affine case: $X = \mathrm{Specm}(R)$, where R is a normal integral domain. The field of rational functions $\mathbb{C}(X)$ is the fraction field of R. There is a correspondence

$$\{ \text{ prime divisors of } X \} \xleftrightarrow{1:1} \{ \text{ codimension one prime ideals of } R \}.$$

Let $\mathfrak{p}$ be the prime ideal of the prime divisor $D \subset X$, i.e., $D = V_X(\mathfrak{p})$. The local ring $R_\mathfrak{p}$ with maximal ideal $\mathfrak{p}R_\mathfrak{p}$ is a subring of $\mathbb{C}(X)$. Its explicit definition is

$$R_\mathfrak{p} = \left\{ \frac{g}{h} \in \mathbb{C}(X) \,:\, g, h \in R,\, h \notin \mathfrak{p} \right\} \subset \mathbb{C}(X).$$

The following is a consequence of Proposition 9.2 in [2].

Lemma 9.1.1. *Let R be a normal integral domain. For any codimension one prime ideal $\mathfrak{p} \subset R$, we have that all ideals of $R_\mathfrak{p}$ are principal and of the form $\langle \pi^k \rangle$ for some $\pi \in R_\mathfrak{p}$ and some $k \in \mathbb{N}$. In particular, the unique maximal ideal of $R_\mathfrak{p}$ is $\mathfrak{p}R_\mathfrak{p} = \langle \pi \rangle$.*

The nonzero elements of $\mathbb{C}(X)$ are denoted by $\mathbb{C}(X)^* = \mathbb{C}(X) \setminus \{0\}$. Notice that for any $f \in \mathbb{C}(X)^*$, we can write $f = \frac{a}{b} = ab^{-1}$, where $a, b \in R_{\mathfrak{p}}$.

Definition 9.1.2. *Let R be a normal integral domain, $X = \mathrm{Specm}(R)$ the corresponding affine variety and $\mathfrak{p} \subset R$ a codimension one prime ideal with prime divisor $D = V_X(\mathfrak{p})$. The* order of vanishing *of $f \in R_{\mathfrak{p}} \setminus \{0\}$ along D is*

$$\nu_D(f) \;=\; \max\{k \in \mathbb{N} \,:\, f \in (\mathfrak{p}R_{\mathfrak{p}})^k\}. \tag{9.1.1}$$

We extend this to $f \in \mathbb{C}(X)^*$ by setting $\nu_D(ab^{-1}) = \nu_D(a) - \nu_D(b)$.

By Lemma 9.1.1, if $f \in R_{\mathfrak{p}} \setminus \{0\}$ then $\nu_D(f)$ is the unique integer for which $\langle f \rangle = \langle \pi^{\nu_D(f)} \rangle$ as ideals in $R_{\mathfrak{p}}$. Here are some elementary properties of ν_D.

Proposition 9.1.3. *The map $\nu_D : \mathbb{C}(x)^* \to \mathbb{Z}$ is a surjective group homomorphism. Moreover, if $f, g, f+g \in \mathbb{C}(X)^*$, then we have $\nu_D(f+g) \geq \min(\nu_D(f), \nu_D(g))$.*

Proof. Let π be as in Lemma 9.1.1. One checks that $\nu_D(\pi^k) = k$ for any $k \in \mathbb{Z}$, so that ν_D is indeed surjective. Fix $f, g \in \mathbb{C}(X)^*$ and write $f = ab^{-1}$ and $g = cd^{-1}$, where $a, b, c, d \in R_{\mathfrak{p}}$. By Definition 9.1.2, we have $\nu_D(fg) = \nu_D(ac) - \nu_D(bd)$. It also follows from the definition that $\nu_D(ac) = \nu_D(a) + \nu_D(c)$ for any two elements $a, c \in R_{\mathfrak{p}} \setminus \{0\}$. Hence, we have $\nu_D(fg) = \nu_D(a) + \nu_D(c) - \nu_D(b) - \nu_D(d) = \nu_D(f) + \nu_D(g)$ and ν_D is a group homomorphism. It is also clear from the definition that, if $f, g, f+g \in R_{\mathfrak{p}} \setminus \{0\}$, then $\nu_D(f+g) \geq \min(\nu_D(f), \nu_D(g))$. To see that this holds in general, we calculate

$$\nu_D(ab^{-1} + cd^{-1}) \;=\; \nu((bd)^{-1}) + \nu_D(ad - bc) \;\geq\; \nu_D((bd)^{-1}) + \min(\nu_D(ad), \nu_D(-bc)).$$

Using the fact that ν_D is a group homomorphism and $\nu_D(-1) = 0$, we find that

$$\nu_D(ab^{-1} + cd^{-1}) \;\geq\; \min\left(\nu_D\left(\tfrac{ad}{bd}\right), \nu_D\left(\tfrac{bc}{bd}\right)\right) \;=\; \min(\nu_D(\tfrac{a}{b}), \nu_D(\tfrac{c}{d})). \qquad \square$$

A group homomorphism $\nu : \mathbb{C}(X)^* \to \mathbb{Z}$ satisfying $\nu(f+g) \geq \min(\nu(f), \nu(g))$ is a *discrete valuation* on $\mathbb{C}(X)$. Its *discrete valuation ring* (DVR) is defined as $\{f \in \mathbb{C}(X)^* : \nu(f) \geq 0\} \cup \{0\}$. Proposition 9.1.3 states that ν_D is a discrete valuation on the fraction field $\mathbb{C}(X)$ of the normal domain R, and its discrete valuation ring is $R_{\mathfrak{p}}$.

Example 9.1.4. Let $f = (x - a_1)^{m_1} \cdots (x - a_r)^{m_r} \in R = \mathbb{C}[x]$, where a_i are distinct complex numbers. Each point $\{a_i\}$ is a divisor on $\mathbb{C} = \mathrm{Specm}(R)$. The order of vanishing of f along $\{a_i\}$ is m_i. $\diamond$

Example 9.1.5. The normality assumption in Lemma 9.1.1 and Proposition 9.1.3 cannot be dropped. Consider the (non-normal) integral domain $R = \mathbb{C}[x,y]/\langle x^3 - y^2 \rangle$, whose spectrum is the cuspidal cubic curve. The prime ideal $\mathfrak{p} = \langle x, y \rangle \subset R$ corresponds to the singular point on that curve. This ideal cannot be generated by a single element. We have $y \in \mathfrak{p}R_{\mathfrak{p}}$ and $y^2 \in (\mathfrak{p}R_{\mathfrak{p}})^3$, because of the relation $x^3 = y^2$. Hence, defining ν_D as in (9.1.1), we would have $2\nu_D(y) \neq \nu_D(y^2)$ and ν_D fails to be a group homomorphism. $\diamond$

We switch to the more general case where X is a normal irreducible (abstract) variety. Recall that the field of rational functions $\mathbb{C}(X)$ on X consists of regular functions $f : U \to \mathbb{C}$ on a nonempty Zariski open subset $U \subset X$ modulo the following equivalence relation. Two functions $f : U \to \mathbb{C}$ and $g : U' \to \mathbb{C}$ are equivalent if they agree on a non-empty Zariski open subset contained in $U \cap U'$. To a rational function $f \in \mathbb{C}(X)^*$, we want to associate a $\mathbb{Z}$-linear combination of prime divisors $\sum a_i D_i$ encoding the *order of vanishing of* f *along* D_i.

For a prime divisor $D \subset X$, we define

$$\mathcal{O}_{X,D} = \{f \in \mathbb{C}(X) \mid f \text{ is defined on } U, \text{ with } U \cap D \neq \emptyset\}.$$

These are the rational functions defined *somewhere* on D, and therefore *almost everywhere* on D. Since X is irreducible, if $U \subset X$ is open and nonempty, we have $\mathbb{C}(X) = \mathbb{C}(U)$. Moreover, if $U \cap D \neq \emptyset$, $\mathcal{O}_{X,D} = \mathcal{O}_{U,U\cap D}$. In particular, we can pick U to be an affine open subset, so that its divisor $U \cap D$ corresponds to a codimension-one prime ideal in the coordinate ring $R = \mathbb{C}[U]$. Since X is normal, R is a normal domain. Moreover, one checks that $\mathcal{O}_{X,D} = \mathcal{O}_{U,U\cap D} = R_{\mathfrak{p}}$, where $\mathfrak{p}$ is the prime ideal of the prime divisor $U \cap D \subset U$. Proposition 9.1.3 gives a discrete valuation ν_D on $\mathbb{C}(U) = \mathbb{C}(X)$, with discrete valuation ring $R_{\mathfrak{p}} = \mathcal{O}_{X,D}$. We summarize this discussion in a corollary.

Corollary 9.1.6. *Let X be a normal variety and $D \subset X$ a prime divisor. There is a discrete valuation $\nu_D : \mathbb{C}(X)^* \to \mathbb{Z}$ with discrete valuation ring $\mathcal{O}_{X,D}$.*

For a nonzero rational function $f \in \mathbb{C}(X)^*$, we say that f *vanishes with order* $\nu_D(f)$ along D if $\nu_D(f) > 0$, or that f *has a pole of order* $-\nu_D(f)$ along D if $\nu_D(f) < 0$.

If X is normal, then the map ν_D from Corollary 9.1.6 gives a way of associating a Weil divisor to a rational function $f \in \mathbb{C}(X)^*$. We define

$$\mathrm{div}(f) \;=\; \sum_{D\subset X} \nu_D(f) \cdot D.$$

This is a Weil divisor since only finitely many coefficients are nonzero [34, Chapter II, Lemma 6.1]. Weil divisors arising in this way are called *principal divisors*. Since

$$\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g) \quad \text{and} \qquad \mathrm{div}(f^{-1}) = -\mathrm{div}(f),$$

it is clear that they form a subgroup. We denote this group by $\mathrm{PDiv}(X) \subset \mathrm{Div}(X)$.

Example 9.1.7. We continue Example 9.1.4. Viewed as a rational function on $\mathbb{C}$, f gives a principal divisor $\mathrm{div}(f) = \sum_{i=1}^r m_i \cdot \{a_i\} \in \mathrm{PDiv}(\mathbb{C})$. Viewed as a rational function on $\mathbb{P}^1$, we have $\mathrm{div}(f) = \sum_{i=1}^r m_i \cdot \{a_i\} - (\sum_{i=1}^r m_i) \cdot \{\infty\}$. ◇

Exercise 9.1.8. Show that $\mathrm{Div}(\mathbb{C}) = \mathrm{PDiv}(\mathbb{C})$ (all Weil divisors on $\mathbb{C}$ are principal). The same is not true for $\mathbb{P}^1$ (see Example 9.1.9).

Example 9.1.9. On $X = \mathbb{P}^d$, let D_i be the zero locus of the i-th homogeneous coordinate function x_i, $i = 1, \ldots, d+1$. Rational functions on X are fractions of homogeneous polynomials of the same degree. One checks that $D_i - D_j = \mathrm{div}(x_i/x_j)$ is a principal divisor, but $D_i \in \mathrm{Div}(X) \setminus \mathrm{PDiv}(X)$ for any $i, j \in \{1, \ldots, d+1\}$. ◇

In what follows, we will use the terminology *divisor* for Weil divisors. Two divisors $D, E \in \mathrm{Div}(X)$ are said to be *linearly equivalent* if $D - E \in \mathrm{PDiv}(X)$. This equivalence relation gives two important quotient groups.

Definition 9.1.10 (Class and Picard group)**.** *The* divisor class group *of a normal variety X is* $\mathrm{Cl}(X) = \mathrm{Div}(X)/\mathrm{PDiv}(X)$.

Here is an important result on the class group of certain affine varieties which will help us deal with more complicated cases later.

Theorem 9.1.11. *Let R be a unique factorization domain and $X = \mathrm{Specm}(R)$. We have*

(a) R is normal and every codimension 1 prime ideal is principal.

(b) $\mathrm{Cl}(X) = 0$.

Proof. (a) It is a standard exercise in commutative algebra to show that every unique factorization domain is normal. Let $\mathfrak{p}$ be a codimension 1 prime ideal and $f \in \mathfrak{p} \setminus \{0\}$. Then $f = c\prod_{i=1}^s f_i^{a_i}$, where the f_i are prime and c is a unit. Since $\mathfrak{p}$ is prime, $f_i \in \mathfrak{p}$ for some i. This means that $\langle f_i \rangle \subset \mathfrak{p}$, and since $\mathfrak{p}$ is codimension 1, $\mathfrak{p} = \langle f_i \rangle$.

(b) Let D_i be a prime divisor. Its prime ideal $\mathfrak{p}_i \subset R$ is principal by part (a). We write $\mathfrak{p}_i = \langle f_i \rangle$. It follows that if $D = \sum_{i=1}^s a_i D_i$, then $D = \mathrm{div}(\prod_{i=1}^s f_i^{a_i})$. Indeed, $\mathrm{div}(\prod_{i=1}^s f_i^{a_i}) = \sum_{i=1}^s a_i \nu_{D_i}(f_i) D_i$ and $\nu_{D_i}(f_i) = 1$ since f_i generates $\mathfrak{p}_i R_{\mathfrak{p}_i}$. □

Remark 9.1.12. In fact, R is a unique factorization domain if and only if R is normal and $\mathrm{Cl}(X) = 0$, see [34, Chapter II, Proposition 6.2].

Example 9.1.13. The class group of $\mathbb{C}^d$ is 0, and so is that of a torus $(\mathbb{C}^*)^d$. For $\mathbb{C}$, you have proved this in Exercise 9.1.8. ◇

Below we write $[D]$ for the image of $D \in \mathrm{Div}(X)$ in $\mathrm{Cl}(X)$. Our next theorem uses the exercise below.

Exercise 9.1.14. Check that the restriction $\mathrm{Div}(X) \to \mathrm{Div}(U) : D \mapsto D_{|U}$ induces a well-defined map $\mathrm{Cl}(X) \to \mathrm{Cl}(U) : [D] \mapsto [D_{|U}]$.

For an open subset $U \subset X$ and a Weil divisor $D = \sum a_i D_i \in \mathrm{Div}(X)$, we define the *restriction* of D to U as

$$D_{|U} = \Big(\sum a_i D_i\Big)_{|U} = \sum_{D_i \cap U \neq \emptyset} a_i \cdot (D_i \cap U) \in \mathrm{Div}(U).$$

Theorem 9.1.15. *Let X be a normal variety and $U \subset X$ a nonempty open subset. Let $D_1, \ldots, D_s$ be the irreducible components of $X \setminus U$ that are prime divisors. Then*

$$\bigoplus_{i=1}^s \mathbb{Z} \cdot D_i \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0$$

is exact. The maps are $\sum_{i=1}^s a_i D_i \mapsto [\sum_{i=1}^s a_i D_i]$ and $[D] \mapsto [D_{|U}]$ (Exercise 9.1.14).

Proof. Exactness at $\mathrm{Cl}(U)$ follows from the fact that $D' = \sum a_i D_i' \in \mathrm{Div}(U)$ is the restriction of $D = \sum a_i D_i \in \mathrm{Div}(X)$, where $D_i = \overline{D_i'}$ is the Zariski closure of D_i' in X. It remains to show exactness at $\mathrm{Cl}(X)$. Clearly the composition is zero. Suppose $[D_{|U}] = 0$ in $\mathrm{Cl}(U)$. Then $D_{|U}$ is the divisor of some $f \in \mathbb{C}(U)^*$. Since $\mathbb{C}(X) = \mathbb{C}(U)$, f defines a divisor $\mathrm{div}(f) \in \mathrm{Div}(X)$ which restricts to $D_{|U}$. Hence $(D - \mathrm{div}(f))_{|U} = D_{|U} - \mathrm{div}(f)_{|U} = 0$, and $D - \mathrm{div}(f)$ is supported in $\bigcup_{i=1}^s D_i$. This means that D is linearly equivalent to an element $\sum_{i=1}^s a_i D_i \in \bigoplus_{i=1}^s \mathbb{Z} \cdot D_i$, so that $[D] = [\sum_{i=1}^s a_i D_i]$. □

Here is a first example of how to apply Theorems 9.1.11 and 9.1.15.

Example 9.1.16. Let $X = \mathbb{P}^1$ and $U = U_y = \mathbb{P}^1 \setminus D_y \simeq \mathbb{C}$ is the open set from Example 7.1.1. By Theorem 9.1.15, we have the exact sequence

$$\mathbb{Z} \cdot D_y \longrightarrow \mathrm{Cl}(\mathbb{P}^1) \longrightarrow \mathrm{Cl}(\mathbb{C}) \longrightarrow 0.$$

In this case, the first map is injective, since $k \cdot D_y$ is principal if and only if $k = 0$. By Theorem 9.1.11, $\mathrm{Cl}(\mathbb{C}) = 0$, so that $\mathrm{Cl}(\mathbb{P}^1) \simeq \mathbb{Z}$. ◇

Exercise 9.1.17. Use the strategy in Example 9.1.16 to show that $\mathrm{Cl}(\mathbb{P}^d) = \mathbb{Z}$.

We conclude with a lemma that will be useful later. For a proof, see [19, Prop. 4.0.16].

Lemma 9.1.18. *Let X be a normal variety and $f \in \mathbb{C}(X)^*$.*

(a) The divisor $\mathrm{div}(f)$ *is effective if and only if $f : X \to \mathbb{C}$ is a morphism.*

(b) We have $\mathrm{div}(f) = 0$ *if and only if $f : X \to \mathbb{C}^*$ is a morphism.*

Point (a) reads "a rational function f has no poles if and only if f is a regular function", and point (b) reads "f has no zeros and no poles if and only if f is a regular function whose value is nowhere zero".

9.2 The class group of a normal toric variety

We turn back to the case where $X = \mathcal{X}_\Sigma$ is a normal toric variety coming from a fan Σ in $N_\mathbb{R} \simeq \mathbb{R}^d$. A distinguished set of divisors on $\mathcal{X}_\Sigma$ are those invariant under the action $T \times \mathcal{X}_\Sigma \to \mathcal{X}_\Sigma$ on $\mathcal{X}_\Sigma$. Here we write $T \simeq (\mathbb{C}^*)^d$ for the dense torus of $\mathcal{X}_\Sigma$. By the orbit-cone correspondence (Definition 7.4.13), these torus invariant divisors correspond to rays of Σ. Let $\Sigma(1)$ be the set of rays and $\rho \in \Sigma(1)$. The closure $D_\rho = \overline{\mathcal{O}(\rho)} \subset \mathcal{X}_\Sigma$ is an irreducible codimension-one subvariety, i.e., a prime divisor, which is a union of T-orbits.

The previous section explained how to determine the order of vanishing of a rational function on $\mathcal{X}_\Sigma$. A natural class of rational functions to start with are the characters of T, i.e., the Laurent monomials $t^m, m \in M$. These are regular functions on $T \subset \mathcal{X}_\Sigma$, and hence elements of $\mathbb{C}(\mathcal{X}_\Sigma)$. Since t^m has no zeros or poles on T, the support of $\mathrm{div}(t^m)$ is contained in $\mathcal{X}_\Sigma \setminus T = \bigcup_\rho D_\rho$. We write $\nu_\rho = \nu_{D_\rho}$ for the order of vanishing along D_ρ.

Proposition 9.2.1. *Let u_ρ be the primitive ray generator of $\rho \in \Sigma(1)$. For any $m \in M$, we have $\nu_\rho(t^m) = \langle u_\rho, m\rangle$.*

Proof. Let $e_1 = u_\rho, e_2, \ldots, e_n$ be a basis of the cocharacter lattice N. Note that u_ρ can be extended to such a basis because u_ρ is primitive. The affine toric variety $\mathcal{Y}_\rho$ is isomorphic to

$$\mathbb{C} \times (\mathbb{C}^*)^{n-1} = \mathrm{Specm}(\mathbb{C}[x_1, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]) \tag{9.2.1}$$

(see Example 2.3.4) and $\mathcal{Y}_\rho \cap D_\rho$ is defined by $x_1 = 0$. Therefore $\mathcal{O}_{\mathcal{X}_\Sigma, D_\rho} = \mathcal{O}_{U_\rho, U_\rho \cap D_\rho} = \mathbb{C}[x_1, \ldots, x_n]_{\langle x_1\rangle}$ and $\nu_\rho(f) = k$ where k is such that $f = x_1^k \frac{g}{h}$ for $g, h \in \mathbb{C}[x_1, \ldots, x_n] \setminus \langle x_1\rangle$. Let $m_1, \ldots, m_n$ be the dual basis of M with respect to $e_1, \ldots, e_n$. The variables x_i in (9.2.1) correspond to the characters t^{m_i}. A character $m \in M$ can be written as $\sum_{i=1}^n a_i m_i$, so that $\langle e_i, m\rangle = a_i$. The character $t^m = t^{\sum_{i=1}^n a_i m_i} = \prod_{i=1}^n (t^{m_i})^{a_i}$ is the restriction of $x_1^{a_1} \cdots x_n^{a_n}$ to the torus, so that $\nu_\rho(t^m) = a_1 = \langle e_1, m\rangle = \langle u_\rho, m\rangle$. □

As a consequence, we get the following elegant formula for the divisor of a character.

Corollary 9.2.2. *For $m \in M$, the divisor $\mathrm{div}(t^m)$ is given by $\sum_{\rho\in\Sigma(1)} \langle u_\rho, m\rangle\, D_\rho$.*

This uses the fact that $\mathrm{div}(t^m)_{|T} = 0$, since $t^m : T \to \mathbb{C}^*$ is a morphism (Lemma 9.1.18). Divisors supported on $\bigcup_{\rho\in\Sigma(1)} D_\rho$ are called *torus invariant Weil divisors*. They form a free subgroup $\mathrm{Div}_T(\mathcal{X}_\Sigma) = \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}\cdot D_\rho \subset \mathrm{Div}(\mathcal{X}_\Sigma)$ of finite rank $k = |\Sigma(1)|$.

Exercise 9.2.3. Let $\sigma \in \Sigma$ be any cone of Σ. Show that the restriction $\mathrm{div}(t^m)_{|\mathcal{Y}_\sigma}$ equals $\sum_{\rho\subseteq\sigma} \langle u_\rho, m\rangle\, D_\rho \in \mathrm{Div}_T(\mathcal{Y}_\sigma)$.

Theorem 9.2.4. *Let $M \to \mathrm{Div}_T(\mathcal{X}_\Sigma)$ be the map that sends $m \mapsto \mathrm{div}(t^m)$ and $\mathrm{Div}_T(\mathcal{X}_\Sigma) \to \mathrm{Cl}(\mathcal{X}_\Sigma)$ sends a divisor to its class. There is an exact sequence*

$$M \longrightarrow \mathrm{Div}_T(\mathcal{X}_\Sigma) \longrightarrow \mathrm{Cl}(\mathcal{X}_\Sigma) \longrightarrow 0. \tag{9.2.2}$$

Furthermore, this extends to a short exact sequence

$$0 \longrightarrow M \longrightarrow \mathrm{Div}_T(\mathcal{X}_\Sigma) \longrightarrow \mathrm{Cl}(\mathcal{X}_\Sigma) \longrightarrow 0 \tag{9.2.3}$$

*if and only if the rays of Σ span $N_\mathbb{R}$, i.e., $\mathcal{X}_\Sigma$ has no torus factor (Theorem **??**).*

Proof. Theorem 9.1.15 gives an exact sequence

$$\mathrm{Div}_T(\mathcal{X}_\Sigma) \longrightarrow \mathrm{Cl}(\mathcal{X}_\Sigma) \longrightarrow \mathrm{Cl}(T) \longrightarrow 0.$$

By Theorem 9.1.11, $\mathrm{Cl}(T) = 0$, which proves exactness at $\mathrm{Cl}(\mathcal{X}_\Sigma)$ in (9.2.2). We now show exactness at $\mathrm{Div}_T(\mathcal{X}_\Sigma)$. The composition is clearly zero, since divisors of characters are principal. Suppose $[D] = 0$ for some $D \in \mathrm{Div}_T(\mathcal{X}_\Sigma)$. Then $D = \mathrm{div}(f)$ and $\mathrm{div}(f)_{|T} = 0$. By Lemma 9.1.18, $f : T \to \mathbb{C}^*$ is a morphism, and hence $f = c\,t^m$ for some $m \in M$ and some $c \in \mathbb{C}^*$ (we used this in the proof of Proposition 1.2.5). We conclude that $D = \mathrm{div}(f) = \mathrm{div}(c\,t^m) = \mathrm{div}(t^m)$. Exactness of (9.2.3) at M if and only if $\{u_\rho\}_{\rho\in\Sigma(1)}$ span $N_\mathbb{R}$ is an easy exercise. □

In [60]:
```
1 Cl = class_group(X_Σ)
```

Out[60]: Abelian Group with Invariants: Z^2

Figure 9.1: Computing class groups in `Oscar.jl`.

An important consequence of Theorem 9.2.4 is that, when $\mathcal{X}_\Sigma$ has no torus factors, $\mathrm{Cl}(\mathcal{X}_\Sigma)$ can be computed using elementary linear algebra over $\mathbb{Z}$. After fixing coordinates, we may assume $M = N = \mathbb{Z}^d$. Let $F = [u_1 \ \cdots \ u_k] \in \mathbb{Z}^{d\times k}$ be the matrix whose columns are the primitive ray generators of the rays in $\Sigma(1)$. The corresponding torus invariant divisors are denoted by $D_1, \ldots, D_k$. By Corollary 9.2.2, the map $M \to \mathrm{Div}_T(\mathcal{X}_\Sigma)$ is $F^\top : \mathbb{Z}^d \to \mathbb{Z}^k$. By Theorem 9.2.4 the class group is given by $\mathrm{Cl}(\mathcal{X}_\Sigma) \simeq \mathbb{Z}^k / \operatorname{im} F^\top$. We will use this notation throughout, and refer to F as the *matrix of ray generators*.

Proposition 9.2.5. *If the rank of $F^\top$ is d, then $\mathrm{Cl}(\mathcal{X}_\Sigma) \simeq \mathbb{Z}^k / \operatorname{im} F^\top$,*

Example 9.2.6. If $\mathcal{X}_\Sigma = \mathbb{P}^2$, the matrix F for the fan of Figure 7.4(e) is

$$F = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

The class group is $\mathbb{Z}\cdot[D_1] + \mathbb{Z}\cdot[D_2] + \mathbb{Z}\cdot[D_3]$ modulo the relations $[D_1] - [D_3] = 0$ (first row of F) and $[D_2] - [D_3] = 0$ (second row of F). It is generated by $[D_3]$, as $[aD_1 + bD_2 + cD_3] = (a+b+c)[D_3]$. We conclude that $\mathrm{Cl}(\mathbb{P}^2) \simeq \mathbb{Z}$. ◇

Exercise 9.2.7. Compute the divisor class group of $\mathbb{P}^d$ for general d.

Example 9.2.8. If $\mathcal{X}_\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$, i.e. Σ is the fan of Figure 7.4(f), we obtain

$$F = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

The class group $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1)$ is isomorphic to $\mathbb{Z}^2$, and it is generated by $[D_1]$ and $[D_3]$, as $[aD_1 + bD_2 + cD_3 + dD_4] = (a+c)[D_1] + (b+d)[D_3]$. ◇

Exercise 9.2.9. Consider the complete fan Σ in $\mathbb{R}^2$ with ray generators given by

$$F = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}.$$

This is the normal fan of a diamond. Show that $\mathrm{Cl}(\mathcal{X}_\Sigma)$ is isomorphic to $\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$.

Example 9.2.10. The group $\mathrm{Cl}(\mathcal{X}_\Sigma)$ can be computed in `Oscar.jl` using the command `class_group`. For $\mathcal{X}_\Sigma$ from Example 7.4.14 we find $\mathrm{Cl}(\mathcal{X}_\Sigma) = \mathbb{Z}^2$, see Figure 9.1. ◇

Remark 9.2.11. Our examples show that $\mathrm{Cl}(\mathcal{X}_\Sigma)$ need not be a free abelian group. However, it follows from [19, Propositions 4.2.5 and 4.2.6] that if Σ is complete and $\mathcal{X}_\Sigma$ is smooth, then $\mathrm{Cl}(\mathcal{X}_\Sigma) \simeq \mathbb{Z}^{k-d}$ has no torsion.

9.3 Divisors and polytopes

The torus invariant Weil divisors $\mathrm{Div}(\mathcal{X}_\Sigma)$ generate the class group $\mathrm{Cl}(\mathcal{X}_\Sigma)$ by Theorem 9.2.4. In this section we associate a polyhedron $\mathcal{P}_D$ to a torus invariant divisor $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$. If Σ is complete, then $\mathcal{P}_D$ is a bounded convex polytope. In that case, it should be thought of as the typical Newton polytope of a Laurent polynomial $f \in \mathbb{C}[M]$ whose divisor $\mathrm{div}(f) \in \mathrm{PDiv}(\mathcal{X}_\Sigma)$ has poles "bounded by D", as explained below.

The definition of $\mathcal{P}_D$, where $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$, is as follows:

$$\mathcal{P}_D \;=\; \{m \in M_\mathbb{R} \,:\, \langle u_\rho, m\rangle + a_\rho \geq 0,\, \rho \in \Sigma(1)\}. \tag{9.3.1}$$

Here, u_ρ is the primitive ray generator of the ray ρ. Note that $\mathcal{P}_D$ is indeed a polyhedron in $M_\mathbb{R} = \mathbb{R}^d$; it is defined by finitely many linear inequalities. Writing $F \in \mathbb{Z}^{d \times k}$ for the matrix whose columns are the primitive vectors u_ρ, like in the previous section, and $a = (a_\rho)_{\rho \in \Sigma(1)} \in \mathbb{Z}^k$, we have the more compact notation

$$\mathcal{P}_D \;=\; \{m \in M_\mathbb{R} \,:\, F^\top m + a \geq 0\}.$$

Lemma 9.3.1. *If $D, E \in \mathrm{Div}_T(\mathcal{X}_\Sigma)$ are linearly equivalent, then $\mathcal{P}_E = \mathcal{P}_D + m = \{m' + m \,:\, m' \in \mathcal{P}_D\}$ for some $m \in M$.*

Proof. Let $D = \sum_\rho a_\rho D_\rho$ and $E = \sum_\rho b_\rho D_\rho$. Linear equivalence means $[D - E] = 0$. By Theorem 9.2.4, there exists $m \in M$ such that $a = b - F^\top m$. Hence

$$\mathcal{P}_D \;=\; \{m' \in M_\mathbb{R} \,:\, F^\top(m' - m) + b \geq 0\}.$$

From this, we see that $m' \in \mathcal{P}_D$ if and only if $m' - m \in \mathcal{P}_E$. Hence, $\mathcal{P}_E = \mathcal{P}_D + m$. □

The opposite implication is false in general.

Exercise 9.3.2. Consider the complete fan in $\mathbb{R}^2$ with ray generators $F = \left(\begin{smallmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{smallmatrix}\right)$. Check that the divisors $D = D_3$ and $E = D_3 + D_4$ are such that $\mathcal{P}_D = \mathcal{P}_E$. Show that D and E are not linearly equivalent.

Exercise 9.3.3. The polyhedron of a divisor is not necessarily a lattice polytope. In general, it might be empty or unbounded. Describe the polyhedra $\mathcal{P}_D, \mathcal{P}_E$ for

$$F = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \end{pmatrix},\; D = D_3,\; E = -D_3.$$

The fan Σ is the normal fan of a triangle with vertices $(0,0), (0,1), (2,0)$.

Now let $D = 0 \in \mathrm{Div}_T(\mathbb{C})$ be the zero element. Check that $\mathcal{P}_D$ is unbounded.

Proposition 9.3.4. *If the rays of Σ generate $N_\mathbb{R}$ as a cone, i.e., $\sum_{\rho \in \Sigma(1)} \mathbb{R}_{\geq 0}\, u_\rho = N_\mathbb{R}$, then $\mathcal{P}_D$ is bounded for all $D \in \mathrm{Div}_T(\mathcal{X}_\Sigma)$.*

Proof. Suppose that $\mathcal{P}_D$ is unbounded. Then there is a sequence $m_i \in \mathcal{P}_D$ such that $\|m_i\| \to \infty$ when $i \to \infty$ and $\frac{m_i}{\|m_i\|}$ converges to a point m^* on the unit sphere in $M_\mathbb{R}$. Clearly, for each $\rho \in \Sigma(1)$ and all i, we have that

$$\left\langle u_\rho, \frac{m_i}{\|m_i\|} \right\rangle + \frac{a_\rho}{\|m_i\|} \geq 0.$$

In particular, $\langle u_\rho, m^* \rangle \geq 0$. But the vectors u_ρ generate $M_\mathbb{R}$ as a cone, so $m^* = 0$, which is a contradiction. □

At the beginning of the section, we hinted at the fact that we want to think of $\mathcal{P}_D$ as a Newton polytope (Definition **??**). To that end, we associate the following vector space of Laurent polynomials to a torus invariant divisor $D \in \mathrm{Div}_T(\mathcal{X}_\Sigma)$:

$$\Gamma(\mathcal{X}_\Sigma, D) \;=\; \bigoplus_{m \in \mathcal{P}_D \cap M} \mathbb{C} \cdot t^m. \tag{9.3.2}$$

The poles of an element in $\Gamma(\mathcal{X}_\Sigma, D)$ are "bounded by D", in the sense that $\mathrm{div}(f) + D$ has no poles. Here is the precise statement.

Proposition 9.3.5. *Let Σ be a fan in $N_\mathbb{R}$ and let $\mathcal{X}_\Sigma$ be the corresponding normal toric variety. For any torus invariant divisor $D \in \mathrm{Div}_T(\mathcal{X}_\Sigma)$, we have*

$$\Gamma(\mathcal{X}_\Sigma, D) \;=\; \{f \in \mathbb{C}(\mathcal{X}_\Sigma)^* \;:\; \mathrm{div}(f) + D \geq 0\}. \tag{9.3.3}$$

Proof. A rational function f belongs to the righthand side in (9.3.3) if and only if $\mathrm{div}(f) + D \geq 0$. Then in particular $(\mathrm{div}(f) + D)_{|T} \geq 0$, which implies by Lemma 9.1.18 that $f \in \mathbb{C}[M]$. Hence, the righthand side is indeed a subspace of $\mathbb{C}[M]$. We will prove the inclusion "$\subseteq$". It suffices to show that each character $t^m, m \in \mathcal{P}_D \cap M$ satisfies $\mathrm{div}(t^m) + D \geq 0$. By Corollary 9.2.2, we have $\mathrm{div}(t^m) = \sum_\rho \langle u_\rho, m\rangle D_\rho$. By definition, the characters for which $\sum_\rho \langle u_\rho, m\rangle D_\rho + D \geq 0$ are precisely the points in $\mathcal{P}_D \cap M$. For the other inclusion, we refer to [19, Proposition 4.3.2] or [30, Page 66]. □

Remark 9.3.6. For the reader who is familiar with sheaves, we point out that the notation $\Gamma(\mathcal{X}_\Sigma, D)$ is motivated by Proposition 9.3.5, which says that it equals the vector space of global sections of the sheaf $\mathcal{O}_{\mathcal{X}_\Sigma}(D)$ on $\mathcal{X}_\Sigma$, usually denoted by $\Gamma(\mathcal{X}_\Sigma, \mathcal{O}_{\mathcal{X}_\Sigma}(D))$. Here $\mathcal{O}_{\mathcal{X}_\Sigma}(D)$ is the structure sheaf of $\mathcal{X}_\Sigma$, twisted by the divisor D.

Exercise 9.3.7. Show that, if $D, E \in \mathrm{Div}_T(\mathcal{X}_\Sigma)$ and D is linearly equivalent to E, then $\Gamma(\mathcal{X}_\Sigma, D) \simeq \Gamma(\mathcal{X}_\Sigma, E)$.

A different way of thinking about $\Gamma(\mathcal{X}_\Sigma, D)$ is the following. The *linear system* $|D|$ of a divisor D consists of all effective divisors on $\mathcal{X}_\Sigma$ which are linearly equivalent to D:

$$|D| \;=\; \{E \in \mathrm{Div}(\mathcal{X}_\Sigma) \;:\; E \geq 0 \text{ and } [D - E] = 0\}. \tag{9.3.4}$$

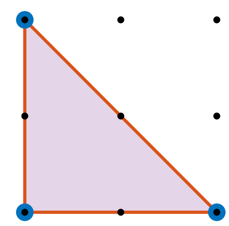

Figure 9.2: A triangle in $\mathbb{R}^2$ whose normal fan is shown in Figure 7.5.

Proposition 9.3.8. *Let Σ be a complete fan in $N_{\mathbb{R}}$ and let $\mathcal{X}_\Sigma$ be the corresponding compact normal toric variety. For any torus invariant divisor $D \in \mathrm{Div}_T(\mathcal{X}_\Sigma)$, we have*

$$|D| \simeq \mathbb{P}\Gamma(\mathcal{X}_\Sigma, D) = \frac{\Gamma(\mathcal{X}_\Sigma, D) \setminus \{0\}}{\mathbb{C}^*}.$$

Proof. The correspondence is $E \mapsto [f]$, where f is such that $E = D + \mathrm{div}(f) \geq 0$ and $[\cdot]$ is its class modulo $\mathbb{C}^*$. Such an f exists since $[D - E] = 0$, and it belongs to $\Gamma(\mathcal{X}_\Sigma, D)$ by Proposition 9.3.5. Moreover, if $f, g \in \Gamma(\mathcal{X}_\Sigma, D) \setminus \{0\}$ satisfy $\mathrm{div}(f) = \mathrm{div}(g)$, then $\mathrm{div}(\frac{f}{g}) = 0$ and $\frac{f}{g} : \mathcal{X}_\Sigma \to \mathbb{C}^*$ is a morphism (Lemma 9.1.18). Since $\mathcal{X}_\Sigma$ is compact, this means that $\frac{f}{g}$ is constant. We have shown that $E \mapsto [f]$ is well-defined. □

Clearly, the linear system $|D|$ only depends on the class of D in $\mathrm{Cl}(\mathcal{X}_\Sigma)$. In the case where $\mathcal{X}_\Sigma = \mathbb{P}^d$, the class group is $\mathbb{Z}$ (Exercise 9.2.7), and the linear system of a general torus invariant divisor $|D| = |a_1 D_1 + \cdots + a_{d+1} D_{d+1}|$ consists of hypersurfaces of degree $a_1 + \cdots + a_{d+1}$. Proposition 9.3.8 is the correspondence between hypersurfaces of degree k and homogeneous polynomials of degree k, considered up to scaling.

Example 9.3.9. Let $\mathcal{X}_\Sigma = \mathbb{P}^2$ and $D = 2D_3$, where D_3 is as in Example 9.2.6. One checks easily that the polyhedron $\mathcal{P}_D$ is the triangle from Figure 9.2. By Proposition 9.3.5, the global sections of $\mathcal{O}_{\mathbb{P}^2}(D)$ are Laurent polynomials supported in this triangle. These correspond to rational functions on $\mathbb{P}^2$ with poles of order up to 2 along D_3, and no poles elsewhere. If D_3 is given by the vanishing of the homogeneous coordinate x_3, such functions are of the form

$$\frac{z_0 x_3^2 + z_1 x_1 x_3 + z_2 x_2 x_3 + z_3 x_1^2 + z_4 x_1 x_2 + z_5 x_2^2}{x_3^2} \sim z_0 + z_1 t_1 + z_2 t_2 + z_3 t_1^2 + z_4 t_1 t_2 + z_5 t_2^2.$$

Note that here Σ is the normal fan of $\mathcal{P}_D$. More generally, the k-dilation $k \cdot \Delta$ of the standard simplex Δ corresponds to the divisor $k \cdot D_3$. This correspondence "reverses" the map $D \mapsto \mathcal{P}_D$. Below, we explain how this works more generally. ◇

Exercise 9.3.10. With the notation of Example 9.2.8, describe $\Gamma(\mathbb{P}^1 \times \mathbb{P}^1, 2D_1 + 5D_3)$.

Exercise 9.3.11. With the notation from Exercise 9.2.9, let $D = D_1 + D_2 \in \mathrm{Div}_T(\mathcal{X}_\Sigma)$. Describe $\mathcal{P}_D$ and $\Gamma(\mathcal{X}_\Sigma, D)$.

We have associated a polyhedron $\mathcal{P}_D$ to a torus invariant divisor $D \in \mathrm{Div}_T(\mathcal{X}_\Sigma)$. Going the other way, one associates a divisor $D_\mathcal{P}$ to a polyhedron $\mathcal{P} \subset M_\mathbb{R} = \mathbb{R}^d$. This construction requires the choice of a normal toric variety $\mathcal{X}_\Sigma$ on which $D_\mathcal{P}$ is a torus invariant divisor.

Definition 9.3.12. *Let Σ, Σ' be complete fans in $N_\mathbb{R} \simeq \mathbb{R}^d$. We say that Σ* refines *Σ' if each d-dimensional cone in $\Sigma'(d)$ is a union of d-dimensional cones in $\Sigma(d)$.*

Lemma 9.3.13. *Let $\mathcal{P} \subset M_\mathbb{R} \simeq \mathbb{R}^d$ be a convex lattice polytope. Let Σ be a complete fan which refines the normal fan $\Sigma_\mathcal{P}$. The ray generators F of $\Sigma(1)$ give a (possibly redundant) facet description of $\mathcal{P}$. More precisely, we have*

$$\mathcal{P} = \{m \in M_\mathbb{R} \,:\, F^\top m + a \geq 0\},$$

with $a = (a_\rho)_{\rho \in \Sigma(1)}$ and $a_\rho = \min\{a'_\rho \,:\, \langle u_\rho, m\rangle + a'_\rho \geq 0 \text{ for all } m \in \mathcal{P}\}$.

Proof. With the given definition of a_ρ, the inclusion $\mathcal{P} \subseteq \{m \in M_\mathbb{R} \,:\, F^\top m + a \geq 0\}$ is clear. The reverse inclusion will follow once we show that a subset of the inequalities $F^\top m + a \geq 0$ gives a minimal facet representation of $\mathcal{P}$. Since Σ refines the normal fan of $\Sigma_\mathcal{P}$, the rays $\Sigma_\mathcal{P}(1)$ are among the rays $\Sigma(1)$. $\square$

Lemma 9.3.13 gives a recipe for constructing a divisor $D_\mathcal{P} \in \mathrm{Div}_T(\mathcal{X}_\Sigma)$ from a polytope $\mathcal{P}$ whose normal fan is refined by Σ: We define

$$D_\mathcal{P}(\Sigma) = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho, \tag{9.3.5}$$

where the coefficients a are as in Lemma 9.3.13. Though we will need this general construction, the most natural scenario is that where $\mathcal{P}$ is a d-dimensional lattice polytope and $\Sigma = \Sigma_\mathcal{P}$ is its normal fan. Then $D_\mathcal{P}(\Sigma_\mathcal{P})$ is a divisor on $\mathcal{X}_{\Sigma_\mathcal{P}} \simeq \mathcal{X}_\mathcal{P}$ (Theorem 7.4.8).

Example 9.3.14. The standard simplex $\mathcal{P} = \Delta = \mathrm{Conv}((0,0),(1,0),(0,1)) \subset \mathbb{R}^2$ defines a divisor on $\mathbb{P}^2$. In this case, $\mathcal{P}$ can uniquely be written in terms of facet inequalities $F^\top m + a \geq 0$, where F is as in Example 9.2.6, and $a = (0,0,1)^\top$. Hence, $D_\mathcal{P}(\Sigma) = D_3$. Similarly, $k \cdot \mathcal{P}$ gives $D_{k\cdot\mathcal{P}}(\Sigma) = k \cdot D_3$. We now switch fans. Let F be as in Example 9.3.2. It contains the primitive ray generators of a complete fan Σ' in $\mathbb{R}^2$. Our simplex $\mathcal{P}$ is given by $F^\top m + a \geq 0$, where $a = (0,0,1,\alpha)$ and α is any nonnegative number. The minimal value for α is 0, so that $D_\mathcal{P}(\Sigma') = D_3 \in \mathrm{Div}_T(\mathcal{X}_{\Sigma'})$. The fan Σ' refines Σ in the sense of Definition 9.3.12: the cone σ_3 in Figure 7.4(e) is a union of two 2-dimensional cones of Σ'. $\diamond$

Exercise 9.3.15. Compute the coefficients of $D_\mathcal{P}(\Sigma) = a_1 D_1 + \cdots + a_4 D_4$, where the D_i and Σ are as in Example 9.2.8 and $\mathcal{P}$ is the rectangle $[0,2] \times [0,7] \subset \mathbb{R}^2$. More generally, compute $D_\mathcal{P}(\Sigma)$ for $\mathcal{P} = [a,b] \times [c,d] \subset \mathbb{R}^2$. Note that the question makes sense when $\mathcal{P}$ is a line segment, i.e., when $a = b$ and/or $c = d$, but in these cases $\Sigma \neq \Sigma_\mathcal{P}$.

Exercise 9.3.16. Consider the quadrilaterals $\mathcal{P}_1 = \mathrm{Conv}((1,0),(0,-1),(-1,0),(0,1))$ and $\mathcal{P}_2 = \mathrm{Conv}((0,0),(2,0),(1,1),(0,1))$ in $\mathbb{R}^2$. Compute the divisors $D_{\mathcal{P}_1}(\Sigma)$ and $D_{\mathcal{P}_2}(\Sigma)$ in $\mathrm{Div}_T(\mathcal{X}_\Sigma)$, where Σ is the complete fan in $\mathbb{R}^2$ with ray generator matrix

$$F = \begin{pmatrix} 1 & 1 & 1 & 0 & -1 & -1 & 0 \\ 1 & 0 & -1 & -1 & -1 & 1 & 1 \end{pmatrix}.$$

Exercise 9.3.17. Show that if $\mathcal{X}_\Sigma \simeq \mathcal{X}_{\mathcal{P}_D}$ for some divisor $D \in \mathrm{Div}_T(\mathcal{X}_\Sigma)$ such that $\mathcal{P}_D$ is full-dimensional, then $D_{\mathcal{P}_D} = D$. Conversely, if $\mathcal{P}$ is a full-dimensional lattice polytope and $D_\mathcal{P} = D_\mathcal{P}(\Sigma_\mathcal{P}) \in \mathrm{Div}_T(\mathcal{X}_\mathcal{P})$ is the corresponding divisor, then $\mathcal{P}_{D_\mathcal{P}} = \mathcal{P}$.

9.4 Hypersurfaces from Laurent polynomials

In this section, we construct codimension-one subvarieties of a complete normal toric variety from a Laurent polynomial. More precisely, our data are

- a nonzero element $f \in \mathbb{C}[M] = \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$,
- a lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ containing the Newton polytope of f and
- a complete fan Σ which refines $\Sigma_\mathcal{P}$ (Definition 9.3.12).

By construction, we have $f \in \Gamma(\mathcal{X}_\Sigma, D_\mathcal{P}(\Sigma)) \setminus \{0\}$. Indeed, by Lemma 9.3.13, the vector space $\Gamma(\mathcal{X}_\Sigma, D_\mathcal{P}(\Sigma))$ consists of all Laurent polynomials whose Newton polytope is contained in $\mathcal{P}$. By Proposition 9.3.8, the class $[f] \in \mathbb{P}\Gamma(\mathcal{X}_\Sigma, D_\mathcal{P}(\Sigma))$ is naturally associated to the effective Weil divisor

$$E_f \;=\; D_\mathcal{P}(\Sigma) + \mathrm{div}(f) \;\in\; \mathrm{Div}(\mathcal{X}_\Sigma). \tag{9.4.1}$$

This divisor represents a (possibly non-reduced) hypersurface in $\mathcal{X}_\Sigma$. Since $D_\mathcal{P}(\Sigma) \in \mathrm{Div}_T(X_\Sigma)$, the restriction $D_\mathcal{P}(\Sigma)_{|T}$ is zero, so that $(E_f)_{|T} = \mathrm{div}(f)_{|T}$. That is, when restricted to the torus T, E_f coincides with the hypersurface $V_T(f)$ defined by f in T.

A more direct way of defining a hypersurface in $\mathcal{X}_\Sigma$ from f is to simply take the Zariski closure of the hypersurface $V_T(f) \subset T \subset \mathcal{X}_\Sigma$ in $\mathcal{X}_\Sigma$. We shall see that this agrees with the support of the divisor E_f under mild assumptions (Proposition 9.4.5).

An advantage of considering the divisor E_f rather than the closure of $V_T(f)$ in $\mathcal{X}_\Sigma$ is that we can easily write down a local equation for E_f on each affine chart of $\mathcal{X}_\Sigma$ using the data Σ and $\mathcal{P}$. This is the content of Proposition 9.4.3, which needs some more notation.

For each d-dimensional cone $\sigma \in \Sigma(d)$, there is a corresponding cone $\tilde{\sigma} \in \Sigma_\mathcal{P}(d)$ such that $\sigma \subseteq \tilde{\sigma}$. In turn, the cone $\tilde{\sigma}$ corresponds to a vertex in $\mathcal{V}(\mathcal{P}) \subset M$, which we shall denote by m_σ. With this notation, for each $\sigma \in \Sigma(d)$, we define a Laurent polynomial

$$f_\sigma \;=\; \frac{1}{t^{m_\sigma}} \cdot f.$$

Example 9.4.1. Let $\mathcal{P} \subset \mathbb{R}^2$ be the triangle $\mathrm{Conv}((0,0),(0,1),(2,1))$ and let Σ be the fan in Figure 7.4(g). We order the rays as follows:

$$F = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & -1 \end{pmatrix}.$$

One checks that Σ refines $\Sigma_{\mathcal{P}}$, $D_{\mathcal{P}}(\Sigma) = D_4$ and

$$m_{\sigma_1} = m_{\sigma_4} = (0,0), \quad m_{\sigma_2} = (0,1), \quad m_{\sigma_3} = (2,1).$$

The Newton polytope of the polynomial $f = t_1 t_2 + t_1^2 t_2$ is a line segment contained in $\mathcal{P}$. We have $f_{\sigma_2} = t_1 + t_1^2$. For $g = 1 + t_2 + t_1^2 t_2$, we have $g_{\sigma_2} = t_2^{-1} + 1 + t_1^2$. $\diamond$

Exercise 9.4.2. Verify that if $\mathcal{P}$ has dimension d and $\Sigma = \Sigma_{\mathcal{P}}$ is its normal fan, then m_σ is the vertex of $\mathcal{P}$ corresponding to σ and $\sigma \mapsto m_\sigma$ is a bijection $\Sigma(d) \to \mathcal{V}(\mathcal{P})$.

Proposition 9.4.3. *Let f, $\mathcal{P}$ and Σ be as above. For any cone $\sigma \in \Sigma(d)$, the Laurent polynomial f_σ belongs to the coordinate ring $\mathbb{C}[\mathsf{S}_\sigma]$ of the affine chart $\mathcal{Y}_\sigma \simeq U_\sigma \subset \mathcal{X}_\Sigma$. Its divisor is the restriction of the effective divisor E_f to U_σ. That is, we have*

$$(E_f)_{|U_\sigma} = (D_{\mathcal{P}}(\Sigma) + \mathrm{div}(f))_{|U_\sigma} = \mathrm{div}(f_\sigma)_{|U_\sigma}.$$

Proof. Since $\sigma \in \Sigma(d)$ is contained in the d-dimensional cone $\tilde{\sigma} \in \Sigma_{\mathcal{P}}$ corresponding to the vertex $m_\sigma \in \mathcal{V}(\mathcal{P})$, we have for each $\rho \in \sigma(1)$ that

$$\langle u_\rho, m_\sigma \rangle = \min_{m \in \mathcal{P}} \langle u_\rho, m \rangle.$$

Therefore $m - m_\sigma \in \sigma^\vee \cap M$ for each exponent m appearing in f, and thus $f_\sigma \in \mathbb{C}[\mathsf{S}_\sigma]$.

Since $\mathrm{div}(f_\sigma) = \mathrm{div}(f) - \mathrm{div}(t^{m_\sigma})$, we must show that $D_{\mathcal{P}}(\Sigma)_{|U_\sigma} = -\mathrm{div}(t^{m_\sigma})_{|U_\sigma}$. By Corollary 9.2.2, we have $\mathrm{div}(t^{m_\sigma}) = \sum_{\rho \in \Sigma(1)} \langle u_\rho, m_\sigma \rangle\, D_\rho$. Restricting to $\mathcal{Y}_\sigma$ gives

$$\mathrm{div}(t^{m_\sigma})_{|U_\sigma} = \sum_{\rho \in \sigma(1)} \langle u_\rho, m_\sigma \rangle\, D_\rho.$$

Using again that $\sigma \subseteq \tilde{\sigma} \in \Sigma_{\mathcal{P}}$, we find for each $\rho \in \sigma(1)$ that

$$\langle u_\rho, m_\sigma \rangle = \min_{m \in \mathcal{P}} \langle u_\rho, m \rangle = -\min\{a'_\rho : \langle u_\rho, m \rangle + a'_\rho \geq 0 \text{ for all } m \in \mathcal{P}\} = -a_\rho.$$

The last equality follows from the definition of a_ρ in Lemma 9.3.13. $\square$

Remark 9.4.4. Proposition 9.4.3 says that the divisor E_f is locally defined by a single equation $f_\sigma = 0$ on each toric affine chart $\mathcal{Y}_\sigma$. Such divisors are called *locally principal* or *Cartier*. Cartier divisors on a variety X form a subgroup of $\mathrm{Div}(X)$ containing the (globally) principal divisors $\mathrm{Div}_0(X)$. The *Picard group* $\mathrm{Pic}(X)$ is given by Cartier divisors modulo linear equivalence. For smooth varieties X, all Weil divisors are Cartier and thus $\mathrm{Pic}(X) = \mathrm{Cl}(X)$. We should also point out that, since $D_{\mathcal{P}}(\Sigma)$ is Cartier, the sheaf $\mathcal{O}_{\mathcal{X}_\Sigma}(D_{\mathcal{P}}(\Sigma))$ from Remark 9.3.6 is the sheaf of sections of a line bundle on $\mathcal{X}_\Sigma$. The section f is given by the local data $(U_\sigma, f_\sigma)_{\sigma\in\Sigma(d)}$, and the transition functions are the Laurent monomials $t^{m_\sigma - m_{\sigma'}}$. The effective divisor E_f is the zero locus of this section. For us, it suffices to work with Cartier divisors arising from a choice of f, $\mathcal{P}$ and Σ as above, and we will not use this terminology. The reader who wants to learn more about Cartier divisors and line bundles on toric varieties should consult [19, Chapters 4 and 6].

Proposition 9.4.5. *If the Newton polytope of f equals $\mathcal{P}$, then the divisor E_f is the (scheme-theoretic) closure of $V_T(f)$ in $\mathcal{X}_\Sigma$.*

Proof. By Proposition 9.4.3, it suffices to show that f_σ defines the closure of $V_T(f)$ in $\mathcal{Y}_\sigma$ for each d-dimensional cone $\sigma \in \Sigma(d)$. For this, we show that the ideal $\mathbb{C}[\mathsf{S}_\sigma]\cdot f_\sigma$ generated by f_σ in $\mathbb{C}[\mathsf{S}_\sigma]$ equals $(\mathbb{C}[M]\cdot f)\cap\mathbb{C}[\mathsf{S}_\sigma]$. The inclusion $\mathbb{C}[\mathsf{S}_\sigma]\cdot f_\sigma \subseteq (\mathbb{C}[M]\cdot f)\cap\mathbb{C}[\mathsf{S}_\sigma]$ is clear. For the opposite inclusion, let $g = h\cdot f \in \mathbb{C}[\mathsf{S}_\sigma]$ and $h \in \mathbb{C}[M]$. Since $g \in \mathbb{C}[\mathsf{S}_\sigma]$, we know that $\mathrm{Newt}(g) \subset \sigma^\vee$. On the other hand, $\mathrm{Newt}(g) = \mathrm{Newt}(h) + \mathrm{Newt}(f_\sigma) + m_\sigma$. Since $0 \in \mathrm{Newt}(f_\sigma)$ by the assumption that $\mathrm{Newt}(f) = \mathcal{P}$, we conclude that $\mathrm{Newt}(h) + m_\sigma \subset \sigma^\vee$, and hence $h\cdot t^{m_\sigma} \in \mathbb{C}[\mathsf{S}_\sigma]$. Thus $g \in \mathbb{C}[\mathsf{S}_\sigma]\cdot f_\sigma$, and the statement is proved. □

Example 9.4.6. We continue Example 9.4.1. The polynomial $f = t_1t_2 + t_1^2t_2$ defines the irreducible curve $V_T(f) = \{(-1, t_2) \,:\, t_2 \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^2$, and $f_{\sigma_1} = f$ defines a curve with three irreducible components inside $\mathcal{Y}_{\sigma_1} \simeq \mathbb{C}^2$. Hence, E_f strictly contains the closure of $V_T(f)$. Notice that $\mathrm{Newt}(f) \neq \mathcal{P}$, so Proposition 9.4.5 does not apply. One checks that $g = 1 + t_2 + t_1^2t_2$ defines an irreducible curve in $(\mathbb{C}^*)^2 \subset \mathcal{X}_\Sigma$ with closure E_g. ◇

The condition $\mathrm{Newt}(f) = \mathcal{P}$ is sufficient but not necessary for the conclusion of Proposition 9.4.5. You will show this in the following exercise.

Exercise 9.4.7. Let F be as in Example 9.4.1, let $\mathcal{P}$ be the quadrilateral with vertices $(0,0)$, $(0,1)$, $(3,1)$, $(1,0)$ and let $f = t_1 + t_2 + t_1^3t_2$. Check that $\mathrm{Newt}(f) \subsetneq \mathcal{P}$, yet E_f is the closure of $V_T(f)$ in X_Σ.

When $\mathcal{P}$ is very ample and $\Sigma = \Sigma_{\mathcal{P}}$, the divisor E_f is a hyperplane section of $\mathcal{X}_{\mathcal{P}}$ in its projective embedding corresponding to $\mathcal{P}\cap M$. We explain this in more detail. Let $A = \mathcal{P}\cap M$ be an integer matrix with d rows and $n = |\mathcal{P}\cap M|$ columns $a_1,\dots,a_n \in \mathbb{Z}^d$ representing the integer points of $\mathcal{P}$. The toric vaiety $\mathcal{X}_{\Sigma_{\mathcal{P}}} = \mathcal{X}_{\mathcal{P}}$ is isomorphic to the projective variety $X_A \subset \mathbb{P}^{n-1}$ obtained as the closure of the image of $\Phi_A(t) = (t^{a_1} : \ldots : t^{a_n})$, see Definition 3.5.8 and Theorem 7.4.8. The Laurent polynomial f is of the form $f = \sum_{i=1}^n z_i\, t^{a_i}$, for some $z_i \in \mathbb{C}$ (not all zero).

Exercise 9.4.8. Show that the map Φ_A identifies the hypersurface $V_T(f) \subset (\mathbb{C}^*)^d$ with the hyperplane section $\mathrm{im}\,\Phi_A \cap H_f$, where $H_f = \{x \in \mathbb{P}^{n-1} \,:\, z_1x_1 + \cdots + z_nx_n = 0\}$.

Corollary 9.4.9. *Let $\mathcal{P}$ be very ample and let $\Sigma = \Sigma_{\mathcal{P}}$ be its normal fan. Let A be as above and let H_f be the hyperplane from Exercise 9.4.8. The divisor E_f from* (9.4.1) *is the (scheme-theoretic) intersection $X_A \cap H_f$.*

Proof. This follows from Proposition 9.4.3. The affine chart $U_\sigma \subset \mathcal{X}_{\mathcal{P}}$ is the intersection of X_A with a standard affine chart $U_i = \{x_i \neq 0\} \simeq \mathbb{C}^{n-1}$ of $\mathbb{P}^{n-1}$, where i is such that $a_i = m_\sigma$. On that affine chart, $X_A \cap H_f \cap U_i$ is the intersection of the affine toric variety Y_{A-a_i} with the affine hyperplane $\ell = z_1 \frac{x_1}{x_i} + \cdots + z_n \frac{x_n}{x_i} = 0$, see Proposition 3.2.1. By our assumption that $\mathcal{P}$ is very ample, there is an isomorphism $\mathbb{C}[Y_A] \to \mathbb{C}[\mathsf{S}_\sigma]$ (Proposition 3.5.2). This isomorphism sends the affine linear function $\ell \in \mathbb{C}[Y_A]$ on Y_A to f_σ. □

We end with an example in projective space.

Example 9.4.10. Following Example 9.3.9, we view the (Laurent) polynomial

$$f \;=\; z_1 + z_2\, t_1 + z_3\, t_2 + z_4\, t_1^2 + z_5\, t_1 t_2 + z_6\, t_2^2$$

as an element of $\Gamma(\mathbb{P}^2, 2D_3)$. The fan Σ is that shown in 7.4(e). The z_i are any complex numbers, but they are not all zero. For most choices of z_i, the Newton polytope of f equals $\mathcal{P}$, which is twice the standard simplex, and $\Sigma = \Sigma_{\mathcal{P}}$. If f is generic, then the equation $f = 0$ defines a conic in $T = (\mathbb{C}^*)^2$, the dense torus of $\mathbb{P}^2$. That conic is $V_T(f) = \{t \in T : f(t) = 0\}$. A candidate for the closure of $V_T(f)$ in $\mathbb{P}^2$ is obtained by homogenizing f and considering the curve in $\mathbb{P}^2$ defined by the resulting ternary quadric. This is what we will explore in the next chapter, once we know what it means to *homogenize* for arbitrary projective normal toric varieties. Here, we describe the closure of $V_T(f)$ locally in each affine chart of $\mathbb{P}^2$.

By Example 7.4.4, the affine charts of $\mathbb{P}^2$ are isomorphic to the affine toric varieties $\mathcal{Y}_{\sigma_1}, \mathcal{Y}_{\sigma_2}, \mathcal{Y}_{\sigma_3}$, each isomorphic to $\mathbb{C}^2$, with coordinate rings

$$\mathbb{C}[\mathcal{Y}_{\sigma_1}] \;=\; \mathbb{C}[t_1^{-1}, t_1^{-1}t_2], \quad \mathbb{C}[\mathcal{Y}_{\sigma_2}] \;=\; \mathbb{C}[t_2^{-1}, t_1 t_2^{-1}], \quad \mathbb{C}[\mathcal{Y}_{\sigma_3}] \;=\; \mathbb{C}[t_1, t_2].$$

Following our construction in this section, we obtain a polynomial $f_i = f_{\sigma_i} \in \mathbb{C}[\mathcal{Y}_{\sigma_i}]$ for each vertex of our triangle $\mathcal{P}$:

$$f_1 \;=\; \frac{f}{t_1^2}, \quad f_2 \;=\; \frac{f}{t_2^2}, \quad f_3 \;=\; \frac{f}{1} \;=\; f.$$

By Proposition 9.4.3, f_i is a function on the torus T which extends to a function on $\mathcal{Y}_{\sigma_i}$. Its zero locus on $\mathcal{Y}_{\sigma_i}$ contains the closure of $V_T(f) = V_T(f_i)$ in $\mathcal{Y}_{\sigma_i}$. If the coefficients z_1, z_4, z_6 are nonzero, then this containment is actually an equality (Proposition 9.4.5).

Proposition 9.4.3 also says that the curves $V_{\mathcal{Y}_{\sigma_i}}(f_i)$ glue together to a curve E_f in $\mathbb{P}^2$. The gluing is the restriction of the gluing morphisms of $\mathbb{P}^2$: the zero loci of the f_i agree on the overlaps of the affine charts. Let $U_\tau \subset \mathbb{P}^2$ be the open subset corresponding to the cone $\tau \in \Sigma$. We have $U_{\sigma_1} \cap U_{\sigma_2} = U_{\sigma_1 \cap \sigma_2} \simeq \mathcal{Y}_{\sigma_1 \cap \sigma_2} \simeq (\mathcal{Y}_{\sigma_1})_{\frac{t_2}{t_1}} \simeq (\mathcal{Y}_{\sigma_2})_{\frac{t_1}{t_2}}$. Both f_1 and f_2 are elements in the coordinate ring of $\mathcal{Y}_{\sigma_1 \cap \sigma_2}$, and they only differ by a unit:

$$f_1 \;=\; \left(\frac{t_2}{t_1}\right)^2 f_2.$$

Hence, the zero loci of f_1 and f_2 agree on the overlap $U_{\sigma_1} \cap U_{\sigma_2}$.

The curve E_f is alternatively obtained as a hyperplane section of $X_A \subset \mathbb{P}^5$, where

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}.$$

This is the Veronese surface $\nu_2(\mathbb{P}^2)$. In that embedding, the equations of E_f are

$$\operatorname{rank} \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{pmatrix} \leq 1, \quad z_1 x_1 + z_2 x_2 + \cdots + z_6 x_6 \;=\; 0.$$

When $z_1 = z_2 = z_4 = 0$, E_f is reducible, while $V_T(f)$ is not. $\diamond$

Further reading

Divisors and line bundles on toric varieties are discussed in [30, Section 3.3] and [19, Chapters 4 and 6]. The class group $\mathrm{Cl}(\mathcal{X}_\Sigma)$ is a graded piece of the Chow ring $A_\bullet(\mathcal{X}_\Sigma)$. Intersection theory on $\mathcal{X}_\Sigma$ is explained in [31], [30, Chapter 5] and [19, Section 12.5].

Chapter 10

The Cox ring of a toric variety

In this chapter, we describe a normal toric variety $\mathcal{X}_\Sigma$ as the quotient of a quasi-affine space by the action of a reductive group. The quotient construction gives global coordinates on $\mathcal{X}$, which are useful for describing its subvarieties. This generalizes the well-known homogeneous coordinates on $\mathbb{P}^d$. Homogeneous coordinates or *Cox coordinates* on $\mathcal{X}_\Sigma$ correspond to the variables of the homogeneous coordinate ring or *Cox ring* of $\mathcal{X}_\Sigma$, which is a polynomial ring endowed with a particular grading and a distinguished ideal called the *irrelevant ideal*. We start with the familiar example $\mathcal{X}_\Sigma = \mathbb{P}^2$.

Example 10.0.1. The complex projective plane $\mathbb{P}^2$ can be defined as

$$\mathbb{P}^2 = \frac{\mathbb{C}^3 \setminus \{0\}}{\mathbb{C}^*}. \tag{10.0.1}$$

The quotient identifies points on the same $\mathbb{C}^*$-orbit, where $\mathbb{C}^*$ acts as follows:

$$\mathbb{C}^* \times (\mathbb{C}^3 \setminus \{0\}) \to (\mathbb{C}^3 \setminus \{0\}) \quad \text{is given by} \quad (\lambda, (y_1, y_2, y_3)) \mapsto (\lambda y_1, \lambda y_2, \lambda y_3).$$

This action extends trivially to an action on $\mathbb{C}^3$. Subvarieties of $\mathbb{P}^2$ are given by homogeneous ideals in the polynomial ring $S = \mathbb{C}[y_1, y_2, y_3]$. Here 'homogeneous' is with respect to the standard $\mathbb{Z}$-grading

$$S = \bigoplus_{\alpha \in \mathbb{Z}} S_\alpha,$$

which is such that for $f \in S$ homogeneous, $V_{\mathbb{C}^3}(f)$ is stable under the $\mathbb{C}^*$-action. Equivalently, $V_{\mathbb{C}^3}(f)$ is a union of $\mathbb{C}^*$-orbits. In the ring S, the ideal $B = \langle y_1, y_2, y_3 \rangle$ plays a special role: it is the largest ideal whose zero locus in $\mathbb{P}^2$ is the empty set. The interplay between algebra and geometry in this construction is summarized in the following table.

$$\begin{array}{ccc} \text{Algebra} & & \text{Geometry} \\ \hline S & \xrightarrow{\text{Specm}(\cdot)} & \mathbb{C}^3 \\ B & \xrightarrow{V_{\mathbb{C}^3}(\cdot)} & \{0\} \\ \mathbb{Z} & \xrightarrow{\text{Hom}_{\mathbb{Z}}(\cdot, \mathbb{C}^*)} & \mathbb{C}^* \end{array} \tag{10.0.2}$$

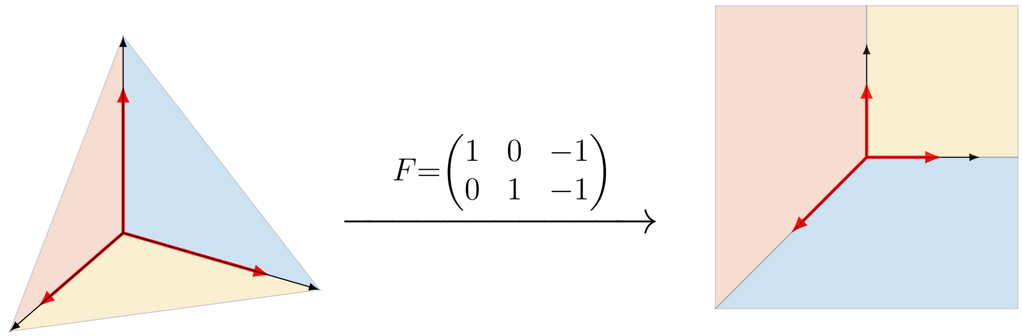

Figure 10.1: An illustration of the $\mathbb{Z}$-linear map $F : N' \to N$ from Example 10.0.1. The ray generators of $\Sigma'(1), \Sigma(1)$ are depicted as red arrows and the two dimensional cones are colored in blue, orange and yellow.

For the purpose of generalizing this construction, we will explain the following claim. The quotient (10.0.1) comes from a toric morphism

$$\pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2, \tag{10.0.3}$$

whose fibers are $\mathbb{C}^*$-orbits. A toric morphism comes from a $\mathbb{Z}$-linear map $N' \to N$ that is compatible with fans Σ' and Σ in $N'_{\mathbb{R}}$ and $N_{\mathbb{R}}$ respectively, see Theorem 8.2.5.

In our case, Σ' is the fan of $\mathbb{C}^3 \setminus \{0\}$ and Σ is the fan of $\mathbb{P}^2$ (Example 7.4.4). The lattices are $N' = \mathbb{Z}^3$ and $N = \mathbb{Z}^2$, and the morphism π comes from $F : N' \to N$ where F is a 2×3 integer matrix whose columns are the primitive ray generators of $\Sigma(1)$. The fans and the matrix F are shown in Figure 10.1. The *compatibility* of the map F with the fans Σ' and Σ comes down to the fact that each cone of Σ' is mapped (under $F_{\mathbb{R}}$) into a cone of Σ. In Figure 10.1 the 2-dimensional cones have matching colors according to this association. Note that the three dimensional cone $\sigma = \mathrm{Cone}(e_1, e_2, e_3)$ of the positive orthant in $\mathbb{R}^3$ is not mapped to a cone of Σ under $F_{\mathbb{R}}$. Therefore, this cone does not belong to Σ'. Removing this three dimensional cone from Σ' corresponds to removing the origin from $\mathbb{C}^3$ (Example 7.4.11). This confirms that $\mathbb{C}^3 \setminus \{0\} = \mathcal{X}_{\Sigma'}$. $\diamond$

10.1 Cox coordinates

In this section we describe the construction of a toric variety $\mathcal{X}_\Sigma$ as the image of a map

$$\pi : \mathbb{C}^k \setminus Z \to \mathcal{X}_\Sigma,$$

where $Z \subset \mathbb{C}^k$ is a subvariety and π is invariant under an algebraic group action $G \times (\mathbb{C}^k \setminus Z) \to (\mathbb{C}^k \setminus Z)$. That is, $\pi(g \bullet y) = \pi(y)$ for any $y \in \mathbb{C}^k \setminus Z$ and $g \in G$. The map π constructs $\mathcal{X}_\Sigma$ as a quotient in the sense of *Geometric Invariant Theorem (GIT)*. We warn the reader who is familiar with GIT that this quotient is not necessarily *geometric*, like in the case of $\mathbb{P}^d$, but it is *almost geometric*. That is, the nicest possible scenario is when points in $\mathcal{X}_\Sigma$ are in one-to-one correspondence with G-orbits in $\mathbb{C}^k \setminus Z$. This happens for *simplicial fans*, and something slightly weaker is true in general. Theorem 10.1.16 gives a precise statement. The quotient construction is due to David Cox [14]. In the analytic category, it had been described by Audin, Delzant and Kirwan, see [3, Chapter 6] and references therein. The GIT quotient generalizes Example 10.0.1 to $\mathcal{X}_\Sigma$.

Let the set of rays of Σ be $\Sigma(1) = \{\rho_1, \dots, \rho_k\}$ and let $u_i \in N = \mathbb{Z}^d$ be the primitive ray generator of ρ_i. Like in the previous chapter, we collect the u_i in a matrix

$$F = (u_1 \ \cdots \ u_k) \in \mathbb{Z}^{d \times k}.$$

Particularly relevant is the case where Σ is the normal fan of a d-dimensional polytope in $\mathbb{R}^d$. When $\mathcal{X}_\Sigma = \mathbb{P}^d$, we have $k = d+1$, $Z = \{0\}$ and $G = \mathbb{C}^*$. The polytope is the d-dimensional standard simplex (Exercise 3.5.9). In what follows, it is instructive to keep the familiar Example 10.0.1 in mind as a reference. However, the discussion applies more generally for fans which are not necessarily complete. We only need the following:

We will assume throughout the chapter that F has rank d.

The matrix F represents a lattice homomorphism $F : N' \to N$ where $N' = \mathbb{Z}^k$. Consider the fan given by the positive orthant in $\mathbb{R}^k$ and all its faces. We let Σ' be the subset of all its cones whose image under $F_\mathbb{R}$ is contained in a cone of Σ.

Exercise 10.1.1. Show that Σ' is a fan.

By construction, the lattice homomorphism F is compatible with the fans Σ' and Σ in $N'_\mathbb{R}$ and $N_\mathbb{R}$ respectively. It follows from Theorem 8.2.5 that F gives a toric morphism $\pi : \mathcal{X}_{\Sigma'} \to \mathcal{X}_\Sigma$. Since Σ' is obtained from the fan of $\mathbb{C}^k$ by removing some of its cones, we have $X_{\Sigma'} = \mathbb{C}^k \setminus Z$, where Z is a union of torus orbits, see Exercises 4.3.6 and 7.4.11. We will refer to the set Z as the *base locus*. Describing it is our first order of business. We emphasize that Z depends on Σ, but we avoid writing $Z(\Sigma)$ to keep notation simple.

The fan of the toric variety $\mathbb{C}^k$ consists of the cones

$$\sigma_C \ = \ \mathrm{Cone}(e_i \ : \ i \in C) \quad \text{for} \quad C \subseteq [k] = \{1, \dots, k\}. \tag{10.1.1}$$

The corresponding $(\mathbb{C}^*)^k$-orbits consist of points whose zero entries are indexed by C:

$$O(\sigma_C) \ = \ \{y \in \mathbb{C}^k \ : \ y_i = 0 \iff i \in C\}.$$

By the above discussion, The base locus Z is a union of such orbits. We have

$$O(\sigma_C) \subseteq Z \iff \sigma_C \notin \Sigma' \iff \mathrm{Cone}(\rho_i \ : \ i \in C) \text{ is not contained in a cone of } \Sigma. \tag{10.1.2}$$

Lemma 10.1.2. *The base locus $Z \subset \mathbb{C}^k$ is a union of coordinate subspaces. That is, Z is a closed subvariety defined by a squarefree monomial ideal.*

Proof. It is clear from (10.1.2) that if $O(\sigma_C) \subseteq Z$, then $O(\sigma_{C'}) \subseteq Z$ for each $C' \supseteq C$. Since $\overline{O(\sigma_C)} = \bigcup_{C' \supseteq C} O(\sigma_{C'})$ is a coordinate subspace, the lemma is proved. □

Here is another immediate consequence of (10.1.2).

Proposition 10.1.3. *Let $\mathcal{C}$ be the set of all subsets $C \subseteq [k] = \{1, \ldots, k\}$ such that* $\mathrm{Cone}(\rho_i)_{i\in C}$ *is not contained in a cone of Σ. The base locus Z is*

$$Z = \bigcup_{C \in \mathcal{C}} V_{\mathbb{C}^k}(y_i : i \in C). \qquad (10.1.3)$$

Example 10.1.4. In Example 10.0.1, the $k = 3$ rays of Σ are generated by the columns of F in Figure 10.1. They are labeled $\{1, 2, 3\}$, and the set $\mathcal{C}$ consists of the unique subset $\{1, 2, 3\}$ for which $\mathrm{Cone}(\rho_1, \rho_2, \rho_3) = \mathbb{R}^2$ is not contained in a cone of Σ. Proposition 10.1.3 gives $Z = \mathbb{V}_{\mathbb{C}^3}(y_1, y_2, y_3)$, which is consistent with $Z = \{0\}$ for $\mathbb{P}^2$ ⋄

We state two easy consequences of Proposition 10.1.3. Proofs are left to the reader.

Corollary 10.1.5. *If $\Sigma = \Sigma_{\mathcal{P}}$ is the normal fan of a d-dimensional polytope $\mathcal{P} \subset \mathbb{R}^d$, so that ρ_i is the normal ray corresponding to the facet $Q_i \preceq \mathcal{P}$, then*

$$Z = \bigcup_{C \in \mathcal{C}} V_{\mathbb{C}^k}(y_i : i \in C),$$

where $\mathcal{C} = \{C \subseteq [k] : \bigcap_{i\in C} Q_i = \emptyset\}$.

Corollary 10.1.6. *The codimension of Z in $\mathbb{C}^k$ is at least two.*

Example 10.1.7. In Example 10.0.1, $\Sigma = \Sigma_{\mathcal{P}}$ where $\mathcal{P}$ is the two-dimensional simplex $\mathrm{Conv}((0, 0), (1, 0), (0, 1))$. The set $\mathcal{C}$ consists of subsets of edges which do not meet. For the triangle, the only such subset is the set of all edges. Again, this leads to the correct description $Z = \mathbb{V}_{\mathbb{C}^3}(y_1, y_2, y_3) = \{0\}$ of our base locus. ⋄

Example 10.1.8. The fan in Figure 7.4(f) has four rays. It is the normal fan of a square $\mathcal{P}$, see Figure 7.6. We use the matrix

$$F = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

for the ordering of the rays (resp. edges). The set $\mathcal{C}$ consists of the following subsets:

$$\{1, 3\}, \quad \{2, 4\}, \quad \{1, 2, 3\}, \quad \{1, 3, 4\}, \quad \{1, 2, 4\}, \quad \{2, 3, 4\}, \quad \{1, 2, 3, 4\}.$$

These are the rays of Σ which do not make cones of Σ, or the edges of $\mathcal{P}$ which do not intersect in a (non-empty) face of $\mathcal{P}$. Via Proposition 10.1.3 (or Corollary 10.1.5) we find

$$Z = V(y_1, y_3) \cup V(y_2, y_4) \cup V(y_1, y_2, y_3) \cup V(y_1, y_3, y_4) \cdots \cup V(y_1, y_2, y_3, y_4),$$

where the variety is taken in $\mathbb{C}^4$. Since this equals $V(y_1, y_3) \cup V(y_2, y_4)$, we see that our description of Z is redundant. Below, we will shrink the set $\mathcal{C}$. Notice that our computation means that we will write the toric variety of Σ, i.e., the surface $\mathbb{P}^1 \times \mathbb{P}^1$ (Exercise 7.4.10), as a quotient of $\mathbb{C}^4 \setminus (V(y_1, y_3) \cup V(y_2, y_4))$. To the experienced reader, this indicates that we are on the right track. ⋄

The following definition aims to remove redundancy in Proposition 10.1.3.

Definition 10.1.9. *A subset $\mathcal{C} \subset [k]$ is a* primitive collection *for Σ if*

1. $\operatorname{Cone}(\rho_i)_{i\in\mathcal{C}} \not\subseteq \sigma$ *for all $\sigma \in \Sigma$ and*
2. *for every proper subset $\mathcal{C}' \subsetneq \mathcal{C}$, there is $\sigma \in \Sigma$ for which* $\operatorname{Cone}(\rho_i)_{i\in\mathcal{C}'} \not\subseteq \sigma$.

In other words, primitive collections are minimal subsets of rays, with respect to inclusion, satisfying the property "$\operatorname{Cone}(\rho_i)_{i\in\mathcal{C}} \not\subseteq \sigma$ for all $\sigma \in \Sigma$".

Proposition 10.1.10. *The irreducible decomposition of the base locus $Z \subset \mathbb{C}^k$ is*

$$Z = \bigcup_{\mathcal{C} \text{ primitive collection}} V_{\mathbb{C}^k}(y_i \,:\, i \in \mathcal{C}).$$

Proof. It suffices to observe that the primitive collections index the maximal varieties, with respect to inclusion, in the union of Equation (10.1.3). □

Exercise 10.1.11. Verify Proposition 10.1.10 in Example 10.1.8.

Now that we understand the base locus Z, what's missing in our description of $\pi : \mathbb{C}^k \setminus Z \to \mathcal{X}_\Sigma$ is the group G under which it is invariant. Naturally, G will be the largest subgroup of $(\mathbb{C}^*)^k$ such that $\pi(g \bullet y) = \pi(y)$ for all $g \in G$. Since π is a toric morphism, it is equivariant with respect to the action of $(\mathbb{C}^*)^k$ on $\mathbb{C}^k \setminus Z$ (Lemma 8.2.2). This means that $\pi(g \bullet y) = \pi(g) \cdot \pi(y)$. We arrive at the following definition:

$$G \;=\; \{g \in (\mathbb{C}^*)^k \,:\, \pi(g) \;=\; 1\} \;=\; \ker\left(\pi_{|(\mathbb{C}^*)^k}(\mathbb{C}^*)^k \to (\mathbb{C}^*)^d\right) .$$

The restriction $\pi_{|(\mathbb{C}^*)^k} : (\mathbb{C}^*)^k \to (\mathbb{C}^*)^d$ is a group homomorphism, which has an easy description based on the matrix F: It is given by the Laurent monomial map

$$\phi_{F^\top} \;=\; (t_1, \ldots, t_k) \mapsto (t^{F_{1,:}}, \ldots, t^{F_{n,:}}) \tag{10.1.4}$$

whose exponents are the rows of F, see Theorem 8.2.5. The kernel of $\pi|_{(\mathbb{C}^*)^k}$ is

$$G = \{g \in (\mathbb{C}^*)^k \mid g^{F_{1,:}} = \cdots = g^{F_{n,:}} = 1\}. \tag{10.1.5}$$

This is a subgroup $G \subset (\mathbb{C}^*)^k$ which acts on $\mathbb{C}^k$ by coordinatewise multiplication:

$$(g_1, \ldots, g_k) \cdot (y_1, \ldots, y_k) \mapsto (g_1 y_1, \ldots, g_k y_k)$$

(this is the restriction of the action of $(\mathbb{C}^*)^k$ on $\mathbb{C}^k$ to G).

Example 10.1.12. For $\mathcal{X}_\Sigma = \mathbb{P}^2$, the matrix $F = \left(\begin{smallmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{smallmatrix}\right)$ is that of Figure 10.1. The group G is $G = \{(g_1, g_2, g_3) \in (\mathbb{C}^*)^3 \,:\, g_1 g_3^{-1} = g_2 g_3^{-1} = 1\} = \{(\lambda, \lambda, \lambda) \,:\, \lambda \in \mathbb{C}^*\} \simeq \mathbb{C}^*$, and the action is the usual coordinate-wise scaling. ◇

Example 10.1.13. For the variety $\mathcal{X}_\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$ from Example 10.1.8, we find that

$$G \;=\; \{(g_1, g_2, g_3, g_4) \in \mathbb{C}^4 \;:\; g_1 g_3^{-1} = g_2 g_4^{-1} = 1\} = \{(\lambda, \mu, \lambda, \mu) \;:\; (\lambda, \mu) \in (\mathbb{C}^*)^2\}.$$

This group acts by scaling the first and third coordinate independently from the second and fourth coordinate: $(\lambda, \mu) \bullet (y_1, y_2, y_3, y_4) = (\lambda y_1, \mu y_2, \lambda y_3, \mu y_4)$. $\diamond$

Example 10.1.14. The normal fan of the diamond from Example 9.2.9 has ray matrix

$$F = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}.$$

The group G is a product of a two-dimensional torus with a finite group:

$$G \;=\; \{(\lambda, \mu, w\lambda, w\mu) \;:\; (\lambda, \mu) \in (\mathbb{C}^*)^2,\; w \in \{-1, 1\}\} \;\simeq\; (\mathbb{C}^*)^2 \times \{-1, 1\}. \qquad \diamond$$

Proposition 10.1.15. *The group G is isomorphic to $W \times (\mathbb{C}^*)^{k-d}$, where W is a finite group. It is a torus precisely when the invariant factors of F are all equal to one.*

Proof. This is a consequence of Proposition 1.2.13, in particular Equation (1.2.1). $\square$

Cox's quotient construction realizes the d-dimensional toric variety $\mathcal{X}_\Sigma$ as a quotient of the k-dimensional space $\mathbb{C}^k \setminus Z$ by the $(k-d)$-dimensional reductive group G.

Theorem 10.1.16 (Cox's theorem)**.** *Consider the action of the group G in* (10.1.5) *on $\mathbb{C}^k \setminus Z$. The toric morphism $\pi : \mathbb{C}^k \setminus Z \to \mathcal{X}_\Sigma$ induces a one-to-one correspondence*

$$\{\text{ closed } G\text{-orbits in } \mathbb{C}^k \setminus Z \;\} \longleftrightarrow \{\text{ points in } \mathcal{X}_\Sigma \;\}.$$

Moreover, the subset $\mathcal{U} \subseteq \mathcal{X}_\Sigma$ for which there is a one-to-one correspondence

$$\{\; G\text{-orbits in } \pi^{-1}(\mathcal{U}) \;\} \longleftrightarrow \{\text{ points in } \mathcal{U} \;\}$$

is open in $\mathcal{X}_\Sigma$ and $\operatorname{codim}_{\mathcal{X}_\Sigma}(\mathcal{X}_\Sigma \setminus \mathcal{U}) \geq 3$*. If each cone $\sigma \in \Sigma$ is simplicial, then $\mathcal{U} = \mathcal{X}_\Sigma$.*

For the definition of a simplicial cone, see Definition 2.2.10. The proof of Theorem 10.1.16 requires some concepts from GIT. It is beyond our scope to introduce these. We refer the interested reader to [14, Theorem 2.1] or [19, Theorem 5.1.11].

The open subset $\mathcal{U} \subseteq \mathcal{X}_\Sigma$ in Theorem 10.1.16 has the following explicit description, which follows from the proof of Theorem 5.1.11 in [19]. We have

$$\mathcal{X}_\Sigma \setminus \mathcal{U} = \bigcup_{\sigma \text{ non-simplicial}} O(\sigma). \tag{10.1.6}$$

The fact that $\mathcal{X}_\Sigma \setminus \mathcal{U}$ has codimension at least 3 in $\mathcal{X}_\Sigma$ now follows immediately: all cones of dimension ≤ 2 are simplicial. We say that Σ is simplicial if all of its cones are simplicial, and $\mathcal{X}_\Sigma$ is called simplicial if Σ is simplicial. If Σ is a simplicial fan, then $\pi : \mathbb{C}^k \setminus Z \to \mathcal{X}_\Sigma$ is a *geometric quotient*, meaning that G-orbits in $\mathbb{C}^k \setminus Z$ are points in $\mathcal{X}_\Sigma$. We write $\mathcal{X}_\Sigma = (\mathbb{C}^k \setminus Z)/G$ in the geometric case, and $\mathcal{X}_\Sigma = (\mathbb{C}^k \setminus Z) /\!\!/ G$ otherwise.

Remark 10.1.17. Any fan in $\mathbb{R}^2$ is simplicial. Hence, Cox's construction gives a geometric quotient representation of any normal toric surface $\mathcal{X}_\Sigma$.

A point $y \in \mathbb{C}^k \setminus Z$ is called a set of *Cox coordinates* for $\pi(y) \in \mathcal{X}_\Sigma$.

Example 10.1.18. The toric variety $\mathcal{X}_\Sigma = \mathbb{P}^2$ is simplicial, so $\mathcal{U} = \mathcal{X}_\Sigma$. The matrix F and the base locus Z were obtained in previous examples. The (real part of the) closure of three G-orbits in $\mathbb{C}^3$ are shown in Figure 10.2. This corresponds to the familiar fact that points in $\mathbb{P}^2$ are lines through the origin in $\mathbb{C}^3$. In symbols, $\mathbb{P}^2 = (\mathbb{C}^3 \setminus \{0\})/\mathbb{C}^*$. The Cox coordinates are the familiar homogeneous coordinates on $\mathbb{P}^2$.

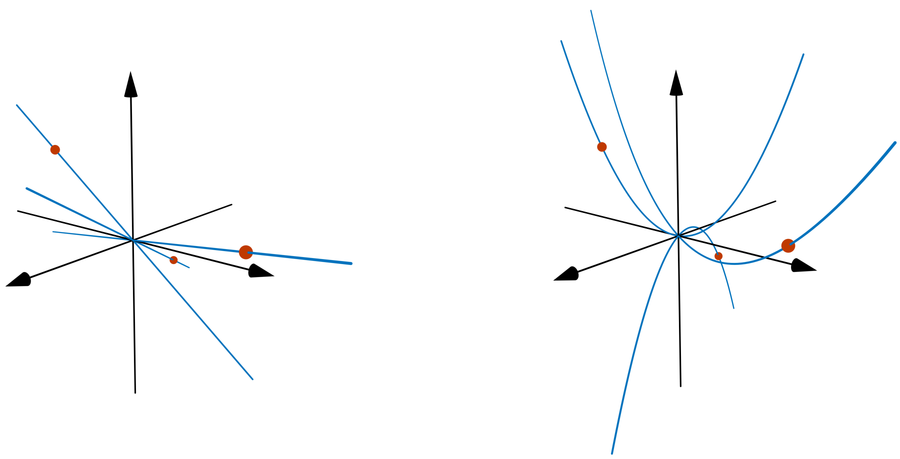

Figure 10.2: Real G-orbits of three points in the quotient construction of $\mathbb{P}^2$ (left) and $\mathbb{P}_{(1,2,1)}$ (right).

We now consider the complete fan Σ in $\mathbb{R}^2$ whose rays are given by

$$F = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{pmatrix}.$$

One checks that $Z = \{0\}$ and $G = \{(\lambda, \lambda^2, \lambda) \mid \lambda \in \mathbb{C}^*\} \simeq \mathbb{C}^*$. Some orbits are shown in the right part of Figure 10.2. The toric variety $\mathcal{X}_\Sigma$ is the *weighted projective space* $\mathbb{P}_{(1,2,1)}$. We can thus think of points in $\mathbb{P}_{(1,2,1)}$ as "curves through the origin in $\mathbb{C}^3$". ◇

Example 10.1.19. The surface $\mathcal{X}_\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$ is a geometric quotient of $\mathbb{C}^4 \setminus Z$, where Z was identified in Example 10.1.8: $Z = V(y_1, y_3) \cup V(y_2, y_4)$. By Example 10.1.13, the quotient is by the $(\mathbb{C}^*)^2$-action $(\lambda, \mu) \bullet (y_1, y_2, y_3, y_4) = (\lambda y_1, \mu y_2, \lambda y_3, \mu y_4)$. The Cox coordinates (y_1, y_2, y_3, y_4) on $\mathbb{P}^1 \times \mathbb{P}^1$ are standard: (y_1, y_3) are homogeneous coordinates on the first copy of $\mathbb{P}^1$, and (y_2, y_4) are homogeneous coordinates on the second factor. ◇

Example 10.1.20. The fan in Figure 7.4(b) gives the toric variety $\mathcal{X}_\Sigma = \mathrm{Bl}_0\mathbb{C}^2$, see Example 7.4.6. We order the rays as follows: $F = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right)$. The group $G = \{(\lambda, \lambda^{-1}, \lambda) : \lambda \in \mathbb{C}^*\}$ acts on $\mathbb{C}^3 \setminus V(y_1, y_3)$ by $\lambda \bullet (y_1, y_2, y_3) = (\lambda y_1, \lambda^{-1} y_2, \lambda y_3)$ and we find

$$\mathrm{Bl}_0\mathbb{C}^2 = \frac{\mathbb{C}^3 \setminus V(y_1, y_3)}{G}.$$

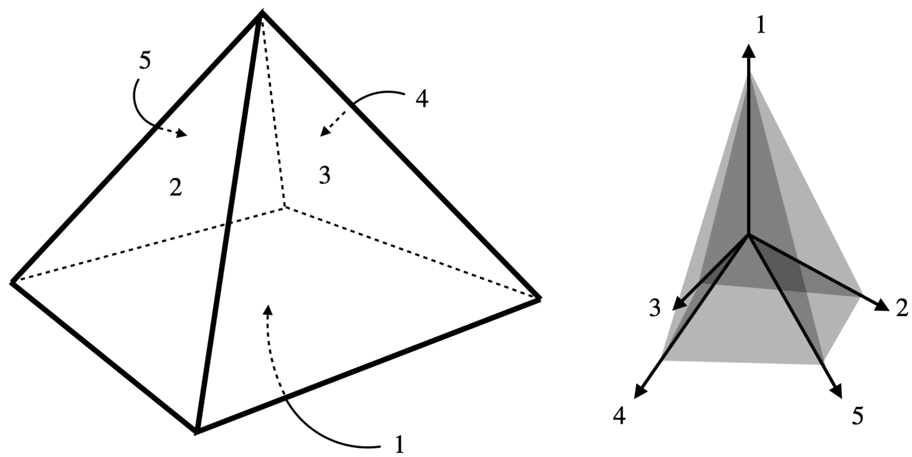

Figure 10.3: A quadrilateral pyramid and its normal fan.

By Proposition 8.2.8, points with $y_2 = 0$ get mapped into the exceptional divisor via π, see Example 8.2.9. $\diamond$

Example 10.1.21 (A non-simplicial example)**.** Consider the toric threefold $\mathcal{X}_\Sigma$ corresponding to a pyramid in $\mathbb{R}^3$ whose normal fan Σ has rays with generators

$$F = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & u_5 \end{pmatrix}.$$

The pyramid and its normal fan are illustrated in Figure 10.3. You have encountered this polytope in Exercise 3.1.6. The base locus $Z \subset \mathbb{C}^5$ is a union of two planes: $Z = V(y_1, y_2, y_4) \cup V(y_1, y_3, y_5)$. The component $V(y_1, y_2, y_4)$ is explained by the fact that the facets labeled 1, 2 and 4 have no intersection point on the pyramid. Dually, the rays labeled 1, 2 and 4 make a primitive collection. The group $G = \ker \phi_{F^\top} \subset (\mathbb{C}^*)^5$ is isomorphic to $(\mathbb{C}^*)^2$. It acts on $\mathbb{C}^5 \setminus Z$ as follows:

$$(\lambda, \mu) \bullet (y_1, y_2, y_3, y_4, y_5) \;=\; (\lambda^2\mu^2 y_1, \lambda y_2, \mu y_3, \lambda y_4, \mu y_5).$$

The normal fan Σ of our quadrilateral pyramid is non-simplicial: Facets 2, 3, 4 and 5 meet in a vertex, which means that rays 2, 3, 4 and 5 generate a non-simplicial cone of Σ. The open subset $\mathcal{U}$ on which the quotient is geometric is $\mathcal{X}_\Sigma \setminus \{p\}$, where p is the torus invariant point corresponding to the top vertex of the pyramid. Notice that $\mathcal{X}_\Sigma \setminus \mathcal{U}$ has codimension three in $\mathcal{X}_\Sigma$. The fact that $\mathcal{U} \subsetneq \mathcal{X}_\Sigma$ means that the quotient $\mathcal{X}_\Sigma = (\mathbb{C}^5 \setminus Z) /\!\!/ G$ is *not geometric*. This is supported by the following observations:

While the G-orbit of a generic point in $\mathbb{C}^5 \setminus Z$ is two-dimensional, the orbit $G \cdot q$ of $q = (1,0,0,0,0) \in \mathbb{C}^5 \setminus Z$ is isomorphic to $\mathbb{C}^*$. By [11, Proposition 1.11], the point q has a one-dimensional stabilizer. Hence, it is not a *stable* point in the sense of [11, Definition 1.25]. We claim that the orbit $G \cdot q$ is the unique closed G-orbit which, under the toric morphism $\pi : \mathbb{C}^k \setminus Z \to \mathcal{X}_\Sigma$, gets mapped to the torus fixed point p. It is clear that $G \cdot q$ is closed in $\mathbb{C}^5 \setminus Z$. To see that $\pi(G \cdot q) = \{p\}$, we apply Proposition 8.2.8. Note that $G \cdot q$ is the $(\mathbb{C}^*)^5$-orbit of $\mathcal{X}_{\Sigma'} = \mathbb{C}^k \setminus Z$ corresponding to the cone $\mathrm{Cone}(e_2, e_3, e_4, e_5)$ of Σ', whose image under the map $F_\mathbb{R}$ is the cone $\mathrm{Cone}(\rho_2, \rho_3, \rho_4, \rho_5) \in \Sigma$, whose orbit is $\{p\}$. By Theorem 10.1.16, $G \cdot q$ is the unique closed G-orbit whose image under π is $\{p\}$.

However, $G \cdot q$ is not the only orbit whose image under π is $\{p\}$. The reader is encouraged to check that the two-dimensional G-orbits $G \cdot (1, 1, 0, 1, 0)$ and $G \cdot (1, 0, 1, 0, 1)$ are not closed in $\mathbb{C}^5 \setminus Z$. Their closure contains $G \cdot q$, and their image under π is $\{p\}$.

The open subset $\mathcal{U} \subsetneq \mathcal{X}_\Sigma$ is the normal toric variety obtained by removing the non-simplicial 3-dimensional cone generated by rays 2, 3, 4 and 5 from Σ. While the group G is unchanged (it depends only on F), the base locus for this new fan is $\tilde{Z} = V(y_3, y_5) \cup V(y_2, y_4)$. The quotient $\mathcal{U} = (\mathbb{C}^5 \setminus \tilde{Z})/G$ is geometric. $\diamond$

Exercise 10.1.22. The polytope $\mathcal{P} = \mathrm{Conv}(A)$ from Example 4.2.4 has five facets, each corresponding to a Cox coordinate on $\mathcal{X}_\mathcal{P} \simeq \mathbb{P}^1 \times \mathbb{P}^2$. Compute Z and G, and confirm that the Cox coordinates are the usual homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^2$.

Exercise 10.1.23. Show that, if G is a torus (see Proposition 10.1.15), then the closure in $\mathbb{C}^5$ of the G-orbit of any point $y \in (\mathbb{C}^*)^5$ is an affine toric variety of dimension $k - d$.

The following proposition will be useful later on.

Proposition 10.1.24. *For $i = 1, \ldots, k$, the image of the set $Y_i = \{y \in \mathbb{C}^k \setminus Z \,:\, y_i = 0\}$ under the map π from Theorem 10.1.16 is the torus invariant divisor D_i. That is, it equals the closure in $\mathcal{X}_\Sigma$ of the orbit $O(\rho_i)$ corresponding to the i-th ray of Σ.*

Proof. By the orbit-cone correspondence, we have that Y_i is the union of all orbits $O(\sigma_C) \subseteq \mathcal{X}_{\Sigma'}$ corresponding to $\sigma_C \in \Sigma'$, where $i \in C$. Here we use notation from (10.1.1). By Proposition 8.2.8, the image of $O(\sigma_C) \subseteq \mathcal{X}_{\Sigma'}$ under π is $O(\mathrm{Cone}(\rho_j \,:\, j \in C)) \subset \mathcal{X}_\Sigma$. Since $i \in C$, that orbit belongs to the closure D_i of $O(\rho_i) \subseteq \mathcal{X}_\Sigma$. This shows that $\pi(Y_i) \subseteq D_i$. For the opposite inclusion, note that for each cone $\sigma \in \Sigma$ containing ρ_i, the subset $C = \{j \,:\, \rho_j \subseteq \sigma\} \subset [k]$ gives a cone $\sigma_C \in \Sigma'$ for which $O(\sigma_C) \subseteq Y_i$. $\square$

10.2 Irrelevant ideal and multigrading

The algebraic counterpart of the base locus Z is its vanishing monomial ideal, called the *irrelevant ideal*. This ideal lives in the coordinate ring $S = \mathbb{C}[y_1, \ldots, y_k]$ of the *total space* $\mathbb{C}^k$. For each $\sigma \in \Sigma$, we define a monomial

$$y^{\hat{\sigma}} = \prod_{\rho_i \not\subseteq \sigma} y_i \quad \in S, \tag{10.2.1}$$

where the product ranges over all $i \in [k] = \{1, \ldots, k\}$ such that $\rho_i = \mathbb{R}_{\geq 0} \cdot u_i \not\subseteq \sigma$. Below, $\Sigma_{\max}$ is the subset of cones in Σ which are maximal with respect to inclusion.

Proposition 10.2.1. *The base locus $Z \subset \mathbb{C}^k$ is the variety $Z = V_{\mathbb{C}^k}(B)$ with*

$$B = \left\langle y^{\hat{\sigma}} \,:\, \sigma \in \Sigma \right\rangle = \left\langle y^{\hat{\sigma}} \,:\, \sigma \in \Sigma_{\max} \right\rangle. \tag{10.2.2}$$

Proof. This follows from the description of Z in Proposition 10.1.3. The second equality in (10.2.2) follows from the fact that $y^{\hat{\sigma}}$ for $\sigma \in \Sigma_{\max}$ are the minimal generators. □

Remark 10.2.2. If Σ is simplicial, then it defines a simplicial complex Δ whose simplices are subsets of $\Sigma(1)$ which generate a cone of Σ. The irrelevant ideal B can be defined as the *Stanley-Reisner ideal* of the Alexander dual of Δ.

Example 10.2.3. Let $\mathcal{X}_\Sigma = \mathbb{P}^2$ and let F be as in Example 10.0.1. Labeling the maximal cones σ_i as in Figure 7.4(e), we have

$$y^{\hat{\sigma}_1} = y_1, \quad y^{\hat{\sigma}_2} = y_2, \quad y^{\hat{\sigma}_3} = y_3,$$

and hence $B = \langle y_1, y_2, y_3 \rangle$ as expected. The only primitive collection is $\mathcal{C} = \{1, 2, 3\}$. ⋄

Exercise 10.2.4. With F as in Exercise 10.1.8, show that for $X = \mathbb{P}^1 \times \mathbb{P}^1$ we have

$$B = \langle y_1y_2, y_1y_4, y_2y_3, y_3y_4 \rangle \;=\; \langle y_1, y_3 \rangle \cap \langle y_2, y_4 \rangle.$$

This matches the irreducible decomposition of the base locus $Z = V(y_1, y_3) \cup V(y_2, y_4)$.

In order to associate the ring S with its distinguished ideal B to our toric variety $\mathcal{X}_\Sigma$, we will equip it with a grading. That grading will be such that homogeneous elements in S define hypersurfaces in $\mathbb{C}^k$ which are stable under the action of G.

Example 10.2.5. The standard grading on $S = \mathbb{C}[y_1, y_2, y_3]$ is the decomposition

$$S \;=\; \bigoplus_{e \in \mathbb{Z}} S_e, \quad \text{where} \quad S_e \;=\; \bigoplus_{\substack{a_1, a_2, a_3 \geq 0 \\ a_1 + a_2 + a_3 = e}} \mathbb{C} \cdot y_1^{a_1}\, y_2^{a_2}\, y_3^{a_3}.$$

Here $S_e = 0$ for $e < 0$. An element $f \in S_e \setminus \{0\}$ is called *homogeneous of degree* α. The hypersurface $V_{\mathbb{C}^k}(f)$ is stable under the action of $\mathbb{C}^* \simeq G \subset (\mathbb{C}^*)^3$ (Example 10.1.12):

$$f(\lambda\, y_1, \lambda\, y_2,\, \lambda y_3) \;=\; \lambda^e\, f(y_1, y_2, y_3) \quad \Longrightarrow \quad f(y) = 0 \;\Leftrightarrow f(\lambda \bullet y) = 0, \forall \lambda \in \mathbb{C}^*. \quad \diamond$$

In general, the grading of S will be by the *(divisor) class group* $\mathrm{Cl}(\mathcal{X}_\Sigma)$ of $\mathcal{X}_\Sigma$, which is the group of Weil divisors modulo linear equivalence, see Chapter 9. That is, the *degree* of a homogeneous polynomial in S will be an element of the class group. This is consistent with Example 10.2.5 because $\mathrm{Cl}(\mathbb{P}^2) \simeq \mathbb{Z}$ (Example 9.2.6).

Definition 10.2.6. *The* $\mathrm{Cl}(\mathcal{X}_\Sigma)$*-grading of* $S = \mathbb{C}[y_1, \ldots, y_k]$ *is defined as follows:*

$$\deg(y^a) \;=\; \deg(y_1^{a_1} \cdots y_k^{a_k}) \;=\; [a_1 D_1 + \cdots + a_k D_k] \in \mathrm{Cl}(\mathcal{X}_\Sigma), \quad \textit{for all } a \in \mathbb{N}^k.$$

Here $D_i = \overline{O(\rho_i)}$ *is the torus invariant prime divisor corresponding to the* i*-th ray of* Σ.

We remind the reader that $\mathrm{Cl}(\mathcal{X}_\Sigma)$ may have torsion, see Remark 9.2.11. By Proposition 9.2.5, we have the explicit description $\mathrm{Cl}(\mathcal{X}_\Sigma) \simeq \mathbb{Z}^k/\mathrm{im}\, F^\top$. We deduced this from

$$0 \longrightarrow M \xrightarrow{F^\top} \mathbb{Z}^k \longrightarrow \mathrm{Cl}(\mathcal{X}_\Sigma) \longrightarrow 0, \tag{10.2.3}$$

where the restriction of the map $\mathbb{Z}^k \to \mathrm{Cl}(\mathcal{X}_\Sigma)$ to $\mathbb{N}^k$ can be thought of as the map that sends an exponent a to the degree of y^a. For $\alpha = [\sum_{i=1}^k a_i D_i] \in \mathrm{Cl}(\mathcal{X}_\Sigma)$, we consider the vector space S_α generated by monomials of degree α:

$$S_\alpha = \bigoplus_{F^\top m + a \geq 0} \mathbb{C} \cdot y^{F^\top m + a} \subset S, \tag{10.2.4}$$

where the sum ranges over all $m \in M$ satisfying $\langle u_i, m\rangle + a_i \geq 0$, for $i = 1, \ldots, k$. Notice that this expression for S_α is independent of the chosen representative for α: setting $a' = a + F^\top m'$ for some $m' \in M$ gives the same vector subspace S_α. Our grading is

$$S = \bigoplus_{\alpha \in \mathrm{Cl}(X_\Sigma)} S_\alpha. \tag{10.2.5}$$

Definition 10.2.7 (Cox ring)**.** *The ring S with its irrelevant ideal B and the grading from Definition 10.2.6 is called the* Cox ring *of $\mathcal{X}_\Sigma$.*

Example 10.2.8. Let $\mathcal{X}_\Sigma = \mathbb{P}^2$. The class group is $\mathrm{Cl}(\mathbb{P}^2) \simeq \mathbb{Z}^3/\operatorname{im} F^\top \simeq \mathbb{Z}$, see Example 9.2.6. Using the identification $\mathbb{Z}^3/\operatorname{im} F^\top \to \mathbb{Z}$ given by $(a_1, a_2, a_3) \sim a_1 + a_2 + a_3 \in \mathbb{Z}$ (the divisors $a_1 D_1 + a_2 D_2 + a_3 D_3$ and $(a_1 + a_2 + a_3) D_3$ are linearly equivalent), we see that the $\mathbb{Z}$-grading on S is the standard grading from Example 10.2.5:

$$\deg(y_1^{a_1} y_2^{a_2} y_3^{a_3}) = a_1 + a_2 + a_3 \quad \text{and} \quad S_{[eD_3]} = \bigoplus_{\substack{m_1 \geq 0 \\ m_2 \geq 0 \\ e - m_1 - m_2 \geq 0}} \mathbb{C} \cdot y_1^{m_1} y_2^{m_2} y_3^{e - m_1 - m_2}.$$

is spanned by monomials of "degree" e, in the classical sense. ◇

Example 10.2.9. The Cox ring of $\mathbb{P}^1 \times \mathbb{P}^1$ is $S = \mathbb{C}[y_1, y_2, y_3, y_4]$ with its irrelevant ideal B from Exercise 10.2.4, and the grading is by $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \simeq \mathbb{Z}^2$. Using the identification $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \simeq \mathbb{Z}^2$ given by $[\sum_{i=1}^4 a_i\, D_i] \mapsto (a_1 + a_3, a_2 + a_4)$ from Example 9.2.8, we have

$$\deg(y_1^{a_1} y_2^{a_2} y_3^{a_3} y_4^{a_4}) \;=\; (a_1 + a_3, a_2 + a_4).$$

Keeping in mind that y_1 and y_3 are coordinates on the first copy of $\mathbb{P}^1$ and y_2 and y_4 are coordinates on the second copy (Example 10.1.19), we see that this is the standard bigrading on the homogeneous coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$. ◇

Exercise 10.2.10. Show that the Cox ring of $\mathcal{X}_\Sigma$ from Example 10.1.18 is $\mathbb{C}[y_1, y_2, y_3]$ with irrelevant ideal $B = \langle y_1, y_2, y_3\rangle$ and grading $\deg(y_1^{a_1} y_2^{a_2} y_3^{a_3}) = a_1 + 2a_2 + a_3$.

Exercise 10.2.11. Show that the class group $\mathrm{Cl}(\mathcal{X}_\Sigma)$ is the character group of the group $G = \ker \phi_{F^\top}$. Hint: show that $G \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(\mathcal{X}_\Sigma), \mathbb{C}^*) \subset (\mathbb{C}^*)^k$ by taking $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ of the short exact sequence (10.2.3).

We are ready to generalize the table in (10.0.2) to summarize the construction:

	Algebra		Geometry	
Cox ring	S	$\xrightarrow{\mathrm{Specm}(\cdot)}$	$\mathbb{C}^k$	total coordinate space
irrelevant ideal	B	$\xrightarrow{V_{\mathbb{C}^k}(\cdot)}$	Z	base locus
divisor class group	$\mathrm{Cl}(\mathcal{X}_\Sigma)$	$\xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(\cdot,\mathbb{C}^*)}$	G	reductive group

As indicated in Example 10.2.5, a hypersurface in $\mathbb{C}^k \setminus Z$ defined by a homogeneous element in the Cox ring of $\mathcal{X}_\Sigma$ is stable under the G-action, and therefore it decends to a hypersurface in $\mathcal{X}_\Sigma$. We end the section by making this precise.

Proposition 10.2.12. *Let $f^h \in S_\alpha \setminus \{0\}$ be a homogeneous element of the Cox ring of $\mathcal{X}_\Sigma$. For any $y \in \mathbb{C}^k$, we have that $f^h(y) = 0$ if and only if $f^h(g \bullet y) = 0$ for all $g \in G$. Here G is the group defined in* (10.1.5). *That is, $V_{\mathbb{C}^k}(f^h)$ is a union of G-orbits.*

Proof. The group G acts by coordinatewise multiplication: $g \bullet y = g \cdot y = (g_1 y_1, \ldots, g_k y_k)$. An element $f^h \in S_\alpha$ can be written as $f^h = \sum_{F^\top m + a \geq 0} z_m \, y^{F^\top m + a}$ by (10.2.4), where $a = (a_1, \ldots, a_k)$ is such that $\alpha = [\sum_{i=1}^k a_i D_i]$. For any $g \in G \subset (\mathbb{C}^*)^k$, we have

$$f^h(g \cdot y) \;=\; \sum_{F^\top m + a \geq 0} z_m (g \cdot y)^{F^\top m + a} \;=\; g^a f^h(y).$$

Here we used $g^{F^\top m} = 1$, see (10.1.5). Hence $V_{\mathbb{C}^k}(f^h)$ is stable under the action of G. $\square$

We define the *zero locus* of a homogeneous element $f^h \in S_\alpha$ in $\mathcal{X}_\Sigma$ as

$$Z_{\mathcal{X}_\Sigma}(f^h) \;=\; \{p \in \mathcal{X}_\Sigma \;:\; f^h(y) = 0 \text{ for some } y \in \pi^{-1}(p)\}. \qquad (10.2.6)$$

Note that this equals $\pi(V_{\mathbb{C}^k \setminus Z}(f^h))$, where $\pi : \mathbb{C}^k \setminus Z \to \mathcal{X}_\Sigma$ is Cox's quotient morphism.

Exercise 10.2.13. Use Proposition 10.1.24 to show that $Z_{\mathcal{X}_\Sigma}(y_i) = D_i$ for $i \in [k]$.

If $\mathcal{X}_\Sigma$ is simplicial, then the points in $Z_{\mathcal{X}_\Sigma}(f^h)$ are in one-to-one correspondence with the G-orbits contained in $V_{\mathbb{C}^k \setminus Z}(f^h)$. This follows from Theorem 10.1.16 and Proposition 10.2.12. In other words, in the simplicial case, we have $\pi^{-1}(Z_{\mathcal{X}_\Sigma}(f^h)) = V_{\mathbb{C}^k \setminus Z}(f^h)$. In the non-simplicial case, the obvious inclusion $\pi^{-1}(Z_{\mathcal{X}_\Sigma}(f^h)) \supseteq V_{\mathbb{C}^k \setminus Z}(f^h)$ might be strict.

Example 10.2.14. We consider the fan Σ from Example 10.1.21. The zero locus of the monomial $f^h = y_2 \in S_{[D_2]}$ is the divisor D_2 by Proposition 10.1.24. That divisor contains the torus invariant point p corresponding to the top of the pyramid in Figure 10.3. The G-orbit $G \cdot (1,1,0,1,0)$ maps to p under the morphism π from Theorem 10.1.16. Hence, that G-orbit belongs to $\pi^{-1}(Z_{\mathcal{X}_\Sigma}(y_2))$, but not to $V_{\mathbb{C}^5 \setminus Z}(y_2)$. $\diamond$

In the next section, we identify the zero locus $Z_{\mathcal{X}_\Sigma}(f^h)$ with the effective divisor E_f of a Laurent polynomial f, see Section 9.4. Passing from f to f^h is *homogenization*.

10.3 Homogenization

In Section 9.3 we associated a polyhedron $\mathcal{P}_D$ to each torus invariant Weil divisor $D \in \mathrm{Div}_T(\mathcal{X}_\Sigma)$. We recall the definition here: For $D = a_1 D_1 + \cdots + a_k D_k$, we set

$$\mathcal{P}_D \;=\; \{m \in M_\mathbb{R} \,:\, F^\top m + a \geq 0\}.$$

The set of lattice points contained in $\mathcal{P}_D$ is $\mathcal{P}_D \cap M$, and its cardinality is $|\mathcal{P}_D \cap M|$. In (9.3.2), we also defined a subspace of the Laurent polynomials $\mathbb{C}[M]$ for each D:

$$\Gamma(\mathcal{X}_\Sigma, D) \;=\; \bigoplus_{m \in \mathcal{P}_D \cap M} \mathbb{C} \cdot t^m.$$

Recall that we keep assuming the matrix $F \in \mathbb{Z}^{d \times k}$ to have rank d.

Proposition 10.3.1. *Fix $\alpha \in \mathrm{Cl}(\mathcal{X}_\Sigma)$ and let $D = a_1 D_1 + \cdots + a_k D_k \in \mathrm{Div}_T(\mathcal{X}_\Sigma)$ be any torus invariant divisor such that $\alpha = [D]$. The linear map*

$$\eta_D : \Gamma(\mathcal{X}_\Sigma, D) \longrightarrow S_\alpha \quad \textit{given by} \quad t^m \longmapsto y^{F^\top m + a} \tag{10.3.1}$$

is an isomorphism of $\mathbb{C}$-vector spaces. In particular, if the rays of Σ are such that $\sum_{\rho \in \Sigma(1)} \mathbb{R}_{\geq 0}\, u_\rho = N_\mathbb{R}$, then we have $\dim_\mathbb{C} S_\alpha = |\mathcal{P}_D \cap M| < \infty$ for any $\alpha \in \mathrm{Cl}(\mathcal{X}_\Sigma)$.

Proof. Since F has rank d, the map $m \mapsto F^\top m + a$ is injective for any a. Hence, the linear map η_D restricts to a bijection of basis elements of $\Gamma_(\mathcal{X}_\Sigma, D)$ and S_α (see (10.2.4)). If the rays of Σ generate $N_\mathbb{R}$ as a cone, then the polyhedron $\mathcal{P}_D$ is bounded by Proposition 9.3.4, and our two isomorphic $\mathbb{C}$-vector spaces are finite-dimensional. $\square$

The map η_D in (10.3.1) is called *homogenization* with respect to D.

Exercise 10.3.2. In Exercise 9.3.7, you showed that if D and D' are linearly equivalent, then there exists $m' \in M$ such that, $\varphi : f \mapsto t^{m'} f$ is an isomorphism $\Gamma(\mathcal{X}_\Sigma, D) \to \Gamma(\mathcal{X}_\Sigma, D')$. Show that this is compatible with homogenization: $(\eta_{D'} \circ \varphi)(f) = \eta_D(f)$.

Example 10.3.3. Let $M = \mathbb{Z}^2$, $\mathbb{C}[M] = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$, $T = (\mathbb{C}^*)^2$ and consider

$$f \;=\; z_0 + z_1\, t_1 + z_2\, t_2 + z_3\, t_1 t_2 + z_4\, t_1^2 + z_5\, t_2^2 \quad \in \mathbb{C}[M] \tag{10.3.2}$$

and let $\mathcal{P} = \mathrm{Conv}((0,0),(2,0),(0,2))$ be the Newton polytope of f. Let $\Sigma = \Sigma_\mathcal{P}$ be the fan of $\mathcal{X}_\Sigma = \mathbb{P}^2$ (Example 7.4.4). The polytope $\mathcal{P}$ is naturally associated to the divisor $D = 2D_3 \in \mathrm{Div}_T(\mathcal{X}_\Sigma)$. The homogenization $\eta_D(f)$ of f with respect to D is

$$f^{h,D} \;=\; \eta_D(f) \;=\; z_0\, y_3^2 + z_1\, y_1 y_3 + z_2\, y_2 y_3 + z_3\, y_1 y_2 + z_4\, y_1^2 + z_5\, y_2^2 \quad \in S_2.$$

Here $S_2 = S_{[2D_3]}$, see Example 10.2.8. This is the usual homogenization $y_3^2 f\big(\frac{y_1}{y_3}, \frac{y_2}{y_3}\big)$. $\diamond$

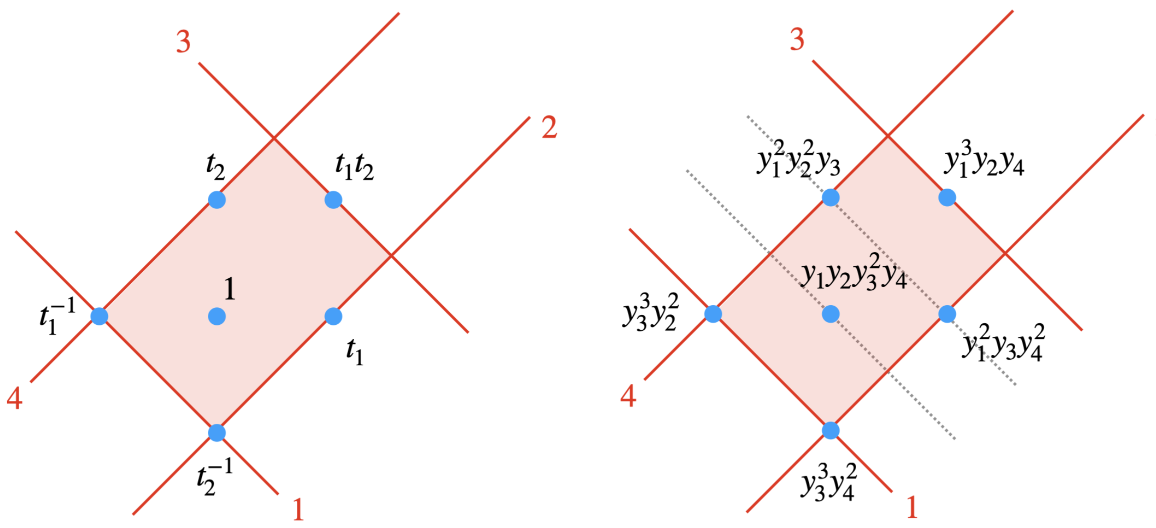

Figure 10.4: Homogenization to the Cox ring of a toric surface.

Example 10.3.4. We revisit Example 10.1.14. The divisor $D = D_1 + D_2 + 2D_3 + D_4$ on this toric surface defines a polygon $\mathcal{P}_D$ which is not a lattice polytope. It is shown in red in Figure 10.4. The coefficients are $a = (1, 1, 2, 1)^\top$. Each ray of the fan defines an inequality $\langle u_i, m\rangle + a_i \geq 0$. The corresponding halfplanes are bounded by the lines labeled $i = 1, \dots, 4$ in the figure. The vector spaces $\Gamma(\mathcal{X}_\Sigma, D)$ and $S_{[D]}$ have dimension six. This is seen from the six blue lattice points contained in $\mathcal{P}_D$. The labels of these lattice points illustrate the homogenization map $\eta_D : t^m \mapsto y^{F^\top m + a}$. $\diamond$

Remark 10.3.5. A useful observation to draw pictures like that in the right part of Figure 10.4 is that the i-th exponent $\langle u_i, m\rangle + a_i$ of $\eta_D(t^m) = y^{F^\top m + a}$ is the *lattice distance* of m to the hyperplane $H_i = \{\langle u_i, m\rangle + a_i = 0\}$. This is the number of parallel translates of H_i we need in the lattice in order for that translate to contain m. For example, the dashed lines in the right part of Figure 10.4 indicate that the lattice distance of $m = 0$ to the line labeled “3” is two, and the distance to line “1” is one. Notice that this means in particular that y_i only appears in the monomial $\eta_D(t^m)$ if m does not lie on H_i.

Example 10.3.6. The normal fan of the heptagon in the left part of Figure 10.5 has rays

$$F = \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 & -1 & -1 \end{pmatrix}.$$

Let us denote this heptagon by $\mathcal{P}$. In the figure, the edge lines of $\mathcal{P}$ are labeled according to the ordering of the columns of F. The heptagon $\mathcal{P}$ defines a torus invariant divisor divisor $D_\mathcal{P}$ on $\mathcal{X}_\mathcal{P} = \mathcal{X}_{\Sigma_\mathcal{P}}$ as in Section 9.3. That divisor is $D_\mathcal{P} = D_1 + D_2 + D_3 + 3D_4 + 3D_5 + 2D_6 + 2D_7$. By Exercise 9.3.17, we have $\mathcal{P}_{D_\mathcal{P}} = \mathcal{P}$. By counting lattice points, we find that $\dim_\mathbb{C} \Gamma(\mathcal{X}_\mathcal{P}, D_\mathcal{P}) = \dim_\mathbb{C} S_{[D_\mathcal{P}]} = 14$.

The polytope $\mathcal{P}_D$ corresponding to $D = D_1 + D_2 + D_3 + D_4 + D_5 + 2D_6 + D_7$ is the diamond shown in the right part of Figure 10.5. The coefficients $a = (1, 1, 1, 1, 1, 2, 1)$

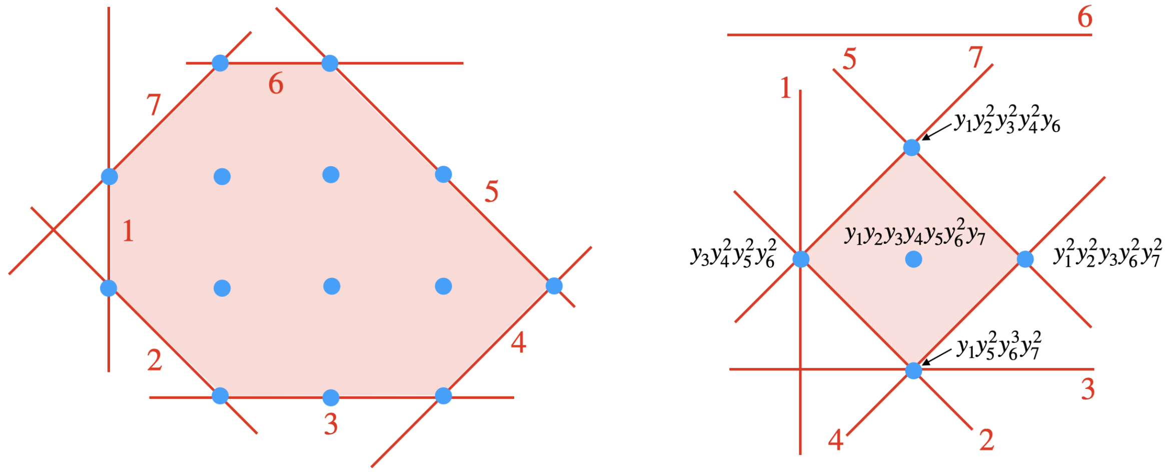

Figure 10.5: Two graded pieces of the Cox ring of $\mathcal{X}_\Sigma$ from Example 10.3.6.

give the seven lines $H_i = \{\langle u_i, m\rangle + a_i = 0\}$ shown in the picture. Notice that the sixth inequality, namely $\langle(0,-1), m\rangle + 2 \geq 0$, holds strictly on $\mathcal{P}_D$. No point of $\mathcal{P}_D \cap M$ lies on the line H_6. Therefore, each monomial of $S_{[D]}$ has y_6 as a factor, see Remark 10.3.5.

Two more examples, for the divisors $E_1 = \sum_{i=1}^{7} D_i$ and $E_2 = 2D_4 + 2D_5 + D_6 + D_7$, are shown in Figure 10.6 (left and right respectively). We encourage the reader to verify the monomial labels of the lattice points in the picture. Notice that $E_1 + E_2 = D_{\mathcal{P}}$. $\diamond$

Below, we write $a_D = (a_{D,1}, \ldots, a_{D,k}) \in \mathbb{Z}^k$ for the coefficients of a torus invariant divisor $D = \sum_{i=1}^{k} a_{D,i} D_i \in \mathrm{Div}_T(\mathcal{X}_\Sigma)$, and we write $\alpha_D = [D] \in \mathrm{Cl}(\mathcal{X}_\Sigma)$ for its class. For a Laurent polynomial $f = \sum_{m \in \mathcal{P}_D \cap M} z_m t^m \in \Gamma(\mathcal{X}_\Sigma, D)$ whose monomial support is contained in $\mathcal{P}_D$, we denote the homogenization of f with respect to D by

$$f^{h,D} = \eta_D(f) = \sum_{m \in \mathcal{P}_D \cap M} z_m \, y^{F^\top m + a_D} \in S_{\alpha_D}.$$

Our next goal is to relate the zero locus $Z_{\mathcal{X}_\Sigma}(f^{h,D})$ from (10.2.6) to the support of the effective divisor E_f defined in (9.4.1). We recall the setup. Our data are as in Section 9.4: a nonzero Laurent polynomial f, a lattice polytope $\mathcal{P}$ containing its Newton polytope, and a complete fan Σ which refines $\Sigma_{\mathcal{P}}$. This leads to the effective divisor

$$E_f^{\mathcal{P}} = D_{\mathcal{P}}(\Sigma) + \mathrm{div}(f) \in \mathrm{Div}(\mathcal{X}_\Sigma),$$

which depends on the choice of $\mathcal{P}$. That dependence was not emphasized in our notation in Section 9.4, but we will write $E_f^{\mathcal{P}}$ here to avoid confusion. The divisor $D_{\mathcal{P}}(\Sigma)$ was defined in Equation 9.3.5. The support $\mathrm{supp}(E_f^{\mathcal{P}})$ is the union of all prime divisors in $\mathcal{X}_\Sigma$ which appear with nonzero coefficient in $E_f^{\mathcal{P}}$.

We want to compare $\mathrm{supp}(E_f^{\mathcal{P}})$ with the zero locus of a homogeneous polynomial. The natural choice is the homogenization of f with respect to $D_{\mathcal{P}}(\Sigma)$:

$$f^{h,\mathcal{P}} = f^{h,D_{\mathcal{P}}(\Sigma)} = \eta_{D_{\mathcal{P}}(\Sigma)}(f) \in S_{[D_{\mathcal{P}}(\Sigma)]}.$$

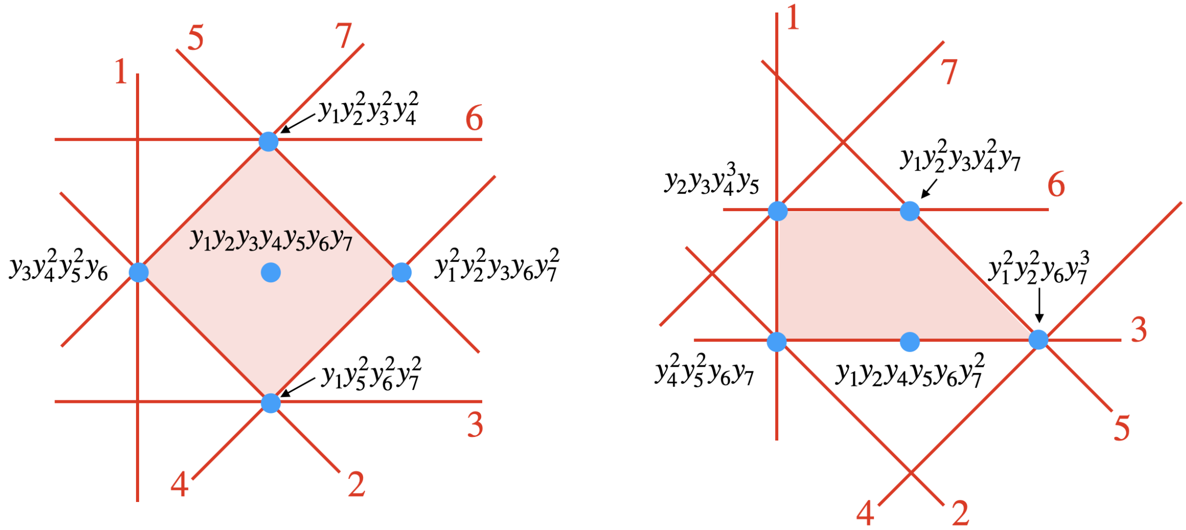

Figure 10.6: Two graded pieces of the Cox ring of $\mathcal{X}_\Sigma$ from Example 10.3.6.

Recall from Equation (10.2.6) that the zero locus of $f^{h,\mathcal{P}}$ in $\mathcal{X}_\Sigma$ is

$$Z_{\mathcal{X}_\Sigma}(f^{h,\mathcal{P}}) \;=\; \{p \in \mathcal{X}_\Sigma \;:\; f^{h,\mathcal{P}}(y) = 0 \text{ for some } y \in \pi^{-1}(p)\}.$$

Theorem 10.3.7. *With the above notation, we have* $Z_{\mathcal{X}_\Sigma}(f^{h,\mathcal{P}}) = \operatorname{supp}(E_f^{\mathcal{P}})$.

Proof. By Proposition 9.4.3, it suffices to show that

$$Z_{\mathcal{X}_\Sigma}(f^{h,\mathcal{P}}) \cap U_\sigma \;=\; V_{U_\sigma}(f_\sigma) \tag{10.3.3}$$

for each maximal cone $\sigma \in \Sigma$. Here $f_\sigma = f/t^{m_\sigma}$, where $m_\sigma \in \mathcal{V}(\mathcal{P})$ is the vertex of $\mathcal{P}$ associated to σ, as in Section 9.4. You will check in Exercise 10.3.8 that the quotient map $\pi : \mathbb{C}^k \setminus Z \to \mathcal{X}_\Sigma$ restricts to $\pi_\sigma = \pi_{|U'_\sigma} : U'_\sigma \to U_\sigma$, where $U'_\sigma = \pi^{-1}(U_\sigma) = \mathbb{C}^k \setminus V(y^{\hat\sigma})$. Writing $f = \sum_{m\in\operatorname{supp}(f)} z_m\, t^m$, the pullback π_σ^* sends f_σ to

$$\pi_\sigma^*(f_\sigma) \;=\; f_\sigma \circ \pi \;=\; \sum_{m\in\operatorname{supp}(f)} z_m\, y^{F^\top(m-m_\sigma)} \;=\; \frac{f^{h,\mathcal{P}}}{y^{F^\top m_\sigma + a_{\mathcal{P}}}} \;\in \mathbb{C}[y]_{y^{\hat\sigma}}, \tag{10.3.4}$$

where $a_{\mathcal{P}}$ is short notation for $a_{D_{\mathcal{P}}(\Sigma)}$. Exercise 10.3.9 asks you to verify this. We now see that, for any point $y \in U'_\sigma$, we have $f^{h,\mathcal{P}}(y) = 0 \Leftrightarrow (f_\sigma \circ \pi)(y) = 0 \Leftrightarrow f_\sigma(\pi(y)) = 0$. This establishes the equality in (10.3.3), and hence the theorem. $\square$

Exercise 10.3.8. Show that, for each cone $\sigma \in \Sigma$, the inverse image $\pi^{-1}(U_\sigma)$ of U_σ under the quotient map π equals $\mathbb{C}^k \setminus V(y^{\hat\sigma})$, with $y^{\hat\sigma}$ as in (10.2.1).

Exercise 10.3.9. Check that the monomial $y^{F^\top m_\sigma + a_{\mathcal{P}}}$ appearing in the proof of Theorem 10.3.7 is a unit in the localization $\mathbb{C}[y]_{y^{\hat\sigma}}$. Verify Equation (10.3.4).

Corollary 10.3.10. *If the Newton polytope of f equals $\mathcal{P}$, then $Z_{\mathcal{X}_\Sigma}(f^{h,\mathcal{P}})$ is the closure of $V_T(f)$ in $\mathcal{X}_\Sigma$.*

Proof. This follows directly from Theorem 10.3.7 and Proposition 9.4.5. $\square$

Corollary 10.3.10 says that homogenization is a practical way of compactifying hypersurfaces in a torus $T = (\mathbb{C}^*)^d$: If $\mathcal{P}$ is the Newton polytope of f, then the equation $f^{h,\mathcal{P}} = 0$ is a *global* description of the closure of $V_T(f)$ in $\mathcal{X}_\Sigma$. This is in contrast with the *local* equations $f_\sigma = 0$ for each $\sigma \in \Sigma_{\max}$ (such local equations exist because, by construction, $E_f^{\mathcal{P}}$ is a Cartier divisor, see Remark 9.4.4).

Example 10.3.11. We reconsider Example 9.4.6 using Cox coordinates. The polytope $\mathcal{P}$ is the convex hull of $(0,0), (0,1), (2,1)$ and $f = t_1t_2 + t_1^2t_2$. The condition $\text{Newt}(f) = \mathcal{P}$ is not satisfied. We have seen in Example 9.4.6 that $\text{supp}(E_f^{\mathcal{P}}) \supsetneq \overline{V_T(f)}$. Using the matrix $F = \left(\begin{smallmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & -1 \end{smallmatrix}\right)$ as before, we find that

$$f^{h,\mathcal{P}} \;=\; y_1y_2y_3 + y_1^2y_2 \;=\; y_1y_2(y_1 + y_3).$$

From this we see that $Z_{\mathcal{X}_\Sigma}(f^{h,\mathcal{P}}) \;=\; D_1 \cup D_2 \cup Z_{\mathcal{X}_\Sigma}(y_1 + y_3)$ is a union of three curves. $\diamond$

With the goals of next chapters in mind, our discussion in this section focused on the zero locus of a single homogeneous polynomial. More generally, one considers a homogeneous ideal $I \subset S$ in the Cox ring of $\mathcal{X}_\Sigma$. Here *homogeneous* means that $I = \langle f_1^h, \ldots, f_s^h \rangle$ is generated by homogeneous polynomials with respect to the grading in Definition (10.2.6). The zero locus of I in $\mathcal{X}_\Sigma$ is

$$Z_{\mathcal{X}_\Sigma}(I) \;=\; \{p \in \mathcal{X}_\Sigma \;:\; \text{for all } f \in I, \text{ there is } y \in \pi^{-1}(p) \text{ such that } f(y) = 0\ \}.$$

The following is a summary of Propositions 5.2.7 and 5.2.8 in [19].

Theorem 10.3.12. *The zero locus $Z_{\mathcal{X}_\Sigma}(I)$ of a homogeneous ideal $I \subset S$ is a closed subvariety of $\mathcal{X}_\Sigma$. All closed subvarieties arise in this way. Moreover, if Σ is simplicial, then $I \mapsto Z_{\mathcal{X}_\Sigma}(I)$ is a one-to-one correspondence between $\{$radical homogeneous ideals contained in the irrelevant ideal $B\}$ and $\{$closed subvarieties of $\mathcal{X}_\Sigma\}$.*

Further reading

We assumed throughout the chapter that F has rank d. The case where F has rank $< d$ is discussed at the end of [19, Section 5.1]. For more on the correspondence between subschemes of $\mathcal{X}_\Sigma$ and homogeneous ideals of S, see [14, Section 3]. In [37], Hu and Keel generalize Cox's GIT construction to *Mori dream spaces*. Such generalized Cox rings are the topic of the textbook [1].

Chapter 11

Toric discriminants and resultants

The most basic example of a *discriminant* arises from the equation $a\,t^2 + b\,t + c \,=\, 0$, with $a, b, c \in \mathbb{C}$ and $a \neq 0$. It is a standard fact that this equation has two solutions $t \in \mathbb{C}$ if and only if $\Delta \neq 0$, where Δ is the *discriminant polynomial*:

$$\Delta \,=\, b^2 - 4\,ac. \tag{11.0.1}$$

That is, the *discriminant hypersurface* $\nabla = \{(a, b, c) \in \mathbb{C}^3 \,:\, \Delta = 0\}$ detects *non-generic* behavior over the complex numbers, where non-generic means having only one root. Moreover, if a, b, c are real numbers, then the number of real solutions is two if $\Delta > 0$, one if $\Delta = 0$ and zero if $\Delta < 0$. Hence, ∇ *discriminates* between qualitatively different behaviors over the real numbers, which justifies the name *discriminant*.

In general, the term *discriminant* is used in complex algebraic geometry for subvarieties of the parameter space of a parametric problem which detect non-generic behavior. In the example above, the parameters are a, b and c, which are coordinates on the natural parameter space $\mathbb{C}^3$ (or $\mathbb{P}^2$). *Resultants* are examples of discriminants in this sense.

This chapter deals with vast generalizations of the quadratic discriminant (11.0.1) which are naturally associated to projective toric varieties. We introduce *A-discriminants* (Section 11.1), *A-resultants* (Section 11.4) and *principal A-determinants* (Section 11.5). The A in these names refers to the matrix $A \in \mathbb{Z}^{d\times n}$ we have been using all along to define the projective toric variety X_A. These objects were studied and popularized by Gel'fand, Kapranov and Zelevinsky, motivated by their work on generalized hypergeometric functions. Their seminal book [33] is the standard reference on this topic.

11.1 A-Discriminants

Our setup in this section is as follows. We fix a matrix $A \in \mathbb{Z}^{d\times n}$ whose columns are $a_1, \ldots, a_n \in \mathbb{Z}^d$. The associated projective toric variety X_A lives in the projective space $\mathbb{P}^{n-1}$, see (1.1.4). We define a Laurent polynomial $f_{A,z} \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ whose monomial support is A, and whose coefficients are complex valued parameters $z = (z_1, \ldots, z_n)$:

$$f_{A,z}(t) \,=\, z_1\, t^{a_1} + z_2\, t^{a_2} + \cdots + z_n\, t^{a_n}. \tag{11.1.1}$$

For any choice of $z \in \mathbb{C}^n$, $f_{A,z}(t)$ defines a (possibly empty) subvariety

$$H_{A,z} \;=\; V_{(\mathbb{C}^*)^d}(f_{A,z}(t)) \;=\; \{t \in (\mathbb{C}^*)^d \;:\; f_{A,z}(t) = 0\} \tag{11.1.2}$$

of the torus $T = (\mathbb{C}^*)^d$. If $n > 1$, then $H_{A,z}$ is a hypersurface for generic z. That hypersurface is invariant under scaling the coefficients z, so it is natural to think of $z = (z_1 : \cdots : z_n)$ as a point in a projective space $\mathbb{P}^{n-1}$. For reasons explained below, we will denote this projective space by $(\mathbb{P}^{n-1})^\vee$ instead. This distinguishes z-space $(\mathbb{P}^{n-1})^\vee$ from $\mathbb{P}^{n-1} \supseteq X_A$, for which we have been using coordinates $x_1, \ldots, x_n$.

A singular point of $H_{A,z}$ is a point $t \in (\mathbb{C}^*)^d$ satisfying the $d+1$ equations

$$f_{A,z}(t) \;=\; \frac{\partial f_{A,z}}{\partial t_1}(t) \;=\; \cdots \;=\; \frac{\partial f_{A,z}}{\partial t_d}(t) \;=\; 0. \tag{11.1.3}$$

We say that $H_{A,z}$ is singular if there exists such a singular point.

Definition 11.1.1 (A-discriminant variety)**.** *The* A-discriminant variety $\nabla_A \subset (\mathbb{P}^{n-1})^\vee$ *associated to the matrix* $A \in \mathbb{Z}^{d\times n}$ *is the Zariski closure of the set*

$$\nabla_A^\circ \;=\; \{z \in (\mathbb{P}^{n-1})^\vee \;:\; H_{A,z} \textit{ is singular }\}.$$

Example 11.1.2. Let $A = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix} \in \mathbb{Z}^{1\times 3}$. The polynomial $f_{A,z}$ is

$$f_{A,z}(t) \;=\; z_1 + z_2\, t + z_3\, t^2.$$

The equations (11.1.3) are $f_{A,z}(t) = f'_{A,z}(t) = 0$. They have a solution if and only if $f_{A,z}$ has a double root in the torus. We find that $\nabla_A = \{(z_1 : z_2 : z_3) \in (\mathbb{P}^2)^\vee \;:\; z_2^2 - 4\, z_3 z_1 = 0\}$. Substituting $(z_1, z_2, z_3) = (c, b, a)$, this is the curve defined by (11.0.1). $\diamond$

Exercise 11.1.3. If $A \in \mathbb{Z}^{d\times\binom{d+2}{2}}$ consists of all nonnegative integer d-vectors whose sum is at most two, then $f_{A,z}(t) = \frac{1}{2}\sum_{i,j=0}^{d} z_{ij}\, t_i t_j$, where $t_0 = 1$. There is a symmetric $(d+1)\times(d+1)$ matrix such that $f_{A,z} = \frac{1}{2} v^\top M(z) v$ where v is the column vector $(1, t_1, \ldots, t_d)$. Show that $\nabla_A = \{z \in (\mathbb{P}^{n-1})^\vee \;:\; \det M_z = 0\}$. For instance, in Example 11.1.2, we have $d = 1$ and $M(z) = \left(\begin{smallmatrix} 2z_1 & z_2 \\ z_2 & 2z_3 \end{smallmatrix}\right)$.

Toric varieties enter the picture in the following geometric interpretation of ∇_A.

Proposition 11.1.4. *Let* $A \in \mathbb{Z}^{d\times n}$ *be such that* $X_A \subsetneq \mathbb{P}^{n-1}$*. The* A*-discriminant variety* $\nabla_A \subseteq (\mathbb{P}^{n-1})^\vee$ *is the projective dual variety (see below for a definition) of the projective toric variety* $X_A \subseteq \mathbb{P}^{n-1}$*. In particular,* ∇_A *is irreducible of dimension* $\leq n-2$*.*

The assumption that $X_A \subsetneq \mathbb{P}^{n-1}$ is justified in Exercise 11.3.2 below. We recall the definition of the *projective dual variety* of a projective variety $X \subset \mathbb{P}^{n-1}$. The points of the dual projective space $(\mathbb{P}^{n-1})^\vee$ are the hyperplanes in $\mathbb{P}^{n-1}$:

$$z = (z_1 : \cdots : z_n) \in (\mathbb{P}^{n-1})^\vee \;\sim\; \Lambda_z = \{x \;:\; z_1\, x_1 + \cdots + z_n\, x_n = 0\} \subset \mathbb{P}^{n-1}.$$

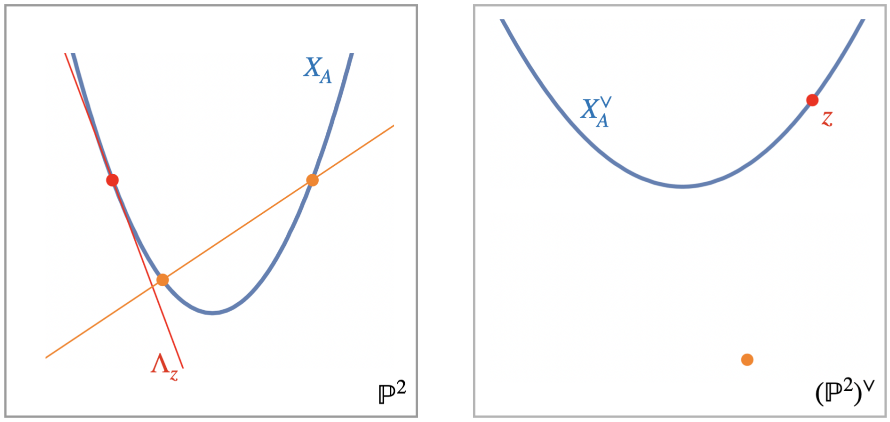

Figure 11.1: Geometric interpretation of the discriminant $b^2 - 4\,ac$.

Let $X_{\rm sm} \subseteq X$ denote the smooth points of X. The projective dual variety of X is

$$X^\vee \;=\; \overline{\{z \in (\mathbb{P}^{n-1})^\vee \,:\, \Lambda_z \supseteq T_x X \text{ for some } x \in X_{\rm sm}\}}.$$

Here $T_x X$ is the tangent space of X at $x \in X$. The condition $\Lambda_z \supseteq T_x X$ reads "Λ_z is tangent to X at x". Proposition 11.1.4 states that $X_A^\vee = \nabla_A$. Before stating its proof, we present an example which may give some intuition about the statement.

Example 11.1.5. The toric variety X_A corresponding to A from Example 11.1.2 is a smooth conic in $\mathbb{P}^2$, given by $x_1x_3 - x_2^2 = 0$. Its dual curve $\nabla_A = X_A^\vee$ is defined by the discriminant $z_2 - 4\,z_1z_3 = 0$. These curves are plotted, in the charts $x_1 = 1$ and $z_1 = 1$ respectively, in Figure 11.1. A point $z \in (\mathbb{P}^2)^\vee$ defines a line $\Lambda_z \subset \mathbb{P}^2$ and a quadratic polynomial $f_{A,z} = z_1 + z_2\,t + z_3\,t^2$. If z is generic, then Λ_z intersects X_A in the points $\Phi_A(t_1), \Phi_A(t_2)$, where t_1, t_2 are the roots of f. If $z \in \nabla_A$, then Λ_z is tangent to X_A at the image under Φ_A of the double root of $f_{A,z}$. In Figure 11.1, the red point $z \in \nabla_A$ is $(1:2:1)$, for which $f_{A,z} = (t+1)^2$ with double root $t = -1$. The red line $\Lambda_z \subset \mathbb{P}^2$ is given by $x_1 + 2x_2 + x_3 = 0$. It is tangent to X_A at the point $\Phi_A(-1) = (1:-1:1)$. $\diamond$

Proof of Proposition 11.1.4. The *conormal variety* of X_A is the Zariski closure of

$$\mathrm{con}(X_A)^\circ \;=\; \{(x,z) \in \mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^\vee \,:\, x \in (X_A)_{\rm sm},\; \Lambda_z \supseteq T_x X_A\}.$$

We denote it by $\mathrm{con}(X_A) = \overline{\mathrm{con}(X_A)^\circ}$. Let $\mathrm{pr}_2 : \mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^\vee \to (\mathbb{P}^{n-1})^\vee$ be the coordinate projection $(x,z) \mapsto z$. By definition, we have

$$X_A^\vee \;=\; \overline{\mathrm{pr}_2(\mathrm{con}(X_A)^\circ)} \;=\; \mathrm{pr}_2(\mathrm{con}(X_A)).$$

We claim that $\mathrm{con}(X_A)^\circ$, and hence also $\mathrm{con}(X_A)$, is irreducible of dimension $n-2$. The other coordinate projection $\mathrm{pr}_1 : \mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^\vee \to \mathbb{P}^{n-1}$ is such that $\mathrm{pr}_1(\mathrm{con}(X_A)^\circ) = (X_A)_{\rm sm}$. Each fiber is a linear subspace of $(\mathbb{P}^{n-1})^\vee$ of dimension $n - 2 - \dim X_A$. Hence, $\mathrm{con}(X_A)^\circ$ is a $\mathbb{P}^{n-2-\dim X_A}$-bundle over the irreducible variety $(X_A)_{\rm sm}$, which implies that it is irreducible of dimension $n-2$ [59, Chapter 1, §6, Theorem 8]. By Remark 4.5.8, the points $\operatorname{im}\Phi_A = X_A \cap (\mathbb{C}^*)^{n-1}$ form a dense subset of $(X_A)_{\mathrm{sm}}$. The same argument as above shows that the following variety is irreducible of dimension $n-2$:

$$\operatorname{con}(X_A)^{\circ\circ} \;=\; \{(x,z) \in \mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^\vee \;:\; x \in \operatorname{im}\Phi_A,\; \Lambda_z \supseteq T_x X_A\} \;\subseteq\; \operatorname{con}(X_A)^\circ.$$

In particular, we have $\overline{\operatorname{con}(X_A)^{\circ\circ}} = \overline{\operatorname{con}(X_A)^\circ}$ and thus $X_A^\vee \;=\; \overline{\operatorname{pr}_2(\operatorname{con}(X_A)^{\circ\circ})}$.

The $(\operatorname{rank}\mathbb{Z}'A)$-dimensional tangent space $T_x X_A$ at $x = \Phi_A(t)$ is spanned by the points $\Phi_A(t) = (t^{a_1} : \cdots : t^{a_n})$ and the d partial derivatives $(a_{i1}\, t^{a_1 - e_i} : \cdots : a_{in}\, t^{a_n - e_i})$, $i = 1, \ldots, d$. One checks that $\Lambda_z \supseteq T_{\Phi_A(t)} X_A$ is equivalent to the equations (11.1.3). Hence,

$$\begin{aligned} \operatorname{pr}_2(\operatorname{con}(X_A)^{\circ\circ}) \;&=\; \{z \in (\mathbb{P}^{n-1})^\vee \;:\; \Lambda_z \text{ contains } T_x X_A \text{ for some } x \in \operatorname{im}\Phi_A\} \\ &=\; \{z \in (\mathbb{P}^{n-1})^\vee \;:\; \exists\, t \in (\mathbb{C}^*)^d \text{ satisfying (11.1.3)}\} \;=\; \nabla_A^\circ, \end{aligned}$$

with ∇_A° as in Definition 11.1.1. Since $\nabla_A = \overline{\nabla_A^\circ}$, we have now shown that $\nabla_A = X_A^\vee$.

We conclude that $\nabla_A = \operatorname{pr}_2(\operatorname{con}(X_A))$ is the coordinate projection of an irreducible variety of dimension $n-2$. Hence, ∇_A is irreducible of dimension $\leq n-2$. □

Exercise 11.1.6. Let $X \subset \mathbb{P}^{n-1}$ be any irreducible projective variety. The conormal variety $\operatorname{con}(X) \subseteq \mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^\vee$ of X is the Zariski closure of

$$\operatorname{con}(X)^\circ \;=\; \{(x,z) \in \mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^\vee \;:\; x \in X_{\mathrm{sm}},\; \Lambda_z \supseteq T_x X\}.$$

The techniques in our proof of Proposition 11.1.4 apply to show that $\operatorname{con}(X)$ is irreducible of dimension $n-2$, and hence $X^\vee$ is irreducible of dimension at most $n-2$.

Corollary 11.1.7. *The A-discriminant variety ∇_A only depends on X_A. In particular, it does not change under affine transformations of A. That is, $\nabla_{T\cdot A+m} = \nabla_A$, where T, m are as in Proposition 1.3.14.*

It follows from our proof of Proposition 11.1.4 that the A-discriminant variety ∇_A is a hypersurface in $(\mathbb{P}^{n-1})^\vee$, i.e., it is of dimension $n-2$, if and only if the projection $\operatorname{pr}_2 : \operatorname{con}(X_A) \to \nabla_A$ has generically finite fibers. In that case, the defining polynomial of ∇_A is the *A-discriminant* $\Delta_A \in \mathbb{C}[z_1, \ldots, z_n]$. A more formal definition will be given in Definition 11.3.7. Varieties whose projective dual is not a hypersurface are called *dual defective*. Matrices A for which X_A is dual defective were characterized by Esterov in [28, Corollary 3.20]. As a slogan, ∇_A is *usually* a hypersurface. Here is a non-example.

Example 11.1.8. The matrix A from Example 1.1.7 defines a three-dimensional toric variety $X_A \subset \mathbb{P}^5$ whose points are the rank-one 2×3 matrices. You analyzed its torus orbits in Exercise 4.2.4. We have

$$A \;=\; \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad f_{A,z} \;=\; t_1(z_1\, t_3 + z_2\, t_4 + z_3\, t_5) + t_2(z_4\, t_3 + z_5\, t_4 + z_6\, t_5).$$

A point $t \in (\mathbb{C}^*)^5$ satisfies (11.1.3) if and only if (t_1, t_2) is a left kernel vector of the 2×3 matrix $Z = \left(\begin{smallmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \end{smallmatrix}\right)$, and (t_3, t_4, t_5) is a right kernel vector of Z. In particular, Z has rank one, which implies that ∇_A has codimension two. ⋄

Exercise 11.1.9. Let A be as in Example 11.1.8. Check that the generic fiber of the projection $\mathrm{pr}_2 : \mathrm{con}(X_A) \to \nabla_A$ has dimension one.

The A-*discriminant* Δ_A is the defining equation of ∇_A, in case it is a hypersurface. This only fixes Δ_A up to a nonzero constant multiple. A preferred normalization exploits the fact that Δ_A can be chosen to have integer coefficients, see Definition 11.3.7. For now, our statements do not depend on this normalization.

Proposition 11.1.10. *If ∇_A is a hypersurface and $\Delta_A = 0$ is its defining equation, then for a smooth point $z \in \nabla_A^\circ$ we have:*

1. *The hyperplane Λ_z is tangent to X_A at precisely one point. That point is given by*
$$x = \Big(\frac{\partial \Delta_A}{\partial z_1}(z) : \cdots : \frac{\partial \Delta_A}{\partial z_n}(z)\Big) \in X_A.$$
2. *The singular points of $H_{A,z}$ are the points in the fiber $\Phi_A^{-1}(x) \subset (\mathbb{C}^*)^d$, with x as in point 1. In particular, if Φ_A is one-to-one (see Proposition 3.2.3), then for generic $z \in \nabla_A$, the hypersurface $H_{A,z}$ has only one singular point.*

Proof. Let $X \subseteq \mathbb{P}^{n-1}$ be an irreducible projective variety. The *biduality theorem* [33, Chapter 1, Theorem 1.1] states that $(X^\vee)^\vee = X$ and, if $x \in X_{\mathrm{sm}}$ and $z \in (X^\vee)_{\mathrm{sm}}$, then $H_z \subset \mathbb{P}^{n-1}$ is tangent to X at x if and only if $H_x \subset (\mathbb{P}^{n-1})^\vee$ is tangent to $X^\vee$ at z. Here H_x is the hyperplane in $(\mathbb{P}^{n-1})^\vee$ corresponding to $x \in \mathbb{P}^{n-1}$. We apply this to our situation, with $X = X_A$ and $X^\vee = \nabla_A$ (Proposition 11.1.4). Since $z \in \nabla_A^\circ$, we know that H_z is tangent to X_A at some $x \in \mathrm{im}\,\Phi_A \subseteq (X_A)_{\mathrm{sm}}$. Therefore, H_x is a tangent hyperplane to ∇_A at z. Since ∇_A is a hypersurface by assumption, such a hyperplane is unique and given by the gradient of the defining equation Δ_A. This proves the first point of the proposition, and the second point is an easy consequence. □

11.2 Degree of the A-discriminant

We will see that the A-discriminant polynomial Δ_A can be quite complicated (Exercise 11.3.10). A first step towards computing Δ_A is to understand its degree. In fact, if ∇_A is a hypersurface, then Δ_A is homogeneous with respect to the $\mathbb{Z}^{d+1}$-*multigrading* induced by $\hat{A}$, which is useful for computations. This grading is defined as follows. For $b \in \mathbb{Z}^{d+1}$ and $v \in \mathbb{N}^n$, the monomial $z^v \in \mathbb{C}[z_1, \ldots, z_n]$ is said to have degree b if $\hat{A}\, v = b$. An element $\Delta \in \mathbb{C}[z_1, \ldots, z_n]$ is $\hat{A}$-*homogeneous* of degree b if $\Delta = \sum_{v \in \mathbb{N}^n, \deg(z^v) = b} c_v \, z^v$ for some coefficients $c_v \in \mathbb{C}$. Notice that, since the last row of $\hat{A}$ is a row of ones, $\hat{A}$-homogeneity implies homogeneity in the usual sense.

Proposition 11.2.1. *If A is such that $\nabla_A \subset (\mathbb{P}^{n-1})^\vee$ is a hypersurface, then there exists $b \in \mathbb{Z}^{d+1}$ such that Δ_A is $\hat{A}$-homogeneous of degree b.*

Proof. Let $\Delta_A = \sum_{v \in \mathbb{N}^n} c_v\, z^v$ and consider the torus action $(\mathbb{C}^*)^{d+1} \times \mathbb{C}^n \to \mathbb{C}^n$ given by $(t_1, \ldots, t_d, u) \bullet z = (u\, t^{a_1} z_1, \ldots, u\, t^{a_n} z_n)$, as encoded by the matrix $\hat{A}$. One checks that the hypersurface $H_{A,z}$ from (11.1.2) is singular at some point $t' \in (\mathbb{C}^*)^d$ if and only if $H_{A,(t,u)\bullet z}$ is singular at $t^{-1} \cdot t'$ (this is coordinate-wise multiplication). It immediately follows that $z \in \nabla_A^\circ$ if and only if $(t,u) \bullet z \in \nabla_A^\circ$. Therefore $\Delta_A(z) = 0$ if and only if $\Delta_A((t,u) \bullet z) = 0$, which means that $\Delta_A((t,u) \bullet z)$ is a constant multiple of $\Delta_A(z)$ for each $(t,u) \in (\mathbb{C}^*)^{d+1}$: $\Delta_A((t,u) \bullet z) = K(t,u) \cdot \Delta_A(z)$. Expanding gives

$$\Delta_A((t,u) \bullet z) \;=\; \sum_{v \in \mathbb{N}^n} c_v\, ((t,u) \bullet z)^v \;=\; \sum_{v \in \mathbb{N}^n} c_v\, t^{A\,v}\, u^{\sum_i v_i}\, z^v \;=\; K(t,u) \cdot \sum_{v \in \mathbb{N}^n} c_v\, z^v.$$

This implies that $\hat{A}\, v = (A\, v, \sum_i v_i) = b$ is independent of v for each v with $c_v \neq 0$. □

Proposition 11.2.1 implies the following structural property of Δ_A.

Corollary 11.2.2. *If A is such that $\nabla_A \subset (\mathbb{P}^{n-1})^\vee$ is a hypersurface, then the Newton polytope of Δ_A has dimension at most $n - 1 - \dim(X_A)$.*

Proof. By Proposition 11.2.1, all exponents v of Δ_A lie in the $(n - 1 - \dim(X_A))$-dimensional affine space $\hat{A}\, v = b$ for some $b \in \mathbb{Z}^{d+1}$. This uses Corollary 1.3.18. □

Example 11.2.3. The discriminant Δ_A in Example 11.1.2 is a polynomial in $n = 3$ variables, whose Newton polytope is a line segment. ◇

Determining the $\hat{A}$-degree $b \in \mathbb{Z}^{d+1}$ of the A-discriminant is a hard task in general. An alternating sum formula for the total degree of ∇_A, i.e., its degree as a hypersurface in $(\mathbb{P}^{n-1})^\vee$, is given in [33, Chapter 9, Theorem 2.8]. For a face $Q \preceq \mathcal{P} = \mathrm{Conv}(A)$, we use the notation $\mathrm{vol}_Q(Q)$ for the *normalized volume* of Q in the affine lattice $\mathbb{Z}'(A \cap Q)$; the intersection of $\mathbb{Z}^d$ with the smallest affine space containing Q. Here *normalized* volume means that the volume is scaled so that vol_Q of a standard simplex in that lattice is 1.

Theorem 11.2.4. *If $X_A \subset \mathbb{P}^{n-1}$ is smooth and ∇_A is a hypersurface, then*

$$\deg \nabla_A \;=\; \sum_{Q \preceq \mathrm{Conv}(A)} (-1)^{\dim \mathcal{P} - \dim Q} (\dim Q + 1)\, \mathrm{vol}(Q).$$

Moreover, this sum is nonnegative, and it evaluates to 0 *if and only if* $\dim \nabla_A < n - 2$.

The tools needed to prove Theorem 11.2.4 are beyond our scope. We illustrate the formula with some examples/exercises.

Example 11.2.5. Figure 11.2 shows the polygon $\mathrm{Conv}(A)$ for the matrix

$$A \;=\; \begin{pmatrix} -3 & -3 & -2 & -2 & 2 & 3 \\ 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

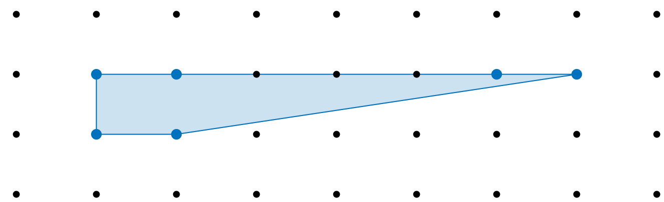

Figure 11.2: The convex hull of A (six blue dots) from Example 11.2.5.

You will check in Exercise 11.2.6 that X_A is a smooth surface, so that Theorem 11.2.4 applies. Our polygon has normalized volume 7: in order for a simplex to have volume 2, we must multiply the Euclidean volume $\mathrm{Vol}(Q)$ by 2. The edge connecting the vertices $(-3,0)$ and $(3,0)$ has normalized volume $\mathrm{vol}_Q(Q) = 6$. All other edges have volume 1 in their affine lattice. A vertex has volume 1 (it is viewed as a standard simplex in its zero-dimensional affine lattice). Therefore, we have

$$\deg \nabla_A \;=\; (-1)^0\cdot(2+1)\cdot 7+(-1)^1\cdot(1+1)\cdot(6+1+1+1)+(-1)^2\cdot(0+1)\cdot(1+1+1+1) \;=\; 7.$$

The fact that this is positive implies that ∇_A is a hypersurface. You will confirm in Exercise 11.3.11 that its defining equation is

$$\Delta_A \;=\; -z_1z_4^6 - z_2^6 z_6 + z_2^5 z_4 z_5 + z_2 z_3 z_4^5 \;=\; 0.$$

The total degree is indeed 7. The $\hat{A}$-degree is $(-15,-6,7)$. The Newton polytope of Δ_A is a tetrahedron of dimension $n-1-\dim(X_A) = 3$ (see Corollary 11.2.2). $\diamond$

Exercise 11.2.6. Show that X_A with A as in Example 11.2.5 is a smooth toric surface.

Exercise 11.2.7. Check that the formula from Theorem 11.2.4 gives 0 for the matrix A in Example 11.1.8. That matrix corresponds to a smooth toric variety whose A-discriminant variety has codimension two.

Exercise 11.2.8. Use Theorem 11.2.4 to show that, if the columns of A are all lattice points in the ℓ-dilated simplex $\ell\cdot\mathrm{Conv}(e_0,e_1,\dots,e_d)\subseteq\mathbb{R}^{d+1}$, then $\deg\Delta_A = (d+1)(\ell-1)^d$. This is the degree of the discriminant of a polynomial of total degree ℓ in d variables. For instance, for $\ell=2, d=1$, we use $A = \left(\begin{smallmatrix} 2 & 1 & 0\\ 0 & 1 & 2\end{smallmatrix}\right)$ and find the same A-discriminant as in Example 11.1.2 (by Corollary 11.1.7). Deduce, by symmetry, that the $\hat{A}$-degree of Δ_A is $(\ell(\ell-1)^d,\dots,\ell(\ell-1)^d,(d+1)(\ell-1)^d)$.

11.3 Computing A-discriminants

We will derive an elimination algorithm for computing the prime vanishing ideal of ∇_A, and we will state a parametric representation known as the *Horn uniformization*. A consequence of these results is that ∇_A is defined over $\mathbb{Q}$, i.e., its ideal in $\mathbb{C}[z_1,\dots,z_n]$ can be generated by polynomials with integer coefficients.

Notice that, since we are interested in singular points $t \in (\mathbb{C}^*)^d$ of $H_{A,z}$ with nonzero coordinates, the equations (11.1.3) are equivalent to the following equations:

$$f_{A,z}(t) \;=\; t_1 \cdot \frac{\partial f_{A,z}}{\partial t_1}(t) \;=\; \cdots \;=\; t_d \cdot \frac{\partial f_{A,z}}{\partial t_d}(t) \;=\; 0. \tag{11.3.1}$$

You will verify in Exercise 11.3.1 that this can be written in matrix form as

$$\hat{A} \cdot \begin{pmatrix} t^{a_1} & 0 & \cdots & 0 \\ 0 & t^{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t^{a_n} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = 0. \tag{11.3.2}$$

Here $\hat{A}$ is the matrix A with a row of ones added, as in Equation (1.3.2).

Exercise 11.3.1. Verify that (11.3.1) and (11.3.2) are equivalent.

Define the following *incidence variety*:

$$V_A \;=\; \left\{(t,z) \in (\mathbb{C}^*)^d \times (\mathbb{P}^{n-1})^\vee \;:\; (t,z) \text{ satisfies } (11.1.3)\text{-}(11.3.2)\,\right\}. \tag{11.3.3}$$

It comes with two natural coordinate projections:

$$\mathrm{pr}_1 : V_A \longrightarrow (\mathbb{C}^*)^d,\;\; \mathrm{pr}_1(t,z) = t \quad \text{and} \quad \mathrm{pr}_2 : V_A \longrightarrow (\mathbb{P}^{n-1})^\vee,\;\; \mathrm{pr}_2(t,z) = z.$$

By definition, we have $\nabla_A^\circ = \mathrm{pr}_2(V_A)$, and the A-discriminant variety is $\overline{\mathrm{pr}_2(V_A)}$.

Exercise 11.3.2. Use (11.3.2) and Corollary 1.3.18 to show that $X_A = \mathbb{P}^{n-1} \Rightarrow \nabla_A = \emptyset$.

Exercise 11.3.3. Show that V_A is irreducible of dimension $n-2$. Hint: use the techniques from the proof of Proposition 11.1.4.

The algorithm we present for computing the prime ideal of ∇_A is based on the equality $\nabla_A = \overline{\mathrm{pr}_2(V_A)}$. For each column a_i of A, we choose nonnegative integer vectors $a_i^+, a_i^- \in \mathbb{N}^d$ such that $a_i = a_i^+ - a_i^-$. We consider new variables $v_1, \ldots, v_d$ and work in the polynomial ring $R = \mathbb{C}[t_1, \ldots, t_d, v_1, \ldots, v_d, z_1, \ldots, z_n]$. We define an ideal $I \subset R$ generated by $2d+1$ polynomials. The first $d+1$ generators are the entries of

$$\begin{pmatrix} g_1 \\ \\ \vdots \\ g_{d+1} \end{pmatrix} = \hat{A} \cdot \begin{pmatrix} t^{a_1^+} v^{a_1^-} & 0 & \cdots & 0 \\ 0 & t^{a_2^+} v^{a_2^-} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t^{a_n^+} v^{a_n^-} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}. \tag{11.3.4}$$

The second group of generators is $h_i = t_i v_i - 1, i = 1, \ldots, d$.

Proposition 11.3.4. *Let $I \subset R$ be the ideal generated by $g_1, \ldots, g_{d+1}, h_1, \ldots, h_d$. The elimination ideal $J = I \cap \mathbb{C}[z_1, \ldots, z_n]$ is the prime vanishing ideal of ∇_A.*

Proof. The variety $V(I)$ defined by the ideal I in $\mathbb{C}^d \times \mathbb{C}^d \times (\mathbb{P}^{n-1})^\vee$ is isomorphic to V_A via the coordinate projection $(t, v, z) \longmapsto (t, z)$. This implies that $V(I \cap \mathbb{C}[z]) = \nabla_A$ as varieties. To show that the elimination ideal is prime, it suffices to show that I is prime itself. For this, Exercise 11.3.6 asks you to check that the Jacobian matrix of the generators (11.3.4) and $t_i v_i - 1$ has rank $\operatorname{rank}(\hat{A}) + d$ at each point of $V(I)$. This implies that, as a scheme, $V(I)$ is smooth, and hence reduced. □

Example 11.3.5. We continue Example 11.1.2. The ideal $I \subset \mathbb{C}[t, v, z]$ is generated by

$$z_2\, t + 2z_3\, t^2, \quad z_1 + z_2\, t + z_3\, t^2, \quad tv - 1.$$

The third generator enforces $t \neq 0, v \neq 0$ for each point in $V(I)$. The Jacobian matrix

$$\begin{pmatrix} z_2 + 4z_3\, t & 0 & 0 & t & 2t^2 \\ z_2 + 2z_3\, t & 0 & 1 & t & t^2 \\ v & t & 0 & 0 & 0 \end{pmatrix}$$

has rank $3 = 1 + \operatorname{rank}(\hat{A})$ for each point $(t, v, z) \in V(I)$, with $\hat{A} = \left(\begin{smallmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{smallmatrix}\right)$. ◇

Exercise 11.3.6. Show that the Jacobian matrix of size $(2d+1) \times (2d+n)$ obtained by taking partial derivatives of all $2d+1$ generators of I with respect to t, v, z has rank $\operatorname{rank}(\hat{A}) + d$ at each point of $V(I)$.

Definition 11.3.7 (A-discriminant)**.** *Fix a matrix $A \in \mathbb{Z}^{d \times n}$. If ∇_A has codimension one, then the* A-discriminant polynomial*, or simply* A-discriminant*, is an irreducible polynomial $\Delta_A \in \mathbb{Z}[z_1, \ldots, z_n]$ with integer coefficients (Exercise 11.3.8) satisfying $\nabla_A = \{z \in \mathbb{P}^n : \Delta_A(z) = 0\}$. We require that its coefficients have greatest common divisor 1, so that Δ_A is uniquely defined up to sign. If ∇_A has codimension > 1, we set $\Delta_A = 1$.*

Exercise 11.3.8. If $k \subset K$ is a field extension and $I \subset K[x_1, \ldots, x_m]$ is an ideal generated by elements in $k[x_1, \ldots, x_m]$, then each elimination ideal of I is generated by polynomials with coefficients in k. Use Proposition 11.3.4 to prove that the vanishing ideal of ∇_A in $\mathbb{C}[z_1, \ldots, z_n]$ can be generated by polynomials with coefficients in $\mathbb{Z}$.

The following code snippet uses `Oscar.jl` to implement the algorithm suggested by Proposition 11.3.4 for computing the prime ideal of ∇_A.

```
function get_aux_variables(A)
    (d, n) = size(A)
    Ahat = [A; ones(Int, 1, n)]
    Ahat = matrix_space(ZZ, size(Ahat)...)(Ahat) # matrix in Oscar format
    a = [A[:, i] for i = 1:n]
    aplus = [[maximum([aa, 0]) for aa in a[j]] for j = 1:n]
    aminus = [[minimum([aa, 0]) for aa in a[j]] for j = 1:n]
    return d, n, Ahat, aplus, aminus
end
```

```
10
11 function get_A_discriminant(A)
12     d, n, Ahat, aplus, aminus = get_aux_variables(A)
13     R, t, v, z = polynomial_ring(QQ, :t => 1:d, :v => 1:d, :z => 1:n)
14     D = diagonal_matrix([prod(t.^aplus[i])*prod(v.^(-aminus[i])) for i=1:n])
15     eqs1 = Ahat * D * z
16     eqs2 = [v[i] * t[i] - 1 for i = 1:d]
17     E = eliminate(ideal([eqs1; eqs2]), [t; v])
18 end
```

The function `get_aux_variables` computes the parameters d, n, the matrix $\hat{A}$ and the exponents a_i^+, a_i^- appearing in (11.3.4). Line 15 in the function `get_A_discriminant` creates the generators $g_1, \ldots, g_{d+1}$, and line 16 defines $h_1, \ldots, h_d$.

Example 11.3.9. The A-discriminant variety from Example 11.1.8 is computed by

```
A = [1 1 1 0 0 0; 0 0 0 1 1 1; 1 0 0 1 0 0; 0 1 0 0 1 0; 0 0 1 0 0 1]
get_A_discriminant(A)
```

The output is an ideal with three generators $-z_2z_6 + z_3z_5, -z_1z_6 + z_3z_4, -z_1z_5 + z_2z_4$. As expected, these are the 2×2 minors of the matrix $Z = \begin{pmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \end{pmatrix}$. ◇

We can use this code to illustrate that ∇_A gets rather complicated, even for "small" A.

Exercise 11.3.10. Check that the variety $\nabla_A \subset \mathbb{P}^4$ for $A = \begin{pmatrix} 4 & 2 & 4 & -3 & -2 \\ 2 & -1 & -1 & 0 & -2 \end{pmatrix}$ is a hypersurface of degree 46. One of the terms of Δ_A is

$$-995628422475629697764741523151987408896\, z_1^{13} z_3^{10} z_5^{23}.$$

By Proposition 11.2.1, Δ_A is $\hat{A}$-homogeneous of degree $(46, -30, 46)$. By Corollary 11.2.2, the Newton polytope of Δ_A has dimension at most two. In fact, it is a pentagon. You can check this using `Oscar.jl` as follows:

```
A = [4 2 4 -3 -2; 2 -1 -1 0 -2]
ΔA = get_A_discriminant(A)
P = newton_polytope(gens(ΔA)[1])
dim(P), vertices(P)
```

Exercise 11.3.11. Use the code above to compute Δ_A from Example 11.2.5.

The A-discriminant variety admits an easy-to-describe parametric representation, called the *Horn uniformization* or *Horn-Kapranov uniformization* of ∇_A. Let B be a kernel matrix of $\hat{A}$. That is, $\hat{A}B = 0$ and $B \in \mathbb{Z}^{n \times (n - \mathrm{rank}(\hat{A}))}$. The rows of B are denoted by $b_1, \ldots, b_n \in \mathbb{Z}^{n-\mathrm{rank}(\hat{A})}$. For $u \in \mathbb{P}^{n-\mathrm{rank}(\hat{A})-1}$, we write $(b_1 \cdot u : \cdots : b_n \cdot u)$ for the image of u under the well-defined linear map $B : \mathbb{P}^{n-\mathrm{rank}(\hat{A})-1} \to \mathbb{P}^{n-1}$.

Theorem 11.3.12. *The A-discriminant variety ∇_A is the closure of the image of*

$$(\mathbb{C}^*)^d \times \mathbb{P}^{n-\mathrm{rank}(\hat{A})-1} \to (\mathbb{P}^{n-1})^\vee, \qquad (t,u) \longmapsto (t^{-a_1}(b_1 \cdot u) : \cdots : t^{-a_n}(b_n \cdot u)).$$

Proof. We use the interpretation from Proposition 11.1.4 of $\nabla_A = X_A^\vee$ as the projective dual variety of X_A. The tangent space $T_{\Phi_A(t)} X_A$ at $x = \Phi_A(t)$ is spanned by the rows of

$$\hat{A} \cdot \begin{pmatrix} t^{a_1} & 0 & \cdots & 0 \\ 0 & t^{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t^{a_n} \end{pmatrix} = \begin{pmatrix} a_{11}t^{a_1} & a_{12}t^{a_2} & \cdots & a_{1n}t^{a_n} \\ a_{21}t^{a_1} & a_{22}t^{a_2} & \cdots & a_{2n}t^{a_n} \\ \vdots & \vdots & & \vdots \\ a_{d1}t^{a_1} & a_{d2}t^{a_2} & \cdots & a_{dn}t^{a_n} \\ t^{a_1} & t^{a_2} & \cdots & t^{a_n} \end{pmatrix}.$$

This follows from Proposition 1.2.20, and the fact that $Y_{\hat{A}}$ is the affine cone over X_A. We conclude that Λ_z is tangent to X_A at $\Phi_A(t)$ if and only if $(t^{a_1}z_1, \ldots, t^{a_n}z_n)^\top$ lies in the kernel of $\hat{A}$. Therefore $z_i = t^{-a_i}(b_i \cdot u)$ for some $u \in \mathbb{C}^{n-\mathrm{rank}(\hat{A})}$. We have showed that the image of the parametrization in the theorem consists of all $z \in \mathbb{P}^{n-1}$ for which Λ_z is tangent to X_A at some point $x \in \mathrm{im}\,\Phi_A$. This is precisely the set $\mathrm{pr}_2(\mathrm{con}(X_A)^{\circ\circ})$ from the proof of Proposition 11.1.4, which we there showed to be equal to ∇_A°. $\square$

Using Theorem 11.3.12, the prime ideal of ∇_A can be computed via *implicitization*, which is the process of computing the vanishing ideal of a variety from a parametric representation. This is implemented by the following function, which uses the auxiliary function `get_aux_variables` defined above. The reader should check correctness.

```
function get_A_discriminant_via_Horn(A)
    d, n, Ahat, aplus, aminus = get_aux_variables(A)
    B = nullspace(Ahat)[2]
    m = size(B,2)
    R, t, v, u, z = polynomial_ring(QQ, :t=>1:d, :v=>1:d, :u=>1:m, :z=>1:n)
    Bu = B*u
    eqs1 = [z[i]*prod(t.^aplus[i])-prod(t.^(-aminus[i]))*Bu[i] for i = 1:n]
    eqs2 = [v[i] * t[i] - 1 for i = 1:d]
    E = eliminate(ideal([eqs1; eqs2]), [t; v; u])
end
```

Exercise 11.3.13. Verify that `get_A_discriminant_via_Horn` gives the same output as `get_A_discriminant` for the matrices A from Example 11.1.8 and Exercise 11.3.10.

Theorem 11.3.12 makes it straightforward to sample points on the A-discriminant variety ∇_A: one simply computes the image of random points (t,u) under the Horn uniformization. If ∇_A is a hypersurface, then the A-discriminant polynomial Δ_A can be obtained using linear algebra via multivariate interpolation. For this one needs sufficiently many sample points, and a good Ansatz for Δ_A. This brings us back to estimating the degree of Δ_A, like in Theorem 11.2.4, or better yet, its $\hat{A}$-degree b from Proposition 11.2.1. Some pointers to the literature are given in the Further reading section.

11.4 A-Resultants and Chow forms

The purpose of the A-discriminant is to detect when the hypersurface of $f_{A,z}$ is singular at some point in $(\mathbb{C}^*)^d$. Instead, the A-*resultant* detects when $d+1$ Laurent polynomials with support A vanish simultaneously at some point in $(\mathbb{C}^*)^d$. To make this precise, we consider complex parameters $z = (z_{ij})_{i=0,\dots,d,j=1,\dots,n} \in \mathbb{C}^{(d+1)\times n}$ and a system of equations

$$f_{i,z} \;=\; \sum_{j=1}^{n} z_{ij}\, t^{a_i} \;=\; 0, \qquad i = 0,\dots,d. \tag{11.4.1}$$

Here $A \in \mathbb{Z}^{d\times n}$ and a_i are its columns, as usual. If $z^{(i)}$ is the i-th row of the matrix z, then $f_{i,z} = f_{A,z^{(i)}}$ in the notation of the previous section. Intuitively, for general choices of z_{ij}, the $d+1$ equations in d variables (11.4.1) do not have any solutions. Special choices of z for which a solution exists lie on the A-*resultant variety.*

Definition 11.4.1 (A-resultant variety)**.** *The A-resultant variety $\mathcal{R}_A \subset \mathbb{C}^{(d+1)\times n}$ associated to the matrix $A \in \mathbb{Z}^{d\times n}$ is the Zariski closure of the set*

$$\mathcal{R}_A^\circ \;=\; \{z \in \mathbb{C}^{(d+1)\times n} \;:\; \textit{there exists } t \in (\mathbb{C}^*)^d \textit{ satisfying } (11.4.1)\}.$$

Example 11.4.2. If $A = \begin{pmatrix} 0 & \mathrm{id}_d \end{pmatrix} \in \mathbb{Z}^{d\times(d+1)}$ is the support of a generic affine-linear equation $z_{i0} + z_{i1}\,t_1 + \cdots + z_{id}\,t_d$, then $\mathcal{R}_A$ consists of the matrices $z \in \mathbb{C}^{(d+1)\times(d+1)}$ with zero determinant. In cases like this, where the resultant variety is a hypersurface, we will write $\mathrm{Res}_A \in \mathbb{C}[z_{ij}]$ for its defining equation. In this example, we have $\mathrm{Res}_A = \det(z)$. $\diamond$

To compute the ideal of $\mathcal{R}_A$, we set up an elimination problem. Define

$$W_A \;=\; \{(t,z) \in (\mathbb{C}^*)^d \times \mathbb{C}^{(d+1)\times n} \;:\; f_{i,z}(t) = 0,\; i = 0,\dots,d+1\}.$$

This is similar to V_A in (11.3.3). The coordinate projections are

$$\mathrm{pr}_1 : W_A \to (\mathbb{C}^*)^d \quad \text{ and } \quad \mathrm{pr}_2 : W_A \to \mathbb{C}^{(d+1)\times n}.$$

By definition, we have $\mathcal{R}_A^\circ = \mathrm{pr}_2(W_A)$ and hence $\mathcal{R}_A = \overline{\mathrm{pr}_2(W_A)}$. This observation suffices to show that, like A-discriminants (Proposition 11.1.4), A-resultants are irreducible.

Theorem 11.4.3. *For any $A \in \mathbb{Z}^{d\times n}$, the A-resultant variety $\mathcal{R}_A$ is an irreducible subvariety of $\mathbb{C}^{(d+1)\times n}$ of codimension at least one.*

Proof. For any point $t \in (\mathbb{C}^*)^d$, the fiber $\mathrm{pr}_1^{-1}(t)$ is defined by $d+1$ linear equations in z. These equations involve different sets of variables, so they are linearly independent. Therefore, $\mathrm{pr}_1 : W_A \to (\mathbb{C}^*)^d$ is a $\mathbb{C}^{(d+1)\times n-(d+1)}$-bundle over $(\mathbb{C}^*)^d$. In particular, W_A is smooth and irreducible of dimension $(d+1)n-1$. This implies that $\mathcal{R}_A = \overline{\mathrm{pr}_2(W_A)}$ is irreducible of dimension at most $(d+1)n-1$. $\square$

To compute the ideal $I(\mathcal{R}_A)$, we work in the ring $R = \mathbb{C}[t_1,\dots,t_d,v_1,\dots,v_d,z_{ij}]$ with $2d+(d+1)n$ variables. Here v_i plays the role of t_i^{-1}, as in (11.3.4). We write $a_i = a_i^+ - a_i^-$ with $a_i^+, a_i^- \in \mathbb{N}^d$ and define $g_i(t,v,z) = \sum_{j=1}^{n} z_{ij}\, t^{a_i^+}\, v^{a_i^-}$, $i = 0,\dots,d$.

Proposition 11.4.4. *Let $I \subset R$ be the ideal generated by $g_0, \dots, g_{d+1}$ and $h_j = t_j v_j - 1, j = 1, \dots, d$. The elimination ideal $J = I \cap \mathbb{C}[z_{ij}]$ is the prime vanishing ideal of $\mathcal{R}_A$.*

Proof. The proof is similar to that of Proposition 11.3.4 and left as an exercise. $\square$

Exercise 11.4.5. Prove Proposition 11.4.4.

Like in the previous section, Proposition 11.4.4 justifies the following definition of the *A-resultant polynomial*.

Definition 11.4.6 (A-resultant)**.** *Fix a matrix $A \in \mathbb{Z}^{d \times n}$. If $\mathcal{R}_A$ has codimension one, then the* A-resultant polynomial, *or simply* A-resultant, *is an irreducible polynomial* $\mathrm{Res}_A \in \mathbb{Z}[z_{ij}]$ *with integer coefficients (Exercise 11.3.8) satisfying $\mathcal{R}_A = \{z \in \mathbb{C}^{(d+1)\times n} : \mathrm{Res}_A(z) = 0\}$. We require that its coefficients have greatest common divisor* 1*, so that* Res_A *is uniquely defined up to sign. If $\mathcal{R}_A$ has codimension > 1, we set* $\mathrm{Res}_A = 1$.

Proposition 11.4.4 leads to the following simple Julia function for computing $\mathcal{R}_A$:

```
function get_A_resultant(A)
    d, n, _, aplus, aminus = get_aux_variables(A)
    R, t, v, z = polynomial_ring(QQ, :t => 1:d, :v => 1:d, :z => (0:d,1:n))
    eqs1 = matrix(z)*[prod(t.^aplus[i]) * prod(v.^(-aminus[i])) for i = 1:n]
    eqs2 = [v[i] * t[i] - 1 for i = 1:d]
    E = eliminate(ideal([eqs1; eqs2]), [t; v])
end
```

This uses the function `get_aux_variables` defined in Section 11.3.

Example 11.4.7. For $A = \begin{pmatrix} 0 & 1 & 2 & 3 \end{pmatrix} \in \mathbb{Z}^{1\times 4}$, the resultant $\mathcal{R}_A$ characterizes when two cubic polynomials have a common root. The function `get_A_resultant` returns a principal ideal generated by the following degree 6 irreducible polynomial:

$$\begin{gathered}
\mathrm{Res}_A \;=\; z_{01}^3 z_{14}^3 - 3z_{01}^2 z_{11} z_{04} z_{14}^2 - z_{01}^2 z_{02} z_{13} z_{14}^2 - 2z_{01}^2 z_{12} z_{03} z_{14}^2 + 3z_{01}^2 z_{12} z_{13} z_{04} z_{14} \\
+z_{01}^2 z_{03} z_{13}^2 z_{14} - z_{01}^2 z_{13}^3 z_{04} + 3z_{01} z_{11}^2 z_{04}^2 z_{14} + 3z_{01} z_{11} z_{02} z_{03} z_{14}^2 - z_{01} z_{11} z_{02} z_{13} z_{04} z_{14} \\
+z_{01} z_{11} z_{12} z_{03} z_{04} z_{14} - 3z_{01} z_{11} z_{12} z_{13} z_{04}^2 - 2z_{01} z_{11} z_{03}^2 z_{13} z_{14} + 2z_{01} z_{11} z_{03} z_{13}^2 z_{04} \\
+z_{01} z_{02}^2 z_{12} z_{14}^2 - 2z_{01} z_{02} z_{12}^2 z_{04} z_{14} - z_{01} z_{02} z_{12} z_{03} z_{13} z_{14} + z_{01} z_{02} z_{12} z_{13}^2 z_{04} \\
+z_{01} z_{12}^3 z_{04}^2 + z_{01} z_{12}^2 z_{03}^2 z_{14} - z_{01} z_{12}^2 z_{03} z_{13} z_{04} - z_{11}^3 z_{04}^3 - 3z_{11}^2 z_{02} z_{03} z_{04} z_{14} + 2z_{11}^2 z_{02} z_{13} z_{04}^2 \\
+z_{11}^2 z_{1,2} z_{03} z_{04}^2 + z_{11}^2 z_{03}^3 z_{14} - z_{11}^2 z_{03}^2 z_{13} z_{04} - z_{11} z_{02}^3 z_{14}^2 + 2z_{11} z_{02}^2 z_{12} z_{04} z_{14} \\
+z_{11} z_{02}^2 z_{03} z_{1,3} z_{14} - z_{11} z_{02}^2 z_{13}^2 z_{04} - z_{11} z_{02} z_{12}^2 z_{04}^2 - z_{11} z_{02} z_{12} z_{03}^2 z_{14} + z_{11} z_{02} z_{12} z_{03} z_{13} z_{04}.
\end{gathered}$$

Below, we shall present two ways of writing this polynomial more compactly. $\diamond$

Next, we relate the A-resultant variety $\mathcal{R}_A$ to the projective toric variety X_A. Recall that $\Phi_A : t \mapsto (t^{a_1} : \cdots : t^{a_n}) \in \mathbb{P}^{n-1}$ parametrizes the dense torus of X_A. Here is an equivalent characterization of $\mathcal{R}_A^\circ$ (see Definition 11.4.1) which uses this map:

$$\mathcal{R}_A^\circ \;=\; \{z \in \mathbb{C}^{(d+1)\times n} \,:\, \text{there exists } x \in \operatorname{im}\Phi_A \subset \mathbb{P}^{n-1} \text{ such that } z \cdot x = 0\}. \tag{11.4.2}$$

Here $z\cdot x$ stands for the matrix-vector product: $(z\cdot x)_i = \sum_{j=1}^n z_{ij}\, x_j$. We use the notation $\Lambda_z \subset \mathbb{P}^{n-1}$ for the linear subvariety defined by $z \cdot x = 0$. If z is a row vector, this agrees with our notation Λ_z in Section 11.1. If z has rank $d+1$, then Λ_z has codimension $d+1$. A point $z \in \mathbb{C}^{(d+1)\times n}$ belongs to $\mathcal{R}_A^\circ$ if and only if Λ_z intersects the torus $\operatorname{im}\Phi_A$ of X_A.

Proposition 11.4.8. *The A-resultant variety $\mathcal{R}_A$ consists of all $z \in \mathbb{C}^{(d+1)\times n}$ for which the intersection $\Lambda_z \cap X_A$ is nonempty. In symbols, we have*

$$\mathcal{R}_A \;=\; \{z \in \mathbb{C}^{(d+1)\times n} \;:\; \Lambda_z \cap X_A \neq \emptyset\}.$$

Hence, $\mathcal{R}_A$ only depends on X_A and $\mathcal{R}_{T\cdot A+m} = \mathcal{R}_A$ with T, m as in Proposition 1.3.14.

Proof. The following incidence variety is analogous to W_A above:

$$W_A' \;=\; \{(x,z) \in \operatorname{im}\Phi_A \times \mathbb{C}^{(d+1)\times n} \;:\; z\cdot x = 0\}.$$

It is irreducible of dimension $(d+1)n - d + 1 + \dim X_A$ by inspection of the fibers of $W_A' \to \operatorname{im}\Phi_A$. Its closure in $X_A \times \mathbb{C}^{(d+1)\times n}$ is

$$\overline{W_A'} \;=\; \{(x,z) \in X_A \times \mathbb{C}^{(d+1)\times n} \;:\; z\cdot x = 0\}.$$

Indeed, the standard fiber argument applied to the projection onto X_A implies that the righthand side is irreducible of the same dimension as W_A'. We have

$$\mathcal{R}_A \;=\; \overline{\operatorname{pr}_2(W_A')} \;=\; \overline{\operatorname{pr}_2(\overline{W_A'})} \;=\; \operatorname{pr}_2(\overline{W_A'}),$$

where the last equality holds because X_A is projective. □

In the proof of the following theorem, we use the notation $\bar{z} \in \mathbb{C}^{d\times n}$ for the submatrix of z consisting of its last d rows. We also write $\Lambda_{\bar{z}} = \{x \in \mathbb{P}^{n-1} \;:\; \bar{z}\cdot x = 0\}$ for the corresponding linear subspace, which is generically of codimension d.

Theorem 11.4.9. *The A-resultant variety $\mathcal{R}_A$ is a hypersurface if and only if the following equivalent conditions hold:*

1. $\dim X_A = d$,
2. $\operatorname{rank}(\hat{A}) = d+1$,
3. $\dim \operatorname{Conv}(A) = d$,
4. *the last d equations $f_{1,z} = \cdots = f_{d,z} = 0$ have a solution in $(\mathbb{C}^*)^d$ for generic z.*

In this situation, the A-resultant Res_A is homogeneous of degree $\deg X_A$ in each of the variable groups $z^{(i)} = (z_{i1}, \ldots, z_{in})$. In particular, its total degree is $(d+1)\deg X_A$.

Proof. We start with the equivalence of the four conditions. The fact that conditions 1, 2 and 3 are equivalent follows from Corollary 1.3.18. For 1 $\Leftrightarrow$ 4, notice that $f_{1,z} = \cdots = f_{d,z} = 0$ has a solution in $(\mathbb{C}^*)^d$ if and only if $\Lambda_{\bar{z}} \cap X_A \neq \emptyset$, where $\Lambda_{\bar{z}} = \{x \in \mathbb{P}^{n-1} : \bar{z} \cdot x = 0\}$. Since $\Lambda_{\bar{z}}$ has codimension d for generic z and $\dim X_A \leq d$, this intersection is generically non-empty if and only if $\dim X_A = d$.

Suppose that $\mathcal{R}_A$ is a hypersurface. We show that $\dim X_A = d$ by contradiction. If $\dim X_A < d$, then the map $\pi : \mathcal{R}_A \to \mathbb{C}^{d\times n}$ given by $z \mapsto \bar{z}$ is not dominant: $\dim(\operatorname{im}\pi) < dn$. That is, a generic linear space of codimension d does not intersect X_A. However, for a generic point $\bar{z} \in \operatorname{im}\pi$, the linear space $\Lambda_{\bar{z}}$ intersects X_A in only one point p. The condition $p \in \Lambda_z \supset \Lambda_{\bar{z}}$ imposes a linear constraint on $z_{01}, \ldots, z_{0n}$, so that a generic fiber $\pi^{-1}(\pi(z))$ has dimension $n-1$. But then $\dim \mathcal{R}_A = \dim(\operatorname{im}\pi) + n - 1 < dn + n - 1 = (d+1)n - 1$, which contradicts the fact that $\mathcal{R}_A$ is a hypersurface. To show the other direction, notice that if $\dim X_A = d$, then π is dominant and for generic $\bar{z} \in \mathbb{C}^{d\times n}$, $\Lambda_{\bar{z}} \cap \operatorname{im}\Phi_A$ consists of finitely many points. To impose that $p \in \Lambda_z \supset \Lambda_{\bar{z}}$ for one of these points p means that $(z_{01}, \ldots, z_{0n})$ lies on a finite union of hyperplanes in $\mathbb{C}^n$. Hence, a generic fiber $\pi^{-1}(\bar{z})$ is a union of finitely many hyperplanes, which has dimension $n-1$. We count that $\dim \mathcal{R}_A = dn + n - 1$, so $\mathcal{R}_A$ is a hypersurface.

Note that it is a priori clear that Res_A is homogeneous in each of the variable groups $z^{(i)}$ separately, as scaling $z^{(i)}$ does not affect the solutions to $f_{0,z} = \cdots = f_{d,z} = 0$. The argument above implies that, if $\dim X_A = d$, then the generic fiber of π is a union of $\deg X_A$-many hyperplanes through the origin in $\mathbb{C}^n$ (one for each intersection point in $\Lambda_{\bar{z}} \cap \operatorname{im}\Phi_A$). Another way of saying this is that, when specializing $z^{(1)}, \ldots, z^{(d)}$ to generic values, Res_A specializes to a product of $\deg X_A$ linear forms in $z^{(0)}$. This shows that Res_A has degree $\deg X_A$ in $z^{(0)}$, and the same argument applies for $z^{(i)}, i = 1, \ldots, d$. $\square$

Recall that a formula for $\deg X_A$ was given in Theorem 3.3.1.

Example 11.4.10. We verify Theorem 11.4.9 in Example 11.4.7. The polynomial Res_A has total degree 6, which is $(d+1)\deg X_A = 2 \cdot 3$. It has degree $3 = \deg X_A$ in the variable groups $z^{(i)} = (z_{i1}, z_{i2}, z_{i3}, z_{i4}), i = 0, 1$. $\diamond$

Exercise 11.4.11. Use the function `get_A_resultant` from above to compute Res_A for

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}.$$

This resultant characterizes when three plane conics meet in $\mathbb{P}^2$. It is a polynomial of degree 12 in 18 variables with 21894 terms. The degree in each group of variables $z^{(0)}, z^{(1)}, z^{(2)}$ is four, which is $2!\operatorname{Vol}(A) = \deg X_A$.

Translating the condition $z \in \mathcal{R}_A^\circ$ into $\Lambda_z \cap X_A \neq \emptyset$ naturally guides us towards the *Chow form* of the toric variety X_A. We write $\operatorname{Gr}(k,n)$ for the Grassmannian of $(k-1)$-dimensional linear subspaces of $\mathbb{P}^{n-1}$. If $\Lambda \subset \mathbb{P}^{n-1}$ is such a subspace, we write $[\Lambda]$ for the corresponding point in $\operatorname{Gr}(k,n)$. The following is proved in [33, Section 3.2B].

Theorem 11.4.12. *Let $X \subset \mathbb{P}^{n-1}$ be an irreducible variety of dimension d. The set*

$$\mathcal{C}_X = \{[\Lambda] \in \mathrm{Gr}(n-d-1, n) : \Lambda \cap X \neq \emptyset\}$$

is an irreducible hypersurface in $\mathrm{Gr}(n-d-1,n)$. Moreover, $\mathcal{C}_X$ is the zero locus of a polynomial Chow_X of degree $\deg X$ in Plücker coordinates, called the Chow form *of X.*

The polynomial Chow_X is defined up to Plücker relations. A point on $\mathcal{C}_X$ is a linear space of codimension $d+1$ which intersects the d-dimensional variety X.

Example 11.4.13. We again consider $X = X_A$ where $A = \begin{pmatrix} 0 & 1 & 2 & 3 \end{pmatrix}$. The Chow hypersurface $\mathcal{C}_{X_A}$ lives in the Grassmannian $\mathrm{Gr}(2,4)$ of lines in $\mathbb{P}^3$. It consists of all lines which intersect the twisted cubic curve X_A. It is convenient for us to write down the equation of $\mathcal{C}_{X_A}$ in *primal Plücker coordinates*. These are obtained as follows. For a line $\Lambda \subset \mathbb{P}^3$, write down a rank-two matrix $z \in \mathbb{C}^{2\times 4}$ such that $\Lambda = \Lambda_z = \{x : z \cdot x = 0\}$. Let p_{ij} be the 2×2 minor of z corresponding to columns i and j. We compute that

$$\mathrm{Chow}_{X_A} = -p_{14}p_{23} + p_{24}p_{13} - p_{34}p_{12}p_{14}^3 - p_{14}^2 p_{23} - 3p_{14}p_{34}p_{12} + p_{24}^2 p_{12} + p_{34}p_{13}^2 - p_{34}p_{23}p_{12}.$$

The line Λ intersects X_A if and only if its primal Plücker coordinates satisfy $\mathrm{Chow}_{X_A} = 0$. The degree in the p_{ij} equals $\deg \mathrm{Chow}_{X_A} = \deg X_A$, as predicted by Theorem 11.4.12. ◇

We will now spell out the relation between the Chow form of X_A and the A-resultant. We start with an easy lemma.

Lemma 11.4.14. *Let $A \in \mathbb{Z}^{d\times n}$ be any integer matrix with $n > d+1$. Let $U \subset \mathbb{C}^{(d+1)\times n}$ be the open subset consisting of rank-$(d+1)$ matrices z. The intersection $U \cap \mathcal{R}_A$ is Zariski dense in $\mathcal{R}_A$. Consequently, $U \cap \mathcal{R}_A^\circ$ is also dense in $\mathcal{R}_A$.*

Proof. Since $\mathcal{R}_A$ is irreducible (Theorem 11.4.3), it suffices to show that $U \cap \mathcal{R}_A \neq \emptyset$. Pick any $t \in (\mathbb{C}^*)^d$. Let $z \in U$ be such that Λ_z is a linear space of codimension $d+1$ containing $\Phi_A(t)$. Then $z \in U \cap \mathcal{R}_A^\circ$ by construction. □

Consider the map $\varphi : \mathbb{C}^{(d+1)\times n} \dashrightarrow \mathrm{Gr}(n-d-1,n)$ which sends z to $[\Lambda_z]$, where $\Lambda_z = \{x \in \mathbb{P}^{n-1} : z \cdot x = 0\}$. This map is well-defined on the open subset U from Lemma 11.4.14. Working with the primal Plücker embedding of $\mathrm{Gr}(n-d-1,n)$ as in Example 11.4.13, the map φ is given by $\varphi(z) = (p_S(z))_{S\subset[n],|S|=d+1}$, where p_S is the $(d+1)\times(d+1)$ minor of the matrix z obtained from the columns indexed by S. The Chow form Chow_{X_A} is a homogeneous polynomial in the primal Plücker coordinates p_S. It pulls back to a polynomial in z_{ij} by substituting $p_S \to p_S(z)$. We denote that polynomial by $\varphi^*\mathrm{Chow}_{X_A}$. Here is what happens for the twisted cubic.

Example 11.4.15. For $A = \begin{pmatrix} 0 & 1 & 2 & 3 \end{pmatrix}$, the map φ is given by

$$\begin{pmatrix} z_{01} & z_{02} & z_{03} & z_{04} \\ z_{11} & z_{12} & z_{13} & z_{14} \end{pmatrix} \mapsto (z_{01}z_{12} - z_{02}z_{11} : z_{01}z_{13} - z_{03}z_{11} : \cdots : z_{03}z_{14} - z_{04}z_{13}) \in \mathbb{P}^5,$$

where we used the lexicographic ordering $(p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34})$ for the primal Plücker coordinates. The polynomial $\varphi^*\mathrm{Chow}_{X_A}$ is computed by substituting $p_{12} = z_{01}z_{12} - z_{02}z_{11}$, $p_{13} = z_{01}z_{13} - z_{03}z_{11}, \ldots$ into the cubic polynomial Chow_{X_A} computed in Example 11.4.13. The result is the degree six polynomial Res_A from Example 11.4.7. ◇

Theorem 11.4.16. *Let $A \in \mathbb{Z}^{d\times n}$ be such that* $\dim X_A = d < n-1$. *For some nonzero constant $c \in \mathbb{C}^*$, we have $\varphi^*\mathrm{Chow}_{X_A} = c\,\mathrm{Res}_A$.*

Proof. First of all, notice that $\varphi^*\mathrm{Chow}_{X_A}$ is not the zero polynomial. Indeed, let U be as in Lemma 11.4.14 and pick any $z \in U$ such that $\Lambda_z \cap X_A = \emptyset$. Then $(\varphi^*\mathrm{Chow}_{X_A})(z) \neq 0$. If $z \in U \cap \mathcal{R}_A^\circ$, then there exists a solution $t \in (\mathbb{C}^*)^d$ to $f_{0,z} = \cdots = f_{d,z} = 0$. Hence $\Phi_A(t) \in \Lambda_z \cap X_A$ and $(\varphi^*\mathrm{Chow}_{X_A})(z) = 0$. Thus $\varphi^*\mathrm{Chow}_{X_A}$ vanishes on $U \cap \mathcal{R}_A^\circ$, and by Lemma 11.4.14 it vanishes on $\mathcal{R}_A$. Therefore Res_A divides $\varphi^*\mathrm{Chow}_{X_A}$. By Theorem 11.4.12, Chow_{X_A} has degree $\deg X_A$. Since each primal Plücker coordinate $p_S(z)$ has degree $d+1$ in z, we see that $\deg \varphi^*\mathrm{Chow}_{X_A} = (d+1)\deg X_A$. By Theorem 11.4.9, this equals $\deg \mathrm{Res}_A$ and the theorem follows. □

11.5 Principal A-determinants

The A-resultant Res_A is a polynomial which depends on $(d+1)n$ variables. One often wants to specialize this polynomial by substituting z_{ij} by the coefficients of $f_0, \ldots, f_d \in R[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$. Since Res_A has integer coefficients (Definition 11.4.6), R can be any ring containing the integers. The polynomials f_i must be such that their monomial support is contained in the A. We will denote such a specialization by $\mathrm{Res}_A(f_0, \ldots, f_d) \in R$.

In this section, our specialization is as follows. Let $R = \mathbb{Z}[z_1, \ldots, z_n]$ and

$$f \;=\; f_{A,z}(t) \;=\; \sum_{i=1}^{n} z_i\, t^{a_i} \quad \in\; R[t_1^{\pm 1}, \ldots, t_d^{\pm 1}].$$

To avoid degeneracies, we will *assume throughout the section that X_A has dimension d.*

Definition 11.5.1 (principal A-determinant)**.** *The* principal A-determinant $E_A \in R$ *is*

$$E_A \;=\; \mathrm{Res}_A\Big(t_1\frac{\partial f}{\partial t_1}, \ldots, t_d\frac{\partial f}{\partial t_d}, f\Big) \quad \in\; R \;=\; \mathbb{Z}[z_1, \ldots, z_n].$$

Example 11.5.2. Let $A = \begin{pmatrix} 0 & 1 & 2 & 3 \end{pmatrix}$. The principal A-determinant is obtained from the A-resultant computed in Example 11.4.7 by substituting

$$\begin{pmatrix} z_{01} & z_{02} & z_{03} & z_{04} \\ z_{11} & z_{12} & z_{13} & z_{14} \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & z_2 & 2z_3 & 3z_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}.$$

The rows of the second matrix are the coefficients of $f = z_1 + z_2\,t + z_3\,t^2 + z_4\,t^3$ and t times its derivative. The result is $E_A = z_1z_4(-27z_1^2z_4^2 + 18z_1z_2z_3z_4 - 4z_1z_3^3 - 4z_2^3z_4 + z_2^2z_3^2)$. ⋄

This simple example illustrates that, unlike the A-resultant, the principal A-determinant is not necessarily irreducible. In fact, it typically breaks up into many factors.

Like in Section 4.2, for a face $Q \preceq \mathrm{Conv}(A)$ of the convex hull of A, let $Q \cap A$ be the submatrix consisting of columns which lie on Q. We write $\mathrm{mult}_Q X_A \geq 1$ for the multiplicity of X_A along its torus orbit $X_{A,Q}^\circ$. If $v \in \mathcal{V}(\mathrm{Conv}(A))$ is a vertex contained in

Q, $\tau = \mathrm{Cone}(A - v)$ and $x \in Y^\circ_{A-v,\tau}$, then this equals the number $\mathrm{mult}_Q X_A = \mathrm{mult}_x Y_{A-v}$ computed in Theorem 4.5.7. Section 4.5 ends with code for computing this number.

Associated to each face Q of dimension ≥ 1 there is a discriminant polynomial $\Delta_{Q\cap A}$ as defined in Definition 11.3.7. This polynomial might be the constant 1 if $\nabla_{Q\cap A}$ is not a hypersurface. In this section, all discriminants $\Delta_{Q\cap A}$ are defined as polynomials in a subset of the variables $z_1, \ldots, z_n$ indexing the columns of A. Namely, the variables of $\Delta_{Q\cap A}$ are those indexing the columns of $Q \cap A$. We leave this implicit in our notation. If $Q = a_i$ is a vertex of $\mathrm{Conv}(A)$, then we define[1] $\Delta_{Q\cap A} = z_i$.

Theorem 11.5.3. *We have the following formula for the principal A-determinant:*

$$E_A = \pm \prod_{Q \preceq \mathrm{Conv}(A)} \Delta_{Q\cap A}^{\mathrm{mult}_Q X_A}.$$

Here $\Delta_{Q\cap A}$ is the $(Q\cap A)$-discriminant from Definition 11.5.1 whose variables are the z_i which are labeled by the columns of $Q \cap A$. In particular, if X_A is smooth, then all multiplicities $\mathrm{mult}_Q X_A$ are one and we have $E_A = \pm \prod_{Q\preceq \mathrm{Conv}(A)} \Delta_{Q\cap A}$.

This is Theorem 1.2 in [33, Chapter 10]. Below, we shall prove a weaker statement:

Corollary 11.5.4. *In the notation of Theorem 11.5.3, for any $z \in \mathbb{P}^{n-1}$, we have that $E_A(z) = 0$ if and only if $\Delta_{Q\cap A}(z) = 0$ for some $Q \preceq \mathrm{Conv}(A)$.*

First, we illustrate these statements with some examples.

Example 11.5.5. The polytope $\mathrm{Conv}(A)$ from Example 11.5.2 has three faces: $Q_0 = \{0\}$, $Q_\infty = \{3\}$ and $Q = [0,3] = \mathrm{Conv}(A)$. These correspond to the submatrices

$$Q_0 \cap A = \begin{pmatrix} 0 \end{pmatrix}, \quad Q_\infty \cap A = \begin{pmatrix} 3 \end{pmatrix}, \quad Q \cap A = \begin{pmatrix} 0 & 1 & 2 & 3 \end{pmatrix}.$$

The associated discriminants are $\Delta_{Q_0\cap A} = z_1$, $\Delta_{Q_\infty\cap A} = z_4$ and $\Delta_{Q\cap A} = \Delta_A = -27z_1^2z_4^2 + 18z_1z_2z_3z_4 - 4z_1z_3^3 - 4z_2^3z_4 + z_2^2z_3^2$. These are the factors appearing in E_A from Example 11.5.2. All exponents are one, because the twisted cubic X_A is smooth. $\diamond$

Example 11.5.6. The matrix A from Exercise 11.4.11 represents a generic degree two polynomial in two variables. Its convex hull is the triangle shown in Figure 3.4 (left). This triangle has seven faces, labeled by the following columns of A:

$$Q_1 \cap A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad Q_4 \cap A = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad Q_6 \cap A = \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$
$$Q_{124} \cap A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_{136} \cap A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \quad Q_{456} \cap A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix},$$

[1] In Section 11.1, we defined ∇_A as a subset of $(n-1)$-dimensional projective space in order to naturally identify ∇_A with the projective dual of $X_A \subset \mathbb{P}^{n-1}$. Alternatively, we could start from

$$\nabla_A^\circ = \{z \in \mathbb{C}^n : f_{A,z} = \frac{\partial f_{A,z}}{\partial t_1} = \cdots = \frac{\partial f_{A,z}}{\partial t_d} = 0 \text{ for some } t \in (\mathbb{C}^*)^d\}$$

and take the closure in $\mathbb{C}^n$. It is an easy exercise to show that, with this definition, ∇_A is the affine cone over our projective version when $n \geq 2$, and the A-discriminant hypersurface of a matrix with only one column is given by $z = 0$ in $\mathbb{C}^1$. This justifies the convention $\Delta_{Q\cap A} = z_i$ for a vertex $Q = a_i$.

and $Q_{123456} = A$. The toric variety X_A is the (smooth) Veronese surface in $\mathbb{P}^5$. We have

$$E_A(z) \;=\; z_1\, z_4\, z_6\, (z_2^2 - 4z_1z_4)(z_3^2 - 4z_1z_6)(z_5^2 - 4z_4z_6)\,\det\begin{pmatrix} 2z_1 & z_2 & z_3 \\ z_2 & 2z_4 & z_5 \\ z_3 & z_5 & 2z_6 \end{pmatrix}.$$

The quadratic factors are instances of the classical discriminant b^2-4ac. This is because the matrix of each of the edges $Q_{124}, Q_{136}, Q_{345}$ gives rise to the same projective toric curve given by $xz-y^2=0$, and the A-discriminant only depends on X_A. The last factor is the A-discriminant from Exercise 11.1.3. The polynomial E_A is alternatively obtained by performing the following substitution in Res_A from Exercise 11.4.11:

$$\begin{pmatrix} z_{01} & z_{02} & z_{03} & z_{04} & z_{05} & z_{06} \\ z_{11} & z_{12} & z_{13} & z_{14} & z_{15} & z_{16} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} & z_{26} \end{pmatrix} \to \begin{pmatrix} 0 & z_2 & 0 & 2z_4 & z_5 & 0 \\ 0 & 0 & z_3 & 0 & z_5 & 2z_6 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \end{pmatrix}. \qquad \diamond$$

Exercise 11.5.7. For $A = \left(\begin{smallmatrix} 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{smallmatrix}\right)$, show that $E_A \;=\; z_1^2\, z_2^2\, z_3^2\,(27z_1z_3z_4 + z_2^3)$. You have encountered this matrix in Exercise 4.5.12. The surface X_A has three singular points.

Proof of Corollary 11.5.4. By Proposition 11.4.8, the A-resultant Res_A vanishes at $\tilde{z} = (\tilde{z}_{ij})$ if and only if $\tilde{z}\cdot x = 0$ has a solution $x \in X_A$. Our specialization of $\tilde{z}$ implies:

$$E_A(z) = 0 \iff \hat{A}\,\mathrm{diag}(z)\,x = 0 \;\text{ for some }\; x \in X_A,$$

which is equivalent to (11.3.2) up to the fact that x need not lie in $\mathrm{im}\,\Phi_A$. For each point $x \in X_A$, there is a unique face $Q \preceq \mathrm{Conv}(A)$ such that $x \in X^\circ_{A,Q}$. We have

$$E_A(z) = 0 \iff \widehat{Q\cap A}\,\mathrm{diag}(z_Q)\,\pi_Q(x) = 0 \;\text{ for some } Q \preceq \mathrm{Conv}(A) \;\text{ and some } x \in X^\circ_{A,Q},$$

where $\widehat{Q\cap A} = \left(\begin{smallmatrix} a_{i_1} & a_{i_2} & \cdots & a_{i_\ell} \\ 1 & 1 & \cdots & 1 \end{smallmatrix}\right)$ is the matrix $Q\cap A$ with an appended row of ones, $z_Q = (z_{i_1}, \dots, z_{i_\ell})$ is the subvector of z consisting of coordinates labeled by $Q\cap A$, and $\pi_Q(x) = (x_{i_1} : \cdots : x_{i_\ell})$ is as in Theorem 4.2.2. But this is precisely the condition for z_Q to lie in the open discriminant locus $\nabla^\circ_{Q\cap A} \subset \mathbb{P}^{|Q\cap A|-1}$ for some face Q. Hence, z lies in the pre-image of $\nabla^\circ_{Q\cap A}$ under the coordinate projection $z \mapsto z_Q$. Therefore, we have shown that the zero locus of $E_A(z)$ in $\mathbb{P}^{n-1}$ equals the union of all such preimages. Since $\dim \nabla^\circ_{Q\cap A} < |Q\cap A| - 1$ by Proposition 11.1.4, $E_A(z)$ is nonzero. Indeed, its zero locus is a strict subset of $\mathbb{P}^{n-1}$. Moreover, since E_A is a linear specialization of the A-resultant polynomial Res_A, a polynomial of positive degree (Theorem 11.4.9), its zero locus must be a hypersurface. Therefore, we alternatively obtain this hypersurface by taking the closure of the union of all discriminant components which are hypersurfaces. $\square$

Corollary 11.5.8. *The principal A-determinant is nonzero and has degree* $(d+1)\deg X_A$.

Theorem 11.5.3 allows to compute the principal A-determinant by combining algorithms from Sections 4.5 and 11.3. First, we give our function `get_A_discriminant` an optional input allowing to specify in which variables the discriminant ideal is returned:

```
function get_A_discriminant(A; vrs = [])
    d, n, Ahat, aplus, aminus = get_aux_variables(A)
    R, t, v, z = polynomial_ring(QQ, :t => 1:d, :v => 1:d, :z => 1:n)
    D = diagonal_matrix([prod(t.^aplus[i])*prod(v.^(-aminus[i])) for i=1:n])
    eqs1 = Ahat * D * z
    eqs2 = [v[i] * t[i] - 1 for i = 1:d]
    E = eliminate(ideal([eqs1; eqs2]), [t; v])
    if !isempty(vrs)
        S = parent(vrs[1])
         = hom(R,S,[S.(ones(Int,2*d));vrs])
        return (E)
    else
        return E
    end
end
```

The following function uses `get_A_discriminant` and `get_multiplicity_proj` (Section 4.5) to return a list of all factors of the principal A-determinant with their multiplicity.

```
function get_principal_A_det(A)
    P = convex_hull(transpose(A))
    d,n = size(A)
    factorlist = [] # Initialize an empty list of factors
    R, z = graded_polynomial_ring(QQ,:z=>1:n)
    if dim(P) != d # This function expects a d-dimensional polytope
        println("P is not full dimensional")
        return []
    end
    for i = 0:dim(P)
        for Q in faces(P,i)
            mult = get_multiplicity_proj(A,Q) # exponent for Q
            Qinds = findall(i->A[:,i] in Q, 1:n)
            AQ = A[:,Qinds]
            Adisc = get_A_discriminant(Array(AQ); vrs = z[Qinds])
            if length(gens(Adisc))>1 # If discriminant has codim > 1
                Adisc = ideal(R(1))
            end
            push!(factorlist, (Adisc,mult))
        end
    end
    return factorlist
end
```

Exercise 11.5.9. Solve Exercise 11.5.7 using the code above.

Further reading

The book [33] remains a standard reference on A-discriminants. Theorem 11.3.12 appeared in [40] and was inspired by work of Horn [36] from about 100 years earlier. It can be read as follows: "The A-discriminant variety is the Hadamard product (entry-wise product) between X_A and the projectivized image of the matrix $\hat{B}$". This was the key insight used in [23] to compute the tropicalization of ∇_A, and hence its Newton polytope. Of course, once we know the Newton polytope, we know the degree and $\hat{A}$-degree of the A-discriminant polynomial Δ_A (see Proposition 11.2.1). Using these tropical data to compute Δ_A via sampling and interpolation is the topic of [58, Sections 3 and 4]. This is part of a more general program called *tropical implicitization* developed in [29, 67, 68, 69]. The book chapter [58] comes with an implementation in `Oscar.jl`. The reader who wants to learn about Grassmannians and their Plücker embedding can consult [52, Chapter 5], and [21] is a nice survey on Chow forms.

Chapter 13

Polyhedral homotopies

Consider d Laurent polynomials $f_1, \dots, f_d$ in d variables $t_1, \dots, t_d$:

$$f_i(t) \;=\; \sum_{a \in A_i} z_{i,a}\, t^a \;\; \in \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}], \quad i = 1, \dots, d.$$

Here $A = (A_1, \dots, A_d)$ is a tuple of finite subsets of $\mathbb{Z}^d$. The coefficients $z_{i,a}$ are nonzero complex numbers. Our goal in this chapter is to describe a specialized homotopy continuation method for solving the system of equations

$$f_1(t) \;=\; f_2(t) \;=\; \cdots \;=\; f_d(t) \;=\; 0. \tag{13.0.1}$$

Specialized means that the homotopy method should optimally exploit the *sparsity* of the equations. This will become clear later in the chapter. We assume that the number of points $t \in (\mathbb{C}^*)^d$ satisfying (13.0.1) is finite. By *solving* we mean computing the coordinates of all these finitely many solutions approximately.

We start by describing series solutions to a lifted version of the system (13.0.1) in Section 13.1. The polyhedral homotopy algorithm is based on this description. It was introduced in [38, 73] and remains one of the most effective numerical methods for solving polynomial systems to date. In modern language, the polyhedral homotopy method is best understood in terms of *tropical geometry* [47]. This translation is our topic in Section 13.5.

13.1 Puiseux series solutions

Let ϵ be a new variable, and consider the *lifted* Laurent polynomials

$$f_{i,w}(t) \;=\; \sum_{a \in A_i} z_{i,a}\, \epsilon^{w_{i,a}}\, t^a \;\in \mathbb{C}\{\{\epsilon\}\}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]. \tag{13.1.1}$$

These are obtained from the f_i by multiplying $z_{i,a}$ by $\epsilon^{w_{i,a}}$, for some *weights* $w_{i,a} \in \mathbb{Z}$. Now $f_{i,w}$ is an element of the Laurent polynomial ring over the field $\mathbb{C}\{\{\epsilon\}\}$ of *Puiseux* *series* in ϵ. This is the algebraically closed field of fractional Laurent series:

$$\mathbb{C}\{\{\epsilon\}\} \;=\; \left\{ \sum_{k \geq k_0, k \in \mathbb{Z}} c_k\, \epsilon^{\frac{k}{m}} \;:\; k_0 \in \mathbb{Z}, c_k \in \mathbb{C}, m \in \mathbb{Z}_{>0} \right\}.$$

The solutions to the *lifted system of equations* $f_{1,w}(t) = \cdots = f_{d,w}(t) = 0$ have Puiseux series coordinates: $t(\epsilon) \in \mathbb{C}\{\{\epsilon\}\}^d$. We will soon investigate their leading terms.

Remark 13.1.1. One can alternatively think of the lifted equations $f_{1,w}(\epsilon, t) = \cdots = f_{d,w}(\epsilon, t) = 0$ as d Laurent polynomial equations on $\mathbb{C}^* \times (\mathbb{C}^*)^d$. Here the first factor $\mathbb{C}^*$ has coordinate ϵ. This defines an algebraic curve, whose projection to that first factor is a branched covering (a generic fiber consists of a constant, finite number of points). The curve is parametrized by algebraic functions in ϵ. This is the typical setup of a homotopy continuation algorithm with *continuation parameter* ϵ.

Passing to the field $\mathbb{C}\{\{\epsilon\}\}$ might confuse the reader, as it seems more complicated to solve equations over $\mathbb{C}\{\{\epsilon\}\}$ than over $\mathbb{C}$. Our ambition is *not* to compute the solutions $t(\epsilon)$ over $\mathbb{C}\{\{\epsilon\}\}$, but only their leading terms. We will see that this is "easy", in the sense that it reduces to combinatorics and solving binomial equations. Here is an example.

Example 13.1.2. For simplicity, let us consider the univariate cubic polynomial $f = f_1 = -\frac{3}{4} + 2t^2 - \frac{3}{4}t^3$. We plug in the variable ϵ in the following manner:

$$f_\epsilon \;=\; -\frac{3}{4}\,\epsilon + 2t^2 - \frac{3}{4}\,\epsilon\, t^3.$$

For $\epsilon = 1$, this equals f. The equation $f_\epsilon(t) = 0$ has three solutions over $\mathbb{C}\{\{\epsilon\}\}$:

$$\begin{aligned}
t^{(1)}(\epsilon) \;&=\; -\frac{1}{2}\sqrt{\frac{3}{2}}\epsilon^{\frac{1}{2}} + \frac{9\epsilon^2}{128} - \frac{135\sqrt{\frac{3}{2}}\epsilon^{7/2}}{8192} + \frac{243\epsilon^5}{32768} - \frac{168399\sqrt{\frac{3}{2}}\epsilon^{13/2}}{67108864} + O\left(\epsilon^{15/2}\right),\\
t^{(2)}(\epsilon) \;&=\; \frac{1}{2}\sqrt{\frac{3}{2}}\epsilon^{\frac{1}{2}} + \frac{9\epsilon^2}{128} + \frac{135\sqrt{\frac{3}{2}}\epsilon^{7/2}}{8192} + \frac{243\epsilon^5}{32768} + \frac{168399\sqrt{\frac{3}{2}}\epsilon^{13/2}}{67108864} + O\left(\epsilon^{15/2}\right),\\
t^{(3)}(\epsilon) \;&=\; \frac{8}{3\epsilon} - \frac{9\epsilon^2}{64} - \frac{243\epsilon^5}{16384} + O\left(\epsilon^{15/2}\right).
\end{aligned}$$

We claim that the first terms in these series, i.e., the terms of order $1/2, 1/2$ and -1, are "easy" to compute. These terms capture the behaviour of the solutions as $\epsilon \to 0$. This is illustrated in Figure 13.1, which shows the curve from Remark 13.1.1 in the real part of $\mathbb{C}^* \times \mathbb{C}^*$. The curve is in black, and the graphs of the functions $-\frac{1}{2}\sqrt{\frac{3}{2}}\epsilon^{\frac{1}{2}}, \frac{1}{2}\sqrt{\frac{3}{2}}\epsilon^{\frac{1}{2}}, \frac{8}{3\epsilon}$ are in green, blue and red respectively. In this example, the leading terms approximate the solution paths quite well, even up to $\epsilon = 1$. Of course, this can in general not be expected. The prediction of the local behaviour near $\epsilon = 0$ is what matters. $\diamond$

A solution to the lifted equations $f_{1,\epsilon}(t) = \cdots = f_{d,\epsilon}(t) = 0$ has the form

$$t(\epsilon) \;=\; (t_{1,0}\,\epsilon^{v_1}, \,\ldots,\, t_{d,0}\,\epsilon^{v_d}) \;+\; \text{higher order terms}, \tag{13.1.2}$$

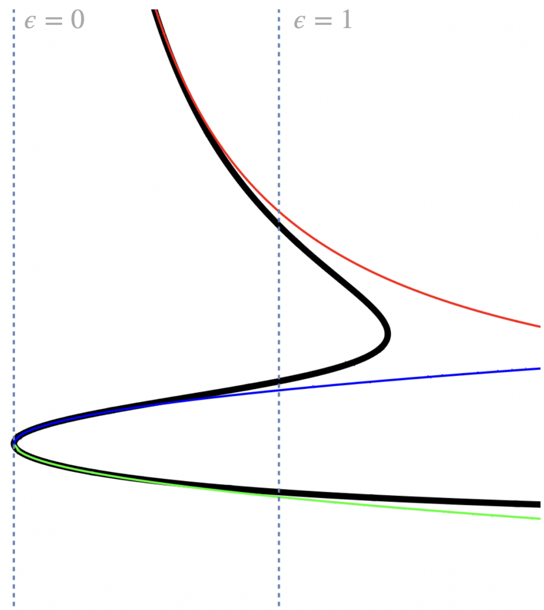

Figure 13.1: Branches of a cubic plane curve in the (ϵ, t)-plane near $\epsilon = 0$.

where $t_{j,0} \in \mathbb{C}^*$ are the (nonzero) *leading coefficients* and $v_j \in \mathbb{Q}$ is the *order*[1] of the Puiseux series $t_j(\epsilon)$, the j-th coordinate of $t(\epsilon)$. A polyhedral homotopy has two steps:

1. Compute the leading terms $(t_{1,0}\,\epsilon^{v_1}, \ldots, t_{d,0}\,\epsilon^{v_d})$ for each Puiseux series solution.

2. Track the solution paths for $\epsilon \to 1$ via numerical homotopy continuation.

These steps are discussed separately in Sections 13.2 and 13.3 (step 1) and 13.4 (step 2).

13.2 Computing leading terms

Our first goal is to determine the possible values for the unknowns $t_{j,0}$ and v_j in (13.1.2). Substituting (13.1.2) into the lifted polynomial $f_{i,w}$ from (13.1.1), we get

$$\begin{aligned} f_{i,w}(t(\epsilon)) &= \sum_{a \in A_i} z_{i,a}\,\epsilon^{w_{i,a}}(t_{1,0}\epsilon^{v_1} + \text{h.o.t.})^{a_1} \cdot \cdots \cdot (t_{d,0}\epsilon^{v_d} + \text{h.o.t.})^{a_d} \\ &= \sum_{a \in A_i} (z_{i,a}\,t_0^a\,\epsilon^{w_{i,a} + \langle v,a\rangle} + \text{h.o.t}). \end{aligned}$$

Here $t_0 = (t_{1,0}, \ldots, t_{d,0}) \in (\mathbb{C}^*)^d$, $\langle v, a\rangle = v_1 a_1 + \cdots + v_d a_d$ and h.o.t. stands for "higher order terms". Since all coefficients $z_{i,a}$ are nonzero, we observe that $f_{i,w}(t(\epsilon))$ is a Puiseux series of order $\mathrm{ord}_{i,v} = \min_{a \in A_i}(w_{i,a} + \langle v, a\rangle) = \min_{a\in A_i}\langle (v,1),(a,w_{i,a})\rangle$. Let us write

$$A_{i,w} = \{(a, w_{i,a}) \,:\, a \in A_i\}, \quad A_{i,w}^v = \{(a, w_{i,a}) \in A_{i,w} \,:\, \langle (v,1),(a,w_{i,a})\rangle = \mathrm{ord}_{i,v}\}.$$

[1] The *order* of a Puiseux series $\sum_{k \geq k_0, k \in \mathbb{Z}} c_k\,\epsilon^{\frac{k}{m}}$ with $c_{k_0} \neq 0$ is $\frac{k_0}{m}$. The *leading term* is $c_{k_0}\,\epsilon^{\frac{k_0}{m}}$.

In words, the points in $A_{i,w} \subset \mathbb{Z}^{d+1}$ are the exponents in A_i, but *lifted* by the weights $w_{i,a}$. The subset $A_{i,w}^v \subseteq A_{i,w}$ consists of those lifted points on which the inner product with the vector $(v, 1)$ is minimized. With this notation, we have

$$f_{i,w}(t(\epsilon)) \;=\; \left(\sum_{(a,w_{i,a}) \in A_{i,w}^v} z_{i,a}\, t_0^a \right) \epsilon^{\mathrm{ord}_{i,v}} + \text{higher order terms}.$$

If $t(\epsilon)$ is a solution to our lifted equations, then this series must be identically zero. In particular, the leading coefficient must be zero, so we find that $t_0 \in (\mathbb{C}^*)^d, v \in \mathbb{Q}^d$ satisfy

$$\sum_{(a,w_{i,a}) \in A_{1,w}^v} z_{1,a}\, t_0^a \;=\; \cdots \;=\; \sum_{(a,w_{d,a}) \in A_{d,w}^v} z_{d,a}\, t_0^a \;=\; 0. \tag{13.2.1}$$

Clearly, these equations have no solutions $t_0 \in (\mathbb{C}^*)^d$ if $A_{i,w}^v$ consists of a single point for some i; a monomial has no zeros on the torus. We have deduced the following statement.

Lemma 13.2.1. *The leading exponents $v = (v_1, \ldots, v_d)$ of a Puiseux series solution $t(\epsilon)$ to the lifted equations $f_{1,w}(t) = \cdots = f_{d,w}(t) = 0$ satisfy*

$$A_{i,w}^v \text{ consists of at least two points, for each } i = 1, \ldots, d. \tag{13.2.2}$$

The corresponding leading coefficients $t_0 = (t_{1,0}, \ldots, t_{d,0}) \in (\mathbb{C}^*)^d$ satisfy (13.2.1).

Example 13.2.2. In Example 13.1.2, there is only one set of lifted exponents: $A_{1,w} = A_w = \{(0,1), (2,0), (3,1)\}$. Its elements correspond to the monomials ϵ, t^2 and ϵt^3. The solution $t^{(1)}(\epsilon)$ has leading exponent $v = \frac{1}{2}$. It satisfies (13.2.2), since the inner product of $(\frac{1}{2}, 1)$ is minimized by both $(0,1)$ and $(2,0)$. That is, $A_w^v = \{(0,1), (2,0)\}$. The corresponding leading coefficient $t_0 = -(\frac{3}{8})^{\frac{1}{2}}$ is a solution to $-\frac{3}{4} + 2t_0^2 = 0$. ◇

So far, we have taken the weights $w_{i,a}$ to be integers, and this is indeed practical for homotopy continuation. Since we work over the field of Puiseux series, one could in fact use rational weights $w_{i,a} \in \mathbb{Q}$. The points in $A_{i,w} \subset \mathbb{Z}^d \times \mathbb{Q}$ now have one rational coordinate. Let $\mathbb{Q}^A = \mathbb{Q}^{|A_1|} \times \cdots \times \mathbb{Q}^{|A_d|}$ be the space of all rational weights $w = (w_{i,a}, i = 1, \ldots, d, a \in A_i)$. In what follows we will show that for almost all choices of $w \in \mathbb{Q}^A$, there are finitely many points $v \in \mathbb{Q}^d$ satisfying (13.2.2), and for each such v, the system (13.2.1) has finitely many solutions. In particular, we will define a set $\Xi(A) \subset \mathbb{Q}^A$ consisting of finitely many linear spaces such that the following holds.

Proposition 13.2.3. *Suppose that $w \in \mathbb{Q}^A$ is such that $w \notin \Xi(A)$. Then we have:*

1. *There are finitely many points $v \in \mathbb{Q}^d$ satisfying* (13.2.2).
2. *For each v satisfying* (13.2.2), *$A_{1,w}^v, \ldots, A_{d,w}^v$ consist of precisely two points each.*
3. *For each v satisfying* (13.2.2), *write $A_{i,w}^v = \{(b_{i,1}, w_{i,1}), (b_{i,2}, w_{i,2})\}$ and let*

$$M_v = \begin{pmatrix} b_{1,2} - b_{1,1} & \cdots & b_{d,2} - b_{d,1} \end{pmatrix} \in \mathbb{Z}^{d \times d}.$$

We have $\det M_v \neq 0$ *and the number of solutions $t_0 \in (\mathbb{C}^*)^d$ to* (13.2.1) *is* $|\det M_v|$.

The condition $w \notin \Xi(A)$ is a genericity condition on the weights w. We will need two easy lemma's before proving Proposition 13.2.3, and we will define $\Xi(A)$ along the way. The definition of $\Xi(A)$ will imply that $\mathbb{Z}^A \setminus \Xi(A)$ is non-empty, so we can always choose integer weights. Let $B = (B_1, \dots, B_d)$ be a tuple of subsets $B_i \subset A_i \subset \mathbb{Z}^d$ and let $B_w = (B_{1,w}, \dots, B_{d,w}) \subset A_{i,w}$ be the corresponding lifted exponent sets. That is,

$$B_i = \{b_{i,1}, \dots, b_{i,k_i}\} \subset A_i \quad \text{and} \quad B_{i,w} = \{(b_{i,1}, w_{i,1}), \dots, (b_{i,k_i}, w_{i,k_i})\} \subset A_{i,w},$$

where $k_i \geq 2$ is the cardinality of B_i and $w_{i,j}$ is the last out of $d+1$ coordinates of $(b_{i,j}, w_{i,j})$. We define matrices $M_i(B)$ and row vectors $q_i(w)$ as follows:

$$\begin{aligned} M_i(B) &= \begin{pmatrix} b_{i,2} - b_{i,1} & \cdots & b_{i,k_i} - b_{i,1} \end{pmatrix} \in \mathbb{Z}^{d \times (k_i - 1)}, \\ q_i(w) &= \begin{pmatrix} w_{i,2} - w_{i,1} & \cdots & w_{i,k_i} - w_{i,1} \end{pmatrix} \in \mathbb{Q}^{k_i - 1}. \end{aligned}$$

Concatenating these for all i, we obtain $M_B = \begin{pmatrix} M_1(B) & \cdots & M_d(B) \end{pmatrix} \in \mathbb{Z}^{d \times \sum_{i=1}^d (k_i - 1)}$, $q_w = \begin{pmatrix} q_1(w) & \cdots & q_d(w) \end{pmatrix} \in \mathbb{Q}^{\sum_{i=1}^d (k_i-1)}$. Notice that, if $B_{i,w} = A^v_{i,w}$ and v satisfies the conditions in Proposition 13.2.3, then $M_B = M_v$, where M_v is as in point 3 of the proposition.

Lemma 13.2.4. *If $B_i = A^v_{i,w}, i = 1, \dots, d$ for some $v \in \mathbb{Q}^d$, then $-v^\top M_B = q_w$. In particular, q_w lies in the row span $\mathrm{Row}_\mathbb{Q}(M_B)$ of M_B over $\mathbb{Q}$.*

Proof. It suffices to check that $-v^\top (b_{i,j} - b_{i,1}) = w_{i,j} - w_{i,1}$. This follows easily from the fact that $\langle (v, 1), (b_{i,j}, w_{i,j}) \rangle = \mathrm{ord}_{i,v}$ for $j = 1, \dots, k_i$. □

Exercise 13.2.5. Notice that the condition $q_w \in \mathrm{Row}_\mathbb{Q}(M_B)$ is invariant under scaling the weights w. If $w \in \mathbb{Z}^A$ is an integer weight vector, then in the notation of Chapter 1, this condition $q_w \in \mathrm{Row}_\mathbb{Q}(M_B)$ can be expressed in terms of *affine lattices* (Definition 1.3.15). The rank of the affine lattice generated by all exponents $B_{1,w} \cup \cdots \cup B_{d,w}$ in $\mathbb{Z}^{d+1}$ must be equal to that of the rank of the affine lattice generated by $B_1 \cup \cdots \cup B_d$ in $\mathbb{Z}^d$. This shows that the condition is independent of the ordering of the elements in B_i, even though q_w and M_B are not.

Here is a direct consequence of Lemma 13.2.4.

Lemma 13.2.6. *If $\mathrm{rank}(M_B) < \sum_{i=1}^d (k_i - 1)$, then there is a linear space $L_B \subset \mathbb{Q}^A$ of codimension $\sum_{i=1}^d (k_i - 1) - \mathrm{rank}(M_B)$ such that for $w \in \mathbb{Q}^A \setminus L_B$, there is no $v \in \mathbb{Q}^d$ such that $B_i = A^v_{i,w}, i = 1, \dots, d$.*

Proof. By Lemma 13.2.4, the linear space L_B can be defined as follows: $L_B = \{w \in \mathbb{Q}^A : q_w \in \mathrm{Row}_\mathbb{Q}(M_B)\}$. We leave the claim about the codimension to the reader. □

Running over all subset tuples B satisfying $\mathrm{rank}(M_B) < \sum_{i=1}^d (k_i - 1)$, we obtain a finite union of subspaces sufficient for Proposition 13.2.3:

$$\Xi(A) = \bigcup_{\mathrm{rank}(M_B) < \sum_{i=1}^d (k_i - 1)} L_B. \tag{13.2.3}$$

Proof of Proposition 13.2.3. Let $B = (B_1, \dots, B_d)$ with $B_{i,w} = A_{i,w}^v$ for some v satisfying (13.2.2). We must have $k_i = |B_i| \geq 2, i = 1, \dots, d$, which means that $d \leq \sum_{i=1}^d (k_i - 1)$. By Lemma 13.2.6 and the assumption $w \notin \Xi(A)$, we have $\sum_{i=1}^d (k_i - 1) \leq \operatorname{rank}(M_B)$. We have shown

$$d \;\leq\; \sum_{i=1}^d (k_i - 1) \leq \operatorname{rank}(M_B) \;\leq\; d,$$

which means $k_i = 2, i = 1, \dots, d$ and $\operatorname{rank}(M_B) = d$. The only choices for $B_{i,w}$ are the two-element subsets of $A_{i,w}$ for which M_B has rank d. For any such choice, the vector v is uniquely determined by the condition $-v^\top M_B = q_w$ from Lemma 13.2.4. This gives a finite list of candidates for v, and one can easily check whether $B_i = A_{i,w}^v$. If v passes this test, then it is such that (13.2.2) holds, and $M_B = M_v$. This proves points 1 and 2. For point 3, notice that the equations (13.2.1) are

$$z_{1,b_{1,1}} t_0^{b_{1,1}} + z_{1,b_{1,2}} t_0^{b_{1,2}} \;=\; \cdots \;=\; z_{d,b_{d,1}} t_0^{b_{d,1}} + z_{d,b_{d,2}} t_0^{b_{d,2}} \;=\; 0. \qquad (13.2.4)$$

Since $t_0 \in (\mathbb{C}^*)^d$, we may equivalently write this as

$$\phi_{M_B}(t_0) = \left(\frac{-z_{1,b_{1,1}}}{z_{1,b_{1,2}}}, \dots, \frac{-z_{d,b_{d,1}}}{z_{d,b_{d,2}}} \right), \qquad (13.2.5)$$

where $\phi_{M_B} : (\mathbb{C}^*)^d \to (\mathbb{C}^*)^d$ is the monomial map whose exponents are the columns of M_B, see (1.1.1). Recall that all coefficients $z_{i,a}$ are assumed to be nonzero. By Corollary 1.2.14, the number of solutions for $t_0 \in (\mathbb{C}^*)^d$ is $|\det M_B|$. $\square$

Example 13.2.7. In Example 13.1.2, the tuple B consists of $d = 1$ subset of $A = \{0, 2, 3\}$, of cardinality at least two. There are four such subsets:

$$\{0, 2\}, \quad \{0, 3\}, \quad \{2, 3\}, \quad \{0, 2, 3\}.$$

In the same order, the corresponding matrices M_B are

$$\begin{pmatrix} 2 - 0 \end{pmatrix} \in \mathbb{Z}^{1\times 1}, \quad \begin{pmatrix} 3 - 0 \end{pmatrix} \in \mathbb{Z}^{1\times 1}, \quad \begin{pmatrix} 3 - 2 \end{pmatrix} \in \mathbb{Z}^{1\times 1}, \quad \begin{pmatrix} 2 - 0 & 3 - 0 \end{pmatrix} \in \mathbb{Z}^{1\times 2}.$$

The last matrix satisfies $1 = \operatorname{rank}(M_B) < |B| - 1 = 2$. It defines a linear space

$$L_B \;=\; \left\{ (w_0, w_2, w_3) \in \mathbb{Q}^A \;:\; \det \begin{pmatrix} 2 & 3 \\ w_2 - w_0 & w_3 - w_0 \end{pmatrix} = 0 \right\}.$$

Our choice $w = (w_0, w_2, w_3) = (1, 0, 1)$ in Example 13.1.2 satisfies $w \notin \Xi(A) = L_B$. The two-element subsets $\{0, 2\}, \{0, 3\}, \{2, 3\}$ each give one candidate for v via $-v \cdot M_B = q_w$. For $B = \{0, 2\}$, we have $-v \cdot 2 = -1$, and $A_w^v = A_w^{1/2} = B_w = \{(0, 1), (2, 0)\}$. The determinant $\det M_B = 2$ counts the two solutions $t^{(1)}(\epsilon), t^{(2)}(\epsilon)$ of order $v = 1/2$. For $B = \{0, 3\}$, we obtain $v = 0$, but $A_w^0 = \{(2, 0)\} \neq B_w$, so there are no solutions of order 0. Finally, the subset $B = \{2, 3\}$ gives one solution $t^{(3)}(\epsilon)$ of order -1.

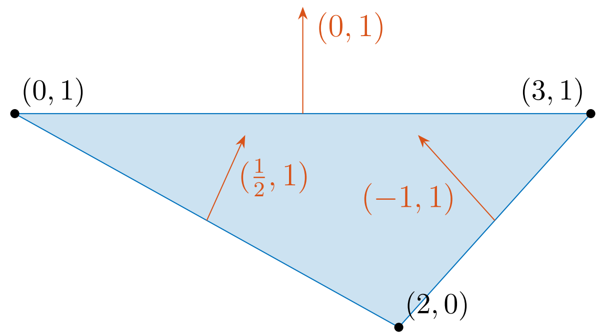

Figure 13.2: A polyhedral interpretation of Example 13.2.7.

We end the example with a *polyhedral* interpretation of these computations, to which the algorithm explained in this chapter owes its name. Figure 13.2 shows the convex hull of the lifted points A_w. The requirement that the inner product of $(v, 1)$ is minimized on two points in A_w means, in geometric terms, that $(v, 1)$ is perpendicular to one of the edges of this triangle. The three values $\frac{1}{2}, 0, -1$ above can be found in this way from the picture. The values of v which actually lead to a solution of (13.2.2) are those for which $(v, 1)$ exposes an edge in the *lower hull* of the triangle. The lower hull is defined as the union of edges whose normal vector points in the positive "y-direction". The condition $w \notin L_B$ ensures that the three black dots in Figure 13.2 are not colinear. $\diamond$

Exercise 13.2.8. Compute the order of all Puiseux series solutions $t(\epsilon)$ to the quartic equation $\epsilon^2 + \epsilon\, t + t^2 + \epsilon\, t^3 + \epsilon^2 t^4 = 0$.

Remark 13.2.9. The condition $w \notin \Xi(A)$ with $\Xi(A)$ as in (13.2.3) is sufficient but in general not necessary for the conclusions of Proposition 13.2.3 to hold. The precise condition on w be phrased nicely in the language of *tropical geometry*: the tropical hypersurfaces defined by the lifted polynomials $f_{i,w}, i = 1, \ldots, d$ must intersect transversely.

13.3 Mixed subdivisions

This section elaborates on the polyhedral interpretation of the computation of the vectors v in Proposition 13.2.3 given at the end of Example 13.2.7. We generalize it to higher dimensions. These insights are crucial for completing step 1 in the polyhedral homotopy algorithm. We need some more definitions first.

Definition 13.3.1 (Lower hull)**.** *The* lower hull *of a convex polytope $\hat{\mathcal{P}}$ in $\mathbb{R}^{d+1}$ is the union of all its faces of the form*

$$\hat{\mathcal{P}}^{(v,1)} \;=\; \{p \in \hat{\mathcal{P}} \;:\; \langle (v,1), p\rangle = v_1 p_1 + \cdots + v_d p_d + p_{d+1} \;=\; \min_{p' \in \hat{\mathcal{P}}} \langle (v,1), p'\rangle\}$$

for some $v \in \mathbb{R}^d$. We say that $\hat{\mathcal{P}}^{(v,1)}$ is the face in the lower hull exposed by $(v, 1)$.

Example 13.3.2. The lower hull of the triangle in Figure 13.2 is the union of the two edges $\mathrm{Conv}((0,1),(2,0))$ and $\mathrm{Conv}((2,0),(3,1))$. These are exposed by $(v,1) = (\frac{1}{2},1)$ and $v = (-1,1)$ respectively. The vertex $(2,0)$ is a face on the lower hull as well. It is exposed by any $(v,1)$, with v in the open interval $(-1,\frac{1}{2}) \subset \mathbb{R}$. ◇

In our context, the relevant polytopes $\hat{\mathcal{P}}$ in $\mathbb{R}^{d+1}$ are obtained from the lifted exponent sets. We use the following notation:

$$\mathcal{P}_i \;=\; \mathrm{Conv}(A_i) \;\subset\; \mathbb{R}^d, \quad \mathcal{P}_{i,w} \;=\; \mathrm{Conv}(A_{i,w}) \;\subset\; \mathbb{R}^{d+1}, \quad i = 1,\dots,d.$$

Notice that, by definition, $\mathcal{P}_i$ is the Newton polytope of the Laurent polynomial f_i. Similarly, $\mathcal{P}_{i,w}$ is the "Newton polytope" of the lifted polynomial $f_{i,w}$, if we allow fractional exponents and consider variables $(t_1,\dots,t_d,\epsilon)$. The coordinate projection $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ which sends $(x_1,\dots,x_d,x_{d+1}) \to (x_1,\dots,x_d)$ is such that $\pi(\mathcal{P}_{i,w}) = \mathcal{P}_i$ for each i.

Exercise 13.3.3. With the notation of the previous section, show that $A^v_{i,w} = A_{i,w} \cap \mathcal{P}^{(v,1)}_{i,w}$ and $\mathcal{P}^{(v,1)}_{i,w} = \mathrm{Conv}(A^v_{i,w})$.

Definition 13.3.4. *Let $\mathcal{P}_{i,w} \subset \mathbb{R}^{d+1}$ and $\mathcal{P}_i = \pi(\mathcal{P}_{i,w}) \subset \mathbb{R}^d$ be as above. The* regular subdivision *of $\mathcal{P}_i$ induced by the weights w is*

$$\mathcal{Q}_{i,w} \;=\; \{\pi(Q) \,:\, Q \textit{ is a face on the lower hull of } \mathcal{P}_{i,w}\}.$$

The following examples will clarify the terminology *subdivision*.

Example 13.3.5. The triangle $\mathcal{P}_w$ in Figure 13.2 induces the subdivision

$$\mathcal{Q}_w \;=\; \{\{0\},\{2\},\{3\},[0,2],[2,3]\}$$

of the interval $\mathcal{P} = [0,3]$. These are the projections of the three vertices and the two edges on the lower hull of $\mathcal{P}_w$, which subdivide the interval $[0,3]$ into two smaller intervals. ◇

Example 13.3.6. We consider the following system of lifted equations:

$$\begin{aligned} f_{1,w} \;&=\; 1 + \epsilon^2\, t_1^{-1} + \epsilon^2\, t_1 + \epsilon^2\, t_2^{-1} + \epsilon^2\, t_2 \;=\; 0, \\ f_{2,w} \;&=\; 1 + \epsilon\, t_1 + \epsilon^4\, t_2 + \epsilon^2\, t_1 t_2 + \epsilon^4\, t_1^2 \;=\; 0. \end{aligned}$$

The Laurent polynomials f_1, f_2 are obtained by setting $\epsilon = 1$. We have

$$A_1 = \{(0,0),(-1,0),(0,1),(0,-1),(0,1)\} \quad \text{and} \quad A_2 = \{(0,0),(1,0),(0,1),(1,1),(2,0)\}.$$

The lifted exponent sets are $A_{1,w} = \{(0,0,0),(-1,0,2),(0,1,2),(0,-1,2),(0,1,2)\}$ and $A_{2,w} = \{(0,0,0),(1,0,1),(0,1,4),(1,1,2),(2,0,4)\}$. We consider the convex hulls $\mathcal{P}_{1,w} = \mathrm{Conv}(A_{1,w})$ and $\mathcal{P}_{2,w} = \mathrm{Conv}(A_{2,w})$ of these lifted exponent sets. The resulting polytopes are shown in Figure 13.3. Their lower hulls induce subdivisions of the Newton polygons of f_1 and f_2. This is illustrated by the projected subdivided polygons in the figure. For f_1, the diamond $\mathcal{P}_1 = \mathrm{Conv}(A_1)$ is subdivided into four triangles. The set $\mathcal{Q}_{1,w}$ consists of these four triangles, eight line segments and five points. ◇

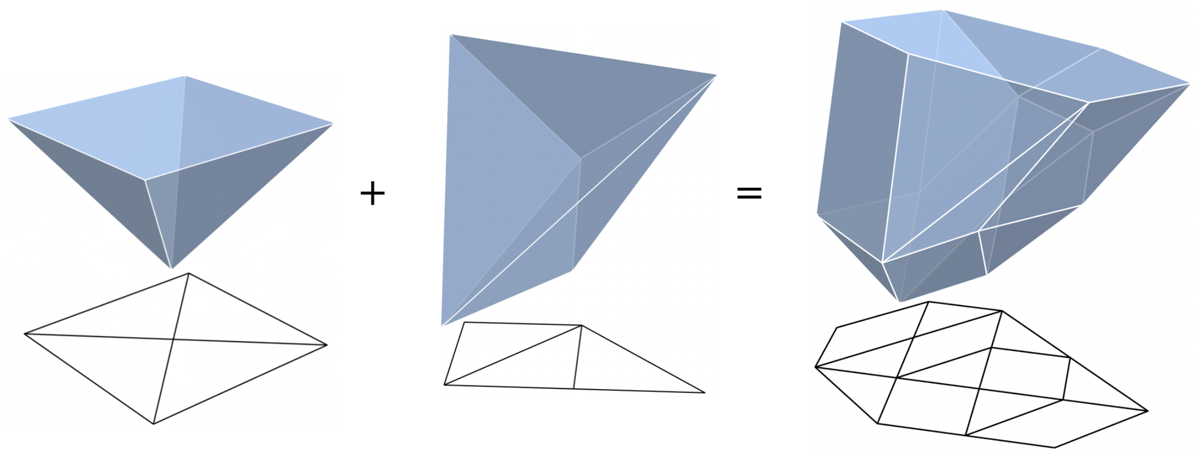

Figure 13.3: A mixed subdivision of the Minkowski sum of two polygons.

The Minkowski sum $\mathcal{P}_w = \mathcal{P}_{1,w} + \cdots + \mathcal{P}_{d,w}$ is a polytope in $\mathbb{R}^{d+1}$. Its projection $\pi(\mathcal{P}_w) \subset \mathbb{R}^d$ equals $\mathcal{P} = \mathcal{P}_1 + \cdots + \mathcal{P}_d$ and the lower hull of $\mathcal{P}_w$ induces a subdivision:

Definition 13.3.7. *Let $\mathcal{P}_w \subset \mathbb{R}^{d+1}$ and $\mathcal{P} = \pi(\mathcal{P}_w) \subset \mathbb{R}^d$ be as above. The* mixed subdivision *of $\mathcal{P}$ induced by the weights w is*

$$\mathcal{Q}_w = \{\pi(\hat{Q}) : \hat{Q} \textit{ is a face on the lower hull of } \mathcal{P}_w\}.$$

Equivalently, we have $\mathcal{Q}_w = \{Q_v : v \in \mathbb{Q}^d\}$, where $Q_v = \pi(\mathcal{P}_w^{(v,1)})$.

Example 13.3.8. The mixed subdivision of $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$ from Example 13.3.6 is shown in Figure 13.3. The two-dimensional elements in $\mathcal{Q}_w$ consist of seven triangles and five quadrilaterals. ◇

Exercise 13.3.9. Show that for any polytopes $\mathcal{P}_1, \ldots, \mathcal{P}_d$, a face $\mathcal{P}^v$ of the Minkowski sum $\mathcal{P} = \mathcal{P}_1 + \cdots + \mathcal{P}_d$ is a sum of faces, given by $\mathcal{P}^v = \mathcal{P}_1^v + \cdots + \mathcal{P}_d^v$.

Definition 13.3.10. *Let $Q_v = \pi(\mathcal{P}_w^{(v,1)})$ be an element of the mixed subdivision $\mathcal{Q}_w$ in Definition 13.3.7. We have $\mathcal{P}_w^{(v,1)} = \mathcal{P}_{1,w}^{(v,1)} + \cdots + \mathcal{P}_{d,w}^{(v,1)}$ by Exercise 13.3.9. We call $Q_v \in \mathcal{Q}_w$ a* mixed cell *if*

1. *Q_v has dimension d and*
2. *$\dim \mathcal{P}_{i,w}^{(v,1)} = 1$ for $i = 1, \ldots, d$.*

That is, mixed cells in a mixed subdivision come from Minkowski sums of edges.

Example 13.3.11. The mixed subdivision in Figure 13.3 has five mixed cells. These are precisely the parallelograms in the rightmost picture. This is illustrated in Figure 13.4, where the mixed cells are highlighted together with the corresponding edges in the regular subdivisions of $\mathcal{P}_1$ and $\mathcal{P}_2$. ◇

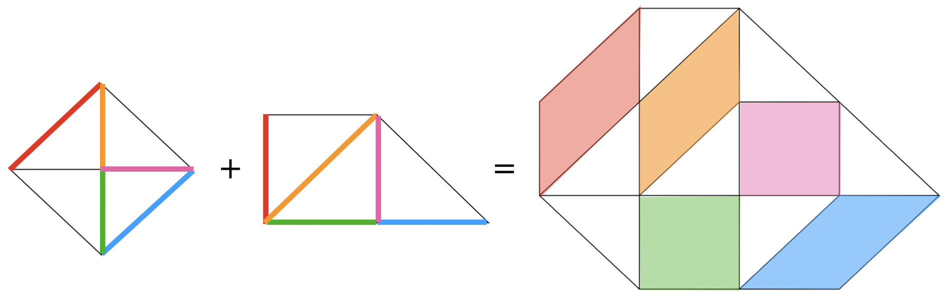

Figure 13.4: Mixed cells in a mixed subdivision.

We are now ready to characterize the vectors v from Proposition 13.2.3 in terms of mixed cells in mixed subdivisions.

Proposition 13.3.12. *Let $w \in \mathbb{Q}^A \setminus \Xi(A)$, with $\Xi(A)$ as defined in* (13.2.3)*. The following are equivalent:*

1. *$v \in \mathbb{Q}^d$ is such that $A_{i,w}^v$ consists of two points for $i = 1, \ldots, d$.*

2. *$\pi(\mathcal{P}_w^{(v,1)})$ is a mixed cell of $\mathcal{Q}_w$.*

Moreover, in this case, the number $|\det M_v|$ equals the volume of the mixed cell $\pi(\mathcal{P}_w^{(v,1)})$.

Proof. We start with $1 \Rightarrow 2$. If $A_{i,w}^v$ consists of two points, then its convex hull forms the edge $\mathcal{P}_{i,w}^{(v,1)}$ of $\mathcal{P}_{i,w}$. The projection $\pi(\mathcal{P}_{i,w}^v)$ of this edge is parallel to the vector $b_{i,2} - b_{i,1}$, where $A_{i,w}^v = \{(b_{i,1}, w_{i,1}), (b_{i,2}, w_{i,2})\}$. By Proposition 13.2.3, the determinant of $M_v = \begin{pmatrix} b_{1,2} - b_{1,1} & \cdots & b_{d,2} - b_{d,1} \end{pmatrix}$ is nonzero. Hence, the Minkowski sum of the projected edges has dimension d, and $\pi(\mathcal{P}_w^{(v,1)})$ is a mixed cell with volume $|\det M_v|$.

For $2 \Rightarrow 1$, we start from the fact that $\mathcal{P}_{i,w}^v$ has dimension one for all i. This means that $A_{i,w}^v = \mathcal{P}_{i,w}^v \cap A_{i,w}$ contains at least two points for all i (Exercise 13.3.3). By Proposition 13.2.3 and the assumption $w \notin \Xi(A)$, $A_{i,w}^v$ contains exactly two points. $\square$

Propositions 13.2.3 and 13.3.12 lead to Algorithm 2 for computing all leading terms $t_0\,\epsilon^v$ of solutions to our lifted equations when $w \in \mathbb{Q}^A$ is sufficiently generic:

This implies in particular that the number of Puiseux series solutions to our lifted equations is at most the sum of the volumes of the mixed cells. To match this with what we saw in **??**, we must relate this to the mixed volume $\mathrm{MV}(\mathcal{P}_1, \ldots, \mathcal{P}_d)$.

Proposition 13.3.13. *Let $w \notin \Xi(A)$, with $\Xi(A)$ as in* (13.2.3)*, and let $\mathcal{Q}_w$ be the mixed subdivision of $\mathcal{P}_w$ induced by w. We have*

$$\sum_{\substack{Q_v \in \mathcal{Q}_w \\ Q_v \, mixed}} \mathrm{vol}(Q_v) \;=\; \mathrm{MV}(\mathcal{P}_1, \ldots, \mathcal{P}_d).$$

Algorithm 2 Compute the leading terms of all solutions to $f_{1,w} = \cdots = f_{d,w} = 0$

Input: $f_{1,w}, \ldots, f_{d,w}$ with $w \in \mathbb{Q}^A$ sufficiently generic
Output: Leading terms of all Puiseux series solutions
$\mathcal{P}_w \leftarrow \mathcal{P}_{1,w} + \cdots + \mathcal{P}_{d,w}$.
$V \leftarrow \emptyset$
for each facet $\mathcal{P}_w^{(v,1)}$ in the lower hull of $\mathcal{P}_w$ **do**
  if $\mathcal{P}_{i,w}^{(v,1)}$ is an edge for each i **then**
    for each solution $t_0 = (t_{1,0}, \ldots, t_{d,0}) \in (\mathbb{C}^*)^d$ of (13.2.5) **do**
      $V \leftarrow V \cup \{(t_{1,0}\epsilon^{v_1}, \ldots, t_{d,0}\epsilon^{v_d})$
    end for
  end if
end for
return V

Proof. We use Definition 12.1.2 and the fact that the volume of $\mathcal{P} = \mathcal{P}_1 + \cdots + \mathcal{P}_d$ is the sum of the volumes of the d-dimensional elements in $\mathcal{Q}_w$. Among those elements are the mixed cells. Suppose $Q_v = \pi(\mathcal{P}_w^{(v,1)}) \in \mathcal{Q}_w$ has dimension d and $\pi(\mathcal{P}_w^{(v,1)}) = \pi(\mathcal{P}_{1,w}^{(v,1)}) + \cdots + \pi(\mathcal{P}_{d,w}^{(v,1)})$. The assumptions $\dim Q_v = d$ and $w \notin \Xi(A)$ ensure that $\sum_{i=1}^d \dim \pi(\mathcal{P}_{i,w}^{(v,1)}) = d$. Let us write $\kappa_i^v = \dim \pi(\mathcal{P}_{i,w}^{(v,1)})$ for the dimensions in this sum. We have

$$\mathrm{vol}(\lambda_1\mathcal{P}_1 + \cdots + \lambda_d\mathcal{P}_d) \;=\; \sum_{\substack{Q_v \in \mathcal{Q}_w \\ \dim(Q_v)=d}} \mathrm{vol}(Q_v)\, \lambda_1^{\kappa_1^v}\lambda_2^{\kappa_2^v}\cdots\lambda_d^{\kappa_d^v}$$

for any $\lambda_i \in \mathbb{R}_{\geq 0}$. This is a homogeneous polynomial of degree d in λ, and the coefficient standing with $\lambda_1 \cdots \lambda_d$ is indeed the sum of the volumes of the mixed cells. $\square$

Exercise 13.3.14. Verify Proposition 13.3.13 in Example 13.3.6. You computed the mixed volume in Exercise **??**.

13.4 Lifting the solutions

Let $w \in \mathbb{Z}^A$ be a sufficiently generic vector of weights, e.g., $w \notin \Xi(A)$. We want to solve

$$F_w(\epsilon, t) \;=\; (f_{1,w}(\epsilon, t), \cdots, f_{d,w}(\epsilon, t)) \;=\; 0,$$

for $\epsilon = 1$. This notation emphasizes the dependence of $f_{i,w}$ on ϵ. As pointed out in Remark 13.1.1, our equations define an algebraic curve $C = \{(\epsilon, t) \in \mathbb{C}^* \times (\mathbb{C}^*)^d : F_w(\epsilon, t) = 0\}$. The projection $p : C \to \mathbb{C}^*$ onto the ϵ-coordinate is a branched cover of $\mathbb{C}^*$ of degree δ. The *branch locus* B of p is the finite set of points $\epsilon \in \mathbb{C}^*$ for which the fiber $p^{-1}(\epsilon)$ does not consist of δ points. We will assume that the coefficients $z_{i,a}$ are *generic* complex numbers, so that the real line segment $(0, 1]$ does not intersect B.

The number δ in our setting is the number of branches of $C \cap p^{-1}(D)$, where $D \subset \mathbb{C}^*$ is a punctured open neighborhood of 0. In previous sections, we have shown that there are at most $\mathrm{MV}(\mathcal{P}_1, \ldots, \mathcal{P}_d)$ such branches. To show that $\delta = \mathrm{MV}(\mathcal{P}_1, \ldots, \mathcal{P}_d)$, it suffices to observe that each of the leading terms returned by Algorithm 2 lifts to a locally convergent Puiseux series solution. For each leading exponent v satisfying the conditions of Proposition 13.2.3, let

$$g_{i,w}(\epsilon, u) \;=\; \epsilon^{-\mathrm{ord}_{i,v}}\, f_{i,w}(\epsilon, u\epsilon^v).$$

Here, $u = (u_1, \ldots, u_d)$ are new variables and $u\epsilon^v = (u_1\epsilon^{v_1}, \ldots, u_d\epsilon^{v_d})$. By construction, the terms of $g_{i,w}$ have nonnegative exponents in ϵ, and for $\epsilon = 0$, $g_{1,w}(0, u) = \cdots = g_{d,w}(0, u)$ is the system of binomial equations (13.2.4). For each leading term $t_0\epsilon^v$ found in Algorithm 2, we have $g_{i,w}(0, t_0) = 0$. Moreover, the Jacobian matrix

$$\mathrm{Jac}_u(g_{1,w}, \ldots, g_{d,w}) \;=\; \left(\frac{\partial g_{i,w}}{\partial u_j}\right)_{i,j},$$

when evaluated at $\epsilon = 0, u = t_0$, is an invertible $d \times d$ matrix (Exercise 13.4.1). The implicit function theorem gives locally holomorphic functions $(\epsilon, u(\epsilon))$ satisfying $g_{i,w}(\epsilon, u(\epsilon)) = 0$ and $u(0) = t_0$. Setting $t(\epsilon) = u(\epsilon)\,\epsilon^v$ gives a branch $(\epsilon, t(\epsilon))$ of C, and the leading term of $t(\epsilon)$ is $t_0\epsilon^v$.

Exercise 13.4.1. Fix $v \in \mathbb{Q}^d$ satisfying the conditions of Proposition 13.2.3 and let $g_{i,w}$ be defined as above. Consider the *toric Jacobian matrix*

$$\mathrm{Jac}_u^*(g_{1,w}, \ldots, g_{d,w}) \;=\; \left(u_j \cdot \frac{\partial g_{i,w}}{\partial u_j}\right)_{i,j}.$$

We write $A_{i,w}^v = \{b_{i,1}, b_{i,2}\}$. Check that, for $\epsilon = 0$, we have $g_{i,w}(0, u) = z_{i,b_{i,1}}u^{b_{i,1}} + z_{i,b_{i,2}}u^{b_{i,2}}$ and recall that a solution t_0 to $g_{1,w}(0, t_0) = \cdots = g_{d,w}(0, t_0) = 0$ (13.2.5). Use this to show that the i-th row of the toric Jacobian matrix, evaluated at $\epsilon = 0, u = t_0$, equals $z_{i,b_{i,1}} \cdot t_0^{b_{i,1}} \cdot (b_{i,1} - b_{i,2})^\top$. Conclude that the determinant of $\mathrm{Jac}_u^*(g_{1,w}, \ldots, g_{d,w})_{|\epsilon=0,u=t_0}$ is a nonzero multiple of $\det M_v \neq 0$. Finally, show that this implies that

$$\det \mathrm{Jac}_u(g_{1,w}, \ldots, g_{d,w})_{|\epsilon=0,u=t_0} \neq 0.$$

The leading terms returned by Algorithm 2 can be used to approximate the branches $t(\epsilon)$ at small ϵ. Let us denote this small value of ϵ by $0 < \epsilon^* \ll 1$. One can improve the approximation $t_0(\epsilon^*)^v \approx t(\epsilon^*)$ by using Newton iteration on the system of equations $F_w(\epsilon^*, t) = 0$ with starting value $t_0(\epsilon^*)^v$. This way, one obtains accurate approximations of all δ solutions in $p^{-1}(\epsilon^*) = \{t \in (\mathbb{C}^*)^d \,:\, F_w(\epsilon^*, t) = 0\}$.

The next step is to increase ϵ^* to the target value $\epsilon = 1$. Observe that the solution paths $(\epsilon, t(\epsilon))$ satisfy the following system of ordinary differential equations:

$$\mathrm{Jac}_t(F_w(\epsilon, t)) \cdot \frac{\partial t}{\partial \epsilon} + \frac{\partial F_w}{\partial \epsilon} = 0. \tag{13.4.1}$$

This is called *Davidenko's equation*. It is obtained by taking the total derivative on both sides of $F_w(\epsilon, t(\epsilon)) = 0$. The final step in the polyhedral homotopy algorithm is to solve (13.4.1) repeatedly with the δ different initial conditions $t(\epsilon^*)$ computed in the previous step. For this, one typically uses *predictor-corrector schemes*. Notice that by the assumption $(0,1] \cap B = \emptyset$, the matrix $\mathrm{Jac}_t(F_w(\epsilon, t))$ is invertible along each solution path over $\epsilon \in (\epsilon^*, 1]$. When the coefficients $z_{i,a}$ are not *generic*, several things can happen. If $1 \notin B$, then one can avoid the branch locus B by considering a different continuation path in $\mathbb{C}^*$ connecting ϵ^* and 1. If $1 \in B$, then one or more solution paths may diverge when $\epsilon \to 1$, or several solution paths may come together. Such standard issues in homotopy continuation are dealt with by algorithms called *end games*. We refer to [62] for more details on numerical homotopy continuation.

13.5 A tropical viewpoint

The polyhedral homotopy algorithm has a natural interpretation in terms of *tropical geometry*. More precisely, step 1 of the algorithm (i.e., computing leading terms) can be seen as *solving a tropical intersection problem*. The tropical data computed in that step are then used to set up the homotopy in step 2. We have not developed the language of tropical geometry in this book. Nonetheless, it is worthwhile to shed some light on this connection with a minimal amount of new tropical terminology.

Each of our lifted Laurent polynomials $f_{i,w} = \sum_{a \in A_i} z_{i,a}\, \epsilon^{w_{i,a}}\, t^a \in \mathbb{C}\{\{\epsilon\}\}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ defines a *tropical hypersurface* defined as follows:

$$T_i \;=\; \mathrm{Trop}(\{f_{i,w} = 0\}) \;=\; \{v \in \mathbb{Q}^d \;:\; A_{i,w}^v \text{ contains at least two points}\}.$$

The intersection $T_1 \cap T_2 \cap \cdots \cap T_d$ consists precisely of all points v satisfying (13.2.2).

In words, T_i consists of all $v \in \mathbb{Q}^d$ such that the minimum of $\langle v, a\rangle + w_{i,a}$ is attained at least twice for $a \in A_i$. If one considers the convex piecewise linear function $f^{\mathrm{tr}} : \mathbb{Q}^d \to \mathbb{Q}$ sending $v \mapsto \min_{a \in A_i}(\langle v, a\rangle + w_{i,a})$, then T_i contains the points at which f^{tr} is not linear.

The equivalence between (13.2.2) and $T_1 \cap \cdots \cap T_d$ implies that the problem of finding the leading exponents v can be regarded as computing the intersection of d tropical hypersurfaces. This is indeed equivalent to computing mixed cells in the mixed subdivision of the Minkowski sum $\mathcal{P}$ induced by the lifted polytope $\mathcal{P}_w$. The volume of a mixed cell $\pi(\mathcal{P}_w^{(v,1)})$ is precisely the *tropical intersection multiplicity* of $T_1 \cap \cdots \cap T_d$ at the point v. Providing all details is beyond our scope. We explain the ideas via our two-dimensional Example 13.3.6.

Example 13.5.1. The lifted polynomials $f_{1,w}, f_{2,w}$ in Example 13.3.6 define tropical curves $T_1, T_2 \subset \mathbb{Q}^2$. Their closure in $\mathbb{R}^2$ is shown in Figure 13.5, where both curves are in black, and T_2 is dashed. The intersection $T_1 \cap T_2$ consists of five points, corresponding to the five mixed cells in Figure 13.4. The colors in the two figures match this correspondence, i.e., the color of a dot $v \in \mathbb{R}^2$ in Figure 13.5 corresponds to the color of the mixed cell in Figure 13.4 obtained as the projection of the face $\mathcal{P}_w^{(v,1)}$ exposed by $(v, 1)$. The branches of the space curve $C = \{(\epsilon, t) \in (\mathbb{C}^*)^3 \;:\; f_{1,w}(\epsilon, t) = f_{2,w}(\epsilon, t) = 0\}$ near $\epsilon = 0$ are locally of the form $(\epsilon, t_{1,0}\epsilon^{v_1}, t_{2,0}\epsilon^{v_2})$, with v one of the intersection points in $T_1 \cap T_2$. In this example, one could (roughly) phrase Proposition 13.2.3 tropically as follows: For

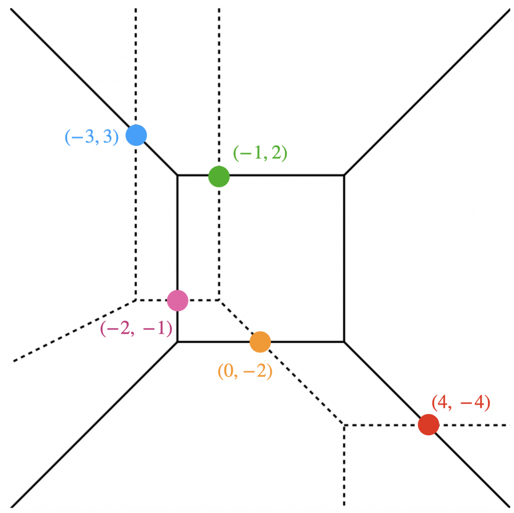

Figure 13.5: Intersecting two tropical plane curves.

generic weights w, our tropical curves intersect in finitely many points, and locally the intersections look like the transverse intersection of two lines. The determinant $|\det M_v|$ is the intersection multiplicity of the tropical curves at v. In this example, all these multiplicities are 1. By Proposition 13.3.13, the sum of the intersection multiplicities (assuming transverse intersections) equals the mixed volume. $\diamond$

Further reading

Early references on polyhedral homotopy methods are [38, 73]. A recent summary is found in [4]. Diverging solution paths in a polyhedral homotopy can be dealt with using Cox coordinates (Chapter 10) [24]. For a complexity analysis, see the series of papers by Malajovich [48, 49, 50]. The polyhedral homotopy method is implemented in several software packages, including [10, 45, 72]. Generalizations of the polyhedral homotopy method exploiting connections with tropical geometry are found, for instance, in [35, 46]. The polyhedral homotopy method solves equations on toric varieties. The papers [8, 12] generalize this to solving equations on varieties which admit a *toric degeneration*, using the theory of SAGBI and Khovanskii bases.

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