Toric Geometry Reading Group

Section 1: Monomial maps and toric ideals
Section 2: Cones and affine toric varieties

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Example 1 (The degree 2 moment curve)

Consider the integer matrix $A = \begin{pmatrix} 1 & 2 \end{pmatrix} \in \mathbb{Z}^{1\times 2}$.
A defines the monomial map $\phi_A : \mathbb{C}^* \to \mathbb{C}^2$ by

$$\phi_A(t) = (t^1, t^2).$$

The image of ϕ_A is

$$\mathrm{im}\ \phi_A = \{(t, t^2) \mid t \neq 0\} = \{(x, y) \in \mathbb{C}^2 \mid y = x^2\} \setminus \{(0, 0)\}.$$

The Zariski closure $\overline{\mathrm{im}\ \phi_A}$ is

$$\overline{\mathrm{im}\ \phi_A} = \{(x, y) \in \mathbb{C}^2 \mid y = x^2\}.$$

The affine toric variety Y_A associated to A is

$$Y_A = \overline{\mathrm{im}\ \phi_A} = \{(x, y) \in \mathbb{C}^2 \mid y = x^2\} = V(x^2 - y).$$

Definition 2 (Affine toric variety)

A toric variety is encoded by an integer matrix $A \in \mathbb{Z}^{d \times n}$.

A defines a monomial map $\phi_A : (\mathbb{C}^*)^d \to \mathbb{C}^n$ where for $t = (t_1, \ldots, t_d) \in (\mathbb{C}^*)^d$:

$$\phi_A(t) = (t^{a_1}, \ldots, t^{a_n}), \quad t^{a_i} = t_1^{a_{1i}} \cdots t_d^{a_{di}}$$

where a_i is the i-th column of the matrix A, i.e., $A = (a_{ij})_{d \times n}$.

The **affine toric variety** Y_A of A is the Zariski closure of the image of ϕ_A:

$$Y_A = \overline{\mathrm{im}\ \phi_A} \subseteq \mathbb{C}^n.$$

Definition 3 (Projective toric variety)

An integer matrix $A \in \mathbb{Z}^{d \times n}$ induces a map $\Phi_A : (\mathbb{C}^*)^d \to \mathbb{P}^{n-1}$ to the $(n-1)$-dimensional projective space by

$$\Phi_A(t) = [t^{a_1} : \cdots : t^{a_n}],$$

where $t = (t_1, \ldots, t_d) \in (\mathbb{C}^*)^d$, and $t^{a_i} = t_1^{a_{1i}} \cdots t_d^{a_{di}}$.

The **projective toric variety** X_A is the Zariski closure of the image of the map Φ_A:

$$X_A = \overline{\mathrm{im}\ \Phi_A} \subseteq \mathbb{P}^{n-1}.$$

Example 4 (Moment curves) [Eg. 1.1.2]

The moment curve of degree δ, Y_δ, arises from

$$A = \begin{pmatrix} 1 & 2 & \cdots & \delta \end{pmatrix} \in \mathbb{Z}^{1\times\delta}$$

is parameterised by the monomial map $\phi_A : \mathbb{C}^* \to \mathbb{C}^\delta$

$$\phi_A(t) = (t^1, t^2, \ldots, t^\delta).$$

The moment curve Y_δ is

$$Y_\delta = \overline{\mathrm{im}\ \phi_A} = \{(x_1, \ldots, x_\delta) \in \mathbb{C}^\delta \mid x_1 = t, \ldots, x_\delta = t^\delta\}.$$

Example 4 (Moment curves) [Ex. 1.1.3]

The moment curve Y_δ is

$$Y_\delta = \overline{\mathrm{im}\ \phi_A} = \{(x_1, \ldots, x_\delta) \in \mathbb{C}^\delta \mid x_1 = t, \ldots, x_\delta = t^\delta\}.$$

Claim: The moment curve Y_δ is defined by $x_1^k - x_k = 0$ for $k = 2, ..., \delta$.
Proof. If $(x_1, \ldots, x_\delta) \in Y_\delta$, then $x_1 = t$ and $\forall k \in \{2, \ldots, \delta\}$, $x_k = t^k$, so

$$x_1^k - x_k = t^k - t^k = 0.$$

Conversely, if $(x_1, \ldots, x_\delta) \in \mathbb{C}^\delta$ satisfies $x_1^k - x_k = 0$ for $k \in \{2, \ldots, \delta\}$, then setting $t = x_1$ gives

$$x_k = x_1^k = t^k \quad \forall k.$$

So $(x_1, \ldots, x_\delta) = (t^1, t^2, \ldots, t^\delta) \in Y_\delta$.

Example 5 (Rational normal curve) [Ex. 1.1.9]

$A = \begin{pmatrix} 0 & 1 & 2 & \cdots & \delta \end{pmatrix} \in \mathbb{Z}^{1\times(\delta+1)}$ induces $\Phi_A : \mathbb{C}^* \to \mathbb{P}^\delta$ for $t \in \mathbb{C}^*$

$$\Phi_A(t) = [t^0 : t^1 : t^2 : \cdots : t^\delta].$$

The projective toric variety X_A is

$$X_A = \overline{\mathrm{im}\ \Phi_A} = \overline{\{[t^0 : t^1 : \cdots : t^\delta] \mid t \in \mathbb{C}^*\}}.$$

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The **rational normal curve** is the locus of points $[x_0 : x_1 : \cdots : x_\delta] \in \mathbb{P}^\delta$,

$$\mathrm{rank} \begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_{\delta-1} \\ x_1 & x_2 & x_3 & \cdots & x_\delta \end{pmatrix} \leq 1.$$

I.e. all the 2×2 minors are 0. For $1 \leq i \leq k \leq \delta - 1$,

$$x_i x_k - x_{i+1} x_{k-1} = 0 \quad (\star)$$

Example 5 (Rational normal curve) [Ex. 1.1.9]

Claim: X_A is the rational normal curve of degree δ.
Proof. For $[x_0 : x_1 : \cdots : x_\delta] \in X_A$,

$$x_i x_k - x_{i+1}x_{k-1} = (t^i)(t^k) - (t^{i+1})(t^{k-1}) = t^{i+k} - t^{i+k} = 0.$$

Conversely, if $(\star)$ holds $\forall i, k$, in particular, we have the adjacent minors condition:

$$x_i x_{i+2} - x_{i+1}^2 = 0.$$

Then there exists λ such that

$$x_1 = \lambda x_0, \quad x_2 = \lambda^2 x_0, \quad \ldots, \quad x_\delta = \lambda^\delta x_0.$$

Hence $[x_0 : x_1 : \cdots : x_\delta] = [x_0 : \lambda x_0 : \cdots : \lambda^\delta x_0] = [1 : \lambda : \cdots : \lambda^\delta] \in X_A$.

Example 5 (Rational normal curve) [Ex. 1.1.9]

Claim: The affine chart of X_A with $x_0 \neq 0$ is the degree δ moment curve.
Proof. The affine chart where $x_0 \neq 0$ is

$$U_0 = \{[x_0 : \cdots : x_\delta] \in \mathbb{P}^\delta \mid x_0 \neq 0\},$$

is isomorphic to $\mathbb{C}^\delta$ via the dehomogenisation map

$$(x_1/x_0, x_2/x_0, \ldots, x_\delta/x_0) = (y_1, \ldots, y_\delta).$$

On X_A, we have $x_i = t^i x_0$. The coordinates in the affine chart become

$$y_i = \frac{x_i}{x_0} = \frac{t^i x_0}{x_0} = t^i.$$

So the curve in the affine chart $\mathbb{C}^\delta$ is parameterised by

$$(y_1, \ldots, y_\delta) = (t^1, t^2, \ldots, t^\delta),$$

which is the **degree δ moment curve**.

Torus

Definition 6. (Torus) [Def. 1.2.2]
A torus T in d-dim is an algebraic variety isometric to $(\mathbb{C}^*)^d$.

Definition 7. (Character) [Def. 1.2.4]
A character of a torus T is a morphism of varieties $\chi : T \to \mathbb{C}^*$ that is also a group homomorphism (i.e., $\chi(t \cdot s) = \chi(t)\chi(s)$).

Remark. The characters of a torus T form a lattice (i.e., a free abelian group of finite rank), the **character lattice**. Denoted $M = \mathrm{Hom}_g(T, \mathbb{C}^*)$.

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Proposition 9. [Prop. 1.2.5]
The characters of $(\mathbb{C}^*)^d$ are the Laurent monomials

$$\chi_{(m_1,\dots,m_d)}(t_1,\dots,t_d) = t_1^{m_1} \cdots t_d^{m_d} = t^m \quad \text{for } m = (m_1,\dots,m_d) \in \mathbb{Z}^d.$$

Definition 10 (Smith Normal Form) [Def. 1.2.7]

Given $A \in \mathbb{Z}^{d \times n}$, a diagonal matrix $S \in \mathbb{Z}^{d \times n}$ is the Smith Normal Form if $\exists P \in \mathbb{Z}^{d \times d}$, $Q \in \mathbb{Z}^{n \times n}$, $\det(P) = \det(Q) = \pm 1$ such that

$$PAQ = S,$$

where the diagonal entries $(s_1, \ldots, s_d)$ of S satisfy $s_i | s_{i+1}$, called the **invariant factors**.

Remark. $\mathrm{rank}(A)$ equals the number of non-zero invariant factors.

Fibers

Definition 11. (Fiber)
The fibers of the map ϕ_A are the set of preimages for a point x in the image

$$\phi_A^{-1}(x) = \{t \in (\mathbb{C}^*)^d \mid \phi_A(t) = x\}, \quad x \in \mathrm{im}\ \phi_A.$$

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Proposition 12. (All fibers of ϕ_A are isomorphic) [Prop. 1.2.13]
Let $A \in \mathbb{Z}^{d\times n}$. $\ker \phi_A = \{t \in (\mathbb{C}^*)^d \mid \phi_A(t) = (1, \ldots, 1) = \mathbf{1}\}$, is the fiber over the identity element.
For $x \in \mathrm{im}\ \phi_A$, the fiber

$$\phi_A^{-1}(x) \cong \ker \phi_A.$$

Remark. Because ϕ_A is a group homomorphism, the fibers are related by multiplication. For t_0 satisfies $\phi_A(t_0) = x$,

$$\phi_A^{-1}(x) = t_0 \cdot \ker(\phi_A).$$

Fibers

Proof.

- The fiber $\phi_A^{-1}(x)$ for $x = \phi_A(t)$ consists of all $t' \in (\mathbb{C}^*)^d$ such that $\phi_A(t) = \phi_A(t')$.
- $\phi_A(t) = \phi_A(t') \iff \phi_A^{-1}(t)\phi_A(t') = \mathbf{1} \iff \phi_A(t^{-1} \cdot t') = \mathbf{1} \iff t^{-1} \cdot t' \in \ker \phi_A$.

$$\begin{aligned} \phi_A^{-1}(x) = \phi_A^{-1}(\phi_A(t)) &= \{t' \in (\mathbb{C}^*)^d \mid t^{-1} \cdot t' \in \ker \phi_A\} \\ &= \{t \cdot u \mid u \in \ker \phi_A\} \\ &= t \ker \phi_A \\ &\cong \ker \phi_A. \end{aligned} \tag{1}$$

Fibers

Proposition 13. (Structure of the kernel) [Prop. 1.2.13]
Let $A \in \mathbb{Z}^{d \times n}$. Let $PAQ = S$ be the SNF of A, $r = \mathrm{rank}(A)$.

$$S = \begin{pmatrix} \mathrm{diag}(s_i) & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{N}^{d \times n} \quad \text{where } i = 1, \dots, r.$$

Then,

$$\ker \phi_A \cong \underbrace{G_{s_1} \times \cdots \times G_{s_r}}_{\text{finite factor}} \times \underbrace{(\mathbb{C}^*)^{d-r}}_{\text{torus factor}}$$

where $G_{s_i} \subseteq \mathbb{C}^*$ is the multiplicative subgroup of s_i-th roots of unity, i.e., $G_{s_i} \cong \mathbb{Z}/s_i\mathbb{Z}$ (a cyclic group of order s_i).

Remark. The structure is completely determined by SNF.
Each fiber $\phi_A^{-1}(x)$ has dimension $d - \mathrm{rank}(A)$.

Fibers

Proof.

Claim 1: If $A = A_1 A_2$, then $\phi_A = \phi_{A_2} \circ \phi_{A_1}$.

Claim 2: If $P \in \mathbb{Z}^{d\times d}$, $\det P = \pm 1$, then $\phi_P : (\mathbb{C}^*)^d \to (\mathbb{C}^*)^d$ is an isomorphism of algebraic groups.

Fibers

Proof. (Cont.)

- A has the SNF $A = P^{-1}SQ^{-1}$. By Claim 1, $\phi_A = \phi_{Q^{-1}} \circ \phi_S \circ \phi_{P^{-1}}$.
- We want to solve for

$$\phi_A(t) = \mathbf{1} \iff \phi_{Q^{-1}}(\phi_S(\phi_{P^{-1}}(t)) = \mathbf{1} \iff \phi_S(\phi_{P^{-1}}(t)) = \mathbf{1}.$$

- Change of variable, let $\tau = \phi_{P^{-1}}(t)$. Then $\phi_S(\tau) = \mathbf{1}$ becomes

$$\phi_S(\tau) = (\tau_1^{s_1}, \ldots, \tau_r^{s_r}, \tau_{r+1}^0, \ldots, \tau_d^0) = (\mathbf{1}, \ldots, \mathbf{1}).$$

  Hence, $\forall i \in \{1, \ldots, r\}$, $\tau_i^{s_i} = 1 \implies \tau_i \in G_{s_i}$. $\tau_{r+1}, \ldots, \tau_d$ are free (i.e., any element of $\mathbb{C}^*$). Thus,

$$\ker \phi_S = G_{s_1} \times \cdots \times G_{s_r} \times (\mathbb{C}^*)^{d-r}.$$

- By Claim 2, $\phi_{P^{-1}}$ is an isomorphism.
  $t \in \ker \phi_A \iff \phi_{P^{-1}}(t) \in \ker \phi_S$. Therefore

$$\ker \phi_A \cong \ker \phi_S \cong G_{s_1} \times \cdots \times G_{s_r} \times (\mathbb{C}^*)^{d-r}.$$

Sublattice

Definition 14. (Sublattice)
The columns $a_1, \ldots, a_n \in \mathbb{Z}^d$ of A generate a sublattice

$$\mathbb{Z}A = \{c_1 a_1 + \cdots + c_n a_n \mid c_i \in \mathbb{Z}\} \subset \mathbb{Z}^d.$$

This is the image of the $\mathbb{Z}$-linear map $A : \mathbb{Z}^n \to \mathbb{Z}^d$ defined by multiplication by A.

Remark.
$\mathrm{rank}(\mathbb{Z}A) = \mathrm{rank}(A)$.

Definition 15. (Lattice index)
If $\mathrm{rank}(A) = d$, then the quotient $\mathbb{Z}^d/\mathbb{Z}A$ has finite order (i.e., finite distinct cosets). The order is the lattice index of $\mathbb{Z}A$ in $\mathbb{Z}^d$, denoted $[\mathbb{Z}^d : \mathbb{Z}A]$.

Sublattice

Proposition 16.
If $\mathrm{rank}(A) = d$, then $[\mathbb{Z}^d : \mathbb{Z}A] = |s_1 \cdots s_d|$.

Proof.

$$\mathbb{Z}A = \{Ac \mid c \in \mathbb{Z}^n\} = \{P^{-1}SQ^{-1}c \mid c \in \mathbb{Z}^n\} = \{P^{-1}Sd \mid d \in \mathbb{Z}^n\}$$

(Let $d = Q^{-1}c$. Since $Q^{-1} \in \mathrm{GL}_n(\mathbb{Z}), d$ still runs over $\mathbb{Z}^n$).
Since S is diagonal, $\mathbb{Z}S = s_1\mathbb{Z} \times \cdots \times s_d\mathbb{Z}$. So $[\mathbb{Z}^d : \mathbb{Z}S] = |s_1 \cdots s_d|$.

$$[\mathbb{Z}^d : \mathbb{Z}A] = [\mathbb{Z}^d : P^{-1}(\mathbb{Z}S)] = [\mathbb{Z}^d : \mathbb{Z}S] = |s_1 \cdots s_d|.$$

Sublattice

Proposition 16.
If $\operatorname{rank}(A) = d$, then $[\mathbb{Z}^d : \mathbb{Z}A] = |s_1 \cdots s_d|$.

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$$[\mathbb{Z}^d : \mathbb{Z}A] = [\mathbb{Z}^d : P^{-1}(\mathbb{Z}S)] = [\mathbb{Z}^d : \mathbb{Z}S] = |s_1 \cdots s_d|.$$

Proposition 17. (Alternative 1-1 map with same image) [Prop. 1.2.17]
Let $A = P^{-1}SQ^{-1}$. Let $\bar{A} \in \mathbb{Z}^{r\times n}$ consisting of the first $r = \operatorname{rank}(A)$ rows of Q^{-1}. Then $\phi_{\bar{A}} : (\mathbb{C}^*)^r \to (\mathbb{C}^*)^n$ is 1-1 and $\operatorname{im}\, \phi_{\bar{A}} = \operatorname{im}\, \phi_A$.

Proposition 18. (Dimension of toric variety) [Prop. 1.2.24]
The affine toric variety Y_A has dimension $r = \operatorname{rank}(A)$.

Toric ideal

Definition 19. (Toric ideal) [Def. 1.3.1]
An ideal of the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ is **toric** if it is the vanishing ideal $I(Y_A)$ of an affine toric variety Y_A for some $A \in \mathbb{Z}^{d \times n}$.

Theorem 20. (Binomial generators) [Thm. 1.3.2]
The toric ideal $I_A = I(Y_A) \subset \mathbb{C}[x_1, \ldots, x_n]$ is generated by binomials

$$\mathcal{B}_A = \{x^u - x^v : u, v \in \mathbb{N}^n, A(u - v) = 0\}.$$

Theorem 21. (Characterisation of toric ideals) [Thm. 1.3.9] An ideal I is prime and generated by binomials if and only if I is toric.

Toric ideal

Example 22. [Eg. 1.3.8]
Compute the toric ideal I_A in `Oscar.jl` of

$$A = \begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 & 1 \end{pmatrix}$$

[2]:

```
A = [2 2 1 0 0 1 1; 1 0 0 1 2 2 1; 0 1 2 2 1 0 1]
I = toric_ideal(transpose(A))
```

[2]:

```
Ideal generated by
  x4*x6 - x5*x7
  x3*x6 - x7^2
  -x1*x7 + x2*x6
  x3*x5 - x4*x7
  x2*x5 - x7^2
  x1*x5 - x6*x7
  x2*x4 - x3*x7
  x1*x4 - x7^2
  x1*x3 - x2*x7
```

Projective toric variety

Definition 23. [Thm. 1.3.11] The vanishing ideal $I(X_A)$ of the projective toric variety X_A is the toric ideal $I_{\hat{A}}$, where

$$\hat{A} = \begin{pmatrix} A \\ \mathbf{1}^T \end{pmatrix} \in \mathbb{Z}^{(d+1)\times n},$$

where $\mathbf{1}^T$ is the row vector of all ones.

Remark $Y_{\hat{A}}$ is the affine cone over X_A.

Definition 24. (Affine Lattice) [Def. 1.3.15]
The affine lattice generated by $A \in \mathbb{Z}^{d\times n}$ is

$$\mathbb{Z}'A = \left\{ \sum_{i=1}^{n} c_i a_i \mid c_i \in \mathbb{Z}, \sum_{i=1}^{n} c_i = 0 \right\} \subset \mathbb{Z}^d.$$

Projective toric variety

Proposition 25. (Dimension of projective toric variety) [Cor. 1.3.18]

$$\dim X_A = \operatorname{rank}(\hat{A}) - 1 = \operatorname{rank}(\mathbb{Z}'A).$$

Proof.
$Y_{\hat{A}}$ has dimension $\operatorname{rank}(\hat{A})$.
X_A has dimension $\operatorname{rank}(\hat{A}) - 1$ since $Y_{\hat{A}}$ is the affine cone over X_A.
Finally, $\operatorname{rank}(\hat{A})$ is the rank of

$$\hat{A} \cdot \begin{pmatrix} 1 & -1 & -1 & \cdots & -1 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 - a_1 & a_3 - a_1 & \cdots & a_n - a_1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$