Chapter 2

Cones and affine toric varieties

Describe the coordinate ring of an affine toric variety Y_A in terms of semigroups.

2.1 Semigroup algebras

Torus $T \simeq (\mathbb{C}^*)^d$. Character lattice of T is $M \simeq \mathbb{Z}^d$.

Simply set $T = (\mathbb{C}^*)^d$, $M = \mathbb{Z}^d$

Def 2.1.1 An affine semigroup in $M = \mathbb{Z}^d$ is a subset $S \subseteq M$ of the form

$$\mathbb{N}\mathcal{A} = \{ \sum_{a \in \mathcal{A}} c_a a : c_a \in \mathbb{N} \},$$

where $\mathcal{A} \subseteq S$ is finite. S generated by $\mathcal{A}$.

In this chapter, we concern about S generated by the columns of the matrix $A \in \mathbb{Z}^{d \times n}$.

$S = \mathbb{N}A \subseteq \mathbb{Z}A \subseteq M$

Ex: $A = id_d$. $S = \mathbb{N}^d$

$A = (id_d, -id_d)$ $S = \mathbb{Z}^d = M$

$A = (2, 3)$ $S = \{ 2c_1 + 3c_2, \ c_1, c_2 \in \mathbb{N} \}$

exercise 2.1.3

Let $c_2 = 0$. c_1 range over $\mathbb{N}$. Then $2c_1 + 3c_2 = 2c_1$ range over even.
Let $c_2 = 1$, c_1 range over $\mathbb{N}$. Then $2c_1 + 3$ range over odd $\setminus \{1\}$.
Hence, $S = \mathbb{N} \setminus \{1\} \subset M$

Def 2.1.4. Let $M = \mathbb{Z}^d$. The semigroup algebra associated to an affine semigroup $S \subseteq M$ is the $\mathbb{C}$-algebra

$$\mathbb{C}[S] = \{ \sum_{m \in S} c_m t^m : c_m \in \mathbb{C}, \text{ finitely many } c_m \text{ are non-zero} \}$$

$$\subseteq \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$$

$S = \mathbb{N}A \quad \mathbb{C}[S] = \mathbb{C}[\mathbb{N}A] = \mathbb{C}[t^{a_1}, \cdots, t^{a_n}]$ where a_i is the i-th column of A.
"monomial subalgebras of $\mathbb{C}[t_1^{\pm 1}, \cdots, t_d^{\pm 1}]$"

Ex: $S = \mathbb{Z}^d$, $\mathbb{C}[S] = \mathbb{C}[M] = \mathbb{C}[t_1^{\pm 1}, \cdots, t_d^{\pm 1}]$

semigroups algebras are coordinate rings of toric variets:

Prop 2.1.7 Let $S = \mathbb{N}A \subset M$ be an affine semigroup. The semigroup algebra $\mathbb{C}[S]$ is a finitely generated $\mathbb{C}$-algebra. Moreover, it is an integral domain. The corresponding affine variety, denoted by $\mathrm{Specm}(\mathbb{C}[S])$, is isomorphic to the affine toric variety Y_A. That is, the coordinate ring of Y_A is isomorphic to $\mathbb{C}[S]$.

$$\begin{array}{cc} Y_A & \mathbb{C}[Y_A] \\ \updownarrow & \updownarrow \\ \mathrm{Specm}(\mathbb{C}[S]) & \mathbb{C}[S] \end{array}$$

Cor 2.1.8 If $\mathbb{N}A \subseteq M$ and $\mathbb{N}A' \subseteq M'$ are isomorphic, then the affine toric varieties Y_A and $Y_{A'}$ are isomorphic.

Ex 2.1.9. two different generating sets of S leads to different embeddings $Y_A \simeq Y_{A'}$.

$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad A' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\mathbb{N}A = \mathbb{N}A' = \mathbb{N}^2$

$Y_A = \{x - yz = 0\} \simeq Y_{A'} = \mathbb{C}^2$

$(x, y, z) \to (y, z)$

$(t_1 t_2, t_1, t_2) \leftarrow (t_1, t_2)$

Projective version:

projective variety $X \subset \mathbb{P}^{n-1}$

Prop 2.1.10. The coordinate ring $\mathbb{C}[X_A]$ of the projective toric variety $X_A \subset \mathbb{P}^{n-1}$ is isomorphic to the semigroup algebra $\mathbb{C}[\mathbb{N}\hat{A}]$, where $\hat{A} = \begin{pmatrix} A \\ 1 \end{pmatrix}$. This is a graded isomorphism, where the grading on $\mathbb{C}[\mathbb{N}\hat{A}]$ is given by

$$\mathbb{C}[\mathbb{N}\hat{A}] = \bigoplus_{k=0}^{\infty} \mathbb{C}[\mathbb{N}\hat{A}]_k = \bigoplus_{k=0}^{\infty} \bigoplus_{\substack{m \in \mathbb{N}\hat{A} \\ m_{d+1} = k}} \mathbb{C} \cdot t^m .$$

In other words, the degree of a monomial t^m in $\mathbb{C}[\mathbb{N}\hat{A}]$ is the last coordinate of m.

$$\phi_{\hat{A}}^* : \mathbb{C}[x_1, \cdots, x_n] \to \mathbb{C}[M] \quad f \mapsto f \circ \phi_{\hat{A}} \quad f(x_1, \cdots, x_n) \to f(t^{a_1}, \cdots, t^{a_n})$$

Exact sequence of graded rings:

$$0 \to I(Y_{\hat{A}}) \hookrightarrow \mathbb{C}[x_1, \cdots, x_n] \xrightarrow{\phi_{\hat{A}}^*} \mathbb{C}[\mathbb{N}\hat{A}] \to 0$$

$\operatorname{im} \phi_{\hat{A}}^* = \mathbb{C}[\mathbb{N}\hat{A}]$: $\quad \mathbb{C}[\mathbb{N}\hat{A}] \ni f = \sum_{m \in \mathbb{N}A} c_m t^m$, $c_m \in \mathbb{C}$, finite many non-zero

$$m = \sum_{i=1}^n c_i a_i \quad , \quad c_i \in \mathbb{N}$$

$$f(t^{a_1}, \cdots t^{a_n}) = \sum_{m \in \mathbb{N}A} c_m \cdot t^{\sum_{i=1}^n c_i a_i}$$

$$\Rightarrow f(x_1, \cdots, x_n) = \sum_{m \in \mathbb{N}A} c_m \cdot \prod_{i=1}^n (x_i)^{c_i} \quad \in \mathbb{C}[x_1, \cdots x_n]$$

$$\ker \phi_{\hat{A}}^* = I(Y_{\hat{A}})$$

$$\mathbb{C}[x_1, \cdots x_n] / I(Y_{\hat{A}}) \simeq \mathbb{C}[\mathbb{N}\hat{A}]$$

$$\begin{array}{c} \| \\ \mathbb{C}[Y_{\hat{A}}] \\ \| \\ \mathbb{C}[X_A] \end{array} \quad ???$$

Hence, $\mathbb{C}[X_A] \simeq \mathbb{C}[\mathbb{N}\hat{A}]$

graded:

Every term in $f(x_1, \cdots, x_n)$ has degree k.

$$\prod_{i=1}^n x_i^{\alpha_i} \quad , \quad \sum_{i=1}^n \alpha_i = k$$

$$\phi_{\hat{A}}^*\left(\prod_{i=1}^n x_i^{\alpha_i}\right) = \prod_{i=1}^n (t^{a_i})^{\alpha_i} = t^{\sum_{i=1}^n a_i \alpha_i}$$

Note that degree of t^m is m_{d+1}.

$$\left(\sum_{i=1}^n a_i \alpha_i\right)_{d+1} = \sum_{i=1}^n \alpha_i = k$$

2.2. Rational polyhedral cones

cone σ $\to$ affine semigroup S_σ $\to$ affine toric variety $\mathrm{Specm}(\mathbb{C}[S_\sigma])$

$M \simeq \mathbb{Z}^d$ lattice.

$S \subseteq M$ consists of the lattice points inside cones of vector space $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^d$

dual vector space $N_{\mathbb{R}} = (M_{\mathbb{R}})^\vee$

begin from a cone σ in $N_{\mathbb{R}}$.

Def 2.2.1 A rational convex polyhedral cone in $N_{\mathbb{R}}$ is a set of the form

$$\sigma = \mathrm{Cone}(S) = \left\{ \sum_{u \in S} \lambda_u u : \lambda_u \in \mathbb{R}_{\geq 0} \right\}.$$

for some finite set $S \subset N$. S is called a set of cone generators of σ.

Def $\sigma \subset N_{\mathbb{R}}$. dual cone $\sigma^\vee = \{ m \in M_{\mathbb{R}} : \langle u, m \rangle \geq 0 \text{ for all } u \in \sigma \}$.

$(\sigma^\vee)^\vee = \sigma$

for $m \in M_{\mathbb{R}}$, $H_m = \{ u \in N_{\mathbb{R}} : \langle u, m \rangle = 0 \}$ $\quad H_m^+ = \{ u \in N_{\mathbb{R}} : \langle u, m \rangle \geq 0 \}$.

Def 2.2.2. A face τ of a cone σ is a subset of the form $\tau = \sigma \cap H_m$ for $m \in \sigma^\vee$.
denote $\tau \preceq \sigma$ and $\tau \prec \sigma$ if $\tau \neq \sigma$

Present cones by a finite set of linear inequalities:

Prop 2.2.4 The set $\{m_i\}_{i=1}^s \subset M$ generates $\sigma^\vee$ if and only if $\sigma = \bigcap_{i=1}^s H_{m_i}^+$

Prop 2.2.5. σ. τ. τ' cones.

1. $\tau \preceq \sigma$ and $\tau' \preceq \sigma$ $\Rightarrow$ $\tau \cap \tau' \preceq \sigma$
2. $\tau \preceq \sigma$ and $\tau' \preceq \tau$ $\Rightarrow$ $\tau' \preceq \sigma$
3. $\tau \preceq \sigma$ and $v, w \in \sigma$. $v + w \in \tau \Rightarrow v \in \tau$ and $w \in \tau$
4. faces of σ and faces of $\sigma^\vee$ are in bijective, inclusion reversing correspondence
   $\tau \preceq \sigma^\vee$,
   $m \in$ relative interior (τ),
   $H_m \cap \sigma \preceq \sigma$

Def: $\sigma \subset N_{\mathbb{R}}$. dimension of σ is the smallest linear subspace of $N_{\mathbb{R}}$ containing σ

Def 2.2.7 σ is called pointed or strongly convex if
$\sigma \cap (-\sigma) = \{0\}$. $\Leftrightarrow \{0\} \preceq \sigma \Leftrightarrow \dim \sigma^\vee = d$

Def: one-dimensional faces $\rho \preceq \sigma$ are called rays of σ
If σ is pointed, for each ρ, there is a unique lattice point u_ρ such that
$\rho \cap N = \mathbb{N} \cdot u_\rho$.

Def 2.2.8: The set of primitive ray generators $\{u_\rho \mid \rho \preceq \sigma, \dim \rho = 1\}$ generate σ.
It's called minimal generators of σ.

Def 2.2.9: A pointed cone σ is smooth if its minimal generators are a subset of a $\mathbb{Z}$-basis of N.

Ex: $\sigma = \mathrm{Cone}\{(0,1), (1,2), (2,1)\}$ is not smooth.
minimal generators $\{(0,1), (1,2)\}$. can not generate $\mathbb{Z}^2$.

Def 2.2.10 A pointed cone σ is simplicial if its minimal generators are linearly independent over $\mathbb{R}$.
Ex: two-dimensional cones are simplicial.

2.3. Affine toric varieties from cones

$\sigma \subset N_{\mathbb{R}}$ be a rational convex polyhedral cone
semigroup $S_\sigma = \sigma^\vee \cap M \subset M$ finitely generated ?

Lemma 2.3.1 (Gordan's Lemma) The semigroup S_σ is finitely generated.

proof: $S_\sigma = \sigma^\vee \cap M$

σ rational polyhedral $\Rightarrow \sigma^\vee$ rational polyhedral

$\Rightarrow \exists$ finite set $T \subset M$ such that $\sigma^\vee = \mathrm{Cone}(T)$

$w \in S_\sigma$

$$w = \sum_{m \in T} \lambda_m m, \quad \lambda_m \geq 0$$

$$\lambda_m = \lfloor \lambda_m \rfloor + \delta_m, \quad \lfloor \lambda_m \rfloor \in \mathbb{N}, \quad \delta_m \in [0,1)$$

$$w = \sum_{m \in T} \lfloor \lambda_m \rfloor m + \sum_{m \in T} \delta_m m.$$

Define $K = \{\sum_{m \in T} \delta_m m \mid 0 \leq \delta_m < 1\}$. It's a bounded region of $M_{\mathbb{R}}$.

So. $K \cap M$ is a finite set.

Hence, $T \cup (K \cap M)$ is a finite set generating S_σ.

Def 2.3.2. $\sigma \subset N_{\mathbb{R}}$ rational cone with associated semigroup $S_\sigma = \sigma^\vee \cap M$. The affine toric variety associated to σ is $\mathcal{Y}_\sigma = \mathrm{Specm}(\mathbb{C}[S_\sigma])$

caligraphic font to denote toric varieties with no specified embedding.

Prop 2.3.3. Let $\mathcal{Y}_\sigma = \mathrm{Specm}(\mathbb{C}[S_\sigma])$ be the affine toric variety of $\sigma \subset N_{\mathbb{R}}$. We have $\dim \mathcal{Y}_\sigma = \dim \sigma^\vee$. In particular, $\dim \mathcal{Y}_\sigma = d$ if and only if σ is pointed.

Ex 2.3.4. $\sigma = \mathrm{Cone}(e_1, \cdots, e_r) \subset \mathbb{R}^d$, where $r < d$, and e_i is the i-th standard basis vector. $\sigma^\vee = \mathrm{Cone}(e_1', \cdots, e_r', \pm e_{r+1}', \cdots \pm e_d')$. e_i' is the dual basis vector of $(\mathbb{R}^d)^\vee$. The associated affine toric variety $\mathcal{Y}_\sigma$ is

$$\mathrm{Specm}(\mathbb{C}[x_1, \cdots, x_r, x_{r+1}^{\pm 1}, \cdots, x_d^{\pm 1}]) \simeq \mathbb{C}^r \times (\mathbb{C}^*)^{d-r}$$

By changing bases in the lattice $\mathbb{Z}^n$, the affine toric variety corresponding to any smooth cone σ is the product of an affine space with a torus.

Unique minimal set of semigroup generators.

Def: irreducible element : $m \in S_\sigma$ is irreducible if $m = m' + m''$ for $m', m'' \in S_\sigma$, implies that $m' = 0$ or $m'' = 0$.

$\mathcal{H}$ (Hilbert basis of S_σ) : the set of all irreducible elements of S_σ.

Prop 2.3.6. $\sigma \subset N_{\mathbb{R}}$ be full-dimensional. Let $\mathcal{H}$ be the Hilbert basis of S_σ. $\mathcal{H}$ is the unique minimal set of generators for S_σ w.r.t. inclusion

$\mathcal{Y}_\sigma$ can be embedded via a monomial map in an affine space of dimension $|\mathcal{H}|$.

If $S_\sigma \subset \mathbb{Z}^d$, monomial map ϕ_H, $H \in \mathbb{Z}^{d \times |\mathcal{H}|}$ with columns $\mathcal{H}$.

$\mathcal{Y}_\sigma$ cannot be embedded in an affine space of dimension smaller than $|\mathcal{H}|$. (prop 2.4.9)

Exercise 2.3.8. $\sigma \subset N_{\mathbb{R}}$ full dimension. $\mathcal{H}$ Hilbert basis of S_σ.

Show that $\mathcal{H}$ contains the minimal generators of $\sigma^\vee$.

$\sigma = \mathrm{Cone}(n_1, \cdots, n_s)$, $\{n_1, \cdots, n_s\}$ is minimal generators of σ.

$\sigma^\vee = \{ u \in M_{\mathbb{R}} \mid \langle u, n_i \rangle \geq 0, \ \forall i = 1, \cdots s \}$.

Define $\rho_j = \sigma^\vee \bigcap_{\substack{i=1 \\ i \neq j}}^{s} H_{n_i}$ — ray of $\sigma^\vee$.

σ full-dimension $\Rightarrow$ $\sigma^\vee$ pointed. $\Rightarrow$ $\rho_j \cap M = \mathbb{N} \cdot u_{\rho_j}$

$\sigma^\vee = \mathrm{Cone}\{ u_{\rho_1}, \cdots, u_{\rho_s} \}$. $\{u_{\rho_1}, \cdots u_{\rho_s}\}$ minimal generators

① $u_{\rho_j} \in \sigma^\vee \cap M = S_\sigma$

② u_{ρ_j} is irreducible: $u_{\rho_j} = a + b$, $a, b \in S_\sigma$.

$\forall i \neq j$, $\langle u_{\rho_j}, n_i \rangle = 0 \Rightarrow \langle a, n_i \rangle + \langle b, n_i \rangle = 0$. (1)

$\because a, b \in S_\sigma$ $\therefore$ $\langle a, n_i \rangle \geq 0$, $\langle b, n_i \rangle \geq 0$.

By (1), $\langle a, n_i \rangle = \langle b, n_i \rangle = 0$.

$\therefore$ $a, b \in \rho_j \Rightarrow a = \alpha u_{\rho_j}$, $b = \beta u_{\rho_j}$. $\alpha, \beta \in \mathbb{N}$

$\therefore$ $u_{\rho_j} = a + b = (\alpha + \beta) u_{\rho_j} \Rightarrow \alpha + \beta = 1$

By $\alpha, \beta \in \mathbb{N}$, one of them equal to 0.

2.4. Normality and smoothness.

Normal affine toric varieties are exactly those affine toric varieties corresponding to cones.

(An irreducible affine variety V is called normal if its coordinate ring $\mathbb{C}[V]$ is normal)

Def 2.4.1. An affine semigroup $S \subset M \simeq \mathbb{Z}^d$ is saturated in M if $km \in S$ for $k \in \mathbb{N}_{>0}$, $m \in M$ implies that $m \in S$.

Thm 2.4.2. Let $Y = \mathrm{Specm}(\mathbb{C}[S])$ be the affine toric variety of an affine semigroup $S \subset M$ such that $\mathbb{Z}S = M$. The following statements are equivalent:

1. Y is normal
2. S is saturated in M
3. $S = S_\sigma$ for some cone $\sigma \subset N_{\mathbb{R}}$

Cor 2.4.4. The affine toric variety Y_A is normal if and only if the affine semigroup $\mathbb{N}A$ is saturated in $\mathbb{Z}A$.

Thm 2.4.6. The affine toric variety $Y_\sigma = \mathrm{Specm}(\mathbb{C}[S_\sigma])$ is smooth if and only if σ is a smooth rational cone.

Prop 2.4.9. The cardinality $|\mathcal{H}|$ of the Hilbert basis for S_σ is the smallest integer n for which Y_σ can be embedded in $\mathbb{C}^n$.

Exercise 2.4.10