Def: T a torus. A co-character of T is a morphism of varieties $\mathbb{C}^* \to T$

Rmk: cocharacters $\leftrightarrow$ lattice $N = \mathbb{Z}^d$.

$u \in \mathbb{Z}^d$: $u = (u_1, \dots, u_d)$.

$\lambda_u : t \mapsto (t^{u_1}, \dots, t^{u_d})$.

$\mathrm{im}(\lambda_u)$ is a 1-dim subtorus of T,

$\mathrm{im}(\lambda_u) \subset Y_A \supset (\mathbb{C}^*)^d$

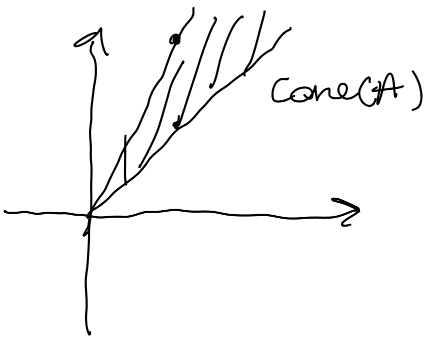
Prop: The limit $\lim_{t \to 0} \phi_A(\lambda_u(t))$ exists $\Leftrightarrow$ $u \in \mathrm{Cone}(A)^\vee$.

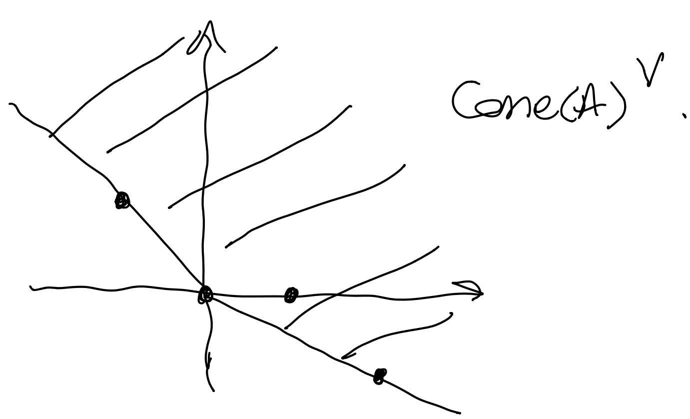
Proof:
$$\phi_A(\lambda_u(t)) = \phi_A(t^{u_1}, \dots, t^{u_d})$$
$$= \left(t^{u_1 a_{11} + u_2 a_{21} + \dots + u_d a_{d1}}, \dots, t^{u_1 a_{1d} + \dots}\right)$$
$$= \left(t^{\langle u, a_1 \rangle}, \dots, t^{\langle u, a_d \rangle}\right).$$

lim exists $\Leftrightarrow$ $\langle u, a_i \rangle \geq 0 \ \forall i$

$\Leftrightarrow$ $u \in \mathrm{Cone}(A)^\vee$.

Ex: $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.





$u=(2,-1)$. $\lambda^u(t) = (t^2, t^{-1})$ $\quad \phi_A(\lambda^u(t)) = (t,1) \xrightarrow[t\to 0]{} (0,1)$.

$u=(-1,1)$ $\quad \phi_A(\lambda^u(t)) = (1,t) \xrightarrow[t\to 0]{} (1,0)$.

$u=(0,0)$ $\quad \phi_A(\lambda^u(t)) = (1,1) \to (1,1)$

$u=(1,0)$ $\quad$ —— $= (t,t) \xrightarrow[t\to 0]{} (0,0)$

$$\rightsquigarrow \quad Y_A = \bigsqcup_{\tau \preceq Cone(A)} Y^{\circ}_{A,\tau}$$

<u>Prop</u>: If $u \in relint(\tau)$ for $\tau \in Cone(A)^{\vee}$, then

$$\lim_{t\to 0} \phi_A(\lambda^u(t)) = \gamma_{\tilde{\tau}},$$

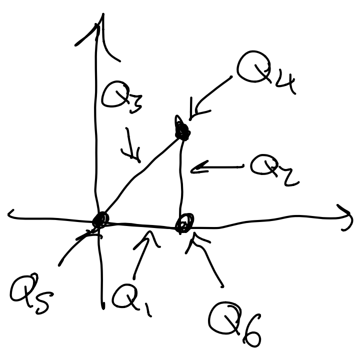
where γ_τ vector whose i-th coordinate is

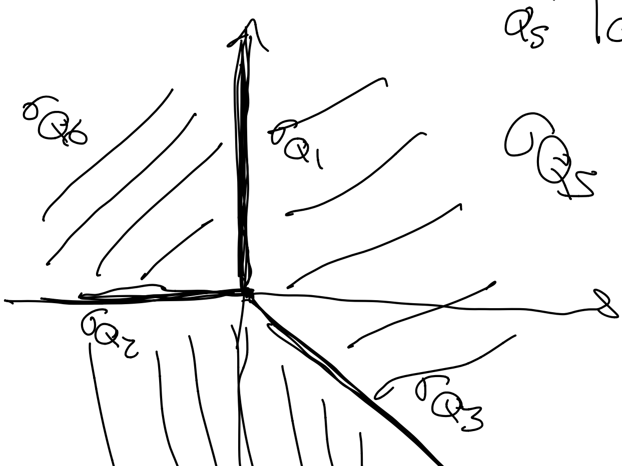
$$\begin{cases} 1 & \text{if } a_i \in \tilde{\tau} \\ 0 & \text{otherwise} \end{cases}$$

<u>Def</u>: For each face $Q \preceq Conv(A)$, let

$$\sigma_Q = \{ u \in N_{\mathbb{R}} : \min_{m \in P} \langle u, m \rangle \text{ is attained for all } m \in Q \}$$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$





σ_{Q_4}

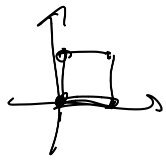
Prop: Fix $u \in N$, and let Q be the smallest face of $\mathrm{conv}(A)$ such $u \in \mathrm{conv}(\sigma_Q)$.

we have $\lim\limits_{t \to 0} \overline{\Phi}_A(\lambda_u(t)) = \gamma_Q \in X_A$.

Ex: (4.4.12) $\quad A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

$\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$

$(t_1 : t_2), (t_3 : t_4) \mapsto (t_1t_3 : t_1t_4 : t_2t_3 : t_2t_4)$



Vertices: $(1:0:0:0) \to (1:0), (1:0)$

$(0:1:0:0) \quad (0:1), (1:0)$

$(0:0:1:0) \quad (1:0), (0:1)$

$(0:0:0:1) \quad (0:1), (0:1)$

Edges: $(1:0:1:0)$

$(1:0:0:1)$

$(0:1:1:0)$

$(0:1:0:1)$

$(1:1:1:1)$

Singularities of Y_A.

$x \in Y_A \to m_x \in \mathbb{C}[Y_A]$.

$$\dim_{\mathbb{C}} \mathfrak{m}_x^k / \mathfrak{m}_x^{k+1} = \frac{\mathrm{mult}_x Y_A}{(d-1)!} k^{d-1} + O(k^{d-2})$$

Ex 4.5.11: $Y_{(2\,3)}$ $\{ y^2 = x^3 \}$

$$\mathrm{mult}_0 Y_{(2\,3)}.$$

$$\mathfrak{m}_0 = (x,y)\,\mathbb{C}[x,y]/(y^2-x^3).$$

Let $k \geq 1$. $\mathfrak{m}_0^k = (x,y)^k\,\mathbb{C}[x,y]/(y^2-x^3)$

$$= \langle x^k, x^{k+1}, x^{k+2}, \dots, x^{k-1}y, x^k y, \dots \rangle$$

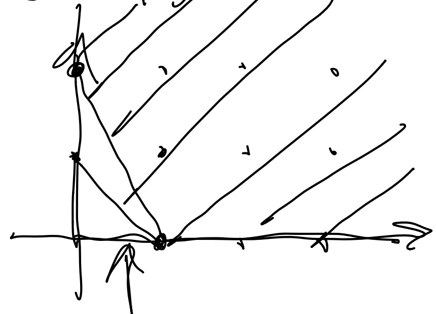
$$\mathfrak{m}_0^{k+1} = \langle x^{k+1}, x^{k+2}, \dots, x^k y, x^{k+1} y, \dots \rangle$$

$$\dim(\mathfrak{m}_0^k / \mathfrak{m}_0^{k+1}) = 2.$$

Def: $A \in \mathbb{Z}^{d \times n}$. The subdiagram volume of A is

$$SDV(A) = \frac{d!\,\mathrm{Vol}(\mathrm{Cone}(A) \setminus \mathrm{Conv}(\mathbb{N}A \setminus \{0\}))}{[\mathbb{Z}^d : \mathbb{Z}A]}$$

Ex: $A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$



volume 1.

$d = 2$, $[\mathbb{Z}^d : \mathbb{Z}A] = 1$

$SDV(A) = \frac{2 \times 1}{1} = 2.$

Prop: Let $A \in \mathbb{Z}^{d \times n}$ such that $\sigma^\vee = \mathrm{Cone}(A)$ is pointed of dim d. Let $0 \in Y_n$. Then we have

$$\mathrm{mult}_0 Y_A = \mathrm{SDV}(A).$$

<u>Thm 4.5.7:</u> $\quad \mathrm{mult}_x Y_A = i(\tau, A) \cdot \mathrm{SDV}(A/\tau)$

<u>Ex 4.5.11:</u> $\quad \mathrm{mult}_0 Y_{(2\ 3)} : \quad A = (2\ 3)$

$$= \mathrm{SDV}((2\ 3)) = \frac{1!\ 2}{1} = 2.$$

$(d = 1,\ \mathbb{Z}A = \mathbb{Z},$