

Chapter 4: Toric boundaries

4.1 The boundary of an affine toric variety

Lemma

Let $A \in \mathbb{Z}^{d \times n}$ and let $Y_A \subset \mathbb{C}^n$ be the corresponding affine toric variety, parametrized by $\phi_A : (\mathbb{C}^\star)^d \rightarrow (\mathbb{C}^\star)^n$ with $\phi_A(t) = (t^{a_1}, \dots, t^{a_n})$. We have $\text{im } \phi_A = Y_A \cap (\mathbb{C}^\star)^n$.

Proof.

$\Rightarrow \text{im } \phi_A \subseteq \overline{\text{im } \phi_A} \cap (\mathbb{C}^\star)^n = Y_A \cap (\mathbb{C}^\star)^n$.

\Leftarrow Let $x \in Y_A \cap (\mathbb{C}^\star)^n$, then x satisfies the following equation (from Proposition 1.2.21)

$$\{x \in (\mathbb{C}^\star)^n : x^{b_1} = \dots = x^{b_{n-r}} = 1\} = \text{im } \phi_A \quad (1)$$

where $B = (b_1, \dots, b_{n-r})$ is a matrix whose columns form a \mathbb{Z} -basis for $\ker A$. □

\Rightarrow The boundary $Y \setminus \text{im } \phi_A$ consists of the points in Y_A with at least one zero coordinate.

Definition (support)

For $x \in (\mathbb{C}^\star)^n$ we define the support of x as $\text{supp}(x) = \{a_i \in A : x_i \neq 0\}$.

Proposition

Let $x \in Y_A \subset (\mathbb{C})^n$. We have $\text{supp}(x) = \tau \cap A$ for some $\tau \preceq \text{Cone}(A) = \{\lambda_1 a_1 + \cdots + \lambda_n a_n : \lambda_i \in \mathbb{R}_{\geq 0}\}$.

Proof.

Idea: choose $\tau \preceq \text{Cone}(A)$ to be the smallest face containing $\text{supp}(x)$ and use the binomial generators of the ideal from Theorem 1.3.2. (generating set $\{x^u - x^v : u, v \in \mathbb{N}^n, A(u - v) = 0\}$) □

Corollary

The affine toric variety Y_A is a disjoint union of open strata

$$Y_A = \bigsqcup_{\tau \preceq \text{Cone}(A)} Y_{A,\tau}^\circ,$$

where $Y_{A,\tau}^\circ = \{x \in Y_A : \text{supp}(x) = \tau \cap A\}$.

Since $\text{im}\phi_A = Y_A \cap (\mathbb{C}^\star)^n$, the open stratum $Y_{A,\text{Cone}(A)}^\circ$ is the image of the map ϕ_A .

⇒ The boundary is

$$Y_A \setminus \text{im}\phi_A = \bigsqcup_{\tau \prec \text{Cone}(A)} Y_{A,\tau}^\circ$$

Proposition

For a face $\tau \preceq \text{Cone}(A)$, let $\tau \cap A = \{a_{i1}, \dots, a_{i\ell}\}$ and let $\pi_\tau : \mathbb{C}^n \rightarrow \mathbb{C}^\ell$ be a coordinate projection $\pi_\tau(x) = (x_{i1}, \dots, x_{i\ell})$. Let $Y_{A,\tau} = \{x \in Y_A : \text{supp}(x) \subseteq \tau \cap A\}$ (closed strata, closure of the open strata). Then

- a. $\pi_\tau(Y_A) = \pi_\tau(Y_{A,\tau}) = Y_{\tau \cap A}$,
- b. the map $(\pi_\tau)|_{Y_{A,\tau}} : Y_{A,\tau} \rightarrow Y_{\tau \cap A}$ is an isomorphism and
- c. $\pi_\tau(Y_{A,\tau}^\circ) = \text{im } \phi_{\tau \cap A}$.

Corollary

The decomposition $Y_A = \bigsqcup_{\tau \preceq \text{Cone}(A)} Y_{A,\tau}^\circ$ is a stratification of Y_A into tori: $Y_{A,\tau}^\circ \simeq \text{im } \phi_{\tau \cap A}$ is a torus of dimension $\dim(\tau)$. For the closed stratum the following equation holds

$$Y_{A,\tau} = \overline{Y_{A,\tau}^\circ} = \bigsqcup_{\tau' \preceq \tau} Y_{A,\tau'}^\circ,$$

where the disjoint union ranges over all faces of $\tau \preceq \text{Cone}(A)$.

Example

The matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ gives rise to the smooth toric surface $Y_A = \{x - yz = 0\} \subset \mathbb{C}^3$. The $\text{Cone}(A)$ is the nonnegative quadrant in \mathbb{R}^2 . Y_A can be decomposed into four pieces, one for each face of the cone.

The open strata of the surface Y_A are

$$Y_{A, \text{Cone}(A)}^\circ = \text{im } \phi_A,$$

$$Y_{A, \mathbb{R}_{\geq 0} \cdot (1,0)}^\circ = \{0\} \times \text{im } \phi_{(1,0)\tau} \times \{0\}$$

$$Y_{A, \mathbb{R}_{\geq 0} \cdot (0,1)}^\circ = \{0,0\} \times \text{im } \phi_{(0,1)\tau}$$

$$Y_{A, \{(0,0)\}}^\circ = \{(0,0,0)\}$$

We have $Y_{A, \mathbb{R}_{\geq 0} \cdot (1,0)} = \overline{Y_{A, \mathbb{R}_{\geq 0} \cdot (1,0)}^\circ} = Y_{A, \mathbb{R}_{\geq 0} \cdot (1,0)}^\circ \sqcup Y_{A, \{(0,0)\}}^\circ$

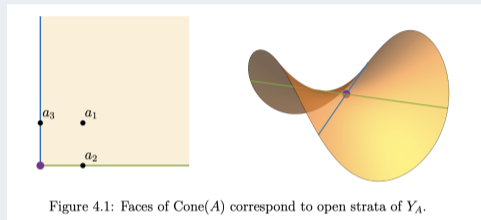


Figure 4.1: Faces of $\text{Cone}(A)$ correspond to open strata of Y_A .

Figure: Figure 4.1. from “Lectures on Toric Geometry”, S. Telen

4.2 The boundary of a projective toric variety

Projective toric varieties are stratified into tori in a very similar manner:

Projective variety X_A :

$$x \in \mathbb{P}^{n-1} \quad \text{supp}(x) = \{a_i \in A : x_i \neq 0\}$$

Let $Q \subset \text{Conv}(A)$ be a face. Then

$$X_{A,Q}^\circ = \{x \in X_A : \text{supp}(x) = Q \cap A\}$$

$$X_{A,Q} = \{x \in X_A : \text{supp}(x) \subseteq Q \cap A\}$$

Affine variety Y_A :

$$x \in (\mathbb{C})^n \quad \text{supp}(x) = \{a_i \in A : x_i \neq 0\}.$$

Let $\tau \preceq \text{Cone}(A)$. Then

$$Y_{A,\tau}^\circ = \{x \in Y_A : \text{supp}(x) = \tau \cap A\}$$

$$Y_{A,\tau} = \{x \in Y_A : \text{supp}(x) \subseteq \tau \cap A\}$$

Lemma

Let $X_A \subset \mathbb{P}^{n-1}$ be parametrized by Φ_A with $\Phi_A(t) = (t^{a_1} : \dots : t^{a_n})$. Then
 $\text{im } \Phi_A = X_A \cap \{x \in \mathbb{P}^{n-1} : x_1 \cdots x_n \neq 0\}$. affine: $\text{im } \phi_A = Y_A \cap (\mathbb{C}^*)^n$

Theorem

Let $X_A \subset \mathbb{P}^{n-1}$ be the projective toric variety of $A \in \mathbb{Z}^{d \times n}$

1. We have $X_A = \bigsqcup_{Q \preceq \text{Conv}(A)} X_{A,Q}^\circ$ and $X_{A,Q} = \bigsqcup_{Q' \preceq Q} X_{A,Q'}^\circ$ affine: $Y_A = \bigsqcup_{\tau \preceq \text{Cone}(A)} Y_{A,\tau}^\circ$, $Y_{A,\tau} = \bigsqcup_{\tau' \preceq \tau} Y_{A,\tau'}^\circ$,
2. For $Q \preceq \text{Conv}(A)$, let $Q \cap A = \{a_{i_1}, \dots, a_{i_\ell}\}$. The map $\pi_Q : X_{A,Q} \rightarrow X_{Q \cap A}$, $x \mapsto (x_{i_1}, \dots, x_{i_\ell})$ is a well defined isomorphism, and $\pi_Q(X_{A,Q}^\circ) = X_{Q \cap A}^\circ = \text{im } \Phi_{Q \cap A}$ affine: $\pi_\tau(Y_{A,\tau}^\circ) = \text{im } \phi_{\tau \cap A}$

Theorem

Let $X_A \subset \mathbb{P}^{n-1}$ be the projective toric variety of $A \in \mathbb{Z}^{d \times n}$

1. We have $X_A = \bigsqcup_{Q \preceq \text{Conv}(A)} X_{A,Q}^\circ$ and $X_{A,Q} = \bigsqcup_{Q' \preceq Q} X_{A,Q'}^\circ$ affine: $Y_A = \bigsqcup_{\tau \preceq \text{Cone}(A)} Y_{A,\tau}^\circ$, $Y_{A,\tau} = \bigsqcup_{\tau' \preceq \tau} Y_{A,\tau'}^\circ$,
2. For $Q \preceq \text{Conv}(A)$, let $Q \cap A = \{a_{i1}, \dots, a_{i\ell}\}$. The map $\pi_Q : X_{A,Q} \rightarrow X_{Q \cap A}$, $x \mapsto (x_{i1} : \dots : x_{i\ell})$ is a well defined isomorphism, and $\pi_Q(X_{A,Q}^\circ) = X_{Q \cap A}^\circ = \text{im } \Phi_{Q \cap A}$ affine: $\pi_\tau(Y_{A,\tau}^\circ) = \text{im } \phi_{\tau \cap A}$

Proof.

Idea: Use results from the affine case, since the faces $Q \preceq \text{Conv}(A)$ are in one-to-one correspondence with positive dimensional faces of the pointed cone $\text{Cone}(\hat{A}) \subset \mathbb{R}^{d+1}$ and the map $\pi : \mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{P}^{n-1}$ given by $\pi(x_1, \dots, x_n) = (x_1 : \dots : x_n)$.

$$\begin{array}{ccc}
 Y_{\hat{A}, \tau_Q} \setminus \{0\} & \xrightarrow{\pi_{\tau_Q}} & Y_{\tau_Q \cap \hat{A}} \setminus \{0\} \\
 \downarrow \pi & & \downarrow \tilde{\pi} \\
 X_{A,Q} & \xrightarrow{\pi_Q} & X_{Q \cap A}
 \end{array}$$

□

Exercise 4.2.4

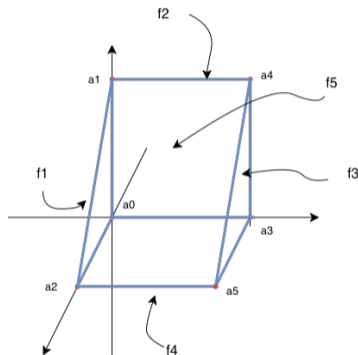
$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

```
julia> B
4×6 Matrix{Int64}:
 0  0  0  1  1  1
 0  1  0  0  1  0
 0  0  1  0  0  1
 1  1  1  1  1  1

julia> toric_ideal(transpose(B))
Ideal generated by
 -x2*x6 + x3*x5
 -x1*x6 + x3*x4
 -x1*x5 + x2*x4
```

```
julia> C
5×6 Matrix{Int64}:
 1  1  1  0  0  0
 0  0  0  1  1  1
 1  0  0  1  0  0
 0  1  0  0  1  0
 0  0  1  0  0  1

julia> toric_ideal(transpose(C))
Ideal generated by
 -x2*x6 + x3*x5
 -x1*x6 + x3*x4
 -x1*x5 + x2*x4
```



Remark 1.2.24: The image of ϕ_A only depends on the row span of A over \mathbb{Q} and the row span of both matrices is identical.

Since $\dim(\text{Conv}(A)) = 3$ we can use Kushnirenko's theorem: $\deg(X_A) = 3! \cdot V = 3$. (A standard simplex in 3 dimensions has volume $1/6$, so $\text{Conv}(A)$ with volume $1/2$ would fit 3)

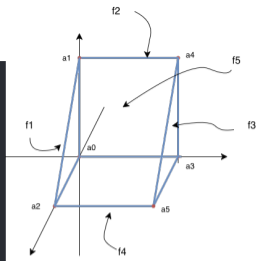
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julia> R, (x1,x2,x3,x4,x5,x6) = graded_polynomial_ring(QQ, [:x1, :x2, :x3, :x4, :x5, :x6])
(Graded multivariate polynomial ring in 6 variables over QQ, MPolyDecRingElem{QQFieldElem, QQMPolyRingElem}[x1, x2, x3, x4, x5, x6])

julia> A, _ = quo(R, ideal(R, [-x2*x6 + x3*x5, -x1*x6 + x3*x4, -x1*x5 + x2*x4]))
(Quotient of multivariate polynomial ring by ideal (-x2*x6 + x3*x5, -x1*x6 + x3*x4, -x1*x5 + x2*x4), Map: R → A)

julia> hilbert_polynomial(A)
1//2*t^3 + 2*t^2 + 5//2*t + 1

```



Theorem 3.3.5 using the Hilbert-polynomial with leading term $\frac{\deg(X)}{d!} k^d$ leads to $1/2 = \frac{\deg(X)}{6}$

Parametrization of the surfaces, edges and vertices can be gained in an analogous way:

$$1 \rightarrow x_0, t_1 \rightarrow x_1, 1 \rightarrow y_0, t_2 \rightarrow y_1, t_3 \rightarrow y_2$$

$$X_{A, \text{Conv}(A)}^\circ = \text{im } \Phi_A = \{(1 : t_2 : t_3 : t_1 : t_1 t_2 : t_1 t_3) : t_1, t_2, t_3 \in \mathbb{C}^\star\}$$

$$X_{A, f_1}^\circ = \{(1 : t_2 : t_3 : 0 : 0 : 0) : t_1, t_2, t_3 \in \mathbb{C}^\star\}$$

$$X_{A, f_2}^\circ = \{(1 : t_2 : 0 : t_1 : t_1 t_2, 0) : t_1, t_2, t_3 \in \mathbb{C}^\star\}$$

$$X_{A, f_3}^\circ = \{(0 : 0 : 0 : t_1 : t_1 t_2 : t_1 t_3) : t_1, t_2, t_3 \in \mathbb{C}^\star\}$$

$$X_{A, f_4}^\circ = \{(1 : 0 : t_3 : t_1 : 0 : t_1 t_3) : t_1, t_2, t_3 \in \mathbb{C}^\star\}$$

$$X_{A, f_5}^\circ = \{(0 : t_2 : t_3 : 0 : t_1 t_2 : t_1 t_3) : t_1, t_2, t_3 \in \mathbb{C}^\star\}$$

$$X_{A, \text{Conv}(A)}^\circ = \{(x_0 : x_1), (y_0 : y_1 : y_2)\}$$

$$X_{A, f_1}^\circ = \{(x_0 : 0), (y_0 : y_1 : y_2)\}$$

$$X_{A, f_2}^\circ = \{(x_0 : x_1), (y_0 : y_1 : 0)\}$$

$$X_{A, f_3}^\circ = \{(0 : x_1), (y_0 : y_1 : y_2)\}$$

$$X_{A, f_4}^\circ = \{(x_0 : x_1), (y_0 : 0 : y_2)\}$$

$$X_{A, f_5}^\circ = \{(x_0 : x_1), (0 : y_1 : y_2)\}$$

for $x_0, x_1, y_0, y_1, y_2 \in \mathbb{C}^\star$

4.3 Torus orbits

Let G be an algebraic group (an affine variety V with a group operation $V \times V \rightarrow V$ which is a morphism) and let X be a variety. An algebraic group action of G on X is a morphism $G \times X \rightarrow X$, $(g, x) \mapsto g \bullet x$ satisfying

1. $e \bullet x = x$ for all $x \in X$ and $e \in G$ the identity
2. $g \bullet (h \bullet x) = (g \cdot h) \bullet x$ for all $x \in X$ and $g, h \in G$

The **orbit** of $x \in X$ under the group action $G \times X \rightarrow X$ is $O_x = \{g \bullet x : g \in G\}$.

Proposition

Let $A \in \mathbb{Z}^{d \times n}$ be such that $\mathbb{Z}A = \mathbb{Z}^d$ (can always be achieved using the Smith normal form). The morphism

$$(\mathbb{C}^*)^d \times Y_A \rightarrow Y_A, \quad (2)$$

$$(t, x) \mapsto \phi_A(t) \cdot x = (t^{a_1} x_1, \dots, t^{a_n} x_n) \quad (3)$$

is an algebraic group action of $(\mathbb{C}^*)^d \simeq \text{im } \phi_A$ on Y_A which extends the action of $\text{im } \phi_A$ on itself.

Proposition

A is such that $\Phi_A : (\mathbb{C}^*)^d \rightarrow \mathbb{P}^{n-1}$ is one-to-one. The action of $(\mathbb{C}^*)^d$ on X_A is defined as:

$$(\mathbb{C}^*)^d \times X_A \rightarrow X_A, \quad (t, x) \mapsto (t^{a_1} x_1 : \dots : t^{a_n} x_n). \quad (4)$$

The morphism above is an algebraic group action of $(\mathbb{C}^*)^d \simeq \text{im } \Phi_A$ on X_A which extends the action of $\text{im } \Phi_A$ on itself.

Theorem

Let $A \in \mathbb{Z}^{d \times n}$ be such that $\mathbb{Z}A = \mathbb{Z}^d$ and let Y_A be the corresponding affine toric variety. The stratification $Y_A = \bigsqcup_{\tau \preceq \text{Cone}(A)} Y_{A,\tau}^\circ$ decomposes Y_A into $(\mathbb{C}^\star)^d$ -orbits, where the action is that of (3).

Theorem

The orbits of morphism (4) are the open strata $X_{A,Q}^\circ$. The disjoint union $X_A = \bigsqcup_{Q \preceq \text{Conv}(A)} X_{A,Q}^\circ$ decomposes X_A into $(\mathbb{C}^\star)^d$ orbits.

Definition

The orbit-cone correspondence for the toric variety Y_A is a bijection between faces of the dual cone $\text{Cone}(A)^\vee$ and $(\mathbb{C}^\star)^d$ -orbits of Y_A , given by $\tau \mapsto Y_{A,\tilde{\tau}}^\circ$.

Definition

Toric Variety A toric variety \mathcal{X} is an irreducible algebraic variety containing a torus $T \simeq (\mathbb{C}^\star)^d$ as a dense open subset, such that the action of T on itself extends to an algebraic action $T \times \mathcal{X} \rightarrow \mathcal{X}$.