

## Chapter 4: Toric boundaries

### 4.1 The boundary of an affine toric variety

#### Lemma

Let  $A \in \mathbb{Z}^{d \times n}$  and let  $Y_A \subset \mathbb{C}^n$  be the corresponding affine toric variety, parametrized by  $\phi_A : (\mathbb{C}^\star)^d \rightarrow (\mathbb{C}^\star)^n$  with  $\phi_A(t) = (t^{a_1}, \dots, t^{a_n})$ . We have  $\text{im } \phi_A = Y_A \cap (\mathbb{C}^\star)^n$ .

#### Proof.

$$\Rightarrow \text{im } \phi_A \subseteq \overline{\text{im } \phi_A} \cap (\mathbb{C}^\star)^n = Y_A \cap (\mathbb{C}^\star)^n.$$

$\Leftarrow$  Let  $x \in Y_A \cap (\mathbb{C}^\star)^n$ , then  $x$  satisfies the following equation (from Proposition 1.2.21)

$$\{x \in (\mathbb{C}^\star)^n : x^{b_1} = \dots = x^{b_{n-r}} = 1\} = \text{im } \phi_A \quad (1)$$

where  $B = (b_1, \dots, b_{n-r})$  is a matrix whose columns form a  $\mathbb{Z}$ -basis for  $\ker A$ . □

$\Rightarrow$  The boundary  $Y \setminus \text{im } \phi_A$  consists of the points in  $Y_A$  with at least one zero coordinate.

#### Definition (support)

For  $x \in (\mathbb{C})^n$  we define the support of  $x$  as  $\text{supp}(x) = \{a_i \in A : x_i \neq 0\}$ .

## Proposition

Let  $x \in Y_A \subset (\mathbb{C})^n$ . We have  $\text{supp}(x) = \tau \cap A$  for some  $\tau \preceq \text{Cone}(A) = \{\lambda_1 a_1 + \cdots + \lambda_n a_n : \lambda_i \in \mathbb{R}_{\geq 0}\}$ .

## Proof.

Idea: choose  $\tau \preceq \text{Cone}(A)$  to be the smallest face containing  $\text{supp}(x)$  and use the binomial generators of the ideal from Theorem 1.3.2. (generating set  $\{x^u - x^v : u, v \in \mathbb{N}^n, A(u - v) = 0\}$ ) □

## Corollary

The affine toric variety  $Y_A$  is a disjoint union of open strata

$$Y_A = \bigsqcup_{\tau \preceq \text{Cone}(A)} Y_{A,\tau}^\circ,$$

where  $Y_{A,\tau}^\circ = \{x \in Y_A : \text{supp}(x) = \tau \cap A\}$ .

Since  $\text{im } \phi_A = Y_A \cap (\mathbb{C}^\star)^n$ , the open stratum  $Y_{A,\text{Cone}(A)}^\circ$  is the image of the map  $\phi_A$ .

⇒ The boundary is

$$Y_A \setminus \text{im } \phi_A = \bigsqcup_{\tau \prec \text{Cone}(A)} Y_{A,\tau}^\circ$$

## Proposition

For a face  $\tau \preceq \text{Cone}(A)$ , let  $\tau \cap A = \{a_{i1}, \dots, a_{i\ell}\}$  and let  $\pi_\tau : \mathbb{C}^n \rightarrow \mathbb{C}^\ell$  be a coordinate projection  $\pi_\tau(x) = (x_{i1}, \dots, x_{i\ell})$ . Let  $Y_{A,\tau} = \{x \in Y_A : \text{supp}(x) \subseteq \tau \cap A\}$  (closed strata, closure of the open strata). Then

- a.  $\pi_\tau(Y_A) = \pi_\tau(Y_{A,\tau}) = Y_{\tau \cap A}$ ,
- b. the map  $(\pi_\tau)|_{Y_{A,\tau}} : Y_{A,\tau} \rightarrow Y_{\tau \cap A}$  is an isomorphism and
- c.  $\pi_\tau(Y_{A,\tau}^\circ) = \text{im } \phi_{\tau \cap A}$ .

## Corollary

The decomposition  $Y_A = \bigsqcup_{\tau \preceq \text{Cone}(A)} Y_{A,\tau}^\circ$  is a stratification of  $Y_A$  into tori:  $Y_{A,\tau}^\circ \simeq \text{im } \phi_{\tau \cap A}$  is a torus of dimension  $\dim(\tau)$ . For the closed stratum the following equation holds

$$Y_{A,\tau} = \overline{Y_{A,\tau}^\circ} = \bigsqcup_{\tau' \preceq \tau} Y_{A,\tau'}^\circ,$$

where the disjoint union ranges over all faces of  $\tau \preceq \text{Cone}(A)$ .

## Example

The matrix  $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$  gives rise to the smooth toric surface  $Y_A = \{x - yz = 0\} \subset \mathbb{C}^3$ . The Cone( $A$ ) is the nonnegative quadrant in  $\mathbb{R}^2$ .  $Y_A$  can be decomposed into four pieces, one for each face of the cone.

The open strata of the surface  $Y_A$  are

$$Y_{A, \text{Cone}(A)}^\circ = \text{im } \phi_A,$$

$$Y_{A, \mathbb{R}_{\geq 0} \cdot (1,0)}^\circ = \{0\} \times \text{im } \phi_{(1,0)^T} \times \{0\}$$

$$Y_{A, \mathbb{R}_{\geq 0} \cdot (0,1)}^\circ = \{0,0\} \times \text{im } \phi_{(0,1)^T}$$

$$Y_{A, \{(0,0)\}}^\circ = \{(0,0,0)\}$$

We have  $Y_{A, \mathbb{R}_{\geq 0} \cdot (1,0)} = \overline{Y_{A, \mathbb{R}_{\geq 0} \cdot (1,0)}^\circ} = Y_{A, \mathbb{R}_{\geq 0} \cdot (1,0)}^\circ \sqcup Y_{A, \{(0,0)\}}^\circ$

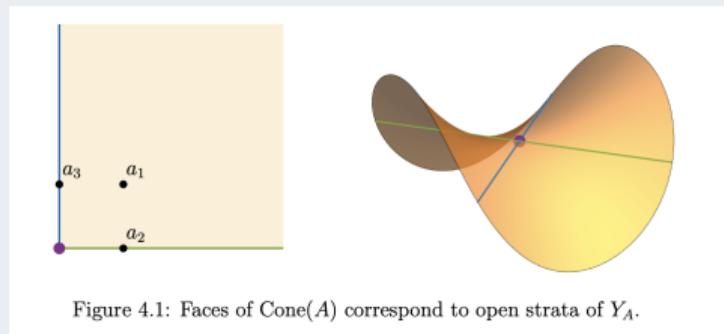


Figure 4.1: Faces of Cone( $A$ ) correspond to open strata of  $Y_A$ .

**Figure:** Figure 4.1. from "Lectures on Toric Geometry", S. Telen

## 4.2 The boundary of a projective toric variety

Projective toric varieties are stratified into tori in a very similar manner:

Projective variety  $X_A$ :

$$x \in \mathbb{P}^{n-1} \quad \text{supp}(x) = \{a_i \in A : x_i \neq 0\}$$

Let  $Q \subset \text{Conv}(A)$  be a face. Then

$$X_{A,Q}^\circ = \{x \in X_A : \text{supp}(x) = Q \cap A\}$$

$$X_{A,Q} = \{x \in X_A : \text{supp}(x) \subseteq Q \cap A\}$$

Affine variety  $Y_A$ :

$$x \in (\mathbb{C})^n \quad \text{supp}(x) = \{a_i \in A : x_i \neq 0\}.$$

Let  $\tau \preceq \text{Cone}(A)$ . Then

$$Y_{A,\tau}^\circ = \{x \in Y_A : \text{supp}(x) = \tau \cap A\}$$

$$Y_{A,\tau} = \{x \in Y_A : \text{supp}(x) \subseteq \tau \cap A\}$$

### Lemma

Let  $X_A \subset \mathbb{P}^{n-1}$  be parametrized by  $\Phi_A$  with  $\Phi_A(t) = (t^{a_1} : \dots : t^{a_n})$ . Then  
 $\text{im } \Phi_A = X_A \cap \{x \in \mathbb{P}^{n-1} : x_1 \cdots x_n \neq 0\}$ . affine:  $\text{im } \phi_A = Y_A \cap (\mathbb{C}^\star)^n$

### Theorem

Let  $X_A \subset \mathbb{P}^{n-1}$  be the projective toric variety of  $A \in \mathbb{Z}^{d \times n}$

1. We have  $X_A = \bigsqcup_{Q \preceq \text{Conv}(A)} X_{A,Q}^\circ$  and  $X_{A,Q} = \bigsqcup_{Q' \preceq Q} X_{A,Q'}^\circ$  affine:  $Y_A = \bigsqcup_{\tau \preceq \text{Cone}(A)} Y_{A,\tau}^\circ$ ,  $Y_{A,\tau} = \bigsqcup_{\tau' \preceq \tau} Y_{A,\tau'}^\circ$ ,
2. For  $Q \preceq \text{Conv}(A)$ , let  $Q \cap A = \{a_{i1}, \dots, a_{i\ell}\}$ . The map  $\pi_Q : X_{A,Q} \rightarrow X_{Q \cap A}$ ,  $x \mapsto (x_{i1}, \dots, x_{i\ell})$  is a well defined isomorphism, and  $\pi_Q(X_{A,Q}^\circ) = X_{Q \cap A}^\circ = \text{im } \Phi_{Q \cap A}$  affine:  $\pi_\tau(Y_{A,\tau}^\circ) = \text{im } \phi_{\tau \cap A}$

## Theorem

Let  $X_A \subset \mathbb{P}^{n-1}$  be the projective toric variety of  $A \in \mathbb{Z}^{d \times n}$

1. We have  $X_A = \bigsqcup_{Q \preceq \text{Conv}(A)} X_{A,Q}^\circ$  and  $X_{A,Q} = \bigsqcup_{Q' \preceq Q} X_{A,Q'}^\circ$  affine:  $Y_A = \bigsqcup_{\tau \preceq \text{Cone}(A)} Y_{A,\tau}^\circ$ ,  $Y_{A,\tau} = \bigsqcup_{\tau' \preceq \tau} Y_{A,\tau'}^\circ$ ,
2. For  $Q \preceq \text{Conv}(A)$ , let  $Q \cap A = \{a_{i1}, \dots, a_{i\ell}\}$ . The map  $\pi_Q : X_{A,Q} \rightarrow X_{Q \cap A}$ ,  $x \mapsto (x_{i1} : \dots : x_{i\ell})$  is a well defined isomorphism, and  $\pi_Q(X_{A,Q}^\circ) = X_{Q \cap A}^\circ = \text{im } \Phi_{Q \cap A}$  affine:  $\pi_\tau(Y_{A,\tau}^\circ) = \text{im } \phi_{\tau \cap A}$

## Proof.

Idea: Use results from the affine case, since the faces  $Q \preceq \text{Conv}(A)$  are in one-to-one correspondence with positive dimensional faces of the pointed cone  $\text{Cone}(\hat{A}) \subset \mathbb{R}^{d+1}$  and the map  $\pi : \mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{P}^{n-1}$  given by  $\pi(x_1, \dots, x_n) = (x_1 : \dots : x_n)$ .

$$\begin{array}{ccc} Y_{\hat{A}, \tau_Q} \setminus \{0\} & \xrightarrow{\pi_{\tau_Q}} & Y_{\tau_Q \cap \hat{A}} \setminus \{0\} \\ \downarrow \pi & & \downarrow \tilde{\pi} \\ X_{A,Q} & \xrightarrow{\pi_Q} & X_{Q \cap A} \end{array}$$

□

## Exercise 4.2.4

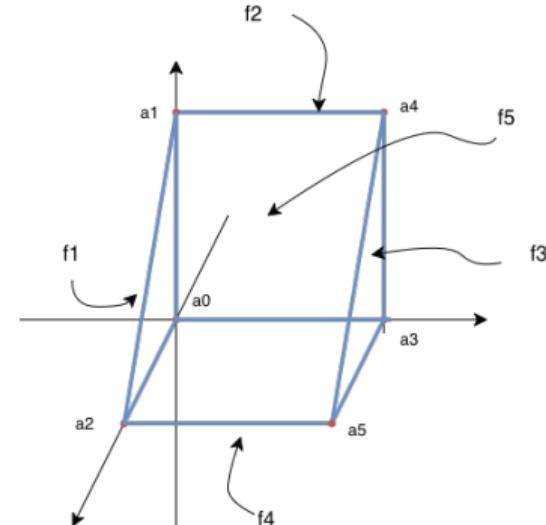
$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

```
julia> B
4×6 Matrix{Int64}:
 0  0  0  1  1  1
 0  1  0  0  1  0
 0  0  1  0  0  1
 1  1  1  1  1  1
```

```
julia> toric_ideal(transpose(B))
Ideal generated by
 -x2*x6 + x3*x5
 -x1*x6 + x3*x4
 -x1*x5 + x2*x4
```

```
julia> C
5×6 Matrix{Int64}:
 1  1  1  0  0  0
 0  0  0  1  1  1
 1  0  0  1  0  0
 0  1  0  0  1  0
 0  0  1  0  0  1
```

```
julia> toric_ideal(transpose(C))
Ideal generated by
 -x2*x6 + x3*x5
 -x1*x6 + x3*x4
 -x1*x5 + x2*x4
```



**Remark 1.2.24:** The image of  $\phi_A$  only depends on the row span of  $A$  over  $\mathbb{Q}$  and the row span of both matrices is identical.

Since  $\dim(\text{Conv}(A)) = 3$  we can use Kushnirenko's theorem:  $\deg(X_A) = 3! * V = 3$ . (A standard simplex in 3 dimensions has volume  $1/6$ , so  $\text{Conv}(A)$  with volume  $1/2$  would fit 3)

```
julia> R, (x1,x2,x3,x4,x5,x6) = graded_polynomial_ring(QQ, [:x1, :x2, :x3, :x4, :x5, :x6])
(Graded multivariate polynomial ring in 6 variables over QQ, MPolyDecRingElem{QQFieldElem, QQMPolyRingElem}[x1, x2, x3, x4, x5, x6])
```

```
julia> A, _ = quo(R, ideal(R, [-x2*x6 + x3*x5, -x1*x6 + x3*x4, -x1*x5 + x2*x4]))
(Quotient of multivariate polynomial ring by ideal (-x2*x6 + x3*x5, -x1*x6 + x3*x4, -x1*x5 + x2*x4),
 Map: R → A)
```

```
julia> hilbert_polynomial(A)
1//2*t^3 + 2*t^2 + 5//2*t + 1
```

Theorem 3.3.5 using the Hilbert-polynomial with leading term  $\frac{\deg(X)}{d!} k^d$  leads to  $1/2 = \frac{\deg(X)}{6}$

Parametrization of the surfaces, edges and vertices can be gained in an analogous way:

$$1 \rightarrow x_0, t_1 \rightarrow x_1, 1 \rightarrow y_0, t_2 \rightarrow y_1, t_3 \rightarrow y_2$$

$$X_{A, \text{Conv}(A)}^\circ = \text{im } \Phi_A = \{(1 : t_2 : t_3 : t_1 : t_1 t_2 : t_1 t_3) : t_1, t_2, t_3 \in \mathbb{C}^*\}$$

$$X_{A, f1}^\circ = \{(1 : t_2 : t_3 : 0 : 0 : 0) : t_1, t_2, t_3 \in \mathbb{C}^*\}$$

$$X_{A, f2}^\circ = \{(1 : t_2 : 0 : t_1 : t_1 t_2, 0) : t_1, t_2, t_3 \in \mathbb{C}^*\}$$

$$X_{A, f3}^\circ = \{(0 : 0 : 0 : t_1 : t_1 t_2 : t_1 t_3) : t_1, t_2, t_3 \in \mathbb{C}^*\}$$

$$X_{A, f4}^\circ = \{(1 : 0 : t_3 : t_1 : 0 : t_1 t_3) : t_1, t_2, t_3 \in \mathbb{C}^*\}$$

$$X_{A, f5}^\circ = \{(0 : t_2 : t_3 : 0 : t_1 t_2 : t_1 t_3) : t_1, t_2, t_3 \in \mathbb{C}^*\}$$

$$X_{A, \text{Conv}(A)}^\circ = \{(x_0 : x_1), (y_0 : y_1 : y_2)\}$$

$$X_{A, f1}^\circ = \{(x_0 : 0), (y_0 : y_1 : y_2)\}$$

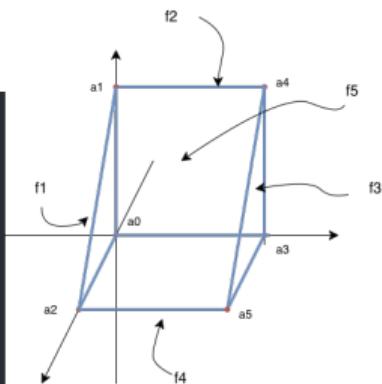
$$X_{A, f2}^\circ = \{(x_0 : x_1), (y_0 : y_1 : 0)\}$$

$$X_{A, f3}^\circ = \{(0 : x_1), (y_0 : y_1 : y_2)\}$$

$$X_{A, f4}^\circ = \{(x_0 : x_1), (y_0 : 0 : y_2)\}$$

$$X_{A, f5}^\circ = \{(x_0 : x_1), (0 : y_1 : y_2)\}$$

for  $x_0, x_1, y_0, y_1, y_2 \in \mathbb{C}^*$



## 4.3 Torus orbits

Let  $G$  be an algebraic group (an affine variety  $V$  with a group operation  $V \times V \rightarrow V$  which is a morphism) and let  $X$  be a variety. An algebraic group action of  $G$  on  $X$  is a morphism  $G \times X \rightarrow X$ ,  $(g, x) \mapsto g \bullet x$  satisfying

1.  $e \bullet x = x$  for all  $x \in X$  and  $e \in G$  the identity
2.  $g \bullet (h \bullet x) = (g \cdot h) \bullet x$  for all  $x \in X$  and  $g, h \in G$

The **orbit** of  $x \in X$  under the group action  $G \times X \rightarrow X$  is  $O_x = \{g \bullet x : g \in G\}$ .

### Proposition

Let  $A \in \mathbb{Z}^{d \times n}$  be such that  $\mathbb{Z}A = \mathbb{Z}^d$  (can always be achieved using the Smith normal form). The morphism

$$(\mathbb{C}^*)^d \times Y_A \rightarrow Y_A, \quad (2)$$

$$(t, x) \mapsto \phi_A(t) \cdot x = (t^{a_1}x_1, \dots, t^{a_n}x_n) \quad (3)$$

is an algebraic group action of  $(\mathbb{C}^*)^d \simeq \text{im } \phi_A$  on  $Y_A$  which extends the action of  $\text{im } \phi_A$  on itself.

### Proposition

$A$  is such that  $\Phi_A : (\mathbb{C}^*)^d \rightarrow \mathbb{P}^{n-1}$  is one-to-one. The action of  $(\mathbb{C}^*)^d$  on  $X_A$  is defined as:

$$(\mathbb{C}^*)^d \times X_A \rightarrow X_A, \quad (t, x) \mapsto (t^{a_1}x_1 : \dots : t^{a_n}x_n). \quad (4)$$

The morphism above is an algebraic group action of  $(\mathbb{C}^*)^d \simeq \text{im } \Phi_A$  on  $X_A$  which extends the action of  $\text{im } \Phi_A$  on itself.

## Theorem

Let  $A \in \mathbb{Z}^{d \times n}$  be such that  $\mathbb{Z}A = \mathbb{Z}^d$  and let  $Y_A$  be the corresponding affine toric variety. The stratification  $Y_A = \bigsqcup_{\tau \preceq \text{Cone}(A)} Y_{A,\tau}^\circ$  decomposes  $Y_A$  into  $(\mathbb{C}^*)^d$ -orbits, where the action is that of (3).

## Theorem

The orbits of morphism (4) are the open strata  $X_{A,Q}^\circ$ . The disjoint union  $X_A = \bigsqcup_{Q \preceq \text{Conv}(A)} X_{A,Q}^\circ$  decomposes  $X_A$  into  $(\mathbb{C}^*)^d$  orbits.

## Definition

The orbit-cone correspondence for the toric variety  $Y_A$  is a bijection between faces of the dual cone  $\text{Cone}(A)^\vee$  and  $(\mathbb{C}^*)^d$ -orbits of  $Y_A$ , given by  $\tau \mapsto Y_{A,\tilde{\tau}}^\circ$ .

## Definition

Toric Variety A toric variety  $\mathcal{X}$  is an irreducible algebraic variety containing a torus  $T \simeq (\mathbb{C}^*)^d$  as a dense open subset, such that the action of  $T$  on itself extends to an algebraic action  $T \times \mathcal{X} \rightarrow \mathcal{X}$ .