Solutions to Some Exercises

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Note: These problems are all solved in the book. I include their solutions here to demonstrate how to typeset them in LaTeX.

Chapter 1 Exercises

Section 1.1

1. $\{5x-1 : x \in \mathbb{Z}\} = \{..., -11, -6, -1, 4, 9, 14, 19, 24, 29, ...\}$

13. $\{x \in \mathbb{Z} : |6x| < 5\} = \{0\}$

Section 1.2

1. (a) $A \times B = \{(1, a), (1, c), (2, a)(2, c), (3, a), (3, c), (4, a), (4, c)\}$

Section 1.4

A. Find the indicated sets.

3. $\mathcal{P}(\{\{a, b\}, \{c\}\}) = \{\emptyset, \{\{a, b\}\}, \{\{c\}\}, \{\{a, b\}, \{c\}\}\}$

B. Suppose that $|A| = m$ and $|B| = n$. Find the indicated cardinalities.

13. $|\mathcal{P}(\mathcal{P}(\mathcal{P}(A)))| = 2^{(2^{(2^m)})}$

15. $|\mathcal{P}(A \times B)| = 2^{mn}$

Section 1.5

3. Suppose $A = \{0, 1\}$ and $B = \{1, 2\}$. Find:

(a) $A \cup B = \{1, 3, 4, 5, 6, 7, 8, 9\}$

(b) $A \cap B = \{4, 6\}$

Section 1.6

1. Suppose $A = \{4, 3, 6, 7, 1, 9\}$ and $B = \{5, 6, 8, 4\}$ have the universal set $U = \{n \in \mathbb{Z} : 0 \leq n \leq 10\}$

$\overline{\overline{A} \cap B} = \{0, 1, 2, 3, 4, 6, 7, 9, 10\}$

Section 1.8

5(a) $\bigcup_{i \in \mathbb{N}} [i, i+1] = [1, \infty)$ or:

$$\bigcup_{i \in \mathbb{N}} [i, i+1] = [1, \infty)$$

5(b) $\bigcap_{i \in \mathbb{N}} [i, i+1] = \emptyset$ or:

$$\bigcap_{i \in \mathbb{N}} [i, i+1] = \emptyset$$

Chapter 2 Exercises

Section 2.2 Express each statement as one of the forms $P \wedge Q$, $P \vee Q$, or $\sim P$. (I will also accept $\neg P$.)

9. $x \in A - B$

$(x \in A) \wedge \neg(x \in B)$

Section 2.5

5. Write a truth table for $(P \wedge \neg P) \vee Q$

P	Q	$(P \wedge \neg P)$	$(P \wedge \neg P) \vee Q$
T	T	F	**T**
T	F	F	**F**
F	T	F	**T**
F	F	F	**F**

Chapter 3 Exercises

Section 3.3

1. Suppose a set A has 37 elements. How may subsets of A have 10 elements? How many subsets have 30 elements? How many have 0 elements?

Answers: $\binom{37}{10} = 348,330,136$; $\binom{37}{30} = 10,295,472$; $\binom{37}{0} = 1$.

Chapter 4 Exercises

7. Suppose $a, b \in \mathbb{Z}$. If $a \mid b$, then $a^2 \mid b^2$.

Proof. Suppose $a \mid b$.

By definition of divisibility, this means $b = ac$ for some integer c.

Squaring both sides of this equation produces $b^2 = a^2c^2$.

Then $b^2 = a^2d$, where $d = c^2 \in \mathbb{Z}$.

By definition of divisibility, this means $a^2 \mid b^2$. ■

Chapter 5 Exercises

9. Proposition Suppose $n \in \mathbb{Z}$. If $3 \nmid n^2$, then $3 \nmid n$.

Proof. (Contrapositive) Suppose it is not the case that $3 \nmid n$, so $3 \mid n$. This means that $n = 3a$ for some integer a. Consequently $n^2 = 9a^2$, from which we get $n^2 = 3(3a^2)$. This shows that there is an integer $b = 3a^2$ for which $n^2 = 3b$, which means $3 \mid n^2$. Therefore it is not the case that $3 \nmid n^2$. ■

21. Proposition Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $a \equiv b \mod n$, then $a^3 \equiv b^3 \mod n$.

Proof. (Direct) Suppose $a \equiv b \mod n$. This means $n \mid (a - b)$, so there is an integer c for which $a - b = nc$. Then:

$$\begin{aligned} a - b &= nc \\ (a-b)(a^2 + ab + b^2) &= nc(a^2 + ab + b^2) \\ a^3 + a^2b + ab^2 - ba^2 - ab^2 - b^3 &= nc(a^2 + ab + b^2) \\ a^3 - b^3 &= nc(a^2 + ab + b^2). \end{aligned}$$

Since $a^2 + ab + b^2 \in \mathbb{Z}$, the equation $a^3 - b^3 = nc(a^2 + ab + b^2)$ implies $n \mid (a^3 - b^3)$, and therefore $a^3 \equiv b^3 \mod n$. ■

Chapter 9 Exercises

27. The equation $x^2 = 2^x$ has three real solutions.

Proof. By inspection, $x = 2$ and $x = 4$ are two solutions of this equation. But there is a third solution. Let m be the real number for which $m2^m = \frac{1}{2}$. Then negative number $x = -2m$ is a solution, as follows.

$$x^2 = (-2m)^2 = 4m^2 = 4\left(\frac{m2^m}{2^m}\right)^2 = 4\left(\frac{\frac{1}{2}}{2^m}\right)^2 = \frac{1}{2^{2m}} = 2^{-2m} = 2^x$$

Chapter 10 Exercises

1. For every integer $n \in \mathbb{N}$, it follows that

$$1 + 2 + 3 + 4 + \ldots + n = \frac{n^2 + n}{2}$$

or

$$\sum_{i=1}^{n} i = \frac{n^2 + n}{2}$$

In this proof I use the second notation. The book shows the solution in the first notation.

Proof. we will prove this with mathematical induction.

(1) Observe that if $n = 1$, this statement is $1 = \frac{1^2+1}{2}$, which is obviously true.

(2) Consider any integer $k \geq 1$. We must show that S_k implies S_{k+1}. In other words, we must show that if

$$\sum_{i=1}^{k} i = \frac{k^2 + k}{2}$$

is true, then

$$\sum_{i=1}^{k+1} i = \frac{(k+1)^2 + (k+1)}{2}$$

is also true. We use direct proof.

Suppose $k \geq 1$ and

$$\sum_{i=1}^{k} i = \frac{k^2 + k}{2}$$

We observe that

$$\begin{aligned}
\sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1) && \text{(isolating the last term in the sum)} \\
&= \frac{k^2 + k}{2} + (k+1) && \text{(by the inductive hypothesis)} \\
&= \frac{k^2 + k + 2(k+2)}{2} \\
&= \frac{k^2 + 2k + 1 + k + 1}{2} \\
&= \frac{(k+1)^2 + (k+1)}{2}
\end{aligned}$$

Therefore we have shown that

$$\sum_{i=1}^{k+1} i = \frac{(k+1)^2 + (k+1)}{2}$$