Book of Proof: Part III, More on Proof

January 16, 2018

If-and-Only-If Proof

Outline for If-and-Only-If Proof

Proposition P if and only if Q.

Proof.

"Only if"

[Prove $P \Rightarrow Q$ by whatever means you can.]

"If"

[Prove $Q \Rightarrow P$ by whatever means you can.]

Equivalent Statements

Theorem Suppose A is an $n \times n$ matrix. The following statements are equivalent:

a. A is invertible.

b. $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$.

c. $Ax = 0$ has only the trivial solution.

d. The reduced row echelong form of A is I_n.

e. $\det(A) \neq 0$.

f. The matrix A does not have 0 as an eigenvector.

Equivalent Statements

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Existence Proofs

Proposition There exists an even prime number.

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Proof.

$$1^3 + 12^3 = 1729$$
$$9^3 + 10^3 = 1729$$

Example

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there exist $k, \ell \in \mathbb{Z}$ for which $\gcd(a, b) = ak + b\ell$.

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Proof. Suppose $a, b \in \mathbb{N}$.
Consider the set $A = \{ax + by : x, y \in \mathbb{Z}\}$.
A contains positive integers and 0.
Let $d \in A$ be the smallest positive integer.
$d = ak + b\ell$ for some $k, \ell \in \mathbb{Z}$.
We will show that $d = \gcd(a, b)$.
First, prove that $d \mid a$ and $d \mid b$.
Then show that it is the largest such number.