Book of Proof: Fundamentals

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Sets: A mathematical structure

$$\{1, 2, 3\}$$
$$\{a, b, c, d\}$$
$$\{cat, dog, pig\}$$
$$\{2, 4, 6, 8, ...\}$$
$$\emptyset = \{\}$$
$$\emptyset \neq \{\emptyset\}$$

Note: $\{1, 2, 3\}$ is not the same as $1, 2, 3$ or $(1, 2, 3)$ or *etc.*

Sets have no order or duplicates

$$
\begin{aligned}
\{1,2,3\} &= \{2,3,1\} \\
&= \{2,1,3\} \\
&= \{1,1,2,2,3,3\} \\
&= \{2,3,3,2,1,1,1,1,2,3,2,2,2,3,1\}
\end{aligned}
$$

Some important sets

The integers, the natural numbers, the nonnegative integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$
$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$
$$\mathbb{N}^0 = \{0, 1, 2, 3, 4, \ldots\}$$

We won't have much use for the real numbers, $\mathbb{R}$.

The size of a finite set

$$
\begin{aligned}
3 &= |\{a, b, c\}| \\
5 &= |\{a, b, c, d, e\}| \\
&= |\{a, b, c, d, e, a, d, b\}| \\
0 &= |\emptyset| \\
1 &= |\{\emptyset\}| \\
1 &= |\{\{\emptyset\}\}|
\end{aligned}
$$

Membership and subsets

$$3 \in \{1, 2, 3, 4, 5\}$$
$$3 \notin \{2, 4, 6, 8\}$$
$$cat \in \{cat, dog, pig\}$$
$$3 \in \mathbb{N}^0$$
$$\pi \notin \mathbb{Z}$$

$$\{2, 5, 8\} \subseteq \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$
$$\{2, 5, 8\} \not\subseteq \{1, 2, 3, 4, 5, 6, 7\}$$
$$\{3\} \subseteq \{1, 2, 3, 4, 5\}$$
$$\{3\} \not\subseteq \{2, 4, 6, 8\}$$
$$\mathbb{N}^0 \subseteq \mathbb{Z}$$
$$\mathbb{R} \not\subseteq \mathbb{N}$$

Set builder notation

$$
\begin{aligned}
\{n : n \text{ is odd and } 4 \leq n \leq 16\} &= \{5, 7, 9, 11, 13, 15\} \\
\{2n + 5 : n \in \{3, 6, 7\}\} &= \{11, 17, 19\} \\
\left\{2n : n \in \mathbb{N}^0\right\} &= \{0, 2, 4, 6, 8, \ldots\} \\
\{n \in \mathbb{N} : n < 5\} &= \{1, 2, 3, 4\} \\
\{3n : n \in \mathbb{N} \text{ and } n < 5\} &= \{3, 6, 9, 12\}
\end{aligned}
$$

Ordered pairs, triples, n-tuples

$$\begin{aligned}
(2,4) &\neq (4,2)\\
(2,2) &\neq (2)\\
(1,2,3) &\neq (3,2,1)\\
(1,1,2) &\neq (1,2)\\
(5,3,2,1,6) &\neq (1,2,3,5,6)
\end{aligned}$$

Cartesian product

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$$
$$\{a, b\} \times \{1, 2, 3\} = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

If A and B are finite sets, then $|A \times B| = |A| \cdot |B|$.

Higher order Cartesian products

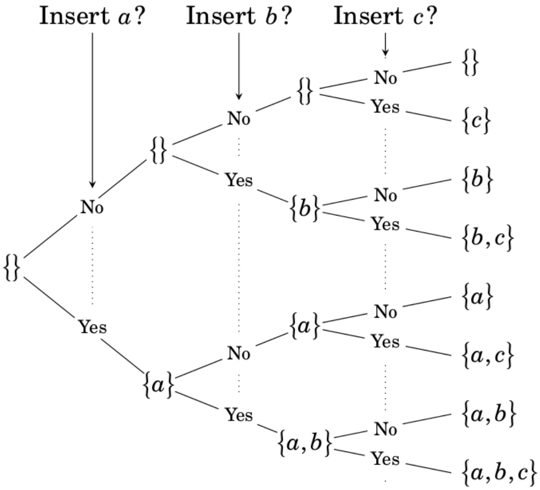
$$\begin{aligned}
A \times B \times C &= \{(a, b, c) : a \in A, b \in B, c \in C\} \\
A^n &= A \times A \times A \times \ldots \times A \\
&= \{(x_1, x_2, x_3, \ldots, x_n) : x_1, x_2, x_3, \ldots, x_n \in A\}
\end{aligned}$$

Power set: the set of all subsets

$$\mathcal{P}(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

How many subsets are there?

If $|A| = n$ then $|\mathcal{P}(A)| = 2^n$



Union, Intersection, Difference

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$
$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$
$$A - B = \{x : x \in A \text{ and } x \notin B\}$$

$$A = \{1, 2, 3, 4, 5\}$$
$$B = \{4, 5, 6, 7, 8\}$$
$$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$$
$$A \cap B = \{4, 5\}$$
$$A - B = \{1, 2, 3\}$$

Complement

$$\overline{A} = \{x : x \notin A\}$$

$$\overline{\{2, 4, 6, 8, \ldots\}} = \{1, 3, 5, 7, \ldots\}$$

Usually relative to some implied **universal set** or **universe**, in this case, $\mathbb{N}$.

Indexed Sets

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n$$

Indexed Sets

$$
\begin{aligned}
A_i &= \{ni : n \in \mathbb{N}\} \\
A_1 &= \{1, 2, 3, 4, ...\} \\
A_2 &= \{2, 4, 6, 8, ...\} \\
A_3 &= \{3, 6, 9, 12, ...\} \\
A_4 &= \{4, 8, 12, 16, ...\} \\
&\ldots
\end{aligned}
$$

$$
\begin{aligned}
\bigcup_{i=2}^{4} A_i &= \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, ...\} \\
\bigcap_{i=2}^{4} A_i &= \{12, 24, 36, 48, 72, ...\}
\end{aligned}
$$

Indexed Sets