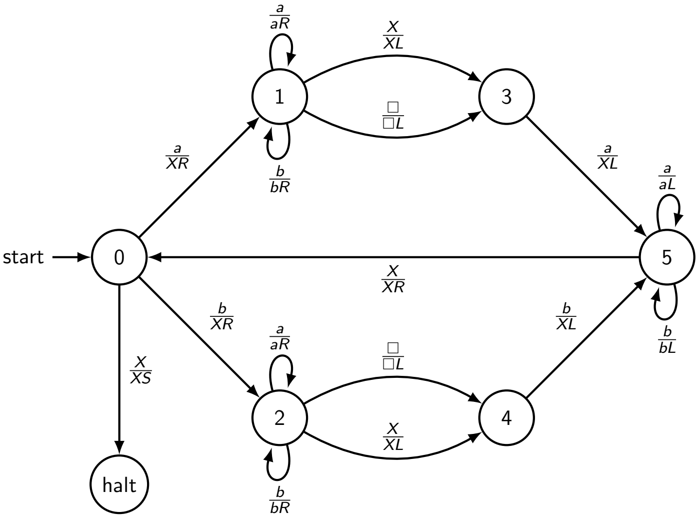
Introduction to Theory of Computation

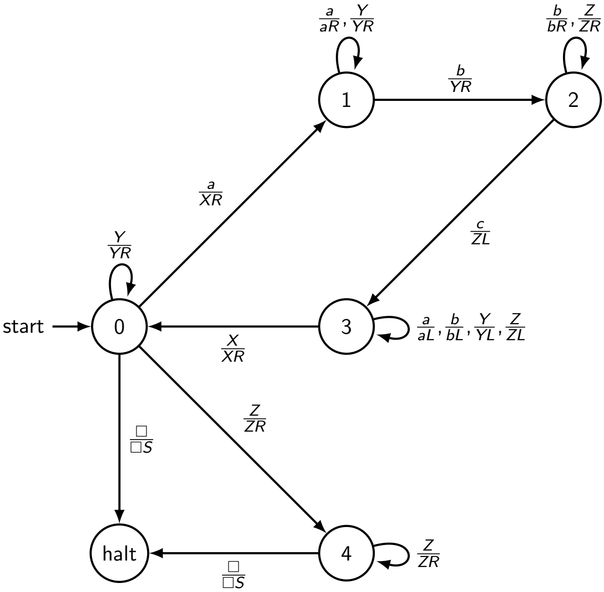
Chapter 4, Turing Machines

March 2, 2017

Even Palindromes



$a^n b^n c^n$



$\frac{a}{aR}, \frac{Y}{YR}$
$\frac{b}{bR}, \frac{Z}{ZR}$
1
$\frac{b}{YR}$
2
$\frac{a}{XR}$
$\frac{c}{ZL}$
$\frac{Y}{YR}$
start
0
3
$\frac{a}{aL}, \frac{b}{bL}, \frac{Y}{YL}, \frac{Z}{ZL}$
$\frac{X}{XR}$
$\frac{Z}{ZR}$
$\frac{\square}{\square S}$
halt
4
$\frac{Z}{ZR}$
$\frac{\square}{\square S}$

Equivalent models:

1. One-tape Turing machines.
2. k-tape Turing machines.
3. Non-deterministic Turing machines.
4. Java programs.
5. Scheme programs.
6. C++ programs.
7. ...

The Church-Turing Thesis

Every computational process that is intuitively considered to be an algorithm can be converted to a Turing machine.

Decidability

A language A over Σ is *decidable* if there exists a Turing machine M such that for every string $w \in \Sigma^*$:

1. If $w \in A$ then M, started on w, halts in the accept state.
2. If $w \notin A$ then M, started on w, halts in the reject state.

Enumerability

A language A over Σ is *enumerable* if there exists a Turing machine M such that for every string $w \in \Sigma^*$:

1. If $w \in A$ then M, started on w, halts in the accept state.
2. If $w \notin A$ then M, started on w, either halts in the reject state or loops forever.

Why do we call it enumerable?

- If we can enumerate the elements with an algorithm, then we can create an algorithm to correctly identify elements of the set.
    - How?

Why do we call it enumerable?

- If we can enumerate the elements with an algorithm, then we can create an algorithm to correctly identify elements of the set.
  - How?
- If we can correctly identify elements of the set, then we can build an algorithm to enumerate them.
  - How?

Decidable *vs.* enumerable

- **Decidable** is also called
  - **computable**
  - **recursive**
- **Enumerable** is also called
  - **semi-decidable**
  - **recognizable**
  - **recursively enumerable**

Describing machines and problems as strings

- We assume any machine (DFA, PDA, TM) can be described by a string M using some alphabet.
- The input to any machine is a string w using some alphabet.
- We can thus describe both a machine M and its input w, with a pair of strings: (M, w).
- This pair can be converted to a single string $\langle M, w \rangle$.
- For convenience, we assume $\langle M, w \rangle$ is encoded in binary.
- In general, $\langle M \rangle$ means: encode M as a binary string.
- We can now define a language A as the set of all strings $\langle M, w \rangle$ such that w is in the computation model of M.

The language A_{DFA} is decidable

$$A_{DFA} = \{\langle M, w\rangle : M \text{ is a DFA that accepts } w\}$$

Proof?

The language A_{DFA} is decidable

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Proof?

- Given input $\langle M, w \rangle$:
  - Run M on w.
  - It must terminate.
  - If it accepts, accept, else reject.

The language A_{NFA} is decidable

$$A_{NFA} = \{\langle M, w\rangle : M \text{ is a NFA that accepts } w\}$$

Proof?

The language A_{NFA} is decidable

$$A_{NFA} = \{\langle M, w\rangle : M \text{ is a NFA that accepts } w\}$$

Proof?

- Given input $\langle M, w\rangle$:
  - Convert NFA M to DFA N.
  - This algorithm terminates.
  - Run N on w.
  - It must terminate.
  - If it accepts, accept, else reject.

The language A_{CGF} is decidable

$$A_{NFA} = \{\langle M, w\rangle : M \text{ is a CFG that accepts } w\}$$

Proof?

The language A_{CGF} is decidable

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Proof?

- Given input $\langle M, w \rangle$:
  - Convert CFG M to Chomsky normal form CFG N.
  - This algorithm terminates.
  - Run N on w for all derivations up to length $2|w|$.
  - There are a finite number of these, so it must terminate.
  - If any derivation accepts, accept, else reject.

The language A_{TM} is not decidable.

$$A_{TM} = \{\langle M, w \rangle : M \text{ is a TM that accepts } w\}$$

Proof?

The language A_{TM} is not decidable.

$$A_{TM} = \{\langle M, w\rangle : M \text{ is a TM that accepts } w\}$$

Proof? By contradiction.

- Assume there is a TM H that decides this language.
- Construct the following TM, D:

  D: On input $\langle M\rangle$:
  - Run H on $\langle M, \langle M\rangle\rangle$.
  - If H accepts, reject, else accept.

- If H accepts $\langle D, \langle D\rangle\rangle$, then D rejects $\langle D\rangle$.
  - Therefore, by definition $\langle D, \langle D\rangle\rangle \notin A_{TM}$.
- If H rejects $\langle D, \langle D\rangle\rangle$, then D accepts $\langle D\rangle$.
  - Therefore, by definition $\langle D, \langle D\rangle\rangle \in A_{TM}$.
- In either case, H does not decide A_{TM}.

Diagonal argument

- Machine H that decides A_{TM} can fill in this table:

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\langle M_5 \rangle$	...
M_0	**accept**	*accept*	*accept*	*reject*	*accept*	*reject*	...
M_1	*accept*	**reject**	*accept*	*accept*	*accept*	*reject*	...
M_2	*accept*	*reject*	**accept**	*accept*	*accept*	*reject*	...
M_3	*accept*	*accept*	*reject*	**reject**	*accept*	*accept*	...
M_4	*reject*	*accept*	*accept*	*reject*	**accept**	*accept*	...
M_5	*reject*	*reject*	*accept*	*accept*	*accept*	**reject**	...
...	...	...	...	...	...	...	...

- D uses H to give the opposite answer on the diagonal.
- H must give the wrong answer somewhere on machine D.

The language A_{TM} is enumerable but not decidable.

$$A_{TM} = \{\langle M, w \rangle : M \text{ is a TM that accepts } w\}$$

Proof?

Hilbert's 10th problem is enumerable but not decidable

$$\begin{aligned} \mathit{Hilbert} = \{\langle p \rangle \quad : \quad & p \text{ is a polynomial with integer coefficients} \\ & \text{that has an integral root}\} \end{aligned}$$

$$15x^3y^2 + 12xy^2z^3 - 17x^9y^2z = 0$$

The language *Halt* is not decidable.

$$Halt = \{\langle M, w\rangle : M \text{ is a TM that terminates on } w\}$$

Proof?

The language *Halt* is not decidable.

$$Halt = \{\langle M, w\rangle : M \text{ is a TM that terminates on } w\}$$

Proof?

By contradiction. Assume there is a TM H that decides this language. Construct the following TM, Q:

Q: On input $\langle M\rangle$:
▶ while $H(\langle M, \langle M\rangle\rangle)$ do end;

- What happens if we run Q on itself?
- $Q(\langle Q\rangle)$ terminates iff $Q(\langle Q\rangle)$ does not terminate.

The language M_a is not decidable.

$$M_a = \{\langle M \rangle \mid \mathcal{L}(M) = \{a\}\}$$

Proof?

The language M_a is not decidable.

$$M_a = \{\langle M\rangle \mid \mathcal{L}(M) = \{a\}\}$$

Proof?

By contradiction.

- Suppose TM A decides M_a.
- Construct the following TM, H:

> H: On input $\langle M, w\rangle$:
> - Construct TM D:
>   > D: On input $\langle s\rangle$:
>   > - Run M on w.
>   > - If $s = a$ accept, else reject.
> - Run A on D. If it accepts, accept, else reject.

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Proof?

By contradiction.

- Suppose TM A decides M_a.
- Construct the following TM, H:

  > H: On input $\langle M, w \rangle$:
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  >   > - Run M on w.
  >   > - If $s = a$ accept, else reject.
  > - Run A on D. If it accepts, accept, else reject.

- $\mathcal{L}(D) = \{a\}$ iff M halts on w.

The language M_a is not decidable.

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Proof?

By contradiction.

- Suppose TM A decides M_a.
- Construct the following TM, H:

> H: On input $\langle M, w \rangle$:
> - Construct TM D:
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>   > D: On input $\langle s \rangle$:
>   > - Run M on w.
>   > - If $s = a$ accept, else reject.
>
> - Run A on D. If it accepts, accept, else reject.

- $\mathcal{L}(D) = \{a\}$ iff M halts on w.
- H decides the language $Halt$. But that's impossible!

Rice's Theorem

Let $\mathcal{T}$ be the set of all binary encoded TMs.
Let $\mathcal{P}$ be a subset of $\mathcal{T}$ such that

1. $\mathcal{P} \neq \emptyset$
2. $\mathcal{P} \neq \mathcal{T}$
3. If $L(M_1) = L(M_2)$, then either both or neither is in $\mathcal{P}$.

Then $\mathcal{P}$ is undecidable.

Rice's Theorem Examples

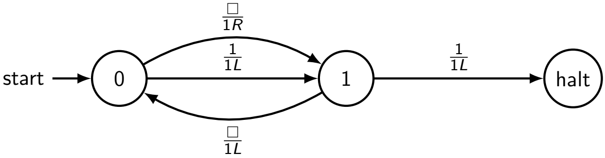
1. $\{\langle M \rangle \mid M$ accepts only inputs in the language $a^* b^*\}$
2. $\{\langle M \rangle \mid M$ accepts only input of length $n^2\}$
3. $\{\langle M \rangle \mid M$ accepts only input of length $k\}$
4. $\{\langle M \rangle \mid M$ accepts all inputs$\}$
5. $\{\langle M \rangle \mid M$ does not accept all inputs$\}$
6. $\{\langle M \rangle \mid M$ accepts some input$\}$
7. $\{\langle M \rangle \mid M$ does not accept any input$\}$

None of these is decideable.

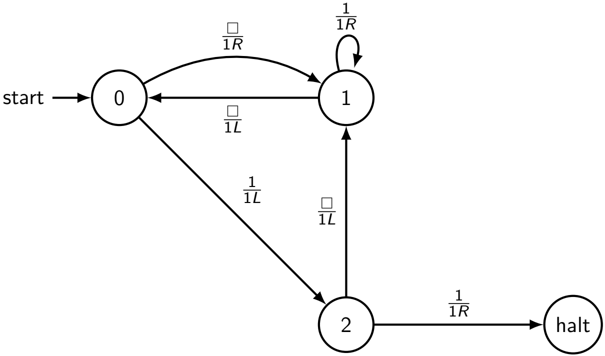
Busy beavers are not enumerable

The nth busy beaver number is the largest (finite) number of 1s that can be output by a Turing machine with n states.

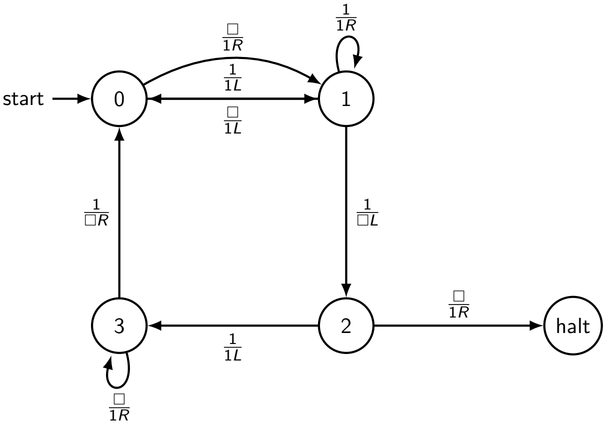
2 State Busy Beaver: four 1s



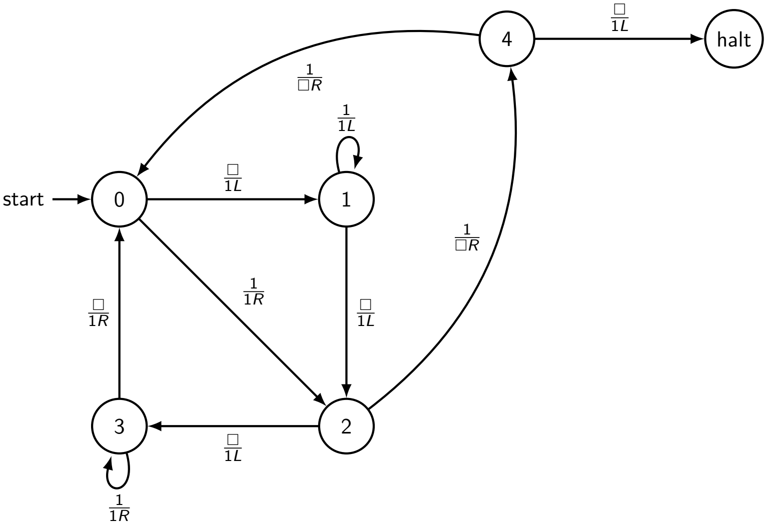
3 State Busy Beaver: six 1s



4 State Busy Beaver: thirteen 1s



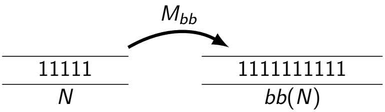
5 State Busy Beaver (?): 4098 1s



Proof Busy Beaver function is uncomputable

Proof by contradiction.

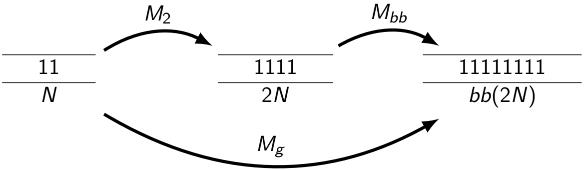
- Let $bb(n)$ be the largest (finite) number of 1's output by a Turing Machine with n states.
- Suppose there is a Turing Machine M_{bb} that computes $bb(n)$, that is, starting with n on the tape, the machine halts with $bb(n)$ on the tape.



- Note: this is a new use of TMs, computing a function from input to output, not recognizing a language.

Busy Beaver proof

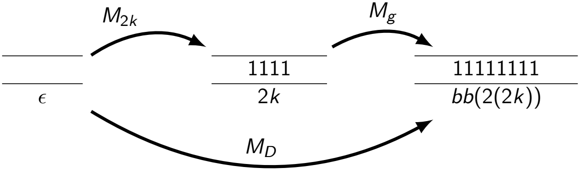
- Let $g(n) = bb(2n)$. We can build a TM for g by starting with a machine that doubles the input, and then runs the machine M_{bb}.



- Suppose the machine for g, M_g has k states.

Busy Beaver proof

- Build a machine M_{2k} with $2k$ states that does nothing but put $2k$ 1s on a blank tape.
- Now build a machine M_D that starts by putting $2k$ 1's on the tape, and then runs the M_g machine.



- M_D can be built with $3k$ states.
- The output of M_D is $g(2k) = bb(2(2k)) = bb(4k)$ 1s.
- Do you see the problem?