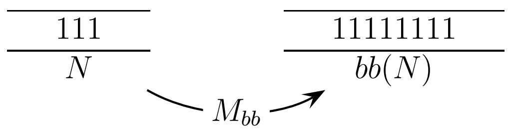
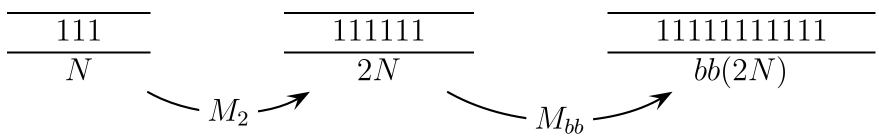
Proof the Busy Beaver function is uncomputable

- Let $bb(n)$ be the largest (finite) number of 1's output by a Turing Machine with n states.

- Suppose there is a Turing Machine M_{bb} that computes $bb(n)$, that is, starting with n on the tape, the machine halts with $bb(n)$ on the tape.

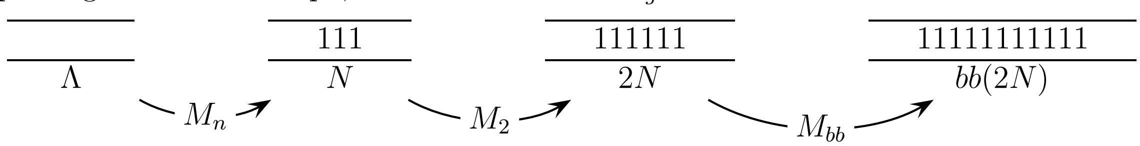


- Let $g(n) = bb(2n)$. We can build a TM for g by starting with a machine that doubles the input, and then runs the machine M_{bb}.



- Suppose the machine for g, M_g, which is a combination of M_2 and M_{bb}, has k states.

- For any natural number n, we could build a new busy beaver machine that starts by putting n 1's on the tape, and then runs the M_g machine.



- Now note:
  - We can build this busy beaver with $n + k$ states.
  - The output of this busy beaver is $g(n) = bb(2n)$ 1's.

- Now pick $n = 2k$. Then we can build a machine with $n + k = 3k$ states and its output is $bb(2(2k)) = bb(4k)$ 1's.

- Therefore, if M_{bb} exists, we could build a machine with $3k$ states that output as many 1's as you can possibly do with $4k$ states.

  See the problem?