

Book of Proof: Part II, Conditionals

January 12, 2018

Proofs

Theorem Something important you want to prove.

Proposition Something not so important you want to prove.

Corollary Something you want to prove in order to prove something else.

Definitions

Definition 4.1 An integer n is **even** if $n = 2a$ for some integer $z \in \mathbb{Z}$.

Definition 4.1 An integer n is **odd** if $n = 2a + 1$ for some integer $z \in \mathbb{Z}$.

Definition 4.3 Two integers have the **same parity** if they are both even or they are both odd. Otherwise they have **opposite parity**.

Definition 4.4 Suppose a and b are integers. We say that a **divides** b , written $a \mid b$, if $b = ac$ for some $c \in \mathbb{Z}$. In this case we also say that a is a **divisor** of b , and that b is a **multiple** of a .

Definition 4.5 A natural number n is **prime** if it has exactly two positive divisors, 1 and n .

Definitions

Definition 4.6 The **greatest common divisor** of integers a and b , denoted $\gcd(a, b)$, is the largest integer that divides both a and b . The **least common multiple** of non-zero integers a and b , denoted $\text{lcm}(a, b)$, is the smallest positive integer that is a multiple of both a and b .

$$\gcd(18, 24) = 6$$

$$\gcd(32, -8) = 8$$

$$\gcd(50, 9) = 1$$

$$\gcd(0, 0) = \text{undefined}$$

$$\text{lcm}(4, 6) = 12$$

$$\gcd(5, 5) = 5$$

$$\gcd(50, 18) = 2$$

$$\gcd(0, 6) = 6$$

$$\text{lcm}(7, 7) = 7$$

Some facts accepted without proof

If $a, b \in \mathbb{Z}$, then $a + b \in \mathbb{Z}$, $a - b \in \mathbb{Z}$, $ab \in \mathbb{Z}$.

The Division Algorithm Given integers a and b with $b > 0$, there exist unique integers q and r for which $a = qb + r$ and $0 \leq r < b$.

Every natural number greater than 1 has a unique factorization into primes.

Direct Proof

If P , then Q .

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Outline for Direct Proof

Proposition If P , then Q .

Proof. Suppose P .

\vdots

Therefore, Q . ■

Example proof development

Proposition If x is odd, then x^2 is odd.

Proof. Suppose x is odd.

\vdots

Therefore x^2 is odd, *for some reason*. ■

Example proof development

Proposition If x is odd, then x^2 is odd.

Proof. Suppose x is odd.

Then $x = 2a + 1$ for some $a \in \mathbb{Z}$, by definition of an odd number.

\vdots

Therefore x^2 is odd, *for some reason*. ■

Example proof development

Proposition If x is odd, then x^2 is odd.

Proof. Suppose x is odd.

Then $x = 2a + 1$ for some $a \in \mathbb{Z}$, by definition of an odd number.

\vdots

Thus $x^2 = 2b + 1$ for an integer b , *for some reason*.

Therefore x^2 is odd, by definition of an odd number. ■

Example proof development

Proposition If x is odd, then x^2 is odd.

Proof. Suppose x is odd.

Then $x = 2a + 1$ for some $a \in \mathbb{Z}$, by definition of an odd number.

Thus

$$x^2 = (2a + 1)^2 \quad \text{by substitution}$$

$$= 4a^2 + 4a + 1 \quad \text{by algebra}$$

$$= 2(2a^2 + 2a) + 1 \quad \text{by algebra}$$

If we let $b = 2a^2 + 2a$ then b is an integer, by math facts.

Thus $x^2 = 2b + 1$ for an integer b , by substitution.

Therefore x^2 is odd, by definition of an odd number. ■

Example proof development

Proposition Let $a, b, c \in \mathbb{Z}$. If $a \mid b$ and $b \mid c$, then $a \mid c$.

Example proof development

Proposition Let $a, b, c \in \mathbb{Z}$. If $a \mid b$ and $b \mid c$, then $a \mid c$.

Proof.

\vdots



Example proof development

Proposition Let $a, b, c \in \mathbb{Z}$. If $a \mid b$ and $b \mid c$, then $a \mid c$.

Proof. Suppose $a \mid b$ and $b \mid c$.

\vdots

Therefore $a \mid c$



Example proof development

Proposition Let $a, b, c \in \mathbb{Z}$. If $a \mid b$ and $b \mid c$, then $a \mid c$.

Proof. Suppose $a \mid b$ and $b \mid c$.

By definition, $a \mid b$ means there exists $d \in \mathbb{Z}$ with

$$b = ad$$

By definition, $b \mid c$ means there exists $e \in \mathbb{Z}$ with

$$c = be$$

\vdots

Thus $c = ax$ for some $x \in \mathbb{Z}$.

Therefore $a \mid c$, by definition. ■

Example proof development

Proposition Let $a, b, c \in \mathbb{Z}$. If $a \mid b$ and $b \mid c$, then $a \mid c$.

Proof. Suppose $a \mid b$ and $b \mid c$.

By definition, $a \mid b$ means there exists $d \in \mathbb{Z}$ with

$$b = ad$$

By definition, $b \mid c$ means there exists $e \in \mathbb{Z}$ with

$$c = be$$

By combining equations these two equations, we get

$$c = be = (ad)e = a(de)$$

Let $x = de$, then $x \in \mathbb{Z}$.

Thus $c = ax$ for some $x \in \mathbb{Z}$.

Therefore $a \mid c$, by definition. ■

Proof by cases

Proposition If $n \in \mathbb{Z}$ then $1 + (-1)^n(2n - 1)$ is a multiple of 4.

Proof. Suppose $n \in \mathbb{Z}$.

Then n is either even or odd.

Case 1. Suppose n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$ and $(-1)^n = 1$. Thus

$$1 + (-1)^n(2n - 1) = 1 + (1)(2 \cdot 2k - 1) = 4k$$

which is a multiple of 4.

Case 2. Suppose n is odd. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$ and $(-1)^n = -1$. Thus

$$1 + (-1)^n(2n - 1) = 1 - (2(2k + 1) - 1) = -4k$$

which is a multiple of 4.

These cases show that $1 + (-1)^n(2n - 1)$ is always a multiple of 4.

Without loss of generality

Proposition If two integers have opposite parity, then their sum is odd.

Proof. Suppose m and n are integers with opposite parity.

Without loss of generality, suppose m is even and n is odd.

Thus $m = 2a$ and $n = 2b + 1$ for some $a, b \in \mathbb{Z}$.

Therefore $m + n = 2a + 2b + 1 = 2(a + b) + 1$, which is odd by definition.

Contrapositive Proof

If P , then Q .
If $\sim Q$, then $\sim P$.

P	Q	$\sim Q$	$\sim P$	$P \Rightarrow Q$	$\sim Q \Rightarrow \sim P$
T	T	F	F	T	T
T	F	T	F	F	F
F	T	F	T	T	T
F	F	T	T	T	T

$$(P \Rightarrow Q) \iff (\sim Q \Rightarrow \sim P)$$

Contrapositive Proof

Outline for Contrapositive Proof

Proposition If P , then Q .

Proof. Suppose $\sim Q$.

\vdots

Therefore, $\sim P$. ■

Example Contrapositive Proof

Proposition Suppose $x \in \mathbb{Z}$. If $7x + 9$ is even, then x is odd.

Proof. (Contrapositive)

Suppose x is not odd.

Thus x is even, and $x = 2a$ for some $a \in \mathbb{Z}$.

Then $7x + 9 = 7(2a) + 9 = 14a + 8 + 1 = 2(7a + 4) + 1$.

Thus $7x + 9 = 2b + 1$ where b is the integer $7a + 4$.

By definition, $7x + 9$ is odd.

Therefore $7x + 9$ is not even.

Proof by Contradiction

P	C	$\sim P$	$C \wedge \sim C$	$(\sim P) \Rightarrow (C \wedge \sim C)$
T	T	F	F	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	F

Outline for Proof by Contradiction

Proposition P .

Proof. Suppose $\sim P$.

\vdots

Therefore, $C \wedge \sim C$. ■

Compare Contrapositive with Contradiction

Outline for Contrapositive Proof

Proposition If P , then Q .

Proof. Suppose $\sim Q$.

\vdots

Therefore, $\sim P$. ■

Outline for Proof by Contradiction

Proposition P .

Proof. Suppose $\sim P$.

\vdots

Therefore, $C \wedge \sim C$. ■

Proving a Conditional Statement with Contradiction

Proposition If P , then Q .

Proof. Suppose P and $\sim Q$.

\vdots

Therefore, $C \wedge \sim C$. ■