Book of Proof: Part IV, Relations, Functions, and Cardinality

January 17, 2018

Relations

$$5 < 10 \quad 3 < 12 \quad 99 < 999$$

$$5 \not< 5 \quad 12 \not< 3 \quad 10 \not< 0$$

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$$5 < 10 \quad 3 < 12 \quad 99 < 999$$

$$5 \not< 5 \quad 12 \not< 3 \quad 10 \not< 0$$

$$R = \{(5,10),(3,12),(99,999),\ldots\}$$

$$(5,10) \in R \quad (3,12) \in R \quad (99,999) \in R$$

$$(5,5) \notin R \quad (12,3) \notin R \quad (10,0) \notin R$$

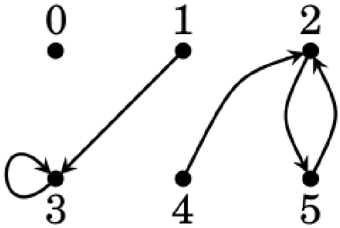
Relations

Definition 11.1 A **relation** on a set A is a subset $R \subseteq A \times A$.
We abbreviate $(x, y) \in R$ as xRy.

Relations in Pictures

Let $B = \{0, 1, 2, 3, 4, 5\}$ and

$$U = \{(1, 3), (3, 3), (5, 2), (2, 5), (4, 2)\} \subseteq B \times B$$



Properties of Relations

Definition 11.2 Suppose R is a relation on set A.

1. R is **reflexive** if xRy for every $x \in A$.

$$\forall x \in A, xRx$$

2. R is **symmetric** if xRy implies yRx for all $x, y \in A$.

$$\forall x, y \in A, xRy \Rightarrow yRx$$

3. R is **transitive** if xRy and yRz imply xRz.

$$\forall x, y, z \in A, ((xRy) \wedge (yRz)) \Rightarrow xRz$$

Pictures of Relation Properties

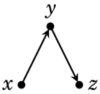
1. A relation is **reflexive** if for each point x …

…there is a loop at x:

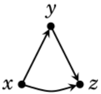
2. A relation is **symmetric** if whenever there is an arrow from x to y …

…there is also an arrow from y back to x:

3. A relation is **transitive** if whenever there are arrows from x to y and y to z …



…there is also an arrow from x to z:



(If $x = z$, this means that if there are arrows from x to y and from y to x …

…there is also a loop from x back to x.)

Relations on $\mathbb{Z}$

Relations on $\mathbb{Z}$:	$<$	$\leq$	$=$	$\mid$	$\nmid$	$\neq$
Reflexive	no	yes	yes	yes	no	no
Symmetric	no	no	yes	no	no	yes
Transitive	yes	yes	yes	yes	no	no

Equivalence relations

Definition 11.3 A relation R on a set A is an **equivalence relation** if it is symmetric, reflexive, and transitive.

Equivalence relations

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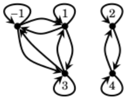
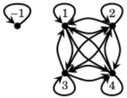
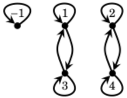
Definition 11.4 Suppose R is an equivalence relation on set A. Given any element $a \in A$, the **equivalence class containing** a is the subset $\{x \in A : xRa\}$ of A consisting of all elements of A that relate to a.
This set is denoted $[a]$:

$$[a] = \{x \in A : xRa\}$$

Pictures of equivalence relations

Relation R	Diagram	Equivalence classes (see next page)
"is equal to" $(=)$ $R_1 = \{(-1,-1),(1,1),(2,2),(3,3),(4,4)\}$		$\{-1\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$
"has same parity as" $R_2 = \{(-1,-1),(1,1),(2,2),(3,3),(4,4),$ $(-1,1),(1,-1),(-1,3),(3,-1),$ $(1,3),(3,1),(2,4),(4,2)\}$		$\{-1,1,3\}$, $\{2,4\}$
"has same sign as" $R_3 = \{(-1,-1),(1,1),(2,2),(3,3),(4,4),$ $(1,2),(2,1),(1,3),(3,1),(1,4),(4,1),$ $(2,3),(3,2),(2,4),(4,2),(1,3),(3,1)\}$		$\{-1\}$, $\{1,2,3,4\}$
"has same parity and sign as" $R_4 = \{(-1,-1),(1,1),(2,2),(3,3),(4,4),$ $(1,3),(3,1),(2,4),(4,2)\}$		$\{-1\}$, $\{1,3\}$, $\{2,4\}$

Congruence as equivalence relations

Example 11.8 proved that $\equiv \pmod{n}$ is an equivalence relation.

$$xRy = \{(x, y) : x \equiv y \pmod{3}\}$$

$$\begin{aligned}
[0] &= \{x \in \mathbb{Z} : x \equiv 0 \pmod{3}\} \\
&= \{x \in \mathbb{Z} : 3 \mid (x - 0)\} = \{x \in \mathbb{Z} : 3 \mid x\} \\
&= \{\ldots, -6, -3, 0, 3, 6, 9, \ldots\} = [3] = [6] \\
[1] &= \{x \in \mathbb{Z} : x \equiv 1 \pmod{3}\} \\
&= \{x \in \mathbb{Z} : 3 \mid (x - 1)\} \\
&= \{\ldots, -5, -2, 1, 4, 7, 10, \ldots\} = [4] = [7] \\
[2] &= \{x \in \mathbb{Z} : x \equiv 2 \pmod{3}\} \\
&= \{x \in \mathbb{Z} : 3 \mid (x - 2)\} \\
&= \{\ldots, -4, -1, 2, 5, 8, 11, \ldots\} = [5] = [7]
\end{aligned}$$

Partitions

Definition 11.5 A **partition** of a set A is a set of non-empty subsets of A, such that the union of all the subsets equals A, and the intersection of any two different subsets is $\emptyset$.

$\{[0],[1],[2]\}$ under the relation $\equiv \pmod 3$, is a partition of $\mathbb{Z}$:

$$\{[0],[1],[2]\} = \{\{\ldots,0,3,6,\ldots\},\{\ldots,1,4,7,\ldots\},\{\ldots,2,5,8,\ldots\}\}$$

Equivalence Relations and Partitions

Theorem 11.2 Suppose R is an equivalence relation on set A. The the set $\{[a] : a \in A\}$ of equivalence classes of R forms a partition of A.

Conversely, any parition of A describes an equivalence relation R where xRy if and only if x and y belong to the same set in the parition.

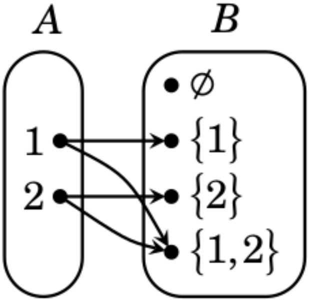
The Integers Modulo n

$$
\begin{aligned}
[0] &= \{x \in \mathbb{Z} : x \equiv 0 \pmod 5)\} &= \{x \in \mathbb{Z} : 5 \mid (x-0)\} &= \{\ldots, -10, -5, 0, 5, 10, 15, \ldots\}, \\
[1] &= \{x \in \mathbb{Z} : x \equiv 1 \pmod 5)\} &= \{x \in \mathbb{Z} : 5 \mid (x-1)\} &= \{\ldots, -9, -4, 1, 6, 11, 16, \ldots\}, \\
[2] &= \{x \in \mathbb{Z} : x \equiv 2 \pmod 5)\} &= \{x \in \mathbb{Z} : 5 \mid (x-2)\} &= \{\ldots, -8, -3, 2, 7, 12, 17, \ldots\}, \\
[3] &= \{x \in \mathbb{Z} : x \equiv 3 \pmod 5)\} &= \{x \in \mathbb{Z} : 5 \mid (x-3)\} &= \{\ldots, -7, -2, 3, 8, 13, 18, \ldots\}, \\
[4] &= \{x \in \mathbb{Z} : x \equiv 4 \pmod 5)\} &= \{x \in \mathbb{Z} : 5 \mid (x-4)\} &= \{\ldots, -6, -1, 4, 9, 14, 19, \ldots\}.
\end{aligned}
$$

$$\mathbb{Z}_5 = \{[0], [1], [2], [3], [4]\}$$

Relations Between Sets

Definition 11.7 A **relation** from a set A to a set B is a subset $R \subseteq A \times B$.
We abbreviate the statement $(x, y) \in R$ as xRy.



Functions