3DMR: Vectors (reminder)

What is the space?

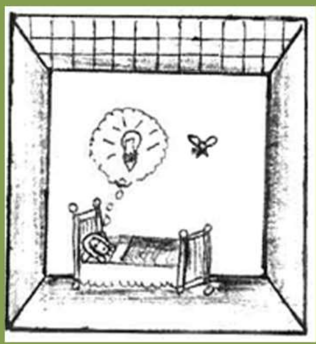
Space: We will not define it for **what it is**

… we will define it for **what it contains**

Defining the 3D space

- **Cartesian coordinates**
  - French philosopher **René Descartes** was lying in bed when a fly went in his room.
  - Descartes' thought:

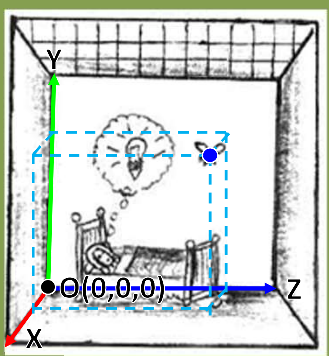
    The **position of the fly** in space can be **defined using 3 numbers,** relative to a given point.



  - This way, **Descartes conceived the coordinate system** that takes his name, and published in 1637.

Defining the 3D space

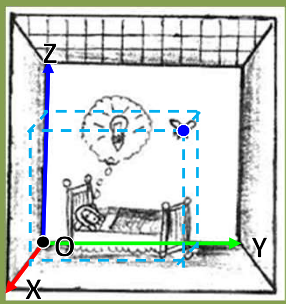
- **Cartesian coordinates:** The **position of the fly** in space can be **defined using 3 numbers,** relative to **a given point**.



- Given an origin point (O)
- And 3 axis (X,Y,Z)
- We can define the position of any point in 3D space (P)
- …but the origin is just as important as the axis!

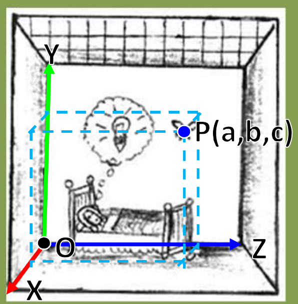
Defining the 3D space

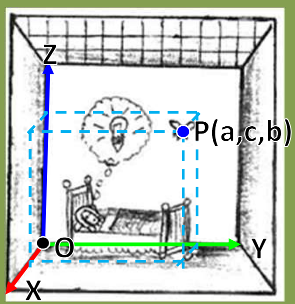
- **Cartesian coordinates:** The **position of the fly** in space can be **defined using 3 numbers,** relative to a given point.
  - If we change the axis, coordinates of P change!



Defining the 3D space

- **Cartesian coordinates:** The **position of the fly** in space can be **defined using 3 numbers,** relative to a given point.



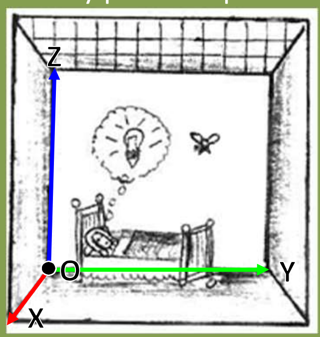


- A point is always a point (P)

  ...but our definition of space, affects how we represent it

Defining the 3D space

- **Cartesian coordinates:**
- A **system of reference** allows us to describe any point in space.
- To do so, we need:
  - An origin point (O)
  - Three axis (X, Y and Z)
  - A unit (cm, meters, feet)
- This allows us to use $\mathbb{R}^3$ to represent any point in space ($P(x,y,z) \in \mathbb{R}^3$).
- We **do not define the space** itself…



….we use a **mathematical construct**, to represent **anything that exists in 3D space**.

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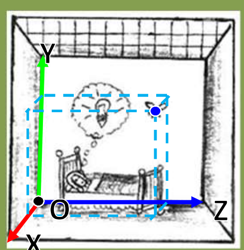
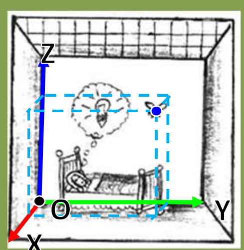
Defining the 3D space

- **Cartesian coordinates:** Takeaways…
- We do not represent the space, we represent **its contents**.
- A **system of reference** allows us to describe any point in space.
- Using a different system of reference → different definition of **the contents** in the space.
  - The space is still the same
  - A point in space is still a point in space





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Defining the 3D space

- **Cartesian coordinates:** Takeaways…
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- Using a different system of reference → different definition of **the contents** in the space.
  - The space is still the same
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- If space is just a **mathematical construct…**

…let's see **how to use it**!!

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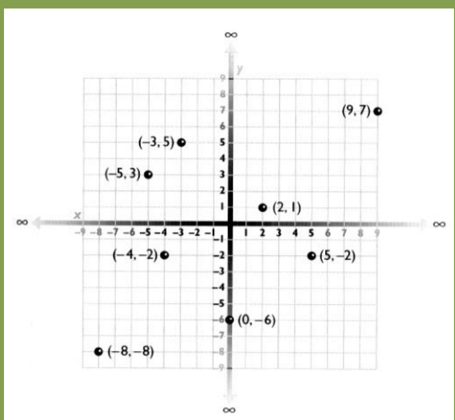
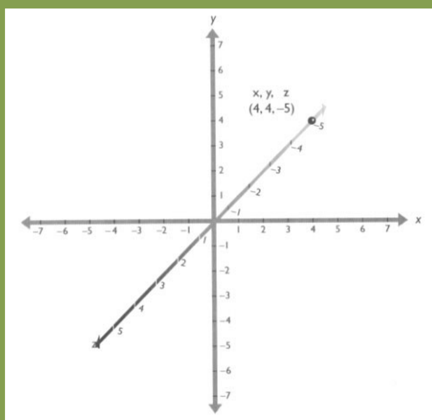
Using the 3D space: Systems of Reference (~ Coordinate Systems)

- 2D Cartesian plane and 3D Cartesian space





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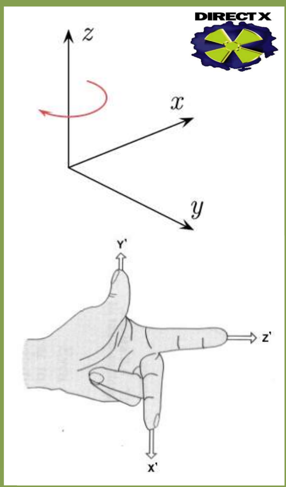
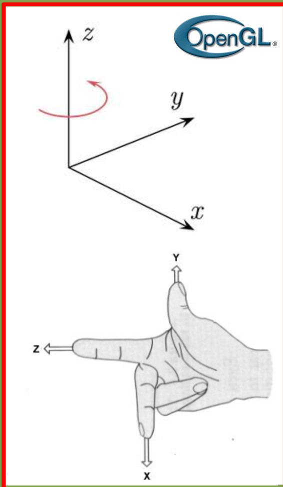
Using the 3D space: Systems of Reference

- Two ways of defining them:



Left handed

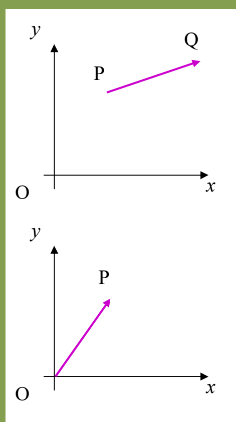


Right handed

Using the 3D Space: Points and vectors

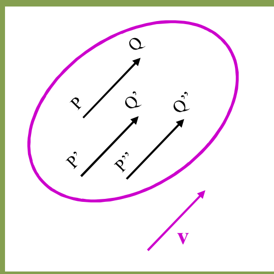
- **Point:** represents a position in space (in a system of reference)
  - We will use **capital letters** to represent points: P, Q, etc …
- **Vector:** difference between the position of two points.
  - **Fixed vector** $\overrightarrow{PQ}$:
    - Initial point (tail) at P
    - End point (head) at Q.
  - **Position vector** $\overrightarrow{OP}$:
    - Tail is the origin O
    - Head at P
- **Point and Vectors are not the same**

  … but are represented in the same way ($\mathbb{R}^3$)



Using the 3D Space: Points and vectors

- Vectors having the same length and direction are called **equipollent**.
- Equipollent vectors represent the same **free vector** (i.e. same distance/ displacement between two points).



- **Notation:** in italics u, v, w
  in bold $\mathbf{u}, \mathbf{v}, \mathbf{w}$
  with arrow $\vec{u}, \vec{v}, \vec{w}$
- **Numeric values:** (1, 2, 3) [1 2 3] $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

components

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Using the 3D Space: vector operations

- Multiplying a vector by a scalar:
  - We will use letters of the Greek alphabet to represent real numbers (scalars): α, β, etc…

$$\alpha \cdot \mathbf{u} = (\alpha \cdot u_1, \alpha \cdot u_2, \alpha \cdot u_3) \in \mathbb{R}$$

  - Same direction (opposite, if the scalar α is negative)
  - Algebraic properties:
    - **Distributive law:** $(\alpha+\beta) \cdot \mathbf{u} = \alpha \cdot \mathbf{u} + \beta \cdot \mathbf{u}$
      $\alpha \cdot (\mathbf{u}+\mathbf{v}) = \alpha \cdot \mathbf{u} + \alpha \cdot \mathbf{v}$
    - **Associative law:** $\alpha \cdot (\beta \cdot \mathbf{u}) = (\alpha \cdot \beta) \cdot \mathbf{u}$
    - **Neutral/Identity element:** $1 \cdot \mathbf{u} = \mathbf{u}$

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Using the 3D Space: vector operations

- **Magnitude of a vector** (i.e. length):
  - The **magnitude** of a vector $\mathbf{v}$ of dimension n is denoted $\|\mathbf{v}\|$ and the result is a positive real number:

$$\|\mathbf{v}\| = \|(v_1, v_1, \ldots, v_n)\| = \sqrt{{v_1}^2 + {v_2}^2 + \cdots + {v_n}^2}$$

- **Normalizing a vector**:

$$\mathbf{v}' = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

$\mathbf{v}'$ has a magnitude of 1, it is a **unit vector**

- **Used in combination:** length and direction of $\mathbf{v}$

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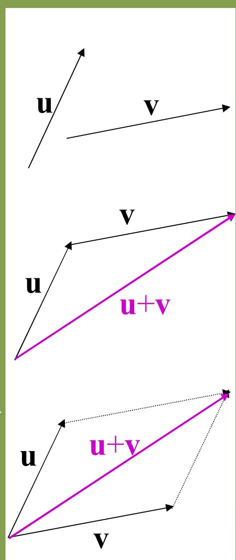
Using the 3D Space: vector operations

- **Vector addition**

$$\mathrm{u} = (u_1, u_2, u_3)$$
$$\mathrm{v} = (v_1, v_2, v_3)$$
$$\mathrm{u} + \mathrm{v} = (u_1+v_1, u_2+v_2, u_3+v_3) \in \mathbb{R}^3 \text{ (a vector)}$$

- Add displacements→ a new displacement
- Parallelogram law
- Algebraic properties:
  - **Commutative law:** $\mathrm{u}+\mathrm{v} = \mathrm{v}+\mathrm{u}$
  - **Associative law:** $\mathrm{u}+(\mathrm{v}+\mathrm{w}) = (\mathrm{u}+\mathrm{v})+\mathrm{w}$
  - **Neutral element($\vec{0}$):** $\mathrm{v}+\vec{0} = \vec{0}+\mathrm{v} = \mathrm{v}$
  - **Inverse element:** $\mathrm{v}+(\text{-}\mathrm{v}) = \vec{0}$



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Using the 3D Space: vector operations

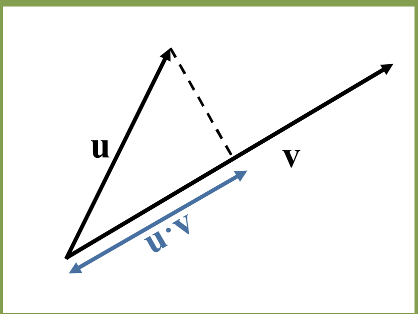
- **Dot (·) product** (i.e. scalar product):
  - The **dot product** of $\mathbf{u}$ and $\mathbf{v}$ is written $\mathbf{u}\cdot\mathbf{v}$ and the result is a real number:

$$\mathbf{u} = (u_1, u_2, u_3) \qquad \mathbf{u}\cdot\mathbf{v} = u_1v_1+u_2v_2+u_3v_3 \in \mathbb{R}$$
$$\mathbf{v} = (v_1, v_2, v_3)$$

  - Algebraic properties:
    - **Commutative law:** $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$
    - **Distributive law (wrt addition):** $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$
    - **Associative law (wrt scalar prod):** $(\alpha\mathbf{u})\cdot(\beta\mathbf{v}) = \alpha\beta(\mathbf{u}\cdot\mathbf{v})$
    - **Null element ($\vec{0}$):** $\mathbf{u}\cdot\vec{0} = \vec{0}\cdot\mathbf{u} = \vec{0}$
    - If $\mathbf{u}$ is orthogonal to $\mathbf{v}$ (written $\mathbf{u}\perp\mathbf{v}$) → $\mathbf{u}\cdot\mathbf{v} = \vec{0}$

Using the 3D Space: vector operations

- **Dot (·) product:** $\mathbf{u}\cdot\mathbf{v} = u_1v_1+u_2v_2+u_3v_3$
- Dot product defines a scalar (value in $\mathbb{R}$)
- What does it represent?



- Projection of $\mathbf{u}$ on $\mathbf{v}$
- If $\mathbf{u}$ and $\mathbf{v}$ are perpendicular:
  - Projection is zero
- We will use it a lot:
  - Compute reflections
  - Compute refractions

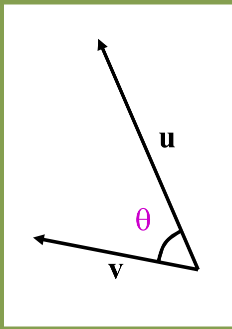
Using the 3D Space: vector operations

- **Dot product and magnitude of a vector**:
  - Additional properties: angle between vectors

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\cdot\|\mathbf{v}\|\cdot\cos\theta \;\Rightarrow\; \cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\cdot\|\mathbf{v}\|}$$

  - If $\|\mathbf{u}\|=1$ and $\|\mathbf{v}\|=1$ (unitary) $\Rightarrow$ $\mathbf{u}\cdot\mathbf{v} = \cos\theta$
  - Also useful:

$$\|\mathbf{v}\| = \sqrt{(\mathbf{v}\cdot\mathbf{v})} \qquad \Rightarrow \qquad \mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2$$



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Using the 3D Space: vector operations

- **Cross (×) product** (i.e. vector product)
  - Def.: The **cross product** of $\mathbf{u}$ and $\mathbf{v}$ is written $\mathbf{u}\times\mathbf{v}$ or $\mathbf{u}\wedge\mathbf{v}$ and the result is a vector that is normal (90º) to the plane containing $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) =$$

$$= (\, u_2v_3 - u_3v_2,\;\; u_3v_1 - u_1v_3,\;\; u_1v_2 - u_2v_1 \,)$$

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Mathematics for computer graphics

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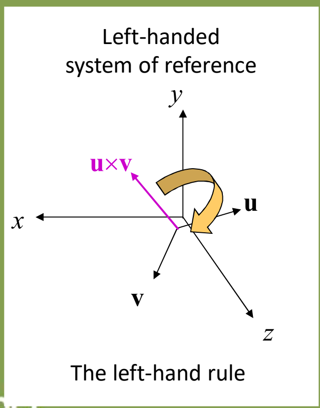
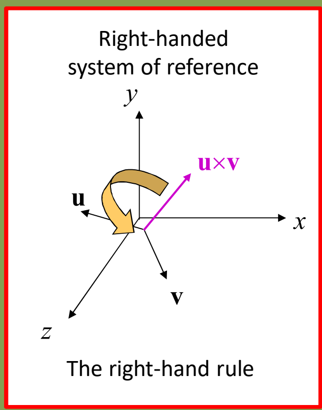
Using the 3D Space: vector operations

- **Cross (×) product** (i.e. vector product)
  - Graphical representation: Normal to **u** and **v**





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Using the 3D Space: vector operations

- **Per component (⊙) product** (Hadamart Product)

$$\mathbf{u} = (u_1, u_2, u_3) \qquad \mathbf{u} \odot \mathbf{v} = (u_1 v_1\,, u_2 v_2\,, u_3 v_3) \in \mathbb{R}$$
$$\mathbf{v} = (v_1, v_2, v_3)$$

- Algebraic properties:
  - **Commutative:** $\mathbf{u} \odot \mathbf{v} = \mathbf{v} \odot \mathbf{u}$
  - **Distributive (wrt addition):** $\mathbf{u} \odot (\mathbf{v}+\mathbf{w}) = (\mathbf{u} \odot \mathbf{v})+(\mathbf{u} \odot \mathbf{w})$
  - **Associative:** $\mathbf{u}\odot(\mathbf{v}\odot\mathbf{w}) = (\mathbf{u}\odot\mathbf{v}) \odot \mathbf{w}$
- Limited application in 3D/Geometry → Some use for shaders
  - **… carefull!!** Default vector multiplication in glm uses Hadamart Product

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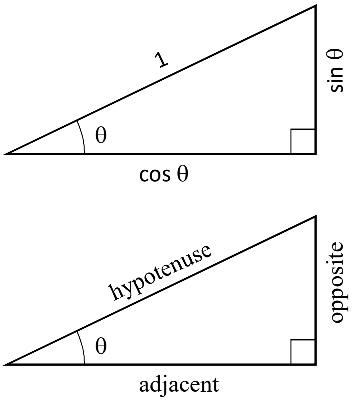
Using the 3D Space: vector operations

- Vector and dot product → compute sin θ and cos θ
- **Very useful!** Let’s remind some trigonometric properties:



$$\sin^2\theta + \cos^2\theta = 1$$

$$\text{tag}\,\theta = \frac{\sin\theta}{\cos\theta}$$

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\text{tag}\,\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Using the 3D Space: vector operations

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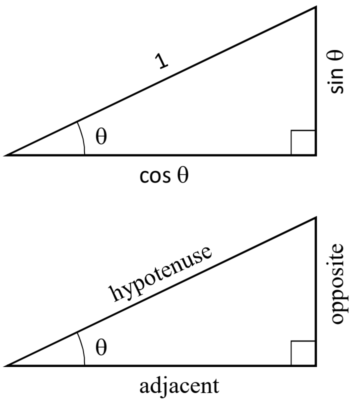

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$$\text{tag}\,\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Trigonometry (II)

- Trigonometric ratios and relationships (cont'd)

	0º	30º	45º	60º	90º
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tag	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞

- Compound angles

$$\sin(\alpha+\beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$$

Using the 3D Space: vector operations

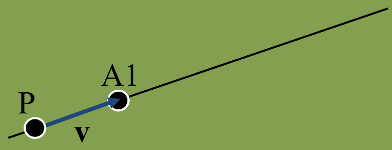
- Linear combination
  - Def.: A **linear combination** of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$,..., $\mathbf{v}_n$ is an expression constructed by multiplying each vector by a constant and adding the results (α_i is a real number)

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$$

- **Lines** and **planes** are a linear combination
  - **Line**: Linear combination of 1 point (origin) and 1 vector
  - **Plane**: Linear combination of 1 point (origin) and 2 vectors
  - **3D Space**: Linear combination of??
    - Let's keep this in mind for the future…

Using the 3D Space: defining lines

- A point P $(p_1, p_2, p_3, 1)$ and a vector $\mathbf{v}$ $(v_1, v_2, v_3, 0)$ define a line
- We want to describe any point A $(x, y, z, 1)$ in this line:
  - We have a line
  - A point in this line P
  - And a vector
  - ...and we need a helper scalar λ



- We can define any point A, with the help of scalar product:

$$\mathbf{A1} = \mathrm{P} + 1 \cdot \mathbf{v}$$

$$\mathbf{A} = \mathrm{P} + \lambda \cdot \mathbf{v}$$

Using the 3D Space: defining lines

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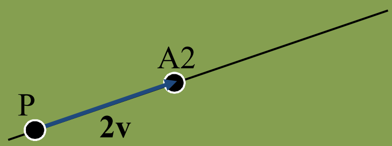

- We can define any point A, with the help of scalar product:

$$\mathbf{A1} = \mathrm{P} + 1 \cdot \mathbf{v}$$

$$\mathbf{A} = \mathrm{P} + \lambda \cdot \mathbf{v} \qquad \mathbf{A2} = \mathrm{P} + 2 \cdot \mathbf{v}$$

Using the 3D Space: defining lines

- A point P $(p_1, p_2, p_3, 1)$ and a vector $\mathbf{v}$ $(v_1, v_2, v_3, 0)$ define a line
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  - We have a line
  - A point in this line P
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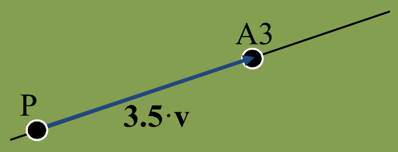
  - …and we need a helper scalar λ



- We can define any point A, with the help of scalar product:

$$\mathbf{A} = \mathrm{P} + \lambda \cdot \mathbf{v} \qquad \begin{aligned} \mathbf{A1} &= \mathrm{P} + 1 \cdot \mathbf{v} \\ \mathbf{A2} &= \mathrm{P} + 2 \cdot \mathbf{v} \\ \mathbf{A3} &= \mathrm{P} + 3.5 \cdot \mathbf{v} \end{aligned}$$

- For any point, there is a value of λ that satisfies: $\mathbf{A} = \mathrm{P} + \lambda \cdot \mathbf{v}$

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Using the 3D Space: defining lines

- This is the vector equation of a line: $\mathbf{A} = \mathrm{P} + \lambda \cdot \mathbf{v}$

$$\mathbf{A} = (p_1,\ p_2,\ p_3, 1) + \lambda \cdot (v_1,\ v_2,\ v_3, 0)$$

- It can be useful to represent it in other ways:

Parametric form

$$x = p_1 + \lambda v_1$$
$$y = p_2 + \lambda v_2$$
$$z = p_3 + \lambda v_3$$

$$\lambda = \frac{x - p_1}{v_1}$$
$$\lambda = \frac{y - p_2}{v_2}$$
$$\lambda = \frac{z - p_3}{v_3}$$

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Using the 3D Space: defining lines

- This is the vector equation of a line: $\mathbf{A} = \mathrm{P} + \lambda\cdot\mathbf{v}$

$$\mathbf{A} = (p_1,\ p_2,\ p_3,1) + \lambda\cdot(v_1,\ v_2,\ v_3,0)$$

- It can be useful to represent it in other ways:

Parametric form		Continuous equations		
$x = p_1 + \lambda v_1$ $y = p_2 + \lambda v_2$ $z = p_3 + \lambda v_3$	⇔	$\frac{x-p_1}{v_1} = \frac{y-p_2}{v_2} = \frac{z-p_3}{v_3}$	⇔	$v_2\cdot(x-p_1) - v_1\cdot(y-p_2) = 0$ $v_3\cdot(x-p_1) - v_1\cdot(z-p_3) = 0$

Using the 3D Space: defining lines

- This is the vector equation of a line: $\mathbf{A} = \mathrm{P} + \lambda\cdot\mathbf{v}$

$$\mathbf{A} = (p_1,\ p_2,\ p_3\ ,1) + \lambda\cdot(v_1,\ v_2,\ v_3,0)$$

- It can be useful to represent it in other ways:

Parametric form		Continuous equations		Implicit equations
$x = p_1 + \lambda v_1$ $y = p_2 + \lambda v_2$ $z = p_3 + \lambda v_3$	⇔	$\frac{x-p_1}{v_1} = \frac{y-p_2}{v_2} = \frac{z-p_3}{v_3}$	⇔	$v_2\cdot x - v_1\cdot y + (v_1\cdot p_2 - v_2\cdot p_1) = 0$ $v_3\cdot x - v_1\cdot z + (v_1\cdot p_3 - v_3\cdot p_1) = 0$

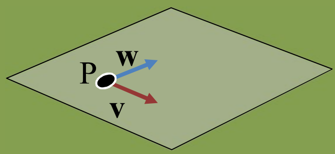
- They are all equivalent and represent the same

  …but some times a representation is more useful than other.

Using the 3D Space: defining planes

- Two vectors and a point represent a plane:

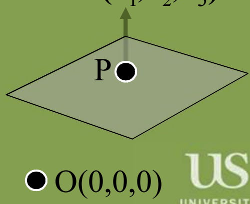
  P (p_1, p_2, p_3)
  v (v_1, v_2, v_3)
  w (w_1, w_2, w_3)



- **Vector equation:** $A = P + \lambda \cdot \mathbf{v} + \mu \cdot \mathbf{w}$
  - Scalars λ and μ will help us "**reach**" points A in the plane

Parametric form	Implicit equation
$x = p_1 + \lambda v_1 + \mu w_1$	$n_1 x + n_2 y + n_3 z - D = 0$
$y = p_2 + \lambda v_2 + \mu w_2$	• Unit normal vector **n**
$z = p_3 + \lambda v_3 + \mu w_3$	• Point **P**

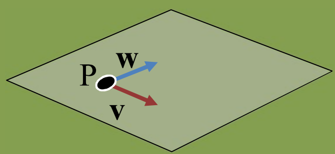
$\mathbf{n} = (n_1, n_2, n_3)$



Using the 3D Space: defining planes

- Two vectors and a point represent a plane:

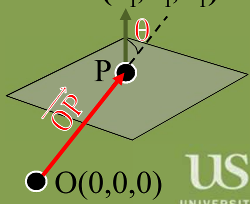
  P (p_1, p_2, p_3)
  v (v_1, v_2, v_3)
  w (w_1, w_2, w_3)



- **Vector equation:** $A = P + \lambda \cdot \mathbf{v} + \mu \cdot \mathbf{w}$
  - Scalars λ and μ will help us "**reach**" points A in the plane

Parametric form	Implicit equation
$x = p_1 + \lambda v_1 + \mu w_1$	$n_1 x + n_2 y + n_3 z - D = 0$
$y = p_2 + \lambda v_2 + \mu w_2$	• D ~ dist. from P to O
$z = p_3 + \lambda v_3 + \mu w_3$	$D = \overrightarrow{OP} \cdot \mathrm{n} = \lVert\overrightarrow{OP}\rVert \cdot \cos\theta$

$\mathbf{n} = (n_1, n_1, n_1)$



3DMR: Vectors (reminder)

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