Exercise 1

a) n couples
Each couple : 2 kids

P(All couples have 1 daughter (F) and 1 son (M)) (1)

Since we assume that each gender (male or female) has the same chance of occuring, and we assume the events are independent, then we have:

(1) Would become for 1 couple: P(MF or FM) where MF is having a son first, daughter second, and FM the reverse

So $P(MF \text{ or } FM) = P(M) \cdot P(F) + P(F) \cdot P(M) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} =$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Lastly, we assume that the birth and genders of other couples are independent events, thus:

$$(1) \rightarrow \left(P(MF \text{ or } FM)\right)^n = \left(\frac{1}{2}\right)^n$$

b) Fair coin: $P(H) = \frac{1}{2}$, $P(T) = \frac{1}{2}$

H : Heads
T : Tails

flips: n ~~rolls~~

P(Heads n times) (1);

We assume each flip is an independent event :

$(1) \rightarrow P(H_1) \cdot P(H_2) \cdots \cdot P(H_n) = \frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2} = \left(\frac{1}{2}\right)^n$, where H_1 first ~~roll~~ flip

H_2 : second flip

H_n : n_{th} flip

c) 3 colours : Red , Green , Blue

$$\text{Red} \downarrow 30\% = P(\text{Red}) \qquad \text{Green} \downarrow 50\% = P(\text{Green}) \qquad \text{Blue} \downarrow 20\% = P(\text{Blue})$$

$\frac{1}{2}$ of Red : Hollow $= P(H|\text{Red})$

$\frac{2}{3}$ of Blue : Hollow $= P(H|\text{Blue})$

$\frac{2}{3}$ of Green : Hollow $= P(H|\text{Green})$

P(Hollow bead); (Randomly pick)

Picking any colour is individual event, so:

$\cancel{P(\text{Hollow bead}) = P(H|\text{Red}) \cdot P + P(H|\text{Green}) + P(H|\text{Blue})}$

$\cancel{P(H|\text{Red})}$

$$P(\text{Hollow bead}) = \overset{1/2}{P(H|\text{Red})} \cdot \overset{30\% = \frac{3}{10}}{P(\text{Red})} + \overset{2/3}{P(H|\text{Green})} \, \overset{50\% = \frac{1}{2}}{P(\text{Green})} + \overset{\frac{2}{3}}{P(H|\text{Blue})} \cdot \overset{\frac{1}{5}}{P(\text{Blue})} =$$

$$= \frac{1}{2} \cdot \frac{3}{10} + \frac{2}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{5} = \frac{3}{20} + \frac{2}{6} + \frac{2}{15} \approx 0{,}6166\ldots$$

Exercise 2

a)
P(photon passes Earth atmosphere) = P(Photon) = 1e-7
Since P(Photon) + P(Not Photon) = 1, then P(Not Photon) = 1 - 1e-7
Detector FPR : 10% --> P(Detection | Not Photon)
Detector TPR : 85% --> P(Detection | Photon)

We want to find P(Photon | Detection) (1)
Using Bayes theorem on (1) we get that:
(1) = P(Detection | Photon)*P(Photon) / P(Detection)
Also we can calculate P(Detection | Photon) = 0.85, P(Photon) = 1*(10**-7), so we just need to find P(Detection).

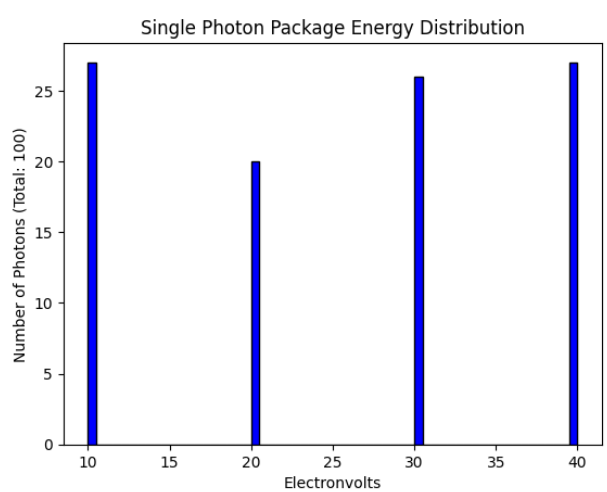
Let Photon + Not_Photon be the only two possible outcomes (We assume mutually exclusive and exhaustive events)
Then from the Law of Total Probability we can say that:
P(Detection) = P(Detection | Photon)*P(Photon) + P(Detection | Not Photon)*P(Not Photon)
We know all the above terms so we just need to substitute now. We just pass variables and equations in the script and calculate results.

Using the script we calculate:
P(Detection) = 0.10000007500000001
and
P(Photon | Detection) = 8.49999362500478e-07 ≈ 0.0000008500

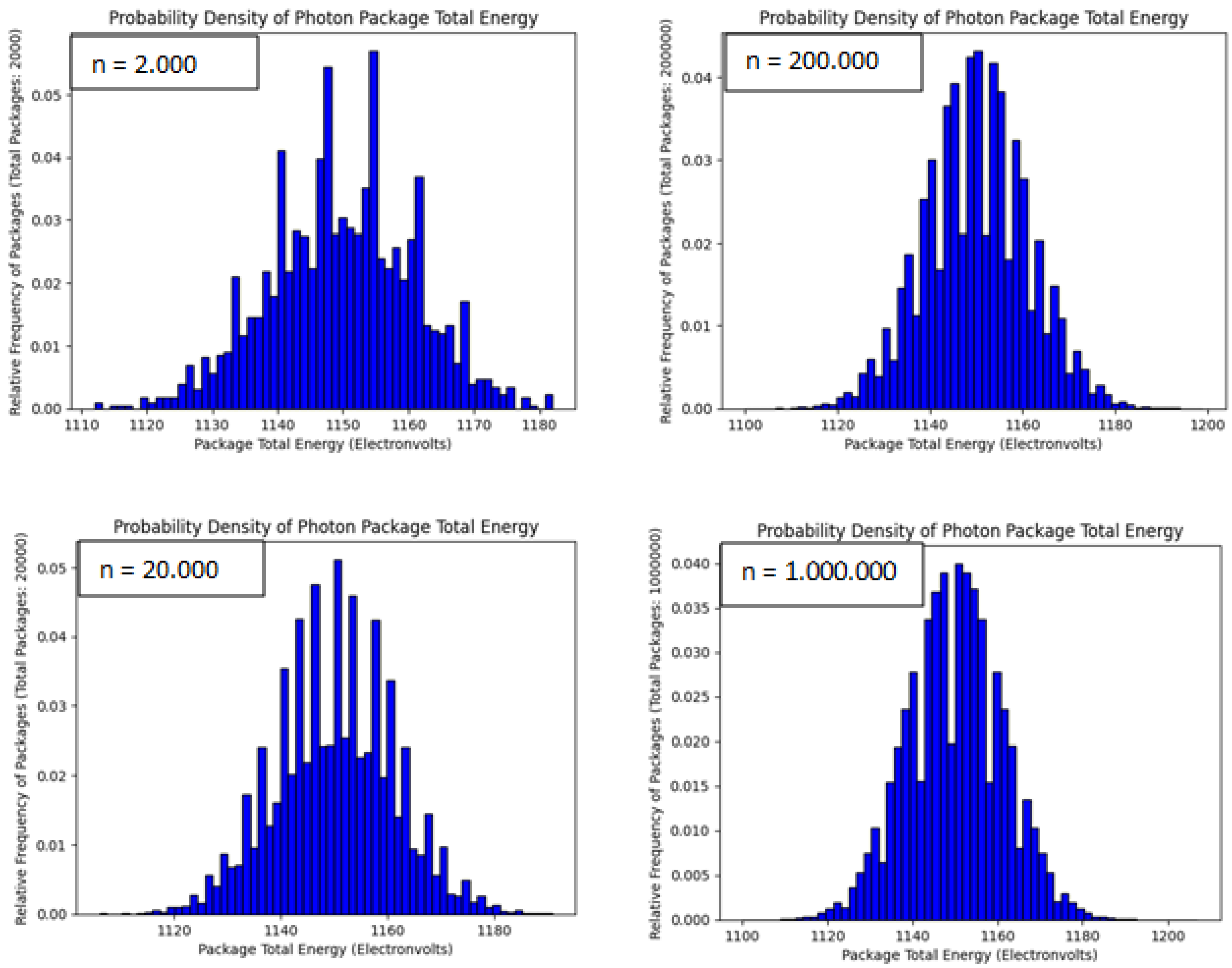
b)
A photon package has 100 photons
Photon possible energies (PE) are 10, 20, 30 or 40 (Electrovolts)

Since each photon has an equal probability to carry any one of the energy values above, we can say that P(PE_10) = P(PE_20) = P(PE_30) = P(PE_40) = 25% = 0.25

Plotting the energy distribution for a single photon package, the distribution seems to indeed resemble uniform distribution (y_axis: Counts for each energy, x_axis: energy level).



To find out what distribution the sum of the energies of photons within a photon package follows, we will uniformly sample energies for different package counts (2.000, 20.000, 200.000 and 1.000.000) and each time we sample, we will sum the energies creating a new vector of summed energies. To visualize this, we will use a histogram with normalized y_axis, so that it forms a probability density (y_axis: Relative frequency of package energy sum, x_axis: Package total energy).



While for lower package counts (***n***) we notice more “spikes” (frequency of certain summed photon energy) in the distributions, when package count increases, they resemble more and more with a normal distribution. This is due to the Central Limit Theorem (CLT), because in theory, the sum of a large number of independent and uniformly distributed random variables, no matter the shape of the original distribution (in our case it was uniform), will be approximately normally distributed.

c)

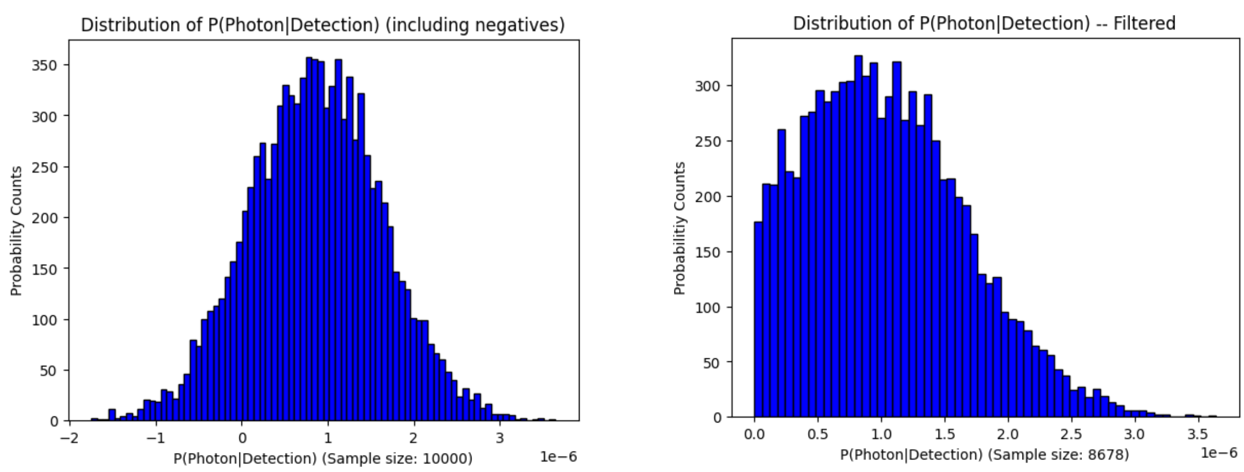
Now we assume the probability of a photon reaching the ground is not fixed (1e-7) but rather follows a normal distribution *N*(μ,σ) with **μ = 1e-7** (mean of normal distribution) and **σ = 9e-8** (standard deviation of normal distribution).

The TPR and FPR do not change, so:
P(Detection | Photon) = 0.85 (True Positive Rate)
PDetection | Not Photon) = 0.1 (False Positive Rate)

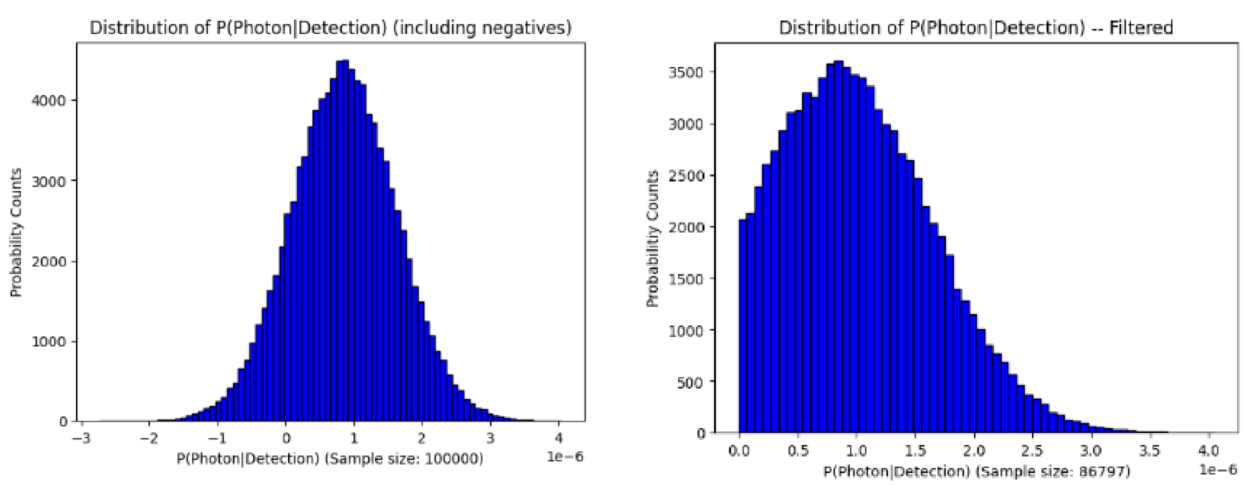
First we randomly sample from the distribution *N*(μ=1e-7, σ=9e-8) for different sample sizes ***n*** (10.000 , 100.000, 1.000.000) to check the impact in our results. After that, we try two different approaches. Both will make sure no negative values are within, since we are talking about probabilities, which cannot be negative. The first approach is to first sample from our normal distribution (***n*** times, for all our different sample sizes) and then filter out the negative values. What we expect is to see a normal distribution starting from 0 with no left tail (no negative values). Below are plots for ***n*** =10.000 (A1), ***n*** =100.000 (B) and ***n*** =1.000.000 (C1).

On the left side is the unfiltered distribution (including negative values) and on the right the same distribution after removing negative values. The amount of values removed is denoted on x_axis label. For the visualization of both approaches, ***density = False*** is chosen as a parameter.
The x_axis is scaled with 1e-6 as seen on the bottom right of the plots.

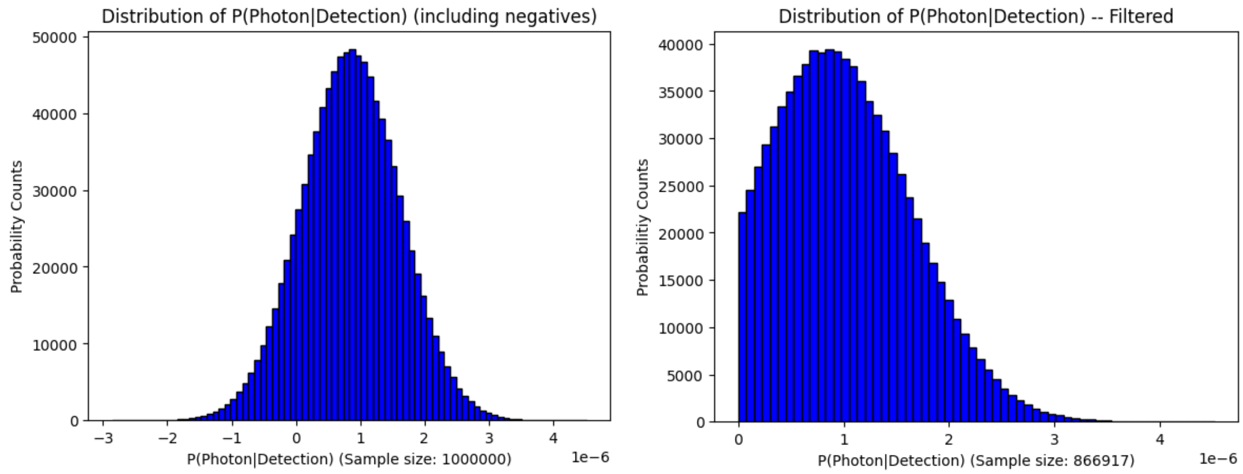
A1)



B1)

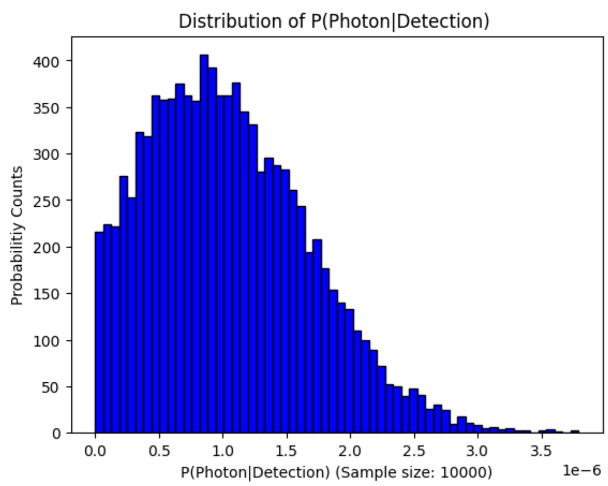


C1)

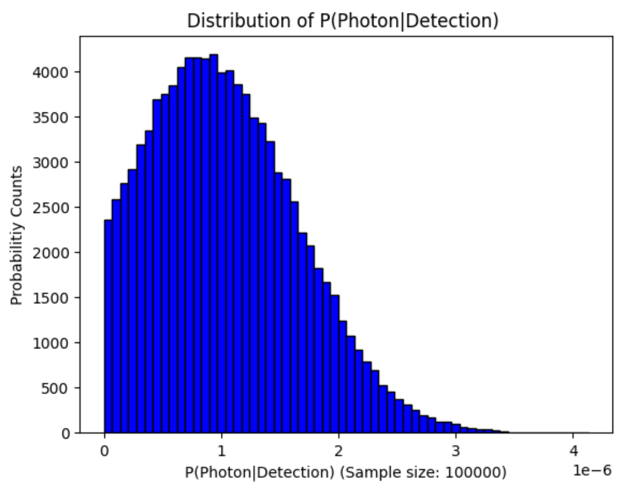


The second approach keeps sampling from our normal distribution until we get only positive values. We can do this by removing negative values after the first sampling, and then re-sample for the amount we filtered out (from the same distribution) until we get to ***n*** P(Photon | Detection) values in our vector. We can then visualize the results using a histogram. Below are plots for ***n*** =10.000 (A2), ***n*** =100.000 (B2) and ***n*** =1.000.000 (C2). The x_axis is scaled with 1e-6 as seen on the bottom right of the plots.

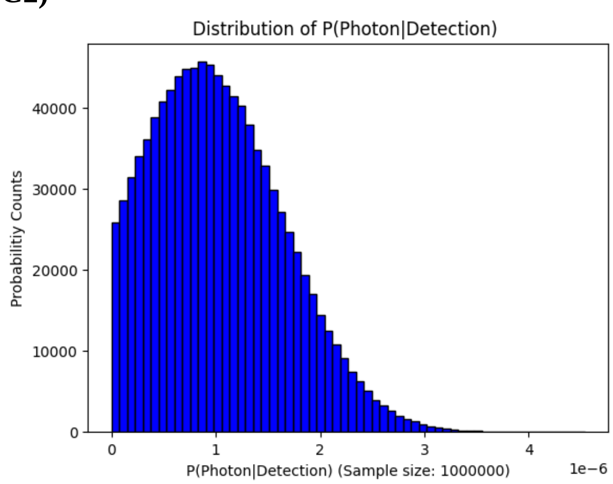
A2)



B2)



C2)



The plots represent the probability of a photon having reached the ground if the detector reported a detection. When comparing the two approaches, we notice the same truncated normal distribution (on the non-negative side) with higher probability counts values with the latter approach (since we have more samples than when filtering). As the sample size ***n*** increases, the histogram resembles more with a normal distribution (truncated). Even though we replace negative values by resampling from $N(\mu,\sigma)$, we do not notice difference and even more so as mentioned and ***n*** increases which is due to the Law of Large Numbers (LLN), converging towards the true mean of the distribution.