Chapter 3

Complex Numbers Continued

In the previous chapter we introduced the concept of imaginary numbers and formed a new number system called complex numbers. We learned how to add, subtract, multiply and divide these numbers and how to represent them graphically.

In this chapter we are going to delve further into some interesting properties of complex numbers and consider how we represent them.

3.1 Polar and Cartesian Co-ordinate Systems

To specify

- the position of a point on a plane, or
- a complex number

we always need two bits of information, for example

1. (x, y) co-ordinates
2. Real and imaginary parts $x + yi$

This system, where we give the distance along, and the distance up, is known as a Cartesian co-ordinate system (although calculators dumb-down, and call it a 'rectangular' co-ordinate system. Those are not the only options, however.

The most common alternative to Cartesian co-ordinates is a system of **Polar Co-ordinates**. In this, we specify an origin and a single axis, which correspond to the origin and x-axis in the Cartesian system. Then to say where the point is, we give its distance from the origin, r, and the angle of rotation from the axis, θ.

Key features of polar co-ordinates are:

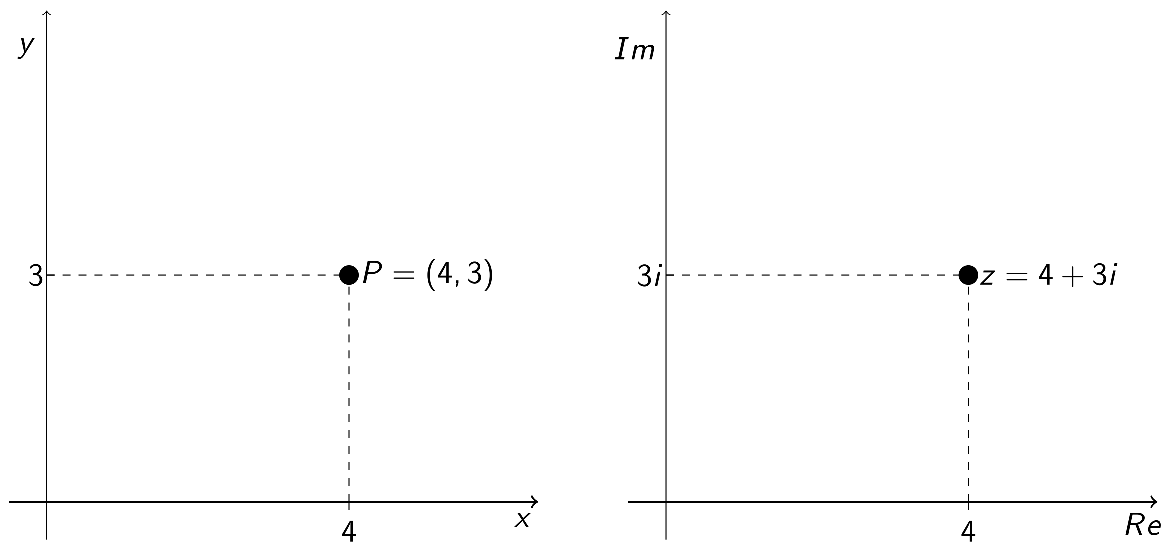


Figure 3.1: Example of complex number drawn in the complex plane

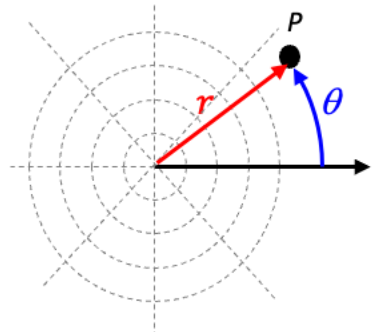


Figure 3.2: Figure showing polar co-ordinates in 2D plane.

- The positive direction for rotation is **anticlockwise**.
- The negative direction for rotation is **clockwise**.
- When we're talking about co-ordinates, we write this (r, θ).
- When we're talking about a complex number, we write $r\angle\theta$.

In our example above we have

$$P = (5, 36.9^\circ) \qquad \text{(3 sig fig)} \quad \text{and} \qquad z = 5\angle 36.9^\circ \qquad \text{(3 sig fig)}$$

Notice that as with Cartesian co-ordinates, we need two bits of information, but here we give the distance from the origin and the angle relative to the horizontal axis.

3.1.1 Modulus and Argument of a Complex Number

For complex numbers, the distance and angle have special names and notation.

Modulus The distance from the origin, r. Denote modulus of $z = |z|$ - vertical lines either side of the complex number.

Argument The angle relative to the axis, θ, denoted $\arg(z)$.

Example 12.

$$|3\angle 45^\circ| = 3 \qquad \text{since} \qquad |r\angle\theta| = r$$

and

$$arg(3\angle 45^\circ) = 45^\circ \qquad \text{since} \qquad arg(r\angle\theta) = \theta$$

Note

1. These names do not apply to co-ordinates of a point, only to complex numbers.
2. Compare the use of 'modulus' with a use you've already met, $|-2| = |2| = 2$. Where are -2 and 2 on the complex plane? How far are they from the origin?

3.1.2 Units for Angles

You know that we have two units for angles, degrees (which we indicate with a little $^\circ$) and radians, which we can indicate by 'rads' or a little c (^{c}).

- A full turn is $360^\circ = 2\pi^c$
- It follows that a half turn is $180^\circ = \pi^c$
- Dividing both sides by π gives $1^c = \left(\frac{180}{\pi}\right)^\circ$ Alternatively, dividing both sides by 180 gives $1^\circ = \left(\frac{\pi}{180}\right)^c$

Example 13.

1. Convert 36.9° from degrees to radians.
2. Convert $\frac{\pi}{2}$ from radians to degrees.

Solution

1. Since $1^\circ = \left(\frac{\pi}{180}\right)^c$, multiplying both sides by 36.9 gives

$$\begin{aligned} 36.9^\circ &= \left(\tfrac{36.9\pi}{180}\right)^c \\ &= 0.644^c \quad \text{3 sig fig (simplifying)} \end{aligned}$$

2. Since $1^c = \left(\frac{180}{\pi}\right)^\circ$, multiplying both sides by $\frac{\pi}{2}$ gives

$$\left(\frac{\pi}{2}\right)^c = \left(\frac{\pi}{2} \times \frac{180}{\pi}\right)^\circ = 90^\circ \quad \text{(simplifying)}$$

When we specify a point in Polar co-ordinates (r, θ) or a complex number in Polar form $r\angle\theta$ it doesn't matter whether we use degrees or radians (as long as we give the units). In our example above

$$\begin{aligned} P &= (5, 36.9^\circ) &= (5, 0.644^c) \quad \text{3 sig fig} \\ z &= 5\angle 36.9^\circ &= 5\angle 0.644^c \quad \text{3 sig fig} \end{aligned}$$

However, some maths (crucially, calculus, and soon, the exponential form of a complex number) work best with radians . For this reason, **mathematicians almost always USE RADIANS**.

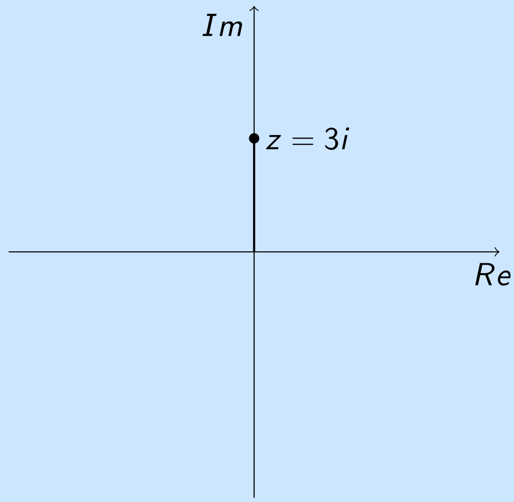
3.2 Converting between Polar and Cartesian form

3.2.1 Part 1: Graphically

For some simple cases, conversion between the two forms is best done with a quick sketch.

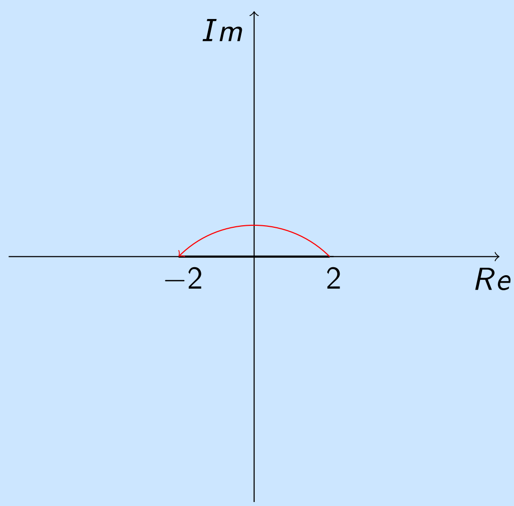
Example 14.

1. Convert $3i$ to polar form



$3i \equiv 3\angle\frac{\pi}{2}$

2. Convert $2\angle\pi$ to Cartesian form



$2\angle\pi \equiv -2$

Exercise 7. **Converting between polar and Cartesian 1**

1. Sketch these numbers on an Argand diagram, and hence convert those that in in polar form to Cartesian, and vice versa (no calculations should be necessary, except, perhaps, in $\sqrt{2}\angle\frac{-\pi}{4}^c$)

   (a) $1\angle 180^\circ$ (b) -2 (c) $8i$ (d) $\sqrt{2}\angle\left(-\frac{\pi}{4}\right)^c$

2. For each of the complex numbers given above, state the modulus and argument. Use statements with the correct notation, for example

$$|1\angle 180^\circ| = 1 \qquad \text{and} \qquad \arg(1\angle 180^\circ) = 180^\circ.$$

3. Now consider the sketch shown here, which shows a complex number

$$\begin{aligned} z &= a + bi \\ &= r\angle\theta. \end{aligned}$$

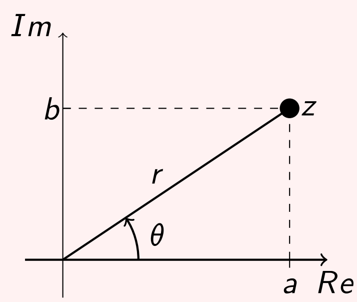
4. Use Pythagoras and trigonometry to find relationships (ie formulae) for:

   (a) $Re(z) = a$ in terms of r and θ

   (b) $Im(z) = b$ in terms of r and θ

   (c) $|z| = r$ in terms of a and b

   (d) $\arg(z) = \theta$ in terms of a and b.



3.2.2 Part 2: Algebraically

3.3 Exponential Form of Complex Numbers

3.3.1 Laws of Indices

3.3.2 Euler's relationship

3.3.3 Multiplication in Polar/Exponential Form

3.3.4 Division in Polar/Exponential Form

3.3.5 Powers of Complex Numbers in Exponential and Polar Form

3.3.6 Roots of Complex Numbers in Exponential and Polar Form

3.4 De Moivre's Theorem