Chapter 4

Complex Numbers: Exponential Form and De Moivre's Theorem

4.1 Exponential Form of Complex Numbers

4.1.1 Euler's relationship

Today's material all stems from one amazing fact, discovered by Leonhard Euler around 1740 (assisted by Johann Bernoulli's and Roger Cotes' earlier work) and first published in 1748. It is extraordinarily useful in many applications, and astonishing because it connects areas of mathematics (trigonometry, exponentials/logs, and complex numbers) which we first meet entirely independently of each other. This relationship is now named after Euler, and is

Definition 9. Eulers Relationship

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{4.1}$$

where

e is the base of the natural logarithm

i is the imaginary unit and

θ is given in radians

Proving Euler's relationship is best done using power series for the exponential function, cosine and sine; we will meet power series in in a few weeks, and you will get to prove this relationship for yourself then. Until then, you just have to trust me.

We now have four different ways of writing a complex number. These are

$$\begin{array}{ll} \text{Cartesian} & z = a + bi \\ \text{Cartesian using } r, \theta & z = r(\cos\theta + i\sin\theta) \\ \text{Polar} & z = r\angle\theta \\ \text{Exponential} & z = r\mathrm{e}^{\mathrm{i}\theta} \end{array} \tag{4.2}$$

Exercise 11. Practice with different forms of a complex number

1. Write the following complex numbers in the different form specified
   (a) $3\angle\frac{\pi}{3}$ in exponential and Cartesian form.
   (b) $5\angle 30^\circ$ in exponential and Cartesian form.
   (c) $4\mathrm{e}^{\mathrm{i}\pi/2}$ in polar and Cartesian form.
   (d) -6 in polar and exponential form.
2. Write down the following
   (a) $|21.4e^{0.625i}|$
   (b) $\arg 2e^{i\pi/7}$

4.1.2 Multiplication in Polar/Exponential Form

Exponential Form

We can multiply together complex numbers which are in exponential form by using our existing knowledge about multiplication and the power laws.

Example 17.

Evaluate the product of $3\mathrm{e}^{\mathrm{i}\pi/3}$ and $2\mathrm{e}^{\mathrm{i}\pi/4}$

Solution

$$\begin{aligned} (3\mathrm{e}^{\mathrm{i}\pi/3})(2\mathrm{e}^{\mathrm{i}\pi/4}) &= (3 \times 2).\mathrm{e}^{\mathrm{i}\pi/3}.\mathrm{e}^{\mathrm{i}\pi/4} \\ &= 6\mathrm{e}^{\mathrm{i}\pi/3}.\mathrm{e}^{\mathrm{i}\pi/4} \end{aligned}$$

Now using the power law $a^n a^m = a^{n+m}$ we get

$$\begin{aligned}(3e^{i\pi/3})(2e^{i\pi/4}) &= 6e^{i\left(\frac{\pi}{3}+\frac{\pi}{4}\right)} \\ &= 6e^{i\frac{7\pi}{12}}\end{aligned}$$

We can find a more general rule for this process by considering two general complex numbers, $r_1e^{i\theta_1}$ and $r_2e^{i\theta_2}$:

$$(r_1e^{i\theta_1})(r_2e^{i\theta_2}) = r_1r_2e^{i\theta_1}e^{i\theta_2}$$

and using the power laws we get

Definition 10.

$$(r_1e^{i\theta_1})(r_2e^{i\theta_2}) = r_1r_2e^{i(\theta_1+\theta_2)} \tag{4.3}$$

Note

- The resulting modulus is the product of the moduli we started with.
- The resulting argument is the sum of the arguments we started with
- so to calculate a product in exponential form, we multiply the moduli together, to get the modulus add the arguments to get the argument

Polar form

Since we know from equation (4.2) that

$$re^{i\theta} \equiv r\angle\theta$$

we can rewrite equation (4.3) as

Definition 11.

$$(r_1\angle\theta_1)(r_2\angle\theta_2) \equiv r_1r_2\angle(\theta_1+\theta_2) \tag{4.4}$$

In other words, we can use the result we derived in exponential form to do quick multiplication in polar form too. In fact, in this form, it doesn't matter whether we use degrees or radians (remember, in exponential form, we definitely need to use radians).

Example 18.

Evaluate $(3\angle 100^\circ)(6\angle 120^\circ)$.

Solution

$$\begin{aligned}(3\angle 100^\circ)(6\angle 120^\circ) &= (3\times 6)\angle(100+120)^\circ \\ &= 18\angle 220^\circ\end{aligned}$$

Oh, but hang on! We always give complex numbers with the smallest possible argument. quick sketch then

$$18\angle 220^\circ \equiv 18\angle -140^\circ$$

4.1.3 Division in Polar/Exponential Form

Division of complex numbers in exponential and polar form follows a similar and predictable pattern:

Definition 12.

In exponential form:

$$\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \tag{4.5}$$

In polar form:

$$\frac{r_1\angle\theta_1}{r_2\angle\theta_2} = \frac{r_1}{r_2}\angle(\theta_1-\theta_2) \tag{4.6}$$

Example 19.

Evaluate $3e^{i\pi/3}$ divided by $2e^{i\pi/4}$.

Solution

$$\frac{3e^{i\pi/3}}{2e^{i\pi/4}} = \frac{3}{2}e^{i\left(\frac{\pi}{3}-\frac{\pi}{4}\right)} = \frac{3}{2}e^{i\frac{\pi}{12}}$$

Exercise 12.

Given

$$z_1 = 2\angle\frac{\pi}{6}, \quad z_2 = 3\angle\frac{3\pi}{4}, \quad z_3 = 4e^{2i\pi/3}, \quad z_4 = 5e^{-i\pi/6}$$

calculate the following

1. $z_1 z_2$
2. $\frac{z_3}{z_4}$
3. $z_2 z_3$
4. $|z_1 z_2 z_3|$
5. $\arg\left(\frac{z_2 z_4}{z_1}\right)$

4.1.4 Powers of Complex Numbers in Exponential and Polar Form

You know that powers are simply a shorthand way of writing repeated multiplication, ie

$$z^4 = z.z.z.z$$

. It follows that finding powers of a complex number in polar or exponential form follows the same pattern as multiplication, for example

$$\begin{aligned}(r\angle\theta)^3 &= (r\angle\theta)(r\angle\theta)(r\angle\theta) \\ &= r.r.r\angle(\theta+\theta+\theta) \\ &= r^3\angle 3\theta.\end{aligned}$$

We can see this more directly if we use exponential form

$$(re^{i\theta})^n \equiv r^n(e^{i\theta})^n$$

and apply the power law

$$(re^{i\theta})^n \equiv r^n(e^{in\theta}) \tag{4.7}$$

The general result written in polar form is

$$(r\angle\theta)^n \equiv r^n\angle n\theta \tag{4.8}$$

We can see the graphical effects of multiplication and division on an Argand diagram. Remember the key results: we multiply or divide the moduli (distance from the origin), and we add or subtract the arguments (angle relative to the axis) – equations (4.3, 4.4)). This is most easily demonstrated if we consider powers of a complex number:

Example 20.

Consider $z = 4\angle\frac{\pi}{3}$, Calculate z^2, z^3, z^4.

Solution

$$\begin{aligned} z^2 &= 2^2\angle 2\times\frac{pi}{3} &&= 4\angle\frac{2\pi}{3} \\ z^3 &= 2^3\angle 3\times\frac{pi}{3} &&= 8\angle\pi \\ z^4 &= 2^4\angle 4\times\frac{pi}{3} &&= 16\angle\frac{4\pi}{3} \equiv 16\angle-\frac{2\pi}{3}\end{aligned}$$

Exercise 13. **Powers of complex numbers**

1. Given
$$z_1 = 2\angle\frac{\pi}{6}, \qquad z_2 = 3\angle\frac{3\pi}{4} \qquad \text{and} \qquad z_3 = 4e^{2i\pi/3}$$
calculate the following
   (a) z_1^2
   (b) z_3^3
   (c) z_2^{10}
   (d) $\frac{1}{z_2}$
      i. by writing 1 in polar form and using the rule for dividing complex numbers, and
      ii. by remembering that $\frac{1}{x} = x^{-1}$ and using the rule for powers.
2. Sketch $z_1, z_1^2, z_1^3, z_1^4, z_1^5, z_1^6, z_1^7$ on an Argand diagram. Hence write z^7 down in the best version of polar and exponential form.

4.1.5 Roots of Complex Numbers in Exponential and Polar Form

In the task above, you should have found

$$\begin{aligned} z_3^3 &= (4e^{2i\pi/3})^3 \\ &= 4^3 e^{3(2i\pi/3)} \\ &= 64e^{2\pi i} \\ &= 64 \end{aligned}$$

But hang on... if $(4\angle\frac{2\pi}{3})^3 = 64$ then

$$\sqrt[3]{64} = 4\angle\frac{2\pi}{3}$$

but we know that

$$\sqrt[3]{64} = 4$$

What is happening?

Algebraic interpretation Recall that

$$r\angle\theta \equiv r\angle(\theta + 2n\pi), \qquad \text{where } n \text{ is any integer}$$

In exponential form

$$re^{\theta i} = re^{i(\theta+2n\pi)}$$

so if we are looking for a k'th root, we are trying to find

$$(re^{\theta})^{\frac{1}{k}} = (re^{(\theta+2n\pi)})^{\frac{1}{k}}$$

Applying the power laws we know, we can say

$$\begin{aligned}(re^{\theta})^{\frac{1}{k}} &= r^{\frac{1}{k}} e^{\frac{\theta+2n\pi}{k}} \\ &= r^{\frac{1}{k}} e^{\frac{\theta}{k}+\frac{2n\pi}{k}}\end{aligned}$$

Writing this in polar form makes it clearer

$$(r\angle\theta)^{\frac{1}{k}} \equiv (r\angle\left(\frac{\theta}{k}+\frac{2n\pi}{k}\right) \qquad \text{where } n \in \mathbb{Z} \tag{4.9}$$

Now let's return to our earlier example, $\sqrt[3]{64}$. We can say that

$$64 = 64\angle 0 = 64\angle(0 + 2n\pi)$$

Now applying equation (4.9), we have

$$\begin{aligned}(64\angle 0)^{\frac{1}{3}} &= 64^{\frac{1}{3}}\angle\left(\tfrac{1}{3}+\tfrac{2n\pi}{3}\right) \\ &= 4\angle\tfrac{2n\pi}{3} \qquad \text{where } n \in \mathbb{Z}\end{aligned}$$

Now consider different values of n:

n	$4\angle\frac{2n\pi}{3}$
0	$4\angle 0$
1	$4\angle\frac{2\pi}{3}$
2	$4\angle\frac{4\pi}{3} = 4\angle - \frac{2n\pi}{3}$
3	$4\angle\frac{6\pi}{3} = 4\angle 0$
4	$4\angle\frac{8\pi}{3} = 4\angle\frac{2n\pi}{3}$

Some keypoints

- When finding $z^{1/k}$ there are always k possible answers;
- these correspond to any k successive values of n in the formula above.
- We could therefore write our formula (4.9) more precisely as

$$(r\angle\theta)^{\frac{1}{k}} \equiv r^{\frac{1}{k}}\angle\left(\frac{\theta}{k}+\frac{2n\pi}{k}\right) \qquad \text{where } n = 0, 1, 2, ..., (k-1)$$

- The results are evenly distributed around the argand diagram;

- The $\sqrt{z}$ notation means the principal value, ie the version with the smallest angle.

Compare this last comment with what you already know. $\sqrt{4} = 2$, not ± 2, even though $(-2)^2 = 4$. $+2 = 2\angle 0$ so this is the version with the smallest angle.

Exercise 14. Roots of complex numbers in exponential and polar form

Find all the solutions of the equation $z^3 = i$.

Hint Write i in polar or exponential form.

4.2 De Moivre's Theorem

We have defined a number of ways to define a complex number and how to perform basic arithmetic.

From equation (4.2) we have the number of ways in which a complex number can be written:

$$z = a + bi \equiv r(\cos\theta + i\sin\theta) \equiv r\angle\theta \equiv re^{i\theta}$$

where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1} ba$ and equations (4.7) and (4.8) show how we find powers of complex numbers. Rewriting the power relationship using the $r(\cos\theta + i\sin\theta)$ form, we get De Moivre's Theorem.

Definition 13. De Moivre's Theorem

$$z^n = r^n e^{in\theta} = [\mathrm{r}(\cos\theta + \mathrm{i}\sin\theta)]^\mathrm{n} = \mathrm{r}^\mathrm{n}(\cos \mathrm{n}\theta + \mathrm{i}\sin \mathrm{n}\theta) \tag{4.10}$$

In its trigonometric form, equation (4.10), it can be used to prove many of the trigonometric identities. You will explore this further in exercises.

Exercise 15.

Recall the trig identities for $\sin 2\theta$ and $\cos 2\theta$ as:

$$\sin 2\theta = 2sin\theta\cos\theta \quad \text{and} \quad \cos 2\theta = \cos^2\theta - \sin^2\theta.$$

1. Use De Moivre's theorem with $n = 2$ and $r = 1$ to prove the identities. [Hint: consider that in an equation with real and imaginary parts, the real parts either side of the equals sign and the imaginary parts

either side of the equals sign must balance. In other words, we can deduce two equations from a single complex equation. Amazing!]

2. Now find trig identities for $\sin 3\theta$ and $\cos 3\theta$ in terms of $\sin\theta$ and $\cos\theta$.

Exercise 16.

Do as many of these as you need to in order to be confident that you (a) can get the right answer and (b) understand what you're doing. If you don't want to do all the questions, then at least try the ones that look like they are the hardest ones.

1. Express the following in exponential form:

   (a) $3\angle -\frac{2\pi}{3}$
   (b) $1\angle\frac{\pi}{12}$
   (c) $2\left(\cos\left(\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{12}\right)\right)$
   (d) $-\frac{1}{2}-\frac{\sqrt{3}}{2}$
   (e) i
   (f) $-2i$
   (g) -4.

2. Substitute into Euler's relationship and hence prove arguably the most elegant equation in mathematics, which includes the five most important numbers, namely

$$e^{i\pi}+1=0.$$

3. Write these in Cartesian form [no calculation should be necessary but a sketch might help]

   (a) $4e^{\pi i/2}$
   (b) $3e^{-\pi i/2}$
   (c) $5e^{i\pi}$

4. Write down $|3e^{2i}|$ and $\arg(3e^{2i})$ [No calculation should be necessary]. Sketch $3e^{2i}$ on an Argand diagram.

5. Calculate the following making sure your answer is given in the simplest form possible.

(a) $2\angle\pi \times 1\angle -\pi$ [what real numbers do these represent? Check that your result is consistent with what we would get with simple real number multiplication]

(b) $2\angle 150^\circ \times 4\angle 120^\circ$

(c) $3e^{i\pi/2}.4e^{i\pi/6}$

6. If

$$z_1 = 2\angle\frac{\pi}{3}, \quad z_2 = 10\angle\frac{\pi}{2} \qquad \text{and} \qquad z_3 = 5\angle\pi$$

find

(a) $\left|\frac{z_1 z_2}{z_3}\right|$ (b) $\arg\left(\frac{z_1 z_2}{z_3}\right)$ (c) $|z_1^5|$ (d) $\arg(z_1^5)$

7. If

$$z_1 = r_1\angle\theta_1 \qquad \text{and} \qquad z_2 = r_2\angle\theta_2,$$

write down

(a) $|z_1 z_2|$ (b) $\arg(z_1 z_2)$ (c) $\left|\frac{z_1}{z_2}\right|$ (d) $\arg\left(\frac{z_1}{z_2}\right)$

8. Find

(a) $\left(3\angle\frac{2\pi}{3}\right)^6$ (b) $\left(2e^{\frac{i\pi}{2}}\right)^6$ (c) $(1-i)^6$

9. Solve

(a) $z^2 = 3\angle\frac{\pi}{3}$ (b) $z^3 = 8$

and sketch your answers on an Argand diagram. If you had found, say $\sqrt[3]{8}$ before learning about complex numbers, what answer would you have got? Is this consistent with the complex answers you get?

Solutions to exercises Answer for Exercise 15 part (2)

$$\cos 3\theta \equiv \cos^3\theta - 3\cos\theta\sin^2\theta \qquad \text{and} \qquad \sin 3\theta \equiv -\sin^3\theta + 3\cos^2\theta\sin\theta.$$

Answers for Exercise 16:

1. (a) $3e^{-2\pi i/3}$ (b) $e^{i\pi/12}$ (c) $2e^{i\pi/12}$ (d) $e^{-2\pi i/3}$ (e) $e^{i\pi/2}$ (f) $2e^{-i\pi)/2}$ (g) $4e^{\pi i}$.

2.

3. (a) $4i$ (b) $-3i$ (c) -5

4. (a) 3 (b) 2

5. (a) 2 (b) $-8i$ (c) $12e^{2\pi i/3}$.

6. (a) 4 (b) $-\pi/6$ (c) 32 (d) $5\pi/3$

7. (a) $r_1 r_2$ (b) $\theta_1 + \theta_2$ (c) r_1/r_2 (d) $\theta_1 - \theta_2$.

8. (a) 729 (b) -64 (c) $8i$

9. (a) $\sqrt{3}\angle\frac{\pi}{6}$ and $\sqrt{3}\angle-\frac{5\pi}{6}$ (b) 2 and $2\angle\frac{2\pi}{3}$ and $2\angle-\frac{2\pi}{3}$.