Chapter 2

Introduction to Complex Numbers

Over the millennia of our existence our understanding of the world and with it mathematics has improved. This is demonstrated with our understanding and classification of numbers. We started with the counting numbers - whole positive numbers, called Natural numbers that has the special symbol $\mathbb{N}$. We define natural numbers as the set of all positive integers:

$$\mathbb{N} = \{1, 2, 3, 4, ...\}.$$

These numbers, by themselves enable us to count how many things we have but do not allow loss. Ie we cannot always find solutions to equations of the form

$$x + n = m \qquad (m, n \in \mathbb{N}).$$

To counter this we add negative whole numbers and zero of give us the set of integers:

$$\mathbb{Z} = \{... - 3, -2, -1, 0, 1, 2, 3, ...\},$$

and so $x = m - n$ has a solution even if $n > m$. The concept of negative numbers and zero came later than some of the number systems below.

Dividing goods between people probably led to the invention of rational numbers where some equations of the form

$$nx = m, \qquad (m, n \in \mathbb{Z}, n \neq 0)$$

cannot be solved in the integers. Thus

$$\mathbb{Q} = \left\{ \frac{M}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}.$$

Now every linear equation of the form $ax = b, (a, b \in \mathbb{Q}, a \neq 0)$ has a solution $x = \frac{a}{b}$ where $\frac{a}{b} \in \mathbb{Q}$. However there is no solution to

$$x^2 = 2$$

in $\mathbb{Q}$, so another extension is added to include numbers that cannot be represented as whole numbers or quotients. These are the real numbers, denoted $\mathbb{R}$. Real numbers include all the numbers along the number line including e, $\sqrt{2}$ and π. However, we still do not have a number system that allows for the square root of a negative number. For example, how do we solve an equation of the form

$$x^2 = -1?$$

To include solutions to such an equation we introduce the concept of a complex number.

Definition 4. Imaginary Unit

We define the imaginary unit to be

$$i = \sqrt{-1} \quad \Longleftrightarrow \quad i^2 = -1.$$

Note In some text books, especially physics or engineering ones, j is used to denote the imaginary unit.

The imaginary unit i multiplied by a real number (other than zero), is called an **imaginary number**.

We can deduce the square root of other negative numbers, by considering what happens when we square an imaginary number.

$$\begin{aligned} (2i)^2 &= 2 \times 2 \times \times i \times i \\ &= 4i^2 \\ &= 4(-1) \\ \Rightarrow &\quad (2i)^2 = -4 \\ \text{Hence} &\quad \sqrt{-4} = 2i. \end{aligned}$$

In general, we can say that if $\beta > 0$, then

$$\sqrt{-\beta} = isqrt\beta$$

You know, of course, that $(-2)^2 = (2)^2 = 4$. The same idea works with imaginary numbers

$$\begin{aligned} (-i)^2 &= (-1) \times i \times (-1) \times i \\ &= (-1)^2 i^2 \\ &= 1i^2 \\ &= -1 = i^2 \\ (-2i)^2 = (-2) \times (-2) \times i \times i &= 4i^2 = 4(-1) = -4 \end{aligned}$$

The square root sign always means "the positive (or principal) square root of". If we want both the positive and negative one, we need to include $\pm$.

To solve $x^2 = 4$ we say $x = \pm\sqrt{4} = \pm 2$.

To solve $x^2 = -4$ we say $x = \pm\sqrt{-4} = \pm 2i$.

When we add or subtract a (non-zero) real number with a (non-zero) imaginary number, we get something which is neither real nor imaginary.

Definition 5. Complex Number

A **complex number** is an expression of the form

$$a + ib \qquad \text{or} \qquad a + bi$$

where a and b are real numbers, and i is the imaginary unit.

The "complex" part of complex number refers to the the number being made up of different components. Examples of complex numbers include:

$3 + 2i$	$\frac{7}{3} + \frac{2}{3}i$	$-3 + 0i = -3$
$4 - i$	$0 + i\pi = i\pi$	

It is often convenient to represent a complex number by a single letter; w or z are frequently used. If a, b, x and y are all real numbers, and

$$w = a + bi \qquad \text{and} \qquad z = x + yi$$

then we can refer to the complex numbers, w and z.

Definition 6.

If $z = x + yi$ is a complex number, where x and y are real, then

- x is the real part of z, denoted $Re(z)$.

$$Re(z) = Re(x + yi) = x$$

- y is the imaginary part of z, donoted $Im(z)$.

$$Im(z) = Im(x + yi) = y$$

Example 6.

The real part $Re(3+2i) = 3$

The imaginary part $Im(3+2i) = 2, Im(4-i) = -1$

Note that the imaginary part is, by convention, the coefficient of the imaginary unit i, and does not include the i itself.

A complex number with a zero imaginary part is a real number (real numbers are a subset of complex numbers). For example, $3 + 0i = 3 \in \mathbb{R}$.

A complex number with a zero real part is an imaginary number (imaginary numbers are a subset of complex numbers). For example, $0 + 4i = 4i$.

Example 7.

Solve the equation

$$4x^2 - 8x + 13 = 0$$

Solution Using the quadratic formula with $a = 4, b = -8, c = 13$ we can say

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(4)(13)}}{2(4)} = \frac{8 \pm \sqrt{-144}}{8}.$$

Now

$$\sqrt{-144} = \sqrt{-1}\sqrt{144} = 12i$$

So, roots of the equation are

$$x = 1 \pm \frac{3}{2}i.$$

Keypoint A quadratic equation where the discriminant is negative has no real roots but it does have two complex roots.

Exercise 3.

Solve the equations

1. $x^2 + 2x + 5 = 0$ 2. $2x^2 - 12x + 24 = 0$ 3. $-4x^2 + 16x - 17 = 0$.

2.1 Complex Arithmetic

2.1.1 The Complex Conjugate

We define the complex conjugate, or conjugate of a complex number, as follows:

Definition 7. Complex conjugate

If $z = a + bi$ then its conjugate is $a - bi$ (ie change the sign of the imaginary part).
We denote the conjugate as $\bar{z}$ or z^*.

Example 8.

If $z = 3 - 4i$ then $\bar{z} = z^* = 3 + 4i$

Notice that conjugates come in pairs. In the example here $(3 + 4i)^* = 3 - 4i = z$.

In our quadratic example, example 7, our roots were a complex conjugate pair, ie conjugates of each other:

$$x = 1 + \frac{3}{2}i \qquad \text{and} \qquad \bar{x} = 1 - \frac{3}{2}i.$$

When we get complex roots of a polynomial, they always occur in conjugate pairs like this.

2.1.2 Complex Number Arithmetic: Addition, Subtraction and Multiplication

Addition, subtraction, and multiplication of complex numbers works exactly as you would expect, using your existing rules of arithmetic and algebra. The key principle is that we add the real and imaginary parts separately - gathering like terms.

Example 9.

Suppose we have two complex numbers:

$$z = 2 - 5i \qquad \text{and} \qquad w = 4 + 6i$$

then

Addition

$$z + w = (2 + 4) + (-5 + 6)i = 6 + i$$

Subtraction

$$z - w = (2 - 4) + (-5 - 6)i = -2 - 11i$$

Multiplication

$$\begin{aligned} zw &= (2-5i)(4+6i) \\ &= 2\times 4 + 2\times 6 + (-5i)\times 4 + (-5i)\times 6i \\ &= 8 + 12i - 20i + (-30)\times(i\times i) \\ &= 8 - 8i + (-30)\times(-1) \\ &= 8 - 8i + 30 \\ &= 38 - 8i. \end{aligned}$$

Note In the case of both addition and subtraction, the key thing is:

- add/subtract the real parts, to get the new real part;
- add/subtract the imaginary parts, to get the new imaginary part.

and multiplication is the only arithmetic so far which is any different from algebra we've met before. With multiplication,

- we always get an i^2, which we can simplify using $i^2 = -1$.

2.1.3 Complex Number Arithmetic: Division

Division by a complex number requires a cunning trick. In order to make progress evaluating a complex quotient, $\frac{z}{w}$, say, we first multiply top and bottom of the fraction by the complex conjugate of the numerator (bottom).

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}}$$

It turns out that a complex number multiplied by its conjugate is always real (proof later), $w\bar{w} \in \mathbb{R}$, and we know how to divide by a real number.

Example 10.

Find $\frac{z}{w}$ where

$$z = 2 - 5i \qquad \text{and} \qquad w = 4 + 6i.$$

Solution To calculate $\frac{z}{w}$ we multiply top and bottom by the complex conjugate of the denominator so

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}}.$$

So

- First we find the complex conjugate of the denominator $\bar{w}$:

$$\bar{w} = 4 - 6i$$

- Calculate $w\bar{w}$:

$$\begin{aligned} w\bar{w} &= (4+6i)(4-6i) \\ &= 4 \times 4 + 4 \times (-6i) + 4 \times 6i + (-6i) \times 6i \\ &= 16 + 36 \\ &= 52. \end{aligned}$$

- Calculate $z\bar{w}$:

$$\begin{aligned} z\bar{w} &= (2-5i)(4-6i) \\ &= 2 \times 4 + 2 \times (-6i) + 4 \times (-5i) + (-5i) \times (-6i) \\ &= 8 - 12i - 20i + 30i^2 \\ &= -22 - 32i. \end{aligned}$$

- Calculate $\frac{z\bar{w}}{w\bar{w}}$

$$\begin{aligned} \frac{z\bar{w}}{w\bar{w}} &= \frac{-22-32i}{52} \\ &= \frac{-22}{52} + \frac{-32i}{52} \\ &= -\frac{11}{26} - \frac{8}{13}i \end{aligned}$$

Exercise 4.

1. Go back to Exercise 3 and check your answers by substituting them into the quadratic expression and checking that it simplifies to zero.

2. Given

$$z_1 = 4 + 2i, \quad z_2 = 2 - 3i, \quad z_3 = 2 - i,$$

find:

(a) $z_1 + z_2$

(b) $z_1 - z_2$

(c) $Im(z_2 - 2z_3)$

(d) $z_1 z_2$

(e) $\bar{z}_2.\bar{z}_3$

(f) $\frac{z_1}{z_2}$

2.1.4 Every Quadratic Factorises

You may have been told previously that only some quadratic will factorise. This is not true. Every quadratic will factorise, but only some can be factorised 'by inspection'. For others, we need to find the roots first.

Suppose we think about a quadratic equation, containing a quadratic which does factorise easily, (in fact, we did earlier today)

$$2x^2 + 7x - 4 = 0.$$

Factorising gives us

$$(2x - 1)(x + 4) = 0$$

We could go one stage further, and take the 2 out of the first bracket so that the coefficients of the x's are both 1:

$$2\left(x - \frac{1}{2}\right)(x + 4) = 0 \tag{2.1}$$

Now we know that to solve this, we say

$$x - \frac{1}{2} = 0 \qquad \text{or} \qquad x + 4 = 0$$

which means that the roots are

$$x = \frac{1}{2} \quad \text{and} \quad x = -4. \tag{2.2}$$

Comparing equations (2.1) and (2.2), we can see that the roots appear in the factors we had in equation (2.1) (with the opposite sign).

This is always the case. We can always write a quadratic as follows:

$$ax^2 + bx + c = a(x - \beta)(x - \gamma) \tag{2.3}$$

where the roots are $+\beta$ and $+\gamma$. Notice though that the a must be outside the factors.

So, if we cannot factorise a quadratic by inspection, we can instead:

1. Find the roots of the quadratic by our preferred method;

2. Substitute the roots (and the coefficient of x) into equation (2.3).

Example 11.

Factorise the quadratic $4x^2 - 8x + 13$.

Solution We cannot factorise this easily, but we found the roots in an earlier example. They are

$$x = 1 + \frac{3}{2}i \quad \text{and} \quad x = 1 - \frac{3}{2}i.$$

Substituting these for β and γ, and $a = 4$ (coefficient of the x^2) into equation (2.3) gives us

$$4x^2 - 8x + 13 = 4\left(x - \left(1 + \frac{3}{2}i\right)\right)\left(x - \left(1 - \frac{3}{2}i\right)\right).$$

Multiplying out the inner sets of brackets gives us

$$4x^2 - 8x + 13 = 4\left(x - 1 - \frac{3}{2}i\right)\left(x - 1 + \frac{3}{2}i\right).$$

When we get rid of fractions, it's good to do so. Here we can say

$$4x^2 - 8x + 13 = 2.\left(x - 1 - \frac{3}{2}i\right).2.\left(x - 1 + \frac{3}{2}i\right). = (2x - 2 - 3i)(2x - 2 + 3i).$$

Exercise 5. **Factorising quadratics**

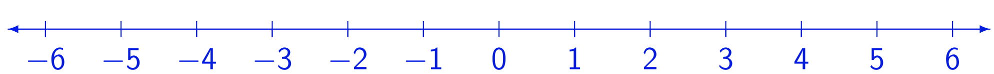
Factorise each of the following quadratic expressions into two linear factors. (Note that you already found the roots in earlier questions)

1. $2x^2 - 20x + 44$
2. $x^2 + 2x + 5$
3. $2x^2 - 12x + 24$
4. $-4x^2 + 16x - 17$

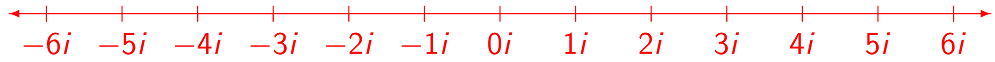
Check your answers by carefully expanding the brackets and confirming that you get the original expression.

2.2 Graphical Representation of Complex Numbers

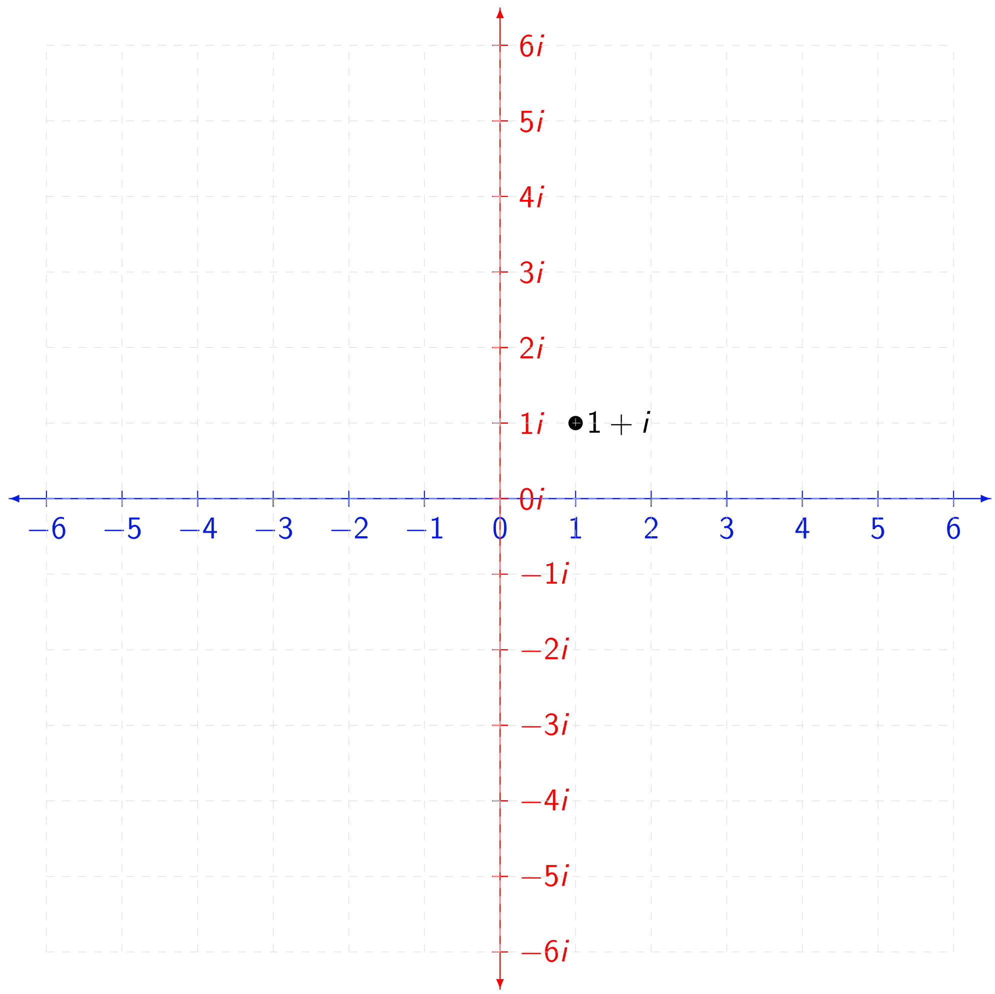
When you were young, you probably met the concept of a number line for real numbers:



We could have another number line for imaginary numbers:



However, we know that zero multiplied by anything is zero, so 0i=0. This means our real number line must intersect at 0. The easiest way of doing this is to imagine rotating the imaginary number line through 90 degrees, which gives us something like a pair of axes:



Now, as well as the two axes representing real and imaginary numbers, we can represent a complex number as a point somewhere else on the plane.

This plane is known as the **Complex Plane**.

A picture of the complex plane is known as an **Argand diagram**.

Every complex number lies somewhere on this plane, and every point on the plane represents a complex number.

One way of thinking about complex numbers is that they are 2-dimensional numbers, which can describe points anywhere on a flat plane, whereas real numbers which are 1-dimensional and can describe a point along a line. Notice the similarities with the co-ordinate system you're used to. A point (x, y) on Cartesian axes is similar to a complex number $x + yi$.

This similarity can be exploited in various applications of complex numbers; however, that thinking only gets you so far, and it's important to realise that co-ordinates and complex numbers are not simply interchangeable things. We can represent addition and subtraction of complex numbers on an Argand diagram, as you can represent addition/subtraction on real numbers on a number line.

Exercise 6.

Given $z = 3 + 2i$ and $w = 4 - 3i$,

1. Mark z and w on the complex plane below.
2. Mark $w - z$ on the complex plane.
3. Mark z^* and w^* on the complex plane. What do you notice about the relationship between a complex number and its conjugate?