Chapter 3

Complex Numbers: Polar and Cartesian Coordinates

In the previous chapter we introduced the concept of imaginary numbers and formed a new number system called complex numbers. We learned how to add, subtract, multiply and divide these numbers and how to represent them graphically.

In this chapter we are going to delve further into some interesting properties of complex numbers and consider how we represent them.

3.1 Polar and Cartesian Co-ordinate Systems

To specify

- the position of a point on a plane, or
- a complex number

we always need two bits of information, for example

1. (x, y) co-ordinates
2. Real and imaginary parts $x + yi$

This system, where we give the distance along, and the distance up, is known as a Cartesian co-ordinate system (although calculators dumb-down, and call it a 'rectangular' co-ordinate system. Those are not the only options, however.

The most common alternative to Cartesian co-ordinates is a system of **Polar Co-ordinates**. In this, we specify an origin and a single axis, which correspond to the origin and x-axis in the Cartesian system. Then to say where the point is, we give its distance from the origin, r, and the angle of rotation from the axis, θ.

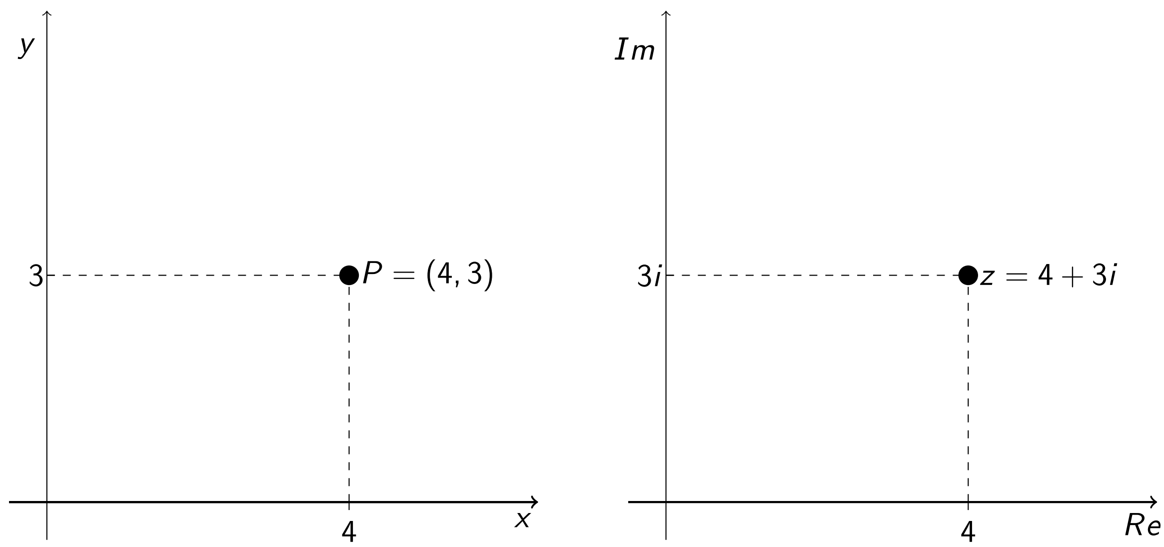


Figure 3.1: Example of complex number drawn in the complex plane

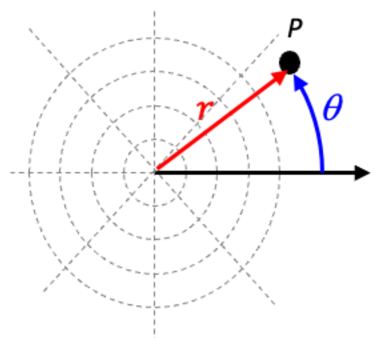


Figure 3.2: Figure showing polar co-ordinates in 2D plane.

Key features of polar co-ordinates are:

- The positive direction for rotation is **anticlockwise**.
- The negative direction for rotation is **clockwise**.
- When we're talking about co-ordinates, we write this (r, θ).
- When we're talking about a complex number, we write $r\angle\theta$.

In our example above we have

$$P = (5, 36.9^\circ) \qquad \text{(3 sig fig)} \qquad \text{and} \qquad z = 5\angle 36.9^\circ \qquad \text{(3 sig fig)}$$

Notice that as with Cartesian co-ordinates, we need two bits of information, but here we give the distance from the origin and the angle relative to the horizontal axis.

3.1.1 Modulus and Argument of a Complex Number

For complex numbers, the distance and angle have special names and notation.

Definition 8.

Modulus The distance from the origin, r. Denote modulus of $z = |z|$ - vertical lines either side of the complex number.

Argument The angle relative to the axis, θ, denoted $\arg(z)$.

Example 12.

$$|3\angle 45^\circ| = 3 \qquad \text{since} \qquad |r\angle\theta| = r$$

and

$$arg(3\angle 45^\circ) = 45^\circ \qquad \text{since} \qquad arg(r\angle\theta) = \theta$$

Note

1. These names do not apply to co-ordinates of a point, only to complex numbers.
2. Compare the use of 'modulus' with a use you've already met, $|-2| = |2| = 2$. Where are -2 and 2 on the complex plane? How far are they from the origin?

3.1.2 Units for Angles

You know that we have two units for angles, degrees (which we indicate with a little $^\circ$) and radians, which we can indicate by 'rads' or a little c (^{c}).

- A full turn is $360^\circ = 2\pi^c$
- It follows that a half turn is $180^\circ = \pi^c$
- Dividing both sides by π gives $1^c = \left(\frac{180}{\pi}\right)^\circ$ Alternatively, dividing both sides by 180 gives $1^\circ = \left(\frac{\pi}{180}\right)^c$

Example 13.

1. Convert 36.9° from degrees to radians.
2. Convert $\frac{\pi}{2}$ from radians to degrees.

Solution

1. Since $1^\circ = \left(\frac{\pi}{180}\right)^c$, multiplying both sides by 36.9 gives

$$\begin{aligned} 36.9^\circ &= \left(\tfrac{36.9\pi}{180}\right)^c \\ &= 0.644^c \quad \text{3 sig fig (simplifying)} \end{aligned}$$

2. Since $1^c = \left(\frac{180}{\pi}\right)^\circ$, multiplying both sides by $\frac{\pi}{2}$ gives

$$\left(\frac{\pi}{2}\right)^c = \left(\frac{\pi}{2} \times \frac{180}{\pi}\right)^\circ = 90^\circ \quad \text{(simplifying)}$$

When we specify a point in Polar co-ordinates (r, θ) or a complex number in Polar form $r\angle\theta$ it doesn't matter whether we use degrees or radians (as long as we give the units). In our example above

$$\begin{aligned} P &= (5, 36.9^\circ) &&= (5, 0.644^c) \quad \text{3 sig fig} \\ z &= 5\angle 36.9^\circ &&= 5\angle 0.644^c \quad \text{3 sig fig} \end{aligned}$$

However, some maths (crucially, calculus, and soon, the exponential form of a complex number) work best with radians . For this reason, **mathematicians almost always USE RADIANS**.

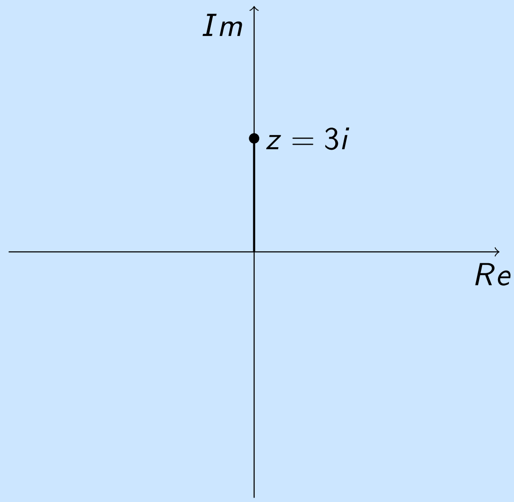
3.2 Converting between Polar and Cartesian form

3.2.1 Part 1: Graphically

For some simple cases, conversion between the two forms is best done with a quick sketch.

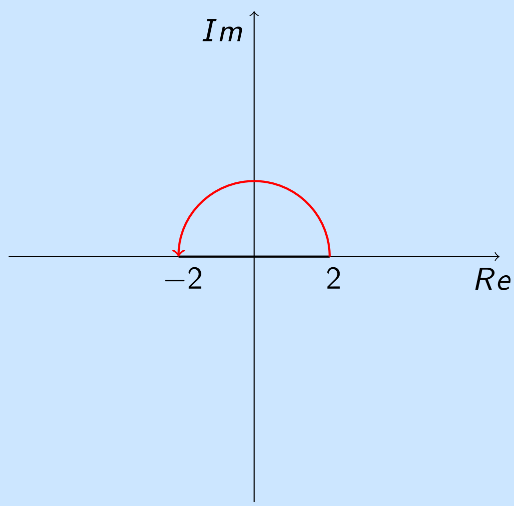
Example 14.

1. Convert $3i$ to polar form



$3i \equiv 3\angle\frac{\pi}{2}$

2. Convert $2\angle\pi$ to Cartesian form



$2\angle\pi \equiv -2$

Exercise 7. **Converting between polar and Cartesian 1**

1. Sketch these numbers on an Argand diagram, and hence convert those that in in polar form to Cartesian, and vice versa (no calculations should be necessary, except, perhaps, in $\sqrt{2}\angle-\frac{\pi}{4}^c$)

   (a) $1\angle180°$ (b) -2 (c) $8i$ (d) $\sqrt{2}\angle-\left(\frac{\pi}{4}\right)^c$

2. For each of the complex numbers given above, state the modulus and argument. Use statements with the correct notation, for example
$$|1\angle 180^\circ| = 1 \qquad \text{and} \qquad \arg(1\angle 180^\circ) = 180^\circ.$$

3. Now consider the sketch shown here, which shows a complex number
$$\begin{aligned} z &= a + bi \\ &= r\angle\theta. \end{aligned}$$

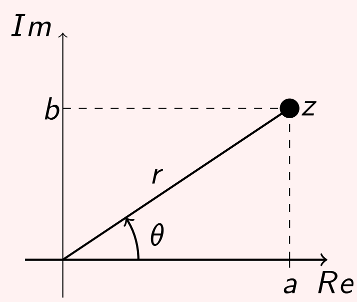
4. Use Pythagoras and trigonometry to find relationships (ie formulae) for:

   (a) $Re(z) = a$ in terms of r and θ

   (b) $Im(z) = b$ in terms of r and θ

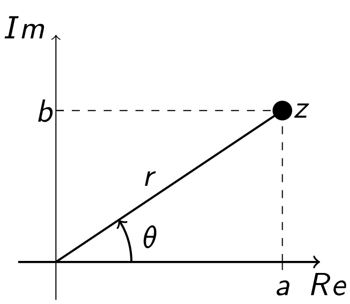
   (c) $|z| = r$ in terms of a and b

   (d) $\arg(z) = \theta$ in terms of a and b.



3.2.2 Part 2: Algebraically

For everything other than the easy cases, to convert between different forms, we have Pythagoras and trigonometry.



Let
$$z = a + bi = r\angle\theta$$
where
$$r = \sqrt{a^2 + b^2}$$
and
$$a = r\cos\theta \text{ and } b = r\sin\theta$$
by rules of right angled triangles. Thus, substituting a and b into z we have:
$$a + bi = r\cos\theta + i\sin\theta$$

Example 15.

1. Convert $4\angle\frac{\pi}{4}$ to Cartesian form.

   Solution Here we have that $r = 4$ and $\theta = \frac{\pi}{4}$ so

$$\begin{aligned} a &= Re(z) \\ &= r\cos\theta \\ &= 4\cos(\pi/4) \\ &= 4\times\tfrac{\sqrt{2}}{2} \\ &= 2\sqrt{2} \end{aligned}$$

   Similarly

$$\begin{aligned} b &= Im(z) \\ &= r\sin\theta \\ &= 4\sin(\pi/4) \\ &= 4\times\tfrac{\sqrt{2}}{2} \\ &= 2\sqrt{2} \end{aligned}$$

   So

$$4\angle\frac{\pi}{4} = 2\sqrt{2}(1+i)$$

2. Convert $2+3i$ to polar form.

   Solution By definition, the modulus is given by:

$$r = |2+3i| = \sqrt{a^2+b^2} = \sqrt{2^2+3^2} = \sqrt{13}$$

   and the argument:

$$\begin{aligned} \arg 2+3i &= \tan^{-1}\left(\tfrac{3}{2}\right) \\ &= 0.983^c \quad \text{(3sf)} \end{aligned}$$

   Thus $2+3i = \sqrt{13}\angle 0.983^c$.

Complex numbers outside the first quadrant

For any complex number where the real or imaginary part is negative, the number lies outside the first quadrant of the complex plane (see Figure 3.3). In these cases, the formulae we have above for finding the real and imaginary parts from the polar form, and for finding

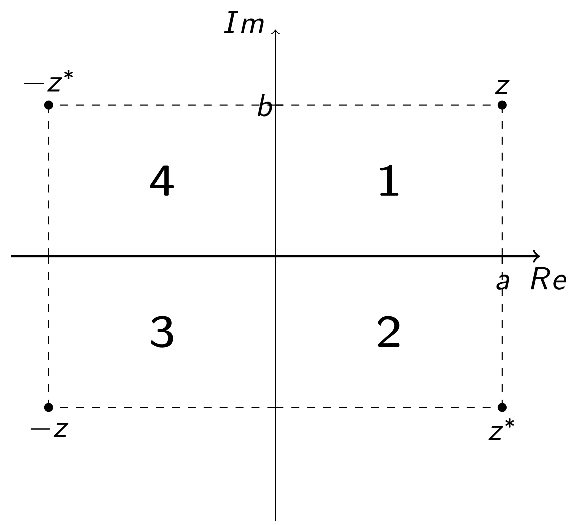


Figure 3.3: Plot showing the different quadrants of the Argand diagram.

the modulus from the Cartesian form are unchanged, but we have to be careful about the argument, remembering that this is **the angle relative to the horizontal axis. Always draw a picture.**

Example 16.

Convert -2+3i to polar form.

Solution The modulus is given as in the previous example

$$|-2+3i| = \sqrt{(-2)^2+3^2} = \sqrt{13}$$

Note that it doesn't matter if we use the real part -2 or just 2.
The angle β is given by

$$\beta = \tan^(-1)\left(\frac{3}{2}\right) = 0.983^c \quad \text{(calc,3 sf)}$$

where we use the length of the triangle's sides, rather than the real and imaginary parts with minus signs attached. However, this is not the angle we require. Instead, we want θ which is given in this example by

$$\theta = \pi - \beta = \pi - 0.983 = 2.16 \quad \text{(3sf)}$$

Hence

$$-2+3i = \sqrt{13}\angle 2.16^c.$$

Using your calculator to convert between polar and Cartesian

A few calculators can deal with complex numbers, in which case they can convert between polar and Cartesian form. Almost all scientific calculators can convert between polar and Cartesian co-ordinates, however.

Because every calculator is different, it is not possible to write notes on how to do it here. although we can say that most calculators refer to Cartesian co-ordinates as 'rectangular' co-ordinates.

Exercise 8. **Converting between polar and Cartesian 2**

1. Convert the following to Cartesian form. All angles are in radians:

   (a) $3\angle -\frac{2\pi}{3}$ (b) $3\angle 1$

2. Convert the following to Polar form, giving your answers in radians:

   (a) $5+2i$ (b) $-3-2i$

3. Check your calculator manual (Google to find it online, if you've lost it) for how to convert between polar and Cartesian ("rectangular"), and check your answers to the questions above.

3.2.3 Alternative forms for polar form

A curious thing about polar form is that, unlike with Cartesian co-ordinates, the way of specifying a point/complex number is not unique.

Consider a complex number $z = 4\angle\frac{3\pi}{2}$.

An alternative version of this would be $z = 4\angle -\frac{\pi}{2}$. Another alternative would be $z = 4\angle\frac{3\pi}{2} + 2\pi = \frac{7\pi}{2}$.

In fact, we could always add or subtract any number of full turns $z = 4\angle\frac{3\pi}{2} + 2n\pi$ where $n \in \mathbb{Z}$.

In general, we can say that for any complex number

$$r\angle\theta = r\angle(\theta + 2n\pi).$$

We could, if we wanted to, give a negative distance for r (imagine an instruction "turn 45 degrees to the left, and walk backwards 5 paces".) In this case, we could say

$$r\angle\theta = -r\angle(\theta \pm 2n\pi.$$

) We always give positive distance r, the smallest possible angle, ie $-\pi < \theta \leq \pi$.

Exercise 9.

Write these complex numbers with a more sensible argument (all arguments are in radians)

1. $4\angle\frac{5\pi}{2}$
2. $3\angle - 3\pi$

Exercise 10.

1. Three complex numbers are defined as $z_1 = 4+2i$, $z_2 = 2-3i$, $z_3 = 2-1$. Mark the numbers on an Argand diagram. Write down z_1^*, z_2^* and z_3^* and mark these on your Argand diagram.
2. Write these complex numbers with a more sensible argument (all arguments are in radians)
   (a) $2\angle\frac{7\pi}{6}$
   (b) $1\angle\frac{215\pi}{23}$
3. (a) Given that $z = 1 + i$, mark z on an Argand diagram and convert it to polar form.
   (b) Use this (and your Argand diagram) to write down the following in polar form (without further calculations):
      i. $1 - i$
      ii. $-1 + i$
      iii. $-1 - i$.
4. Convert the following to Cartesian form. All angles are in radians:
   (a) $2\angle\pi/6$
   (b) $2\angle 13\pi/6$
5. Convert the following to Polar form, giving your answers in radians.
   (a) $-5 - 2i$
   (b) $-5 + 2i$

6. Find the following

   (a) $|-2+6i|$

   (b) $\arg(-3+4i)$

   (c) $Re(6\angle 2\pi/3)$

   (d) $Im(\sqrt{3}\angle -3\pi/4)$

7. (a) Calculate i^2, i^3, i^4, i^5, i^6.

   (b) Plot the following numbers on an Argand diagram: $i, i^2, i^3, i^4, i^5, i^6$

   (c) What do you notice about their positions on the complex plane? Use the pattern to quickly give the values of i^23, i^24, i^25.

8. Given that $z = 1 + 1i$,

   (a) calculate z^2, z^3, z^4, z^5

   (b) plot z and the numbers you've calculated in (a) on an Argand diagram. What do you notice about the pattern?

   (c) without having to multiply out lots of sets of brackets, use the pattern to get z^8, z^10, z^100