Chapter 1

Functions

1.1 Formal Definition of a Function

Whenever one quantity depends on another quantity then we can say that the former quantity is a function of the latter.

For example, the area of a circle depends on its radius so the area is a function of the radius:

$$A = \pi r^2.$$

The formula given is a rule that tells us how to calculate a unique (single) output value of the area, A, for each possible input value of the radius, r.

The set of all possible input values for the radius, called the **domain**, maps to a set of output values, called the **codomain** or **range**. In this example, the domain and range are the set of all positive real numbers ($[0, \infty)$).

Other examples include: the temperature at which water boils depends on its altitude above sea level and the interest paid depends of cash investment on the length of time for which the investment is made.

In calculus, we often generalise and refer to a generic function without having any particular formula in mind. To denote that y is a function of x we write

$$y = f(x),$$

which we read as "y equals f of x." Here the function is represented by f with x being the independent input variable and y being the dependent output variable.

Definition 1. Function

A function is a mathematical relation between two sets, X and Y, such that for each element x in the set X, there is a unique element y in the set Y.

$$\begin{aligned} f : X &\to Y \\ x &\mapsto y \\ y = &\ f(x) \end{aligned}$$

- f represents the function.
- X is the domain, the set of input values.
- Y is the codomain, the set of possible output values.

1.2 Properties of a Function:

- Each element x in X (written as $x \in X$ or "x belongs to X") maps to a unique element y in Y.
- Not every element of Y needs to have a pre-image in X.
- A function can be represented as ordered pairs: $\{(x, y) \mid x \in X, y \in Y\}$.
- The graph of a function is the set of all ordered pairs $(x, f(x))$.

Example 1.

1. The area of a circle is given by the function A such that $A(r) = \pi r^2$, for $R \geq 0$.
2. The volume of a ball of radius r is given by the function $V(r) = \frac{4}{3}\pi r^3$. for $r \geq 0$. The volume of a ball of radius $6cm$ is

   $$V(6) = \frac{4}{3}\pi 6^3 = 288\pi \text{cm}^3.$$

   Note how the variable r is replaced by the special value $r = 6$ in the formula defining the volume to obtain the value of the function at $r = 6$.
3. A function F is defined for all real numbers t by $F(t) = 2t+3$. Find the output values fo F that correspond to the input values of $0, 2, x + 2$ and $F(2)$.

Solution

$$\begin{aligned}
F(0) &= 2(0)+3=3\\
F(2) &= 2(2)+3=7\\
F(x+2) &= 2(x+2)+3=2x+7\\
F(F(2)) &= 2(2(2)+3)+3=17
\end{aligned}$$

1.3 The Domain Convention

For a function to be properly defined then we must specify its domain. For example, let

$$f(x)=x^2 \quad \text{for all} \quad x \geq 0 \qquad \text{and} \qquad g(x)=x^2 \quad \text{for all} \quad x.$$

The domain convention states that

When a function f is defined without specifying its domain, we assume that the domain consists of all real numbers x for which the value $f(x)$ of the of the function is a real number.

Thus, if the function above is specified without a domain then we assume that we are talking about $g(x)$.

Example 2.

1. The square root function: The domain of $f(x)=\sqrt{x}$ is the interval $[0,\infty)$, since negative numbers do not have real square roots. Note that, by definition, a function has one unique output for each input. Therefore, to consider the square root to be a function then we only take the positive output. For example, if $x=4$ then there are two possible answers, $x=-2$ or $x=2$. Thus the square root function $\sqrt{x}$ always denotes the nonnegative square root of x. The two solutions of the equation $x^2=4$ are $x=\sqrt{4}=2$ and $x=-\sqrt{4}=-2$.

2. The domain of the function $g(x)=\frac{x}{x^2-1}$ consists of all real numbers **except** $x=\pm1$. Expressed in terms of intervals we have

$$\text{dom}(g)=(-\infty,-1)\cup(-1,1)\cup(1,\infty).$$

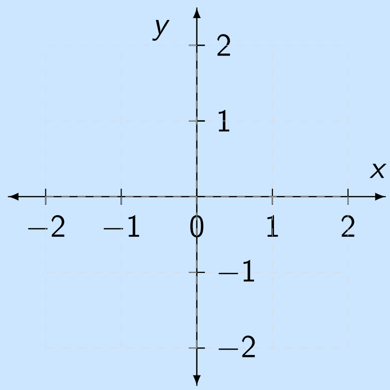
3. The domain of $h(t)=\sqrt{4-t^2}$ is defined only on the interval where $t-t^2\geq 0$ so

$$\text{dom}(h)=[-2,2].$$

Note Curves that you can draw are not necessarily functions. By definition, a function, f, has one unique output thus no **vertical line** can intersect the graph of a function at more than one point.

Example 3.

Plot the graph of $x^2 + y^2 = 1$. Show that this graph is not a function.



Exercise 1.

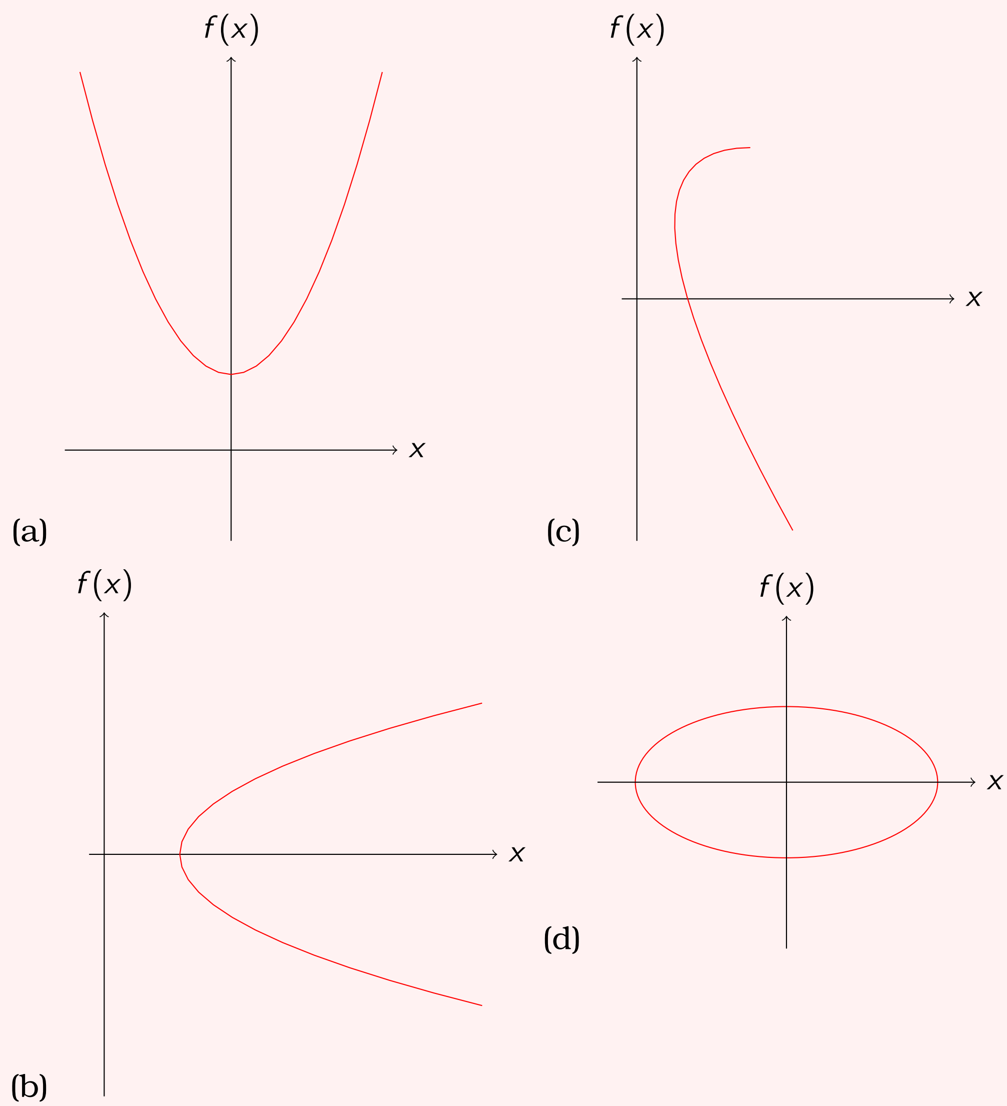
1. Find the domain and range of each of the following functions:

   (a) $f(x) = 1 + x^2$ (b) $g(x) = \sqrt{4 - x}$ (c) $h(x) = \frac{1}{x-2}$

2. By calculating the values of $f(x)$ at $x = 0, \pm 1/2, \pm 1, \pm 3/2, \pm 2$, sketch the graphs of the following functions:

   (a) $y = c$
   (b) $y = x$
   (c) $y = x^2$
   (d) $y = \sqrt{x}$
   (e) $y = x^3$
   (f) $y = x^{\frac{1}{3}}$
   (g) $y = \frac{1}{x}$
   (h) $y = \frac{1}{x^2}$
   (i) $y = \sqrt{1 - x^2}$
   (j) $y = |x|$
   (k) $y = 1 + \sqrt{x - 4}$
   (l) $y = \frac{2-x}{x-1}$

3. Are the following graphs functions? Explain your reasoning.

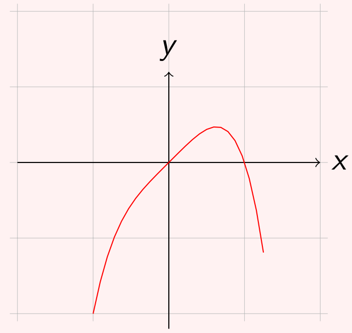


4. Match graphs (i) -(iv) with the following functions:

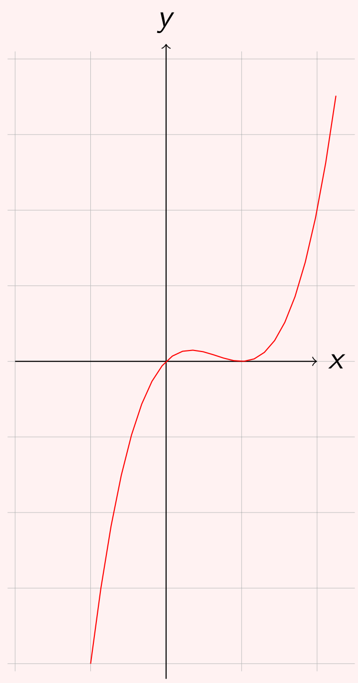
(a) $x - x^4$ (b) $x^3 - x^4$ (c) $x(1-x)^2$ (d) $x^2 - x^3$.

i)

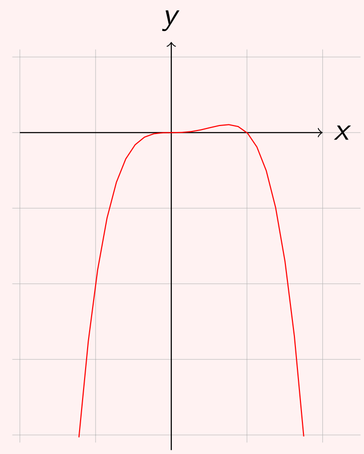
iii)



ii)



iv)



1.4 Polynomials

Definition 2. **Polynomial function**

A polynomial function is one that *only* involves non-negative integer powers of x, for example:

- $y = 7x + 4$ (polynomial of order/degree 1, linear)

- $y = 3x^2 - 5x - 1$ (polynomial of order/degree 2, quadratic)

- $y = -x^3 + 5x^2 - 7x + 12.01$ (poly. of order/degree 3, cubic)

Functions containing negative or non-integer powers, or other functions, (such as trigonometric) are **not** polynomials, e.g.

- $y = x^2 + 4\sqrt{x} - 5$ $\qquad y = \dfrac{5}{x^2} - 7x^3 + 6x - 4$
- $y = x^2 + \sin(x)$

Definition 3. **Equation of a straight line**

$$y = mx + c$$

where m and c are constants, represents a straight line.

m is the **gradient** (slope) of the line and can be calculated as

$$m = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\Delta y}{\Delta x}$$

c is the value of y when the line crosses the y-axis (at $x = 0$), known as the y-**intercept**.

Note: to find where the line crosses the x-axis, simply let $y = 0$.

1.4.1 Quadratic Equations

A quadratic is a polynomial expression that can be written in the form

$$ax^2 + bx + c,$$

where x is a variable, $a \neq 0$ and a, b, c are usually constants (but strictly, not functions of x).

A quadratic expression can be written in different forms. The two most common alternative forms are when we factorise or complete the square.

Example 4.

1. **Factorising**

$$2x^2 + 7x - 4 = (2x - 1)(x + 4)$$

Notice that this is not the only version we could write. One variation which is sometimes useful is to rewrite it so that the coefficients of the x in each set of brackets is one, eg

$$2x^2 + 7x - 4 = (2x - 1)(x + 4) = 2(x - 1/2)(x + 4)$$

2. **Completing the square** To “complete the square”, we should factories to take out the coefficient of the x first, so that in the brackets we have a quadratic where the coefficient of x is 1, eg

$$2x^2 + 7x + 4 = 2(x^2 + 7/2x + 2)$$

Next, we are aiming for something which includes something like $(x + A)^2$. Considering expanding that bracket, we know that

$$(x + A)^2 = x^2 + 2Ax + A^2$$

so we can see that A must be equal to half the coefficient or x in the quadratic inside the brackets above.

$$\begin{aligned} 2x^2 + 7x + 4 &= 2\left(x^2 + \tfrac{7}{2}x + 2\right) \\ &= 2\left(\left(x + \tfrac{1}{2}\tfrac{7}{2}\right)^2 + \cdots\right) \\ &= = 2((x + 7/4)^2 + \ldots\ldots) \end{aligned}$$

Next we have to work out what goes in the gap above. There are two parts to this:

- the constant we had in the quadratic in brackets on the first line, ie 2
- the constant we get when we expand $(x + 7/4)^2$, ie $(7/4)^2$ which is already included, so we need to subtract it.

$$2x^2 + 7x + 4 = 2\left(\left(x + \frac{7}{4}7/4\right)^2 + 2 - \left(\frac{7}{4}\right)^2\right)$$

Now all that's left to do is simplify the expression

$$\begin{aligned} 2x^2 + 7x + 4 &= 2\left(\left(x + \tfrac{7}{4}\right)^2 + \tfrac{36}{16} - \tfrac{49}{16}\right), \\ &= 2\left(\left(x + \tfrac{7}{4}\right)^2 - \tfrac{17}{16}\right), \end{aligned}$$

and finally, expanding the outside brackets we get

$$2\left(\left(x + \frac{7}{4}\right)^2 - \frac{17}{8}\right).$$

Checking We should always check our answers when we can. Since expanding brackets is usually easier than factorising or completing the square, we will most likely find any error if we do this in these cases.

Quadratic Graphs

If we plot a quadratic function

$$ax^2 + bx + c$$

then we get one of the two shapes illustrated in Figure 1.1.

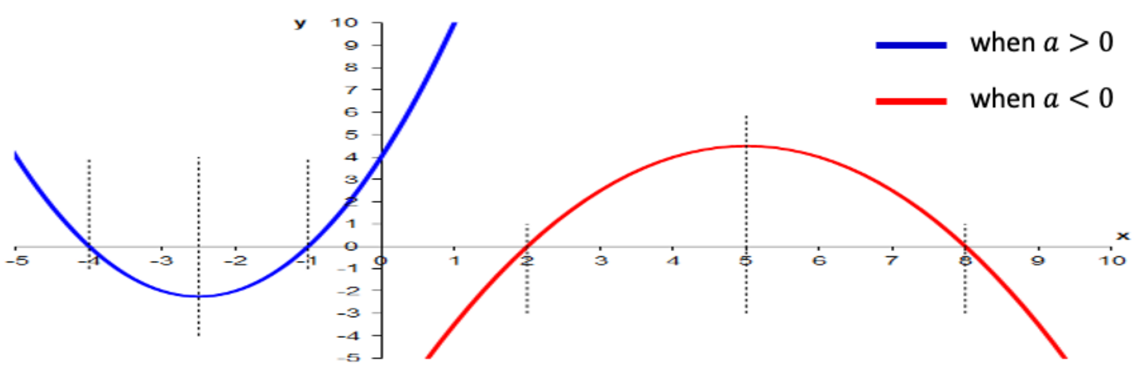


Figure 1.1: Plot showing two quadratic functions with $a < 0$, red and $a > 0$, blue.

Quadratic equations/ roots of a quadratic

The roots of any polynomial, including a quadratic, are when the polynomial evaluates to zero. So in the case of a quadratic, the roots are the solutions of the quadratic equation

$$ax^2 + bx + c = 0.$$

This is equivalent to $y = 0$ on the graphs above, ie where the graph crosses the x-axis. To find the solution of a quadratic equation, we have lots of possibilities:

- Factorising, and then using the fact that if two factors multiplied together equal zero, then at least one of them must be zero. (Only useful if we can factorise easily.)
- Completing the square, and then rearranging the resulting equation.
- Plotting the graph, and looking for the x-intercepts. (Only as accurate as your plot.)
- Using your calculator (depends on the calculator you have).
- Using a computer algebra package (eg Desmos, Geogebra, Matlab, Python)
- **The quadratic formula** If $ax^2 + bx + c = 0$ and $a \neq 0$ then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The expression $b^2 - 4ac$ is known as the discriminant.

Example 5.

Sketch the graph of $y = -3x^2 - 4x + 5$, showing all the important features.

Solution What do we mean by "important features"?

General shape We have $a < 0$ so we are looking at an $\cap$ shape.

Where it crosses the axes The y-intercept occurs when $x = 0$. For any quadratic $y = ax^2 + bx + c$ when $x = 0$ we have $y = c$ so here the y-intercept is $y = 5$.

The x-intercepts are the roots of $-3x^2 - 4x + 5$, ie the solution of $-3x^2 - 4x + 5 = 0$. Using the quadratic formula, with $a = -3, b = -4, c = 5$ we can say

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(-3)(5)}}{2(-3)} = \frac{4 \pm \sqrt{76}}{-6}.$$

We can rewrite $\sqrt{76} = \sqrt{19 \times 4} = 2\sqrt{19}$ which gives us $x = \frac{4 \pm 2\sqrt{19}}{-6}$ which means our two roots are

$$x = -\frac{1}{3}(2 + \sqrt{19}) \qquad \text{and} \qquad x = -\frac{1}{3}(2 - \sqrt{19}).$$

In this form, the answers are exact, but for the purposes of sketching, it is useful to have the decimal approximations $x = 0.7863$ and $x = -2.120$ (4 sig fig)

Where the minimum or maximum is For all quadratics, the minimum or maximum point occurs midway between the roots, at

$$x = -\frac{b}{2a}.$$

We can easily see this by rewriting the quadratic formula slightly

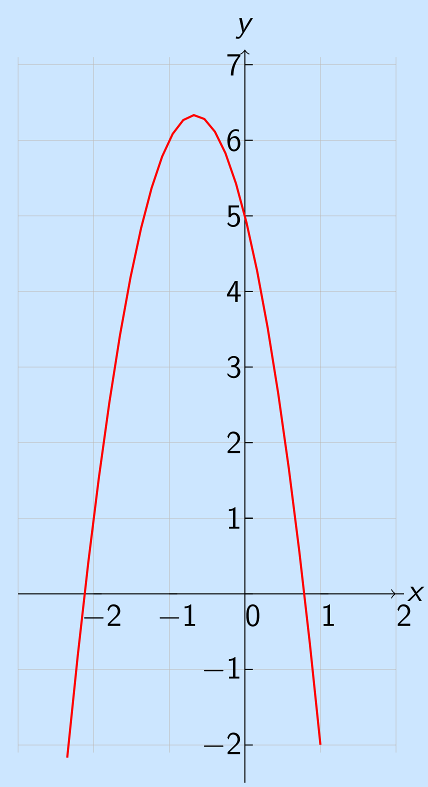
$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

In this example, we can therefore say the minimum/maximum is at $x = -\frac{2}{3}$. We can also say, since a= -3<0, we know the curve is hill-shaped, so we are looking for a maximum.

The height of the min/max Substituting $x = -\frac{2}{3}$ into the quadratic function, we can find y at the maximum:

$$y_{max} = -3\left(-\frac{2}{3}\right)^2 - 4(-\frac{2}{3}) + 5 = \frac{19}{3}$$

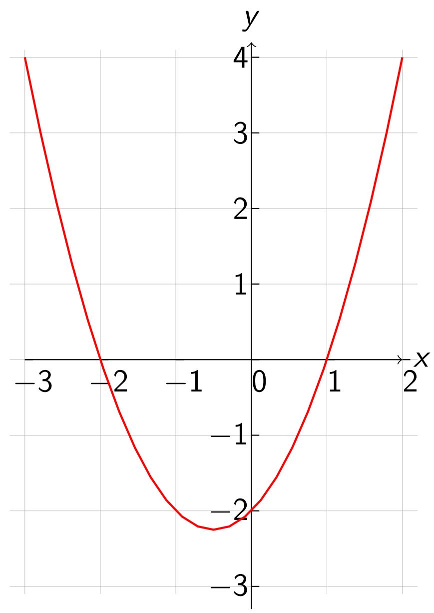
Now we can sketch the graph



The discriminant and roots

You will be familiar with the idea that a quadratic can have two, one, or no real roots. We can see this easily from either the graph, or the quadratic formula.

Graphically



Quadratic Formula Given that roots are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we can see that if the discriminant

- is positive, ie $b^2 - 4ac > 0$ then we have two real roots, because adding and subtracting the square root gives different results.

- if the discriminant is zero, ie $b^2 - 4ac = 0$ then the square root is zero, which means the $\pm$ makes no difference, and the only root is at $x = -b/2a$.

- if the discriminant is negative, ie $b^2 - 4ac < 0$ in which case, we're looking for the square root of a negative number. Since we know that positive numbers and negative numbers squared are both positive, and zero squared is zero, the we know that there are no **real** roots because we cannot find the square root of a negative number... or can we?

In next weeks session we will explore the square root to negative numbers i.e. what are called **complex numbers**.

Exercise 2.

1. Complete the square for the following quadratics:

   (a) $x^2 + 5x + 6$ (b) $2x^2 - 20x + 44$ (c) $-4x^2 + 16x - 17$

   Check your answers by carefully expanding the brackets again. Is your answer equal to the original quadratic?

2. Solve the equations (where possible)

   (a) $x^2 + 5x + 6 = 0$

   (b) $2x^2 - 20x + 44 = 0$

   (c) $-4x^2 + 16x - 17 = 0$

   Check your answers by substituting the values back into the quadratic expression and checking that you get zero. If you believe there is no (real) solution, how can you check that this is correct?

3. For each of the above equations, find the y-intercept and the value of x for which we have a minimum or maximum. Is it a maximum or minimum? Sketch the three quadratic functions and annotate with as much information as you can.

   Check your answers by plotting the functions in software such as www.desmos.com. Was your plot correct?