Preparation for Week 2

1 Derivation of Fourier coefficients

Definition 1. **Some useful integrals in Fourier Analysis**

$$\int_{-L}^{L} \cos\frac{n\pi t}{L}\,dt = 0, \qquad \int_{-L}^{L} \sin\frac{n\pi t}{L}\,dt = 0$$

$$\int_{-\pi}^{\pi} \cos(mt)\cos(nt)\,dt = \begin{cases} 0 & \text{if} \quad m \neq n \\ \pi & \text{if} \quad m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mt)\sin(nt)\,dt = \begin{cases} 0 & \text{if} \quad m \neq n \\ \pi & \text{if} \quad m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mt)\cos(nt)\,dt = 0$$

Sine and cosine are called *orthogonal functions* because of this set of integral properties they have.

1.0.1 Derivation of a_0

We integrate both sides of (??) with respect to t over one full period. Note that when $T = 2\pi$ the range of integration is between $-\pi$ and π and that $\frac{2n\pi t}{T} = nt$. So,

$$\int_{-\pi}^{\pi} f(t)\,dt = \int_{-\pi}^{\pi} \frac{1}{2}a_0\,dt + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty}(a_n\cos(nt) + b_n\sin(nt))dt$$

Assume you can integrate term by term along the series (take care with this sort of thing with infinite series, as long as the Dirichlet conditions are satisfied then it is fine).

$$\int_{-\pi}^{\pi} f(t)\,dt = \int_{-\pi}^{\pi} \frac{1}{2}a_0\,dt + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos(nt)dt + b_n\int_{-\pi}^{\pi}\sin(nt)\,dt\right)$$

performing the integrals:

$$\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{2}a_0(\pi - (-\pi)) + \sum_{n=1}^{\infty}(a_n \times 0 + b_n \times 0)$$

and simplify for a_0

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, dt$$

as it says. (Note that an intuitive way of thinking of a_0 is that is the average value of the function.)

1.0.2 Derivation of $a_n\,(n = 1, 2, \cdots)$

For a_n we multiply throughout by $\cos(mt)$, then integrate throughout by t, over one time period.

$$\int_{-\pi}^{\pi} f(t)\, \cos(mt)\, dt = \int_{-\pi}^{\pi} \frac{1}{2}a_0\, \cos(mt)\, dt + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty}(a_n \cos(mt)\cos(nt) + b_n \cos(mt)\sin(nt))\, dt$$

Again assume you can integrate term by term along the series

$$\int_{-\pi}^{\pi} f(t)\, \cos(mt)\, dt = \int_{-\pi}^{\pi} \frac{1}{2}a_0\, \cos(mt)\, dt + \sum_{n=1}^{\infty}(a_n \int_{-\pi}^{\pi} \cos(mt)\cos(nt)dt + b_n \int_{-\pi}^{\pi} \cos(mt)\sin(nt))\, dt$$

We perform the integrals (this involves some tricky trignometric integrations: there is the need to define m and n as integers, and the cases $m \neq n$ and $m = n$ will need to be treated separately). The result is:

$$\int_{-\pi}^{\pi} f(t)\, \cos(mt)\, dt = \int_{-\pi}^{\pi} \frac{1}{2}a_0 \times 0 + \sum_{n=1}^{\infty}(a_n \times 0) + a_m \times \pi + \sum_{n=1}^{\infty}(b_n \times 0)$$

simplifying

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, \cos(mt)\, dt$$

1.0.3 Derivation of $b_n\,(n = 1, 2, \cdots)$

This is the same as the method shown for finding a_n but using sines instead of cosines.

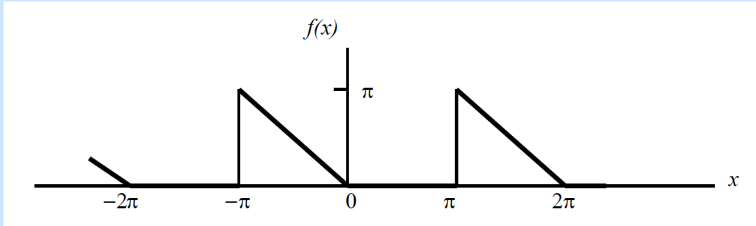
Exercise 1.

Perform the necessary steps to obtain expressions for a_n and b_n.

1.1 Fourier Series of function $f(x)$, Examples

Example 1. **Find the Fourier series**

Find the Fourier series for the function defined in the following figure



where the analytical solution is given by

$$\begin{aligned} f(x) &= \begin{cases} -x & \text{where} \quad -\pi \le x < 0 \\ 0 & \text{where} \quad 0 \le x < \pi \end{cases} \\ &= f(x+2\pi) \end{aligned} \tag{1}$$

Solution: The Fourier series can be deduced using equations (??)-(??) in Definition ?? where $L = \pi$.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(nx)) + b_n \sin(nx)\right] \tag{2}$$

where the Fourier coefficients are given by:

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \end{aligned}$$

Find a_0 Substitute $f(x)$ into a_0 above:

$$\begin{aligned} a_0 &= \frac{1}{\pi} \textstyle\int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{\pi} \left(\textstyle\int_0^{\pi} 0 dx + \int_{-\pi}^{0} -x dx \right) \\ &= \frac{1}{\pi} \left[-\frac{x^2}{2} \right]_{-\pi}^{0} \\ &= \frac{\pi}{2} \end{aligned} \tag{3}$$

Find a_n To find a_n we have:

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0}(-x)\cos nx dx \\
&= -\frac{1}{\pi}\int_{-\pi}^{0} x\cos nx dx \quad \text{Using integration by parts} \\
&= -\frac{1}{\pi}\left(\left[x\frac{\sin nx}{n}\right]_{-\pi}^{0} - \frac{1}{n}\int_{-\pi}^{0}\sin nx dx\right) \\
&= -\frac{1}{\pi}\left((0-0) - \frac{1}{n}\left[-\frac{\cos nx}{n}\right]_{-\pi}^{0}\right) \\
&= \frac{1}{\pi n^2}(1-\cos n\pi)
\end{aligned} \tag{4}
$$

But $\cos n\pi = \pm 1$ (1 for even and -1 for odd). Thus

$$a_n = -\frac{2}{\pi n^2} \quad (n \text{ odd}) \qquad a_n = 0 \quad (n \text{ even})$$

Find b_n To find b_n we use the same method as a_n to obtain:

$$b_n = -\frac{1}{n} \quad (n \text{ odd}) \qquad b_n = \frac{1}{n} \quad (n \text{ even})$$

Thus we have

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \ldots\right) - \left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\right)$$

The partial sums are shown in figure 1

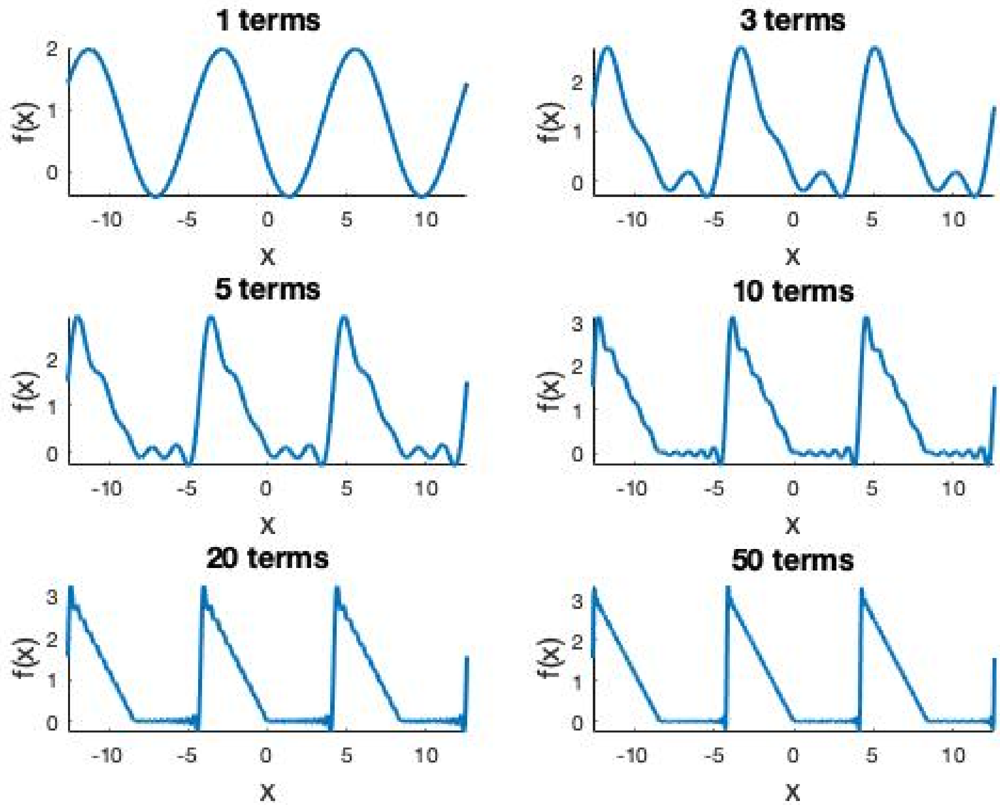
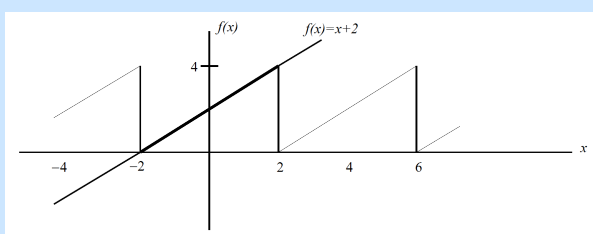


Figure 1: Plot of Example 2 for n=1, 3, 5, 10, 20 and 50 plotted

Example 2.

Find the Fourier series of $f(x) = x + 2$ in the range $-2 \leq x \leq 2$



Solution

Find a_0 To find a_0 we have:

$$a_0 = \tfrac{1}{2}\int_{-2}^{2}(x+2)dx = \tfrac{1}{2}\left[\frac{x^2}{2}+2x\right]_{-2}^{2} = \tfrac{1}{2}\left[(2+4)-(2-4)\right] = 4$$

Find a_n To find a_n:

$$a_n = \tfrac{1}{2}\int_{-2}^{2}(x+2)\cos\frac{n\pi x}{2}\,dx \qquad - \qquad \text{integrate by parts}$$

$$\text{Let } u = x+2 \quad \frac{dv}{dx} = \cos\frac{n\pi x}{2}$$

$$\frac{du}{dx} = 1 \qquad v = \frac{2}{n\pi}\sin\frac{n\pi x}{2}$$

$$\therefore\ a_n = \tfrac{1}{2}\left\{\frac{2(x+2)}{n\pi}\sin\frac{n\pi x}{2} - \frac{2}{n\pi}\int_{-2}^{2}\sin\frac{n\pi x}{L}\right\}$$

$$= \frac{1}{2}\left\{\frac{2(x+2)}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right\}_{-2}^{2} = \frac{2}{n^2\pi^2}\left[\cos n\pi - \cos(-n\pi)\right] = 0$$

Find b_n To find b_n:

$$b_n = \frac{1}{2}\int_{-2}^{2}(x+2)\sin\frac{n\pi x}{2}\,dx \qquad u = x+2\ \frac{dv}{dx} = \sin\frac{n\pi x}{2} \qquad v = -\frac{2}{n\pi}\cos\frac{n\pi x}{2}$$

$$= \tfrac{1}{2}\left[-\frac{2(x+2)}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_{-2}^{2}$$

$$= \frac{1}{n\pi}\left[-4\cos n\pi + 0\right] = -\frac{4}{n\pi}(-1)^n$$

Deduce $f(x)$ The final answer is given as:

$$\therefore \quad x+2=\frac{1}{2}a_0+\sum_1^\infty\left[a_n\cos\frac{n\pi x}{2}+b_n\sin\frac{n\pi x}{2}\right]$$

$$=2-\sum_1^\infty\frac{4}{n\pi}(-1)^n\sin\frac{n\pi x}{2}$$

$$=2+\frac{4}{\pi}\left\{\sin\frac{\pi x}{2}-\frac{1}{2}\sin\pi x+\frac{1}{3}\sin\frac{3\pi x}{2}\ldots\right\}$$

The partial sums are shown in Figure 2.

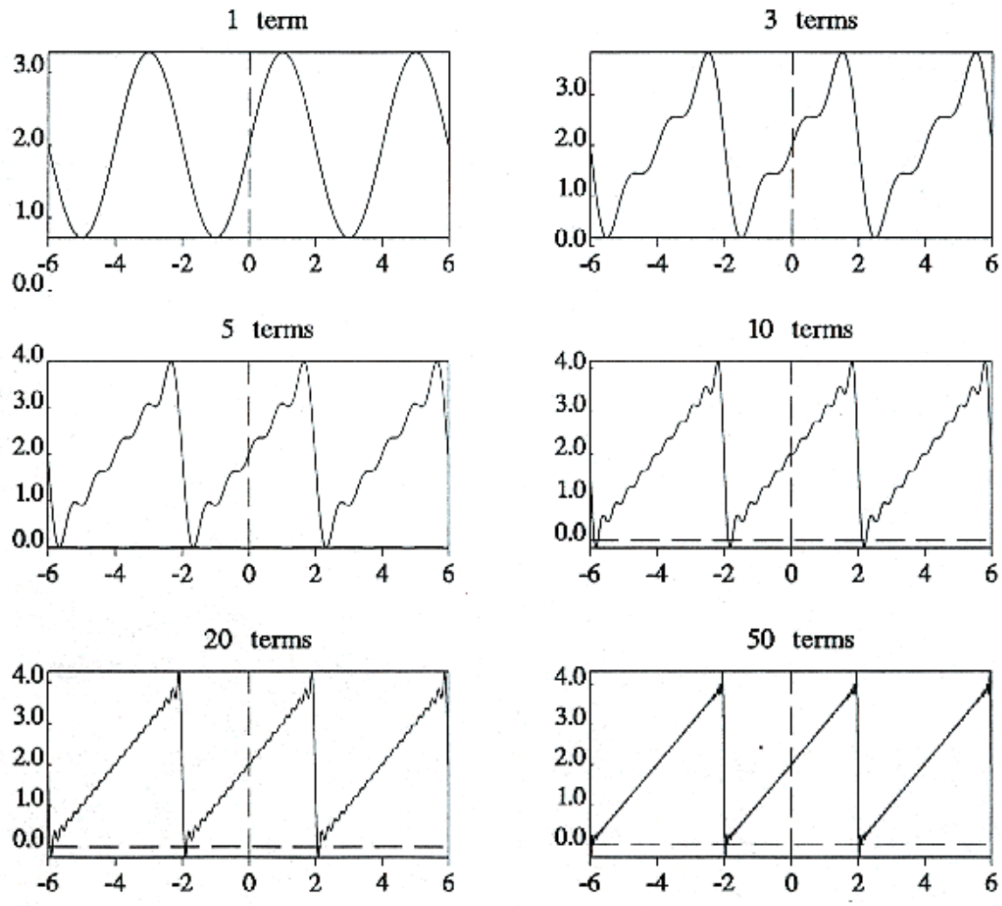
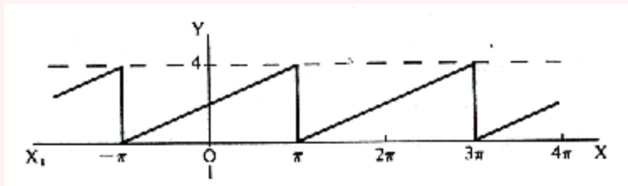


Figure 2: Partial sums for n=1, 3, 5, 10, 20 and 50 plotted where $f(x) = 2 - \sum_{n=1}^{n} \frac{4}{n\pi}(-1)^n \sin\left(\frac{n\pi x}{2}\right)$

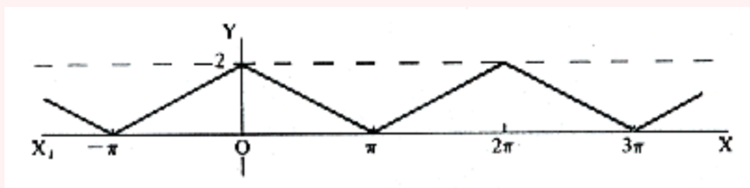
Exercise 2.

Establish the Fourier Series for the following functions:

1. $f(x) = f(x + 2\pi)$



2. $f(x) = f(x + 2\pi)$



3. $f(x) = f(x + 2\pi)$, and

$$f(x) = \begin{cases} 6 & -\pi < x < 0 \\ 2 & 0 < x < \pi \end{cases}$$

Exercise - answers

1. $f(x) = 2 + \frac{4}{\pi}\left(\sin x - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) + \cdots\right)$
2. $f(x) = 1 + \frac{8}{\pi^2}\left(\cos x + \frac{1}{3^2}\cos(3x) + \frac{1}{5^2}\cos(5x) + \cdots\right)$
3. $f(x) = 4 - \frac{8}{\pi}\left(\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \cdots\right)$