Preparation for Week 1

Infinite series and partial sums

A review of infinite series and partial sums is necessary before continuing with the characteristics of trigonometric functions.

Series turn out to be very important in lots of different places and consist of a string of terms added together. An infinite series is a series that never ends. The notation used is the traditional sum symbol which sums over an interval given by the first term found by using $n = a$ and ends with $n = b$ i.e., $\sum_{n=a}^{b}$. An example of this is the following infinite series

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \tag{1}$$

Definition 1. Maclaurin series and Taylor series

If $f(X)$ has a derivative for all orders at $x = c$ (that is, if $f^{(n)}(c)$ exists for all $n = 0, 1, 2, 3, \ldots$) then

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + \ldots$$

is called the **Taylor Series** of f about c. If $c = 0$ then we have the **Maclaurin Series**.

These series are slightly different forms of a similar idea, namely, they are infinite series whose sum is equal to some function.

Exercise 1.

Show that the Maclaurin series of $\cos(x)$ and $\sin(x)$ can be given by:

$$\begin{aligned}
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \\
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots
\end{aligned}$$

provided x is in radians.

These series are only truly valid when we include an infinite number of terms.

However, in practice, it is not possible to write down or plot infinite numbers of terms. Therefore, this is approximated by working out the first n terms. Such approximations are known as partial sums (because they're only part of the total sum required).

A series is said to be **converging** if it gets closer and closer to the original function as more and more terms are added.

Exercise 2. **Showing convergence**

The exponential function may be approximated as follows:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{2}$$

1. Plot, using appropriate technology, a graph of $f(x) = e^x$ and also, on the same axes, the following partial sums of the power series:
   (a) $p_0(x) = 1$
   (b) $p_1(x) = 1 + x$
   (c) $p_2(x) = 1 + x + \frac{x^2}{2!}$
   (d) $p_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$
2. Following the established pattern, obtain and plot $p_4(x)$ and $p_5(x)$.
3. How do the partial sums relate to the function $f(x) = e^x$?

Transforming Functions and Graphs

Applying certain changes to a function affects it's graph in predictable ways. Here I want you to revise that and try to explain it.

Exercise 3. Transforming Functions and Graphs

Our aim here is to provide a *precise* description of the transformations to a function when we apply certain changes, ie comparing the graph of $y = f(t)$ with

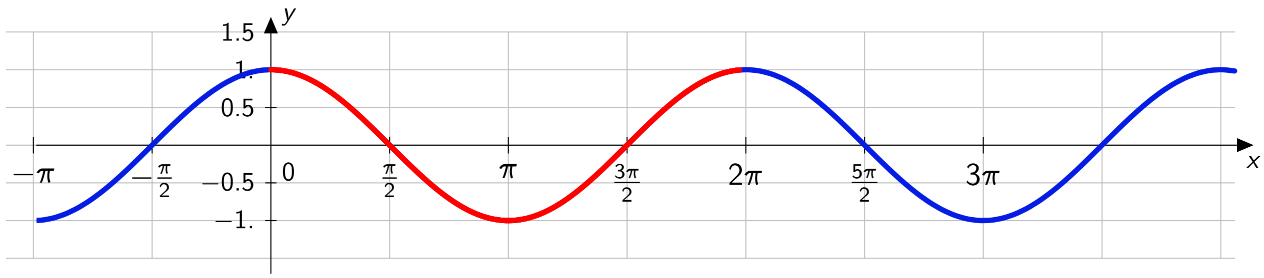
$$y = f(t+\phi), \quad y = f(t) + B, \quad y = f(\omega t) \quad \text{and} \quad y = A.f(t)$$

where ϕ, B, ω and A are constants. For example, if $f(t) = \sin(t)$ then we want to compare $y = \sin(t)$ with $y = \sin(t+\phi)$, $y = B + \sin(t)$, $y = \sin(\omega t)$ and $y = A\sin(t)$.

1. Try to recall the transformation and make a note of them.
2. Plot some test cases to check whether you've remembered correctly. For example, you could compare $y = t^2$ with $y = 3t^2$, $y = (3t)^2$, $y = (t+3)^2$, $y = t^2 + 3$ etc. Revise your answers to question 1 if necessary.
3. Now consider special cases; is your description of what happens still accurate if the constants ϕ, B, ω and A are between 0 and 1, equal to 0, equal to 1, negative. Again, revise your answer to question 1 if necessary.
4. Now for the hard part: can you explain **why** each of the transformations has the effect it does? Write a summary of the affect that ϕ, B, ω and A have on the function.

Periodic Functions and Graphs

A periodic function is one which has the exact same shape/ behaviour repeated over and over. For example, $y = \cos x$ below:



Exercise 4.

1. Define
   (a) Period, T
   (b) Frequency, f
   (c) Angular frequency, ω
2. For the cosine wave shown above
   (a) what is the period?
   (b) what is the frequency?
   (c) what is the angular frequency?
3. Think about the relationship $f(x) = f(x+T)$. What does it mean? Why is it the algebraic definition of a periodic function? Use the cosine wave as an example to think about it.
4. For each of the functions below, sketch the function and calculate both the period and amplitude:
   (a) $y = \sin x$
   (b) $y = 3\sin 2x$
   (c) $y = \frac{1}{2}\sin 2x$,
   (d) $y = \frac{1}{2}\sin 3x$
   (e) $y = -3\sin\left(\frac{x}{2}\right)$

   [Remember that if we write $\sin 2x$ we mean $\sin(2x)$]
5. Calculate the period of $y = \sin\left(\frac{2x}{3}\right)$ and then sketch it.