211

1. Studiați convergența seriei cu termeni pozitivi. $\sum_{n=2}^{\infty} n \cdot tg\frac{\pi}{2^n}$

2. Fie $f(x,y)=2\ln\left(tg\frac{x}{y}\right)$. Calculați $\nabla f\left(\frac{\pi}{2},2\right)$

3. Studiați convergența integralei $J(\alpha)=\int_1^{\infty}\frac{1}{x^{\alpha}(1+x^{\alpha})}dx$. Calculați $J(2)$.

4. Calculați valorile extreme ale funcției $f(x,y)=x^2y$ relativ la $A=\{(x,y)\in\mathbb{R} \mid x^2+y^2\le 1\}$

C.RAP.

1. $$\lim \frac{x_n}{x_{n+1}} = \lim \frac{n\, tg\frac{\pi}{2^n}}{(n+1)\, tg\frac{\pi}{2^{n+1}}} = \frac{n}{n+1}\cdot\frac{tg\frac{\pi}{2^n}}{\frac{\pi}{2^n}}\cdot\frac{\frac{\pi}{2^{n+1}}}{tg\frac{\pi}{2^{n+1}}}\cdot\frac{\frac{\pi}{2^n}}{\frac{\pi}{2^{n+1}}} = \frac{2^{n+1}}{2^n} = 2 > 1 \Rightarrow \text{convergentă}$$

2. $f(x,y)=2\ln\left(tg\frac{x}{y}\right)$ $\qquad a=\left(\frac{\pi}{2},2\right)$

$\nabla f(a)=?$

$\nabla f(a)=\left(\frac{\partial f}{\partial x}(a), \frac{\partial f}{\partial y}(a)\right)$

$$\frac{\partial f}{\partial x}=\left(2\ln\left(tg\frac{x}{y}\right)\right)' = \frac{2}{tg\frac{x}{y}}\cdot\left(tg\frac{x}{y}\right)'$$
$$=\frac{2}{tg\frac{x}{y}}\cdot\frac{1}{\cos^2\frac{x}{y}}\cdot\left(\frac{x}{y}\right)'$$
$$=\frac{2}{tg\frac{x}{y}}\cdot\frac{1}{\cos^2\frac{x}{y}}\cdot\frac{1}{y}$$
$$=2\cdot\frac{\cancel{\cos\frac{x}{y}}}{\sin\frac{x}{y}}\cdot\frac{1}{\cos^{\cancel{2}}\frac{x}{y}}\cdot\frac{1}{y}$$
$$=\frac{2}{y}\cdot\frac{1}{\sin\frac{x}{y}\cdot\cos\frac{x}{y}}=\frac{4}{y}\cdot\frac{1}{2\sin x\cos x}=\frac{4}{y\cdot\sin 2\frac{x}{y}}$$
$$\Rightarrow \frac{\partial f}{\partial x}\left(\frac{\pi}{2},2\right)=\frac{4\cdot 2}{2\cdot\sin 2\cdot\frac{\frac{\pi}{2}}{2}}=\frac{2}{\sin\frac{\pi}{2}}=2$$
x y

acelasi lucru
$$\frac{\partial f}{\partial y}=\dots=\frac{4}{x\cdot\sin 2\frac{x}{y}}$$
$$\Rightarrow \frac{\partial f}{\partial y}\left(\frac{\pi}{2},2\right)=\frac{4}{\frac{\pi}{2}\cdot\sin 2\cdot\frac{\frac{\pi}{2}}{2}}=\frac{8}{\pi}$$
$$\Rightarrow \nabla f(a)=\left(2,\frac{8}{\pi}\right)$$

aici e greșit

$$\frac{\partial f}{\partial x}=\frac{2}{tg\frac{x}{y}}\left(tg\frac{x}{y}\right)'=\frac{2}{tg\frac{x}{y}}\cdot\frac{1}{\cos^2\frac{x}{y}}\cdot\left(\frac{x}{y}\right)'=\frac{2}{tg\frac{x}{y}\cos^2\frac{x}{y}}\cdot\frac{x'y-y'x}{y^2}=\frac{2}{\dots}\cdot\frac{y'}{y^2}=\frac{2}{\dots}\cdot\frac{1}{y}$$
$$=2\cdot\frac{\cancel{\cos\frac{x}{y}}}{\sin\frac{x}{y}}\cdot\frac{1}{\cos^{\cancel{2}}\frac{x}{y}}\cdot\frac{1}{y}=\frac{2}{y}\cdot\frac{1}{\sin\frac{x}{y}\cos\frac{x}{y}}=\frac{4}{y}\cdot\frac{1}{2\sin\frac{x}{y}\cos\frac{x}{y}}$$
$$=\frac{4}{y}\cdot\frac{1}{2\sin\frac{x}{y}\cos\frac{x}{y}}=\frac{4}{y}\cdot\frac{1}{\sin 2\frac{x}{y}}=\frac{4}{y\cdot\sin 2\frac{x}{y}}$$
$$\frac{\partial f}{\partial x}\left(\frac{\pi}{2},2\right)=\frac{4\cdot 2}{2\cdot\sin 2\frac{\pi}{4}}=\frac{2}{\sin\frac{\pi}{2}}=2$$

$$\frac{x'y-y'x}{y^2}=\frac{-x}{y^2}$$

$$\frac{\partial f}{\partial y}=\frac{2}{tg\frac{x}{y}}\left(tg\frac{x}{y}\right)'=\frac{2}{tg\frac{x}{y}}\cdot\frac{1}{\cos^2\frac{x}{y}}\cdot\left(\frac{x}{y}\right)'=$$
$$=\frac{2}{tg\frac{x}{y}}\cdot\frac{1}{\cos^2\frac{x}{y}}\cdot\left(-\frac{x}{y^2}\right)=$$
$$=2\cdot\frac{\cos\frac{x}{y}}{\sin\frac{x}{y}}\cdot\frac{1}{\cos^2\frac{x}{y}}\cdot\left(-\frac{x}{y^2}\right)=$$
$$=-\frac{2x}{y^2}\cdot\frac{1}{\sin\frac{x}{y}\cdot\cos\frac{x}{y}}=-\frac{4x}{y^2\sin 2\frac{x}{y}}$$
$$=-\frac{4x}{y^2}\cdot\frac{1}{\sin 2\frac{x}{y}}$$
$$\frac{\partial f}{\partial y}\left(\frac{\pi}{2},2\right)=-\frac{4\frac{\pi}{2}}{2^2\cdot\sin 2\cdot\frac{\pi}{2}/2}=-\frac{\frac{\pi}{2}}{\sin\frac{\pi}{2}}=-\frac{\pi}{2}$$

Fie $f(x,y) = 2\ln\left(tg\frac{x}{y}\right)$. Calculați $\nabla f\left(\frac{\pi}{2}, 2\right)$

$$\frac{\partial f}{\partial y} = \left(2\ln\left(tg\frac{x}{y}\right)\right)' = \frac{2}{tg\frac{x}{y}} \cdot \frac{1}{\cos^2\frac{x}{y}} \cdot \left(\frac{x}{y}\right)'$$

$$\left(\frac{x}{y}\right)' = \frac{x' \cdot y - y' \cdot x}{y^2}$$

$$= \frac{2}{tg\frac{x}{y}} \cdot \frac{1}{\cos^2\frac{x}{y}} \cdot \left(-\frac{x}{y^2}\right)$$

$$= 2 \cdot \frac{\cancel{\cos\frac{x}{y}}}{\sin\frac{x}{y}} \cdot \frac{1}{\cos^2\frac{x}{y}} \cdot \left(-\frac{x}{y^2}\right)$$

$$= \frac{-2x}{y^2} \cdot \frac{1}{\sin\frac{x}{y} \cdot \cos\frac{x}{y}}$$

$$= \frac{-4x}{y^2} \cdot \frac{1}{\sin 2\frac{x}{y}}$$

$$= \frac{-4x}{y^2 \cdot \sin 2\frac{x}{y}}$$

$$\frac{\partial f}{\partial y}\left(\frac{\pi}{2}, 2\right) = \frac{-\cancel{4}\frac{\pi}{2}}{\cancel{4} \cdot \sin \cancel{2} \cdot \frac{\pi}{\cancel{4}\,2}}$$

$$= -\frac{\frac{\pi}{2}}{\sin\frac{\pi}{2}} \rightarrow 1$$

$$= -\frac{\pi}{2}$$

3) $\mathcal{I}(\alpha) = \int_1^\infty \frac{1}{x^\alpha (1+x^\alpha)} dx$

$\mathcal{I}(2) = ?$

$$f(x) = \frac{1}{x^\alpha (1+x^\alpha)} = \frac{1}{x^\alpha + x^{2\alpha}}$$

$$g(x) = \frac{1}{x^p}$$

$$\int_1^\infty \frac{1}{x^p} dx \begin{cases} \text{convergentă pt. } p>1 \\ \text{divergentă pt. } p\le 1 \end{cases}$$

$$f(x) \le g(x)$$

$$\left.\begin{array}{l} \frac{1}{x^\alpha + x^{2\alpha}} \le \frac{1}{x^p} \iff 2\alpha \ge p \Rightarrow \alpha \ge \frac{p}{2} \\ \int_1^\infty g(x)\,dx \text{ convergentă} \Rightarrow p>1 \end{array}\right\} \Rightarrow \alpha > \frac{1}{2} \Rightarrow \int_1^\infty \frac{1}{x^\alpha(1+x^\alpha)} dx \text{ convergentă pt. } \alpha \in \left(\frac{1}{2}, \infty\right)$$

$$\mathcal{I}(2) = \int_1^\infty \frac{1+x^2-x^2}{x^2(1+x^2)} dx = \int_1^\infty \frac{1}{x^2} dx - \int_1^\infty \frac{1}{1+x^2} dx$$

$$= \lim_{v\to\infty} \int_1^v \frac{1}{x^2} dx - \lim_{v\to\infty} \int_1^v \frac{1}{1+x^2} dx$$

$$= \lim_{v\to\infty} \frac{x^{-1}}{-1}\Big|_1^v - \lim_{v\to\infty} (\overset{\frac{\pi}{2}}{\operatorname{arctg} v} - \operatorname{arctg} 1)$$

$$= \underset{-\frac{1}{v}+\frac{1}{1}}{0+1} - \overset{2)}{\frac{\pi}{2}} + \frac{\pi}{4}$$

$$= 1 - \frac{3\pi}{4}$$

4. $f(x,y) = x^2 y$

$A = \{(x,y) \in \mathbb{R} \mid x^2+y^2 \le 1\}$

$\left.\begin{array}{l} A\text{- compactă} \\ f\text{- cont} \end{array}\right\} \xRightarrow{T.W.}$ f își atinge min și max relativ la A

$\text{int } A = \{(x,y) \in \mathbb{R} \mid x^2+y^2 < 1\}$

$\text{fr } A = \{(x,y) \in \mathbb{R} \mid x^2+y^2 = 1\}$

$A = \text{int } S \cup \text{fr } S$

1) int S:

$\nabla f(x,y) = (0,0) \Rightarrow \begin{cases} y=0 \\ x=0 \end{cases} \Rightarrow P(0,0)$ p. critic

$H(f)(x,y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad H(f)(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \Delta_1 = 0$, $\Delta_2 = -1$ $\Rightarrow$ nu putem det. natura

$d^2 f(0,0)(u_1,u_2) = -2u_1u_2$ $\quad d^2 f(0,0)(1,1) = -2 < 0$, $d^2 f(0,0)(1,-1) = 2 > 0$ $\Rightarrow d^2 f$ indef. $\Rightarrow P(0,0)$ pa

2) fr S:

Fie $F(x,y) = x^2+y^2-1$

$\text{fr } A = \{(x,y) \in A \mid F(x,y) = 0\}$

! x^2y în loc de xy

$L(x,y,\lambda) = f(x,y) + \lambda F(x,y) = xy + \lambda(x^2+y^2-1)$

$= xy + \lambda x^2 + \lambda y^2 - \lambda$

$\begin{cases} \frac{\partial L}{\partial x} = 0 \\ \frac{\partial L}{\partial y} = 0 \\ \frac{\partial L}{\partial \lambda} = 0 \end{cases} \Rightarrow \begin{cases} y + 2x\lambda = 0 \\ x + 2y\lambda = 0 \\ x^2+y^2-1 = 0 \end{cases} \Rightarrow$

$\nabla L(x,y,\lambda) = 0$

$y = -2x\lambda$

$x + 2\lambda(-2x\lambda) = 0$

$x - 4x\lambda^2 = 0$

$x(1-4\lambda^2) = 0$

I. $x=0 \Rightarrow y=0$ impos.

II. $1-4\lambda^2 = 0$

$4\lambda^2 = 1 \Rightarrow \lambda^2 = \frac{1}{4} \Rightarrow \lambda = \pm \frac{\sqrt{4}}{4}$

$\lambda = \pm \frac{2}{4} = \frac{1}{2}$

I. $\lambda = \frac{1}{2} \Rightarrow y = -\cancel{2}x \cdot \frac{1}{\cancel{2}} = -x$

$x^2 + x^2 - 1 = 0$

$2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{2} \Rightarrow y = \mp\frac{1}{2}$

II. $\lambda = -\frac{1}{2} \Rightarrow y = -\cancel{2}x \cdot \frac{1}{\cancel{2}} = x$

$x^2 + x^2 - 1 = 0$

$2x^2 = 1 \Rightarrow x = \pm\frac{1}{2} \Rightarrow y = \pm\frac{1}{2}$

Pct. critice: $P_1\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$; $P_2\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$P_3\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$; $P_4\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$

$f\left(\frac{1}{2}, \frac{1}{2}\right)$

$f\left(-\frac{1}{2}, -\frac{1}{2}\right)$

$f\left(-\frac{1}{2}, \frac{1}{2}\right)$

$f\left(\frac{1}{2}, -\frac{1}{2}\right)$

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① $\sum \frac{3+(-1)^n}{2^{n+2}}$

② $f(x,y) = \arcsin \frac{x^2}{y+x}$ derivate parțiale + $\frac{\partial f}{\partial x}(1,1)$ și $\frac{\partial f}{\partial y}(1,1)$

③ Seria Taylor + mulțimea de convergență

$f(x) = \frac{x+3}{1+3x}$

⑤ $J(p,q) = \int_0^1 x^{p-1} q(1-x)^{q-1}\,dx$ $\forall p>0,\ q>0$

convergența integralei + $J(\frac{1}{2},\frac{1}{2})$

1. $$\sum_{n=0}^{\infty} \frac{3+(-1)^n}{2^{n+2}} = \frac{4}{2^2} + \frac{2}{2^3} + \frac{4}{2^4} + \frac{2}{2^5} + \frac{4}{2^6} + \frac{2}{2^7} + \dots$$

$$= 1 + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^4} + \dots$$

$$= 1 + \underbrace{\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5}}_{\frac{1}{2} \cdot \frac{1-\left(\frac{1}{2^2}\right)^n}{1-\frac{1}{2^2}}} + \dots$$

$$\left(\frac{1}{2^2}\right)^n \to 0$$

$$= 1 + \frac{1}{2} \cdot \frac{1}{\frac{3}{4}}$$

$$= 1 + \frac{2}{3}$$

$$= \frac{5}{3}$$

+ convergența

$$\sum_{n=0}^{\infty} \frac{3+(-1)^n}{2^{n+2}} = \sum_{n=0}^{\infty} \frac{3}{2^{n+2}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \cdot 2^2} = \frac{3}{4}\sum_{n=0}^{\infty} \frac{1}{2^n} + \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{1}{2^n}\right)^n$$

$S_1 = \frac{3}{4}\sum_{n=0}^{\infty} \frac{1}{2^n}$ e serie geometrică

rație: $\left(\frac{1}{2}\right) < 1 \Rightarrow$ seria e convergentă

$S_2 = \frac{1}{4}\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n$ e serie geometrică

$-\frac{1}{2} \in (-1, 1) \Rightarrow$ seria e convergentă

(2) $f(x,y) = \arcsin \frac{x^2}{y+x}$ derivate partiale $\frac{\partial f}{\partial x}(1,1)$ și $\frac{\partial f}{\partial y}(1,1)$

2. $f(x,y) = \arcsin \frac{x^2}{y+x}$

$$\frac{\partial f}{\partial x} = \left(\arcsin \frac{x^2}{y+x}\right)' = \frac{1}{\sqrt{1-\left(\frac{x^2}{y+x}\right)^2}} \cdot \left(\frac{x^2}{y+x}\right)'$$

$$= \frac{1}{\sqrt{1-\frac{x^4}{(y+x)^2}}} \cdot \frac{(x^2)' \cdot (y+x) - x^2 \cdot (y+x)'}{(y+x)^2}$$

$$= \frac{2x(y+x) - x^2}{(y+x)^2} \cdot \frac{1}{\sqrt{\dots}}$$

$$= \frac{2xy + 2x^2 - x^2}{(y+x)^2} \cdot \frac{1}{\dots}$$

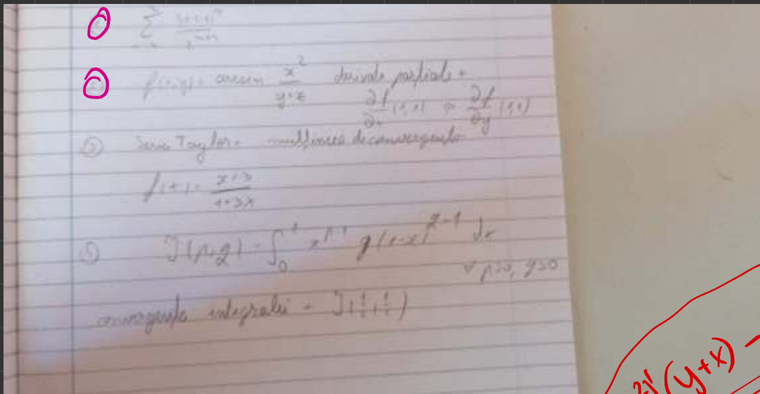
$$= \frac{x(2y+x)}{\sqrt{1-\frac{x^4}{(y+x)^2}} \cdot (y+x)^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{\dots}} \cdot \frac{-x^2}{(y+x)^2} = \frac{-x^2}{\sqrt{1-\frac{x^4}{(y+x)^2}}\,(y+x)^2}$$

$$\frac{\partial f}{\partial x}(1,1) = \frac{3}{\sqrt{1-\frac{1}{4}} \cdot 4} = \frac{3}{\sqrt{\frac{3}{4}} \cdot 4} = \frac{1}{\sqrt{\frac{3}{4}}} \cdot \frac{3}{4} = \frac{1}{\frac{\sqrt{3}}{2}} \cdot \frac{3}{4} = \frac{\cancel{2}^{1}}{\sqrt{3}} \cdot \frac{3}{\cancel{4}_{2}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\frac{\partial f}{\partial y}(1,1) = \frac{-1}{\sqrt{1-\frac{1}{4}} \cdot 4} = \frac{-1}{\sqrt{\frac{3}{4}} \cdot 4} = \frac{1}{\frac{\sqrt{3}}{2}} \cdot -\frac{1}{4} = -\frac{\cancel{2}^{1}}{\sqrt{3}} \cdot \frac{1}{\cancel{4}_{2}} = -\frac{\sqrt{3}}{6}$$

Examen 212



① $\sum \frac{3+(-1)^n}{2^{n+2}}$

② $f(x,y) = \arcsin \frac{x^2}{y+x}$ derivate parțiale + $\frac{\partial f}{\partial x}(1,1)$ și $\frac{\partial f}{\partial y}(1,1)$

③ Seria Taylor + mulțimea de convergență

$f(x) = \frac{x+3}{1+3x}$

④ $J(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$ $\forall p>0, q>0$

convergența integralei + $J(\frac{1}{2},\frac{1}{2})$

1. $$\sum_{n=0}^{\infty} \frac{3+(-1)^n}{2^{n+2}}$$

$$\left(\frac{x^2}{y+x}\right)' = \frac{(x^2)'(y+x) - x^2\cdot(y+x)'}{(y+x)^2} = \frac{2x\cdot(y+x) - x^2}{(y+x)^2} = \frac{2xy+2x^2-x^2}{(y+x)^2} = \frac{2xy+x^2}{(y+x)^2}$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{3+(-1)^n}{2^{n+2}}} = \lim_{n\to\infty} \frac{\sqrt[n]{3+(-1)^n}}{4} \;..\; \text{nu}$$

$\Rightarrow$ criteriul comparației ...

2. $f(x,y) = \arcsin \frac{x^2}{y+x}$ derivate parțiale

$+ \ \frac{\partial f}{\partial x}(1,1), \frac{\partial f}{\partial x}(1,1)$

$$\frac{\partial f}{\partial x} = \left(\arcsin \frac{x^2}{y+x}\right)' = \frac{1}{\sqrt{1-\left(\frac{x^2}{y+x}\right)^2}} \cdot \left(\frac{x^2}{y+x}\right)'$$

$$= \frac{1}{\sqrt{\ldots}} \cdot \left(x^2 \cdot \frac{1}{y+x}\right)'$$

$$= \frac{1}{\sqrt{\ldots}} \cdot \left(2x \cdot \frac{1}{y+x} + x^2 \cdot \left(-\frac{1}{(x+y)^2}\right)\right)$$

$$= \frac{1}{\sqrt{1-\frac{x^4}{(x+y)^2}}} \cdot \left(\frac{^{x+y)}2x}{x+y} - \frac{x^2}{(x+y)^2}\right)$$

$$= \frac{x(x+2y)}{(x+y)^2\sqrt{1-\frac{x^4}{(x+y)^2}}}$$

$$\frac{\partial f}{\partial y} = \ldots = -\frac{x^2}{(y+x)^2\sqrt{1-\frac{x^4}{(y+x)^2}}}$$

$$\Rightarrow \frac{\partial f}{\partial x}(1,1) = \frac{1(1+2)}{3^2\sqrt{1-\frac{1}{4}}} = \frac{1}{3\cdot\sqrt{\frac{3}{4}}}$$

$$= \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9}$$

$$\frac{\partial f}{\partial y}(1,1) = -\frac{1}{4\sqrt{1-\frac{1}{4}}} = \frac{\cancel{2}}{\cancel{4}\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

$$= \frac{\sqrt{3}}{6}$$

3. $f(x) = \dfrac{x+3}{1+3x}$ seria Taylor + mulțimea de conv

$$\lim_{n\to\infty} \left(f(x) - T_n(x) \right) = 0$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$$

???

4. $\mathcal{I}(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} \, dx$ $\qquad \mathcal{I}\left(\frac{1}{2}, \frac{1}{2}\right)$

- de calculat integrala improprie
- apoi după înlocuire facem schimbare de variabilă $\sqrt{x} \overset{not}{=} t \Rightarrow x = t^2$

Subiect 212

1. Studiați convergența și calculați suma. $\sum_{n=0}^{\infty} \frac{3+(-1)^n}{2^{n+2}}$

$= \sum_{n=0}^{\infty} \frac{3}{2^{n+2}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+2}}$ serie alternantă, $\lim_{n\to\infty} \frac{1}{2^{n+2}} = 0 \Rightarrow$ e convergentă ②

$\Downarrow$

$3\sum_{n=0}^{\infty} \frac{1}{2^{n+2}}$ - convergentă ① pt. că $\lim_{n\to\infty} \sum = 0$

①, ② $\Rightarrow \sum_{n=0}^{\infty} \frac{3+(-1)^n}{2^{n+2}}$ - convergentă.

$$\sum_{n=0}^{\infty} \frac{3+(-1)^n}{2^{n+2}} = 3\sum_{n=0}^{\infty} \frac{1}{2^{n+2}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+2}} = \frac{3}{4}\sum_{n=0}^{\infty} \frac{1}{2^n} + \frac{1}{4}\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n$$

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \dots + \frac{1}{2^n} = 1 \cdot \frac{\left(\frac{1}{2}\right)^n - 1}{\frac{1}{2} - 1} = \left(\left(\frac{1}{2}\right)^n - 1\right) \cdot (-2) = 2 \ (n \to \infty).$$

$$\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \dots \left(-\frac{1}{2}\right)^n = 1 \cdot \frac{\left(-\frac{1}{2}\right)^n - 1}{\underbrace{-\frac{1}{2} - 1}_{-\frac{3}{2}}} = -\frac{2}{3}\left(\left(-\frac{1}{2}\right)^n - 1\right) = \frac{2}{3} \ (n \to \infty)$$

$$\frac{3}{4}\sum_{n=0}^{\infty} \frac{1}{2^{n+2}} + \frac{1}{4}\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = \frac{3}{\cancel{4}_2} \cdot \cancel{2} + \frac{1}{\cancel{4}_2} \cdot \frac{\cancel{2}}{3} = {}^{3/}\frac{3}{2} + \frac{1}{6} = \frac{10}{6} = \frac{5}{3}.$$

2. Calculați derivatele parțiale $f(x,y) = \arcsin \frac{x^2}{y+x}$ în $(1,1)$

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{\sqrt{1 - \left(\frac{x^2}{y+x}\right)^2}} \cdot \frac{2x(y+x) - x^2}{(y+x)^2} =$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{1}{\sqrt{1 - \left(\frac{x^2}{x+y}\right)^2}} \cdot \frac{-x^2}{(y+x)^2}$$

$$\frac{\partial f}{\partial x}(1,1) = \frac{1}{\sqrt{1 - \left(\frac{1}{2}\right)^2}} \cdot \frac{2\cdot 2 - 1}{2^2} = \frac{1}{\sqrt{1 - \frac{1}{4}}} \cdot \frac{3}{4} = \frac{\cancel{2}}{\sqrt{3}} \cdot \frac{3}{\cancel{4}_2} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}.$$

$$\frac{\partial f}{\partial y}(1,1) = \frac{\cancel{2}}{\sqrt{3}} \cdot \left(\frac{-1}{2^{\cancel{2}}}\right) = -\frac{1}{2\sqrt{3}}$$

S.t.p: $\sum_{n=0}^{\infty} a_n$ - convergentă $\Rightarrow \lim_{n\to\infty} a_n = 0$

dacă nu e 0 $\Rightarrow$ e divergentă

3. Calculați seria Taylor și mulțimea de convergență.

$$f(x) = \frac{x+3}{1+3x} = \frac{x+3}{3\left(x+\frac{1}{3}\right)} = \frac{1}{3}\cdot\frac{x+\frac{1}{3}-\frac{1}{3}+3}{x+\frac{1}{3}} = \frac{1}{3}\left(\frac{\cancel{x+\frac{1}{3}}}{\cancel{x+\frac{1}{3}}} + \frac{\frac{8}{3}}{x+\frac{1}{3}}\right)$$

$$f(x) = \frac{1}{3}+\frac{8}{9}\cdot\frac{1}{x+\frac{1}{3}} = \frac{1}{3}+\frac{8}{9}\underbrace{\left(x+\frac{1}{3}\right)^{-1}}$$ – notăm $(x+c)^{-1}$

calculăm derivata de ordin n

$f'(x) = \left((x+c)^{-1}\right)' = -1(x+c)^{-2}$

$f''(x) = 2(x+c)^{-3}$ $\quad\dots\quad$ $f^{(n)}(x) = (-1)^n\cdot n!\,(x+c)^{-n-1} = P(n)\ \forall n\geq 1$

I Verificăm $P(1)$

II Presupunem $P(k)$

III Demonstrăm $P(k+1)$

$$f^n(x) = \begin{cases} \frac{1}{3}+\frac{8}{9}\left(x+\frac{1}{3}\right)^{-1}, & n=0 \\ \frac{8}{9}\left[(-1)^n\cdot n!\left(x+\frac{1}{3}\right)^{-n-1}\right], & n\neq 0. \end{cases}$$

$$f^n(0) = \begin{cases} \frac{1}{3}+\frac{8}{\cancel{9}_3}\cdot\cancel{3} = 3, & n=0 \\ \frac{8}{9}\left((-1)^n\, n!\cdot 3^{n+1}\right), & n\neq 0 \end{cases}$$

seria Taylor: $\sum\limits_{n=0}^{\infty}\frac{f^n(0)}{n!}\cdot(x-x_0)^n = \sum\limits_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\cdot x^n$

$$3+\sum_{n=1}^{\infty}\frac{\frac{8}{9}(-1)^n\cancel{n!}\cdot 3^{n+1}}{\cancel{n!}}\cdot x^n = 3+\frac{8}{9}\sum_{n=1}^{\infty}(-1)^n\cdot 3^{n+1}\cdot x^n$$

$$= 3+8\sum_{n=1}^{\infty}(-1)^n\, 3^{n-1}\, x^n$$

aflăm raza de convergență.

$$r = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{3^{n-1}}{3^n} = \lim_{n\to\infty}\frac{1}{3}\cdot\frac{\cancel{3^n}}{\cancel{3^n}} = \frac{1}{3}.$$

$(x_0-r,\ x_0+r) \subseteq J \subseteq [x_0-r,\ x_0+r]$.

$\left(-\frac{1}{3},\frac{1}{3}\right) \subseteq J \subseteq \left[-\frac{1}{3},\frac{1}{3}\right]$, e converg. pe $\left(-\frac{1}{3},\frac{1}{3}\right)$, verificăm în $\frac{1}{3}$ și $-\frac{1}{3}$

pt $x=\frac{1}{3} \Rightarrow \sum\limits_{n=1}^{\infty}(-1)^n 3^{n-1}\left(\frac{1}{3}\right)^n = \sum\limits_{n=1}^{\infty}(-1)^n\cdot 3^{n-1-n} = \sum\limits_{n=1}^{\infty}(-1)^n\cdot 3^{-1} = \frac{1}{3}\sum\limits_{n=1}^{\infty}(-1)^n$

pt n=par $\lim\limits_{n\to\infty} 1-1+1-1\dots 1 = 0$
pt n-impar $\lim\limits_{n\to\infty} 1-1+1-1+\dots-1 = 1$
} avem două subșiruri ale lui S_n care nu au aceeași limită

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Pt $x = -\frac{1}{3}$ la fel $\Rightarrow$ seria Taylor e divergentă.

4. Studiați convergența lui $J(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$ și $J\left(\frac{1}{2},\frac{1}{2}\right)$

$J(p,q)$ nu e definită în 1 dacă $q-1<0$.
nu e definită în 0 dacă $p-1<0$

luăm $p,q \in (0,1)$ pentru a acoperi toate cazurile.

$$\int_0^1 x^{p-1}(1-x)^{q-1}dx = \underbrace{\int_0^{\frac{1}{2}} x^{p-1}(1-x)^{q-1}dx}_{J_1} + \underbrace{\int_{\frac{1}{2}}^1 x^{p-1}\cdot(1-x)^{q-1}dx}_{J_2}$$

$$J_1 = \int_0^{\frac{1}{2}} x^{p-1}(1-x)^{q-1}dx$$

I, $p-1 \geq 0 \Rightarrow J_1$ - definită $\Rightarrow$ convergentă

II $p-1<0$, proprietățile cu p și λ

P3: $(0,\frac{1}{2}]$

$\lim\limits_{x \searrow 0} x^{p'}\cdot x^{p-1}(1-x)^{q-1}$ găsim p' pt care $\lim \in (0,\infty)$

alegem $p' = -p+1 \Rightarrow \lim\limits_{x\searrow 0} x^{-p+1}\cdot x^{p-1}(1-x)^{q-1} = 1 \in (0,\infty)$.

$1-p<1 \Rightarrow p>0$ „A" $\Rightarrow J_1$ - convergentă $\forall p>0$.

$$J_2 = \int_{\frac{1}{2}}^1 x^{p-1}(1-x)^{q-1}dx$$

I $q-1 \geq 0$ - definită $\Rightarrow$ e convergentă

II $q-1<0$, propr. cu p și λ

P1. $[\frac{1}{2},1)$

$\lim\limits_{x\nearrow 1} (1-x)^{p'}\cdot \overset{=1}{x^{p-1}}\cdot(1-x)^{q-1} =$

alegem $p' = 1-q \Rightarrow \lim\limits_{x\nearrow 1}(1-x)^{1-q+q-1}\cdot x^{p-1} = \lim\limits_{x\nearrow 1} x^{p-1} = 1 \in (0,\infty)$

$1-q<1 \Rightarrow q>0 \Rightarrow J_2$ - convergentă $\forall q>0$.

$J_1+J_2 \Rightarrow$ convergentă $\forall q,p>0$.

$J\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}dx = \int_0^1 \frac{1}{\sqrt{x(1-x)}}dx =$ $\quad t=\sqrt{x}$, $dt = \frac{1}{2\sqrt{x}}dx$

$= 2\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = 2\lim\limits_{u\searrow 0} \arcsin\sqrt{x}\Big|_u^{\frac{1}{2}} + \lim\limits_{\mu\nearrow 1}\arcsin\sqrt{x}\Big|_{\frac{1}{2}}^{\mu} = -\frac{\pi}{4}$

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1. Raza + mulțimea de conv. $\sum_{n=0}^{\infty} \frac{\ln(n+1)}{n+1} \cdot x^n$

2. convergența: $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx$ ✓

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3. $f(x) = \frac{1}{4x^2+1}$ dezv. în serie Taylor în $x_0 = 0$

$f^{(10)}(x_0) = ?$

4. pct. critice + natura lor $f:(0,+\infty)^2 \to \mathbb{R}$, $f(x,y) = x^3 \cdot y^2 \cdot (5-x-y)$ ✓

1. $\sum_{n=0}^{\infty} \frac{\ln(n+1)}{n+1} \cdot x^n$

$$a_n = \frac{\ln(n+1)}{n+1}$$

$$r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left| \frac{\frac{\ln(n+1)}{n+1}}{\frac{\ln(n+2)}{n+2}} \right| = \lim_{n\to\infty} \left| \left(\frac{\ln(n+1)}{n+1} \cdot \frac{n+2}{\ln(n+2)} \right) \right| =$$

$$= \lim_{n\to\infty} \left| \underbrace{\frac{n+2}{n+1}}_{\downarrow\, 1} \cdot \underbrace{\frac{\ln(n+1)}{\ln(n+2)}}_{\downarrow\, 1} \right| = 1$$

$$\Rightarrow (-1,1) \subseteq \mathcal{J} \subseteq [-1,1]$$

pt. $x = -1$

$$\sum_{n=0}^{\infty} \frac{\ln(n+1)}{n+1} \cdot (-1)^n$$

Obs. că a_n e șir desc. de termeni pozitivi (1)

$$\lim_{n\to\infty} \frac{\ln(n+1)}{n+1} = 0 \quad (2)$$

$$(\ln(n+1) \ll n+1)$$

$$\xrightarrow[\text{C. Leibnitz}]{(1)(2)} \sum_{n=0}^{\infty} (-1)^n \cdot a_n \text{ - conv.}$$

pt. $x = 1$

$$\sum_{n=0}^{\infty} \frac{\ln(n+1)}{n+1} > \sum_{n=0}^{\infty} \frac{1}{n+1} \quad (*)$$

Știm: $\sum_{n=0}^{\infty} \frac{1}{n+1} \sim \sum_{n=0}^{\infty} \frac{1}{n}$ serie canonică cu $p=1$ $\Rightarrow$ divergentă

(*) C. COMP.

$$\Rightarrow \sum_{n=0}^{\infty} \frac{\ln(n+1)}{n+1} \text{ - divergentă}$$

$$\Rightarrow \mathcal{J} = [-1,1)$$

2. $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx$

VAR. I

$x^2 \overset{not.}{=\!=} t$

$$\lim_{t \nearrow 1} \int_0^1 \frac{t}{\sqrt{1-t^2}}\,dt = -\sqrt{1-t^2}\,\Big|_0^1 = -0+1 = 1 < +\infty \Rightarrow \text{convergentă}$$

VAR. II

$f : [0,1) \to [0,+\infty)$

$f(x) = \dfrac{x^2}{\sqrt{1-x^4}}$ → funcție poz. și local integrabilă pe $[0,1)$ (1)

$$\lim_{x \nearrow 1} (1-x)^p \cdot \frac{x^2}{\sqrt{1-x^4}}$$

$$= \lim_{x \nearrow 1} (1-x)^p \cdot \frac{x^2}{\sqrt{1-x^2}\cdot\sqrt{1+x^2}}$$

$$= \lim_{x \nearrow 1} (1-x)^p \frac{x^2}{\sqrt{1-x}\cdot\sqrt{1+x}\cdot\sqrt{1+x^2}}$$

$$= \lim_{x \nearrow 1} (1-x)^p \cdot \frac{x^2}{(1-x)^{\frac{1}{2}}\cdot\sqrt{1+x}\cdot\sqrt{1+x^2}}$$

$$\underline{\underline{p \overset{not.}{=} \tfrac{1}{2}}} \lim_{x \nearrow 1} \cancel{(1-x)^{\frac{1}{2}}} \cdot \frac{x^2}{\cancel{(1-x)^{\frac{1}{2}}}\cdot\sqrt{1+x}\cdot\sqrt{1+x^2}}$$

$$\left.\begin{aligned} &= \frac{1}{\sqrt{2\cdot 2}} = \frac{1}{2} < +\infty \\ &p = \frac{1}{2} < 1 \end{aligned}\right\} \overset{(1)}{\Longrightarrow} \int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx \text{ convergentă}$$

3. $f(x) = \dfrac{1}{4x^2+1}$ serie Taylor $x_0 = 0$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$f^{10}(x_0) = ?$

$f'(x) = -\dfrac{\overset{=8\cdot 1}{8x}}{(4x^2+1)^2}$ $=8^0$

$f''(x) = \dfrac{\overset{=128}{96x^2} - \overset{=8\cdot 1}{8}}{(4x^2+1)^3}$

$f'''(x) = -\dfrac{384x\,(4x^2-1)}{(4x^2+1)^4} = \underset{96\cdot 16}{1536x^3} - \underset{\substack{32\cdot 12 \\ 48\cdot 8}}{384x}$

$\vdots$

$f^n(x) = \dfrac{}{(4x^2+1)^{n+1}} \cdot (-1)^n$

4. $f : (0, +\infty)^2 \to \mathbb{R}$

$$f(x,y) = x^3 \cdot y^2 \cdot (5-x-y)$$

$$\frac{\partial f}{\partial x} = y^2 \cdot \frac{\partial f}{\partial x}\left(x^3 \cdot (5-x-y)\right) = y^2 \cdot \left(3x^2 \cdot (5-x-y) + x^3 \cdot \underbrace{(5-x-y)'}_{=-1}\right)$$

$$= y^2 \cdot \left(15x^2 - 3x^3 - 3x^2y - x^3\right)$$

$$= y^2 \left(15x^2 - 4x^3 - 3x^2y\right) = 15x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$\frac{\partial f}{\partial y} = x^3 \cdot \frac{\partial f}{\partial y}\left(y^2 \cdot (5-x-y)\right) = x^3 \cdot \left(2y \cdot (5-x-y) + y^2 \cdot \underbrace{(5-x-y)'}_{=-1}\right)$$

$$= x^3 \cdot \left(10y - 2yx - 2y^2 - y^2\right)$$

$$= x^3 \left(10y - 2yx - 3y^2\right) = 10x^3y - 2x^4y - 3x^3y^2$$

$$\Delta f(x,y) = \left(15x^2y^2 - 4x^3y^2 - 3x^2y^3,\ 10x^3y - 2x^4y - 3x^3y^2\right)$$

$$\Leftrightarrow \begin{cases} 15x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0 \\ 10x^3y - 2x^4y - 3x^3y^2 = 0 \end{cases} \Leftrightarrow \begin{cases} x^2y^2(15 - 4x - 3y) = 0 \\ x^3y(10 - 2x - 3y) = 0 \end{cases}$$

$$x, y \in (0, \infty) \Rightarrow x^2y^2,\ x^3y \neq 0$$

$$\Leftrightarrow \begin{cases} 15 - 4x - 3y = 0 \\ 10 - 2x - 3y = 0 \end{cases} \Leftrightarrow \begin{cases} 4x + 3y = 15 \\ 2x + 3y = 10 \end{cases} \Leftrightarrow \begin{cases} 3y = 15 - 4x \\ 3y = 10 - 2x \end{cases} \Leftrightarrow 15 - 4x = 10 - 2x$$

$$5 = 2x \Rightarrow x = \frac{5}{2} \Rightarrow y = \frac{5}{3}$$

$\Rightarrow \left(\frac{5}{2}, \frac{5}{3}\right)$ puncte critice

$$a = \left(\frac{5}{2}, \frac{5}{3}\right)$$

$$a = \left(\frac{5}{2}, \frac{5}{3}\right)$$

$$\frac{\partial f}{\partial x} = 15x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$\frac{\partial f}{\partial y} = 10x^3y - 2x^4y - 3x^3y^2$$

$$\frac{\partial^2 f}{\partial x^2} = (15x^2)' \cdot y^2 + 15x^2(y^2)' - \left((4x^3)' \cdot y^2 + 4x^3(y^2)'\right) - \left((3x^2)' \cdot y^3 + 3x^2(y^3)'\right)$$

$$= 30xy^2 - 12x^2y^2 - 6xy^3$$

$$\frac{\partial^2 f}{\partial y^2} = (10x^3)'y + 10x^3 \cdot y' - \left((2x^4)' \cdot y + 2x^4 \cdot y'\right) - \left((3x^3)' \cdot y^2 + 3x^3(y^2)'\right)$$

$$= 10x^3 - 2x^4 - 6x^3y$$

$$\frac{\partial f}{\partial x \partial y} = (15x^2)' \cdot y^2 + 15x^2(y^2)' - \left((4x^3)' \cdot y^2 + 4x^3(y^2)'\right) - \left((3x^2)' \cdot y^3 + 3x^2(y^3)'\right)$$

$$= 30x^2y - 8x^3y - 9x^2y^2$$

$$= \frac{\partial f}{\partial y \partial x}$$

$$\frac{\partial^2 f}{\partial x^2}(a) = 30 \cdot \frac{5}{2}\left(\frac{5}{3}\right)^2 - 12\left(\frac{5}{2}\right)^2\left(\frac{5}{3}\right)^2 - 6 \cdot \frac{5}{2}\left(\frac{5}{3}\right)^3$$

$$= \frac{\cancel{150}^{75}}{\cancel{2}_1} \cdot \frac{25}{9} - \cancel{12}^{\cancel{3}\,1} \cdot \frac{25}{\cancel{4}_1} \cdot \frac{25}{\cancel{9}_3} - \frac{\cancel{30}^{15}}{\cancel{2}_1} \cdot \frac{125}{27}$$

$$= \frac{^{3)}1875}{9} - \frac{^{9)}625}{3} - \frac{1875}{27}$$

$$= \frac{\cancel{5625} - \cancel{5625} - 1875}{27}$$

$$= -\frac{1875}{27}$$

• • •

I. Dacă $\Delta_i > 0$ $\forall i = \overline{1,2}$ $\Rightarrow$ pozitiv definită $\Rightarrow$ $a(a_1, a_2)$ – punct de minim

II. Dacă $\Delta_i \cdot (-1)^k > 0$ $\forall i = \overline{1,2}$ $\Rightarrow$ negativ definită $\Rightarrow$ $a(a_1, a_2)$ – punct de maxim

III. Altfel, a – punct șa (doar punct critic)

4. pct. critice + natura lor $f:(0,+\infty)^2 \to \mathbb{R}$, $f(x,y) = x^3 \cdot y^2 \cdot (5-x-y)$

$$f:(0,\infty)^2 \to \mathbb{R}$$

$$f(x,y) = x^3y^2(5-x-y)$$

$$\frac{\partial f}{\partial x} = y^2 \cdot \frac{\partial f}{\partial x} x^3(5-x-y) = y^2\left(3x^2(5-x-y) + x^3(-1)\right) = 15x^2y^2 - 3x^3y^2 - 3x^2y^3 - y^2x^3$$
$$= 15x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$\frac{\partial f}{\partial y} = x^3 \cdot \frac{\partial f}{\partial y} y^2(5-x-y) = x^3\left(2y(5-x-y) + y^2(-1)\right) = x^3(10y - 2xy - 2y^2 - y^2)$$
$$= 10x^3y - 2x^4y - 3x^3y^2$$

$$\Delta f(x,y) = (15x^2y^2 - 4x^3y^2 - 3x^2y^3, 10x^3y - 2x^4y - 3x^3y^2)$$

$$\Leftrightarrow \begin{cases} 15x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0 \\ 10x^3y - 2x^4y - 3x^3y^2 = 0 \end{cases} \Leftrightarrow \begin{cases} x^2y^2(15-4x-3y) = 0 \\ x^3y(10-2x-3y) = 0 \end{cases} \Leftrightarrow \begin{cases} 15-4x-3y=0 \\ 10-2x-3y=0 \end{cases}$$

$$x,y \in (0,\infty) \Rightarrow x^2y^2, x^3y \neq 0$$

$$\Leftrightarrow \begin{cases} 4x+3y=15 \\ 2x+3y=10 \end{cases} \Leftrightarrow \begin{cases} 3y = 4x-15 \\ 3y = 2x-10 \end{cases} \Rightarrow 4x-15 = 2x-10$$
$$2x = 5 \Rightarrow x = \frac{5}{2} \Rightarrow y = -\frac{5}{3}$$

$\Rightarrow \left(\frac{5}{2}, -\frac{5}{3}\right)$ puncte critice

$$\frac{\partial f}{\partial x} = 15x^2y^2 - 4x^3y^2 - 3x^2y^3 \qquad \frac{\partial f}{\partial y} = 10x^3y - 2x^4y - 3x^3y^2$$

$$\frac{\partial^2 f}{\partial x^2} = 15 \cdot 2xy^2 + 0 \quad - 4 \cdot 3x^2y^2 + 0 \quad - 3 \cdot 2xy^3 + 0$$
$$= 30xy^2 - 12x^2y^2 - 6xy^3$$

$$\frac{\partial^2 f}{\partial y^2} = 10x^3 - 2x^4 - 6x^3y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 10 \cdot 3x^2y - 2 \cdot 4x^3y - 3 \cdot 3x^2y^2$$
$$= 30x^2y - 8x^3y - 9x^2y^2$$

$$\frac{\partial^2 f}{\partial x^2}(a) = \overset{15}{\cancel{30}} \cdot \frac{5}{\underset{1}{\cancel{2}}}\left(-\frac{5}{3}\right)^2 - 12\left(\frac{5}{2}\right)^2\left(-\frac{5}{3}\right)^2 - \overset{3}{\cancel{6}}\,\frac{5}{\underset{1}{\cancel{2}}}\left(-\frac{5}{3}\right)^3$$

$$= \frac{15 \cdot 5 \cdot 25}{9} - \overset{3}{\cancel{12}} \cdot \frac{25}{\underset{1}{\cancel{4}}} \cdot \frac{25}{9} - 15 \cdot \frac{25}{9}$$

$$= \cancel{\frac{1875}{9}} - \cancel{\frac{1875}{9}} - \frac{375}{9}$$

$$= -\frac{375}{9}$$

$$= -\frac{125}{3}$$

$$\frac{\partial^2 f}{\partial y^2}(a) = 10\left(\frac{5}{2}\right)^3 - 2\left(\frac{5}{2}\right)^4 - 6\left(\frac{5}{2}\right)^3\left(-\frac{5}{3}\right)$$

$$= \overset{5}{\cancel{10}} \cdot \frac{125}{\underset{4}{\cancel{8}}} - \overset{1}{\cancel{2}} \cdot \frac{625}{\underset{8}{\cancel{16}}} + \overset{2\,1}{\cancel{6}} \cdot \frac{125}{\underset{4}{\cancel{8}}} \cdot \frac{5}{\underset{1}{\cancel{3}}}$$

$$= \frac{5 \cdot 125}{4} - \frac{625}{8} + \frac{125 \cdot 5}{4}$$

$$= \frac{625}{4} - \frac{625}{8} + \frac{625}{4}$$

$$= \frac{1250 - 625 + 1250}{4}$$

$$= \frac{1875}{4}$$

$$\frac{\partial^2 f}{\partial x \partial y}(a) = 30\left(\frac{5}{2}\right)^2 \cdot \left(-\frac{5}{3}\right) - 8\left(\frac{5}{2}\right)^3 \cdot \left(-\frac{5}{3}\right) - 9\left(\frac{5}{2}\right)^2\left(-\frac{5}{3}\right)^2$$

$$= -\overset{5\ 10}{\cancel{30}} \cdot \frac{25}{\underset{2}{\cancel{4}}} \cdot \frac{5}{\underset{1}{\cancel{3}}} + \cancel{8}\,\frac{125}{\cancel{8}} \cdot \frac{5}{3} - \cancel{9} \cdot \frac{25}{4} \cdot \frac{25}{\cancel{9}}$$

$$= -\frac{25 \cdot 25}{2} + \frac{625}{3} - \frac{625}{4}$$

$$= -\frac{625}{2} + \frac{625}{3} - \frac{625}{4}$$

$$= \frac{-3750 + 1250 - 1875}{12}$$

$$= \frac{4375}{12}$$

$$H(f)(x,y) = \begin{pmatrix} -\frac{125}{3} & \frac{4375}{12} \\ \frac{4375}{12} & \frac{1875}{4} \end{pmatrix}$$

$$D_1 = -\frac{125}{3}$$

$$D_2 = -\frac{125}{3} \cdot \frac{1875}{4} - \frac{4375}{12} \cdot \frac{4375}{12}$$

$$D_2 = -\frac{234375}{12} - \frac{19140625}{12}$$

$$= -\frac{18906250}{12}$$

$\Rightarrow$ a punct şa

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① Studiați natura seriei cu termeni pozitivi

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{2^{n^2}}$$

② Determinați punctele critice și pct de extrem local (precizând tipul acestora) ale funcției:

$$f:(0,+\infty)^2 \to \mathbb{R},\ f(x,y)=xy+\frac{27}{x}+\frac{27}{y}$$

③ Se dă fct.

$$f(x,y)=(x^2+y^2)\cdot\left(\operatorname{arctg}\frac{y}{x}\right)^2.$$

Determinați constantele $\alpha,\beta\in\mathbb{R}$ a.î.:

$$\frac{\partial^2 f}{\partial x^2}(x,y)+\frac{\partial^2 f}{\partial y^2}(x,y)=\alpha+\beta\left(\operatorname{arctg}\frac{y}{x}\right)^2,\ \forall x,y\in(0,+\infty)^2$$

④ Determinați valorile parametrului $a>0$, pentru care

$$\int_0^{\infty}\frac{1-e^{-x}}{x^a}\,dx$$ este convergentă

① $$\lim_{n\to\infty}\frac{x_n}{x_{n+1}}=\lim_{n\to\infty}\frac{\frac{(n!)^2}{2^{n^2}}}{\frac{((n+1)!)^2}{2^{(n+1)^2}}}=\lim_{n\to\infty}\frac{\cancel{(n!)^2}}{2^{n^2}}\cdot\frac{2^{(n+1)^2}}{\frac{\cancel{((n+1)!)^2}}{(n+1)^2}}=\lim_{n\to\infty}\frac{2^{n^2+2n+1}}{2^{n^2}\cdot(n+1)^2}=\lim_{n\to\infty}\frac{\cancel{2^{n^2}}\cdot 2^{2n}\cdot 2}{\cancel{2^{n^2}}\cdot(n+1)^2}=$$

$$=\infty>1\Rightarrow \text{convergentă}$$

②

$f:(0,\infty)^2 \to \mathbb{R}$

$f(x,y) = xy + \frac{27}{x} + \frac{27}{y}$

$\frac{\partial f}{\partial x} = y + \frac{-27}{x^2} = y - \frac{27}{x^2}$

$\Delta f(x,y) = \left(y - \frac{27}{x^2}, x - \frac{27}{y^2}\right)$

$\frac{\partial f}{\partial y} = x + \frac{-27}{y^2} = x - \frac{27}{y^2}$

$\begin{cases} y - \frac{27}{x^2} = 0 \\ x - \frac{27}{y^2} = 0 \end{cases} \iff \begin{cases} y = \frac{27}{x^2} \\ x = \frac{27}{y^2} \end{cases} \iff \begin{cases} x^2 y = 27 \Rightarrow x=3 \ y=3; \ x=-3 \ y=3 \\ xy^2 = 27 \Rightarrow x=3 \ y=3; \ x=3 \ y=-3 \end{cases} \Rightarrow x=y=3$

$(3,3)$ punct critic

$\frac{\partial^2 f}{\partial x^2} = \left(y - \frac{27}{x^2}\right)'_x = -\frac{27 \cdot 2x}{x^4} = -\frac{54x}{x^4} = -\frac{54}{x^3}$

$\frac{\partial^2 f}{\partial y^2} = \left(x - \frac{27}{y^2}\right)'_y = -\frac{27 \cdot 2y}{y^4} = -\frac{54y}{y^4} = -\frac{54}{y^3}$

$\frac{\partial f}{\partial x \partial y} = \left(y - \frac{27}{x^2}\right)'_y = 1 = \frac{\partial f}{\partial y \partial x}$

$\frac{\partial^2 f}{\partial x^2}(3,3) = -\frac{54}{3^3} = -\frac{54}{81} = -\frac{2}{3}$

$\frac{\partial^2 f}{\partial y^2}(3,3) = -\frac{54}{3^3} = -\frac{2}{3}$

$H(f)(x,y) = \begin{pmatrix} -\frac{2}{3} & 1 \\ 1 & -\frac{2}{3} \end{pmatrix}$ Δ_2, Δ_1

$\Delta_1 = -\frac{2}{3} < 0$

$\Delta_2 = -\frac{5}{9} < 0$ $\Rightarrow$ punct sa

$$f(x,y) = (x^2+y^2)\left(\operatorname{arctg}\frac{y}{x}\right)^2$$

③

$$\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = \alpha + \beta\left(\operatorname{arctg}\frac{y}{x}\right)^2 \quad \forall x,y \in (0,+\infty)^2$$

$$\frac{\partial f}{\partial x} = (x^2+y^2)'\left(\operatorname{arctg}\frac{y}{x}\right)^2 - (x^2+y^2)\left(\left(\operatorname{arctg}\frac{y}{x}\right)^2\right)'$$

$$= 2x\left(\operatorname{arctg}\frac{y}{x}\right)^2 - (x^2+y^2)\cdot 2\operatorname{arctg}\left(\frac{y}{x}\right)\left(\operatorname{arctg}\frac{y}{x}\right)'$$

$$= 2x\left(\operatorname{arctg}\frac{y}{x}\right)^2 - (x^2+y^2)\cdot 2\operatorname{arctg}\left(\frac{y}{x}\right)\cdot\frac{1}{\left(\frac{y}{x}\right)^2+1}\cdot\left(\frac{y}{x}\right)'$$

$$= 2x\left(\operatorname{arctg}\frac{y}{x}\right)^2 - \frac{(x^2+y^2)\cdot 2\cdot\operatorname{arctg}\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)^2+1}\cdot\frac{-y}{x^2}$$

$$= 2x\left(\operatorname{arctg}\frac{y}{x}\right)^2 + \frac{2y(x^2+y^2)\operatorname{arctg}\left(\frac{y}{x}\right)}{\left(\left(\frac{y}{x}\right)^2+1\right)x^2}$$

$$\frac{\partial f}{\partial x} = 2\operatorname{arctg}\left(\frac{y}{x}\right)\cdot\left(-y + x\operatorname{arctg}\left(\frac{y}{x}\right)\right)$$

$$\frac{\partial f}{\partial y} = 2\operatorname{arctg}\left(\frac{y}{x}\right)\left(x + y\operatorname{arctg}\left(\frac{y}{x}\right)\right)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2\left(y^2 - 2xy\operatorname{arctg}\left(\frac{y}{x}\right) + (x^2+y^2)\left(\operatorname{arctg}\left(\frac{y}{x}\right)\right)^2\right)}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{2\left(x^2 + 2xy\operatorname{arctg}\left(\frac{y}{x}\right) + (x^2+y^2)\left(\operatorname{arctg}\left(\frac{y}{x}\right)\right)^2\right)}{x^2+y^2}$$

③ $f(x,y) = (x^2+y^2)\cdot\left(\operatorname{arctg}\frac{y}{x}\right)^2$

$$\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = \alpha + \beta\left(\operatorname{arctg}\frac{y}{x}\right)^2 \quad \forall x,y \in (0,+\infty)^2$$

$$\frac{\partial f}{\partial x} = 2x\left(\operatorname{arctg}\frac{y}{x}\right)^2 - 2y\left(\operatorname{arctg}\frac{y}{x}\right)$$

$$\frac{\partial f}{\partial y} = 2y\left(\operatorname{arctg}\frac{y}{x}\right)^2 + 2x\left(\operatorname{arctg}\frac{y}{x}\right)$$

$$\frac{\partial^2 f}{\partial x} = \dots$$

④ $\displaystyle\int_0^\infty \frac{1-e^{-x}}{x^a}\,dx \qquad a>0$

$\lim\limits_{x\to\infty} x^p\cdot f(x) = \lambda \qquad \lim\limits_{x\to\infty} x^p\cdot\frac{1-e^{-x}}{x_a} = \lambda$

$a>1$ și $a=p \Rightarrow \lim\limits_{x\to\infty} = 1 < +\infty$

I. $a<p \Rightarrow \lim = \infty$ – nu e ok

II. $a>p \Rightarrow \lim = 0$ – e ok că e $<\infty$

1. Convergență stp

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$$\sum_{n=0}^{\infty}\left(\frac{n}{n+2}\right)^{\sqrt{n^4+n}}$$

2. Puncte critice

$f:\mathbb{R}^3\to\mathbb{R}$ $\quad f(x) = (y+z-x)\cdot e^{-x^2-y^2-z^2}$

$f(x,y,z)$

✓

3. Se dă

$$J(\alpha)=\int_1^{\infty} x^{\alpha}\cdot \operatorname{arctg}(x^2)\,dx \quad ,\alpha\in\mathbb{R}$$

Converg + $J(-3)$

?4. $A=\bar{B}(0_2,1)\setminus\{(0,0)\}$ $\quad f:A\to\mathbb{R}$ $\quad f(x,y)=\dfrac{x^2+y^2}{|x|+|y|}$

a) $\exists \lim\limits_{(x,y)\to 0_2} f(x,y)$?

b) $\sup$ ~~(f)~~ $f(A)$, $\inf(f(A))=$?

1. criteriul radical

$$\sum_{n=0}^{\infty}\left(\frac{n}{n+2}\right)^{\sqrt{n^4+n}}$$

$$\lim_{n\to\infty}\sqrt[n]{x_n}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{n+2}\right)^{\sqrt{n^4+n}}}=\lim_{n\to\infty}\left(\frac{n}{n+2}\right)^{\frac{\sqrt{n^4+n}}{n}}=\lim_{n\to\infty}\left(\frac{n+2-2}{n+2}\right)^{\frac{\sqrt{n^4+n}}{n}}=\lim_{n\to\infty}\left(1-\frac{2}{n+2}\right)^{\frac{\sqrt{n^4+n}}{n}}$$

$\frac{\sqrt{n^4+n}}{n} \to \sqrt{n^4\left(1+\frac{n}{n^4}\right)}$

$$=\lim_{n\to\infty}\left(1-\frac{2}{n+2}\right)^{\left(-\frac{n+2}{2}\right)\cdot\left(-\frac{2}{n+2}\right)\cdot\frac{n^2\sqrt{1+\frac{1}{n^3}}}{n}}$$

$$=e^{\lim\limits_{n\to\infty}\frac{-2n^2\sqrt{1+\frac{1}{n^3}}}{n(n+2)}}$$

$\sqrt{1+\frac{1}{n^3}} \to 1$

$$=e^{-2}=\frac{1}{e^2}<1 \Rightarrow \text{convergentă}$$

2. $f:\mathbb{R}^3 \to \mathbb{R}$

$$f(x,y,z) = (y+z-x)\cdot e^{-x^2-y^2-z^2}$$

$$\frac{\partial f}{\partial x} = -1\cdot e^{-x^2-y^2-z^2} + (y+z-x)\, e^{-x^2-y^2-z^2}\cdot\underbrace{(-x^2-y^2-z^2)'}_{=-2x}$$

$$= -e^{-x^2-y^2-z^2} + e^{-x^2-y^2-z^2}\cdot(-2xy-2xz+2x^2)$$

$$= e^{-x^2-y^2-z^2}(2x^2-2xy-2xz-1)$$

$$\frac{\partial f}{\partial y} = e^{-x^2-y^2-z^2}(-2y^2+2yx-2yz+1)$$

$$\frac{\partial f}{\partial z} = e^{-x^2-y^2-z^2}(-2z^2+2zx-2zy+1)$$

$$\Delta f(x,y,z) = \left(e^{-x^2-y^2-z^2}(2x^2-2xy-2xz-1),\ e^{-x^2-y^2-z^2}(-2y^2+2yx-2yz+1),\ e^{-x^2-y^2-z^2}(-2z^2+2zx-2zy+1)\right)$$

$$\begin{cases} e^{-x^2-y^2-z^2}(2x^2-2xy-2xz-1)=0 \\ e^{-x^2-y^2-z^2}(-2y^2+2yx-2yz+1)=0 \\ e^{-x^2-y^2-z^2}(-2z^2+2zx-2zy+1)=0 \end{cases} \iff \begin{cases} 2x^2-2xy-2xz-1=0 \\ -2y^2+2yx-2yz+1=0 \\ -2z^2+2zx-2zy+1=0 \end{cases}$$

$$\iff \begin{cases} 2x(x-y-z)=1 & (1) \\ 2y(x-y-z)=-1 & (2) \\ 2z(x-y-z)=-1 & (3) \end{cases}$$

$$\frac{(1)}{(2)} \iff \frac{2x}{2y} = \frac{1}{-1} \iff \frac{x}{y} = -1 \Rightarrow x=-y$$

$$\frac{(2)}{(3)} \iff \frac{2y}{2z} = \frac{-1}{-1} \iff \frac{y}{z} = 1 \Rightarrow y=z$$

(1): $2(-y)(-y-y-y)=1$

$-2y\cdot -3y = 1$

$6y^2=1 \Rightarrow y=\pm\sqrt{\frac{1}{6}} = \pm\frac{1}{\sqrt{6}} = \pm\frac{\sqrt{6}}{6}$

CAZ I	CAZ II
$y=\frac{\sqrt{6}}{6}$	$y=-\frac{\sqrt{6}}{6}$
$x=-\frac{\sqrt{6}}{6}$	$x=\frac{\sqrt{6}}{6}$
$z=\frac{\sqrt{6}}{6}$	$z=-\frac{\sqrt{6}}{6}$

3. $\mathcal{I}(\alpha) = \int_1^\infty \underbrace{x^\alpha \cdot \operatorname{arctg}(x^2)}_{f(x)}\, dx, \qquad \alpha \in \mathbb{R}$

$\mathcal{I}(-3) = ?$

$\lim_{x\to\infty} x^p \cdot (x^\alpha \underbrace{\operatorname{arctg}(x^2)}_{\frac{\pi}{2}})\, dx$

$p = -\alpha$

$\frac{\pi}{2} \in (0,\infty) = \ell$

$\mathcal{I}(\alpha)$ - conv.

$\Leftrightarrow \begin{cases} p > 1 \\ \ell < \infty \end{cases} \Rightarrow \begin{matrix} -\alpha > 1 \\ \ell = \frac{\pi}{2} < \infty \end{matrix} \Rightarrow \alpha < 1$

$\mathcal{I}(\alpha)$ - convergentă pt. $\alpha \in (-\infty, 1)$

$\mathcal{I}(-3) = \int_1^\infty x^{-3} \cdot \operatorname{arctg}(x^2)\, dx$

$(x^{-2})' = -2 \cdot x^{-3}$

$= -\frac{1}{2} \cdot \int_1^\infty (x^{-2})' \cdot \operatorname{arctg}(x^2)\, dx$

$= -\frac{1}{2} \cdot x^{-2} \cdot \operatorname{arctg} x^2 \Big|_1^\infty + \frac{1}{2} \int_1^\infty x^{-2} \cdot \frac{-1}{x^2+1} \cdot 2x\, dx$

$= 0 + \frac{\pi}{8} + \frac{1}{2} \int_1^\infty \frac{1}{x^2+1}\, dx - \frac{1}{2} \int_1^\infty \frac{1}{x^2}\, dx$

$= \frac{\pi}{8} + \frac{1}{2} \operatorname{arctg} x \Big|_1^\infty - \frac{1}{2} \left(\frac{x^{-2+1}}{x^{-2+1}} \right) \Big|_1^\infty$

$= \cancel{\frac{\pi}{8}} + \frac{\pi}{4} - \cancel{\frac{\pi}{8}} + \frac{1}{2x} \Big|_1^\infty$

$= \frac{\pi}{4} + 0 - \frac{1}{2}$

$= \frac{\pi}{4} - \frac{1}{2}$

2022

1. Studiati convergenta si absolut convergenta seriei

$$\sum_{n=2}^{\infty}(-1)^n \frac{n}{n\sqrt{n}-1}$$

2. Studiati convergenta integralei improprii

$$\int_0^1 \frac{1}{e^{\sqrt{x}}-1}\,dx$$

3. Justificati egalitatea

$$\frac{1}{(1+x^2)^2} = \sum_{n=0}^{\infty}(-1)^n(n+1)x^{2n}, \quad \forall x \in I \subseteq \mathbb{R}$$

si determinati multimea de convergenta I a seriei din membrul drept.

4. Fie $a, b, c > 0$ constante, functia $f : (0,\infty)^3 \to \mathbb{R}$, $f(x,y,z) = x^a y^b z^c$ si multimea $S = \{(x,y,z) \in (0,\infty)^3 \,|\, x+y+z = 1\}$. Determinati $\sup f(S)$ stiind ca functia f atinge aceasta valoare.

1. $\sum_{n=2}^{\infty}(-1)^n \frac{n}{n\sqrt{n}-1}$

$S = \sum_{n=2}^{\infty}\left|(-1)^n \frac{n}{n\sqrt{n}-1}\right| = \sum_{n=2}^{\infty}\frac{n}{n\sqrt{n}-1}$

$\frac{n}{n\sqrt{n}-1} > \frac{n}{n\sqrt{n}-n} = \frac{\cancel{n}}{\cancel{n}(\sqrt{n}-1)} = \frac{1}{\sqrt{n}-1} > \frac{1}{n}$

$\sum \frac{1}{n}$ e divergentă, fiind serie armonică

$\Rightarrow \sum \frac{n}{n\sqrt{n}-1}$ e D, nu abs. conv.

Folosim crit. Leibniz: dacă $(a_n)_{n\in\mathbb{N}}$ e un şir descrescător de nr. poz. cu $\lim_{n\to\infty} a_n = 0$, atunci $\sum_{n=0}^{\infty}(-1)^n \cdot a_n$ e convergentă.

$a_n - a_{n+1} > 0 \iff \frac{n}{n\sqrt{n}-1} - \frac{n+1}{(n+1)\sqrt{n+1}-1} > 0$

$\iff n((n+1)\sqrt{n+1}-1) - (n+1)(n\sqrt{n}-1) > 0$

$\iff n(n+1)\sqrt{n+1} - n - (n(n+1)\sqrt{n} - n - 1) > 0$

$\iff n(n+1)\sqrt{n+1} - n > n(n+1)\sqrt{n} - n - 1 \quad | + n$

$\iff n(n+1)\sqrt{n+1} > n(n+1)\sqrt{n} - 1$

... > 0, adev pt. $n > \ldots$

$\Rightarrow$ şirul e $\searrow$, deci S convergentă

2. $\int_0^1 \frac{1}{e^{\sqrt{x}} - 1}\,dx$ - convergence

Avem pb. in $x = 0$

$\lambda = \lim_{x \searrow 0} (x-a)^p \cdot \frac{1}{\quad}$

3. $\frac{1}{(1+x^2)^2} = \sum_{n=0}^{\infty} (-1)^n (n+1) x^{2n} \quad \forall x \in I \subseteq \mathbb{R}$

$$\sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2} \qquad \forall x \in (-1,1)$$

$$\sum_{n=0}^{\infty} (-x)^{2n} = \sum_{n=0}^{\infty} (-1)^{2n} \cdot x^{2n} = \frac{1}{1+x^2} \quad \Big| \text{derivăm}$$

$$\sum_{n=0}^{\infty} (-1)^{2n} \cdot 2n \cdot x^{2n-1} = -\frac{1}{(1+x^2)^2} \cdot 2x \quad \Big| \cdot (-1)$$

$$\sum_{n=0}^{\infty} (-1)^{2n-1} \cdot 2n \cdot x^{2n-1} = \frac{1}{(1+x^2)^2}$$

$m = 2n-1$

$$\sum_{n=0}^{\infty} (-1)^{m} (m+1) \cdot x^{2m} = \frac{1}{(1+x^2)^2}$$

divergentă

2021

1. Studiati convergenta si absolut convergenta seriei

$$\sum_{n=1}^{\infty}(-1)^{n+1}\sin\frac{\pi}{\sqrt{n}}$$

2. Calculati integrala improprie

$$\int_1^{\infty}\frac{1}{x^3+x}\,\mathrm{d}x$$

3. Determinati punctele critice si punctele de extrem local (specificand tipul acestora) pentru functia

$$f:\mathbb{R}^2\to\mathbb{R},\quad f(x,y)=(x+xy+y^2)\sqrt{e^x}$$

4. Fie $g:(0,+\infty)^2\to\mathbb{R}$ o functie de clasa C^1. Exprimati relatia

$$-u\frac{\partial g}{\partial u}(u,v)+\frac{v}{1+2u}\frac{\partial g}{\partial v}(u,v)=1,\quad \forall(u,v)\in(0,+\infty)^2$$

in variabilele $(x,y)\in(0,+\infty)^2$, efectuand transformarea $u=\frac{y}{x}$ si $v=x+2y$. Determinati apoi o functie g cu proprietatile de mai sus. Verificare.

(1) $$\sum_{n=1}^{\infty}(-1)^{n+1}\sin\frac{\pi}{\sqrt{n}}$$

C. COMPARAȚIEI

$$\sum_{n=1}^{\infty}\left|(-1)^{n+1}\sin\frac{\pi}{\sqrt{n}}\right|=\sum_{n=1}^{\infty}\left|\sin\frac{\pi}{\sqrt{n}}\right|$$

$\frac{\pi}{\sqrt{n}}=\frac{\pi\cdot 1}{n^{\frac{1}{2}}}$ comparăm cu $\frac{1}{n}$ – serie geometrică

$\Downarrow$

$\sum_{n=1}^{\infty}\left|\sin\frac{\pi}{\sqrt{n}}\right|$ divergentă $\Rightarrow$ nu e abs. convergentă

$\frac{\pi}{\sqrt{n}}$ desc. sin în cadranul I e crescătoare

$\Rightarrow \sin\frac{\pi}{\sqrt{n}}$ descrescător pt. $n\geq 4$

$\xRightarrow{\text{Leibnitz}} \sum_{n=1}^{\infty}(-1)^{n+1}\sin\frac{\pi}{\sqrt{n}}$ convergentă

② $$\int_1^{\infty} \frac{1}{x^3+x}\,dx = \int_1^{\infty} \frac{1}{x(x^2+1)}\,dx = \int_1^{\infty} \frac{1}{x}\cdot\frac{1}{x^2+1}\,dx$$

$$f' = \frac{1}{x^2+1} \Rightarrow f = \operatorname{arctg} x$$

$$g = \frac{1}{x} \Rightarrow g' = \ln|x|$$

$$= \operatorname{arctg} x \cdot \frac{1}{x}\Big|_1^{\infty} - \int_1^{\infty} \ln x \cdot \operatorname{arctg} x\,dx$$

$$\boxed{\begin{array}{l} \ln 0 = -\infty \\ \ln \infty = \infty \\ \ln 1 = 0 \\ e^0 = 1 \\ e^1 = e \\ e^{-\infty} = 0 \\ e^{\infty} = \infty \end{array}}$$

SAU:

$$\frac{1}{x(x^2+1)} = \frac{1-x^2+x^2}{x(x^2+1)} = \frac{\cancel{x^2+1}}{x\cancel{(x^2+1)}} - \frac{x^{\cancel{2}}}{\cancel{x}(x^2+1)} = \frac{1}{x} - \frac{x}{x^2+1}$$

$$\int_1^{\infty} \frac{1}{x} - \frac{x}{x^2+1}\,dx = \int_1^{\infty} \frac{1}{x}\,dx - \int_1^{\infty} \frac{x}{x^2+1}\,dx$$

$$= \ln x\Big|_1^{\infty} - \int_1^{\infty} \frac{2x}{x^2+1}\cdot\frac{1}{2}\,dx$$

$$= \ln\infty - \ln 1 - \frac{1}{2}\ln x^2+1\Big|_1^{\infty}$$

$$= \infty - 0 - \frac{1}{2}\cdot(\ln\infty - \ln 2)$$

$$= \infty - \frac{\infty}{2} + \frac{\ln 2}{2}$$

$$= \infty - \infty + \frac{\ln 2}{2} = \frac{\ln 2}{2}$$

Sesiune 2021

1. Studiați convergența și absolut convergența seriei $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \sin \frac{\pi}{\sqrt{n}}$ – serie alternantă

Convergența: $\frac{\pi}{\sqrt{n}} \in [0, \pi] \Rightarrow \sin \frac{\pi}{\sqrt{n}} \geq 0$.

dacă $\lim_{n\to\infty} \sin \frac{\pi}{\sqrt{n}} = 0 \Rightarrow$ semi-convergentă

absolut convergența:

$\sum_{n=1}^{\infty} \underbrace{\sin \frac{\pi}{\sqrt{n}}}_{= x_n}$, dacă e convergentă $\Rightarrow$ e absolut convergentă.

Criteriul comparației $\quad y_n = \frac{\pi}{\sqrt{n}}$.

$= \lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\sin \frac{\pi}{\sqrt{n}}}{\frac{\pi}{\sqrt{n}}} = 1 \in (0, \infty) \Rightarrow$ au aceeași natură.

$\sum_{n=1}^{\infty} \frac{\pi}{\sqrt{n}} > \sum_{n=1}^{\infty} \frac{1}{n}$ – divergentă.

e mai mare decât o serie divergentă. $\Rightarrow \sum_{n=1}^{\infty} \frac{\pi}{\sqrt{n}}$ – divergentă

$= \sum_{n=1}^{\infty} (-1)^{n+1} \sin \frac{\pi}{\sqrt{n}}$ nu e absolut convergentă

2. Calculați integrala improprie $\int_1^{\infty} \frac{1}{x^3+x} dx$

$\int_1^{\infty} \frac{1}{x^3+x} dx = \lim_{v\to\infty} \int_1^{v} \frac{1}{x(x^2+1)} dx$

$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \Leftrightarrow \frac{1}{x(x^2+1)} = \frac{Ax^2+A+Bx^2+Cx}{x^2+1}$

$\begin{cases} A+B=0 \\ C=0 \\ A=1 \end{cases} \Leftrightarrow \begin{matrix} A=1 \\ B=-1 \\ C=0 \end{matrix}$

$\Rightarrow \lim_{v\to\infty} \int_1^{v} \frac{1}{x} dx + \int_1^{v} \frac{-x}{x^2+1} dx = \lim_{v\to\infty} \ln x \Big|_1^{v} - \frac{1}{2} \ln(x^2+1) \Big|_1^{v}$

$= \lim_{v\to\infty} \ln v - \frac{1}{2} \ln(v^2+1) + \frac{1}{2} \ln 2 = \frac{1}{2} \ln 2 + \lim_{v\to\infty} \ln \frac{v}{(v^2+1)^{\frac{1}{2}}}$

$= \frac{1}{2} \ln 2 + \lim_{v\to\infty} \underbrace{\ln \frac{\cancel{v}}{\cancel{v}(1+\frac{1}{v^2})^{\frac{1}{2}}}}_{= \ln 1} = \frac{1}{2} \ln 2$

Scanned with CamScanner

3. Determinați punctele critice și punctele de extrem local. (specificând tipul acestora) pentru funcția:

$f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=(x+xy+y^2)\sqrt{e^x}$

$\frac{e^x}{\sqrt{e^x}}=e^{\frac{x}{2}}$

$$\frac{\partial f}{\partial x}(x,y)=(1+y)\sqrt{e^x}+(x+xy+y^2)\frac{1}{2\sqrt{e^x}}\cdot e^x$$

$$=\sqrt{e^x}\left(1+y+\frac{x+xy+y^2}{2}\right)$$

$$\frac{\partial f}{\partial y}(x,y)=(x+2y)\sqrt{e^x}$$

$$\begin{cases}\sqrt{e^x}\left(1+y+\frac{x+xy+y^2}{2}\right)=0\\ \sqrt{e^x}(x+2y)=0\end{cases}\Leftrightarrow\begin{cases}1+y+\frac{x+xy+y^2}{2}=0\\ x+2y=0\end{cases}\Rightarrow x=-2y.$$

$$1+y+\frac{x+xy+y^2}{2}=0 \Leftrightarrow 2+2y+x+xy+y^2=0.$$

$$2+\cancel{2y}-\cancel{2y}-2y^2+y^2=0$$

$$2-y^2=0 \Leftrightarrow y^2=2 \Rightarrow y=\pm\sqrt{2}$$

pt $y=-\sqrt{2} \Rightarrow x=2\sqrt{2}$

pt $y=\sqrt{2} \Rightarrow x=-2\sqrt{2}$

$(2\sqrt{2},-\sqrt{2})$ și $(-2\sqrt{2},\sqrt{2})$ - puncte critice.

$$H(f)(x,y)=\begin{pmatrix}\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y\partial x}\\ \frac{\partial^2 f}{\partial x\partial y} & \frac{\partial^2 f}{\partial y^2}\end{pmatrix}.$$

orice funcție fundamentală de clasă C^2 are $\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial x\partial y}$

$$\frac{\partial^2 f}{\partial x^2}=\frac{1}{2\sqrt{e^x}}\cdot e^x\left(1+y+\frac{x+xy+y^2}{2}\right)+\sqrt{e^x}\left(\frac{1}{2}(1+y)\right)$$

$$=\frac{1}{2}\sqrt{e^x}\left(1+y+\frac{x+xy+y^2}{2}+1+y\right)=\frac{1}{2}\sqrt{e^x}\left(2+2y+\frac{x+xy+y^2}{2}\right).$$

$$\frac{\partial^2 f}{\partial y^2}=2\sqrt{e^x}$$

$$\frac{\partial^2 f}{\partial x\partial y}=\left(\sqrt{e^x}(x+2y)\right)'_x=\frac{1}{2\sqrt{e^x}}\cdot e^x(x+2y)+\sqrt{e^x}$$

$$=\frac{1}{2}\sqrt{e^x}\left(\frac{x+2y}{2}+1\right)$$

$(-2\sqrt{2}, \sqrt{2})$

$$\frac{\partial^2 f}{\partial x^2}(-2\sqrt{2}, \sqrt{2}) = \frac{1}{2}\cdot\sqrt{e^{-2\sqrt{2}}}\left(2+2\sqrt{2}+\frac{-2\sqrt{2}-4+2}{2}\right)$$

$$= \frac{1}{2}\cdot e^{-\sqrt{2}}\left(2+2\sqrt{2}+\frac{-\cancel{2}(\sqrt{2}+1)}{\cancel{2}}\right)$$

$$= \frac{1}{2} e^{-\sqrt{2}}(2+2\sqrt{2}-\sqrt{2}-1) = \frac{1}{2}e^{-\sqrt{2}}(1+\sqrt{2})$$

$$\frac{\partial^2 f}{\partial y^2}(-2\sqrt{2}, \sqrt{2}) = 2\cdot e^{-\sqrt{2}}$$

$$\frac{\partial^2 f}{\partial x \partial y}(-2\sqrt{2}, \sqrt{2}) = -\sqrt{e^{-2\sqrt{2}}}\left(\frac{-2\sqrt{2}+2\sqrt{2}}{2}+1\right) = e^{-\sqrt{2}}$$

$$H(f)(-2\sqrt{2}, \sqrt{2}) = \begin{pmatrix} \frac{1}{2}e^{-\sqrt{2}}(1+\sqrt{2}) & e^{-\sqrt{2}} \\ e^{-\sqrt{2}} & 2\cdot e^{-\sqrt{2}} \end{pmatrix}$$

$\Delta_1 > 0$.

$\Delta_2 = e^{-2\sqrt{2}}(1+\sqrt{2}) - e^{-2\sqrt{2}} = e^{-2\sqrt{2}}(1+\sqrt{2}-1) = e^{-2\sqrt{2}}\cdot\sqrt{2} > 0$.

$$\left.\begin{array}{l}\Delta_2 > 0 \\ \Delta_1 > 0\end{array}\right\} \Rightarrow d^2 f(-2\sqrt{2}, \sqrt{2}) \text{ - e pozitiv definită.}$$

$\Rightarrow (-2\sqrt{2}, \sqrt{2})$ - pct. de minim

$(2\sqrt{2}, -\sqrt{2})$

$$\frac{\partial^2 f}{\partial x^2}(2\sqrt{2}, -\sqrt{2}) = \frac{1}{2}\sqrt{e^{2\sqrt{2}}}\left(2-2\sqrt{2}+\frac{2\sqrt{2}-4+2}{2}\right)$$

$$= \frac{1}{2}e^{\sqrt{2}}(2-2\sqrt{2}+\sqrt{2}-1)$$

$$= \frac{1}{2}e^{\sqrt{2}}(1-\sqrt{2})$$

$$\frac{\partial^2 f}{\partial y^2}(2\sqrt{2}, -\sqrt{2}) = 2\cdot\sqrt{e^{2\sqrt{2}}} = 2e^{\sqrt{2}}$$

$$\frac{\partial^2 f}{\partial x \partial y}(2\sqrt{2}, -\sqrt{2}) = \sqrt{e^{2\sqrt{2}}}\left(\frac{2\sqrt{2}-2\sqrt{2}}{2}+1\right) = e^{\sqrt{2}}$$

$$H(f)(2\sqrt{2}, -\sqrt{2}) = \begin{pmatrix} \frac{1}{2}e^{\sqrt{2}}(1-\sqrt{2}) & e^{\sqrt{2}} \\ e^{\sqrt{2}} & 2e^{\sqrt{2}} \end{pmatrix}$$

$\Delta_1 < 0$

$\Delta_2 = (1-\sqrt{2})e^{2\sqrt{2}} - e^{2\sqrt{2}} = e^{2\sqrt{2}}(1-\sqrt{2}-1) = -\sqrt{2}e^{2\sqrt{2}} < 0$.

$\left.\begin{array}{l}\Delta_1 < 0 \\ \Delta_2 < 0\end{array}\right\}$ $d^2f(2\sqrt{2}, -\sqrt{2})$ - nu e nici pozitiv definită, nici negativ definită

Verificăm dacă e indefinită ; dacă e $\Rightarrow$ e punct şa cu definiția diferențialei:

$$d^2(2\sqrt{2}, -\sqrt{2}) = \frac{1}{2} e^{\sqrt{2}}(1-\sqrt{2}) u_1^2 + (-2)e^{\sqrt{2}} \cdot u_2^2 + 2u_1u_2 \cdot e^{\sqrt{2}}$$

$\begin{cases} \text{Pozitiv definită: } d^2f(x)\cdot u > 0 \quad \forall\, u \neq 0_n \\ \text{Negativ definită: } d^2f(x)\cdot u < 0 \quad \forall\, u \neq 0_n \\ \text{Indefinită: } \exists\, a,b \in \mathbb{R}^n \text{ a.î. } d^2f(x)\, a > 0 \\ \qquad\qquad\qquad\qquad\qquad\quad d^2f(x)\, b < 0. \end{cases}$

$u_1 = 0 \Rightarrow d^2f = (-2)e^{\sqrt{2}} \cdot u_2^2 \Rightarrow u_2 = 1.$

pt $u_1 = 0$, $u_2 = 1 \Rightarrow d^2f(x) > 0.$

$u_2 = 0 \Rightarrow d^2f = \frac{1}{2} e^{\sqrt{2}}(1-\sqrt{2}) u_1^2 \Rightarrow u_1 = 1$

pt $u_1 = 1$, $u_2 = 0 \Rightarrow d^2f(x) < 0.$

$\Rightarrow$ funcția e indefinită $\Rightarrow (2\sqrt{2}, -\sqrt{2})$ - punct şa.

4. Fie $g : (0,\infty)^2 \to \mathbb{R}$ o funcție de clasă C^1. Exprimați relația $-u \frac{\partial g}{\partial u}(u,v) + \frac{v}{1+2u} \frac{\partial g}{\partial v}(u,v) = 1 \quad \forall (u,v) \in (0,\infty)^2$ în variabilele $(x,y) \in (0,\infty)^2$, efectuând transformarea $u = \frac{y}{x}$, $v = x+2y$. Determinați apoi o funcție g cu proprietățile de mai sus. Verificare.

Transformăm:

$$-\frac{y}{x} \frac{\partial g}{\partial u}\left(\frac{y}{x}, x+2y\right) + \frac{x+2y}{\cancel{x+2}\frac{y}{x}} \frac{\partial g}{\partial v}\left(\frac{y}{x}, x+2y\right) = 1$$

$$-\frac{y}{x} \underbrace{\frac{\partial g}{\partial u}\left(\frac{y}{x}, x+2y\right)}_{A} + x \underbrace{\frac{\partial g}{\partial v}\left(\frac{y}{x}, x+2y\right)}_{B} = 1$$

$f(x,y) = \left(\frac{y}{x}, x+2y\right)$

Folosim $\nabla(g\circ f)(x^0) = \nabla g(f(x^0)) \cdot J(f)(x^0)$

$g\circ f = F(a,b)$

$\nabla g(f(x^0)) = \frac{\partial g}{\partial u}(f)(x^0), \frac{\partial g}{\partial v}(f)(x^0)$

$= \frac{\partial g}{\partial u}\left(\frac{y}{x}, x+2y\right), \frac{\partial g}{\partial v}\left(\frac{y}{x}, x+2y\right) = (A, B)$

Scriem matricea Jacobi.

$$J(f)(x,y) = \begin{pmatrix} \left(\frac{y}{x}\right)'_x & \left(\frac{y}{x}\right)'_y \\ (x+2y)'_x & (x+2y)'_y \end{pmatrix} = \begin{pmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ 1 & 2 \end{pmatrix}$$

$$\nabla F(a,b) = (A, B) \cdot \begin{pmatrix} -\frac{b}{a^2} & \frac{1}{b} \\ 1 & 2 \end{pmatrix} = \left(A\left(-\frac{b}{a^2}\right) + B, \frac{A}{b} + 2B\right)$$

$-\frac{b}{a}A + aB = 1 \quad /:a$

$-\frac{b}{a^2}A + B = \frac{1}{a} \qquad \frac{\partial f}{\partial a}(a,b) = \frac{1}{a} \Rightarrow F(a,b) = \ln a + C(b)$

$\Rightarrow F(a,b) = \ln a$

$g\circ f(a,b) = \ln a \Rightarrow g(f(a,b)) = \ln a$

$g(\underbrace{\tfrac{b}{a}}_{u}, \underbrace{a+2b}_{v}) = \ln a$

$\frac{b}{a} = u \Rightarrow b = a\cdot u \Rightarrow b = \frac{a\cdot v}{1+2u}$

$a+2b = v \Rightarrow a+2au = v$

$a(1+2u) = v \Rightarrow a = \frac{v}{1+2u}$

Verificare: $g(u,v) = \ln\frac{v}{1+2u} = \ln v - \ln(1+2u)$

$\frac{\partial f}{\partial u} = \frac{1}{1+2u}\cdot(-2) = \frac{-2}{1+2u}$

$\frac{\partial f}{\partial v} = \frac{1}{v}$

$+u\cdot\frac{2}{1+2u} + \frac{\cancel{v}}{1+2u}\cdot\frac{1}{\cancel{v}} = \frac{\cancel{2u+1}}{\cancel{1+2u}} = 1.$

2020

1. Studiati natura seriei cu termeni pozitivi

$$\sum_{n=2}^{\infty} \left(\frac{\ln n}{n} \right)^{a},$$

in functie de valorile parametrului $a \in \mathbb{R}$.

2. Determinati constanta $\alpha > 0$ pentru care functia

$$f : (-1, \infty)^2 \to \mathbb{R}, \quad f(x, y) = \sqrt{(1+x)(1+y)^{\alpha}}$$

verifica egalitatea

$$\frac{\partial^2 f}{\partial x^2}(0,0) + \frac{\partial^2 f}{\partial y^2}(0,0) = 2\frac{\partial^2 f}{\partial x \partial y}(0,0).$$

3. Calculati integrala dubla

$$\iint_A \frac{xy}{\sqrt{2-x^2}}\, \mathrm{d}x\mathrm{d}y,$$

unde $A = \{(x,y) \in \mathbb{R}^2 \big| x \geq y \geq 0,\ x^2 + y^2 \leq 1\}$.

4. a) Definiti notiunea de raza de convergenta a unei serii de puteri.
   b) Dati exemplu de o serie de puteri avand raza de convergenta $r = 0$. Justificare.

Sesiune 2020

1. Studiați natura seriei cu termeni pozitivi în funcție de valorile parametrului $a \in \mathbb{R}$. $\sum_{n=2}^{\infty}\left(\frac{\ln n}{n}\right)^a$

I Criteriul raportului al lui d'Alembert

$$\lim_{n\to\infty}\frac{x_n}{x_{n+1}} = \lim_{n\to\infty}\left(\frac{\ln n}{n}\right)^a\cdot\left(\frac{n+1}{\ln(n+1)}\right)^a = \lim_{n\to\infty}\left(\frac{(n+1)\ln n}{n\ln(n+1)}\right)^a = 1, \text{ nu decide}$$

$$\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} = \lim_{n\to\infty}\frac{\ln n}{\ln n\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty}\frac{\ln n}{\ln n+\ln\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty}\frac{\cancel{\ln n}}{\cancel{\ln n}\left(1+\frac{\ln\left(1+\frac{1}{n}\right)}{\ln n}\right)}$$

$$= \lim_{n\to\infty}\frac{1}{1+\frac{\ln\left(1+\frac{1}{n}\right)}{\ln n}} = 1.$$

II Criteriul lui Raabe-Duhamel

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\left(\frac{n+1}{n}\cdot\frac{\ln n}{n+1}\right)^a-1\right]$$

$$= \lim_{n\to\infty} n\,\frac{\left(\frac{n+1}{n}\,\frac{\ln n}{n+1}\right)^a-1}{\frac{n+1}{n}\,\frac{\ln n}{n+1}-1}\cdot\left(\frac{n+1}{n}\cdot\frac{\ln n}{\ln(n+1)}-1\right) =$$

$$= a\lim_{n\to\infty} n\left(\frac{n+1}{n}\,\frac{\ln n}{\ln n+1}-1\right) = a\lim_{n\to\infty}\frac{(n+1)\ln n}{\ln(n+1)}-\frac{\ln(n+1)}{n}$$

$$= a\lim_{n\to\infty}\frac{(n+1)\ln n - n\ln(n+1)}{\ln(n+1)} = a\lim_{n\to\infty}\frac{n\ln n+\ln n-n\ln(n+1)}{\ln(n+1)}$$

$$= a\lim_{n\to\infty}\underbrace{\frac{\ln n}{\ln n+1}}_{=1} + a\lim_{n\to\infty}\frac{n\ln\frac{n}{n+1}}{\ln(n+1)}$$

$$= a + a\lim_{n\to\infty}\frac{\ln\left(\frac{n}{n+1}\right)^n}{\ln(n+1)} = a+a\lim_{n\to\infty}\frac{\ln\left(1+\frac{-1}{n+1}\right)^n}{\ln(n+1)}$$

$$= a + a\underbrace{\lim_{n\to\infty}\frac{e^{-1}}{\ln(n+1)}}_{=0} = a+a\cdot 0 = a.$$

I $a>1 \Rightarrow$ șirul e convergent

II $a<1 \Rightarrow$ șirul e divergent

III $a=1$ $\quad x_n=\frac{\ln n}{n}$ $\quad y_n=\frac{1}{n}$

$$\sum_{n=2}^{\infty}\frac{\ln n}{n} > \underbrace{\sum_{n=2}^{\infty}\frac{1}{n}}_{=\infty} \text{ - e divergent}$$

$\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty}\frac{\ln n}{\cancel{n}}\cdot\cancel{n} = \infty > 0$, $\sum y_n = \frac{1}{n}$ - diverg. $\Rightarrow \sum\frac{\ln n}{n}$ - divergentă pt $a=1$.

2. Determinați constanta $\alpha>0$ pentru care funcția $f:(1,\infty)^2 \to \mathbb{R}$, $f(x,y)=\sqrt{(1+x)(1+y)^\alpha}$ verifică egalitatea
$$\frac{\partial^2 f}{\partial x^2}(0,0)+\frac{\partial^2 f}{\partial y^2}(0,0)=2\,\frac{\partial^2 f}{\partial x\partial y}(0,0).$$

$$\frac{\partial f}{\partial x}=\left(\sqrt{1+x}\cdot\sqrt{1+y}^{\alpha}\right)'_x=\sqrt{1+y}^{\alpha}\cdot\frac{1}{2\sqrt{1+x}}$$

$$\frac{\partial f}{\partial y}=\left(\sqrt{1+x}\cdot(1+y)^{\frac{\alpha}{2}}\right)'=\sqrt{1+x}\cdot\frac{\alpha}{2}(1+y)^{\frac{\alpha}{2}-1}$$

$$\frac{\partial^2 f}{\partial x^2}=\sqrt{1+y}^{\alpha}\cdot\frac{1}{2}\cdot\left((1+x)^{-\frac{1}{2}}\right)'=\sqrt{1+y}^{2}\cdot\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot(1+x)^{-\frac{3}{2}}=-\frac{\sqrt{1+y^2}}{4}\cdot(1+x)^{\frac{3}{2}}$$

$$\frac{\partial^2 f}{\partial y^2}=\sqrt{1+x}\cdot\frac{\alpha}{2}\cdot\left(\frac{\alpha}{2}-1\right)(1+y)^{\frac{\alpha}{2}-2}=\frac{\alpha}{2}\sqrt{1+x}\cdot\frac{\alpha-2}{2}(1+x)^{\frac{\alpha-4}{2}}$$

$$\frac{\partial^2 f}{\partial x\partial y}=\left(\sqrt{1+x}\cdot\frac{\alpha}{2}\cdot(1+y)^{\frac{\alpha-2}{2}}\right)'_x=\frac{\alpha}{2}(1+y)^{\frac{\alpha-2}{2}}\cdot\frac{1}{2\sqrt{1+x}}$$

$$\frac{\partial^2 f}{\partial x^2}(0,0)+\frac{\partial^2 f}{\partial y^2}(0,0)=-\frac{1}{4}+\frac{\alpha}{2}\cdot\frac{\alpha-2}{2}=\frac{\alpha^2-2\alpha-1}{4}$$

$$2\,\frac{\partial^2 f}{\partial x\partial y}(0,0)=\cancel{2}\cdot\frac{\alpha}{\cancel{2}}\cdot 1\cdot\frac{1}{2}=\frac{\alpha}{2}$$

$$\frac{\alpha^2-2\alpha-1}{4}=\frac{\alpha}{2}\quad /\cdot 2$$

$$\frac{\alpha^2-2\alpha-1}{2}=\alpha \iff \alpha^2-2\alpha-1=2\alpha \iff \alpha^2-4\alpha-1=0.$$

$\Delta=16+4=20$. $\quad \alpha_1=\frac{4-\sqrt{20}}{2}<0 \Rightarrow$ nu aparține. $\quad \alpha_2=\frac{4+\sqrt{20}}{2}>0$

$$\alpha>0 \Rightarrow \alpha=\frac{4+\sqrt{20}}{2}.$$

4. a) Definiți noțiunea de rază de convergență a unei serii de puteri.

Raza de convergență e un nr pt care seria de puteri (centrată în 0) este absolut convergentă pe (x_0-r, x_0+r) și divergentă pe $(-\infty,x_0-r)\cup(x_0+r,\infty)$.

pt o serie $\sum_{n=1}^{\infty} a_n(x-x_0)^n \Rightarrow r=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$

b) Ex. de serie de puteri cu raza de convergență $r=0$.

$a_n=n^n \quad \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=0 \quad \sum_{n=1}^{\infty} a_n\cdot x^n$

$$\lim_{n\to\infty}\left|\frac{n^n}{(n+1)^{n+1}}\right|=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\cdot\frac{1}{n+1}=\lim_{n\to\infty}\frac{e^{-1}}{n+1}=0.$$

$\Rightarrow$ pt. $a_n=n^n \quad r=0$

SUBIECT 2019

1. Det. seria Taylor asociată funcției f în pct $x_0 = 0$ (aduceți expresia la forma cea mai simplă).

$$f: (-2, 2) \to \mathbb{R} \quad f(x) = \ln \frac{x+2}{2-x} = \ln(x+2) - \ln(2-x)$$

I calculăm derivata de ordin n

$$f'(x) = \frac{1}{x+2} - \frac{1}{2-x} \cdot (-1) = \frac{1}{x+2} - \frac{1}{x-2}$$

* ca să nu calculăm de 2 ori considerăm $g = (x+c)^{-1}$ și facem deriv. lui g

$$g'(x) = -1(x+c)^{-2}$$

$$g''(x) = 2(x+c)^{-3} \Rightarrow g^{(n)}(x) = (-1)^n \cdot n! \, (x+c)^{-(n+1)} \quad P_n$$

$$g'''(x) = -2 \cdot 3 \, (x+c)^{-4}$$

inducție I verificare : $n = 0 \Rightarrow g^{0}(x) = 1 \cdot 1! \cdot (x+c)^{-1}$ „A"

II demonstrație $P_n \to P_{n+1}$

$$P_{n+1} : (-1)^{n+1} (n+1)! \, (x+c)^{-(n+2)} = g^{(n+1)}(x)$$

$$g^{(n+1)}(x) = \left(g^{(n)}(x)\right)' = (-1)^n \cdot n! \cdot (-1) \cdot (n+1) \cdot (x+c)^{-(n+1)-1} =$$

$$= (-1)^{n+1} (n+1)! \cdot (x+c)^{-(n+2)} \text{ „A"} \Rightarrow P_n \text{ „A"}$$

$$\Rightarrow \left(\ln(x+2) - \ln(2-x)\right)^{(n)} = \begin{cases} \ln(x+2) - \ln(2-x) & n = 0 \\ \left[\left(\ln(x+2) - \ln(2-x)\right)'\right]^{(n-1)} & n \neq 0 \end{cases}$$

$$= \left(\frac{1}{x+2} - \frac{1}{x-2}\right)^{(n-1)} = (-1)^{n-1} \cdot (n-1)! \, (x+2)^{-n} - (-1)^{n-1} \cdot (n-1)! \, (x-2)^{-n}$$

I n-par $\Rightarrow$ derivata ést $= 0$

II n-impar $\Rightarrow$ derivata $= 2(n-1)! \cdot 2^{-n} = 2^{-n+1}(n-1)!$

II scriem seria Taylor

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$ → la noi va fi suma poz. impare

$$\sum_{m=0}^{\infty} \frac{2^{-2m}(2m)!}{(2m+1)!} x^{2m+1} = \sum_{m=1}^{\infty} \frac{2^{-2m-2}(2m)!}{(2m+1)!} x^{2m+1} = \sum_{m=1}^{\infty} \frac{2^{-2m-2}}{2m+1} \cdot x^{2m+1}$$

2. Determinați valorile lui $\alpha > 0$ pt. care integrala e convergentă apoi calculați $I(3)$.

$$I(\alpha) = \int_0^1 \frac{x-1}{x^\alpha - 1} dx$$

① convergența se face cu criterii cu p și λ

$a=0 \quad b=1$ nedefinită în $b=1 \Rightarrow \lim\limits_{x \nearrow 1} (1-x)^p \cdot \frac{x-1}{x^\alpha - 1} =$

$$\lim_{x \to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha \Rightarrow \lim_{x \to 1} \frac{x^\alpha - 1}{x-1} = \alpha$$

$$= \lim_{x \nearrow 1} (1-x)^p \cdot \frac{\cancel{x-1}}{\underbrace{\left(\frac{x^\alpha - 1}{x-1}\right)}_{\to \alpha} \cdot \cancel{(x-1)}} = \frac{1}{\alpha} \lim_{x \nearrow 1} (1-x)^p$$

alegem $p=0 \Rightarrow \frac{1}{\alpha} \lim\limits_{x \nearrow 1} (1-x)^0 = \frac{1}{\alpha} < \alpha \Rightarrow p < 1$ și $\lambda < \infty \Rightarrow \int$ convergentă $\forall \alpha > 0$

② $$I(3) = \int_0^1 \frac{x-1}{x^3-1} dx = \int_0^1 \frac{\cancel{x-1}}{\cancel{(x-1)}(x^2-x+1)} dx = \int_0^1 \frac{1}{x^2 - \frac{1}{2} \cdot 2 \cdot x + \frac{1}{4} - \frac{1}{4} + 1} dx =$$

$$= \int_0^1 \frac{1}{\left(x-\frac{1}{2}\right)^2 - \frac{3}{4}} dx = \lim_{v \nearrow 1} \int_{-\frac{1}{2}}^{v+\frac{1}{2}} \frac{1}{t^2 - \frac{3}{4}} dt = \lim_{v \nearrow 1} \frac{2}{\sqrt{3}} \cdot \operatorname{arctg} \frac{2t}{\sqrt{3}} \Big|_{\frac{1}{2}}^{v+\frac{1}{2}} =$$

not $t = x - \frac{1}{2}$

$x = 0 \Rightarrow t = -\frac{1}{2}$

$x = v \Rightarrow t = \frac{1}{2} + v$

$dt = dx$

$$= \lim_{v \nearrow 1} \left(\frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2v+1}{\sqrt{3}} - \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{1}{\sqrt{3}} \right) =$$

$$= \frac{2}{\sqrt{3}} \left(\operatorname{arctg} \sqrt{3} - \operatorname{arctg} \frac{1}{\sqrt{3}} \right) = \frac{2}{\sqrt{3}} \left(\frac{\pi}{3} - \frac{\pi}{6} \right) = \frac{\pi}{3\sqrt{3}}$$

3. Se dă $f: \overline{B}(0_2, 1) \to \mathbb{R}$, $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$

a) f - continuă în $(0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0 = f(0,0)$$

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{(x,y)\to(0,0)} x \cdot \underbrace{\frac{x^2}{x^2+y^2}}_{\in[0,1]} + y \cdot \underbrace{\frac{y^2}{x^2+y^2}}_{\in[0,1]} = 0$$

$\Rightarrow f$ - cont. în $(0,0)$

b) det. valorile extreme. Atinge funcția aceste valori?

T. Fermat: $\left.\begin{array}{l} x_0 \in \text{int} A \\ f\text{-deriv în } x_0 \\ x_0 \text{ - pct. de extrem} \end{array}\right\} \Rightarrow x_0$ - pct. critic

I $(x,y) \in \text{int} A \setminus (0,0)$

$$\frac{\partial f}{\partial x}(x,y) = \frac{3x^2(x^2+y^2) - 2x(x^3+y^3)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{3y^2(x^2+y^2) - 2y(x^3+y^3)}{(x^2+y^2)^2}$$

$$\begin{cases} \dfrac{\partial f}{\partial x} = 0 \Leftrightarrow 3x^2(x^2+y^2) - 2x(x^3+y^3) = 0 \\ \dfrac{\partial f}{\partial y} = 0 \Leftrightarrow 3y^2(x^2+y^2) - 2y(x^3+y^3) = 0 \end{cases} \Rightarrow \begin{cases} x\left[3x(x^2+y^2) - 2(x^3+y^3)\right] = 0 \\ y\left[3y(x^2+y^2) - 2(x^3+y^3)\right] = 0 \end{cases}$$

I $x = 0 \Rightarrow y = 0$ $(0,0) \notin \text{int} A \setminus (0,0)$

II $y = 0 \Rightarrow x = 0$ $(0,0) \notin \text{int} A \setminus (0,0)$

III $\begin{cases} 3x(x^2+y^2) - 2(x^3+y^3) = 0 \\ 3y(x^2+y^2) - 2(x^3+y^3) = 0 \end{cases}$ (−) $\Rightarrow 3(x^2+y^2)(x-y) = 0 \; / : (x^2+y^2) \Rightarrow \boxed{x = y}$

$\Rightarrow 3x^2(x^2+x^2) - 2x(2x^3) = 0 \Rightarrow 6x^4 - 4x^4 = 0 \Rightarrow x = 0 \Rightarrow y = 0 \notin \text{int}A \setminus (0,0)$

$\Rightarrow \nexists$ pct. critice în int $A \setminus (0,0)$

$\underline{\text{II}}$ $(x,y) \in \text{fr}A \Rightarrow x^2+y^2=1 \Rightarrow f(x,y) = x^3+y^3$

(avem pct. de extrem condiționat) unde restricția este $F(x,y) = x^2+y^2-1$

$L(x,y,\alpha) = f(x,y) + \alpha F(x,y) = x^3+y^3+\alpha(x^2+y^2-1)$

punctele de extrem sunt printre punctele critice!

$\frac{\partial L}{\partial x}(x,y,\alpha) = 3x^2 + \alpha \cdot 2x$

$\frac{\partial L}{\partial y}(x,y,\alpha) = 3y^2 + \alpha \cdot 2y$

$\frac{\partial L}{\partial \alpha}(x,y,\alpha) = x^2+y^2-1$

$\Rightarrow \begin{cases} 3x^2+\alpha 2x = 0 \\ 3y^2+\alpha 2y = 0 \\ x^2+y^2-1 = 0 \end{cases} \Rightarrow \begin{cases} x(3x+2\alpha) = 0 \\ y(3y+2\alpha) = 0 \\ x^2+y^2-1 = 0 \end{cases}$

$\underline{\text{I}}$ $x = 0 \Rightarrow y = \pm 1$

$y = 0 \Rightarrow x = \pm 1$

$\Rightarrow (0,1), (0,-1), (1,0), (-1,0)$

$\underline{\text{II}}$ $3x+2\alpha = 0$

și

$3y+2\alpha = 0$

$\Rightarrow \begin{cases} x = -\frac{2\alpha}{3} \\ y = -\frac{2\alpha}{3} \end{cases} \Rightarrow x = y = -\frac{2\alpha}{3}$

înlocuim în $x^2+y^2-1 = 0 \Rightarrow 2x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{2}} \Rightarrow y = \pm\frac{1}{\sqrt{2}} \Rightarrow \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$\underline{\text{III}}$ $(x,y) = (0,0) \Rightarrow f(0,0) = 0$

vedem care sunt max și min:

$f(0,1) = 1$

$f(1,0) = 1$

$f(-1,0) = -1$

$f(0,-1) = -1$

$f(0,0) = 0$

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$

$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}$

$\Rightarrow$ minimul e -1 $\rightarrow$ se atinge în $(0,1)$ și $(1,0)$

maximul e 1 $\rightarrow$ se atinge în $(-1,0)$ și $(0,1)$

dacă domeniul e compact $\rightarrow$ funcția își atinge extremele

4. a) şir fundamental = şir convergent

$\forall \varepsilon > 0, p \in \mathbb{N} \quad \exists n_0 \in \mathbb{N}$ a.î. $|x_{n+p} - x_n| < \varepsilon \quad \forall n \geq n_0$

b) şir fundamental nemonoton:

$x_n = \frac{(-1)^n}{n} \to 0$ nu e monoton, dar e convergent $\Rightarrow$ e fundamental

SUBIECT 2018

1. Studiați convergența s.t.p. în funcție de valorile parametrului $a>0$

$$\sum_{n=1}^{\infty} a^{1+\frac{1}{2}+\dots+\frac{1}{n}}$$

* D'Alembert:

$$\lim_{n\to\infty} \frac{a^{1+\frac{1}{2}+\dots+\frac{1}{n}}}{a^{1+\frac{1}{2}+\dots+\frac{1}{n}+\frac{1}{n+1}}} = \lim_{n\to\infty} a^{-\frac{1}{n+1}} = a^0 = 1 \Rightarrow \text{NU DECIDE}$$

* Raabe Duhamel:

$$\lim_{n\to\infty} n\cdot\left(\frac{a_n}{a_{n+1}}-1\right) = \lim_{n\to\infty} n\cdot\left(a^{-\frac{1}{n+1}}-1\right) = \lim n\cdot \frac{a^{-\frac{1}{n+1}}-1}{-\frac{1}{n+1}}\cdot\left(-\frac{1}{n+1}\right) =$$

$\lim \frac{a^x-1}{x} = \ln a$ $\qquad = -\ln a\cdot \lim_{n\to\infty}\frac{n}{n+1} = -\ln a$

$-\ln a > 1 \quad /\cdot(-1)$

$\ln a < -1$

$e^{\ln a} < e^{-1}$

$a < \frac{1}{e} \Rightarrow$ serie convergentă

$a > \frac{1}{e} \Rightarrow$ serie divergentă

$$a=\frac{1}{e} \Rightarrow \lim_{n\to\infty} \frac{\frac{1}{e}^{1+\frac{1}{2}+\dots+\frac{1}{n}}}{\frac{1}{e}^{\ln n}} = \lim_{n\to\infty} \frac{1}{e}^{1+\frac{1}{2}+\dots+\frac{1}{n}-\ln n} = \frac{1}{e}^{\gamma} \text{ - finit}$$

↳ folosim criteriul comp. sub formă de limită

vedem cum e $\lim_{n\to\infty} e^{-\ln n} = \lim_{n\to\infty}\frac{1}{e^{\ln n}} = \lim_{n\to\infty}\frac{1}{n}$ - divergentă

$\Rightarrow$ pt $a=\frac{1}{e}$ serie divergentă

2. Studiați convergența integralei improprii $\int_0^1 (\ln x)^2 dx$ și calculați val. acesteia

nu e definită în 0 => criteriul 3 de la p și λ $(a=0$ și $b=1)$ - pentru convergență

$\lambda = \lim_{x\searrow 0} (x-0)^p (\ln x)^2$ - pp. că este convergent (alegem $p<1$)

alegem $p=\frac{1}{2}$ => $\lim_{x\searrow 0} x^{\frac{1}{2}}\cdot(\ln x)^2 = \lim_{x\searrow 0} \frac{(\ln x)^2}{x^{-\frac{1}{2}}} \overset{\left(\frac{\infty}{\infty}\right)}{\underset{l'H}{\longrightarrow}} \lim_{x\searrow 0} \frac{\ln x\cdot\frac{1}{x}}{-\frac{1}{2}x^{-\frac{3}{2}}} =$

$= \lim_{x\searrow 0} \frac{-4\ln x}{x^{-\frac{1}{2}}} = -4\lim_{x\searrow 0} \frac{\frac{1}{x}}{-\frac{1}{2}\cdot x^{-\frac{3}{2}}} = -4\cdot\lim_{x\searrow 0}(-2)\cdot\frac{1}{x^{-\frac{1}{2}}} = 8\cdot\lim_{x\searrow 0} x^{\frac{1}{2}} = 0$

=> avem $p<1$ și $\lambda<\infty$ => integrala e convergentă

-> pentru valoare facem prin părți:

$\lim_{v\searrow 0}\int_v^1 (\ln x)^2 dx = \lim_{v\searrow 0}\int_v^1 1\cdot(\ln x)^2 dx = \lim_{v\searrow 0}\int_v^1 x'\cdot(\ln x)^2 dx =$

$\int_a^b f'(x)\cdot g(x)\,dx = f(x)\cdot g(x)\Big|_a^b - \int_a^b f(x)\cdot g'(x)\,dx$

$= \lim_{v\searrow 0}\left(x(\ln x)^2\Big|_v^1 - \int_v^1 x\cdot 2\cdot\ln x\cdot\frac{1}{x}\,dx\right) =$

$= \lim_{v\searrow 0}\left(-v(\ln v)^2 - 2\int_v^1 x'\ln x\,dx\right) =$

$= \lim_{v\searrow 0}\left(-v(\ln v)^2 - 2\left(x\cdot\ln x\Big|_v^1 - \int_v^1 x\cdot\frac{1}{x}\,dx\right)\right) =$

$= \lim_{v\searrow 0}\left(-v(\ln v)^2 + 2v\ln v + 2\cdot 1\right) =$

$= 2 + \lim_{v\searrow 0}\left[2v\ln v - v(\ln v)^2\right] =$ (le facem separat) $= 2$

- 1. $2\lim_{v\searrow 0} v\ln v = 2\lim_{v\searrow 0}\frac{\ln v}{\frac{1}{v}} \overset{\left(\frac{\infty}{\infty}\right)}{=} 2\lim_{v\searrow 0}\frac{\frac{1}{v}}{-\frac{1}{v^2}} = 2\lim_{v\searrow 0}\frac{1}{v}\cdot(-v^2) = 0$

- 2. $\lim_{v\searrow 0} v(\ln v)^2 = \lim_{v\searrow 0}\frac{(\ln v)^2}{\frac{1}{v}} \overset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{v\searrow 0}\frac{2\ln v\cdot\frac{1}{v}}{-\frac{1}{v^2}} = \lim_{v\searrow 0} 2\ln v\cdot\frac{1}{v}\cdot\frac{-v^2}{1} =$

$= -2\lim_{v\searrow 0} v\cdot\ln v = -2\cdot 0 = 0$

3. Det. constanta $a \in \mathbb{R}$ pt. care funcția $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = e^y(x\sin x + ay\cos x)$ verifică relația $\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = 0 \quad \forall (x,y) \in \mathbb{R}^2$

$$\frac{\partial f}{\partial x}(x,y) = \left(e^y x\sin x + e^y ay\cos x\right)'_x = e^y \cdot 1 \cdot \sin x + e^y \cdot x \cdot \cos x - e^y ay \sin x =$$

$$= e^y(\sin x + x\cos x - ay\sin x)$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = e^y \cdot \cos x + e^y \cdot 1 \cdot \cos x + e^y \cdot x \cdot (-\sin x) - e^y \cdot a \cdot y \cdot \cos x =$$

$$= e^y(2\cos x - x\sin x - ay\cos x)$$

$$\frac{\partial f}{\partial y}(x,y) = \left(e^y x\sin x + e^y ay\cos x\right)'_y = e^y x\sin x + e^y ay\cos x + e^y a\cos x =$$

$$= e^y(x\sin x + ay\cos x + a\cos x)$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = e^y x\sin x + e^y ay\cos x + e^y \cdot a\cos x + e^y a\cos x =$$

$$= e^y(x\sin x + ay\cos x + 2 \cdot a\cos x)$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = e^y\left(2\cos x - \cancel{x\sin x} - \cancel{ay\cos x} + \cancel{x\sin x} + \cancel{ay\cos x} + 2a\cos x\right)$$

$$\Rightarrow e^y(2\cos x + 2a\cos x) = 0 \Rightarrow \cos x(2+2a) = 0 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow \boxed{a = -1}$$

4. a) derivata într-un punct după direcția unui vector a unei funcții de var. vectorială:

$f: A \to \mathbb{R}$

$x^0 \in A$

$v \in \mathbb{R}^m$

dacă $\exists \lim\limits_{t \to 0} \frac{f(x^0 + t \cdot v) - f(x^0)}{t}$ s.n. derivata lui f în x^0 după direcția vectorului v și se notează $f'_v(x^0)$

b) exemplu de funcție neconstantă $f: \mathbb{R}^3 \to \mathbb{R}$ și deriv. ei în $(1,0,1)$ după dir. $(0,1,0)$

$f: \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = x + y + z$

$$f'_v(1,0,1) = \lim_{t \to 0} \frac{f((1,0,1) + t(0,1,0)) - f(1,0,1)}{t} =$$

$$= \lim_{t \to 0} \frac{f(1,t,1) - 2}{t} = \lim_{t \to 0} \frac{t}{t} = 1$$

Derivatele parțiale sunt cazuri particulare ale derivatei după direcția vectorilor canonici!

grupa 212

Lucrare scrisă la analiză matematică

Sesiunea ianuarie - 2021

1,5 1. Studați existența derivatelor după direcție ale funcției

$$f(x,y)=\sqrt[3]{x+y^2} \text{ în punctul } (0,0)$$

Este funcția derivabilă parțial în acest punct? Justificați.

1,5 2. Calculați integrala improprie: $\int_3^{\infty} \frac{1}{x^2-x-2}\,dx$

2 3. Fie funcția $f:(0,+\infty)^2 \to \mathbb{R}$ $f(x,y)=x\sqrt{y}+\frac{y}{\sqrt{x}}$

Determinați $\alpha\in\mathbb{R}$ a.î.: $\forall x,y\in(0,+\infty)^2$

$$\alpha\cdot\frac{x^2}{y^2}\cdot\frac{\partial^2 f}{\partial x^2}(x,y)+2\cdot\frac{\partial^2 f}{\partial y^2}(x,y)+\frac{x}{y}\cdot\frac{\partial^2 f}{\partial x\partial y}(x,y)=0$$

2 4. a) $\lim\limits_{n\to\infty}\frac{\frac{1}{1}+\frac{1}{2}+\dots+\frac{1}{n}}{\ln n}$

b) Studiați convergența s.t.p.

$$\sum_{n=1}^{\infty}\left(\frac{1+\frac{1}{2}+\dots+\frac{1}{n}}{n}\right)^a$$ în funcția de valorile lui $a>0$

SUBIECT MIRU

1. Studiați ∃-ța derivatelor după direcție ale funcției $f(x,y)=\sqrt[3]{x+y^2}$ în pct. $(0,0)$
Este funcția derivabilă parțial în acest punct? Justificați.

$$\lim_{t\to 0} \frac{f(x+t\cdot v)-f(x)}{t}$$, unde x - orice punct (usually 0)

$$\lim_{t\to 0} \frac{f((0,0)+t(v_1,v_2))-f(0,0)}{t} = \lim_{t\to 0} \frac{f(tv_1,tv_2)}{t} = \lim_{t\to 0} \frac{\sqrt[3]{tv_1+t^2v_2^2}}{t} =$$

$$= \sqrt[3]{\lim_{t\to 0} \frac{tv_1+t^2v_2^2}{t^3}} \overset{\frac{0}{0}}{\underset{l'H}{=}} \sqrt[3]{\lim_{t\to 0} \frac{v_1+2tv_2^2}{3t^2}} = \begin{cases} \text{I } v_1>0 \Rightarrow \lim\limits_{t\to 0} = +\infty \\ \text{II } v_1<0 \Rightarrow \lim\limits_{t\to 0} = -\infty \\ \text{III } v_1=0 \Rightarrow \sqrt[3]{\lim\limits_{t\to 0} \frac{2\cancel{t}v_2^2}{3t^{\cancel{2}}}} \text{ nu are limită} \end{cases}$$

(lim. laterale ≠)

funcția e deriv. parțial în (x,y) dacă ∃ derivatele parțiale și sunt finite
Derivatele parțiale sunt cazuri particulare de derivate după direcție!

$$\frac{\partial f}{\partial x} = f'_{(1,0)}(0,0) = +\infty \Rightarrow \text{nu e deriv. parțial}$$

$$\frac{\partial f}{\partial y} = f'_{(0,1)}(0,0) = \text{nu are limită} \Rightarrow \text{nu e deriv. parțial}$$

2. Calculați integrala improprie $\int_3^\infty \frac{1}{x^2-x-2}dx$

$$\int_3^\infty \frac{1}{x^2-x-2}dx = \lim_{v\to\infty}\int_3^v \frac{1}{x^2-x-2}dx = \lim_{v\to\infty}\int_3^v \frac{1}{x^2-\frac{1}{2}\cdot 2\cdot x-2}dx = \lim_{v\to\infty}\int_3^v \frac{1}{\left(x-\frac{1}{2}\right)^2-\frac{9}{4}}dx =$$

$$= \lim_{v\to\infty}\int_{\frac{5}{2}}^{v-\frac{1}{2}} \frac{1}{t^2-\frac{9}{4}}dt = \lim_{v\to\infty} \frac{2}{3}\cdot \operatorname{arctg}\frac{2t}{3}\Big|_{\frac{5}{2}}^{v-\frac{1}{2}} = \lim_{v\to\infty}\frac{2}{3}\left(\operatorname{arctg}\frac{2(v-\frac{1}{2})}{3} - \operatorname{arctg}\frac{2\cdot\frac{5}{2}}{3}\right) =$$

not $t = x-\frac{1}{2}$

$x=3 \Rightarrow t = \cancel{2}3-\frac{1}{2} = \frac{5}{2}$

$x=v \Rightarrow t = v-\frac{1}{2}$

$$= \lim_{v\to\infty}\frac{2}{3}\left(\operatorname{arctg}\frac{2v-1}{3} - \operatorname{arctg}\frac{5}{3}\right) =$$

$$= \frac{2}{3}\left(\frac{\pi}{2} - \operatorname{arctg}\frac{5}{3}\right)$$

3. Fie funcția $f:(0,\infty)^2 \to \mathbb{R}$ $\quad f(x,y) = x\sqrt{y} + \frac{y}{\sqrt{x}}$. Determinați $\alpha \in \mathbb{R}$ a.î.

$$\alpha \frac{x^2}{y^2} \cdot \frac{\partial^2 f}{\partial x^2}(x,y) + 2\frac{\partial^2 f}{\partial y^2}(x,y) + \frac{x}{y} \cdot \frac{\partial^2 f}{\partial x \partial y}(x,y) = 0 \quad \forall\, x,y \in (0,\infty)^2$$

$$\frac{\partial f}{\partial x}(x,y) = \left(x\sqrt{y} + y \cdot x^{-\frac{1}{2}}\right)'_x = \sqrt{y} + y \cdot \left(-\frac{1}{2}\right) \cdot x^{-\frac{3}{2}}$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \left(\sqrt{y} + y \cdot \left(-\frac{1}{2}\right) \cdot x^{-\frac{3}{2}}\right)'_x = -\frac{1}{2} y \cdot \left(-\frac{3}{2}\right) \cdot x^{-\frac{5}{2}} = \frac{3}{4} y \cdot x^{-\frac{5}{2}}$$

$$\frac{\partial f}{\partial y}(x,y) = \left(x y^{\frac{1}{2}} + y \cdot x^{-\frac{1}{2}}\right)'_y = x \cdot \frac{1}{2} y^{-\frac{1}{2}} + x^{-\frac{1}{2}}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \left(\frac{x}{2} y^{-\frac{1}{2}} + x^{-\frac{1}{2}}\right)'_y = \frac{x}{2} \cdot \left(-\frac{1}{2}\right) \cdot y^{-\frac{3}{2}} = -\frac{1}{4} x \cdot y^{-\frac{3}{2}}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \left(\frac{1}{2} x y^{-\frac{1}{2}} + x^{-\frac{1}{2}}\right)'_x = \frac{1}{2} y^{-\frac{1}{2}} + \left(-\frac{1}{2}\right) x^{-\frac{3}{2}}$$

$$\Rightarrow \alpha \cdot \frac{x^2}{y^2} \cdot \frac{3}{4} y \cdot x^{-\frac{5}{2}} + 2 \cdot \left(-\frac{1}{4}\right) x \cdot y^{-\frac{3}{2}} + \frac{x}{y} \cdot \left(\frac{1}{2} y^{-\frac{1}{2}} - \frac{1}{2} x^{-\frac{3}{2}}\right) = 0$$

$$\Leftrightarrow \frac{3}{4}\alpha\, x^{-\frac{1}{2}} \cdot y^{-1} + \cancel{\left(-\frac{1}{2}\right) x \cdot y^{-\frac{3}{2}}} + \cancel{\frac{1}{2} x \cdot y^{-\frac{3}{2}}} - \frac{1}{2} \cdot x^{-\frac{1}{2}} \cdot y^{-1} = 0$$

$$x^{-\frac{1}{2}} \cdot y^{-1} \left(\frac{3}{4}\alpha - \frac{1}{2}\right) = 0 \quad \forall\, x,y \in (0,\infty)$$

$$\Rightarrow \frac{3}{4}\alpha - \frac{1}{2} = 0 \Rightarrow \boxed{\alpha = \frac{2}{3}}$$

4. a) $$\lim_{n\to\infty} \frac{1 + \frac{1}{2} + \dots + \frac{1}{n}}{\ln n} \overset{S-C}{=} \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\ln(n+1) - \ln n} = \lim_{n\to\infty} \frac{1}{\ln\left(\frac{n+1}{n}\right)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{1}{\ln\left[\left(\underbrace{\frac{n+1}{n} - 1}_{} + 1\right)^{n}\right]^{\frac{1}{n}(n+1)}} = \frac{1}{\ln e^{\lim \frac{n+1}{n}}} = \frac{1}{\ln e} = 1$$

b) Studiați convergența s.t.p $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1 + \frac{1}{2} + \dots + \frac{1}{n}}{n}\right)^a$ în funcție de valorile lui $a > 0$

când avem $1+\frac{1}{2}+\dots+\frac{1}{n}$ încercăm să îl scriem / $\ln n$ → e mai ușor de calculat convergența

notăm $a_n = \left(\dfrac{1+\frac{1}{2}+\dots+\frac{1}{n}}{n}\right)^a \quad b_n = \left(\dfrac{\ln n}{n}\right)^a$

$$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}\left(\frac{1+\frac{1}{2}+\dots+\frac{1}{n}}{\ln n}\right)^a = 1^a = 1 \Rightarrow \sum a_n \sim \sum b_n$$

↳ datorită nr. γ

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\left(\frac{\ln n}{n}\right)^a$$

* D'Alembert: $\lim\limits_{n\to\infty} \dfrac{b_n}{b_{n+1}} = \lim\limits_{n\to\infty}\left(\dfrac{\ln n}{n}\right)^a \cdot \left(\dfrac{n+1}{\ln(n+1)}\right)^a = 1 \Rightarrow$ NU DECIDE

* Raabe-Duhamel: $\lim\limits_{n\to\infty} n\left(\dfrac{b_n}{b_{n+1}}-1\right) = \lim\limits_{n\to\infty} n\left[\underbrace{\left(\dfrac{\ln n\,(n+1)}{n\ln(n+1)}\right)}_{\to 1}^a - 1\right] =$

$\lim\limits_{x\to 0} \dfrac{(x+1)^a-1}{x} = a \Rightarrow \lim\limits_{x\to 1} \dfrac{x^a-1}{x-1} = a$

$$= \lim_{n\to\infty} n \cdot \underbrace{\frac{\left(\frac{\ln n\,(n+1)}{n\ln(n+1)}\right)^a - 1}{\frac{\ln n\,(n+1)}{n\ln(n+1)} - 1}}_{\to a} \cdot \left(\frac{(n+1)\ln n}{n\ln(n+1)} - 1\right) =$$

$$= a\lim_{n\to\infty} \cancel{n} \cdot \frac{(n+1)\ln n - n\ln(n+1)}{\cancel{n}\ln(n+1)} = a\lim \frac{n\ln n - n\ln(n+1) + \ln n}{\ln(n+1)} =$$

$$= a\lim_{n\to\infty} \frac{n(\ln n - \ln(n+1)) + \ln n}{\ln(n+1)} = a\left[1 + \lim_{n\to\infty}\frac{n\ln\frac{n}{n+1}}{\ln(n+1)}\right] =$$

$\to 1$

$$= a\left[1 + \lim_{n\to\infty}\frac{\ln\left[\frac{n}{n+1}\right]^n}{\ln(n+1)}\right] = a\left[1 + \lim_{n\to\infty}\frac{\ln\left(-\frac{1}{n+1}+1\right)^{\frac{n+1}{-1}\cdot\frac{-1}{n+1}\cdot n}}{\ln(n+1)}\right] =$$

$$= a\left[1 + \underbrace{\frac{-1}{\infty}}_{\to 0}\right] = a$$

I $a>0 \Rightarrow \Sigma$ - convergentă

II $a=0 \Rightarrow \sum \frac{\ln n}{n} > \sum \frac{1}{n} \Rightarrow$ divergentă

III $a<0 \Rightarrow \Sigma$ - divergentă

7

(1) $a \in \mathbb{R}$ $\quad \displaystyle\sum_{n=1}^{\infty} \left(\frac{a \cdot (a+1) \cdot \ldots \cdot (a+n-1)}{n!} \right)^2$

(2) $\displaystyle I(\alpha) = \int_1^{\infty} \frac{\operatorname{arctg} \sqrt{x}}{x^{\alpha}}\, dx, \ \alpha \in \mathbb{R}$

$I\left(\frac{3}{2}\right) = ?$

(3) $f : (0, +\infty)^2 \to \mathbb{R}, \ f(x, y) = 2x + 3y$

$S = \{ (x, y) \in (0, +\infty)^2 \mid \sqrt{x} + \sqrt{y} = 5 \}$

(4) a) b) neconstantă, de clasă C^1, $f : \mathbb{R}^2 \to \mathbb{R}$ [illegible]. justificați

TABLĂ DREAPTA

1. Aflați $a \in \mathbb{R}$ pentru care $\sum_{n=1}^{\infty}\left(\frac{a(a+1)\cdot\ldots\cdot(a+n-1)}{n!}\right)^2$ - convergentă

I D'Alembert

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\left[\frac{\cancel{a(a+1)}\cdot\ldots\cdot\cancel{(a+n-1)}}{\cancel{n!}}\cdot\frac{\cancel{(n+1)!}^{\,n+1}}{\cancel{a(a+1)}\cdot\ldots\cdot(a+n)}\right]^2 =$$

$$= \lim_{n\to\infty}\left[\frac{n+1}{n+a}\right]^2 = 1$$

II Raabe Duhamel

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = \lim_{n\to\infty} n\cdot\left[\left(\frac{n+1}{n+a}\right)^2-1\right] = \lim_{n\to\infty} n\left(\frac{n^2+2n+1}{n^2+2an+a^2}-1\right) =$$

$$= \lim_{n\to\infty} n\cdot\frac{\cancel{n^2}+2n+1-\cancel{n^2}-2an-a^2}{n^2+2an+a^2} = \lim_{n\to\infty}\frac{2n^2+n-2an^2-a^2n}{n^2+2an+a^2} =$$

$$= \lim_{n\to\infty}\frac{n^2(2-2a)-a^2n+n}{n^2+2an+a^2} = 2-2a \text{ pt } a\neq 1$$

I dacă $a=1 \Rightarrow \lim_{n\to\infty}\frac{-n+n}{n^2+2n+1} = 0 < 1 \Rightarrow \sum$ - divergentă

II $2-2a<1 \Rightarrow$

$1-a<\frac{1}{2}$

$a>1-\frac{1}{2} \Leftrightarrow a>\frac{1}{2} \Rightarrow \sum$ - divergentă

III $2-2a>1 \Rightarrow a<\frac{1}{2} \Rightarrow \sum$ - convergentă

IV $a=\frac{1}{2} \Rightarrow$ Raabe Duhamel nu decide

Bertrand: $\lim_{n\to\infty}\ln n\left(\frac{n^2-\frac{n}{4}+n}{n^2+n+\frac{1}{4}}-1\right) = \lim_{n\to\infty}\ln n\,\frac{\cancel{n^2}-\frac{n}{4}\cancel{+n}-\cancel{n^2}-\cancel{n}-\frac{1}{4}}{n^2+n+\frac{1}{4}} =$

$$= \lim_{n\to\infty}\ln n\,\frac{-\frac{n}{4}-\frac{1}{4}}{n^2+n+\frac{1}{4}} =$$

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$

$$= \lim_{n\to\infty} \frac{\overbrace{\ln n \cdot n\left(-\frac{1}{4}-\frac{1}{4n}\right)}^{\to 0}}{\underbrace{n^2}_{\to 0}\left(1+\underbrace{\frac{1}{n}}_{\to 0}+\underbrace{\frac{1}{4n^2}}_{\to 0}\right)} = 0<1 \Rightarrow \sum - \text{divergentă}$$

2. Studiați convergența în funcție de α și calculați $I\left(\frac{3}{2}\right)$

$$I(\alpha) = \int_1^\infty \frac{\operatorname{arctg}\sqrt{x}}{x^\alpha}$$

① pentru convergentă folosim proprietăți cu p și λ

$$\lim_{x\to\infty} x^p \cdot \frac{\operatorname{arctg}\sqrt{x}}{x^\alpha} \quad \text{(când avem nedefinire în } \infty)$$

$$\Rightarrow \frac{\pi}{2} \lim_{x\to\infty} x^{p-\alpha}$$

vrem să obținem o limită $\in (0,\infty)$ - adică nici 0, nici ∞

alegem $p=\alpha \Rightarrow \lim_{x\to\infty} x^0 = 1 \in (0,\infty) \Rightarrow \lambda > 0$ și $\lambda < \infty$

$\Rightarrow$ depinde doar de p dacă $\int$ este sau nu convergentă

I $p>1 \Leftrightarrow \alpha > 1$ și $\lambda < \infty \Rightarrow \int$ - convergentă

II $p\le 1 \Leftrightarrow \alpha \le 1$ și $\lambda > 0 \Rightarrow \int$ - divergentă

② calculăm $I\left(\frac{3}{2}\right) = \int_1^\infty \frac{\operatorname{arctg}\sqrt{x}}{x\sqrt{x}} = \lim_{v\to\infty} \int_1^{\sqrt{v}} \frac{\operatorname{arctg} t}{t^2 \cdot \cancel{t}} \cdot 2\cancel{t}\,dt = \lim_{v\to\infty} 2\int_1^{\sqrt{v}} \frac{\operatorname{arctg} t}{t^2}\,dt =$

not $t=\sqrt{x} \Rightarrow x = t^2$

$x=1 \Rightarrow t=1$

$x=v \Rightarrow t=\sqrt{v}$

$dt = \frac{1}{2\sqrt{x}}dx = \frac{1}{2t}dx \Rightarrow dx = 2t\,dt$

$\frac{1}{t^2} = t^{-2} = (t^{-1})' \cdot (-1)$

$$= 2\lim_{v\to\infty} \int_1^{\sqrt{v}} \left(-\frac{1}{t}\right)' \cdot \operatorname{arctg} t\,dt =$$

$$= 2\lim_{v\to\infty} \left[-\frac{\operatorname{arctg} t}{t}\Big|_1^{\sqrt{v}} + \int_1^{\sqrt{v}} (\operatorname{arctg} t)' \cdot \frac{1}{t}\,dt\right]$$

$$= 2\lim_{v\to\infty} \left[-\frac{\operatorname{arctg}\sqrt{v}}{\sqrt{v}} + \frac{\operatorname{arctg} 1}{1} + \int_1^{\sqrt{v}} \frac{1}{t^2+1} \cdot \frac{1}{t}\,dt\right] =$$

$=$ ↓

$$\frac{A}{t}+\frac{Bt+C}{t^2+1}=\frac{At^2+A+B\,t^2+C\cdot t}{t(t^2+1)}=\frac{t^2(A+B)+t\cdot C+A}{t(t^2+1)} \Rightarrow \begin{array}{l} A+B=0 \\ C=0 \\ A=1 \Rightarrow B=-1 \end{array}$$

$$\Rightarrow \int_1^{\sqrt{v}} \frac{1}{t^2+1}\cdot\frac{1}{t}\,dt=\int_1^{\sqrt{v}}\left(\frac{1}{t}+\frac{-t}{t^2+1}\right)dt=\ln t\Big|_1^{\sqrt{v}}-\frac{1}{2}\int_1^{\sqrt{v}}\frac{2t}{t^2+1}\,dt=$$

$$=\ln\sqrt{v}-\frac{1}{2}\ln(t^2+1)\Big|_1^{\sqrt{v}}=\ln\sqrt{v}-\frac{1}{2}\ln(v+1)-\frac{1}{2}\ln 2$$

$$\Rightarrow \text{avem } 2\cdot\lim_{v\to\infty}\left(-\frac{\operatorname{arctg}\sqrt{v}}{\sqrt{v}}+\frac{\overset{=\frac{\pi}{4}}{\operatorname{arctg}1}}{1}+\ln\sqrt{v}-\frac{1}{2}\ln(v+1)-\frac{1}{2}\ln 2\right)=$$

$$=2\cdot\frac{\pi}{4}-\ln 2+2\lim_{v\to\infty}\left[\overset{=\frac{\pi/2}{\infty}=0}{-\frac{\operatorname{arctg}\sqrt{v}}{\sqrt{v}}}+\ln\sqrt{v}-\frac{1}{2}\ln(v+1)\right]=$$

$$=\frac{\pi}{2}-\ln 2+2\lim_{v\to\infty}\underbrace{\ln\frac{\sqrt{v}}{\sqrt{v+1}}}_{0}=\frac{\pi}{2}-\ln 2$$

3. Puncte de extrem condiționat relativ la S.

$f:(0,\infty)^2\to\mathbb{R} \qquad f(x,y)=2x+3y$

$S=\{(x,y)\in(0,\infty)^2 \mid \sqrt{x}+\sqrt{y}=5\} \Rightarrow F(x)=\sqrt{x}+\sqrt{y}-5$

$L(x,y,\lambda)=f(x,y)+\lambda F(x,y)=2x+3y+\lambda(\sqrt{x}+\sqrt{y}-5)=2x+3y+\lambda\sqrt{x}+\lambda\sqrt{y}-5\lambda$

$$\frac{\partial L}{\partial x}=2+\lambda\frac{1}{2\sqrt{x}} \qquad \frac{\partial L}{\partial y}=3+\lambda\frac{1}{2\sqrt{y}} \qquad \frac{\partial L}{\partial \lambda}=\sqrt{x}+\sqrt{y}-5$$

$$\Rightarrow \begin{cases} 2+\lambda\frac{1}{2\sqrt{x}}=0 \Rightarrow 2+\lambda\frac{1}{2(5-\sqrt{y})}=0 \\ 3+\lambda\frac{1}{2\sqrt{y}}=0 \Rightarrow \lambda=-3\cdot 2\sqrt{y} \\ \sqrt{x}+\sqrt{y}-5=0 \Rightarrow \sqrt{x}=5-\sqrt{y} \end{cases}$$

$$\Rightarrow 2+(-6\sqrt{y})\cdot\frac{1}{10-2\sqrt{y}}=0 \Leftrightarrow 2=\frac{6\sqrt{y}}{10-2\sqrt{y}} \Rightarrow \dots \Rightarrow y=4 \Rightarrow x=9$$

$\Rightarrow (9,4)$ – punct critic   sigur punctele de extrem sunt printre punctele critice

$f(9,4)=30$ minimul funcției f restricționat la S se atinge în $(9,4)$

(știm că e minim pt. că maxim e când x sau $y\to 0$)

2023

2. $f(x,y,z) = \frac{1}{x+y+z}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$, Aflați $p \in \mathbb{R}$ a.î.

$$x \cdot \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \cdot \frac{\partial f}{\partial z} = p \cdot f(x,y,z)$$

---

3. $A = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\quad 0 \le y \le \sqrt{2-x^2}\}$

a) Desen

b) $\iint_A xy^2 \, dx\, dy$

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1. a) $\lim_{n \to \infty} a_n = ?$, $a_n = n^2\left(\frac{1}{n} - \ln\left(1+\frac{1}{n}\right)\right)$

b) Studiază convergența seriei în funcție de $a > 0$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \ln\left(1+\frac{1}{n}\right)\right)^a$$

(1) a) $a_n = n^2\left(\frac{1}{n} - \ln\left(1+\frac{1}{n}\right)\right)$

$$\lim_{n\to\infty} a_n =$$

$$\lim_{n\to\infty}\left[n^2\left(\frac{1}{n} - \ln\left(1+\frac{1}{n}\right)\right)\right]$$

$$= \lim_{n\to\infty}\left(\frac{\frac{1}{n} - \ln\left(1+\frac{1}{n}\right)}{\frac{1}{n^2}}\right)$$

$$\overset{\frac{0}{0}}{\underset{LH}{=}} \lim_{n\to\infty} \frac{-\frac{1}{n^2} + \frac{1}{\left(1+\frac{1}{n}\right)n^2}}{-\frac{2}{n^3}}$$

$$= \lim_{n\to\infty} \frac{n}{2(n+1)}$$

$$= \frac{1}{2}\lim_{n\to\infty}\frac{n}{n+1}$$

$$= \frac{\lim\limits_{n\to\infty}\frac{1}{1+\frac{1}{n}}}{2}$$

$$= \frac{\lim\limits_{n\to\infty} 1}{2}$$

$$= \frac{1}{2}$$

$$\frac{-\frac{1}{n^2} + \frac{1}{\left(1+\frac{1}{n}\right)n^2}}{-\frac{2}{n^3}} = \frac{{}^{n+1)}-\frac{1}{n^2} + {}^{n)}\frac{1}{\left(\frac{n+1}{\cancel{n}}\right)n^2}}{-\frac{2}{n^3}} = \frac{\frac{\cancel{n}-\cancel{n}-1}{n^2(n+1)}}{-\frac{2}{n^3}} = -\frac{1}{\cancel{n^2}(n+1)}\cdot -\frac{\cancel{n^3}}{2} = \frac{n}{2(n+1)}$$

b) $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \ln\left(1+\frac{1}{n}\right)\right)^a$ – convergența seriei în funcție de a

2) $f(x,y,z) = \dfrac{1}{x+y+z}\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right)$ $\qquad p \in \mathbb{R}$ a.î.

$$x\cdot\frac{\partial f}{\partial x} + y\cdot\frac{\partial f}{\partial y} + z\cdot\frac{\partial f}{\partial z} = p\cdot f(x,y,z)$$

$$\frac{\partial f}{\partial x} = \left(\frac{1}{x+y+z}\right)'\cdot\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) + \frac{1}{x+y+z}\cdot\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)'$$

$$= -\frac{1}{(x+y+z)^2}\cdot\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) + \frac{1}{x+y+z}\cdot\frac{-1}{x^2}$$

$$= -\frac{1}{x^2(x+y+z)} - \frac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}{(x+y+z)^2}$$

$$\frac{\partial f}{\partial y} = -\frac{1}{y^2(x+y+z)} - \frac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}{(x+y+z)^2}$$

$$\frac{\partial f}{\partial z} = -\frac{1}{z^2(x+y+z)} - \frac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}{(x+y+z)^2}$$

$$x\cdot\left(-\frac{1}{x^2(x+y+z)} - \frac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}{(x+y+z)^2}\right) = -\frac{\cancel{x}\,1}{x^{\cancel{2}}(x+y+z)} - x\cdot\frac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}{(x+y+z)^2}$$

$$-\frac{\overset{yz}{1}}{x(x+y+z)}\;-x\cdot\frac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}{(x+y+z)^2} - \frac{\overset{xz}{1}}{y(x+y+z)}\;-y\cdot\frac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}{(x+y+z)^2} - \frac{\overset{xy}{1}}{z(x+y+z)}\;-z\cdot\frac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}{(x+y+z)^2}$$

$$= \frac{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)(-x-y-z)}{(x+y+z)^2} + \frac{-yz-xz-xy}{xyz(x+y+z)} \Rightarrow \boxed{p=-2}$$

$$= -\boxed{\frac{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\cancel{(x+y+z)}}{(x+y+z)^{\cancel{2}}}}_{f(x,y,z)} + \frac{\cancel{xyz}\left(-\frac{1}{x}-\frac{1}{y}-\frac{1}{z}\right)}{\cancel{xyz}(x+y+z)}$$

$$= -f(x,y,z) + \frac{-\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)}{(x+y+z)} \to -f(x,y,z)$$

3. $A = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1,\ 0 \leq y \leq \sqrt{2-x^2}\}$

a) desen

b) $\iint_A xy^2 \, dx\, dy$

b) $$\int_0^1 \left(\int_0^{\sqrt{2-x^2}} xy^2 \, dy \right) dx = \int_0^1 x \cdot \frac{y^3}{3} \Big|_0^{\sqrt{2-x^2}} dx = \int_0^1 x \cdot \frac{(\sqrt{2-x^2})^3}{3} dx = \int_0^1 x \cdot \frac{(2-x^2)\sqrt{2-x^2}}{3}$$

$$= \int_0^1 \frac{2x\sqrt{2-x^2} - x^3\sqrt{2-x^2}}{3} dx = \frac{1}{3}\int_0^1 2x\sqrt{2-x^2}\, dx - \frac{1}{3}\int_0^1 x^3\sqrt{2-x^2}\, dx$$

$$\int_0^1 2x\sqrt{2-x^2}\, dx = 2\int_0^1 x\sqrt{2-x^2}\, dx \quad (t = 2-x^2) = 2\int_0^1 -\frac{1}{2}\sqrt{t}\, dt = -\int_0^1 t^{\frac{1}{2}}\, dt = -\frac{2\sqrt{t}\,|t|}{3}$$

$$t = 2-x^2 \Rightarrow \frac{-2\sqrt{2-x^2}\,|2-x^2|}{3}$$

$= \ldots$ calcule

$$= \frac{-1+4\sqrt{2}}{15}$$

$$\int_0^{\sqrt{2-x^2}} \int_0^1 xy^2 \, dx \, dy$$

$$y^2 \cdot \frac{x^2}{2}\Big|_0^1$$

$$y^2 \cdot \frac{1}{2}$$

$$\int_0^{\sqrt{2-x^2}} \frac{y^2}{2}\, dy$$

$$\frac{1}{2}\int_0^{\sqrt{2-x^2}} y^2\, dy = \frac{1}{2} \cdot \frac{y^3}{3}\Big|_0^{\sqrt{2-x^2}}$$

$$= \frac{1}{2} \cdot \frac{(\sqrt{2-x^2})^3}{3}$$

a)