Seminarii Prob. și Statistică

Andrei

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# Seminar 1

(1) 5 culegeri mate
3 culegeri info
4 romane

a) $3! \cdot 4! \cdot 5!$

b) $4! \cdot \cancel{8!}$ $9!\,(5+3+1)!$
↳ o entitate de 4 romane

c) $3! \cdot 5! \cdot 4!$

(2) a) coduri de 5 caractere 3 cifre și 2 majuscule

26 litere
10 cifre

$10^3 \cdot 26^2$

01DDO
A21 20
999XX

b) coduri de 5 caractere 3 cifre și 2 majuscule (distincte)

$A_{26}^{2} \cdot A_{10}^{3} = 25 \cdot 26 \cdot 8 \cdot 9 \cdot 10$

(3) m fete
n băieți

m+n locuri totale

A: x are 2 vecine

$p(A) = ?$

_ x _

x _

$m_1 \times m_2$ __ __ . $(m-1+n)!$

grupul $f \times f \Rightarrow A_m^2$

$\Rightarrow A_m^2 \cdot (m-1+n)!$

$$p(A) = \frac{A_m^2 \cdot (m+n-1)!}{(m+n)!}$$

(4) 3A 2B 3C 1 → perm. cu repetiții $\frac{9!}{3! \cdot 2! \cdot 3! \cdot 1!}$

pt cerc este același principiu, dar împărțim n de rotații posib $\Rightarrow$

$$\Rightarrow \frac{9!}{3! \cdot 2! \cdot 3! \cdot 9}$$

(5) X 00, 000, Q, grup 3 lit ce încep cu O sau I

41 județe + Buc

99 numere , pt Buc poate fi 999.
↳ 01

__ ↳ 23 __ ↳ 25 __ ↳ 25

$23 \cdot 25^2$ litere

$41 \cdot 99 \cdot 23 \cdot 25^2 + 999 \cdot 25^2 \cdot 23$

(6) 7 călușari $(c_1, c_2, \dots \to c_7)$ în care $A: c_1, c_7$ vecini

$P(A) = ?$

6! moduri

5! moduri + 2! $\quad P = \frac{5! + 2!}{6!} = \frac{2}{5}$

(7) a) $x_1 + \dots + x_k = n$

$k-1$ valori de „+"

$k$ valori $(x_1 \dots x_k)$

$\sum_{i=1}^{k} x_i = n \Rightarrow C_{n-1}^{k-1}$ soluții

$\to$ + uri

$\hookrightarrow$ spații libere

$1 \_ 1 \_ 1 \_ \dots \_ 1$

b) $x_1 + \dots + x_k = n \; | + k$ $\quad$ pt a fi sigur $n \geq k$

$(x_1+1) + (x_2+1) + \dots + (x_k+1) = n+k$

$y_1 + \dots + y_k = n + k \Rightarrow C_{n+k-1}^{k-1}$ soluții

(8) 10 biți 0, $k$ biți de 1

$\_ 0 \_ 0 \_ \dots \_ 0$

$1 + C_{10}^1 + C_{10}^2 + \dots + C_6^5$

$k \in \{0, \dots, 5\}$

## Seminarul 2

(1) 25 de componente

6 componente alese aleator

4 din 25 defecte

A: „pachetul să fie returnat"

$\bar{A}$: „pachetul este acceptat"

6 OK $\Rightarrow$ se livrează

$P(A) = 1 - P(\bar{A})$

$P(\bar{A}) = \frac{C_{21}^6}{C_{25}^6}$

(2) 5 bile

a) 2, 4 $\quad \underline{2}\ 1\ 3\ 5\ \underline{4}$ $\quad P(A) = \frac{3! \cdot 2!}{5!} \Rightarrow \frac{2}{20} = \frac{1}{10}$

b) 1 3 2 4 5 $\to 2! \cdot 3!$

1 5 2 4 3 $\to 2! \cdot 3!$

3 5 2 4 1 $\to 2! \cdot 3!$

$P(A) = \frac{3 \cdot 3! \cdot 2!}{5} = \frac{\cancel{6}^{3}}{\cancel{20}_{10}} = \frac{3}{10}$

c) 2 4 3 1 5

$$P(C) = \frac{(3+1)! \cdot 2!}{5!} = \frac{2}{5}$$

d) $P(D) = 1 - P(\overline{D})$ $\quad P(\overline{D}) = 2! \cdot 3!$ $\quad P(D) = 1 - \frac{1}{10} = \frac{9}{10}$

(3) 10 mașini distincte
1 destinatar dintr-o listă 20

$f: \{m_1, \dots, m_{10}\} \to \{p_1, \dots, p_{20}\}$

$$P(A) = \frac{C_{10}^5 \cdot 19^5}{20^{10}}$$

$n_p = 20^{10}$ $\quad n_f = C_{10}^5 \cdot 19^5$

(4) M - submulț. cu 3 elem. alese aleator ale mulț. $\{2, 3, 4, 5, 6\}$
N - submulț. cu 2 elem. alese aleator ale mulț. $\{0, 1, 7, 8\}$
$U = M \cup N$

A: „U conține doar numere impare" $\Rightarrow$ eveniment imposibil $\Rightarrow P(A) = 0$

B: „U conține doar numere consecutive" $\quad P(A) = \frac{2}{C_5^3 \cdot C_4^2}$

C: „$\{0, 4\} \subset U$" $\quad P(A) = \frac{C_3^1 \cdot \cancel{C_4^2}}{C_5^3 \cdot \cancel{C_4^2}} = \frac{3}{2 \cdot 5} = \frac{1}{10}$

D: „minim 2 numere pare"
$\overline{D}$: „cel mult un număr par"

2 3 5
3 4 5
3 5 6

01 81
07 87
08

$$P(D) = 1 - \frac{\cancel{3}^{1}}{\cancel{60}_{20}} = \frac{19}{20}$$

(5) 9 pers $\to$ 3 vagoane $\quad f: \{p_1, \dots, p_9\} \to \{v_1, \dots, v_3\}$

A: „I vagon 3 pers" $\quad P(A) = \frac{C_9^3 \cdot 2^6}{3^9}$

B: „în fiecare vagon să fie 3 pers"

$$n_f = C_9^3 \cdot C_6^3 \cdot C_3^3 \Rightarrow P(A) = \frac{C_9^3 \cdot C_6^3}{3^9}$$

C: „I – 1 pers, II, III – 4 pers"

$n_f = 9 \cdot C_8^4 \cdot 3$ $\qquad P(C) = \dfrac{C_8^4 \cdot 9 \cdot 3}{3^9}$

D: „în fiecare vagon să fie cel puțin o persoană"

$\overline{D}$: „cel puțin un vagon este gol"

$P(\overline{D}) = 1 - \dfrac{3+3\cdot(2^9-2)}{3^9}$

2 vagoane goale → 3 moduri

1 vagon gol → $3\cdot(2^9-2)$ moduri

(6) 8 f, 8 b

16 fotolii

a) b f b f ...

f b f b —

$P(A) = \dfrac{2\cdot 8!\cdot 8!}{16!}$

b) b f ... BF b f — 8 cazuri

f b ... FB f b — 8 cazuri

f b f BF b f b f b — 7 cazuri

b f b FB f b f b — 7 cazuri

30 ⇒ $P(B) = \dfrac{30\cdot 7!\cdot 7!}{16!}$

(7) 4 programe antivirus

$p_1 = \frac{3}{4},\ p_2 = \frac{1}{4},\ p_3 = \frac{2}{4},\ p_4 = \frac{1}{4}$

a) $A = V_1 \cap V_2 \cap V_3 \cap V_4 \Rightarrow P(A) = P(V_1)\cdot P(V_2)\cdot P(V_3)\cdot P(V_4) = \dfrac{6}{4^4}$

b) $B = (V_1 \cap \overline{V_2} \cap \overline{V_3} \cap \overline{V_4}) \cup (\overline{V_1} \cap V_2 \cap \overline{V_3} \cap \overline{V_4}) \cup (\overline{V_1} \cap \overline{V_2} \cap V_3 \cap \overline{V_4})$

$\cup (\overline{V_1} \cap \overline{V_2} \cap \overline{V_3} \cap V_4) = \frac{3}{4}\left(1-\frac{1}{4}\right)\left(1-\frac{2}{4}\right)\left(1-\frac{1}{4}\right) + \underline{\left(1-\frac{3}{4}\right)\frac{1}{4}\left(1-\frac{2}{4}\right)\left(1-\frac{1}{4}\right)}$

$+ \left(1-\frac{3}{4}\right)\left(1-\frac{1}{4}\right)\frac{2}{4}\left(1-\frac{1}{4}\right) + \underline{\left(1-\frac{3}{4}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{2}{4}\right)\frac{1}{4}}$

$= \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{2}{4}\cdot\frac{3}{4} + \cancel{2}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{\cancel{2}}{4}\cdot\frac{3}{4} + \frac{\cancel{2}}{\cancel{4}_2}\cdot\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}$

$= \frac{27}{4^3\cdot 2} + \frac{3}{4^3} + \frac{3^2}{4^3\cdot 2} = \frac{27+6+9}{4^3\cdot 2} = \frac{21}{4^3}$

c) $P(C) = (v_1 \cap v_2 \cap v_3 \cap \overline{v_4}) \cup (v_1 \cap v_2 \cap \overline{v_3} \cap v_4) \cup (v_1 \cap \overline{v_2} \cap v_3 \cap v_4) \cup (\overline{v_1} \cap v_2 \cap v_3 \cap v_4)$

$$= \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{2}{4}\left(1-\frac{1}{4}\right) + \frac{3}{4} \cdot \frac{1}{4} \cdot \left(1-\frac{2}{4}\right) \cdot \frac{1}{4} + \frac{3}{4}\left(1-\frac{1}{4}\right) \cdot \frac{2}{4} \cdot \frac{1}{4} + \left(1-\frac{3}{4}\right) \cdot \frac{1}{4} \cdot \frac{2}{4} \cdot \frac{1}{4}$$

$$= \frac{3 \cdot 1 \cdot 3 \cdot 2}{4^3 \cdot 2} + \frac{3 \cdot 1 \cdot 1}{4^3 \cdot 2} + \frac{1 \cdot 1 \cdot 1}{4^3 \cdot 2} = \frac{18+3+1}{4^3 \cdot 2} = \frac{11}{4^3}$$

d) $P(D) = B \cup \overline{A} = P(B) + P(\overline{A})$

e) $P(E) = 1 - P(\overline{E}) = 1 - P(\overline{A})$

## Seminar 3

(1) $m_1$ – 50 angajați – 50% femei
$m_2$ – 45 angajați – 60% femei
$m_3$ – 100 angajați – 70% femei

$A \cap B$: „angajatul norocos este bărbat și lucrează la $m_3$"

A: „angajat la $m_3$"

B: „este bărbat"

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$m_1$ – 25 bărbați

$m_2$ – 30 bărbați

$m_3$ – 30 bărbați

$$P(B) = \frac{25+30 \cdot 2}{150+\cancel{45}} = \frac{85}{225}$$

$$P(A \cap B) = \frac{30}{225}$$

$$P(A \mid B) = \frac{30^{(5}}{85} = \frac{6}{17}$$

(2) 2 zaruri roșii
3 zaruri albastre } se alege unul aleator

Dacă zarul este roșu → aruncă zarul ales de 3 ori
este albastru → aruncă zarul ales de 2 ori

prob. ca suma punctelor obț. în urma aruncărilor să fie 10

formula probabilității totale

$$P(S) = P(S \mid A)\,P(A) + P(S \mid B)\,P(B)$$

S: „Suma totală este 10" $\quad P(A) = 3/5 \quad P(B) = 2/5$

A: „zarul ales este albastru"

B: „zarul ales este roșu"

$$P(S\backslash B) = P(X=10 \backslash B) = \frac{3!\cdot 3 + 3\cdot 3}{6^3} = \frac{18+9}{216} = \frac{27}{216}$$

$\hookrightarrow$ 6 numere, 3 aruncări

comb. posib:

$$\begin{cases} 1+4+5 \to 3! \text{ var.} \\ 1+3+6 \to 3! \text{ var} \\ 2+4+4 \to 3 \text{ var} \\ 2+3+5 \to 3! \text{ var} \\ 2+2+6 \to 3 \text{ var} \\ 3+3+4 \to 3 \text{ var} \end{cases}$$

$$P(S\backslash A) = P(X=10 \backslash A) = \frac{3}{6^2} = \frac{1}{12}$$

$$\begin{cases} \text{cazuri posib} \\ 6+4 \\ 5+5 \\ 4+6 \end{cases}$$

$$P(S) = \frac{1}{\cancel{12}_4} \cdot \frac{\cancel{3}}{5} + \frac{\cancel{27}^1}{\cancel{216}\,\cancel{8}_4} \cdot \frac{\cancel{2}}{5} = 2 \cdot \frac{1}{20} = \frac{1}{10}$$

(3) N – nr. care a apărut la aruncarea unui zar

zarul este aruncat de N+1 ori

prob ca N=3:

a) nr. obț în urma celor N aruncări să fie diferite

b) nr. obț în cele N aruncări să fie egale.

$P(N=3 \backslash A)$ $\quad$ A: „nr obț. dif"

$P(N=3 \backslash B)$ $\quad$ B: „nr. obț egale"

Formula lui Bayes $\quad P(N=3 | A) = \dfrac{P(A \backslash N=3) \cdot P(N=3)}{P(A)}$

$P(N=3) = \frac{1}{6}$.

$$P(A \backslash N=3) = \frac{P(A \cap N=3)}{P(N=3)} = \frac{\frac{1}{\cancel{6}}\cdot\frac{\cancel{6}}{\cancel{6}} \cdot \frac{5}{\cancel{6}} \cdot \frac{4}{6}}{\frac{1}{\cancel{6}}} = \frac{6}{20}$$

$P(A) = P(A \mid N=3) \cdot P(N=3) = \frac{\cancel{6}}{20} \cdot \frac{1}{\cancel{6}} = \frac{1}{20}$

$P(N=3 \mid A) = \dfrac{\frac{\cancel{6}}{20} \cdot \frac{1}{\cancel{6}}}{\frac{1}{20}} = 1$

? ? ?

(4) ① ② ② ③ ③ ③ ④ ④ ④ ④ $\rightarrow 1+2+3+4 = 10$ bile

Se extrag aleator 4 bile pt a forma un cod x.

$X = \overline{x_1 x_2 x_3 x_4}$

Prob. evenimentelor A: „X = 1234" B: „X = 4321"

$P(A) = P(X = 1234) = P(x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4)$

$= P(x_1 = 1) \cdot P(x_2 = 2 \mid x_1 = 1) \cdot P(x_3 = 3 \mid x_2 = 2, x_1 = 1)$

$\cdot P(x_4 = 4 \mid x_3 = 3, x_2 = 2, x_1 = 1)$

$= \frac{1}{10} \cdot \frac{\cancel{2}}{\cancel{9}_3} \cdot \frac{\cancel{3}}{\cancel{8}_{\cancel{4}}} \cdot \frac{\cancel{4}}{7} = \frac{1}{210}$

$P(B) = P(X = 4321) = P(x_1 = 4, x_2 = 3, x_3 = 2, x_4 = 1)$

$= P(x_1 = 4) \cdot P(x_2 = 3 \mid x_1 = 4) \cdot P(x_3 = 2 \mid x_2 = 3, x_1 = 4) \cdot P(x_4 = 1 \mid x_3 \ldots)$

$= \frac{4}{10} \cdot \frac{3}{9} \cdot \frac{2}{8} \cdot \frac{1}{7} = \frac{1}{210}$

(5) D: „cipul este defect"

$P(D) = 0{,}06$

O comp. are 12 cipuri $\rightarrow$ funcț dacă 11 cipuri OK
$\downarrow$
4 comp. indepen. într-un calculator

A: „o comp funcț"

B: „2 comp funcț"

C: „cel puțin o comp funcț"

Model binomial:

A – succes $\quad p = P(A)$

$\overline{A}$ – insucces $\quad 1-p = P(\overline{A})$

se repetă de n ori experimentul
de k ori succes

$b(k, n) = C_n^k \cdot p^k \cdot (1-p)^{n-k}$

$X \sim \begin{pmatrix} k \\ C_n^k \cdot p^k \cdot (1-p)^{n-k} \end{pmatrix}$

$1 - p = 0{,}06 \Rightarrow p = 0{,}94$

a) $P(A) = b(11, 12) + b(12, 12) \rightarrow q$

↳ necesar doar minim 11 cipuri funcț pt ca comp să meargă

$b(11, 12) = C_{12}^{11} \cdot (0{,}94)^{11} \cdot (0{,}06)^1$

$b(12, 12) = C_{12}^{12} \cdot (0{,}94)^{12} \cdot (0{,}06)^0$

b) $P(B) = b(2,4) = C_4^2 \cdot q^2 \cdot (1-q)^2$

c) $P(C) = b(1,4) + b(2,4) + b(3,4) + b(4,4)$

$= C_4^1 \cdot q^1 \cdot (1-q)^3 + C_4^2 \cdot q^2 \cdot (1-q)^2 + C_4^3 \cdot q^3 \cdot (1-q)^1 + C_4^4 \cdot q^4$

sau $P(C) = 1 - P(\overline{C}) \to$ niciuna

$= 1 - b(0,4)$

(6) 11 litere

A: „abracadabra"

↳ permutare

26 litere $\Rightarrow p_i = \frac{1}{26}$, $i = \overline{1,26}$

modelul urnei cu n culori și bilă returnată

n extrageri

$p_i$ = prob. de a extrage bilă de culoarea i, $i = \overline{1,12}$

$k_i$ bile de culoare i

$b(k_1, k_n; n) = \frac{n!}{k_1! \cdots k_n!} \cdot p_1^{k_1} \cdots p_n^{k_n}$

5 a

2 b

2 r

1 c

1 d

$P(A) = b(5,2,2,1,1;11) = \frac{11!}{5! \cdot 2! \cdot 2! \cdot 1! \cdot 1!} \cdot \left(\frac{1}{26}\right)^5 \cdot \left(\frac{1}{26}\right)^2 \cdot \left(\frac{1}{26}\right)^2 \cdot \frac{1}{26^2}$

$= \frac{11!}{5! \cdot 4} \cdot \frac{1}{26^{11}}$

(7) 52 cărți → extrag 13

A: „nu se extrage treflă"

B: „se extrag 5 inimi"

C: „se extrage doar un as".

4 tipuri de cărți → $52 : 4 = 13$ c.

distribuția hipergeometrică

$p(k;n) = \frac{C_{m_1}^{k} \cdot C_{m_2}^{n-k}}{C_{m_1+m_2}^{n}}$ $n \le m_1 + m_2$

bile roșii + bile negre

$P(A) = \frac{C_{13}^0 \cdot C_{39}^{13}}{C_{52}^{13}}$

$p(0,13)$

$P(B) = p(5,13) = \frac{C_{13}^5 \cdot C_{39}^8}{C_{52}^{13}}$

$P(C) = p(1,13) + p(0,13) = \frac{C_{13}^1 \cdot C_{39}^{12}}{C_{52}^{13}} + \frac{C_{13}^0 \cdot C_{39}^{13}}{C_{52}^{13}}$

(8) 4 cercetători A: „2 mat, 1 info, 1 fiz"

4 matem, 3 info, 5 fizicieni

$$P(A) = \frac{C_4^2 \cdot C_3^1 \cdot C_5^1}{C_{12}^4}$$

(9) Un ~~zar~~ zar aruncat de 5 ori.

A: „2 nr pare"

B: „1 - 2 ori, 3 - o dată, 6 - 2 ori"

C: „2 nr prime, 1 nr = 1, 2 nr = 4"

$\to$ 2, 4, 6 - 50% din mulț $\{1, 2, ..., 6\}$

a) $Bino\left(5, \frac{1}{2}\right) = C_5^2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(1 - \frac{1}{2}\right)^3$

b) $b(2, 1, 2; 5) = \frac{5!}{2! \cdot 1! \cdot 2!} \cdot \left(\frac{1}{6}\right)^5$

c) $2, 3, 5 \Rightarrow p = \frac{1}{2}$

$b(2, 1, 2; 5) = \frac{5!}{2! \cdot 1! \cdot 2!} \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{6}\right)^3$

(10) 4N pers din 4 orașe
N pers. din fiecare oraș $\}$ $\to$ aleg 5 pers.

A: „exact 4 pers. sunt din același oraș"

$$\underbrace{A_4^2}_{2 \text{ orașe}} \cdot \frac{C_N^4 \cdot C_N^1 \cdot C_N^0 \cdot C_N^0}{C_{4N}^5}$$

B: „3 pers alese dintr-un oraș, 2 alt oraș"

$$A_4^2 \cdot \frac{C_N^3 \cdot C_N^2 \cdot C_N^0 \cdot C_N^0}{C_{4N}^5}$$

C: „3 pers − oraș, 1, 1"

$$A_4^3 \cdot \frac{C_N^3 \cdot C_N^1 \cdot C_N^1 \cdot C_N^0}{C_{4N}^5}$$

(11) B: „Înt. pe ploaie" $P(B) = 0,2$ $P_3$: „plouă"

C: „Înt. senin" $P(C) = 0,1$ $P(P_3) = 0,8$

a) A: „pers să ajungă la timp"

$P(A) = P(A|B) \cdot P(B) + P(A|C) \cdot P(C) = 0.8 \cdot 0.8 + 0,2 \cdot 0,9$

↳ prob condiționată

b) A: „ploaie, pers ajunge la timp"

$$P(C|A) = \frac{P(A|C) \cdot P(C)}{P(A)} = \frac{0,8 \cdot 0,8}{0,82}$$

$P(A) = P(A|C)\, P(C) + P(A|\bar{C}) \cdot P(C)$

## Seminarul 4

(1) S − mult. nr. nat. cel mult egale cu 50 cu 2 cifre de parități diferite

$\Rightarrow \{12, 14, 16, 18, 21, 23, 25, 27, 29, 30, 32, \dots, 50\}$

Se alege un nr. aleator

X − suma cifrelor nr. ales

distribuția lui X, E(X) → val medie

10 − 18 → 5
21 − 29 → 5
30 − 38 → 5
41 − 49 → 5

$5 \cdot 4 + 1 = 21$ nr.

$S = \{\overline{ab} \in \mathbb{N} \mid \overline{ab} \le 50, a+b \text{ impar}\}$

$X : \Omega \to \mathbb{R}$

↳ v.a. discretă finită

$$X \sim \begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 \\ 1/21 & 2/21 & 5/21 & 4/21 & 4/21 & 3/21 & 1/21 \end{pmatrix}$$

$$E(X) = \sum_{i=1}^{n} x_i \cdot p_i = \frac{1}{21} + \frac{3 \cdot 2}{21} + \frac{5 \cdot 5}{21} + \frac{7 \cdot 4}{21} + \frac{9 \cdot 4}{21} + \frac{11 \cdot 3}{21} + \frac{13}{21}$$

$$= \frac{141}{21} = 6,71$$

(2) clasificarea naivă Bayes a unor restaurante (R) în clasele recom./nerec.
în funcț de cost, timp, mâncare
R, C, T, M – var. aleatoare

| R=r | R=n |
|---|---|
| 10 | 10 |

| P(R=r) | P(R=n) |
|---|---|
| 1/2 | 1/2 |

| C | R=r | R=n | P(C=_\|R=r) | P(C=_\|R=n) |
|---|---|---|---|---|
| i | 3 | 4 | 3/10 | 4/10 |
| m | 4 | 3 | 4/10 | 3/10 |
| s | 3 | 3 | 3/10 | 3/10 |

| T | R=r | R=n | P(T=..\|R=r) | P(T=..\|R=n) |
|---|---|---|---|---|
| p | 6 | 3 | 6/10 | 3/10 |
| m | 3 | 2 | 3/10 | 2/10 |
| i | 1 | 5 | 1/10 | 5/10 |

| M | R=r | R=n | P(M=..\|R=r) | P(M=..\|R=n) |
|---|---|---|---|---|
| f | 1 | 3 | 1/10 | 3/10 |
| a | 1 | 4 | 1/10 | 4/10 |
| b | 3 | 2 | 3/10 | 2/10 |
| d | 5 | 1 | 5/10 | 1/10 |

ii) vector de atribute : $E = (C=s) \cap (T=m) \cap (M=b)$.
Alegeți o clasă pt E, care prob e mai mare $P(R=r|E)$ sau $P(R=n|E)$

$$P(R=r|E) = \frac{P(E|R=r) \cdot P(R=r)}{P(E)} = \frac{\frac{3 \cdot 3 \cdot 3}{10^3} \cdot \frac{1}{2}}{P(E)}$$

$$E = (C=s) \cap (T=m) \cap (M=b) = \frac{3}{10} \cdot \frac{3}{10} \cdot \frac{3}{10} \qquad = \frac{27}{2000} \cdot \frac{1}{P(E)}$$

$$P(R=m \mid E) = \frac{1}{P(E)} \cdot \frac{3}{10} \cdot \frac{2}{10} \cdot \frac{2}{10} \cdot \frac{1}{2} = \frac{12}{2000} \cdot \frac{1}{P(E)}$$

$$\Rightarrow P(R=n \mid E) > P(R=m \mid E)$$

$$1 - P(R=n \mid E) = P(R=m \mid E) \Leftrightarrow P(R=n \mid E) + P(R=m \mid E) = 1$$

$$\Rightarrow \frac{12+24}{2000} \cdot \frac{1}{P(E)} = 1$$

$$\Rightarrow P(E) = \frac{12+24}{2000} = \frac{39}{2000}$$

(4) $a, b \in \mathbb{Z}$
$a \leq b$
$e \in (0,1)$

$$X \sim \begin{pmatrix} a & a+1 & \dots & b \\ e & e & \dots & e \end{pmatrix}$$

a) $\sum_{i=a}^{b} P(X=i) = 1.$

$$c(\underbrace{b-a+1}_{\text{nr. apariții } e}) = 1 \Rightarrow c = \frac{1}{b-a+1}$$

b) $a = 3 \quad b = 21$ $\qquad [a, b]$

$$P\left(\left\{X \leq \frac{a+b}{2}\right\} \cup \left\{\frac{a+b}{6} \leq X\right\}\right) = ?$$

$$P(\{X \leq 12\} \cup \{4 \leq X\}) = ?$$

$$P(\{X \leq 12\} \cap \{4 \leq X\})$$

$$X \sim \begin{pmatrix} 3 & \dots & 21 \\ 1/19 & \dots & 1/19 \end{pmatrix}$$

$$P\{X \in [4 \dots 12)\} = \frac{12-4+1}{19} = \frac{9}{19}$$

$$P(\{X \leq 12\} \cup \{X \geq 4\}) = P(X \in \{3, \dots, 12\}) + P(X \in \{4, \dots, 21\}) - P\{X \in (4 \dots 12)\}$$

$$= \frac{12-3+1}{19} + \frac{21-4+1}{19} - \frac{9}{19} = \frac{19}{19} = 1.$$

e) Det $a, b$, $P(X=a)=\frac{1}{3}$ $E(X)=1$.

$$E(X)=\sum_{i=a}^{b} g(x_i)\cdot p_i = a\cdot c+(a+1)\cdot c+\ldots+b\cdot c$$

$$P(X=a)=\frac{1}{3} \Rightarrow c=\frac{1}{3}=\frac{1}{b-a+1}$$

$$3=b-a+1 \Rightarrow b-a=2 \Rightarrow b=a+2$$

$$=\frac{1}{3}(a+a+1+\ldots+b)$$

$$=\frac{1}{3}(a+a+1+a+2)$$

$$=\frac{1}{3}(3a+3)=a+1$$

$$\left.\begin{array}{l} E(X)=1 \\ E(X)=a+1 \end{array}\right\} \Rightarrow a=0, b=2.$$

(3) $X\sim Geo(p)$, $P(X=k)=p\cdot(1-p)^k$, $k=0,1,\ldots$

↳ de k ori fail până la primul succes

$$E(X)=\sum_{k=0}^{\infty} k\cdot p(1-p)^k$$

$$=p(1-p)+2p(1-p)^2+3p(1-p)^3+\ldots+np(1-p)^n+\ldots$$

$$=(1-p)p\,[1+2(1-p)+3(1-p)^2+\ldots+n(1-p)^{n-1}+\ldots]$$

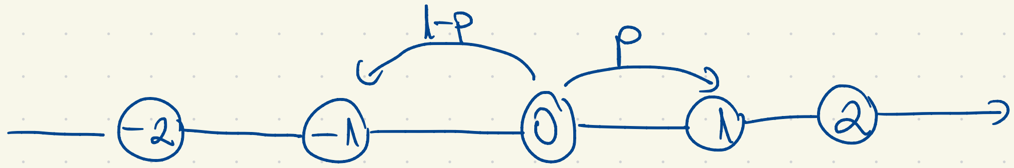
$$=(1-p)\sum_{k=0}^{\infty}(k+1)\cdot p(1-p)^k=(1-p)\underbrace{\sum_{k=0}^{\infty} k\,p(1-p)^k}_{E(X)}+(1-p)\sum_{k=0} p(1-p)^k$$

$$E(X)-(1-p)E(X)=(1-p)\cdot p\sum_{k=0}^{\infty}(1-p)^k$$

$$p\cdot E(X)=(1-p)\cdot p\cdot\frac{1}{1-(1-p)}$$

$$pE(X)=1-p \Rightarrow E(X)=\frac{1-p}{p}$$

(5)




X- v.a. => indică poz. finală după n pași (depl. începe de la nodul 0)

distribuție + val. medie

$Y_i \sim \begin{pmatrix} -1 & 1 \\ 1-p & p \end{pmatrix}$ → distribuția unui pas

$X = Y_1 + Y_2 + \dots + Y_n$

Bernoulli $(p) \sim \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix} = X_1 = X_i$

Bino $(p) \sim \begin{pmatrix} k \\ C_n^k \, p^k (1-p)^{n-k} \end{pmatrix} \sim X_1 + X_2 + \dots X_n$

pasul n: $Y_n \sim 2X_n - 1$

$X \sim \begin{pmatrix} 2k-n \\ C_n^k \, p^k (1-p)^{n-k} \end{pmatrix}$ $\qquad E(X) = 2np - n$

(6)

| X\Y | -2 | 1 | 2 |
|---|---|---|---|
| 1 | 0,2 | 0,1 | 0,2 |
| 2 | 0,1 | 0,1 | 0,3 |

a) distr. de prob.

$P(X = x_p) = \sum_{j \in J} p_{pj}, \forall i \in I.$ vector aleator discret

$X \sim \begin{pmatrix} 1 & 2 \\ 0,5 & 0,5 \end{pmatrix}$ $\qquad Y = \begin{pmatrix} -2 & 1 & 2 \\ 0,3 & 0,2 & 0,5 \end{pmatrix}$

$Y^2 = \begin{pmatrix} 4 & 1 & 4 \\ 0,3 & 0,2 & 0,5 \end{pmatrix}$

b) $A$: „$|X-Y|$" $=1$. $\quad Y>0$

$P(A)=?$

$$P(|X-Y|=1 \mid Y>0)=\frac{P(|X-Y|=1 \cap Y>0)}{P(Y>0)}=\frac{P(X=1,Y=2)+P(X=2,Y=1)}{P(Y>0)}$$

$$=\frac{0{,}2+0{,}1}{0{,}7}=\frac{0{,}3}{0{,}7}.$$

c) evenimentele $X=2$ și $Y=1$ independente $\Leftrightarrow P(X=x, Y=y)=P(X=x)\cdot P(Y=y)$

$0{,}1=0{,}5\cdot 0{,}2 \Leftrightarrow 0{,}1=0{,}1$ „A"

d) var. aleatoare $X$ și $Y$ indepen.

$$\left.\begin{array}{l} P(X=x, Y=y)=P(X=x)\cdot P(Y=y). \\ P(X=1, Y=2)=0{,}2 \\ P(X=1)\cdot P(Y=2)=0{,}5\cdot 0{,}5=0{,}25 \end{array}\right\} \text{2) nu sunt.}$$

e) $X=1$, $Y=1$ cond. indepen. $\quad X+Y=2.$

$$P(X=1, Y=1 \mid X+Y=2)=\frac{P(X=1,Y=1)}{P(X+Y=2)}=\frac{0{,}5\cdot 0{,}2}{}$$

$P(X=1 \mid X+y=2)=$

$P(Y=1 \mid X+Y=2)=$

f) $$P(X=1, Y=2 \mid X+Y=3)=\frac{P(X=1,Y=2)}{P(X+X=3)}=$$

g) $E(2X+Y^2) = 2E(X) + E(Y^2) = 2(1\cdot 0{,}5 + 2\cdot 0{,}5) +$

nu pot trece în afară $\quad (-2)^2 \cdot 0{,}3 + 1^2\cdot 0{,}2 + 2^2\cdot 0{,}5$

$= 6{,}4.$

(4) o monedă se aruncă de 10 ori.

X – v.a. dif dintre nr. de capete și nr. de pajuri obț.

i) C – v.a. indică nr. de capete

$$C \sim \begin{pmatrix} k \\ C_{10}^{k} \cdot \left(\frac{1}{2}\right)^{k} \cdot \left(\frac{1}{2}\right)^{10-k} \end{pmatrix} \qquad k \in \{0, \dots, 10\}$$

↳ Bino pt că e o experiență repetată cu succese + insuccese

$$C \sim \begin{pmatrix} k \\ C_{10}^{k} \cdot \frac{1}{2^{10}} \end{pmatrix} \qquad P + C = 10 \Rightarrow P = 10 - C$$

P – v.a., indică nr de ap. pajură $\quad P \sim \begin{pmatrix} k \\ C_{10}^{k} \cdot \frac{1}{2^{10}} \end{pmatrix}$

$$X = C - P = C - (10 - C) = 2C - 10 \Rightarrow X \sim \begin{pmatrix} 2k-10 \\ C_{10}^{k}\, \frac{1}{2^{10}} \end{pmatrix}$$

$E(X) = E(C) - E(P) = 0$

au aceeași val medie

Seminar 5

(1) X - v.a. continuă

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 0, & x \le 0 \\ cxe^{-x}, & x > 0. \end{cases}$ — Det c

a) fct de repartiție

$$\int_{-\infty}^{\infty} f(x)dx = c\int_{-\infty}^{\infty} t\cdot e^{-t}dt = c\int_{-\infty}^{\infty} t\cdot e^{-t}dt$$

$$= c\left(t\cdot(-e^{-t})\Big|_0^{\infty} + \int_0^{\infty} e^{-t}dt\right)$$

$$\lim_{t\to\infty} (-te^{-t}) = \lim_{t\to\infty} \frac{-t}{e^t} \to 0$$

$$\lim_{t\to\infty} -e^{-t} = -\lim_{t\to\infty} \frac{1}{e^t} \to 0$$

$$= c\left(-te^{-t}\Big|_0^{\infty} (-e^{-t})\Big|_0^{\infty}\right) = 1.$$

$c \cdot 1 = 1 \Rightarrow c = 1.$

a) $F(x) = P(X \le x) = \int_{-x}^{x} f(t)\,dt = \begin{cases} 0, & x \le 0 \\ \int_0^x te^{-t}\,dt \end{cases}$

$$\int_0^x t\, e^{-t}dt = -t\cdot e^{-t} - e^{-t}\Big|_0^x$$

$$= -x\cdot e^{-x} - e^{-x} + 1,\ x > 0$$

$\to 1 - P(X \le 5)$

b) $P(\{|X-3| > 2\}) = P(X < 1) + P(X > 5) = F(1) + 1 - F(5)$

$|X-3| > 2$ $\quad \begin{matrix} X-3 > 2 \\ X-3 < -2 \end{matrix} \Rightarrow \begin{matrix} X > 5 \\ X < 1 \end{matrix}$

$$= 1 - \frac{2}{e} + 1 - 1 + \frac{6}{e^5}$$

$$= 1 - \frac{2}{e} + \frac{6}{e^5}$$

c) $P(X < 3 \mid X > 1) = \dfrac{P(1 < X < 3)}{P(X > 1)} = \dfrac{F(3) - F(1)}{1 - F(1)}$

(2) $T \sim Exp(\lambda)$, $\lambda > 0$

$$f_T(x) = \begin{cases} 0, & x \le 0 \\ \lambda e^{-\lambda x}, & x > 0 \end{cases}$$

$E(T) = 5 \Rightarrow \lambda = \frac{1}{5}$

$P(E) = ?$

$$\int_{-\infty}^{\infty} t\, f_T(t)\, dt \Leftrightarrow \int_0^{\infty} t\, \lambda e^{-\lambda t} dt =$$

$$= \left(-t e^{-\lambda t} - \frac{e^{-\lambda t}}{\lambda}\right)\Big|_0^{\infty} = \frac{1}{\lambda}$$

$$P(T \ge 4) = 1 - P(T < 4) = 1 - F_T(4) = 1 - \int_0^4 \frac{1}{5} \cdot e^{-\frac{1}{5}t} dt$$

(3) 3 condensatoare

$T \sim Exp(\lambda)$

$E(T) = 3 \Leftrightarrow \lambda = \frac{1}{3}$

? ?

(5) vect. aleator continuu $(X, Y)$

$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$f(x,y) = \begin{cases} 2e^{-x-2y}, & x > 0,\ y > 0 \\ 0, & altfel \end{cases}$$

a) fct. de repartiție

$$P(X \le x, Y \le y) = \int_{-\infty}^{x} \left( \int_{-\infty}^{y} f_{(X,Y)}(s,t)\, dt \right) ds$$

$$\int_0^x \left( \int_0^y 2e^{-s-2t}\, dt \right) ds$$

$$= \int_0^x \left(2e^{-s} \left(-\frac{e}{2}\right)^{-2t}\right)\Big|_0^y \Big) ds$$

$$= \cancel{2}\left(-\frac{1}{\cancel{2}}\right)(e^{-2y} - 1) \int_0^x e^{-s} ds.$$

$$= (1 - e^{-2y})(1 - e^{-x})$$

b) $F_X(x) = \lim_{y\to\infty} F_{(X,Y)}(x,y)$

$$= \begin{cases} \lim_{y\to\infty}(1-e^{-2y})(1-e^{-x}), & x>0 \\ 0, & x\le 0 \end{cases} = \begin{cases} 1-e^{-x}, & x>0 \\ 0, & x\le 0 \end{cases}$$

$$F_Y(y) = \lim_{x\to\infty} F_{(X,Y)}(x,y) = \begin{cases} 1-e^{-2y}, & y>0 \\ 0, & y\le 0 \end{cases}$$

c) $f_X(x) = \begin{cases} e^{-x}, & x>0 \\ 0, & x\le 0 \end{cases}$ $\rightarrow (1-e^{-x})'$  $f_Y(y) = \begin{cases} 2e^{-2y}, & y>0 \\ 0, & y\le 0 \end{cases}$ $\rightarrow (1-e^{-2y})'$

d) $X, Y$ indep. sau depen.

$f_{(X;Y)}(x,y) = f_X(x)\cdot f_Y(y)$, $\forall x,y\in\mathbb{R} \Rightarrow$ indepen.

$$2e^{-x-2y} = e^{-x}\cdot 2e^{-2y} \quad \checkmark$$

(6) $F:\mathbb{R}\to\mathbb{R}$,

$$F(x) = \begin{cases} ax^2+bx+c, & 0\le x<2 \\ d, & x<0 \\ e, & x\ge 2 \end{cases}$$

$a, b, c, d, e = ?$

i) $P(1<X<2) = \frac{1}{2}$.

ii) $E(X) = 1$.

$P(1<X<2) = F(2)-F(1)$

$= e - ax^2+bx+c = \frac{1}{2}$

$\lim_{x\to-\infty} F(x) = 0 \Rightarrow d=0$

$\lim_{x\to\infty} F(x) = 1 \Rightarrow d=1$

$\lim_{x\nearrow 0} F(x) = F(0) = c$

$\lim_{x\nearrow 0} F(x) = d = 0$

Seminar 7

(1) a) estimare $p \in (0,1)$ pt $X \sim Bino(N, p)$

$x_1, \dots, x_n$ - date statistice pt caract. $X \sim Bino(N, p)$

$X_1, \dots, X_n$ — variabile de selecție

Met. momentelor $\quad E(X) = \frac{1}{n} \sum_{i=1}^{n} x_i = N \cdot p$

$$\Rightarrow N \cdot p = \frac{1}{n} \sum_{i=1}^{n} x_i \Rightarrow p = \frac{1}{N \cdot n} \sum_{i=1}^{n} x_i \Rightarrow \hat{p} = \frac{1}{N \cdot n} \sum_{i=1}^{n} x_i$$

Met. verosimilității maxime

$$L(x_1, \dots, x_n, p) = P(X = x_1) \cdot P(X = x_2) \dots P(X = x_n)$$

$$\left.\begin{array}{l} P(X = x_i) = C_N^{x_i} \cdot p^{x_i} \cdot (1-p)^{N - x_i} \\ x_i \in \{0, 1, \dots, N\} \end{array}\right\} \Rightarrow L(x_1, \dots, x_n, p) = \prod_{i=1}^{n} C_N^{x_i} p^{x_i} (1-p)^{N-x_i}$$

$$L(x_1, \dots, x_n, p) = \prod_{i=1}^{n} C_N^{x_i} \cdot p^{\sum_{i=1}^{n} x_i} \cdot (1-p)^{n \cdot N - \sum_{i=1}^{n} x_i} \quad \Big| \ln$$

$$\ln L(x_1, \dots, x_n, p) = \ln \prod_{i=1}^{n} C_N^{x_i} + \ln p \cdot \sum_{i=1}^{n} x_i + \left(n \cdot N - \sum_{i=1}^{n} x_i\right) \ln(1-p)$$

$$\frac{d}{dp} L(x_1, \dots, x_n, p) = \underbrace{\sum_{i=1}^{n} x_i \cdot \frac{1}{p} - \left(nN - \sum_{i=1}^{n} x_i\right) \cdot \frac{1}{1-p} = 0}$$

$$(1-p) \sum_{i=1}^{n} x_i = p\left(nN - \sum_{i=1}^{n} x_i\right)$$

$$\sum_{i=1}^{n} x_i - \cancel{p \sum_{i=1}^{n} x_i} = pnN - \cancel{p \sum_{i=1}^{n} x_i}$$

$$\sum_{i=1}^{n} x_i = pnN \Rightarrow p = \frac{\sum_{i=1}^{n} x_i}{nN} = \hat{p}(x_1, \dots, x_n)$$

$$\frac{d^2}{d^2 p} \ln L(x_1, \dots, x_n, p) = -\sum_{i=1}^{n} x_i \frac{1}{p^2} - \left(Nn - \sum_{i=1}^{n} x_i\right)\frac{1}{(1-p)^2} \leq 0$$

$$\hat{p}(X_1, \dots, X_n) = \frac{1}{Nn} \cdot \sum_{i=1}^{n} x_i$$

! $\hat{p}(x_1, \dots, x_n)$ nedeplasat dacă $E(\hat{p}(x_1, \dots, x_n)) = p$.

! $\hat{p}(x_1, \dots, x_n)$ consistent dacă $\hat{p}(x_1, \dots, x_n) \xrightarrow{a.s.} p$.

$$E(\hat{p}(x_1, \dots, x_n)) = E\left(\frac{1}{N \cdot n} \cdot \sum_{i=1}^{n} x_i\right) = \frac{1}{N \cdot n} \cdot \sum_{i=1}^{n} \underbrace{E(X)}_{N \cdot p}$$

$$= \frac{1}{\cancel{Nn}} \cdot \cancel{nN}p = p \ \checkmark$$

LTNM $\Rightarrow \frac{1}{n}(x_1 + \dots + x_n) \xrightarrow{a.s.} E(X)$

$$\hat{p}(x_1, \dots, x_n) = \frac{1}{N} \cdot \frac{1}{n}(x_1 + \dots + x_n) \xrightarrow{a.s.} \frac{1}{N} \qquad E(X) = \frac{1}{\cancel{N}} \cancel{N} p = p$$

ii) bile albe $p \in (0,1)$ – necun.

$n = 6$ serii a câte $N = 5$ extrageri cu returnare

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | |
|---|---|---|---|---|---|---|
| 3 | 4 | 2 | 0 | 2 | 1 | bile albe. |

$p = ?$, ambele metode

$$\hat{p}(x_1, \dots, x_n) = \frac{1}{N \cdot n} \cdot \sum_{i=1}^{n} x_i = \frac{1}{6 \cdot 5} \cdot (3+4+2+0+2+1) = \frac{2}{5}$$

(2) $f_X(x) = \begin{cases} \lambda^2 \cdot x \cdot e^{-\lambda x}, & x > 0, \\ 0, & x \leq 0 \end{cases}$ $\quad \lambda > 0$ fixat $\quad \lambda = ?$ ambele met.

$x_1 \dots x_n$ date statistice

$X_1 \dots X_n$ var. de selecție corespunzătoare

Metoda momentelor

$E(X) = \frac{1}{n}(x_1 + \dots + x_n)$ $\qquad E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \frac{2}{\lambda}$

$\frac{1}{n} \cdot \sum_{i=1}^{n} x_i = \frac{2}{\lambda} \Rightarrow \hat{\lambda}(x_1, \dots, x_n) = \frac{2n}{\sum_{i=1}^{n} x_i}$ val. estimată

Met. vers. max.

$L(x_1, \dots, x_n, \lambda) = \prod_{i=1}^{n} \lambda^2 x_i \cdot e^{-\lambda x_i} = \lambda^{2n} \cdot e^{-\lambda \sum_{i=1}^{n} x_i} \cdot \prod_{i=1}^{n} x_i$

$\ln e = 1$

$\ln(L(x_1, \dots, x_n, \lambda)) = 2n \ln \lambda - \lambda \sum_{i=1}^{n} x_i \cdot \ln\left(\prod_{i=1}^{n} x_i\right)$ $\quad (\ln x)' = \frac{1}{x}$

$$\left.\begin{aligned} \frac{d}{d\lambda} L(x_1, \dots, x_n, \lambda) &= \frac{2n}{\lambda} - \sum_{i=1}^{n} x_i = 0 \\ \frac{d^2}{d^2\lambda} \ln(L(x_1, \dots, x_n, \lambda)) &= -\frac{2n}{\lambda} < 0 \end{aligned}\right\} \Rightarrow \frac{2n}{\sum_{i=1}^{n} x_i} = \hat{\lambda}(x_1, \dots, x_n)$$

$\hat{\lambda}(x_1, \dots, x_n) = \frac{2n}{\sum_{i=1}^{n} x_i} \xrightarrow{a.s.} \frac{2}{\frac{2}{\lambda}} = \lambda$ consistent

b) $\hat{\lambda}(1, \frac{3}{2}, 3, 2, 3, \frac{5}{2}, 1, 2) = \frac{2 \cdot 8}{\sum_{i=1}^{8} x_i} = \frac{16}{16} = 1.$

$1 + \cancel{\frac{3}{2}} + 3 + 2 + 3 + \cancel{\frac{5}{2}} + 1 + 2 = 4 + 10 + 2 = 16$

(3) a) $V(X) = 20$ $\qquad V(X) = E(X^2) - E^2(X) = \sigma^2$

$\sigma = \sqrt{V(X)}$

$\hookrightarrow$ nivel satisfacție

$1 - \alpha = 95\%$ nivel încredere $\Rightarrow \alpha = 5\% = 0{,}05$

$$\left(\overline{X_n} - \frac{\sigma}{\sqrt{n}} \cdot z_{1-\frac{\alpha}{2}}, \overline{X_n} + \frac{\sigma}{\sqrt{n}} \cdot z_{1-\frac{\alpha}{2}}\right)$$

$\overline{X_n} = \frac{1}{n} \sum_{i=1}^{n} x_i$ $\qquad \underbrace{z_{1-\frac{\alpha}{2}}}_{\text{cuantila}} = 1{,}96$

$$\overline{X_5} = \frac{71+68+77+68+65}{5} = 70 \qquad \mu = E(X) \text{ val medie}$$

$$\left(70 - \frac{\sqrt{20}}{\sqrt{5}} \cdot 1{,}96 \ , \ 70 + \frac{\sqrt{20}}{\sqrt{5}} \cdot 1{,}96\right) \Rightarrow (66{,}08 ; 73{,}92)$$

$H_0 : \mu = 75 \qquad H_1 : \mu \neq 75$

$75 \notin (66, 68, 77, 65, 65) \Rightarrow H_1$ se acceptă.