Sorting: Homework

- *By using the code at:*

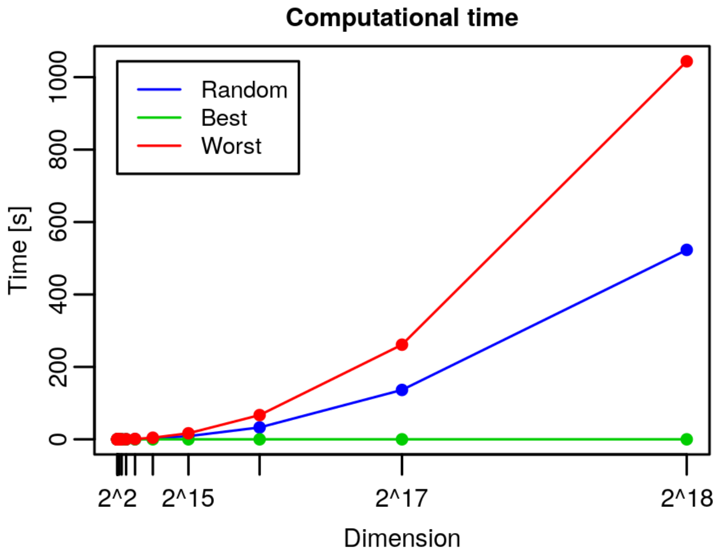
  [*https://github.com/albertocasagrande/AD_sorting*](https://github.com/albertocasagrande/AD_sorting)

  implement Insertion Sort, Quick Sort, Bubble Sort, Selection Sort, and Heap Sort.

  The solution with the implemented code can be found in the files `insertion_sort.c`, `quick_sort.c`, `bubble_sort.c`, `selection_sort.c`, `heap_sort.c` in the folder Sorting.

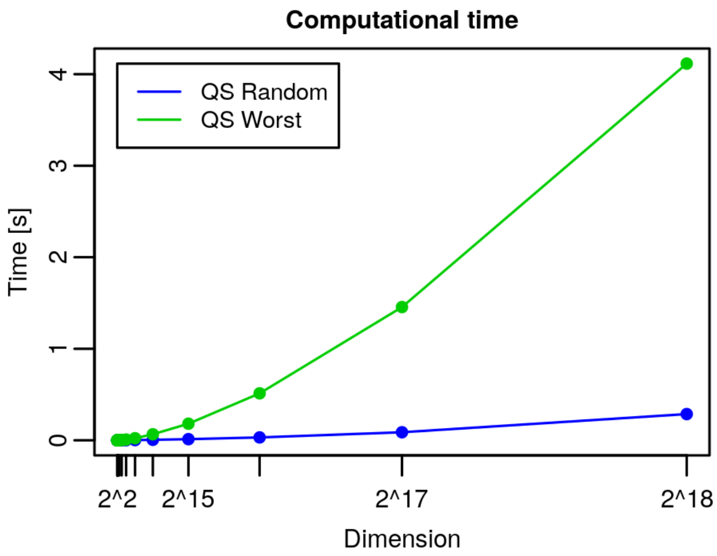
- *For each of the implemented algorithm, draw a curve to represent the relation between the input size and the execution-time.*

  For **Insertion Sort** we have:



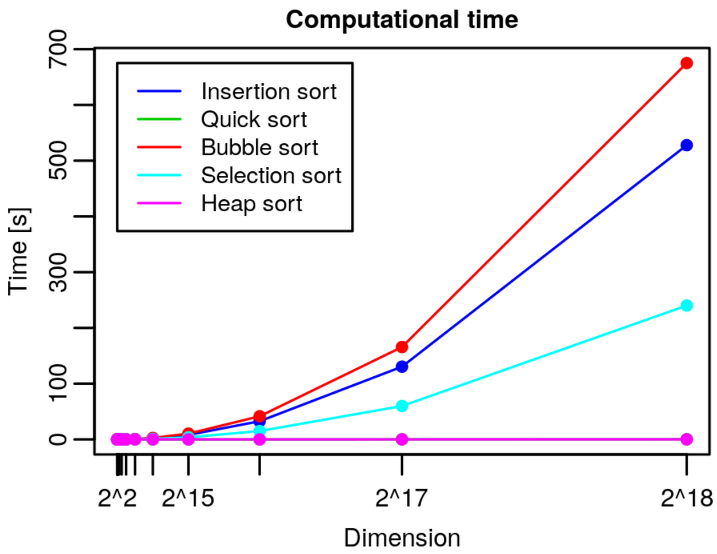
  We can see that, as expected, the worst case (red line), in which the array is sorted in the opposite way, is the one that performs the worst, since it is $\Theta(n^2)$; the random case (blue line) is the one in the middle, since it is $O(n^2)$; and the best case (green line), in which the array is already sorted, is the one that performs better, since it is $\Theta(n)$.

  For **Quick Sort** we have:



Again, as expected, the worst case, in which the array is already sorted, takes more time than a random case, since the first takes $\Theta(n^2)$ wile the second takes $\Theta(n \log n)$.

For **Insertion Sort, Quick Sort, Bubble Sort, Selection Sort, and Heap Sort** altogether we have



We can see that the faster ones are Heap Sort and Quick Sort (its green line it's below Heap Sort's line on the bottom), while the worst is Bubble Sort.

- *Argue about the following statement and answer the questions*

  1. *Heap Sort on an array A whose length is n takes time $O(n)$.*

     FALSE. The overall complexity of Heap Sort is $O(n \log n)$: `build_max_heap` costs $\Theta(n)$ and `extract_min` costs $O(\log i)$ per iteration and in total

     $$T_H(n) = \Theta(n) + \sum_{i=2}^{n} O(\log i) \leq O(n) + O\left(\sum_{i=2}^{n} \log n\right) = O(n \log n).$$

     Hence in the worst case and in the average case $T_H(n) \notin O(n)$, this anyway happens if the vector is made of all equal elements, since `extract_min` would cost $\Theta(1)$ because we won't have to restore the heap property by calling `heapify`.

  2. *Heap Sort on an array A whose length is n takes time $\Omega(n)$.*

     TRUE. From above we have that

     $$T_H(n) = O(n \log n) \geq \Omega(n)$$

     since taken a representative $cn \log n$ of $O(n \log n)$ we have that $cn \log n \geq dn$ for $n > 2$ and $c \geq d$.

  3. *What is the worst case complexity for Heap Sort?*

     The worst case complexity of Heap sort is $O(n \log n)$, since the worst-case running time of `heapify` on a binary heap of size n is $\Omega(\log n)$, as we have seen in the homework 02 01.

  4. *Quick Sort on an array A whose length is n takes time $O(n^3)$.*

     TRUE. We have that in the worst case Quick Sort takes time $\Theta(n^2)$:

     $$T_Q(|A|) = T_Q(|S|) + T_Q(|G|) + \Theta(|A|)$$

     and if $|G| = 0$ or $|S| = 0$ for all recursive call

     $$T_Q(n) = T_Q(n-1) + \Theta(n) = \sum_{i=0}^{n} \Theta(i) = \Theta\left(\sum_{i=0}^{n} i\right) = \Theta(n^2).$$

So we have that $T_Q \in O(n^3)$ since $\Theta(n^2) \subseteq O(n^3)$.

5. *What is the complexity of Quick Sort?*

   In point 4 we have seen that in the worst case $T_Q \in \Theta(n^2)$. In the best case, with a balanced partition, and in the average case we have that $T_Q \in \Theta(n \log n)$.

6. *Bubble Sort on an array A whose length is n takes time $\Omega(n)$.*

   TRUE. Since Bubble Sort works by pair-wise swapping the maximum to the right, even if the vector is ordered we have to scan all the vector, so the cost is at least n, so $T_B(n) \in \Omega(n)$.

7. *What is the complexity of Bubble Sort?*

   We have that one swap-block costs $\Theta(1)$ and that the nested for-loop costs $\Theta(i)$, we have that

$$T_B(n) = \sum_{i=2}^{n} \Theta(i) \cdot \Theta(1) = \Theta\left(\sum_{i=2}^{n} i\right) = \Theta(n^2).$$

- *Solve the following recursive equation:*

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 32 \\ 3 * T\left(\frac{n}{4}\right) + \Theta(n^{3/2}) & \text{otherwise} \end{cases}$$

Using the **recursion tree**, we have that our stopping condition is when $n = 32$, so when in our recursion $\frac{n}{4^i} = 32 \iff n = 2^5 \cdot 2^{2i} \iff n = 2^{5+2i} \iff i = \frac{\log_2 n - 5}{2}$ and choosing $cn^{3/2}$ as representative for $\Theta(n^{3/2})$ we have that

$$\begin{aligned}
T(n) &= 3 * T\left(\frac{n}{4}\right) + \Theta(n^{3/2}) \\
&\leq 3 * T\left(\frac{n}{4}\right) + cn^{3/2} \\
&\leq 3 * \left(3 * T\left(\frac{n}{16}\right) + c\left(\frac{n}{4}\right)^{3/2}\right) + cn^{3/2} \\
&= 9 * T\left(\frac{n}{16}\right) + 3c\left(\frac{n}{4}\right)^{3/2} + cn^{3/2} \\
&\leq 9 * \left(3 * T\left(\frac{n}{64}\right) + c\left(\frac{n}{16}\right)^{3/2}\right) + 3c\left(\frac{n}{4}\right)^{3/2} + cn^{3/2} \\
&\leq 27 * T\left(\frac{n}{64}\right) + 9c\left(\frac{n}{16}\right)^{3/2} + 3c\left(\frac{n}{4}\right)^{3/2} + cn^{3/2} \\
&\leq \dots \\
&\leq 3^{\frac{\log_2 n - 5}{2}} T(32) + \sum_{i=0}^{\log_2 n} 3^i c\left(\frac{n}{4^i}\right)^{3/2} = 3^{\frac{\log_2 n - 5}{2}} \Theta(1) + \sum_{i=0}^{\frac{\log_2 n - 5}{2}} 3^i c \frac{n^{3/2}}{4^{3/2i}} \\
&\leq 3^{\frac{\log_2 n - 5}{2}} c' + \sum_{i=0}^{\frac{\log_2 n - 5}{2}} \frac{3^i}{2^{3i}} cn^{3/2} = 3^{\frac{\log_2 n - 5}{2}} c' + cn^{3/2} \sum_{i=0}^{\frac{\log_2 n - 5}{2}} \left(\frac{3}{2^3}\right)^i \\
&= 3^{\frac{\log_2 n - 5}{2}} c' + cn^{3/2} \cdot \frac{(3/8)^{\left(\frac{\log_2 n - 5}{2}\right)+1} - 1}{3/8 - 1} = 3^{\frac{\log_2 n - 5}{2}} c' + cn^{3/2} \cdot \frac{(3/8)^{\frac{\log_2 n - 3}{2}} - 1}{(3-8)/8} \\
&= 3^{\frac{\log_2 n - 5}{2}} c' - \frac{8}{5} cn^{3/2} \cdot \frac{3^{\frac{\log_2 n - 3}{2}} - 8^{\frac{\log_2 n - 3}{2}}}{8^{\frac{\log_2 n - 3}{2}}}
\end{aligned}$$

since $3^{\frac{\log_2 n - 3}{2}} = (3^{\log_2 n - 3})^{1/2} = \left(\frac{3^{\log_2 n}}{3^3}\right)^{1/2} = \left(\frac{3^{\log_2 3 \log_3 n}}{3^3}\right)^{1/2} = \left(\frac{n^{\log_2 3}}{3^3}\right)^{1/2} = \frac{n^{\frac{\log_2 3}{2}}}{3^{3/2}}$ (since $\log_b n = \log_b a \cdot \log_a n$ we have $\log_2 n = \log_2 3 \cdot \log_3 n$) and $8^{\frac{\log_2 n - 3}{2}} = 2^{\frac{3}{2}(\log_2 n - 3)} = (2^{\log_2 n - 3})^{3/2} = \left(\frac{2^{\log_2 n}}{2^3}\right)^{3/2} = \left(\frac{n}{8}\right)^{3/2}$ we have that

$$
\begin{aligned}
&= \frac{n^{\frac{\log_2 3}{2}}}{3^{3/2}} c' - \frac{8}{5} c n^{3/2} \cdot \frac{\frac{n^{\frac{\log_2 3}{2}}}{3^{3/2}} - (n/8)^{3/2}}{(n/8)^{3/2}} = \frac{n^{\frac{\log_2 3}{2}}}{3^{3/2}} c' - \frac{8}{5} 8^{3/2} c n^{3/2} \cdot \frac{\frac{n^{\frac{\log_2 3}{2}}}{3^{3/2}} - \frac{n^{3/2}}{8^{3/2}}}{n^{3/2}} \\
&= \frac{n^{\frac{\log_2 3}{2}}}{3^{3/2}} c' - \frac{8}{5} 8^{3/2} c \cdot \left(\frac{n^{\frac{\log_2 3}{2}}}{3^{3/2}} - \frac{n^{3/2}}{8^{3/2}} \right) = \frac{n^{\frac{\log_2 3}{2}}}{3^{3/2}} c' - \frac{8}{5} 8^{3/2} c \cdot \frac{n^{\frac{\log_2 3}{2}}}{3^{3/2}} + \frac{8}{5} 8^{3/2} c \frac{n^{3/2}}{8^{3/2}} \\
&= \frac{n^{\frac{\log_2 3}{2}}}{3^{3/2}} \left(c' - \frac{8^{5/2}}{5} c \right) + \frac{8}{5} c n^{3/2} \in O(n^{3/2})
\end{aligned}
$$

So we have that $T(n) \in O(n^{3/2})$. Since all the inequalities work also with the $\geq$, we have that $T(n) \in \Omega(n^{3/2})$, so we have that $T(n) \in \Theta(n^{3/2})$