Binary Heaps: Homework

- *Implement the array-based representation of binary heap together with the functions* `HEAP_MIN`*,* `REMOVE_MIN`*,* `HEAPIFY`*,* `BUILD_HEAP`*,* `DECREASE_KEY`*, and* `INSERT_VALUE`*.*

  The solution can be found in the file `binheap.c` in the folder [03 Binary heaps](), with the corresponding names `min_value`, `extract_min`, `heapify`, `build_heap`, `decrease_key` and `insert_value`. The functions `is_heap_empty`, `swap_keys`, `find_the_max`, `delete_heap`, and `print_heap` are also added.

- *Implement an iterative version of* `HEAPIFY`*.*

  The solution can be found in the function `heapify`, contained in the file `binheap.c` in the folder [03 Binary heaps]().

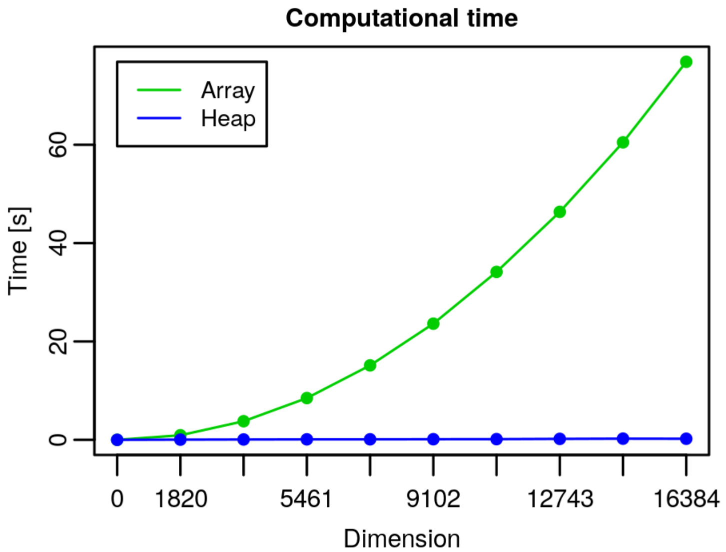
- *Test the implementation on a set of instances of the problem and evaluate the execution time.*

  After running the program [`test_delet_min`](), contained in the folder [03 Binary heaps/tests](), the following output was generated:

```
$ ./test_delete_min

n        On Heaps         On Arrays
0        0.000014         0.000005
1820     0.039135         0.931853
3640     0.070701         3.782944
5461     0.098816         8.491803
7281     0.105832         15.135461
9102     0.124539         23.618608
10922    0.139834         34.143487
12743    0.193662         46.353465
14563    0.239347         60.489976
16384    0.219621         76.866161
```

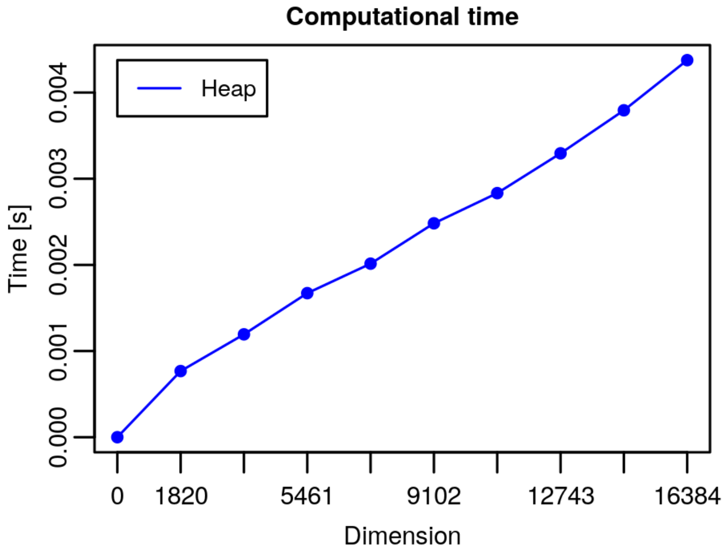
  In the following graph we can see the execution time of the program [`test_delet_min`]():



  We can clearly see that the implementation that uses heaps instead of arrays is much much more efficient.

We have that for the heaps the time of extraction is $O(\log n)$: replacing the root's key costs $\Theta(1)$ and the total cost of $\texttt{HEAPIFY}$ is the height of the heap, $O(\log n)$, so deleting the minimum with the heaps costs $O(\log n)$. However, in this plot it is hard to see.

In the following graph we can better see the execution time using the heaps:



Here we can see a bit better the logarithmic trend of the program.

- *(Ex. 6.1-7 in [1]) Show that, with the array representation, the leaves of a binary heap containing n nodes are indexed by $\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, n$.*

  Let us consider a binary heap containing n nodes. Since the children of a node i are the nodes $2i$ (the left child) and $2i + 1$ (the right child), we have that the "last" parent, meaning the right-most parent in the second-last level, can at most be the node $\lfloor n/2 \rfloor$. That is because if we suppose that $\lfloor n/2 \rfloor + 1$ is a parent node, then its children are the nodes $2(\lfloor n/2 \rfloor + 1)$ and $2(\lfloor n/2 \rfloor + 1) + 1$, but the left child would be

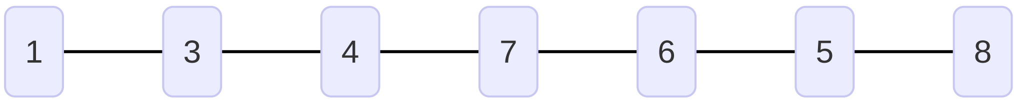
  $$2 \cdot (\lfloor n/2 \rfloor + 1) = 2 \cdot \lfloor n/2 \rfloor + 2 = \begin{cases} 2 \cdot n/2 + 2 & n \text{ even} \\ 2 \cdot (n-1)/2 + 2 & n \text{ odd} \end{cases} = \begin{cases} n+2 & n \text{ even} \\ n+1 & n \text{ odd} \end{cases}$$

  that is out of the boundary of the array, that is impossible. So all the nodes after the node $\lfloor n/2 \rfloor$ must be leaves. Then we have the proof that the nodes $\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, n$ are leaves.

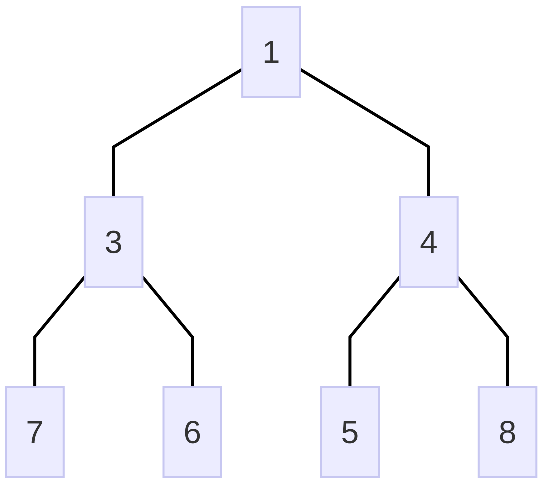
- *(Ex. 6.2-6 in [1]) Show that the worst-case running time of $\texttt{HEAPIFY}$ on a binary heap of size n is $\Omega(\log n)$.*
  *(**Hint**: For a heap with n nodes, give node values that cause $\texttt{HEAPIFY}$ to be called recursively at every node on a simple path from the root down to a leaf.)*

  Let us consider a binary heap of size n. Since a simple path from the root down to a leaf is the height of the tree h, we have to estimate h in terms of n. In a binary tree the root has 2 children at depth 1, each of which has 2 children at depth 2, etc. Thus, the number of nodes ($\equiv$ leaves) at depth h is 2^h. Consequently, the height of a complete binary tree with m leaves, which is the length of the simple path from the root to a leaf, is $\log_2 m$. Since the number of leaves in a complete binary tree is $(n+1)/2$ (n is always odd in a complete binary tree and the number of leaves is $\lceil n/2 \rceil$, as seen in the following exercise), we have that the height is $\log_n (n+1)/2 = \log_2(n+1) - \log_2 2 = \log_2(n+1) - 1$. So in the worst case, if $\texttt{HEAPIFY}$ is called for all the height of the tree, it is called $\Omega(\log n)$ times.

  For example, with the following array in a Min_heap, when the minimum is deleted $\texttt{HEAPIFY}$ is called recursively at every node on a simple path from the root down to a leaf:



that corresponds to the tree



- *(Ex. 6.3-3 in [1]) Show that there are at most $\lceil n/2^{h+1} \rceil$ nodes of height h in any n-element binary heap.*

  > The height of a node in a tree is the number of edges on the longest simple downward path from the node to a leaf, and the height of a tree is the height of its root. The height of a tree is also equal to the largest depth of any node in the tree.

  Let us consider a n-element binary heap with height h. We have that the height of the root is h and the height of the children of the root is $h-1$, while the height of a leaf is 0 and the height of the parents of the leaves is 1.

  Since all the nodes after $\lfloor n/2 \rfloor$ are leaves, we have $\lceil n/2 \rceil$ leaves, so we have $\lceil n/2^{0+1} \rceil$ nodes at height 0. We have a binary heap, so we have a binary tree, thus the number of parents of the leaves will be half the number of the leaves, so we will have $\lceil (n/2)/2 \rceil = \lceil n/4 \rceil = \lceil n/2^{1+1} \rceil$ nodes at height 1. Let's assume that we have $\lceil n/2^{(i-1)+1} \rceil = \lceil n/2^i \rceil$ nodes at height $i-1$, their parents will be half of them, so they will be $\lceil (n/2^i)/2 \rceil = \lceil n/2^{i+1} \rceil$, so we have $\lceil n/2^{i+1} \rceil$ nodes at height i. So by induction we have $\lceil n/2^{h+1} \rceil$ nodes at height h, and we have proved our thesis. Besides, the root is at height $h = \log_2 n$, so we have $\lceil n/2^{\log_2 n+1} \rceil = \lceil n/2n \rceil = \lceil 0,\ldots \rceil = 1$ node at height h.

References

[1] T.H. Cormen, C.E. Leiserson, R.L. Rivest, and C. Stein. *Introduction to Algorithms*. The MIT Press. MIT Press, 2009.