Sorting: Homework

- *By using the code at:*

  https://github.com/albertocasagrande/AD_sorting

  implement Insertion Sort, Quick Sort, Bubble Sort, Selection Sort, and Heap Sort.

- *For each of the implemented algorithm, draw a curve to represent the relation between the input size and the execution-time.*

- *Argue about the following statement and answer the questions*

  1. *Heap Sort on a array A whose length is n takes time $O(n)$.*
  2. *Heap Sort on a array A whose length is n takes time $\Omega(n)$.*
  3. *What is the worst case complexity for Heap Sort?*
  4. *Quick Sort on a array A whose length is n takes time $O(n^3)$.*
  5. *What is the complexity of Quick Sort?*
  6. *Bubble Sort on a array A whose length is n takes time $\Omega(n)$.*
  7. *What is the complexity of Bubble Sort?*

- *Solve the following recursive equation:*

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 32 \\ 3 * T\left(\frac{n}{4}\right) + \Theta(n^{3/2}) & \text{otherwise} \end{cases}$$

Using the **recursion tree**, we have that our stopping condition is when $n = 32$, so when in our recursion $\frac{n}{4^i} = 32 \iff n = 2^5 \cdot 2^{2i} \iff n = 2^{5+2i} \iff i = \frac{\log_2 n - 5}{2}$ and choosing $cn^{3/2}$ as representative for $\Theta(n^{3/2})$ we have that

$$
\begin{aligned}
T(n) &= 3*T\left(\frac{n}{4}\right)+\Theta(n^{3/2}) \\
&\le 3*T\left(\frac{n}{4}\right)+cn^{3/2} \\
&\le 3*\left(3*T\left(\frac{n}{16}\right)+c\left(\frac{n}{4}\right)^{3/2}\right)+cn^{3/2} \\
&= 9*T\left(\frac{n}{16}\right)+3c\left(\frac{n}{4}\right)^{3/2}+cn^{3/2} \\
&\le 9*\left(3*T\left(\frac{n}{64}\right)+c\left(\frac{n}{16}\right)^{3/2}\right)+3c\left(\frac{n}{4}\right)^{3/2}+cn^{3/2} \\
&\le 27*T\left(\frac{n}{64}\right)+9c\left(\frac{n}{16}\right)^{3/2}+3c\left(\frac{n}{4}\right)^{3/2}+cn^{3/2} \\
&\le \dots \\
&\le 3^{\frac{\log_2 n-5}{2}}T(32)+\sum_{i=0}^{\log_2 n}3^i c\left(\frac{n}{4^i}\right)^{3/2} = 3^{\frac{\log_2 n-5}{2}}\Theta(1)+\sum_{i=0}^{\frac{\log_2 n-5}{2}}3^i c\frac{n^{3/2}}{4^{3/2i}} \\
&\le 3^{\frac{\log_2 n-5}{2}}c'+\sum_{i=0}^{\frac{\log_2 n-5}{2}}\frac{3^i}{2^{3i}}cn^{3/2} = 3^{\frac{\log_2 n-5}{2}}c'+cn^{3/2}\sum_{i=0}^{\frac{\log_2 n-5}{2}}\left(\frac{3}{2^3}\right)^i \\
&= 3^{\frac{\log_2 n-5}{2}}c'+cn^{3/2}\cdot\frac{(3/8)^{\left(\frac{\log_2 n-5}{2}\right)+1}-1}{3/8-1} = 3^{\frac{\log_2 n-5}{2}}c'+cn^{3/2}\cdot\frac{(3/8)^{\frac{\log_2 n-3}{2}}-1}{(3-8)/8} \\
&= 3^{\frac{\log_2 n-5}{2}}c'-\frac{8}{5}cn^{3/2}\cdot\frac{3^{\frac{\log_2 n-3}{2}}-8^{\frac{\log_2 n-3}{2}}}{8^{\frac{\log_2 n-3}{2}}}
\end{aligned}
$$

since $3^{\frac{\log_2 n-3}{2}}=(3^{\log_2 n-3})^{1/2}=\left(\frac{3^{\log_2 n}}{3^3}\right)^{1/2}=\left(\frac{3^{\log_2 3\log_3 n}}{3^3}\right)^{1/2}=\left(\frac{n^{\log_2 3}}{3^3}\right)^{1/2}=\frac{n^{\frac{\log_2 3}{2}}}{3^{3/2}}$ (since $\log_b n=\log_b a\cdot\log_a n$ we have $\log_2 n=\log_2 3\cdot\log_3 n$) and $8^{\frac{\log_2 n-3}{2}}=2^{\frac{3}{2}(\log_2 n-3)}=(2^{\log_2 n-3})^{3/2}=\left(\frac{2^{\log_2 n}}{2^3}\right)^{3/2}=\left(\frac{n}{8}\right)^{3/2}$ we have that

$$
\begin{aligned}
&= \frac{n^{\frac{\log_2 3}{2}}}{3^{3/2}}c'-\frac{8}{5}cn^{3/2}\cdot\frac{\frac{n^{\frac{\log_2 3}{2}}}{3^{3/2}}-(n/8)^{3/2}}{(n/8)^{3/2}} = \frac{n^{\frac{\log_2 3}{2}}}{3^{3/2}}c'-\frac{8}{5}8^{3/2}cn^{3/2}\cdot\frac{\frac{n^{\frac{\log_2 3}{2}}}{3^{3/2}}-\frac{n^{3/2}}{8^{3/2}}}{n^{3/2}} \\
&= \frac{n^{\frac{\log_2 3}{2}}}{3^{3/2}}c'-\frac{8}{5}8^{3/2}c\cdot\left(\frac{n^{\frac{\log_2 3}{2}}}{3^{3/2}}-\frac{n^{3/2}}{8^{3/2}}\right) = \frac{n^{\frac{\log_2 3}{2}}}{3^{3/2}}c'-\frac{8}{5}8^{3/2}c\cdot\frac{n^{\frac{\log_2 3}{2}}}{3^{3/2}}+\frac{8}{5}8^{3/2}c\frac{n^{3/2}}{8^{3/2}} \\
&= \frac{n^{\frac{\log_2 3}{2}}}{3^{3/2}}\left(c'-\frac{8^{5/2}}{5}c\right)+\frac{8}{5}cn^{3/2}\in O(n^{3/2})
\end{aligned}
$$

So we have that $T(n)\in O(n^{3/2})$. Since all the inequalities work also with the $\ge$, we have that $T(n)\in\Omega(n^{3/2})$, so we have that $T(n)\in\Theta(n^{3/2})$