Best approximation

Approximation

Approximating a set of data or a function in $[a, b]$ consists in finding a suitable function f that represents them with enough accuracy.

We can use Taylor polynomials to approximate complex functions, but they require many computations and have unpredictable behaviors on the side of the domain.

Definition. If X is a Banach space and $M \subseteq X$ is a subset, we say that $p \in M$ is the **best approximation** of a function $f \in M$ when

$$||f - p|| = E(f) := \inf_{q \in M} ||f - q||$$

Theorem (Existence theorem) **.** *If M is a finite-dimensional subspace of X, so if $\exists \{v_n\}$ s.t. $M = \operatorname{span}\{v_n\}$, then $\exists p$ the best approximation of f in M.*

Definition. A vector space X is **strictly convex** if $f \neq g$ with $||f|| = ||g|| = 1$ and $0 < \theta < 1$, then $||\theta f + (1 - \theta)g|| < 1$. Geometrically, this means that if any x, y on the unit sphere ∂B are joined by a segment that touches ∂B only in x and y.

Theorem (Uniqueness theorem) **.** *If X is **strictly convex**, then the best approximation p is **unique**.*

Proof. We should prove the existence and the in Hilbert spaces uniqueness. Let's prove the uniqueness: let $p_1 \neq p_2$, we have that

$$\begin{aligned} E(f) &= ||f - p_1|| = ||f - p_2|| \\ &\leq \left|\left|f - \frac{1}{2}(p_1 + p_2)\right|\right| = \left|\left|\frac{1}{2}(f - p_1) + \frac{1}{2}(f - p_2)\right|\right| \\ &< \frac{1}{2}||f - p_1|| + \frac{1}{2}||f - p_2|| = E(f) \end{aligned}$$

(the first inequality holds because $\frac{1}{2}(p_1 + p_2)$ is any function while p_1 and p_2 are the best approximations) that is impossible, so it must be $p_1 = p_2$.

$\square$

Best approximation in Hilbert Spaces

Recall that an Hilbert space is a Banach space plus a scalar product $(\cdot, \cdot)$ with norm defined by $||u||^2 := (u, u)$ and that p is B.A. of f w.r.t. a chosen norm if we have that $||f - p|| \leq ||f - q||$ $\forall q \in \mathbb{P}^n$.

Theorem (Best approximation theorem)**.** *Let H be a Hilbert space. Given a function $f \in H$, then p is best approximation of f in H (from $V \subset H$) if and only if*

$$(f, q) = (p, q) \; \forall q \in V.$$

Proof. ($\Rightarrow$) If p is B.A. of f in H, then $||f - p||^2 = E(f)^2 := \inf_{q \in V} ||f - q||^2$ and so we have that for any perturbation $p + tq$ of p, it holds

$$||f - p||^2 \leq ||f - p + tq||^2 \quad \forall t > 0, \forall q \in V$$

Consider that $||a + b||^2 - ||a - b||^2 = 4(a, b)$, then we have that, if $a = f - p + \frac{t}{2}q$ and $b = \frac{t}{2}q$,

$$0 \leq ||f-p+tq||^2 - ||f-p||^2 = \left|\left|f-p+\frac{t}{2}q+\frac{t}{2}q\right|\right|^2 - \left|\left|f-p+\frac{t}{2}q-\frac{t}{2}q\right|\right|^2$$
$$= 4\left(f-p+\frac{t}{2}q, \frac{t}{2}q\right) = 4\left(f-p, \frac{t}{2}q\right) + 4\left(\frac{t}{2}q, \frac{t}{2}q\right) = 2t\,(f-p,q) + t^2||q||^2$$

so we have that $(f-p,q) \geq -\frac{t}{2}||q||^2$.

By adding instead of tq the term $-tq$, with the same reasoning we have that $(f-p,q) \leq \frac{t}{2}||q||^2$, so we have that $\forall t>0, \forall q \in V$

$$-\frac{t}{2}||q||^2 \leq (f-p,q) \leq \frac{t}{2}||q||^2$$

which implies that $(f-p,q)=0$ since a t can be chosen to bound it on both sides. Then we have that

$$(f-p,q)=0 \iff (f,q)-(p,q)=0 \iff (f,q)=(p,q)$$

for each $q \in V$, so we have our thesis.

($\Leftarrow$) If $(f,q)=(p,q)\ \forall q \in V \iff (f-p,q)=0\ \forall q \in V$, then we have that

$$||f-q||^2 = ||f-p+p-q||^2 = ||f-p||^2 + ||p-q||^2 + 2(f-p,p-q)$$

where $2(f-p,p-q)=0$ since $p-q \in V$ (both $p,q \in V$).

So we have that $||f-q||^2 = ||f-p||^2 + ||p-q||^2\ \forall q \in V$, which implies that $||f-p||^2 \leq ||f-q||^2$ and so $||f-p|| \leq ||f-q||\ \forall q \in V$, and this means that p is B.A. of f in H. $\square$

Now consider $L^2(0,1)$, that is a Hilbert space where the scalar product between vectors is $(a,b) := \int_0^1 ab\,ds$ for $a,b \in L^2(0,1)$ and the norm is $||a|| := \sqrt{\int_0^1 |a|^2 ds}$, and take the space $V = \mathrm{span}\{v_i\}_{i=1}^n$.

The best approximation in L^2. We want the best approximation (in Hilbert Spaces) of the function f, on the space $V=\mathrm{span}\{v_i\}$. Then we have seen that $p \in V$ is best approximation of f if and only if:

$$(f,v)=(p,v), \quad \forall v \in V.$$

In particular, for every basis functions $v_i \in V$ we have $(p,v_i)=(f,v_i)$, for $i=1,\dots,n$. Since $p \in V$, we have that it can be expressed as a linear combination of the basis functions v_i and is uniquely defined by the coefficients p_j: $p=\sum_{j=1}^n p_j v_j$, so we have that

$$(p,v_i) = (\sum_j p_j v_j, v_i) = \sum_j p_j (v_j, v_i).$$

Collecting this informations together we get:

$$\sum_{j=1}^n p_j(v_j,v_i) = (f,v_i),\ \forall v_i \in V, i=1,\dots,n \quad \iff \quad Mp=F$$

where M and F are matrices such that $M_{ij} := (v_j,v_i) = \int_0^1 v_j v_i\,ds$ and $F_i := (f,v_i) = \int_0^1 f v_i\,ds$.

If we set $V^n := \mathrm{span}\{x^i\}_{i=0}^{n-1}$ we will then have that $v_i = x^i$, then

$$M_{ij} := \int_0^1 x^j x^i dx = \left.\frac{x^{j+i+1}}{j+i+1}\right|_0^1 = \frac{1}{j+i+1}.$$

that is called the $n \times n$ **Hilbert matrix** H, which is invertible but it is very ill conditioned, so it is difficult to invert, because of collinear lines. Its condition number is

$$K(H) = ||H||\,||H^{-1}|| \sim O\left((1+\sqrt{2})^{4n}/\sqrt{n}\right).$$

When n increases K explodes, which is very bad. We would like to have $H = I$ the identity, which means $M_{ij} = \delta_{ij}$, so we use the **Legendre basis function**, which are orthonormal basis, to make it orthonormal (perpendicular, a.k.a. diagonal) w.r.t. L^2.

We want $v_i \in \mathbb{P}^n = V^n = \text{span}\{x^i\}$ s.t. $M_{ij} = (v_i, v_j) = \delta_{ij}$. To build it, we use the **Gram-Schmidt process**:

$$\begin{cases} v_0 = 1 & f \text{ s.t. } \int_0^1 f = 1 \\ v_{i+1} = \frac{k_{i+1}}{||k_{i+1}||} & \text{where } k_{i+1} = [x\, v_i - \sum_i (x\, v_i, v_i) v_i] \end{cases}$$

or, alternatively

$$\begin{cases} p_0(x) = 1 \\ p_k(x) = x^k - \displaystyle\sum_{j=0}^{k-1} \frac{(x^k, p_j(x))}{(p_j(x), p_j(x))} \\ \qquad\quad = x\, p_{k-1}(x) - \displaystyle\sum_{j=0}^{k-1} \frac{(x p_{k-1}(x), p_j(x))}{(p_j(x), p_j(x))} \end{cases}$$

The set of additive basis having unity as first element. This ensures orthogonality between basis functions.

As the degree i increases, we have that $v_{i+1} = \frac{k_{i+1}}{||k_{i+1}||} \to \infty$ since $x^{i+1} \to \infty$ and thus $k^{i+1} \to 0$. We can avoid instability by using $v_{i+1} = \frac{k_{i+1}}{k_{i+1}(0)}$ instead.

The points created with Gram-Schmidt represent the **Legendre basis**. The make the best approximation p easy to compute, since M becomes easy to invert and we have a diagonal matrix formed by orthogonal basis $p = M^{-1}F$.