15210: Parallel and Sequential Data Structures and Algorithms

BabbleLab

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7.1

For union of two sets S_1, S_2, define

$$m = min(S_1, S_2)$$

$$n = max(S_1, S_2)$$

The work and span are

$$Work = O(mlog\frac{n+m}{m})$$

$$Span = O(log(n+m))$$

For integer, $\sqrt{n} \leq n$, apply $m = \sqrt{n}$ to the above equation, we have

$$Work = O(\sqrt{n}log\frac{n+\sqrt{n}}{\sqrt{n}}) = O(\sqrt{n}log(\sqrt{n}+1)) = O(\sqrt{n}log\sqrt{n})$$

$$Span = O(log(n+\sqrt{n}))$$

7.2

It is not necessary to take $\Omega(n)$ time to generate a whole new set. Since original set won't be changed, we can simply pass a new reference of that set and do not need to worry about any change by this operation. So we can use something like Seq.subseq to perform faster copy operation.

7.3

Define the 3 elements as a, b, c, (as the same order) respectively.

If we pick b as the pivot, a will be compared with b but never be compared with c.

If we pick a as the pivot, it will be compared with both b and c.

If we pick c as the pivot, a will be compared to c, and then a and b would enter next recursive call of quicksort. In next round, the probability of a and b are compared are as shown in class.

Therefore,

$$
\begin{aligned}
&Pr[a \text{ is compared to both } b \text{ and } c] \\
&\quad = Pr[a \text{ is the pivot}] + Pr[c \text{ is the pivot}] \\
&\quad \cdot Pr[a \text{ and } b \text{ are compared in next call}] \\
&\quad = Pr[a \text{ is the pivot}] + Pr[c \text{ is the pivot}] \\
&\quad \cdot Pr[a \text{ or } b \text{ is the pivot in next call}] \\
&\quad = \frac{1}{\frac{3n}{4} - \frac{n}{4} + 1} + \frac{1}{\frac{3n}{4} - \frac{n}{4} + 1} \cdot \frac{1}{\frac{n}{2} - \frac{n}{4} + 1} = \frac{n+12}{(n+2)(n+4)}
\end{aligned}
$$

7.4

Number of calls to lt is the same as number of comparisons

We have a fixed input b at the beginning. To make this function reasonable, b should be less than any number in S.

The first 3 numbers from S are special:

1st → has to be 3 comparisons

2nd → has to be 3 comparisons

3rd → 2 or 3 comparisons, with probability 1/2 vs. 1/2

Define

$$X_i = \begin{cases} 0 \text{ if we did } 1 \text{ comparisons for ith number} \\ 1 \text{ if we did } 2 \text{ comparisons for ith number} \\ 2 \text{ if we did } 3 \text{ comparisons for ith number} \end{cases}$$

$$Y = \text{total comparisons}$$

Then we have

$$Y = 8.5 + (n\text{-}3) + \sum_{i=4}^{n\text{-}1} X_i$$

$$E[Y] = n + 5.5 + \sum_{i=4}^{n\text{-}1} E[X_i]$$

Consider $E[X_i]$

$$E[X_i] = Pr[X_i = 1] + 2 \cdot Pr[X_i = 2]$$
$$= Pr[\text{i th } num\ is\ 3rd\ largest] + 2$$
$$\cdot Pr[\text{i th } num\ is\ 1st\ or\ 2nd\ largest] = \frac{1}{i} + \frac{2}{i} = \frac{3}{i}$$

Therefore,

$$E[Y] = n + 5.5 + 3\sum_{i=4}^{n\text{-}1} \frac{1}{i+1} = n + 5.5 + 3\sum_{i=5}^{n} \frac{1}{i} = n + 5.5 + 3H_n - 3(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4})$$
$$= n - 0.75 + 3H_n$$