Bijection

A powerful tool in mathematics

by M Ahsan Al Mahir
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Bijection

* New way of solving
* Defining Bijection
* Bijection in Action

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Because that would be REALLY hard to say the least.

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So we start by seeing what these numbers actually mean. We can interpret $\binom{n}{k}$ as choosing k items from a box of n balls, for all k.

Now we begin to notice that we can actually describe the sum as:

> Choosing some number of balls (any number, $0, 1, 2$ or n) from the set of n balls.

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Because the sum is *adding all* $\binom{n}{k}$'s.

Now the interesting part. How do we *actually* count it? There *are* many ways to do this, but we will use Bijection.

What if we think about selecting a ball as labeling it with 1, and not selecting means marking it with 0.

For example, selecting b_2, b_3, b_5 from a set of 5 balls is the same is marking them like the following:

$$\begin{array}{ccccc} b_1 & b_2 & b_3 & b_4 & b_5 \\ 0 & 1 & 1 & 0 & 1 \end{array}$$

So we have n balls, each labeled with either 1 or 0, which is just a binary number with lenght n !!

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Can you see why?

That means the number of ways to select a set of balls is the same as the number of binary numbers of length n. Which is precisely

$$\boxed{2^{n}}$$

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And so we have:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

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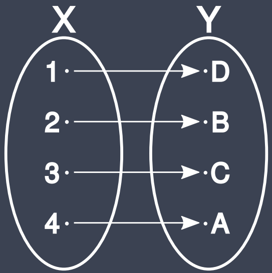
Suppose we have two sets X, Y. And for all elements of X, we can connect it with exactly one element of Y. And also for all element of Y, we can connect it with exactly one element of X. Then we say that there is a ``bijection'' between X and Y.



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So if we want to find the number of elements of X, then we can find a set Y that has a bijection with X and count Y instead!!

In our earlier example, we found a bijection between

The number of ways to select a set of balls from a box of n balls $\Leftrightarrow$ The number of binary numbers of length n

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Ponder for a moment how we would solve this without computation...

We solve it by finding a bijection between `choosing` k `balls from a set of` n `balls` and `removing` $n-k$ `balls from the set of` n `balls to be left with` k `balls`.

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Let's start by seeing another easy application.