math in the times of corona, exam 1

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Problem 1 : Suppose that a and b are two distinct positive real numbers such that $\lfloor na \rfloor$ divides $\lfloor nb \rfloor$ for any positive integer n. Prove that a and b are positive integers.

Problem 2 : We call a $n * n$ table *selfish* if we number the row and column with $0, 1, 2, 3, \dots n-1$ (from left to right an from up to down), and for every $i, j \in \{0, 1, 2, ..., n-1\}$ the value of cell (i, j) is equal to the number of i in row j.

For example we have a *selfish* table for $n = 5$:

$$\begin{matrix} 1 & 0 & 3 & 3 & 4 \\ 1 & 3 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{matrix}$$

Prove that for $n > 5$ there is no *selfish* table.

Problem 3 : Let n be a positive integer and let $k_0, k_1, \dots, k_{2n}$ be nonzero integers such that $k_0 + k_1 + \dots + k_{2n} \neq 0$. Is it always possible to find a permutation $(a_0, a_1, \dots, a_{2n})$ of $(k_0, k_1, \dots, k_{2n})$ so that the equation

$$a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \dots + a_0 = 0$$

has no integer roots?

Problem 4 : Let $U = \{1, 2, \dots, n\}$, where $n \geq 3$. A subset S of U is said to be split by a permutation of the elements of U if an element not in S occurs in the permutation somewhere between two elements of S.

For example, 13542 splits $\{1, 2, 3\}$ and $\{1, 5\}$ but not $\{3, 4, 5\}$.

Prove that for any $n-2$ subsets of U, each containing at least 2 and at most $n-1$ elements, there is an permutation of the elements of U which splits all of them.