0.1 Sets

0.1.1 Lemmas

Lemma 0.1.1 — *Let S be a set with n elements, and let F be a family of subsets of S such that for any pair A, B in F, $A \cap B \neq \varnothing$. Then $|F| \leq 2^{n-1}$.*

Theorem 0.1.2 (Erdos Ko Rado theorem) — *Suppose that A is a family of distinct subsets of $\{1, 2 \ldots n\}$ such that each subset is of size r and each pair of subsets has a nonempty intersection, and suppose that $n \geq 2r$. Then the number of sets in A is less than or equal to the binomial coefficient*

$$\binom{n-1}{r-1}$$

Lemma 0.1.3 — *Let S be a set with n elements, and let F be a family of subsets of S such that for any pair A, B in F, S is not contained by $A \cup B$. Then $|F| \leq 2^{n-1}$.*

Lemma 0.1.4 (Kleitman lemma) — *A set family F is said to be downwards closed if the following holds: if X is a set in F, then all subsets of X are also sets in F. Similarly, F is said to be upwards closed if whenever X is a set in F, all sets containing X are also sets in F. Let F_1 and F_2 be downwards closed families of subsets of $S = \{1, 2, ..., n\}$, and let F_3 be an upwards closed family of subsets of S. Then we have*

$$|F_1 \cap F_2| \geq \frac{|F_1| \cdot |F_2|}{2^n} \tag{1}$$

$$|F_1 \cap F_3| \leq \frac{|F_1| \cdot |F_3|}{2^n} \tag{2}$$

Lemma 0.1.5 — *Let S be a set with n elements, and let F be a family of subsets of S such that for any pair A, B in F, $A \cap B \neq \varnothing$ and $A \cap B \neq S$. Then $|F| \leq 2^{n-2}$.*

Solution. Using the sets in lemma 1 and lemma 2, defining upwards and downwards sets like in Kleitman's Lemma. $\square$

Lemma 0.1.6 (The Sunflower Lemma) — *A sunflower with k petals and a core X is a family of sets $S_1, S_2, \ldots, S_k$ such that $S_i \cap S_j = X$ for each $i \neq j$. (The reason for the name is that the Venn diagram representation for such a family resembles a sunflower.) The sets $S_i \setminus X$ are known as petals and must be nonempty, though X can be empty. Show that if F is a family of sets of cardinality s, and $|F| > s!(k-1)^s$, then F contains a sunflower with k petals.*

Solution. Applying induction and considering the best case where $|X| = 0$ $\square$

0.1.2 Extremal Set Theory

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Theorem 0.1.7 (Mirsky Theorem) — *A set S with a chain of height h can't be partitioned into t anti-chains if $t < h$. In other words, the minimum number of sets in any anti-chain partition of S is equal to the maximum height of the chains in S. (And Vice Versa)*

Theorem 0.1.8 () — *In any poset, the largest cardinality of an antichain is at most the smallest cardinality of a chain-decomposition of that poset.*

Theorem 0.1.9 (Dilworth's Theorem) — *Let P be a poset. Then there exist an antichain A and a chain decomposition $\mathcal{C}$ of P such that $|A| = |\mathcal{C}|$*

Theorem 0.1.10 (Erdos-Szekeres Theorem) — *Any sequence of $ab + 1$ real numbers contains either a monotonically decreasing subsequence of length $a + 1$ or a monotonically increasing subsequence of length $b + 1$. The more useful case is when $a = b = n$.*

Problem 0.1.1 () **:** Let $n \geq 1$ be an integer and let X be a set of $n^2 + 1$ positive integers such that in any subset of X with $n + 1$ elements there exist two elements $x \neq y$ such that $x|y$. Prove that there exists a subset $\{x_1, x_2 \ldots x_{n+1} \in X$ such that $x_i|x_{i+1}$ for all $i = 1, 2, \ldots n$.

0.1.3 Problems

Problem 0.1.2 (USA TST 2005 P1) **:** Let n be an integer greater than 1. For a positive integer m , let $S_m = \{1, 2, \ldots, mn\}$. Suppose that there exists a $2n$ -element set T such that

1. each element of T is an m -element subset of S_m
2. each pair of elements of T shares at most one common element
3. each element of S_m is contained in exactly two elements of T

Determine the maximum possible value of m in terms of n.

Solution. We use double counting to find the ans, after that the rest is easy. □

Problem 0.1.3 (Iran TST 2008 D3P1) **:** Let S be a set with n elements, and F be a family of subsets of S with 2^{n-1} elements, such that for each $A, B, C \in F$, $A \cap B \cap C$ is not empty. Prove that the intersection of all of the elements of F is not empty.

Solution. Using Induction with this lemma. □

Problem 0.1.4 (Romanian TST 2016 D1P2) **:** Let n be a positive integer, and let $S_1, S_2, \ldots S_n$ be a collection of finite non-empty sets such that

$$\sum_{1 \le i < j \le n} \frac{|S_i \cap S_j|}{|S_i||S_j|} < 1$$

Prove that there exist pairwise distinct elements $x_1, x_2 \ldots x_n$ such that x_i is a member of S_i for each index i.

Solution. The Inductive proof reduces the problem to American Mathematical Monthly problem E2309 □

Solution. The other approach is to focus on the given weird condition, and interpolate it to something nice, like probabilistic condition. □

Problem 0.1.5 (American Mathematical Monthly E2309) **:** If $A_1, A_2, \ldots A_n$ are n nonempty subsets of the set $\{1, 2, ..., n-1\}$, then prove that

$$\sum_{1 \le i < j \le n} \frac{|A_i \cap A_j|}{|A_i| \cdot |A_j|} \ge 1$$

Problem 0.1.6 (CGMO 2010 P1) **:** Let n be an integer greater than two, and let $A_1, A_2, \cdots, A_{2n}$ be pairwise distinct subsets of $\{1, 2, \ldots n\}$. Determine the maximum value of

$$\sum_{i=1}^{2n} \frac{|A_i \cap A_{i+1}|}{|A_i| \cdot |A_{i+1}|}$$

Where $A_{2n+1} = A_1$ and $|X|$ denote the number of elements in X.

Problem 0.1.7 (ISL 2002 C5) **:** Let $r \ge 2$ be a fixed positive integer, and let F be an infinite family of sets, each of size r, no two of which are disjoint. Prove that there exists a set of size $r-1$ that meets each set in F.

HMMT 2016 Team Round: Fix positive integers $r > s$, and let $\mathcal{F}$ be an infinite family of sets, each of size r, no two of which share fewer than s elements. Prove that there exists a set of size $r-1$ that shares at least s elements with each set in F.

> ***Solution*** [Focus on a set]**.** If we take an arbitrary set, we can say that there exists infinitely many sets $\in \mathbb{F}$ which includes a fixed element from our test set. If we do this argument for $r-1$ times, we get a set X of $r-1$ elements, and an infinte family of sets that contains X completely. At this point the problem is trivial. □

> ***Solution*** [Adding Elements]**.** Since it's tricky to work with one family, why not introduce another family, like the second monk. This solution generalizes the problem as such. □

Problem 0.1.8 (ISL 1988 P10) **:** Let $N = \{1, 2 \ldots, n\}, n \ge 2$.

A collection $F = \{A_1, \ldots, A_t\}$ of subsets $A_i \subseteq N$, $i = 1, \ldots, t$, is said to be **separating**, if for every pair $\{x, y\} \subseteq N$, there is a set $A_i \in F$ so that $A_i \cap \{x, y\}$ contains just one element.

F is said to be **covering**, if every element of N is contained in at least one set $A_i \in F$.

What is the smallest value $f(n)$ of t, so there is a set $F = \{A_1, \ldots, A_t\}$ which is simultaneously separating and covering.

> ***Solution*** [Binary Representation]**.** Using Binary Representations for the elements as in or not in, we get an easy bijection. □

Problem 0.1.9 (Iran TST 2013 D1P2) **:** Find the maximum number of subsets from $\{1, ..., n\}$ such that for any two of them like A, B if $A \subset B$ then $|B - A| \geq 3$. (Here $|X|$ is the number of elements of the set X.)

> ***Solution.*** By partitioning the maximum set of subsets into groups which contain the number n and which don't and Induction on n we can show that the maximum number of subset is
>
> $$\frac{2^n - (-1)^n}{3}$$
>
> . □

Problem 0.1.10 (Putnam 2005 B4) **:** For positive integers m and n, let $f(m, n)$ denote the number of n-tuples $(x_1, x_2, \dots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \leq m$. Show that $f(m, n) = f(n, m)$.

> ***Solution.*** Try to show Bijection between the result and choosing m or n objects from $m + n$ objects or show that the result is $\binom{m+n}{n}$. □

Problems

Theorems

Categories

Strategies