Bijection

A powerful tool in mathematics

by M Ahsan Al Mahir
on August 21, 2020

Bijection

* New way of solving
* Defining Bijection
* Bijection in Action
* One More Problem
* Conclusion

Bijection

* New way of solving
* Defining Bijection
* Bijection in Action
* One More Problem
* Conclusion

Suppose we want to compute the following sum

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$$

How would you do it?

Suppose we want to compute the following sum

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$$

How would you do it?

Definitely we wouldn't actually start computing by hand!
Because that would be REALLY hard to say the least.

But we can try to find some values for smaller n's:

But we can try to find some values for smaller n's:

$$1 : \binom{1}{0} + \binom{1}{1} = 2$$

But we can try to find some values for smaller n's:

$$1 : \binom{1}{0} + \binom{1}{1} = 2$$

$$2 : \binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 4$$

But we can try to find some values for smaller n's:

$$1 : \binom{1}{0} + \binom{1}{1} = 2$$

$$2 : \binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 4$$

$$3 : \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 8$$

But we can try to find some values for smaller n's:

$$1 : \binom{1}{0} + \binom{1}{1} = 2$$

$$2 : \binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 4$$

$$3 : \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 8$$

Do you see any patterns here? Can you guess why?

Let us start by seeing what these numbers actually mean. We can interpret $\binom{n}{k}$ as choosing k items from a box of n balls, for all k.

Let us start by seeing what these numbers actually mean. We can interpret $\binom{n}{k}$ as choosing k items from a box of n balls, for all k.

And $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$ means...

Let us start by seeing what these numbers actually mean. We can interpret $\binom{n}{k}$ as choosing k items from a box of n balls, for all k.

And $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$ means...

Choosing some number of balls (any number, $0, 1, 2, n-1$ or n) from the set of n balls.

Let us start by seeing what these numbers actually mean. We can interpret $\binom{n}{k}$ as choosing k items from a box of n balls, for all k.

And $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$ means...

> Choosing some number of balls (any number, $0, 1, 2, n-1$ or n) from the set of n balls.

Now the interesting part. How do we actually count it? There are many ways to do this, but we will use Bijection.

What if we think about selecting a ball as labeling it with 1, and not selecting means marking it with 0.

What if we think about selecting a ball as labeling it with 1, and not selecting means marking it with 0.

For example, selecting b_2, b_3, b_5 from a set of 5 balls is the same is marking them like the following:

$$\begin{array}{ccccc} b_1 & b_2 & b_3 & b_4 & b_5 \\ 0 & 1 & 1 & 0 & 1 \end{array}$$

So we have n balls, each labeled with either 1 or 0, which is just a binary number with lenght n !!

And every binary number represents a different set of balls.

So we have n balls, each labeled with either 1 or 0, which is just a binary number with lenght n !!

And every binary number represents a different set of balls.

Can you see why?

That means the number of ways to select *a* set of balls is the same as the number of binary numbers of length n. Which is precisely

$$\boxed{2^{\mathsf{n}}}$$

(Because we have two options, $0, 1$, for each of the n positions)

That means the number of ways to select a set of balls is the same as the number of binary numbers of length n. Which is precisely

$$\boxed{2^n}$$

(Because we have two options, $0, 1$, for each of the n positions)

And so we have:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

Bijection

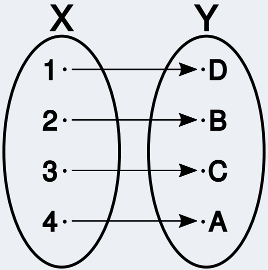
* New way of solving
* Defining Bijection
* Bijection in Action
* One More Problem
* Conclusion

Just as we said earlier, when we need to count something, we can change our interpretation to count something else, that is way easier than the one we had to before.

Just as we said earlier, when we need to count something, we can change our interpretation to count something else, that is way easier than the one we had to before.

That's exactly what bijection does. It gives us *a* way to turn something hard into something easier.

Suppose we have two sets X, Y. And for all elements of X, we can connect it with exactly one element of Y. And also for all element of Y, we can connect it with exactly one element of X. Then we say that there is a ``bijection'' between X and Y.



And how does that help us?

And how does that help us?

It tells us that the number of elements of X is equal to the number of elements of Y!!

And how does that help us?

> It tells us that the number of elements of X is equal to the number of elements of Y!!

So if we want to find the number of elements of X, then we can find a set Y that has a bijection with X and count Y instead!!

And how does that help us?

> It tells us that the number of elements of X is equal to the number of elements of Y!!

So if we want to find the number of elements of X, then we can find a set Y that has a bijection with X and count Y instead!!

In our earlier example, we found a bijection between

The number of ways to select a set of balls from a box of n balls	$\Leftrightarrow$	The number of binary numbers of length n

Before we jump off to seeing some problems, here is another trivial example.

Before we jump off to seeing some problems, here is another trivial example.

Prove that

$$\binom{n}{k} = \binom{n}{n-k}$$

Before we jump off to seeing some problems, here is another trivial example.

Prove that

$$\binom{n}{k} = \binom{n}{n-k}$$

Ponder for a moment how we would solve this without computation...

Before we jump off to seeing some problems, here is another trivial example.

Prove that

$$\binom{n}{k} = \binom{n}{n-k}$$

Ponder for a moment how we would solve this without computation...

We solve it by finding a bijection between `choosing k balls from a set of n balls` and `removing n` $-$ `k balls from the set of n balls to be left with k balls.`

Bijection

* New way of solving
* Defining Bijection
* Bijection in Action
* One More Problem
* Conclusion

Well that's nice, but how do we solve problems with bijection??

Well that's nice, but how do we solve problems with bijection??

Let's start by seeing another easy application.

In how many ways can n be written as sum of integers? For example, 3 can be written in 4 ways

$$1 + 1 + 1 = 1 + 2 = 2 + 1 = 3$$

In how many ways can n be written as sum of integers? For example, 3 can be written in 4 ways

$$1 + 1 + 1 = 1 + 2 = 2 + 1 = 3$$

I will first give you a hint:

$$\begin{array}{ccccccccccc}
(& 1 &) & + & (& 1 &) & + & (& 1 &) & = & 3 \\
(& 1 &) & + & (& 1 & & + & & 1 &) & = & 3 \\
(& 1 & & + & & 1 &) & + & (& 1 &) & = & 3 \\
(& 1 & & + & & 1 & & + & & 1 &) & = & 3
\end{array}$$

Now we are off to the solution.

Now we are off to the solution.

Consider the n -1 spaces between n 1's in the following equation:

$$(1\_\_1\_\_1\_\_1\ldots\_\_1)$$

Now we are off to the solution.

Consider the $n - 1$ spaces between n 1's in the following equation:

$$(1\_\_1\_\_1\_\_1 \ldots \_\_1)$$

If we place $+$ in some of the spaces and $) + ($ in the other spaces, then we get a "partition" like the one we saw before.

Now we are off to the solution.

Consider the n -1 spaces between n 1's in the following equation:

$$(1\_\_1\_\_1\_\_1\dots\_\_1)$$

If we place $+$ in some of the spaces and $)+($ in the other spaces, then we get a "partition" like the one we saw before.

$$\begin{array}{ccccccccccccc}
(& 1 &) & + & (& 1 &) & + & (& 1 &) & = & 3 \\
(& 1 &) & + & (& 1 & & + & & 1 &) & = & 3 \\
(& 1 & & + & & 1 &) & + & (& 1 &) & = & 3 \\
(& 1 & & + & & 1 & & + & & 1 &) & = & 3
\end{array}$$

Now I assume you can tell me the answer to the question? Write in the chat if you've found the answer.

Now I assume you can tell me the answer to the question? Write in the chat if you've found the answer.

Exactly! The answer is 2^{n-1}.

Now I assume you can tell me the answer to the question? Write in the chat if you've found the answer.

Exactly! The answer is 2^{n-1}.

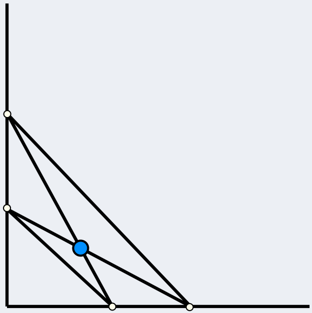
That's because we have $n-1$ places where we can put either $+$ or $)+($, so two options.

Can you explain the bijection here?

Can you explain the bijection here?

Yes, we found a bijection from the set of ways to write n between the set of binary numbers of length $n-1$. And the second set is MUCH easier to compute.

Ten points are selected on the positive x-axis and five points are selected on the positive y -axis. The fifty segments connecting the ten points on x-axis to the five points on y-axis are drawn. What is the maximum possible number of points of intersection of these fifty segments in the interior of the first quadrant?



The question we have to ask here is, when do two segments intersect? Can you answer it?

The question we have to ask here is, when do two segments intersect? Can you answer it?

Yes, they intersect whenever two segments form an $\times$.

The question we have to ask here is, when do two segments intersect? Can you answer it?

Yes, they intersect whenever two segments form an $\times$.

No brainer right? But now answer, whend does an $\times$ appear?

The question we have to ask here is, when do two segments intersect? Can you answer it?

Yes, they intersect whenever two segments form an $\times$.

No brainer right? But now answer, whend does an $\times$ appear?

An unique cross appears when we select two points from the x axis and two points from the y axis.

So we have *a* bijection from the `number of intersection points` to the `number of crosses` to the `number of pairs of pairs` from x-axis and pairs of points from y-axis.

So we have a bijection from the `number of intersection points` to the `number of crosses` to the `number of pairs of pairs` from x-axis and pairs of points from y-axis.

Now we are ready to count the answer.

So we have a bijection from the `number of intersection points` to the `number of crosses` to the `number of pairs of pairs` from x-axis and pairs of points from y-axis.

Now we are ready to count the answer.

There are a total of $\binom{10}{2}$ ways to select two points from x-axis. And there are $\binom{5}{2}$ ways to select two points from y-axis.

So we have a bijection from the `number of intersection points` to the `number of crosses` to the `number of pairs of pairs` from x-axis and pairs of points from y-axis.

Now we are ready to count the answer.

There are a total of $\binom{10}{2}$ ways to select two points from x-axis. And there are $\binom{5}{2}$ ways to select two points from y-axis.

So the number of ways to select two pairs from the two axes is

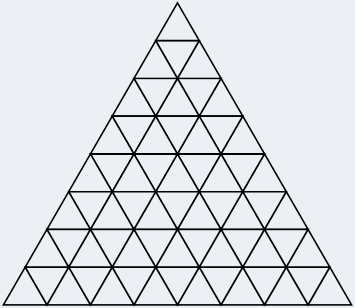
$$\boxed{\binom{10}{2}\binom{5}{2}}$$

So the total number of intersection points is $\binom{10}{2}\binom{5}{2}$.

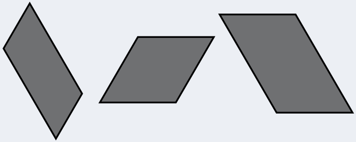
Bijection

* New way of solving
* Defining Bijection
* Bijection in Action
* One More Problem
* Conclusion

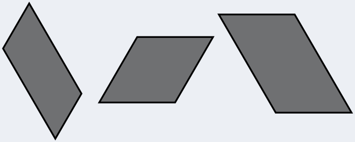
A triangular grid is obtained by tiling an equilateral triangle of side length n by n^2 equilateral triangles of side length 1. Determine the number of *parallelograms* bounded by line segments of the grid.



First we have to see what the parallelograms might look like:

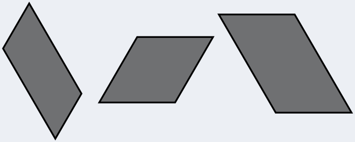


First we have to see what the parallelograms might look like:



If I told you that there were three different orientations of these parallelograms, would you buy it?

First we have to see what the parallelograms might look like:



If I told you that there were three different orientations of these parallelograms, would you buy it?

That's because if you extend those parallelograms' sides, they become parallel to two different sides of the triangle.

Now what we do is, we work with only one orientation. Because if we can count how many parallelograms there are of the first orientation, then we can apply symmetry to count the other orientations.

Now what we do is, we work with only one orientation. Because if we can count how many parallelograms there are of the first orientation, then we can apply symmetry to count the other orientations.

Do you see why?

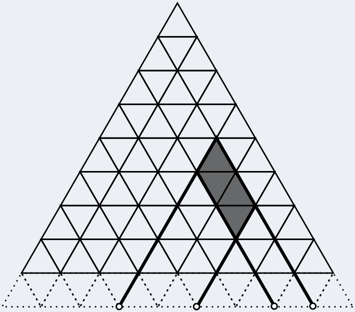
Now what we do is, we work with only one orientation. Because if we can count how many parallelograms there are of the first orientation, then we can apply symmetry to count the other orientations.

Do you see why?

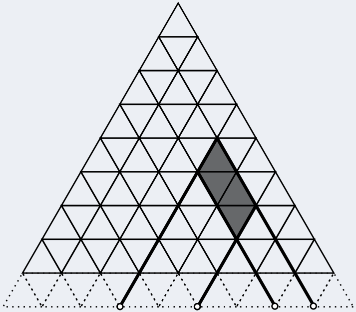
Now, a parallelogram is defined by its parallel sides, right? What if we extend those sides?

If we extend the sides of the parallelogram, and add one more layer at the bottom of the grid, we end up with a picture like this

If we extend the sides of the parallelogram, and add one more layer at the bottom of the grid, we end up with a picture like this

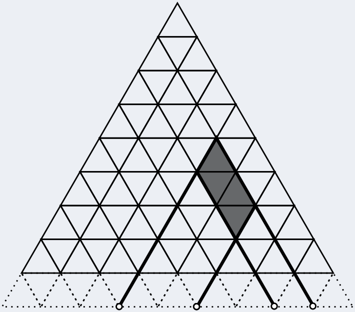


If we extend the sides of the parallelogram, and add one more layer at the bottom of the grid, we end up with a picture like this



What's special about this picture?

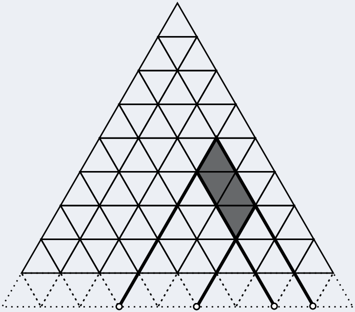
If we extend the sides of the parallelogram, and add one more layer at the bottom of the grid, we end up with a picture like this



What's special about this picture?

The extended lines intersect the edge in 4 different points. And those 4 different points define one unique parallelogram!

If we extend the sides of the parallelogram, and add one more layer at the bottom of the grid, we end up with a picture like this



What's special about this picture?

The extended lines intersect the edge in 4 different points. And those 4 different points define one unique parallelogram!

That's a lot to take in, so I will give you 2 minutes to think about why this happens.

As you have seen, 4 points on the side of the triangle defines one parallelogram.

As you have seen, 4 points on the side of the triangle defines one parallelogram.

And how many “quadruple” of points are there on the extended side?

As you have seen, 4 points on the side of the triangle defines one parallelogram.

And how many “quadruple” of points are there on the extended side?

$$\boxed{\binom{n+1}{4}}$$

As you have seen, 4 points on the side of the triangle defines one parallelogram.

And how many "quadruple" of points are there on the extended side?

$$\boxed{\binom{n+1}{4}}$$

So there are a total of $\binom{n+1}{4}$ parallelograms of this orientation.

As you have seen, 4 points on the side of the triangle defines one parallelogram.

And how many “quadruple” of points are there on the extended side?

$$\boxed{\binom{n+1}{4}}$$

So there are a total of $\binom{n+1}{4}$ parallelograms of this orientation.

The same goes for the other orientations as well!

So there are a total of $3 \times \binom{n+1}{4}$ parallelograms!

So there are a total of $3 \times \binom{n+1}{4}$ parallelograms!

Can you explain where we used bijection?

So there are a total of $3 \times \binom{n+1}{4}$ parallelograms!

Can you explain where we used bijection?

Yes we used bijection to move from the set of parallelograms to the set of quadruples of points on the extended edge, and it became very easy to count.

Bijection

* New way of solving
* Defining Bijection
* Bijection in Action
* One More Problem
* Conclusion

» Further Reader

The Path to Combinatorics for Undergraduate is a really nice boof for cominatorics and Bijection in specific.

» Further Reader

The Path to Combinatorics for Undergraduate is a really nice boof for cominatorics and Bijection in specific.

Yufei Zhao's Note

`http://yufeizhao.com/olympiad/bijections.pdf` is a really nice resource for bijection related problems.

In short, the technique to move from one hard to count set to an easy to count set is called Bijection, it makes your life easier.

In short, the technique to move from one hard to count set to an easy to count set is called Bijection, it makes your life easier.

So whenever possible, think about applying bijection to problems (after induction though, always apply induction at the very beginning) and see if you can get anything nice :D

In short, the technique to move from one hard to count set to an easy to count set is called Bijection, it makes your life easier.

So whenever possible, think about applying bijection to problems (after induction though, always apply induction at the very beginning) and see if you can get anything nice :D

Happy Problem Solving and Good Luck for TST
(you will need it, a lot)