Bijection

A powerful tool in mathematics

by M Ahsan Al Mahir
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Bijection

* New way of solving
* Defining Bijection
* Bijection in Action
* One More Problem
* Conclusion

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Definitely we wouldn't actually start computing by hand!
Because that would be REALLY hard to say the least.

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Do you see any patterns here? Can you guess why?

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> Choosing some number of balls (any number, $0, 1, 2, n-1$ or n) from the set of n balls.

Now the interesting part. How do we actually count it? There are many ways to do this, but we will use Bijection.

What if we think about selecting a ball as labeling it with 1, and not selecting means marking it with 0.

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For example, selecting b_2, b_3, b_5 from a set of 5 balls is the same is marking them like the following:

$$\begin{array}{ccccc} b_1 & b_2 & b_3 & b_4 & b_5 \\ 0 & 1 & 1 & 0 & 1 \end{array}$$

So we have n balls, each labeled with either 1 or 0, which is just a binary number with lenght n !!

And every binary number represents a different set of balls.

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Can you see why?

That means the number of ways to select a set of balls is the same as the number of binary numbers of length n. Which is precisely

$$\boxed{2^n}$$

(Because we have two options, $0, 1$, for each of the n positions)

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And so we have:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

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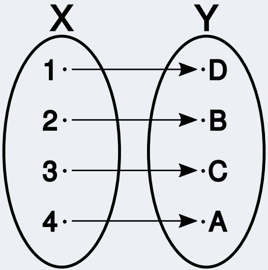
That's exactly what bijection does. It gives us a way to turn something hard into something easier.

Suppose we have two sets X, Y. And for all elements of X, we can connect it with exactly one element of Y. And also for all element of Y, we can connect it with exactly one element of X. Then we say that there is a ``bijection'' between X and Y.



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In our earlier example, we found a bijection between

The number of ways to select a set of balls from a box of n balls	$\Leftrightarrow$	The number of binary numbers of length n

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Ponder for a moment how we would solve this without computation...

We solve it by finding a bijection between choosing k balls from a set of n balls and removing $n - k$ balls from the set of n balls to be left with k balls.

Another example would be the following identity:

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$

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We can solve it by thinking about taking $k+1$ balls from $n+1$ balls as taking the ball no. 1 and then taking k balls from the rest of the n balls, or not taking the ball no. 1 and taking $k+1$ balls from the n balls.

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Where did we use bijection?

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Let's start by seeing another easy application.

In how many ways can n be written as sum of integers? For example, 3 can be written in 4 ways

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I will first give you a hint:

(	1	)	+	(	1	)	+	(	1	)	=	3
(	1	)	+	(	1		+		1	)	=	3
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That's because we have $n - 1$ places where we can put either $+$ or $) + ($, so two options.

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Yes, we found a bijection from the set of ways to write n between the set of binary numbers of length $n-1$. And the second set is MUCH easier to compute.

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Why?

Because we also need to count the sum when some of the integers are 0, and $)+()+($ will produce a 0 in the middle.

That's why, we first make $n + k - 1$ spaces:

$$\underbrace{\_\_ \quad \_\_ \quad \cdots \_\_}_{n+k-1 \text{ dashes}}$$

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And if we put *a* 1 in each of the __, we will get k different nonnegative integers adding upto n!

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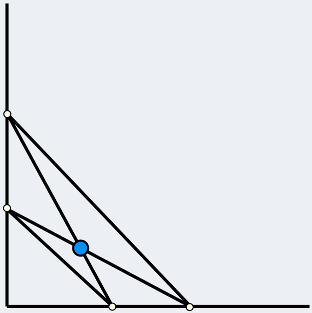
Where did we use bijection here?

We found *a* bijection between the set of k non-negative integers adding up to n and the number of ways to select $k-1$ items from $n+k-1$ choices.

Ten points are selected on the positive x-axis and five points are selected on the positive y -axis. The fifty segments connecting the ten points on x-axis to the five points on y-axis are drawn. What is the maximum possible number of points of intersection of these fifty segments in the interior of the first quadrant?



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An unique cross appears when we select two points from the x axis and two points from the y axis.

So we have *a* bijection from the `number of intersection points` to the `number of crosses` to the `number of pairs of pairs` from x-axis and pairs of points from y-axis.

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And there are $\binom{5}{2}$ ways to select two points from y-axis.

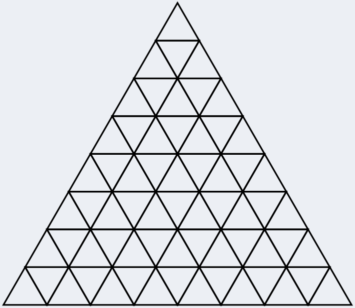
So the number of ways to select two pairs from the two axes is

$$\boxed{\binom{10}{2}\binom{5}{2}}$$

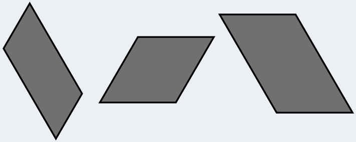
So the total number of intersection points is $\binom{10}{2}\binom{5}{2}$.

Bijection

A triangular grid is obtained by tiling an equilateral triangle of side length n by n^2 equilateral triangles of side length 1. Determine the number of parallelograms bounded by line segments of the grid.



First we have to see what the parallelograms might look like:



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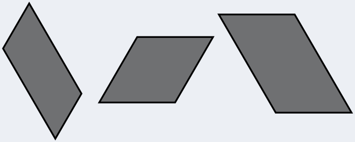

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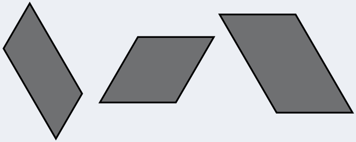

If I told you that there were three different orientations of these parallelograms, would you buy it?

That's because if you extend those parallelograms' sides, they become parallel to two different sides of the triangle.

Now what we do is, we work with only one orientation. Because if we can count how many parallelograms there are of the first orientation, then we can apply symmetry to count the other orientations.

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Do you see why?

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Do you see why?

Now, a parallelogram is defined by its parallel sides, right? What if we extend those sides?

If we extend the sides of the parallelogram, and add one more layer at the bottom of the grid, we end up with a picture like this

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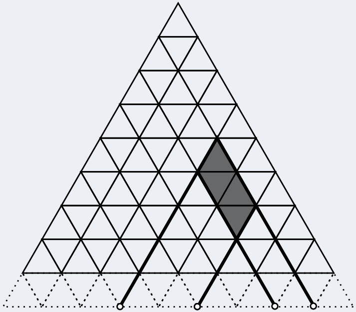

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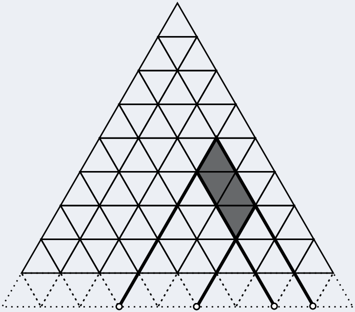

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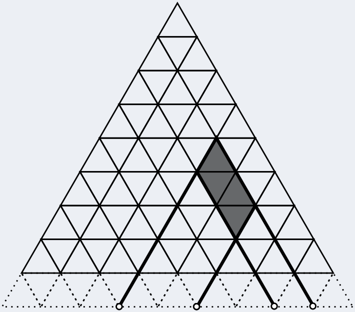

What's special about this picture?

The extended lines intersect the edge in 4 different points. And those 4 different points define one unique parallelogram!

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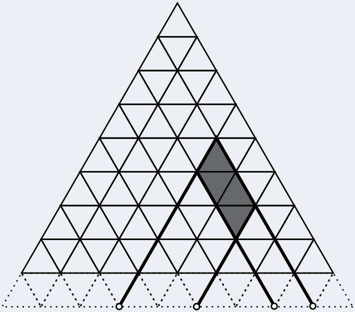

What's special about this picture?

The extended lines intersect the edge in 4 different points. And those 4 different points define one unique parallelogram!

That's a lot to take in, so I will give you 2 minutes to think about why this happens.

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The same goes for the other orientations as well!

So there are a total of $3 \times \binom{n+1}{4}$ parallelograms!

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Can you explain where we used bijection?

So there are a total of $3 \times \binom{n+1}{4}$ parallelograms!

Can you explain where we used bijection?

Yes we used bijection to move from the set of parallelograms to the set of quadruples of points on the extended edge, and it became very easy to count.

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» Further Reader

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Yufei Zhao's Note
`http://yufeizhao.com/olympiad/bijections.pdf` is a really nice resource for bijection related problems.

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So whenever possible, think about applying bijection to problems (after induction though, always apply induction at the very beginning) and see if you can get anything nice :D