

Bijection

A powerful tool in mathematics

by M Ahsan Al Mahir

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Bijection

- * New way of solving
- * Defining Bijection
- * Bijection in Action
- * One More Problem
- * Conclusion

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Definitely we wouldn't actually start computing by hand!
Because that would be **REALLY** hard to say the least.

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$$1 : \binom{1}{0} + \binom{1}{1} = 2$$

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Now the interesting part. How do we actually count it? There are many ways to do this, but we will use Bijection.

What if we think about selecting a ball as labeling it with 1, and not selecting means marking it with 0.

For example, selecting b_2, b_3, b_5 from a set of 5 balls is the same as marking them like the following:

b_1	b_2	b_3	b_4	b_5
0	1	1	0	1

And every binary number represents a different set of balls.

Can you see why?

2^n

[6/27]

2^n

And so we have:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

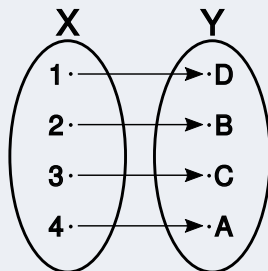
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Just as we said earlier, when we need to count something, we can change our interpretation to count something else, that is way easier than the one we had to before.

That's exactly what bijection does. It gives us a way to turn something hard into something easier.

Suppose we have two sets X, Y . And for all elements of X , we can connect it with exactly one element of Y . And also for all element of Y , we can connect it with exactly one element of X . Then we say that there is a ``bijection'' between X and Y .



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In our earlier example, we found a bijection between

The number of ways to select a set of balls from a box of n balls	\Leftrightarrow	The number of binary numbers of length n
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We solve it by finding a bijection between choosing k balls from a set of n balls and removing $n-k$ balls from the set of n balls to be left with k balls.

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Let's start by seeing another easy application.

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I will first give you a hint:

$$\begin{array}{rclclclcl}
 (& 1 &) & + & (& 1 &) & + & (& 1 &) & = & 3 \\
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That's because we have $n - 1$ places where we can put either $+$ or $) + ($, so two options.

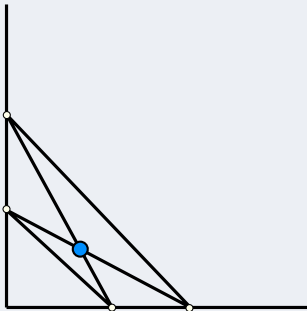


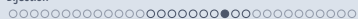
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Yes, we found a bijection from the set of ways to write n between the set of binary numbers of length $n - 1$. And the second set is MUCH easier to compute.

Ten points are selected on the positive x -axis and five points are selected on the positive y -axis. The fifty segments connecting the ten points on x -axis to the five points on y -axis are drawn. What is the maximum possible number of points of intersection of these fifty segments in the interior of the first quadrant?





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An unique cross appears when we select two points from the x axis and two points from the y axis.

So we have a bijection from the number of intersection points to the number of crosses to the number of pairs of pairs from **x-axis and pairs of points from y-axis**.

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There are a total of $\binom{10}{2}$ ways to select two points from x-axis.
And there are $\binom{5}{2}$ ways to select two points from y-axis.

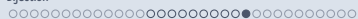
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So the number of ways to select two pairs from the two axes is

$$\boxed{\binom{10}{2} \binom{5}{2}}$$

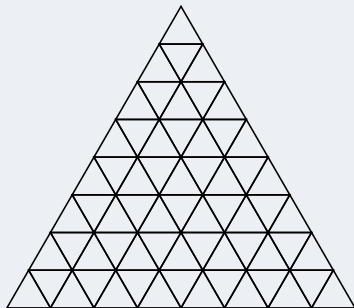


So the total number of intersection points is $\binom{10}{2} \binom{5}{2}$.

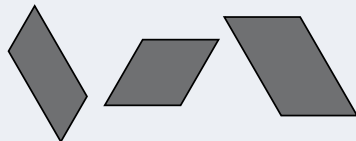
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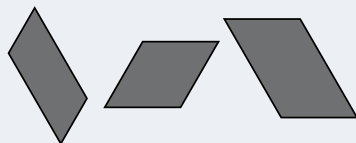
A triangular grid is obtained by tiling an equilateral triangle of side length n by n^2 equilateral triangles of side length 1. Determine the number of parallelograms bounded by line segments of the grid.



First we have to see what the parallelograms might look like:

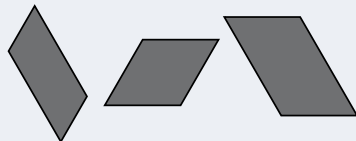


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That's because if you extend those parallelograms' sides, they become parallel to two different sides of the triangle.

Now what we do is, we work with only one orientation. Because if we can count how many parallelograms there are of the first orientation, then we can apply symmetry to count the other orientations.

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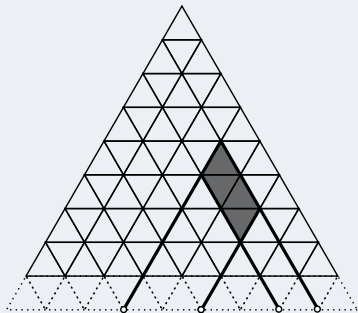
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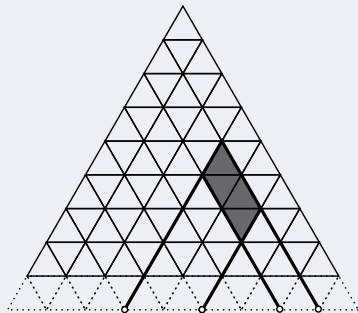
Now, a parallelogram is defined by its parallel sides, right? What if we extend those sides?

If we extend the sides of the parallelogram, and add one more layer at the bottom of the grid, we end up with a picture like this

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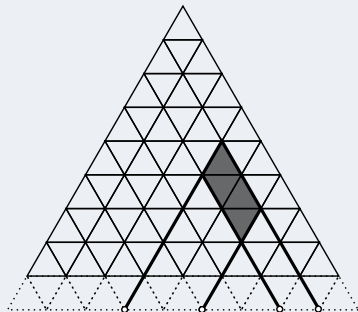


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The extended lines intersect the edge in 4 different points. And those 4 different points define one unique parallelogram!

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The same goes for the other orientations as well!

So there are a total of $3 \times \binom{n+1}{4}$ parallelograms!

Can you explain where we used bijection?

Yes we used bijection to move from the set of parallelograms to the set of quadruples of points on the extended edge, and it became very easy to count.

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» Further Reader

The Path to Combinatorics for Undergraduate is a really nice book for combinatorics and Bijection in specific.

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Yufei Zhao's Note

<http://yufeizhao.com/olympiad/bijections.pdf> is a really nice resource for bijection related problems.

In short, the technique to move from one hard to count set to an easy to count set is called Bijection, it makes your life easier.

So whenever possible, think about applying bijection to problems (after induction though, always apply induction at the very beginning) and see if you can get anything nice :D

