BCS Question Bank

AHSAN

August 17, 2020

Contents

Chapter 1

Algebra

1.1 Functional Equations

Can't Start? Try These

1 GUESS THE POSSIBLE SOLUTIONS.

2 SUBSTITUTION

a Try EVERY possible substitutions, and write them in a list, dont think during this time.

b Now think what these results give you.

c Find values of $f(0), f(1), f(2), f(-x)$ etc.

d Tweak the function a little bit, do substitution again.

e Assume some other functions according to the solutions, substitute them to make the fe easier to get info out of.

3 PROPERTIES OF THE FUNCTION

a Try proving INJECTIVITY, SURJECTIVITY etc.

b Look for Injectivity or Surjectivity of $f(x) - f(y)$.

4 Assume for the sake of contradiction that the value of the function is greater or smaller than the estimated value at some point.

5 Sometimes consider the difference of two values of f.

Stuck? Try These

1 Proving that $f(x) - x$ is injective might come handy in some cases.

2 If you're *NOT* able to make one side of the equation equal to 0, try to make it equal to any real or some particular real. (pco 169 P11)

3 Sometimes in integer functions, divisibility of the type $f(1)^{k-1} \mid f(x)^k$ helps.

4 Durr... I want things to cancel.

1.1.1 Problems

Problem 1.1.1 (EGMO 2012 P3) **:** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(yf(x+y)+f(x)) = 4x + 2yf(x+y)$$

for all $x, y \in \mathbb{R}$.

Problem 1.1.2 (pco 169 P11) **:** Find all $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x, y the following holds:

$$f(x)^2 + 2yf(x) + f(y) = f(y + f(x))$$

Problem 1.1.3 (IMO 1994 P5) **:** Let S be the set of all real numbers strictly greater than -1. Find all functions $f : S \to S$ satisfying the two conditions:

1. $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all x, y in S;
2. $\frac{f(x)}{x}$ is strictly increasing on each of the two intervals $-1 < x < 0$ and $0 < x$.

Problem 1.1.4 (ISL 1994 A4) **:** Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{R}^+$ the subset of all positive ones. Let α and β be given elements in $\mathbb{R}$, not necessarily distinct. Find all functions $f : \mathbb{R}^+ \mapsto \mathbb{R}$ such that:

$$f(x)f(y) = y^{\alpha} f\left(\frac{x}{2}\right) + x^{\beta} f\left(\frac{y}{2}\right) \forall x, y \in \mathbb{R}^+.$$

Problem 1.1.5 (IMO 2017 P2) **:** Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for any real numbers x and y,

$$f(f(x)f(y)) + f(x+y) = f(xy).$$

Problem 1.1.6 (ISL 2008 A1) **:** Find all functions $f : (0, \infty) \mapsto (0, \infty)$ (so f is a function from the positive real numbers) such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers w, x, y, z, satisfying $wx = yz$.

Problem 1.1.7 (pco 169 P15) **:** Find all $a \in \mathbb{R}$ for which there exists a non-constant function $f : (0, 1] \to \mathbb{R}$ such that

$$a + f(x + y - xy) + f(x)f(y) \leq f(x) + f(y)$$

for all $x, y \in (0, 1]$

Problem 1.1.8 (pco 168 P18) **:** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

for all $x, y \in \mathbb{R}$

Problem 1.1.9 (ISL 2011 A3) **:** Determine all pairs (f, g) of functions from the set of real numbers to itself that satisfy

$$g(f(x + y)) = f(x) + (2x + y)g(y)$$

for all real numbers x and y.

Problem 1.1.10 (ISL 2005 A2) **:** We denote by $\mathbb{R}^+$ the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ which have the property:

$$f(x)f(y) = 2f(x + yf(x))$$

for all positive real numbers x and y.

> ***Solution.*** Let's first substitute. If there existed some x such that $f(x) < 1$, we could find a nice substitution. But that leads to a contradiction. So what if we could do something like this for the other cases, $f(x) < 2$ and $f(x) > 2$? □

Problem 1.1.11 (ISL 2005 A4) **:** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) + f(x)f(y) = f(xy) + 2xy + 1$ for all real numbers x and y.

> ***Solution.*** Substitution. □

Problem 1.1.12 (Iran TST T2P1) **:** Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the following conditions:

1. $x + f(y + f(x)) = y + f(x + f(y)) \quad \forall x, y \in \mathbb{R}$
2. The set $I = \left\{ \frac{f(x)-f(y)}{x-y} \mid x, y \in \mathbb{R}, x \neq y \right\}$ is an interval.

Problem 1.1.13 (169 P20) **:** Let a be a real number and let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying: $f(0) = \frac{1}{2}$ and

$$f(x+y) = f(x)f(a-y) + f(y)f(a-x)$$

$\forall x, y \in \mathbb{R}$. Prove that f is constant

Problem 1.1.14 (Vietnam 1991) **:** Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which

$$\frac{1}{2}f(xy) + \frac{1}{2}f(xz) - f(x)f(yz) \geq \frac{1}{4}$$

Solution. Just substitute. □

Problem 1.1.15 () **:** Suppose that f and g are two functions defined on the set of positive integers and taking positive integer values. Suppose also that the equations $f(g(n)) = f(n) + 1$ and $g(f(n)) = g(n) + 1$ hold for all positive integer n. Prove that $f(n) = g(n)$ for all positive integer n.

Solution. Durrr... I want things to cancel... Hint: You want to show $f(n) - g(n) = 0$. □

Problem 1.1.16 (ISL 2002 A1) **:** Find all functions f from the reals to the reals such that

$$f\left(f(x) + y\right) = 2x + f\left(f(y) - x\right)$$

for all real x, y.

Solution. On one of our substitution, we see that there is surjectivity in the equation. So trying to show injectivity is the most intuitive move after that. Again, we have x on the outside, so we need to make x, a once and b once. but we have $f(y) - x$ which we need to eleminate, keeping y constant. We can make it either a or b since we already have $f(a) = f(b)$. And again we can take whatever value we want for $f(y)$. □

Problem 1.1.17 (ISL 2001 A1) **:** Let T denote the set of all ordered triples (p, q, r) of nonnegative integers. Find all functions $f : T \to \mathbb{R}$ satisfying

$$f(p,q,r) = \begin{cases} 0 & \text{if } pqr = 0, \\ 1 + \frac{1}{6}(f(p+1, q-1, r) + f(p-1, q+1, r) & \\ +f(p-1, q, r+1) + f(p+1, q, r-1) & \\ +f(p, q+1, r-1) + f(p, q-1, r+1)) & \text{otherwise} \end{cases}$$

for all nonnegative integers p, q, r.

Solution. First let us guess the ans. For all points on the 3 sides, our function gives 0. We get $f(1,1,1)=1$. We get $f(1,1,2)=f(1,2,1)=f(2,1,1)=\frac{3}{2}$. We get $f(1,1,3)=\frac{9}{5}$. We get $f(1,2,2)=\frac{12}{5}$. Now, since for $pqr=0$, we have $f=0$, we need the expression pqr on the numerator. And we kinda guess that the denominator is $p+q+r$. From here the guess is obvious.
Now proving that this solution is the only solution. Let the solution be g. Define, $h:=f-g$. Our aim is to prove that $h=0$ for all inputs. □

Problem 1.1.18 (RMM 2019 P5) **:** Determine all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(x+yf(x))+f(xy)=f(x)+f(2019y),$$

for all real numbers x and y.

Solution. After getting $f(yf(0))=f(y2019)$, one should think of proving that either f is constant, all zero except 0, or linear. How to do this? □

Problem 1.1.19 (APMO 2015 P2) **:** Let $S=\{2,3,4,\ldots\}$ denote the set of integers that are greater than or equal to 2. Does there exist a function $f:S\to S$ such that

$$f(a)f(b)=f(a^2b^2) \text{ for all } a,b\in S \text{ with } a\neq b?$$

Solution. Try to break the symmetry, add another variable. □

Problem 1.1.20 (ISL 2015 A2) **:** Determine all functions $f:\mathbb{Z}\to\mathbb{Z}$ with the property that

$$f(x-f(y))=f(f(x))-f(y)-1$$

holds for all $x,y\in\mathbb{Z}$.

Solution. It just flows. □

Problem 1.1.21 (ISL 2015 A4) **:** Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ that satisfy the equation

$$f(x+f(x+y))+f(xy)=x+f(x+y)+yf(x)$$

for all real numbers x and y.

Solution. When you don't know any heavy techniques, just plug in simple values into the function, and write down all of the equations in a list. □

Problem 1.1.22 (ISL 2012 A5) **:** Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the conditions

$$f(1+xy) - f(x+y) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R},$$

and $f(-1) \neq 0$.

> ***Solution.*** In FE, always look back to what you have, and what things can you make from those. □

Problem 1.1.23 (ISL 2012 A1) **:** Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a + b + c = 0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

(Here $\mathbb{Z}$ denotes the set of integers.)

> ***Solution.*** Go with the flow. □

Problem 1.1.24 (ISL 2012 A1) **:** Find all function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following equality holds

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

where $\lfloor a \rfloor$ is greatest integer not greater than a.

> ***Solution.*** Go with the flow. □

1.2 Weird Ones

Problem 1.2.1 (ISL 2009 A3) **:** Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths

$$a, f(b) \text{ and } f(b + f(a) - 1).$$

(A triangle is non-degenerate if its vertices are not collinear.)

> ***Solution.*** $f(1) > 1 \implies f$ is periodic $\implies$ repeatation $\implies$ contradiction.
> $f(2) > 2 \implies$ strictly increasing $\implies$ repeatation. □

Problem 1.2.2 (USA TST 2018 P2) **:** Find all functions $f : \mathbb{Z}^2 \to [0, 1]$ such that for any integers x and y,

$$f(x, y) = \frac{f(x-1, y) + f(x, y-1)}{2}.$$

> ***Solution.*** We know that the function has to be a constant function. So it is a intuitive idea considering the difference of two values of the function. Again as we wish to show that this difference is 0, we have to use either equality of limit. As equality is quite ambiguous in this problem, we approach with limits. We see that $f(x, y)$ can be written as a term depending on the values of the 3rd quarter of the plane with (x, y) as its origin. With infinite values in our hand, we try bounding. □

1.3 FE cantonmathguy Seclected Problems

1. Determine all functions $f : \mathbb{Q} \to \mathbb{Q}$ satisfying $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{Q}$.

2. Let $a_1, a_2, \ldots$ be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer n the numbers $a_1, a_2, \ldots, a_n$ leave n different remainders upon division by n. Prove that every integer occurs exactly once in the sequence $a_1, a_2, \ldots$.

3. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying
$$f(x)f(y) = f(x+y) + xy$$
for all real x and y.

4. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a+b+c=0$, the following equality holds:
$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

5. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying
$$f(x) + f(y) = f(x+y) \quad \text{and} \quad f(xy) = f(x)f(y)$$
for all $x, y \in \mathbb{R}$.

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$, the following equality holds
$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$
where $\lfloor a \rfloor$ is the greatest integer not greater than a.

7. ★ Let k be a real number. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying
$$|f(x) - f(y)| \leq k(x-y)^2$$
for all real x and y.

8. Let $f : \mathbb{N} \to \mathbb{N}$ be a function, and suppose that positive integers k and c satisfy
$$f^k(n) = n + c$$
for all $n \in \mathbb{N}$, where f^k denotes f applied k times. Show that $k \mid c$.

9. Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying
$$f(f(f(n))) + f(f(n)) + f(n) = 3n$$
for every positive integer n.

10. Let S be the set of integers greater than 1. Find all functions $f : S \to S$ such that (i) $f(n) \mid n$ for all $n \in S$, (ii) $f(a) \geq f(b)$ for all $a, b \in S$ with $a \mid b$.

11. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all pairs of real numbers x and y.

12. ★ Let T denote the set of all ordered triples (p, q, r) of nonnegative integers. Find all functions $f : T \to \mathbb{R}$ satisfying

$$f(p,q,r) = \begin{cases} 0 & \text{if } pqr = 0, \\ 1 + \frac{1}{6}(f(p+1,q-1,r) + f(p-1,q+1,r) & \\ +f(p-1,q,r+1) + f(p+1,q,r-1) & \\ +f(p,q+1,r-1) + f(p,q-1,r+1)) & \text{otherwise} \end{cases}$$

for all nonnegative integers p, q, r.

13. Determine all strictly increasing functions $f : \mathbb{N} \to \mathbb{N}$ satisfying $nf(f(n)) = f(n)^2$ for all positive integers n.

14. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$.

15. Find all real-valued functions f defined on pairs of real numbers, having the following property: for all real numbers a, b, c, the median of $f(a, b), f(b, c), f(c, a)$ equals the median of a, b, c.

16. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that, for all positive integer n, we have $f(f(n)) < f(n+1)$.

17. Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that, for any $w, x, y, z \in N$,

$$f(f(f(z)))f(wxf(yf(z))) = z^2 f(xf(y))f(w).$$

Show that $f(n!) \geq n!$ for every positive integer n.

18. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(n!) = f(n)!$ for all positive integers n and such that $m - n$ divides $f(m) - f(n)$ for all distinct positive integers m, n.

19. Find all functions f from the reals to the reals such that

$$(f(a) + f(b))(f(c) + f(d)) = f(ac + bd) + f(ad - bc)$$

for all real a, b, c, d.

20. Determine all functions f defined on the natural numbers that take values among the natural numbers for which

$$(f(n))^p \equiv n \pmod{f(p)}$$

for all $n \in \mathbb{N}$ and all prime numbers p.

21. Let $n \geq 4$ be an integer, and define $[n] = \{1, 2, \ldots, n\}$. Find all functions $W : [n]^2 \to \mathbb{R}$ such that for every partition $[n] = A \cup B \cup C$ into disjoint sets,

$$\sum_{a \in A} \sum_{b \in B} \sum_{c \in C} W(a, b)W(b, c) = |A||B||C|.$$

22. ★ Find all infinite sequences $a_1, a_2, \ldots$ of positive integers satisfying the following properties: (a) $a_1 < a_2 < a_3 < \cdots$, (b) there are no positive integers i, j, k, not necessarily distinct, such that $a_i + a_j = a_k$, (c) there are infinitely many k such that $a_k = 2k - 1$.

23. Show that there exists a bijective function $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that for all $m, n \in \mathbb{N}_0$,

$$f(3mn + m + n) = 4f(m)f(n) + f(m) + f(n)$$

24. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f(f(m) + n) + f(m) = f(n) + f(3m) + 2014$$

for all integers m and n.

25. Let $n \geq 3$ be a given positive integer. We wish to label each side and each diagonal of a regular n-gon $P_1 \ldots P_n$ with a positive integer less than or equal to r so that:

   a) every integer between 1 and r occurs as a label;
   b) in each triangle $P_iP_jP_k$ two of the labels are equal and greater than the third.

   Given these conditions:

   a) Determine the largest positive integer r for which this can be done.
   b) For that value of r, how many such labellings are there?

26. ★ Suppose that f and g are two functions defined on the set of positive integers and taking positive integer values. Suppose also that the equations $f(g(n)) = f(n) + 1$ and $g(f(n)) = g(n) + 1$ hold for all positive integer n. Prove that $f(n) = g(n)$ for all positive integer n.

27. Find all the functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ satisfying the relation

28. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$

for all real numbers x and y.

29. Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the sub-sequences

$$s_{s_1}, s_{s_2}, s_{s_3}, \ldots \qquad \text{and} \qquad s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \ldots$$

are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

30. Find all functions f from $\mathbb{N}_0$ to itself such that

$$f(m+f(n)) = f(f(m)) + f(n)$$

for all $m, n \in \mathbb{N}_0$.

31. ★ Consider a function $f : \mathbb{N} \to \mathbb{N}$. For any $m, n \in \mathbb{N}$ we write $f^n(m) = \underbrace{f(f(\ldots f}_{n}(m)\ldots))$.

Suppose that f has the following two properties:

a) if $m, n \in \mathbb{N}$, then $\frac{f^n(m)-m}{n} \in \mathbb{N}$;
b) The set $\mathbb{N} \setminus \{f(n) \mid n \in \mathbb{N}\}$ is finite.

Prove that the sequence $f(1)-1, f(2)-2, f(3)-3, \ldots$ is periodic.

32. Let $\mathbb{N}$ be the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ that satisfy the equation

$$f^{abc-a}(abc) + f^{abc-b}(abc) + f^{abc-c}(abc) = a+b+c$$

for all $a, b, c \geq 2$.

33. Let $2\mathbb{Z}+1$ denote the set of odd integers. Find all functions $f : \mathbb{Z} \to 2\mathbb{Z}+1$ satisfying

$$f(x+f(x)+y) + f(x-f(x)-y) = f(x+y) + f(x-y)$$

for every $x, y \in \mathbb{Z}$.

1.4 Polynomials

1.4.1 Techniques to remember

Stuck? Try These

1 A polynomial with odd degree always has at least one real root.

2 If a polynomial with even degree has a negative value on its graph, then it has at least one real root.

3 Roots of unity divide a polynomial in parts like congruence classes.

4 MODULUS SIGN: use Triangle Inequality.

5 They say, In Poly, chase ROOTS.

6 $x^n f(\frac{1}{x})$ has the same coefficients as $f(x)$, but in opposite order.

Theorem 1.4.1 (Lagrange Interpolation Theorem) — *Given n real numbers, there exist a polynomial with at most $n-1$ degree such that the graph of the polynomial goes through all of the points.*

Theorem 1.4.2 (Finite Differences) — *This is the descrete form of derivatives. The first finite difference of a function f is defined as $g(x) := f(x+1) - f(x)$.*
$n+1$th finite difference of a n degree polynomial: For any polynomial $P(x)$ of degree at most n the following equation holds:

$$\sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} P(i) = 0$$

Remark. This can be used

1. to reduce the degree of a polynomial, manipulate the coefficients etc.

2. to solve recurrences, where the recurrence equation is a bit messy and contains a lot of

previous values. Like this recurrence is quite messy to solve as it is, but if we take the first finite difference here, it becomes easy:

$$f(x) = \frac{1}{3}f(x+1) + \frac{2}{3}f(x-1) + 1$$

It's like solving for the first derivative and then finding the original function.

1.4.2 General Problems

Problem 1.4.1 (USA TST 2014 P4) **:** Let n be a positive even integer, and let $c_1, c_2, \ldots, c_{n-1}$ be real numbers satisfying
$$\sum_{i=1}^{n-1} |c_i - 1| < 1.$$
Prove that
$$2x^n - c_{n-1}x^{n-1} + c_{n-2}x^{n-2} - \cdots - c_1x^1 + 2$$
has no real roots.

> ***Solution.*** A polynomial has no real root means that the polynomial completely lies in either of the two sides of the x-axis. So in this case, we have to prove that $P(x) > 0$. So we gotta try to bound. Again, to make $|c_i - 1|$ a little bit more approachable, we assign $b_i = c_i - 1$ and write $P(x)$ in terms of b_i. Now, how to bring the modulus sign in our polynomial? Oh, we have triangle ineq for those kinda work :0 □

Problem 1.4.2 (USAMO 2002 P3) **:** Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

> ***Solution.*** (i) If we have $n+1$ points, we have an unique polynomial through them.
> (ii) If we have one positive value of a polynomial and one negative value, then there exists a real root between that two values. □

Problem 1.4.3 (China TST 1995 P5) **:** A and B play the following game with a polynomial of degree at least 4:
$$x^{2n} + \_\ x^{2n-1} + \_\ x^{2n-2} + \ldots + \_\ x + 1 = 0$$
A and B take turns to fill in one of the blanks with a real number until all the blanks are filled up. If the resulting polynomial has no real roots, A wins. Otherwise, B wins. If A begins, which player has a winning strategy?

> ***Solution.*** Not always (actually in very few cases) the first move decides the winning strategy. In this case, if B could make the last move, he would definitely win. But as he can't, consider the final two moves. Again "Waves". □

Problem 1.4.4 (Zhao Polynomials) **:** A set of n numbers are considered to be k-cool if $a_1 + a_{k+1} \cdots = a_2 + a_{k+2} \cdots = \cdots = a_k + a_{2k} \ldots$. Suppose a set of 50 numbers are $3, 5, 7, 11, 13, 17$-cool. Prove that every element of that set is 0.

Solution. Equivalence class :0 roots of unity :0 :0 :0 □

Problem 1.4.5 (All Russian Olympiad 2016, Day2, Grade 11, P5) **:** Let n be a positive integer and let $k_0, k_1, \dots, k_{2n}$ be nonzero integers such that $k_0 + k_1 + \dots + k_{2n} \neq 0$. Is it always possible to find a permutation $(a_0, a_1, \dots, a_{2n})$ of $(k_0, k_1, \dots, k_{2n})$ so that the equation

$$a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \dots + a_0 = 0$$

has no integer roots?

Solution. The degree is $2n$, and we have to find a zero, so proving/disproving the existence of negative value of $P(x)$ is enough. If all the values of $P(x)$ are to be positive, the leading coefficient must be very big... □

Problem 1.4.6 (Zhao Poly) **:** Let $f(x)$ be a monic polynomial with degree n with distinct zeroes $x_1, x_2, \dots, x_n$. Let $g(x)$ be any monic polynomial of degree $n-1$. Show that

$$\sum_{j=1}^{n} \frac{g(x_j)}{f'(x_j)} = 1$$

where $f'(x_i) = \prod_{j \neq i} (x_i - x_j)$

Solution. Lagrange's Interpolation □

Problem 1.4.7 (ARO 2018 P11.1) **:** The polynomial $P(x)$ is such that the polynomials $P(P(x))$ and $P(P(P(x)))$ are strictly monotone on the whole real axis. Prove that $P(x)$ is also strictly monotone on the whole real axis.

Problem 1.4.8 (Serbia 2018 P4) **:** Prove that there exists a uniqe $P(x)$ polynomial with real coefficients such that

$$xy - x - y \mid (x+y)^{1000} - P(x) - P(y)$$

for all real x, y.

Solution. Substitution. □

1.4.3 Root Hunting

Problem 1.4.9 (Putnam 2017 A2) **:** Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

1.4.4 NT Polynomials

Problem 1.4.10 (Iran TST 2009 P4) **:** Find all polynomials f with integer coefficient such that, for every prime p and natural numbers u and v with the condition:

$$p \mid uv - 1$$

we always have

$$p \mid f(u)f(v) - 1$$

> ***Solution.*** Notice that we can disregard v by considering it $\frac{1}{u}$, and the condition won't be affected, because primes allow multiplicative inverses. After this observation the problem is almost solved. □

Problem 1.4.11 (Iran TST 2004 P6) **:** p is a polynomial with integer coefficients and for every natural n we have $p(n) > n$. x_k is a sequence that: $x_1 = 1, x_{i+1} = p(x_i)$ for every N one of x_i is divisible by N. Prove that $p(x) = x + 1$

> ***Solution.*** Notice that $\{x_i\}$ becomes periodic mod any prime. Now, we start by showing that $P(1) = 2$. We have, $P(x)$ has to be even. If it is > 2 then what happens? what if we take $N = P(1) - 1$? □

Problem 1.4.12 (ISL 2006 N4) **:** Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

> ***Solution.*** Suppose that there are more than n fixed points. So at least one of them can't be a fixed point of P. Use that. Follow. □

Problem 1.4.13 (ISL 2012 A4) **:** Let f and g be two nonzero polynomials with integer coefficients and $\deg f > \deg g$. Suppose that for infinitely many primes p the polynomial $pf + g$ has a rational root. Prove that f has a rational root.

Solution. dunno □

1.4.5 Fourier Transformation

Lemma 1.4.3 (Dealing with binomial terms with a common factor) — *Let n, k be two integers, and let z be a kth root of unity other than 1. Then,*

$$\binom{n}{0} + \binom{n}{k} + \binom{n}{2k} + \cdots = \frac{(1+z^1)^n + (1+z^2)^n + \cdots + (1+z^k)^n}{k}$$

Proof. For j not divisible by k,

$$z^j \left(\sum_{i=1}^{k} z^i \right) = 0$$

□

1.4.6 Irreducibility

- Summer Camp 2015 Handout
- Yufei Zhao's Handout

Stuck? Try These: What can be showed to prove Irreducibility

- Writing $f = g \cdot h$ and equating coefficients
- If the polynomial involves some prime, it's often useful to try factoring modulo that prime
- If the last coefficient is a prime, then there are some obvious bounds on the roots
- If there are bounds on the coefficients, then try root bounding

Lemma 1.4.4 (Bounds On Roots) — *P is a monic polynomial. Suppose $P(0) \neq 0$ and at most one complex root of P has absolute value at least 1. Then P is irreducible.*

Lemma 1.4.5 — *P is a monic polynomial. Suppose that $|P(0)|$ is prime, and all complex roots of P have absolute value greater than 1. Then P is irreducible.*

Lemma 1.4.6 (Leading Coefficient is LARGE) — *Let $P(x) = b_nx^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0$ such that*

$$|b_n| > |b_{n-1}| + |b_{n-2}| + \cdots + |b_0|$$

Then every root α of P is **strictly inside of the unit circle**, *i.e.* $|\alpha| < 1$.

i.e. If the first coefficient of the polynomial is very large, then all of the roots lie inside the unit circle.

Lemma 1.4.7 (Coefficients form a Decreasing Sequence) — *Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a real polynomial. Such that,*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0$$

Then any complex z of $P(x)$ satisfies $|z| \leq 1$

i.e. If the coefficients form a decreasing sequence then all of the roots lie on or inside the unit circle.

Lemma 1.4.8 (Constant is LARGE) — *Let $P(x) = a_nx^n + a_{n-1}x^{n-1} \cdots + a_1x + a_0$ be a polynomial over integers. Where, a_0 is a prime, and*

$$|a_0| > |a_n| + |a_{n-1}| + \cdots + |a_1|$$

Prove that $P(x)$ is irreducible.

Theorem 1.4.9 (Rouché's Theorem) — *Let f, g be analytic functions on and inside a simple closed curve $\mathcal{C}$. Suppose that*

$$|f(z)| > |g(z)|$$

for all points z on $\mathcal{C}$. Then f and $f + g$ have the same number of zeroes (counting multiplicities) interior to $\mathcal{C}$

Theorem 1.4.10 (Perron's Criterion) — *Let $P(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \dots a_1x + a_0$ be a polynomial over integers such that*

$$|a_{n-1}| > 1 + |a_{n-2}| + |a_{n-3}| + \dots + |a_1| + |a_0|$$

Then $P(x)$ is irreducible.

Remark. The crucial idea behind the proof is that $|a_0| \geq 1$, and if the polynomial is reducible, then there are at least two roots $|z| \geq 1$.

Proof [Bounding Roots]**.** Let $P(z) = 0$ for some $|z| = 1, z \in \mathbb{C}$. That means we have

$$\begin{aligned} -a_{n-1}z^{n-1} &= z^n + a_{n-2}z^{n-2} \dots a_1z + a_0 \\ \implies |a_{n-1}| &= |z^n + a_{n-2}z^{n-2} \dots a_1z + a_0| \\ &\leq |1| + |a_{n-2}| \dots + |a_0| \end{aligned}$$

Which is a contradiction. So $|z| \neq 1$.

We know that there exist a root z that has an absolute value greater than 1. We prove that there is only one such root of $P(x)$.

First, let $P(x) = (x - z)Q(x)$, where $|z| > 1$, and $Q(x) = x^{n-1} + b_{n-2}x^{n-2} + \dots + b_1x + b_0$. So we have,

$$P(x) = (x - z)Q(x)$$
$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \dots a_1x + a_0 = x^n + (b_{n-2} - z)x^{n-1} + \dots + (b_0 - zb_1)x + zb_0$$

$$\begin{aligned} \implies |b_{n-2} - z| &> 1 + |b_{n-3} - zb_{n-2}| + \dots + |b_0 - zb_1| + |zb_0| \\ |b_{n-2}| + |z| &> 1 - |b_{n-3}| + |z||b_{n-2}| + \dots |b_0| - |z||b_1| + |z|b_0| \\ |b_{n-2}| + |z| &= (|z| - 1)(|b_{n-2}| + |b_{n-3} \dots |b_0|) + |b_{n-2}| + 1 \\ 1 &> |b_{n-2}| + |b_{n-3} \dots |b_0| \end{aligned}$$

And by Lemma 1.4.6, $Q(x)$ does not have any root $|z| > 1$. □

Theorem 1.4.11 (Perron's Criterion's Generalization (Dominating Term)) — *Let $P(z) = a_nz^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0$ be a complex polynomial, such that its a_k term is dominant, that is,*

$$|a_k| > |a_0| + |a_1| + \dots |a_{k-1}| + |a_{k+1}| + \dots + |a_n|$$

for some $0 \leq k \leq n$. Then exactly k roots of P lies strictly inside of the unit circle, and the other

$n - k$ roots of P lies strictly outside of the unit circle.

Proof. A direct application of Theorem 1.4.9. $\square$

Lemma 1.4.12 (Bound on roots) — *Let $f(x) = a_n x^n + a_{n-1}x^{n-1} \cdots + a_1 x + a_0$ be an integer polynomial. Suppose that $a_n \geq 1, a_{n-1} \geq 0$ and $a_i \leq H$ for some positive constant H and $i = 0, 1, \ldots n-2$. Then any complex zero α of $f(x)$ has either nonpositive real part, or satisfies*

$$|\alpha| < \frac{1 + \sqrt{1 + 4H}}{2}$$

Proof. Suppose z is a root such that $|z| > 1$ and Re $z > 0$. Then we have

$$\begin{aligned}
\left|\frac{f(z)}{z^n}\right| &\geq \left(a_n - \frac{a_{n-1}}{z}\right) - H\left(\frac{1}{|z|^2} + \frac{1}{|z|^3} \cdots \frac{1}{|z|^n}\right) \\
&> \text{Re}\ \left(a_n + \frac{a_{n-1}}{z}\right) - \frac{H}{|z|^2 - |z|} \\
&\geq 1 - \frac{H}{|z|^2 - |z|} \\
&= \frac{|z|^2 - |z| - H}{|z|^2 - |z|} \\
&\geq 0
\end{aligned}$$

Whenever

$$|z| \geq \frac{1 + \sqrt{1 + 4H}}{2}$$

□

Theorem 1.4.13 (Cohn's Criterion) — *Suppose p is a prime number, expressed as $\overline{p_n p_{n-1} \ldots p_1 p_0}$ in base $b \geq 2$. Then the polynomial*

$$f(x) = p_n x^n + p_{n-1}x^{n-1} \cdots + p_1 x + p_0$$

is irreducible.

Problem 1.4.14 (ISL 2005 A1) **:** Find all pairs of integers a, b for which there exists a polynomial $P(x) \in \mathbb{Z}[X]$ such that product $(x^2 + ax + b) \cdot P(x)$ is a polynomial of a form

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$$

where each of $c_0, c_1, \ldots, c_{n-1}$ is equal to 1 or -1.

Solution. The idea of bounding the roots using the coefficients. □

Problem 1.4.15 () **:** Let $P(x)$ be a polynomial with real coefficients, and $P(x) \geq 0$ for all $x \in \mathbb{R}$. Prove that there exists two polynomials $R, S \in \mathbb{Q}$ such that

$$P(x) = R(x)^2 + Q(x)^2$$

Problem 1.4.16 (Romanian TST 2006 P2) **:** Let p a prime number, $p \geq 5$. Find the number of polynomials of the form

$$x^p + px^k + px^l + 1, \quad k > l, \quad k, l \in \{1, 2, \cdots, p-1\},$$

which are irreducible in $\mathbb{Z}[X]$.

> ***Solution.*** Taking mod p, we have that $x^p + 1 \equiv (x+1)^p (\text{mod } p)$. Now we can try equating terms or plug in some values to check for equality. □

Problem 1.4.17 (Romanian TST 2003 P5) **:** Let $f \in \mathbb{Z}[X]$ be an irreducible polynomial over the ring of integer polynomials, such that $|f(0)|$ is not a perfect square. Prove that if the leading coefficient of f is 1 (the coefficient of the term having the highest degree in f) then $f(X^2)$ is also irreducible in the ring of integer polynomials.

> ***Solution.*** If $f(x^2) = g(x)h(x)$, plugging $-x$ gives us $g(x)h(x) = g(-x)h(-x)$. So we should look at the common roots of $h(x)$ and $h(-x)$. And it is straightforward from here. □

> ***Solution.*** □

1.5 Inequalities

- Olympiad Inequalities - Thomas J. Mildorf
- $A < B$ - Keran Kedlaya
- Convexity - Po Shen Loh
- Brief Intro to Ineqs - Evan Chen

Definition (Majorizes) — Given two sequences of real numbers $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$, we say (x_n) *majorizes* (y_n), written $(x_n) \succ (y_n)$ if

$$\begin{aligned} &x_1 + x_2 \cdots + x_n = y_1 + y_2 \cdots + y_n \quad \text{and,} \\ &x_1 + x_2 \cdots + x_k \geq y_1 + y_2 \cdots + y_k \quad \forall 1 \leq k \leq n-1 \end{aligned}$$

Definition (Mean Values) — Given n *positive reals* $a_2 \ldots a_n$, we have

$$\text{Arithmetic Mean :} \quad \frac{\sum a_i}{n} = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

$$\text{Geometric Mean :} \quad \sqrt[n]{\prod a_i} = \sqrt[n]{a_1 a_2 \ldots a_n}$$

$$\text{Quadratic Mean :} \quad \sqrt{\frac{\sum a_i^2}{n}} = \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}$$

$$\text{Harmonic Mean :} \quad \frac{n}{\sum \frac{1}{a_i}} = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

1.5.1 Basic Inequalities

Theorem 1.5.1 (Muirhead's Inequality) — *If $a_1, a_2, \ldots a_n$ are real positive reals, and $(x_n) \succ (y_n)$, then we have,*

$$\sum_{sym} a_1^{x_1} a_2^{x_2} \cdots a_n^{x_n} \geq \sum_{sym} a_1^{y_1} a_2^{y_2} \cdots a_n^{y_n}$$

Theorem 1.5.2 (Jensen's Inequality) — *If f is convex, then*

$$\frac{f(a_1) + f(a_2) \cdots + f(a_n)}{n} \geq f\left(\frac{a_1 + a_2 \cdots + a_n}{n}\right)$$

The reverse inequality holds if f is concave.

Theorem 1.5.3 (Karamata's Inequality) — *If f is convex and $(x_n) \succ (y_n)$, then*

$$f(x_1) + f(x_2) \cdots + f(x_n) \geq f(y_1) + f(y_2) \cdots + f(x_n)$$

The reverse inequality holds if f is concave.

Theorem 1.5.1 (title=Tangent Line Trick: Best-Nice bound for function)
Even if a fucntion is not convex or concave for us to use Jensen's Inequality, we can still find a number a such that for our required interval I, f stays above (or below) the tangent line to f at $(a, f(a))$, that is

$$f(x) \geq f(a) + f'(a)(x - a)$$

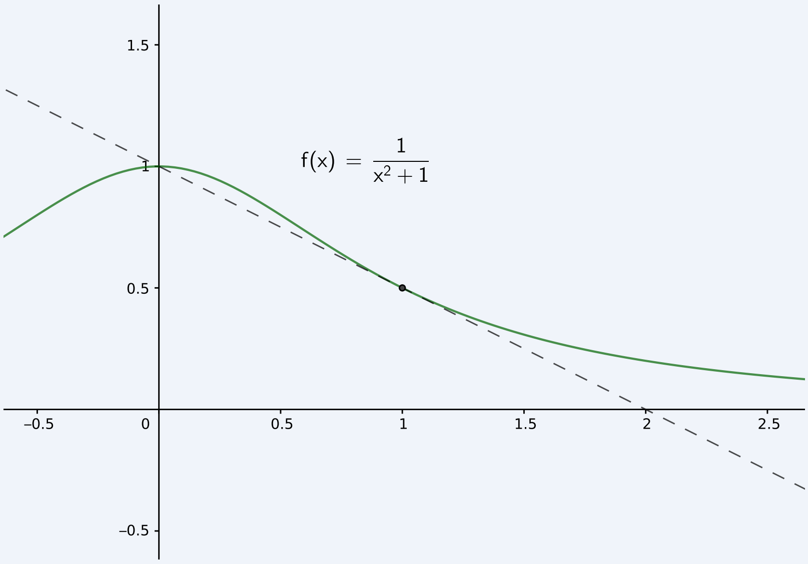

Figure 1.1: Here the tangent line at $(1, f(1))$ is the best to bound with in the interval $[0, 2]$

Theorem 1.5.4 ($n-1$ Equal Values) — *Let $a_1, a_2, \dots a_n$ be real numbers with $a_1 + a_2 \dots a_n$ fixed. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with exactly one* inflection point. *If*

$$f(a_1) + f(a_2) + \cdots + f(a_n)$$

achieves a maximal or minimal value, then $n-1$ of the a_i are equal to each other.

Lemma 1.5.5 (Power function convexity) — *If $(x_i), (y_i), (m_i)$ be three sequences of real numbers, $x, y \in \mathbb{R}$ and $p > 1$. Then for $\alpha \in (0, 1)$,*

$$(x+y)^p \ \leq \ \alpha^{1-p}x^p \ + \ (1-\alpha)^{1-p}y^p$$

$$\sum (x_i + y_i)^p m_i \ \leq \ \alpha^{1-p}\sum x_i^p m_i \ + \ (1-\alpha)^{1-p}\sum y_i^p m_i$$

Equality holds iff

$$\frac{x}{y} = \frac{x_i}{y_i} = \frac{\alpha}{1-\alpha}$$

Proof. Because $f(x) = x^p$ is a convex function,

$$(\alpha a + (1-\alpha)b)^p < \alpha a^p + (1-\alpha)b^p$$

So setting $\alpha a = x$ and $(1-\alpha)b = y$, we get the first inequality. The second one is just an extention of the first one. $\square$

Theorem 1.5.6 (Minkowski) — *If $x_1, x_2 \dots x_n$, $y_1, y_2, \dots y_n$ and $m_1, m_2, \dots m_n$ be three sequence of real numbers and $p > 1$, then*

$$\left(\sum (x_i + y_i)^p \, m_i\right)^{1/p} \leq \left(\sum x_i^p m_i\right)^{1/p} + \left(\sum y_i^p m_i\right)^{1/p}$$

Equality holds iff (x_i) and (y_i) are proportional.

Proof. Let us write,

$$A = \left(\sum x_i^p m_i\right)^{1/p}, \quad B = \left(\sum y_i^p m_i\right)^{1/p}$$

From the second equation of Lemma 1.5.5, for $\alpha \in (0, 1)$ we have

$$\sum (x_i + y_i)^p m_i \leq \alpha^{1-p} A^p + (1-\alpha)^{1-p} B^p$$

Setting α be such that $A/B = \alpha/(1-\alpha)$, by the equality case of the first equation of the lemma we get,

$$\sum (x_i + y_i)^p m_i \ \leq \ \alpha^{1-p} A^p + (1-\alpha)^{1-p} B^p = (A+B)^p$$

$\square$

Theorem 1.5.7 (Young's Inequality) — *If $a, b > 0$ and $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, then we have*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Equality occurs when $a^p = b^q$.

Proof. Consider the function $f(x) = e^x$, it is convex. So we have

$$e^{\frac{1}{p}x + \frac{1}{q}y} \leq e^{\frac{1}{p}x} + e^{\frac{1}{q}y}$$

If we let $a = e^{\frac{x}{p}}$, $b = e^{\frac{y}{q}}$, we are done. Equality occurs when $x = y$. $\square$

Lemma 1.5.8 —

Theorem 1.5.9 (Weighted Power Mean) — *Let $a_1, a_2, \ldots a_n$ and $w_1, w_2, \cdots + w_n$ be positive real numbers with $w_1 + w_2 \ldots w_n = 1$. For real number r, we define,*

$$P(r) = \begin{cases} \left(w_1 a_1^r + w_2 a_2^r \cdots + w_n a_n^r\right)^{1/r} & r \neq 0 \\ a_1^{w_1} a_2^{w_2} \ldots a_n^{w_n} & r = 0 \end{cases}$$

If $r > s$, then $P(r) \geq P(s)$. Equality occurs iff $a_1 = a_2 = \cdots = a_n$

Proof. First we show that, $\lim_{r\to 0} P(r) = P(0)$. Using L'Hopital's law,

$$\begin{aligned} \lim_{r\to 0} \ln P(r) = \lim_{r\to 0} \frac{\ln \sum w_i a_i^r}{r} &= \lim_{r\to 0} \frac{\frac{\sum w_i a_i^r \ln a_i}{\sum w_i a_i^r}}{1} \\ &= \lim_{r\to 0} \frac{\sum w_i a_i^r \ln a_i}{\sum w_i a_i^r} \\ &= \sum \ln a_i^{w_i} \\ &= \ln P(0) \end{aligned}$$

Now we have,

$\square$

1.5.2 Theorems

'Mean' Inequalities

Theorem 1.5.10 (Triangle Inequality) — *For any complex numbers $a_1, a_2 \ldots a_n$ the following holds:*

$$|a_1 + a_2 \cdots + a_k| \leq |a_1| + |a_2| + \cdots + |a_k|$$

§1 One Mean Ineq – QM-AM-GM-HM

Given n positive real numbers $x_1, x_2, \ldots x_n$, the following relation holds:

$$\boxed{\sqrt{\frac{x_1^2+\cdots+x_n^2}{n}} \geq \frac{x_1+\cdots+x_n}{n} \geq \sqrt[n]{x_1\cdots x_n} \geq \frac{n}{\frac{1}{x_1}+\cdots+\frac{1}{x_n}}}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Theorem 1.5.11 (Weighted AM-GM) — *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are nonnegative real numbers (the "weights") which sum to 1, then*

$$\boxed{\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n \geq a_1^{\lambda_1} a_2^{\lambda_2} \ldots a_n^{\lambda_n}}$$

Equality holds if and only if $a_i = a_j$ for all integers i, j such that $\lambda_i \neq 0$ and $\lambda_j \neq 0$. We obtain the unweighted form of AM-GM by setting

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = \frac{1}{n}$$

Theorem 1.5.12 (Cauchy-Schward Inequality) — *For any real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$,*

$$\boxed{\left(a_1^2 + a_2^2 \ldots a_n^2\right)\left(b_1^2 + b_2^2 \ldots b_n^2\right) \geq \left(a_1 b_1 + a_2 b_2 \ldots a_n b_n\right)^2}$$

with equality when there exist constants μ, λ not both zero such that for all $1 \leq i \leq n$, $\mu a_i = \lambda b_i$.

The inequality sometimes appears in the following form.

Theorem 1.5.13 (Cauchy-Schwarz Inequality Complex form) — *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be complex numbers. Then*

$$\boxed{\left(|a_1^2| + |a_2^2| \ldots |a_n^2|\right)\left(|b_1^2| + |b_2^2| \ldots |b_n^2|\right) \geq |a_1 b_1 + a_2 b_2 \ldots a_n b_n|^2}$$

Theorem 1.5.14 (Titu's Lemma) — *For positive reals $a_1, a_2 \ldots a_n$ and $b_1, b_2 \ldots b_n$ the following* *holds:*

$$\boxed{\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+\cdots+\frac{a_n^2}{b_n}\geq\frac{(a_1+a_2+\cdots+a_n)^2}{(b_1+b_2+\cdots+b_n)}}$$

Theorem 1.5.15 (Holder's Inequality) — *If* $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, \ldots, z_1, z_2, \ldots, z_n$ *are nonnegative real numbers and* $\lambda_a, \lambda_b, \ldots, \lambda_z$ *are nonnegative reals with sum of 1, then*

$$a_1^{\lambda_a} b_1^{\lambda_b} \ldots z_1^{\lambda_z} + \cdots + a_n^{\lambda_a} b_n^{\lambda_b} \ldots z_n^{\lambda_z} \leq (a_1+\cdots+a_n)^{\lambda_a}(b_1+\cdots+b_n)^{\lambda_b}\ldots(z_1+\cdots+z_n)^{\lambda_z}$$

.

1.5.3 Tricks

Some tricks to try

1 Replace trigonometric functions by reals, and translate the problem

2 Smoothing, replace two variable while keeping something invariant, to make the inequality sharper.

3 Convexity, differentiate to check convexity, if the second derivative is positive on some interval, then the function is convex on that interval except probably at the endpoints, and concave otherwise.

An example is $\ln \frac{1-x}{x}$. It is convex in $(0, \frac{1}{2}]$ and concave in $[\frac{1}{2}, 1)$

4 If there is product, and if the problem is 'ad-hoc'y, then apply $AM - GM$ and ln to see if there is something to play with.

Definition (Homogeneous Expression) — Expression $F(a_1, a_2 \dots a_n)$ is said to be homogeneous of degree k if and only if there exists real k such that for every $t > 0$ we have

$$t^k F(a_1, a_2 \dots a_n) = F(ta_1, ta_2 \dots ta_n)$$

If an expression is homogeneous, then the following can be assumed (one at a time):

$$\sum_{i=1}^{n} a_i = 1 \tag{1.1}$$

$$\prod_{i=1}^{n} a_i = 1 \tag{1.2}$$

$$a_1 = 1 \text{ or for some } i,\ a_i = 1 \tag{1.3}$$

$$\sum_{i=1}^{n} a_i^2 = 1 \tag{1.4}$$

$$\sum_{Cyc} a_i a_{i+1} = 1 \tag{1.5}$$

Definition (Substitutions) —

- For the condition $abc = 1$, set
$$a = \frac{x}{y},\ b = \frac{y}{z},\ c = \frac{z}{x}$$

-
$$xyx = x + y + z + 2 \implies \frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1$$
implies the existence of a, b, c such that
$$x = \frac{b+c}{a},\ y = \frac{a+c}{b},\ z = \frac{b+a}{c}$$
.

- $2xyz + xy + yz + zx = 1$ is just the inverse of the previouss condition.

-
$$x^2 + y^2 + z^2 = xyz + 4 \text{ and } |x|,\ |y|,\ |z| \geq 2$$
implies the existence of a, b, c such that
$$abc = 1 \text{ and } x = a + \frac{1}{a},\ y = b + \frac{1}{b},\ z = c + \frac{1}{c}$$
In fact even if only $\max(|x|, |y|, |z|) > 2$ is given, the result still holds.

Lemma 1.5.16 — *The following inequality holds for every positive integer n*
$$2\sqrt{n+1} - 2\sqrt{n} < \sqrt{\frac{1}{n}} < 2\sqrt{n} - 2\sqrt{n-1}$$

Lemma 1.5.17 — *Given 4 posititve real numbers $a < b < c < d$. Call the score of a permutation a_1, a_2, a_3, a_4 of the four given reals be equal to the real*
$$|\frac{a_1}{a_2} - \frac{a_3}{a_4}|$$
Prove that the minimum the score can get is equal to
$$|\frac{a}{c} - \frac{b}{d}|$$

1.5.4 Problems

Problem 1.5.1 (APMO 1991 P3) **:** Let $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$. Show that

$$\frac{a_1^2}{a_1+b_1} + \frac{a_2^2}{a_2+b_2} + \cdots + \frac{a_n^2}{a_n+b_n} \geq \frac{a_1+a_2+\cdots+a_n}{2}$$

Problem 1.5.2 (ISL 2009 A2) **:** Let a, b, c be positive real numbers such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = a+b+c$. Prove that:

$$\frac{1}{(2a+b+c)^2} + \frac{1}{(a+2b+c)^2} + \frac{1}{(a+b+2c)^2} \leq \frac{3}{16}.$$

Problem 1.5.3 (ARO 2018 P11.2) **:** Let $n \geq 2$ and $x_1, x_2, \ldots, x_n$ positive real numbers. Prove that

$$\frac{1+x_1^2}{1+x_1x_2} + \frac{1+x_2^2}{1+x_2x_3} + \ldots + \frac{1+x_n^2}{1+x_nx_1} \geq n$$

> ***Solution.*** The inequality says sum is greater, so if the product is greater, then we are done by AM-GM (**??**). □

Problem 1.5.4 (Turkey TST 2017 P5) **:** For all positive real numbers a, b, c with $a + b + c = 3$, show that

$$a^3b + b^3c + c^3a + 9 \geq 4(ab + bc + ca)$$

> ***Solution.*** Always try the most simple ineq possible, AM-GM □

Problem 1.5.5 (IMO 2012 P2) **:** Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2a_3\cdots a_n = 1$. Prove that

$$(1+a_2)^2(1+a_3)^3\cdots(1+a_n)^n > n^n.$$

> ***Solution.*** The main idea is to look for the ans of the ques, $(1 + a_k)^k$ >=?. We have k^{th} power. So if we can get a k term sum inside of the brackets, we can get a clean term for ? from AM-GM. And 1 seems like it's crying to be partitioned. So we write the term as $\left(a_k + \frac{1}{k-1} + \cdots + \frac{1}{k-1}\right)$ □

Solution. *Looks at the $a_2a_3 \ldots a_n = 1$ condition*
Hey, we have a substitution for this one, why not try it out...
darn it, i still have to do the partition thing to cancel out the powers > ($\square$

Problem 1.5.6 (ISL 1998 A1) **:** Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n < 1$. Prove that

$$\frac{a_1 a_2 \cdots a_n \left[1 - (a_1 + a_2 + \cdots + a_n)\right]}{(a_1 + a_2 + \cdots + a_n)(1 - a_1)(1 - a_2) \cdots (1 - a_n)} \le \frac{1}{n^{n+1}}.$$

Solution. Simplifying and making it symmetric, we get to the inequality

$$\prod_{i=1} n \frac{1 - a_i}{a_i} \ge n^{n+1}$$

Now approaching similarly as this problem, we get to the solution. $\square$

Problem 1.5.7 (ISL 2001 A1) **:** Let a, b, c be positive real numbers so that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \le 1.$$

Solution. Substitute. $\square$

Problem 1.5.8 (ISL 1999 A1) **:** Let $n \ge 2$ be a fixed integer. Find the least constant C such the inequality

$$\sum_{i<j} x_i x_j \left(x_i^2 + x_j^2\right) \le C \left(\sum_i x_i\right)^4$$

holds for any $x_1, \ldots, x_n \ge 0$ (the sum on the left consists of $\binom{n}{2}$ summands). For this constant C, characterize the instances of equality.

Solution. Follow the ineq sign and remember AM-GM. $\square$

Problem 1.5.9 (ISL 2017 A1) **:** Let $a_1, a_2, \ldots a_n, k$, and M be positive integers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = k \quad \text{and} \quad a_1 a_2 \cdots a_n = M$$

If $M > 1$, prove that the polynomial

$$P(x) = M(x+1)^k - (x+a_1)(x+a_2)\cdots(x+a_n)$$

has no positive roots.

> ***Solution.*** The same idea used in this and this, spreading an expression to perform AM-GM on it. ☐

Problem 1.5.10 (ISL 2016 A1) **:** Let a, b, c be positive real numbers such that $\min(ab, bc, ca) \geq 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1.$$

> ***Solution.*** Try the simpler version with two variables first. Now you can use this discovery with a little bit of cleverness to solve the problem. The clever part is to notice that 4 variable ineq is more solvable than a 3 variable one. ☐

Problem 1.5.11 (ISL 2016 A2) **:** Find the smallest constant $C > 0$ for which the following statement holds: among any five positive real numbers a_1, a_2, a_3, a_4, a_5 (not necessarily distinct), one can always choose distinct subscripts i, j, k, l such that

$$\left|\frac{a_i}{a_j} - \frac{a_k}{a_l}\right| \leq C.$$

> ***Solution.*** Simplify the problem to get the ans first. Think about what is the smallest such value for any given 4 positive reals. ☐

Problem 1.5.12 (ISL 2004 A1) **:** Let $n \geq 3$ be an integer. Let t_1, t_2, ..., t_n be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n}\right).$$

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

> ***Solution.*** Easy solution by induction. For a more elegant solution, write the right side as sum of paired factors. Finding when the inequality breaks and relating it to the end statement. ☐

Problem 1.5.13 (ISL 1996 A2) **:** Let $a_1 \geq a_2 \geq \ldots \geq a_n$ be real numbers such that for all integers $k > 0$,

$$a_1^k + a_2^k + \ldots + a_n^k \geq 0.$$

Let $p = \max\{|a_1|, \ldots, |a_n|\}$. Prove that $p = a_1$ and that

$$(x - a_1) \cdot (x - a_2) \cdots (x - a_n) \leq x^n - a_1^n$$

for all $x > a_1$.

Solution. After the first part, apply AM-GM on the whole left side, this not gonna work, since we can't bound $\sum a_i$ wrt a_1. So what if we divide both side by $(x - a_1)$ and then apply AM-GM? □

1.5.4.1 Smoothing And Convexity

Some usual tricks

1. Bring x, y closer, keeping $x + y$ constant.
2. If we need to smoothen up the value xy, then take ln on both side.
3. Work with different variables.

Theorem 1.5.18 (Convexity) — *1. The function is convex in interval I iff for all $a, b \in I$ and for all $t < 1$,*

$$tf(a) + (1-t)f(b) \geq f(ta + (1-t)b)$$

Which if put in words, means that the line segment joining $(a, f(a))$ and $(b, f(b))$ lies completely above the graph of the function.

2. The function is convex in interval I if f' is increasing in I or f'' is positive in I.

Theorem 1.5.19 (Jensen's Inequality) — *Let $x_1, \ldots, x_n \in \mathbb{R}$ and let $\alpha_1, \ldots, \alpha_n \geq 0$ satisfy $\alpha_1 + \cdots + \alpha_n = 1$.*

If f is a Convex Function, we have:

$$\alpha_1 f(x_1) + \alpha_2 f(x_2) \cdots + a_n f(x_n) \geq f(\alpha_1 x_1 + \alpha_2 x_2 \cdots + \alpha_n x_n)$$

If f is a Concave Function, we have:

$$\alpha_1 f(x_1) + \alpha_2 f(x_2) \cdots + a_n f(x_n) \leq f(\alpha_1 x_1 + \alpha_2 x_2 \cdots + \alpha_n x_n)$$

Theorem 1.5.20 (Popoviciu's inequality) — *Let f be a convex function on and interval $I \in \mathbb{R}$. Then for any numbers $x, y, z \in I$,*

$$f(x) + f(y) + f(z) \ + 3f\left(\frac{x+y+z}{3}\right) \ \geq \ 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{y+z}{2}\right) + 2f\left(\frac{z+x}{2}\right)$$

Problem 1.5.14 (USAMO 1998 P3) **:** Let $a_0, a_1, \cdots, a_n$ be numbers from the interval $(0, \pi/2)$

such that

$$\tan(a_0 - \frac{\pi}{4}) + \tan(a_1 - \frac{\pi}{4}) + \cdots + \tan(a_n - \frac{\pi}{4}) \geq n - 1.$$

Prove that

$$\tan a_0 \tan a_1 \cdots \tan a_n \geq n^{n+1}.$$

> ***Solution.*** Get rid of the tan's. AM-GM, Jensen doesn't works, so try **smoothing**. The conventional smoothing trick fails at one case, but works for all other cases. Means we have to deal with that case specially. □

Problem 1.5.15 (USAMO 1974 P2) **:** Prove that if a, b, and c are positive real numbers, then

$$a^a b^b c^c \geq (abc)^{(a+b+c)/3}.$$

> ***Solution.*** In □

Problem 1.5.16 (India 1995) **:** Let $x_1, x_2, ..., x_n > 0$ be real numbers such that $x_1+x_2+x_3+...+x_n = 1$. Prove the inequality

$$\frac{x_1}{\sqrt{1-x_1}} + \frac{x_2}{\sqrt{1-x_2}} + ... + \frac{x_n}{\sqrt{1-x_n}} \geq \sqrt{\frac{n}{n-1}}$$

> ***Solution.*** easy smoothing □

Problem 1.5.17 (Vietnam 1998) **:** $x_1, x_2...x_n$ are real numbers such that

$$\frac{1}{x_1 + 1998} + \cdots + \frac{1}{x_n + 1998} = \frac{1}{1998}$$

Prove that

$$\frac{\sqrt[n]{x_1 \ldots x_n}}{n-1} \geq 1998$$

> ***Solution.*** Translate the given expression in a nicer way with new variables... □

Problem 1.5.18 (IMO 1974 P2) **:** The variables a, b, c, d, traverse, independently from each other, the set of positive real values. What are the values which the expression

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

takes?

Solution. $\dfrac{x}{y+c} \leq \dfrac{x}{y} \leq \dfrac{x}{y-c}$ □

Problem 1.5.19 (Bulgaria 1995) **:** Given n real number $x_1, x_2, ..., x_n \in [0, 1]$. Prove the following inequality

$$(x_1 + x_2 + \cdots + x_n) - (x_1x_2 + x_2x_3 + \cdots + x_nx_1) \leq \left\lfloor \frac{n}{2} \right\rfloor$$

1.6 Calculus

Lemma 1.6.1 —

$$(f(x) + g(x))' = f'(x) + g'(x)$$

$$(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

$$(\frac{f(x)}{g(x)})' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

$$(f(g(x)))' = f'(g(x)) + g'(x)$$

Lemma 1.6.2 (The Derivative of an Odd function is always an Even function, and vice versa.) —

1.7 Ad-Hocs

Problem 1.7.1 (ISL 2014 A2) **:** Define the function $f : (0, 1) \to (0, 1)$ by

$$f(x) = x^2, \text{ for } x \geq \frac{1}{2} \text{ and } x + \frac{1}{2}, \text{ for } x < \frac{1}{2}$$

Let a and b be two real numbers such that $0 < a < b < 1$. We define the sequences an and bn by $a_0 = a, b_0 = b$, and $a_n = f(a_{n-1}), b_n = f(b_{n-1})$ for $n > 0$. Show that there exists a positive integer n such that

$$(a_n - a_{n-1})(b_n - b_{n-1}) < 0$$

.

Problem 1.7.2 (ARO 2018 P10.1) **:** Determine the number of real roots of the equation

$$|x| + |x + 1| + \cdots + |x + 2018| = x^2 + 2018x - 2019$$

Problem 1.7.3 (European Mathematics Cup 2018 P3) **:** Find all $k > 1$ such that there exists a set S such that,

1. There exists $N > 0$ such that, if $x \in S, then x < N$.
2. If $a, b \in S$, and $a > b$, then $k(a - b) \in S$

Solution. Find some constraints such as, $k(a - b) \not> a$, S has a smallest element. These two combined with a sequence of decreasing elements of S is enough to solve this problem. □

Problem 1.7.4 (APMO 2018 P2) **:** Let $f(x)$ and $g(x)$ be given by

$$f(x) = \frac{1}{x} + \frac{1}{x-2} + \frac{1}{x-4} + \cdots + \frac{1}{x-2018}$$

$$g(x) = \frac{1}{x-1} + \frac{1}{x-3} + \frac{1}{x-5} + \cdots + \frac{1}{x-2017}$$

Prove that $|f(x) - g(x)| > 2$ for any non-integer real number x satisfying $0 < x < 2018$.

Solution. Substract, manipulate, see that for $\epsilon < 1$, for $x = 2k + \epsilon$ it's true, if it's true for $x = 2k + 1 + \epsilon$. Then for $x = 2k + 1 + \epsilon$, sustitute the value to find a common term appearing in all of those equations. So if that term were to be greater than 2, we would be done. How do we test that? Take the first derivative to find the minima. □

Problem 1.7.5 (ISL 2015 A1) **:** Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \geq \frac{ka_k}{a_k^2 + (k-1)}$$

for every positive integer k. Prove that $a_1 + a_2 + \ldots + a_n \geq n$ for every $n \geq 2$.

Solution. Simplify the inequality. And then sum it up. □

Problem 1.7.6 (ISL 2015 A3) **:** Let n be a fixed positive integer. Find the maximum possible value of

$$\sum_{1 \leq r < s \leq 2n} (s - r - n)x_r x_s,$$

where $-1 \leq x_i \leq 1$ for all $i = 1, \cdots, 2n$.

Solution. The expression is weird, and beautiful. Now if we write the expression as a single variable function, we see that $x_i \in 1, -1$. Now, there is $x_i x_j$ in the expression. So we need to multiply two expressions. Again, see that $s - r - n$ can be rewritten as $-(n - s) - (r)$. Now, how do we get an expression like $x_i x_j$ which can be found in squares, with a coefficient $(n - s)$ and r? By summing it up r times, simple. □

Problem 1.7.7 (ISL 2010 A3) **:** Let $x_1, \ldots, x_{100}$ be nonnegative real numbers such that $x_i + x_{i+1} + x_{i+2} \leq 1$ for all $i = 1, \ldots, 100$ (we put $x_{101} = x_1, x_{102} = x_2$). Find the maximal possible value of the sum $S = \sum_{i=1}^{100} x_i x_{i+2}$.

Solution. Bound a small portion of the large sum. □

Problem 1.7.8 (ISL 2005 A3) **:** Four real numbers p, q, r, s satisfy $p + q + r + s = 9$ and $p^2 + q^2 + r^2 + s^2 = 21$. Prove that there exists a permutation (a, b, c, d) of (p, q, r, s) such that $ab - cd \geq 2$.

Solution. Put p, q, r, s in order, find which permutation must satisfy the condition. Since we know, $\sum_{sym} pq = 30$, what can we say about the largest sum? How do we get $pq - rs$ with the equations given to us? What can we do to make the conditions met? □

1.7.1 Factorization

Problem 1.7.9 (USAMO 2013 P4) **:** Find all real numbers $x, y, z \geq 1$ satisfying

$$\min(\sqrt{x+xyz}, \sqrt{y+xyz}, \sqrt{z+xyz}) = \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

> ***Solution.*** **Replacement is never a bad idea to try out.** But the main part is not replacement, but it's factorization. I don't yet know how to find such factorization, but let's find out. □

1.7.2 Bounding

Problem 1.7.10 (ISL 2004 A2) **:** Let $a_0, a_1, a_2, \ldots$ be an infinite sequence of real numbers satisfying the equation

$$a_n = |a_{n+1} - a_{n+2}|$$

for all $n \geq 0$, where a_0 and a_1 are two different positive reals. Can this sequence $a_0, a_1, a_2 \ldots$ be bounded?

Solution. In bounding problems, name the bounds, then focus on them. **Another thing: In reals, a variable does not necessarily need to be equal to the bound.** □

1.7.3 Manipulation

Problem 1.7.11 (ISL 2011 A2) **:** Determine all sequences $(x_1, x_2, \dots, x_{2011})$ of positive integers, such that for every positive integer n there exists an integer a with

$$\sum_{j=1}^{2011} jx_j^n = a^{n+1} + 1$$

Solution. Manipulate the data.
Since for all n the statement holds, we can guess there is bounding involved. Can we bound x_i or a? Tweak the terms and see if there is something nice to work with. □

Problem 1.7.12 (ISL 2014 A1) **:** Let $a_0 < a_1 < a_2 \dots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + a_2 + \cdots + a_n}{n} \leq a_{n+1}.$$

Solution. Manipulate the data.
Either directly, or using the "Δ method" □

Problem 1.7.13 (Putnam 2011 A2) **:** Let $a_1, a_2, \dots$ and $b_1, b_2, \dots$ be sequences of positive real numbers such that $a_1 = b_1 = 1$ and $b_n = b_{n-1}a_n - 2$ for $n = 2, 3, \dots$. Assume that the sequence (b_j) is bounded. Prove that

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 \cdots a_n}$$

converges, and evaluate S

Solution. Look for partial sum. And in limit problems on contests, it is always a good idea to think about $\epsilon_n = l - S_n$ □

Problem 1.7.14 (Putnam 2013 A3) **:** Suppose that the real numbers $a_0, a_1, \dots, a_n$ and x, with $0 < x < 1$, satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} = 0.$$

Prove that there exists a real number y with $0 < y < 1$ such that

$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

> ***Solution.*** How do you show $\exists$ root $\in I$ if you don't want to construct it? Also those geometric sums are begging to be expanded... $\square$

Problem 1.7.15 (GQMO 2020 P3) **:** We call a set of integers *special* if it has 4 elements and can be partitioned into 2 disjoint subsets $\{a, b\}$ and $\{c, d\}$ such that $ab - cd = 1$. For every positive integer n, prove that the set $\{1, 2, \ldots, 4n\}$ cannot be partitioned into n disjoint special sets.

> ***Solution*** [Multiply 'em All]**.** Each special set must have exactly two evens and two odds. Now, consider the products of all even numbers and all odd numbers. Clearly the product of the odd parts of each set will be much smaller than the product of the even parts. $\square$

Problem 1.7.16 (Korean Summer Program TST 2016 1) **:** Find all real numbers $x_1, \ldots, x_{2016}$ that satisfy the following equation for each $1 \leq i \leq 2016$. (Here $x_{2017} = x_1$.)

$$x_i^2 + x_i - 1 = x_{i+1}$$

1.8 Tricks and Lemmas

Theorem 1.8.1 (Minkowski's theorem) — *Any convex set in $\mathbb{R}^n$, which is symmetric with respect to the origin and with volume greater than $2^n d(L)$ contains a non-zero lattice point.*

1.8.1 Ad Hocs

1. $x^2 + 1 = (x + i)(x - i)$ [USAMO 2014 P1]
2. Add. Everything. Up.
3. Send SHIT to the infinity.