Problem 0.1: **E**

Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

Problem 0.2:

Alice and Bob play a game. There is a threshold n, Initially, the game starts with the number 1. In a move the player replaced the number m with either $m+1$ or $2m$. Assuming optimal play, count the number of threshold under 2019 where Alice wins.

Problem 0.3:

For an integer n, let $a_1, a_2, \ldots a_{\phi(n)}$ be the integers less than and coprime to n. Determine all possible values of

$$\prod_{i=1}^{\phi(n)} a_i \pmod{n}$$

and for which values of n they appear.

Problem 0.4:

Let O and N be the circumcenter and nine-point-center of $\triangle ABC$ respectively. Let I_b and I_c be the B and C excenters of $\triangle ABC$ respectively. Prove that

$$\angle I_bOI_c = 180^\circ - \frac{1}{2}\angle I_bNI_c$$

Problem 0.5: Putnam 2001 A5 **E**

Prove that there are unique positive integers a, n such that

$$a^{n+1} - (a+1)^n = 2001$$

Problem 0.6:

Find all polynomials $P(x)$ with real coefficients such that for all real numbers $x+y+z=0$, then the determinant of the following matrix is 0

$$\begin{bmatrix} 1 & x & P(x) \\ 1 & y & P(y) \\ 1 & z & P(z) \end{bmatrix}$$

Problem 0.7: IMO 1971 P5 **M**

Prove that for every positive integer m we can find a finite set S of points in the plane, such that given any point A of S, there are exactly m points in S at unit distance from A.

Problem 0.8: ISL 1971 **M**

Consider a sequence of polynomials $P_0(x), P_1(x), P_2(x), \ldots, P_n(x), \ldots$, where $P_0(x) = 2, P_1(x) = x$ and for every $n \geq 1$ the following equality holds:

$$P_{n+1}(x) + P_{n-1}(x) = xP_n(x).$$

Prove that there exist three real numbers a, b, c such that for all $n \geq 1$,

$$(x^2 - 4)[P_n^2(x) - 4] = [aP_{n+1}(x) + bP_n(x) + cP_{n-1}(x)]^2.$$

Problem 0.9: IMO 1974 P3

Prove that for any n natural, the number

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^3 k$$

cannot be divided by 5.

Problem 0.10: Putnam 1999 B6 **M**

Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that $\gcd(s, n) = 1$ or $\gcd(s, n) = s$. Show that there exist $s, t \in S$ such that $\gcd(s, t)$ is prime.

Problem 0.11: Sharygin 2009 P4 **E**

Let P and Q be the common points of two circles. The ray with origin Q reflects from the first circle in points A_1, A_2,... according to the rule "the angle of incidence is equal to the angle of reflection". Another ray with origin Q reflects from the second circle in the points B_1, B_2,... in the same manner. Points A_1, B_1 and P occurred to be collinear. Prove that all lines A_iB_i pass through P.

Problem 0.12: Greece **M**

Find all surjective functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $m, n \in \mathbb{N}$

$$m|n \iff f(m)|f(n)$$

Problem 0.13: USA TST 2010 P9 **H**

Determine whether or not there exists a positive integer k such that $p = 6k + 1$ is a prime and

$$\binom{3k}{k} \equiv 1 \pmod{p}$$

Problem 0.14: USA TST 2011 P6 **H**

A polynomial $P(x)$ is called nice if $P(0) = 1$ and the nonzero coefficients of $P(x)$ alternate between 1 and -1 when written in order. Suppose that $P(x)$ is nice, and let m and n be two relatively prime positive integers. Show that

$$Q(x) = P(x^n) \cdot \frac{(x^{mn} - 1)(x - 1)}{(x^m - 1)(x^n - 1)}$$

is nice as well.

Problem 0.15:

Find all pairs of positive integers (m, n) such that $mn - 1$ divides $(n^2 - n + 1)^2$.

Problem 0.16:

In isosceles $\triangle ABC$, $AB = AC$, points D, E, F lie on segments BC, AC, AB such that $DE \parallel AB$, $DF \parallel AC$. The circumcircle of $\triangle ABC$ ω_1 and the circumcircle of $\triangle AEF$ ω_2 intersect at A, G. Let DE meet ω_2 at $K \neq E$. Points L, M lie on ω_1, ω_2 respectively such that $LG \perp KG, MG \perp CG$. Let P, Q be the circumcenters of $\triangle DGL$ and $\triangle DGM$ respectively. Prove that A, G, P, Q are concyclic.

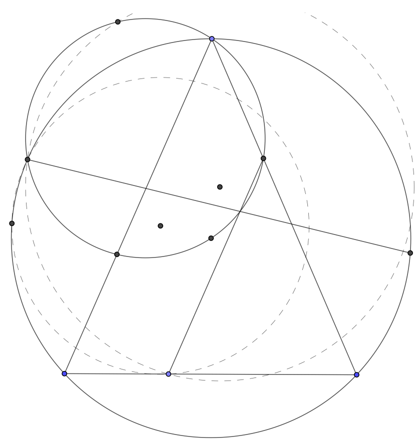

Figure 1

Problem 0.17:

Given a polynomial $f(x)$ with rational coefficients, of degree $d \geq 2$, we define the sequence of sets $f^0(\mathbb{Q}), f^1(\mathbb{Q}), \ldots$ as $f^0(\mathbb{Q}) = \mathbb{Q}$, $f^{n+1}(\mathbb{Q}) = f(f^n(\mathbb{Q}))$ for $n \geq 0$. (Given a set S, we write $f(S)$ for the set $\{f(x) \mid x \in S\}$). Let $f^\omega(\mathbb{Q}) = \bigcap_{n=0}^\infty f^n(\mathbb{Q})$ be the set of numbers that are in all of the sets $f^n(\mathbb{Q})$, $n \geq 0$. Prove that $f^\omega(\mathbb{Q})$ is a finite set.

Problem 0.18:

Define the polymonial sequence $\{f_n(x)\}_{n\geq 1}$ with $f_1(x) = 1$,

$$f_{2n}(x) = xf_n(x),\ f_{2n+1}(x) = f_n(x) + f_{n+1}(x),\ n \geq 1.$$

Look for all the rational number a which is a root of certain $f_n(x)$.

Problem 0.19: Sharygin 2013 Final Round 10.8

Two fixed circles are given on the plane, one of them lies inside the other one. From a point C moving arbitrarily on the external circle, draw two chords CA, CB of the larger circle such that they tangent to the smaller one. Find the locus of the incenter of triangle ABC.

Problem 0.20: IMC 2014 P3

Let n be a positive integer. Show that there are positive real numbers $a_0, a_1, \ldots, a_n$ such that for each choice of signs the polynomial

$$\pm a_n x^n \pm a_{n-1}x^{n-1} \pm \cdots \pm a_1 x \pm a_0$$

has n distinct real roots.

Problem 0.21: Dunno

Suppose that the real numbers $a_0, a_1, ..., a_n$ and x, with $0 < x < 1$, satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + ... + \frac{a_n}{1-x^{n+1}} = 0$$

Prove that there exist a real number y with $0 < y < 1$ such that

$$a_0 + a_1 y + a_2 y^2 + ... + a_n y^n = 0$$

Problem 0.22: RMM 2013 P6

A token is placed at each vertex of a regular $2n$-gon. A move consists in choosing an edge of the $2n$-gon and swapping the two tokens placed at the endpoints of that edge. After a finite number of moves have been performed, it turns out that every two tokens have been swapped exactly once. Prove that some edge has never been chosen.

Problem 0.23: Sharygin 2018 Final Round 8.4

Find all sets of six points in the plane, no three collinear, such that if we partition the set into two sets, then the obtained triangles are congruent.