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Contents

Chapter 1

Combinatorics

1.1 Problems

Problem 1.1.0.1: OC Chap2 P2 M

Arutyun and Amayak perform a magic trick as follows. A spectator writes down on a board a sequence of N (decimal) digits. Amayak covers two adjacent digits by a black disc. Then Arutyun comes and says both closed digits (and their order). For which minimal N can this trick always work? NOTE: Arutyun and Amayak have a strategy determined beforehand.

Idea We have to actually find a bijection between all of the combinations the spectator can create, and all of the combinations that Arutyun might see when he comes back. Which tells us to use "Perfect Matching" tricks. □

Idea Existential proof: for this trick to always work, they have to make a bijection from a set of N digits with two covered, to an unique set of N digits. Consider a bijection from the set of $0-9$ strings with length N to the set of $0-9$ strings with length N with 2 adjacent digits unknown. There exist a bijection iff the two sets satisfy Hall's Marriage Theorem. By double counting we get the value of N from here. □

Problem 1.1.0.2: Polish OI E

Given n jobs, indexed from $1, 2 \ldots n$. Given two sequences of reals, $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n$ where, $0 \leq a_i, b_i \leq 1$. If job i starts at time t , then the job takes $h_i(t) = a_i t + b_i$ time to finish. Order the jobs in a way such that the total time taken by all of the jobs is the minimum.

Idea Example of a problem which is solved by investigating two adjacent objects in the optimal arrangement. □

Problem 1.1.0.3: CodeForces 960/C E

Pikachu had an array with him. He wrote down all the non-empty subsequences of the array on paper. Note that an array of size n has $2^n - 1$ non-empty subsequences in it.

Pikachu being mischievous as he always is, removed all the subsequences in which

$$\text{Maximum element of the subsequence} - \text{Minimum element of subsequence} \geq d$$

Pikachu was finally left with X subsequences.

However, he lost the initial array he had, and now is in serious trouble. He still remembers the numbers X and d. He now wants you to construct any such array which will satisfy the above conditions. All the numbers in the final array should be positive integers less than 10^{18}.

Note the number of elements in the output array should not be more than 10^4. If no answer is possible, print -1.

Problem 1.1.0.4: ARO 2005 P10.3, P11.2 M

Given 2005 distinct numbers $a_1, a_2, \ldots, a_{2005}$. By one question, we may take three different indices $1 \leq i < j < k \leq 2005$ and find out the set of numbers $\{a_i, a_j, a_k\}$ (unordered, of course). Find the minimal number of questions, which are necessary to find out all numbers a_i.

Idea The key idea is to ask questions such that it is connected to multiple other questions, and each question uniquely finds out multiple elements together. One by itself immediately after the question has been asked, and one after the next question which is related to this one has been asked. As we find out three elements' values after one question, first, second, third, so, let us find first from the previous question, second from the current question, third from the next question. □

Problem 1.1.0.5: ARO 1993 P10.4 M

Thirty people sit at a round table. Each of them is either smart or dumb. Each of them is asked: "Is your neighbor to the right smart or dumb?" A smart person always answers correctly, while a dumb person can answer both correctly and incorrectly. It is known that the number of dumb people does not exceed F. What is the largest possible value of F such that knowing what the answers of the people are, you can point at at least one person, knowing he is smart?

Generalization 1.1.0.5.1: ARO 1993 P10.4 generalization

There are n people sitting in a circle, of which some are truthful and others are liars (we don't know who is a liar and who isn't though). Each person states whether the person to in front of him is a liar or not. The truthful people always tell the truth, whereas the liars may either lie or tell the truth. The aim is for us to use the information provided to find one person who is definitely truthful. Show that if the number of liars is at most $\lceil 2\sqrt{n} - 3 \rceil$, we can always do this.

Idea We see that the strings of truth only exist either when all people are dumb or the last one is the truthful one. Now we take the longest such string, and this sting has to be of the second kind. To prove this, we use bounding with the given constraint. □

Problem 1.1.0.6: ARO 2005 P9.4 M

100 people from 50 countries, two from each countries, stay on a circle. Prove that one may partition them onto 2 groups in such way that neither no two countrymen, nor three consecutive people on a circle, are in the same group.

Variant: There are 100 people from 25 countries sitting around a circular table. Prove that they can be separated into four classes, so that no two countrymen are in the same class, nor any two people sitting adjacent in the circle.

Problem 1.1.0.7: IOI 2007 P3 M

You are given two sets of integers $A = \{a_1, a_2 \dots a_n\}$ and $B = \{b_1, b_2 \dots b_n\}$ such that $a_i \geq b_i$. At move i you have to pick b_i distinct integerds from the set $A_i = \{1, 2, \dots a_i\}$. In total, $(b_1 + b_2 \dots b_n)$ integers are selected, but not all of these are distinct. Suppose k distinct integers have been selected, with multiplicities $c_1, c_2, c_3 \dots c_k$. Your score is defined as

$$\sum_{i=1}^{k} c_i(c_i - 1)$$

Give an efficient algorithm to select numbers in order to "minimize" your score.

Idea Some investigation shows that if $c_i > c_j + 1$ and $i > j$, then we can always minimize the score. and if $i < j$, then we can minimize the score only when $i, j \in A_k$ but i has been taken at move k , but j hasn't. So in the minimal state, either both i, j has been taken at move k , or $a_k < j$. So the idea is to take elements from A_i as large as possible, and then taking smaller values after wards if the c_i value of a big element gets more than that of a small element. In this algorithm, we see that we greedily manipulate c_i. So it is a good idea to greedily choose c_i 's from the very beginning. □

Idea **Solution Algo:** at step i , take the set $\{c_1, c_2 \dots c_{a_i}\}$ and take the smallest b_i from this set, and add 1 to each of them (in other words, take their index numbers as the numbers to take). □

Problem 1.1.0.8:

Given n numbers $\{a_1, a_2, ..., a_n\}$, you have to select k of them such that no two consecutive numbers are selected and their sum is maximized.

Idea Notice that if a_i is the maximum value, and if a_i is not counted in the optimal solution, then both of a_{i-1}, a_{i+1} must be in the optimal solution, and $a_{i-1} + a_{i+1} > a_i$. And if a_i is counted in the optimal solution, then none of a_{i-1}, a_{i+1} can be counted in the optimal solution. So either way, we can remove these three and replace them by a single element to use induction. So remove a_{i-1}, a_{i+1} and replace a_i by $a_{i-1} + a_{i+1} - a_i$. :3 □

Problem 1.1.0.9: ISL 2009 C1 E

Consider 2009 cards, each having one gold side and one black side, lying on parallel on a long table. Initially all cards show their gold sides. Two player, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those which showed gold now show black and vice versa. The last player who can make a legal move wins.

1. Does the game necessarily end?
2. Does there exist a winning strategy for the starting player?

Idea Always think about the simplest game at first. □

Problem 1.1.0.10: ISL 2009 C3 H

Let n be a positive integer. Given a sequence ε_1 , . . . , ε_{n-1} with $\varepsilon_i = 0$ or $\varepsilon_i = 1$ for each $i = 1$, . . . , $n-1$, the sequences a_0 , . . . , a_n and b_0 , . . . , b_n are constructed by the following rules:

$$a_0 = b_0 = 1, \quad a_1 = b_1 = 7,$$

$$a_{i+1} = \begin{cases} 2a_{i-1} + 3a_i, & \text{if } \varepsilon_i = 0, \\ 3a_{i-1} + a_i, & \text{if } \varepsilon_i = 1, \end{cases} \quad \text{for each } i = 1, \dots, n-1,$$

$$b_{i+1} = \begin{cases} 2b_{i-1} + 3b_i, & \text{if } \varepsilon_{n-i} = 0, \\ 3b_{i-1} + b_i, & \text{if } \varepsilon_{n-i} = 1, \end{cases} \quad \text{for each } i = 1, \dots, n-1.$$

Prove that $a_n = b_n$.

Idea Got the idea, will try later. □

Problem 1.1.0.11: ARO 2018 P11.5 E

On the table, there're 1000 cards arranged on a circle. On each card, a positive integer was written so that all 1000 numbers are distinct. First, Vasya selects one of the card, remove it from the circle, and do the following operation: If on the last card taken out was written positive integer k , count the k^{th} clockwise card not removed, from that position, then remove it and repeat the operation. This continues until only one card left on the table. Is it possible that, initially, there's a card A such that, no matter what other card Vasya selects as first card, the one that left is always card A ?

Problem 1.1.0.12: ARO 2017 P9.1 E

In country some cities are connected by oneway flights (There are no more then one flight between two cities). City A called "available" for city B , if there is flight from B to A , maybe with some transfers. It is known, that for every 2 cities P and Q exist city R , such that P and Q are available from R. Prove, that exist city A , such that every city is available for A.

Problem 1.1.0.13: ARO 2018 10.3 E

A positive integer k is given. Initially, N cells are marked on an infinite checkered plane. We say that the cross of a cell A is the set of all cells lying in the same row or in the same column as A. By a turn, it is allowed to mark an unmarked cell A if the cross of A contains at least k marked cells. It appears that every cell can be marked in a sequence of such turns. Determine the smallest possible value of N.

Idea First find the construction. □

Problem 1.1.0.14: ARO 2018 P9.5 E

On the circle, 99 points are marked, dividing this circle into 99 equal arcs. Petya and Vasya play the game, taking turns. Petya goes first; on his first move, he paints in red or blue any marked point. Then each player can paint on his own turn, in red or blue, any uncolored marked point adjacent to the already painted one. Vasya wins, if after painting all points there is an equilateral triangle, all three vertices's of which are colored in the same color. Could Petya prevent him?

Idea Think of what Petya must do to prevent immediate losing. □

Problem 1.1.0.15: ISL 2004 C2 E

Let n and k be positive integers. There are given n circles in the plane. Every two of them intersect at two distinct points, and all points of intersection they determine are pairwise distinct (i. e. no three circles have a common point). No three circles have a point in common. Each intersection point must be colored with one of n distinct colors so that each color is used at least once and exactly k distinct colors occur on each circle. Find all values of $n \geq 2$ and k for which such a coloring is possible.

Problem 1.1.0.16: ISL 2004 C3 E

The following operation is allowed on a finite graph: Choose an arbitrary cycle of length 4 (if there is any), choose an arbitrary edge in that cycle, and delete it from the graph. For a fixed integer $n \geq 4$, find the least number of edges of a graph that can be obtained by repeated applications of this operation from the complete graph on n vertices's (where each pair of vertices's are joined by an edge).

Idea Walk backwards. or the same thing with Bipartite Graphs. □

Problem 1.1.0.17: Iran TST 2012 P4 E

Consider $m+1$ horizontal and $n+1$ vertical lines ($m, n \geq 4$) in the plane forming an $m \times n$ table. Cosider a closed path on the segments of this table such that it does not intersect itself and also it passes through all $(m-1)(n-1)$ interior vertices's (each vertex is an intersection point of two lines) and it doesn't pass through any of outer vertices. Suppose A is the number of vertices's such that the path passes through them straight forward, B number of the table squares that only their two opposite sides are used in the path, and C number of the table squares that none of their sides is used in the path. Prove that $A = B - C + m + n - 1$.

Problem 1.1.0.18: AoPS E

Given $2n + 1$ irrational numbers, prove that one can pick n from them s.t. no two of the choosen n sum up to a rational number.

Idea Use a graph theory representation. □

Problem 1.1.0.19: Bulgarian IMO TST 2004, Day 3, Problem 3 H

Prove that among any $2n + 1$ irrational numbers there are $n + 1$ numbers such that the sum of any k of them is irrational, for all $k \in \{1, 2, 3, \ldots, n + 1\}$.

Idea We first create a set B such that any linear combination of the elements in it are irrational. Then for convenience, we add 1 to it, so that now the sum equals to 0 of any linear combinations. An algorithm for building it comes into our mind, which leaves some other original elements, which we then later add to the final solution set A along with the elements in the set B except 1. □

Problem 1.1.0.20: ISL 1997 P4 E

An $n \times n$ matrix whose entries come from the set $S = \{1, 2, ..., 2n-1\}$ is called a "silver matrix" if, for each $i = 1, 2, ..., n$, the i -th row and the i -th column together contain all elements of S. Show that:

1 there is no silver matrix for $n = 1997$;
2 silver matrices exist for infinitely many values of n.

Idea Proving that for odd n 's isn't hard. Then A small try-around with $n = 2, 4$, we see a pattern that leads to a construction for 2^n □

Problem 1.1.0.21: E

A rectangle is completely partitioned into smaller rectangles such that each smaller rectangles has at least one integral side. Prove that the original rectangle also has at least one integral side.

Idea Try a special grid system with $.5 \times .5$ boxes. □

Idea Consider the number of corners in the rectangle. □

Problem 1.1.0.22: ISL 2004 C5 M

A and B play a game, given an integer N, A writes down 1 first, then every player sees the last number written and if it is n then in his turn he writes $n+1$ or $2n$, but his number cannot be bigger than N. The player who writes N wins. For which values of N does B win?

Idea Trying with smaller cases, it's easy. Using most important game theory trick. □

Problem 1.1.0.23: ISL 2006 C1 E

We have $n \geq 2$ lamps $L_1, L_2 \dots L_n$ in a row, each of them being either on or off. Every second we simultaneously modify the state of each lamp as follows: if the lamp L_i and its neighbors (only one neighbor for $i = 1$ or $i = n$, two neighbors for other i) are in the same state, then L_i is switched off; otherwise, L_i is switched on. Initially all the lamps are off except the leftmost one which is on.

1 Prove that there are infinitely many integers n for which all the lamps will eventually be off.
2 Prove that there are infinitely many integers n for which the lamps will never be all off

Problem 1.1.0.24: ISL 2006 C4 M

A cake has the form of an $n \times n$ square composed of n^2 unit squares. Strawberries lie on some of the unit squares so that each row or column contains exactly one strawberry; call this arrangement $\mathbb{A}$.
Let $\mathbb{B}$ be another such arrangement. Suppose that every grid rectangle with one vertex at the top left corner of the cake contains no fewer strawberries of arrangement $\mathbb{B}$ than of arrangement $\mathbb{A}$. Prove that arrangement $\mathbb{B}$ can be obtained from $\mathbb{A}$ by performing a number of switches, defined as follows:
A switch consists in selecting a grid rectangle with only two strawberries, situated at its top right corner and bottom left corner, and moving these two strawberries to the other two corners of that rectangle.

Idea When the first approach fails, don't throw that idea yet. Stick to it, as it is most probably the closest to a correct solution. Taking the smallest rectangle with 0 's equal to 1 's, we see that we can 'shrink' the rectangle. Which leads to a solution instantly. □

Problem 1.1.0.25: ISL 2014 C2 E

We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b , then we erase these numbers and write the number $a+b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m.

Idea When you know that the problem can be solved using invariants, go through all of the possible invariants (from the rules of thumb). Don't give up on one so quickly. And product and sum are actually more close than you think. Because if you are told to prove some bound on the sum, then product can come very handy. After all there is AM-GM to connect sum and product. □

Problem 1.1.0.26: ISL 2016 C3 E

Let n be a positive integer relatively prime to 6. We paint the vertices's of a regular n -gon with three colours so that there is an odd number of vertices's of each colour. Show that there exists an isosceles triangle whose three vertices's are of different colours.

Idea Double Count with the number of points of each colors. □

Problem 1.1.0.27: Iran TST 2002 P3 E

A "2-line" is the area between two parallel lines. Length of "2-line" is distance of two parallel lines. We have covered unit circle with some "2-lines". Prove sum of lengths of "2-lines" is at least 2.

Idea Consider the "2-line" of the largest length. □

Problem 1.1.0.28: ARO 2008 P9.5 E

The distance between two cells of an infinite chessboard is defined as the minimum number to moves needed for a king to move from one to the other. On the board are chosen three cells on pairwise distances equal to 100. How many cells are there that are at the distance 50 from each of the three cells?

Problem 1.1.0.29: USAMO 1986 P2 E

During a certain lecture, each of five mathematicians fell asleep exactly twice. For each pair of mathematicians, there was some moment when both were asleep simultaneously. Prove that, at some moment, three of them were sleeping simultaneously.

Problem 1.1.0.30: Mexican Regional 2014 P6 E

Let $A = n \times n$ be a $\{0, 1\}$ matrix, where each row is different. Prove that you can remove a column such that the resulting $n \times (n-1)$ matrix has n different rows.

Idea Try to represent the sets in a nicer way, with graph. or. Induction on the number of columns deleted and the number or different rows being there. □

Problem 1.1.0.31: IMO 2017 P5 M (H)

An integer $N \geq 2$ is given. A collection of $N(N+1)$ soccer players, no two of whom are of the same height, stand in a row. Show that Sir Alex can always remove $N(N-1)$ players from this row leaving a new row of $2N$ players in which the following N conditions hold:
(1) no one stands between the two tallest players,
(2) no one stands between the third and fourth tallest players,
$\vdots$
(N) no one stands between the two shortest players.

Idea $N(N+1)$, rows, removing ... these things just begs for to be arranged in a systematic order. As arranging thing in a matrix is the simplest way, we arrange the bad-bois in a $N \cdot (N+1)$ matrix. Now finding the algorithm is not very hard. □

Problem 1.1.0.32: ISL 1990 P3 E

Let $n \geq 3$ and consider a set E of $2n-1$ distinct points on a circle. Suppose that exactly k of these points are to be colored black. Such a coloring is good if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly n points from E. Find the smallest value of k so that every such coloring of k points of E is good.

Idea Creating a graph and using Alternating Chains Technique □

Problem 1.1.0.33: USAMO 1999 P1 E

Some checkers placed on an $n \times n$ checkerboard satisfy the following conditions:

1 every square that does not contain a checker shares a side with one that does;
2 given any pair of squares that contain checkers, there is a sequence of squares containing checkers, starting and ending with the given squares, such that every two consecutive squares of the sequence share a side.

Prove that at least $(n^2 - 2)/3$ checkers have been placed on the board.

Idea As the problem simply seems to exist, we can't count how much contribution a checker cntaining square contributes to the whole board. So we place one at a time and see the changes. □

Generalization 1.1.0.33.1: USAMO 1999 P1 generalization

Find the smallest positive integer m such that if m squares of an $n \times n$ board are colored, then there will exist 3 colored squares whose centers form a right triangle with sides parallel to the edges of the board.

Problem 1.1.0.34: ISL 2013 C1 E

Let n be an positive integer. Find the smallest integer k with the following property; Given any real numbers $a_1, \cdots, a_d$ such that $a_1 + a_2 + \cdots + a_d = n$ and $0 \leq a_i \leq 1$ for $i = 1, 2, \cdots, d$, it is possible to partition these numbers into k groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.

Idea Think about the worst case where d is the minimum and the ans is d , it would only be possible if each $a_i > \frac{1}{2}$ but this can't be true, so, the ans is $2n - 1$. Now the ques should become obvious. □

Problem 1.1.0.35: Brazilian Olympic Revenge 2014 M

Let n a positive integer. In a $2n \times 2n$ board, $1 \times n$ and $n \times 1$ pieces are arranged without overlap. Call an arrangement maximal if it is impossible to put a new piece in the board without overlapping the previous ones. Find the least k such that there is a maximal arrangement that uses k pieces.

Idea Intuition gives that there is at least one n -mino in each row. But we can easily guess that there is no maximal arrangement with $2n$ minos. Suppose in a maximal arrangement, there are no vertical n -mino, that means there are more than $2n + 1$ n-minos. So suppose that there is at least one vertical suppose that it lies in a column i between 1 and n. Then we have that there is at least one n -mino in each column in between 1 and i. If there is one in between 1 and $2n$, say j , then there is one in each of the columns on the right side of it. Then we count horizontal n -minos, we show that $2n + 1$ is the answer. □

Problem 1.1.0.36: ISL 2008 C1 E

In the plane we consider rectangles whose sides are parallel to the coordinate axes and have positive length. Such a rectangle will be called a box. Two boxes intersect if they have a common point in their interior or on their boundary. Find the largest n for which there exist n boxes $B_1, B_2 \dots B_n$ such that B_i and B_j intersect if and only if $i \not\equiv j \pm 1 \pmod{n}$.

Idea Instead of focusing on building the boxes from only one side (i.e. starting with $1, 2 \dots$, we should include n in our investigation, and follow from both direction, (i.e. $1, 2 \dots$ and $\dots, n - 1, n$). □

Problem 1.1.0.37: USAMO 2008 P4 E

Let $\mathcal{P}$ be a convex polygon with n sides, $n \geq 3$. Any set of $n - 3$ diagonals of $\mathcal{P}$ that do not intersect in the interior of the polygon determine a triangulation of $\mathcal{P}$ into $n - 2$ triangles. If $\mathcal{P}$ is regular and there is a triangulation of $\mathcal{P}$ consisting of only isosceles triangles, find all the possible values of n.

Idea It's not hard after getting the ans. ☐

Problem 1.1.0.38: ARO 2016 P3 M

We have a sheet of paper, divided into 100×100 unit squares. In some squares, we put right-angled isosceles triangles with $leg = 1$ (Every triangle lies in one unit square and is half of this square). Every unit grid segment (boundary too) is under one leg of a triangle. Find maximal number of unit squares, that don't contains any triangles.

Idea What is the minimum number of triangles you can use in a row? Create a good row one at a time ☐

Problem 1.1.0.39: India TST 2013 Test 3, P1 E

For a positive integer n , a *Sum-Friendly Odd Partition* of n is a sequence $(a_1, a_2 \dots a_k)$ of odd positive integers with $a_1 \leq a_2 \leq \cdots \leq a_k$ and $a_1 + a_2 + \cdots + a_k = n$ such that for all positive integers $m \leq n$, m can be uniquely written as a subsum $m = a_{i_1} + a_{i_2} + \cdots + a_{i_r}$. (Two subsums $a_{i_1} + a_{i_2} + \cdots + a_{i_r}$ and $a_{j_1} + a_{j_2} + \cdots + a_{j_s}$ with $i_1 < i_2 < \cdots < i_r$ and $j_1 < j_2 < \cdots < j_s$ are considered the same if $r = s$ and $a_{i_l} = a_{j_l}$ for $1 \leq l \leq r$.) For example, $(1, 1, 3, 3)$ is a *sum-friendly odd partition* of 8. Find the number of sum-friendly odd partitions of 9999.

Idea Firstly we explore one SFOP at a time. Which gives us a way to tell what a_{i+1} is going to be by looking at $a_1 \dots a_i$. ☐

Problem 1.1.0.40: IMO 2011 P2 H

Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A windmill is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q , takes over as the new pivot, and the line now rotates clockwise about Q , until it next meets a point of $\mathcal{S}$. This process continues indefinitely.
Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

Idea Some workaround gives us the idea that the starting line has to be kinda "in between" the points. Formal words could be: the line should divide the set of points in two sets so that the two sets have equal number of points. Once we take a such line, we see that after every move we get a new line which has similar properties of the first line. ☐

Idea So moral of the story is that if you get some vague idea that something has to satisfy something-ish, remove the -ish part, and try with a formal assumption. ☐

Problem 1.1.0.41: IOI 2016 P5 M

A computer bug has a permutation P of length $2^k = N$ that changes any string added to a DS according to the permutation, i.e. it makes $S[i] = S[P[i]]$. Your task it to find the permutation in the following ways:

1 You can add at most $n \log_2 n$ N bit binary strings to the DS.
2 You can ask at most $n \log_2 n$, in the form of N bit binary strings. The answer will be "true" if the string exists in the DS after the Bug had changed the strings and "no" otherwise.

Idea Typical Divide and Conquer approach. You want to do the same thing for $N = \frac{N}{2}$, and to do so you need to tell exactly what the first $\frac{N}{2}$ terms of the permutation are. To do this, you can use at most N questions. This is easy, you first add strings with only one bit present in the first $\frac{N}{2}$ positions, and then ask N questions with only one bit in every N positions. This maps the first $\frac{N}{2}$ numbers of the permutation to a set of $\frac{N}{2}$ integers. And we can proceed by induction now. $\square$

Problem 1.1.0.42: ISL 2001 C6 M

For a positive integer n define a sequence of zeros and ones to be balanced if it contains n zeros and n ones. Two balanced sequences a and b are neighbors if you can move one of the $2n$ symbols of a to another position to form b. For instance, when $n = 4$, the balanced sequences 01101001 and 00110101 are neighbors because the third (or fourth) zero in the first sequence can be moved to the first or second position to form the second sequence. Prove that there is a set S of at most $\frac{1}{n+1}\binom{2n}{n}$ balanced sequences such that every balanced sequence is equal to or is a neighbor of at least one sequence in S.

Problem 1.1.0.43: ISL 1998 C4 M

Let $U = \{1, 2, \ldots, n\}$, where $n \geq 3$. A subset S of U is said to be split by an arrangement of the elements of U if an element not in S occurs in the arrangement somewhere between two elements of S. For example, 13542 splits $\{1, 2, 3\}$ but not $\{3, 4, 5\}$. Prove that for any $n - 2$ subsets of U, each containing at least 2 and at most $n - 1$ elements, there is an arrangement of the elements of U which splits all of them.

Idea If we try to apply induction, we see that the sets with 2 and $n - 1$ elements create problems, so we handle them first. $\square$

Problem 1.1.0.44: USA TST 2009 P1 M

Let m and n be positive integers. Mr. Fat has a set S containing every rectangular tile with integer side lengths and area of a power of 2. Mr. Fat also has a rectangle R with dimensions $2^m \times 2^n$ and a 1×1 square removed from one of the corners. Mr. Fat wants to choose $m + n$ rectangles from S, with respective areas $2^0, 2^1, \ldots, 2^{m+n-1}$, and then tile R with the chosen rectangles. Prove that this can be done in at most $(m + n)!$ ways.

Idea The fact that this can be done in $(m + n)!$ asks for a bijective proof. Now an intuition gives us that we have to sort the tiles wrt the missing square in some way. Now since the numbers $\square$

Problem 1.1.0.45: ARO 2016 P1 E

There are 30 teams in **NBA** and every team play 82 games in the year. Bosses of **NBA** want to divide all teams on Western and Eastern Conferences (not necessarily equally), such that the number of games between teams from different conferences is half of the number of all games. Can they do it?

Idea You want to divide something. Check the parity. □

Problem 1.1.0.46: AoPS M

Each edge of a polyhedron is oriented with an arrow such that at each vertex, there is at least one arrow leaving the vertex and at least one arrow entering the vertex. Prove that there exists a face on the polyhedron such that the edges on its boundary form a directed cycle.

Idea The trick which is used to prove Euler's Polyhedron theorem. □

Problem 1.1.0.47: ISL 2014 C3 M

Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is peaceful if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 unit squares.

Idea Guessing the "Correct" ans is the challenge, think of the worst case you can produce. □

Problem 1.1.0.48: APMO 2012 P2 E

Into each box of a $n \times n$ square grid, a real number greater than or equal to 0 and less than or equal to 1 is inserted. Consider splitting the grid into 2 non-empty rectangles consisting of boxes of the grid by drawing a line parallel either to the horizontal or the vertical side of the grid. Suppose that for at least one of the resulting rectangles the sum of the numbers in the boxes within the rectangle is less than or equal to 1, no matter how the grid is split into 2 such rectangles. Determine the maximum possible value for the sum of all the $n \times n$ numbers inserted into the boxes. Find the ans for k-dimension grids too.

Idea As the maximal rectangle defines other smaller rectangles in it, we take that. □

Problem 1.1.0.49: Indian Postal Coaching 2011 M

Consider 2011^2 points arranged in the form of a 2011×2011 grid. What is the maximum number of points that can be chosen among them so that no four of them form the vertices's of either an isosceles trapezium or a rectangle whose parallel sides are parallel to the grid lines?

Idea Since we need to maintain the relation of perpendicular bisectors, we focus on perp bisectors and the points on one line only and then count. □

Problem 1.1.0.50: ISL 2010 C2 M

On some planet, there are 2^N countries $(N \geq 4)$. Each country has a flag N units wide and one unit high composed of N fields of size 1×1, each field being either yellow or blue. No two countries have the same flag. We say that a set of N flags is diverse if these flags can be arranged into an $N \times N$ square so that all N fields on its main diagonal will have the same color. Determine the smallest positive integer M such that among any M distinct flags, there exist N flags forming a diverse set.

Idea Using induction we see that if we have found the value of M for $N-1$, then possibly the value for M_N is twice as large than M_{N-1}. With some further calculation, we see that if we have $2*M_{N-1}-1 = M_N$, then we can pick half of them and apply induction and still be left with a 'lot' of flags to choose the Nth element of the diverse set.

After that the only work left is to proof for $N = 4$. Which is easy casework. □

Idea Another way to prove the ans, is to prove the bound for any non-diverse set. In this case, we use hall's marriage to prove the contradiction. □

Problem 1.1.0.51: Iran TST 2007 P2 E

Let A be the largest subset of $\{1, \ldots, n\}$ such that for each $x \in A$, x divides at most one other element in A. Prove that

$$\frac{2n}{3} \leq |A| \leq \left\lceil \frac{3n}{4} \right\rceil .$$

Idea Partition the set optimally. □

Problem 1.1.0.52: India IMO Camp 2017 H

Find all positive integers n s.t. the set $\{1, 2, \ldots, 3n\}$ can be partitioned into n triplets (a_i, b_i, c_i) such that $a_i + b_i = c_i$ for all $1 \leq i \leq n$.

Problem 1.1.0.53: ISL 2012 C2 TE

Let $n \geq 1$ be an integer. What is the maximum number of disjoint pairs of elements of the set $\{1, 2, \ldots, n\}$ such that the sums of the different pairs are different integers not exceeding n ?

Idea As Usual, first find the ans. Using double counting is quite natural. Working with small cases easily gives a construction. □

Problem 1.1.0.54: CodeForces 989C E

Problem 1.1.0.55: CodeForces 989B E

Problem 1.1.0.56: ISL 2011 A5 MH

Prove that for every positive integer n , the set $\{2, 3, \ldots, 3n + 1\}$ can be partitioned into n triples in such a way that the numbers from each triple are the lengths of the sides of some obtuse triangle.

Idea What is the best way to choose the side lengths of an obtuse triangle? Obviously by maintaining some strict rules to get the third side from the first two sides and making the rules invariant. One way of doing this is to take $(a, b, a + b - 1)$.

After that, some (literally this is the hardest part of the problem) experiment to find a construction. First, we try to partition the set into tuples of our desired form, but we soon realize that that can't be done so easily. So we try a little bit of different approach and make one tuple different from the others. Luckily this approach gives us a nice construction. □

Problem 1.1.0.57: Iran TST 2017 D1P1 TE

In the country of Sugarland, there are 13 students in the IMO team selection camp. 6 team selection tests were taken and the results have came out. Assume that no students have the same score on the same test. To select the IMO team, the national committee of math Olympiad have decided to choose a permutation of these 6 tests and starting from the first test, the person with the highest score between the remaining students will become a member of the team. The committee is having a session to choose the permutation.
Is it possible that all 13 students have a chance of being a team member?

Idea If a student is in x^{th} place in a test t_y , and he has a chance to get into the team iff the $1^th, 2^th \ldots x - 1^th$ persons in test t_y are already in the team. So $x \leq 5$. Make a $6 \cdot 6$ grid with place $\cdot$ test. WHY?? Because it makes the best sense among other possible choices of the grid. A little bit of work produces a configuration where every student has a chance to get into the team. □

Problem 1.1.0.58: ISL 2009 C2 M

For any integer $n \geq 2$, let $N(n)$ be the maximum number of triples (a_i, b_i, c_i) , $i = 1, 2 \ldots, N(n)$, consisting of nonnegative integers a_i , b_i and c_i such that the following two conditions are satisfied:

1 $a_i + b_i + c_i = n$ for all $i = 1, \ldots, N(n)$,
2 If $i \neq j$ then $a_i \neq a_j$, $b_i \neq b_j$ and $c_i \neq c_j$

Determine $N(n)$ for all $n \geq 2$.

Idea Find an upper bound. It's easy. Then with some experiment, we see that this upper bound is achievable. So our next task is to find a construction. As it is related to 3 , we first try with $n = 3k$. Some experiment and experience gives us a construction. □

Problem 1.1.0.59: M

Let n be an integer. What is the maximum number of disjoint pairs of elements of the set $\{1, 2, \ldots, n\}$ such that the sums of the different pairs are different integers not exceeding n ?

Problem 1.1.0.60: ISL 2002 C6 H

Let n be an even positive integer. Show that there is a permutation $(x_1, x_2 \ldots x_n)$ of $(1, 2 \ldots n)$ such that for every $1 \leq i \leq n$, the number x_{i+1} is one of the numbers $2x_i, 2x_i - 1, 2x_i - n, 2x_i - n - 1$. Hereby, we use the cyclic subscript convention, so that x_{n+1} means x_1.

Some experiments show that our graph has more than 2 incoming and outgoing degree in all vertexes expect the first and last vertexes. So our lemma won't work yet. To make use of our lemma we take a graph with half of the vertexes of our original graph and make each vertex v_{2k} represent two integers: $(2k-1, 2k)$. Simple argument shows that this graph has an Euler Circuit, and surprisingly this itself is sufficient, as we can follow this circuit to get every integers in the interval $[1, n]$.

Problem 1.1.0.61: USA TST 2017 P1 E

In a sports league, each team uses a set of at most t signature colors. A set S of teams is color-identifiable if one can assign each team in S one of their signature colors, such that no team in S is assigned any signature color of a different team in S.

For all positive integers n and t, determine the maximum integer $g(n, t)$ such that: In any sports league with exactly n distinct colors present over all teams, one can always find a color-identifiable set of size at least $g(n, t)$.

Idea First, guess the answer, then try taking the minimal set. □

Problem 1.1.0.62: Putnam 2017 A4 E

$2N$ students take a quiz in which the possible scores are $0, 1 \ldots 10$. It is given that each of these scores appeared at least once, and the average of their scores is 7.4. Prove that the students can be divided into two sets of N student with both sets having an average score of 7.4.

Idea We take a set $S_1 = \{0, 1 \ldots 10\}$. Basically we have to partition the set of $2N$ into two equal sets with equal sum. So we pair S , and other leftovers and see what happens. □

Problem 1.1.0.63: ISL 2005 C3 MH

Consider a $m \times n$ rectangular board consisting of mn unit squares. Two of its unit squares are called adjacent if they have a common edge, and a path is a sequence of unit squares in which any two consecutive squares are adjacent. Two paths are called non-intersecting if they don't share any common squares.

Each unit square of the rectangular board can be colored black or white. We speak of a coloring of the board if all its mn unit squares are colored.

Let N be the number of colorings of the board such that there exists at least one black path from the left edge of the board to its right edge. Let M be the number of colorings of the board for which there exist at least two non-intersecting black paths from the left edge of the board to its right edge.

Prove that $N^2 \geq M \times 2^{mn}$.

Idea Bijective relation problem, the condition has $\times$, means we find a combinatorial model for the R.H.S. which is a pair of boards satisfying conditions. We want to show a surjection from this model to the model on the L.H.S. □

Problem 1.1.0.64: Result by Erdos MH

Given two *different* sequence of integers $(a_1, a_2 \dots a_n), (b_1, b_2, \dots b_n)$ such that two $\frac{n(n-1)}{2}$-tuples

$$a_1 + a_2, a_1 + a_3 \dots a_{n-1}a_n \text{ and } b_1 + b_2, b_1 + b_3 \dots b_{n-1}b_n$$

are equal upto permutation. Prove that $n = 2^k$ for some k.

Problem 1.1.0.65: A reformulation of Catalan's Numbers MH

Let $n \geq 3$ students all have different heights. In how many ways can they be arranged such that the heights of any three of them are not from left to right in the order: medium, tall, short?

Idea The proof uses derivatives to construct a polynomial similar to a **Maclaurin Series**. □

Problem 1.1.0.66: E

There are n cubic polynomials with three distinct real roots each. Call them $P_1(x), P_2(x), \dots, P_n(x)$. Furthermore for any two polynomials P_i, P_j, $P_i(x)P_j(x) = 0$ has exactly 5 distinct real roots. Let S be the set of roots of the equation

$$P_1(x)P_2(x) \dots P_n(x) = 0$$

. Prove that

1 If for each a, b there is exactly one $i \in \{1, \dots n\}$ such that $P_i(a) = P_i(b) = 0$, then $n = 7$.
2 If $n > 7$, $|S| = 2n + 1$.

Problem 1.1.0.67: Serbia TST 2017 P2 E

Initially a pair (x, y) is written on the board, such that exactly one of it's coordinates is odd. On such a pair we perform an operation to get pair $(\frac{x}{2}, y + \frac{x}{2})$ if $2|x$ and $(x + \frac{y}{2}, \frac{y}{2})$ if $2|y$. Prove that for every odd $n > 1$ there is a even positive integer $b < n$ such that starting from the pair (n, b) we will get the pair (b, n) after finitely many operations.

Idea Finding a construction through investigation and realizing that the infos and operations on x only defines the changes are enough for this problem. □

Problem 1.1.0.68: Serbia TST 2017 P4 E

We have an $n \times n$ square divided into unit squares. Each side of unit square is called unit segment. Some isosceles right triangles of hypotenuse 2 are put on the square so all their vertices's are also vertices's of unit squares. For which n it is possible that every unit segment belongs to exactly one triangle (unit segment belongs to a triangle even if it's on the border of the triangle)?

Idea Finding n is even, seeing 4 fails... $\square$

Problem 1.1.0.69: China MO 2018 P2 M

Let n and k be positive integers and let

$$T = \{(x, y, z) \in \mathbb{N}^3 \mid 1 \leq x, y, z \leq n\}$$

be the length n lattice cube. Suppose that $3n^2 - 3n + 1 + k$ points of T are colored red such that if P and Q are red points and PQ is parallel to one of the coordinate axes, then the whole line segment PQ consists of only red points.

Prove that there exists at least k unit cubes of length 1, all of whose vertices's are colored red.

Idea The inductive solution is tedious, and since we have to count the number of "good" boxes, we can try double counting. Explicitly counting all the "good" boxes. $\square$

Problem 1.1.0.70: China MO 2018 P5 MH

Let $n \geq 3$ be an odd number and suppose that each square in a $n \times n$ chessboard is colored either black or white. Two squares are considered adjacent if they are of the same color and share a common vertex and two squares a, b are considered connected if there exists a sequence of squares $c_1, \ldots, c_k$ with $c_1 = a, c_k = b$ such that c_i, c_{i+1} are adjacent for $i = 1, 2, \ldots, k-1$.

Find the maximal number M such that there exists a coloring admitting M pairwise disconnected squares.

Idea It's not hard to get the ans, now that the answer is guesses, and we have tried to prove with induction and couldn't find anything good, we try double counting. We notice that all the connected components in the $n \times n$ are planar graphs. Now we use Euler's theorem on Planar Graphs to find a value of M wrt to other values, and we double count the other values. $\square$

Problem 1.1.0.71: USAMO 2006 P2 E

For a given positive integer k find, in terms of k, the minimum value of N for which there is a set of $2k + 1$ distinct positive integers that has sum greater than N but every subset of size k has sum at most $\frac{N}{2}$.

Idea The best or simple looking set is the set of consecutive integers. So if there are some 'holes', we can fill them up to some extent, this opens two sub-cases. $\square$

Problem 1.1.0.72: USAMO 2005 P1 E

Determine all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.

Problem 1.1.0.73: USAMO 2005 P5 E

A mathematical frog jumps along the number line. The frog starts at 1, and jumps according to the following rule: if the frog is at integer n, then it can jump either to $n+1$ or to $n+2^{m_n+1}$ where 2^{m_n} is the largest power of 2 that is a factor of n. Show that if $k \geq 2$ is a positive integer and i is a nonnegative integer, then the minimum number of jumps needed to reach $2^i k$ is greater than the minimum number of jumps needed to reach 2^i.

Idea The main idea is to notice that the operation only uses powers of 2. And it depends on only the power of 2 in the integers, and in the sequence of 2-powers, the operation is very nice. □

Problem 1.1.0.74: ISL 1991 P10 E

Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2, \ldots, k$ in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is equal to 1.

Problem 1.1.0.75: E

A robot has n modes, and programmed as such: in mode i the robot will go at a speed of $i\text{ms}^{-1}$ for i seconds. At the beginning of its journey, you have to give it a permutation of $\{1, 2, \ldots n\}$. What is the maximum distance you can make the robot go?

Problem 1.1.0.76: E

A slight variation of the previous problem, in this case, the problem goes at a speed of $(n-1)\text{ms}^{-1}$ for i seconds in mode i.

Problem 1.1.0.77: E

m people each ordered n books but because Ittihad was the mailman, he messed up. Everyone got n books but not necessarily the one they wanted you need to fix this. To go to a house from another house it takes one hour. You can carry one book with you during any trip (at most one). You know who has which books and all books are different (i,e, $n * m$ different books). Prove that you can always finish the job in $m * (n + \frac{1}{2})$ hours

Idea Thinking about the penultimate step, when we have to go to a house empty handed. Thinking in this way gives us a way to pair the houses up, and since pairing... □

Idea Another way to do this is to convert it to a multi-graph. Now go to a house and return with a book means removing two edges from that vertex. We play around with it for sometime □

Problem 1.1.0.78: E

There are n campers in a camp and they will try to solve a IMO P6 but everyone has a confidence threshold (they will solve the problem by group solving). For example Laxem has threshold 5. I.e. if he's in the group, the group needs to contain at least 5 people (him included). A group is 'confident' when everyone of the team is confident. Now MM wants to make a list of possible "perfect confident" groups. I.e. groups that are confident but adding anyone else will destroy the confidence. How long can his list be?

Problem 1.1.0.79: timus 1862 ME

Problem 1.1.0.80: ARO 2014 P9.7 E

In a country, mathematicians chose an $\alpha > 2$ and issued coins in denominations of 1 ruble, as well as α^k rubles for each positive integer k. α was chosen so that the value of each coins, except the smallest, was irrational. Is it possible that any natural number of rubles can be formed with at most 6 of each denomination of coins?

Problem 1.1.0.81: Saint Petersburg 2001 MH

The number n is written on a board. A and B take turns, each turn consisting of replacing the number n on the board with $n-1$ or $\lfloor \frac{n+1}{2} \rfloor$. The player who writes the number 1 wins. Who has the winning strategy?

Idea Recursively building the losing positions. □

Problem 1.1.0.82: ARO 2011 P11.6 E

There are more than n^2 stones on the table. Peter and Vasya play a game, Peter starts. Each turn, a player can take any prime number less than n stones, or any multiple of n stones, or 1 stone. Prove that Peter always can take the last stone (regardless of Vasya's strategy).

Problem 1.1.0.83: ARO 2007 P9.7 E

Two players by turns draw diagonals in a regular $(2n+1)$-gon $(n > 1)$. It is forbidden to draw a diagonal, which was already drawn, or intersects an odd number of already drawn diagonals. The player, who has no legal move, loses. Who has a winning strategy?

Idea Turning the diagonals as vertices, and connection being intersections, we get a graph to play the game on. We then count the degrees. □

Problem 1.1.0.84: E

After tiling a 6×6 box with dominoes, prove that a line parallel to the sides of the box can be drawn that this line doesn't cut any dominoes.

Idea Double count how many lines "cut" a domino, and domino number. $\square$

Problem 1.1.0.85: E

There are 100 points on the plane. You have to cover them with discs, so that any two disks are at a distance of 1. Prove that you can do this in such a way that the total diameter of the disks is < 100.

Idea As the number 100 is very random, we suspect that is true for all values. So we can use induction $\square$

Problem 1.1.0.86: ARO 2014 P10.8 M

Given are n pairwise intersecting convex k-gons on the plane. Any of them can be transferred to any other by a homothety with a positive coefficient. Prove that there is a point in a plane belonging to at least $1 + \frac{n-1}{2k}$ of these k-gons.

Idea The most natural such point should be a vertex of a polygon. And these kinda problems use PHP more often, so we will have to divide by k somewhere. Again to find the polygon to use the PHP we will have to divide by n also. So we want to have nk in the denominator. We change the term to achieve this and Ta-Da! we get a fine term to work with. $\square$

Problem 1.1.0.87: IOI 2018 P1 E

Problem 1.1.0.88: German TST 2004 E7P3 M

We consider graphs with vertices colored black or white. "Switching" a vertex means: coloring it black if it was formerly white, and coloring it white if it was formerly black.

Consider a finite graph with all vertices colored white. Now, we can do the following operation: Switch a vertex and simultaneously switch all of its neighbours (i. e. all vertices connected to this vertex by an edge). Can we, just by performing this operation several times, obtain a graph with all vertices colored black?

Problem 1.1.0.89: ISL 2001 C3 E

Define a k-clique to be a set of k people such that every pair of them are acquainted with each other. At a certain party, every pair of 3-cliques has at least one person in common, and there are no 5-cliques. Prove that there are two or fewer people at the party whose departure leaves no 3-clique remaining.

Idea Casework with the point where most of the triangles are joined. □

Problem 1.1.0.90: USAMO 2004, P4 E

Alice and Bob play a game on a 6 by 6 grid. On his or her turn, a player chooses a rational number not yet appearing in the grid and writes it in an empty square of the grid. Alice goes first and then the players alternate. When all squares have numbers written in them, in each row, the square with the greatest number in that row is colored black. Alice wins if she can then draw a line from the top of the grid to the bottom of the grid that stays in black squares, and Bob wins if she can't. (If two squares share a vertex, Alice can draw a line from one to the other that stays in those two squares.) Find, with proof, a winning strategy for one of the players.

Problem 1.1.0.91: USAMO 2005 P1 E

Determine all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.

Problem 1.1.0.92: USAMO 2005 P4 E

Legs L_1, L_2, L_3, L_4 of a square table each have length n, where n is a positive integer. For how many ordered 4-tuples (k_1, k_2, k_3, k_4) of nonnegative integers can we cut a piece of length k_i from the end of leg L_i $(i = 1, 2, 3, 4)$ and still have a stable table?
(The table is stable if it can be placed so that all four of the leg ends touch the floor. Note that a cut leg of length 0 is permitted.)

Problem 1.1.0.93: USAMO 2006 P2 M

For a given positive integer k find, in terms of k, the minimum value of N for which there is a set of $2k+1$ distinct positive integers that has sum greater than N but every subset of size k has sum at most $\frac{N}{2}$.

Problem 1.1.0.94: USAMO 2006 P5 M

A mathematical frog jumps along the number line. The frog starts at 1, and jumps according to the following rule: if the frog is at integer n, then it can jump either to $n+1$ or to $n + 2^{m_n+1}$ where 2^{m_n} is the largest power of 2 that is a factor of n. Show that if $k \geq 2$ is a positive integer and i is a nonnegative integer, then the minimum number of jumps needed to reach $2^i k$ is greater than the minimum number of jumps needed to reach 2^i.

Problem 1.1.0.95: USAMO 2007 P2 E

A square grid on the Euclidean plane consists of all points (m, n), where m and n are integers. Is it possible to cover all grid points by an infinite family of discs with non-overlapping interiors if each disc in the family has radius at least 5?

Problem 1.1.0.96: USAMO 2009 P2 EM

Let n be a positive integer. Determine the size of the largest subset of $\{-n, -n+1, \dots, n-1, n\}$ which does not contain three elements a, b, c (not necessarily distinct) satisfying $a+b+c=0$.

Problem 1.1.0.97: USAMO 2010 P2 EM

There are n students standing in a circle, one behind the other. The students have heights $h_1 < h_2 < \cdots < h_n$. If a student with height h_k is standing directly behind a student with height h_{k-2} or less, the two students are permitted to switch places. Prove that it is not possible to make more than $\binom{n}{3}$ such switches before reaching a position in which no further switches are possible.

Problem 1.1.0.98: ARO 1999 P9.8 M

There are 2000 components in a circuit, every two of which were initially joined by a wire. The hooligans Vasya and Petya cut the wires one after another. Vasya, who starts, cuts one wire on his turn, while Petya cuts one or three. The hooligan who cuts the last wire from some component loses. Who has the winning strategy?

Problem 1.1.0.99: ARO 1999 P10.1 E

There are three empty jugs on a table. Winnie the pooh, Rabbit, and Piglet put walnuts in the jugs one by one. They play successively, with the initial determined by a draw. Thereby Winnie the pooh plays either in the first or second jug, Rabbit in the second or third, and Piglet in the first or third. The player after whose move there are exactly 1999 walnuts loses the games. Show that Winnie the pooh and Piglet can cooperate so as to make Rabbit lose.

Chapter 2

Algebra

Chapter 3

Geometry

3.1 Pending Problems

Problem 3.1.0.1:

In $\triangle ABC$, I is the incenter, D is the touch point of the incenter with BC. $AD \cap \odot ABC \equiv X$. The tangents line from X to $\odot I$ meet $\odot ABC$ at Y, Z. Prove that YZ, BC and the tangent at A to $\odot ABC$ concur.

Problem 3.1.0.2: IRAN TST 2017 Day 1, P3 M

In triangle ABC let I_a be the A-excenter. Let ω be an arbitrary circle that passes through A, I_a and intersects the extensions of sides AB, AC (extended from B, C) at X, Y respectively. Let S, T be points on segments I_aB, I_aC respectively such that $\angle AXI_a = \angle BTI_a$ and $\angle AYI_a = \angle CSI_a$. Lines BT, CS intersect at K. Lines KI_a, TS intersect at Z. Prove that X, Y, Z are collinear.

Problem 3.1.0.3: IRAN TST 2015 Day 3, P2

In triangle ABC(with incenter I) let the line parallel to BC from A intersect circumcircle of $\triangle ABC$ at A_1 let $AI \cap BC = D$ and E is tangency point of incircle with BC let $EA_1 \cap \odot(\triangle ADE) = T$ prove that $AI = TI$.

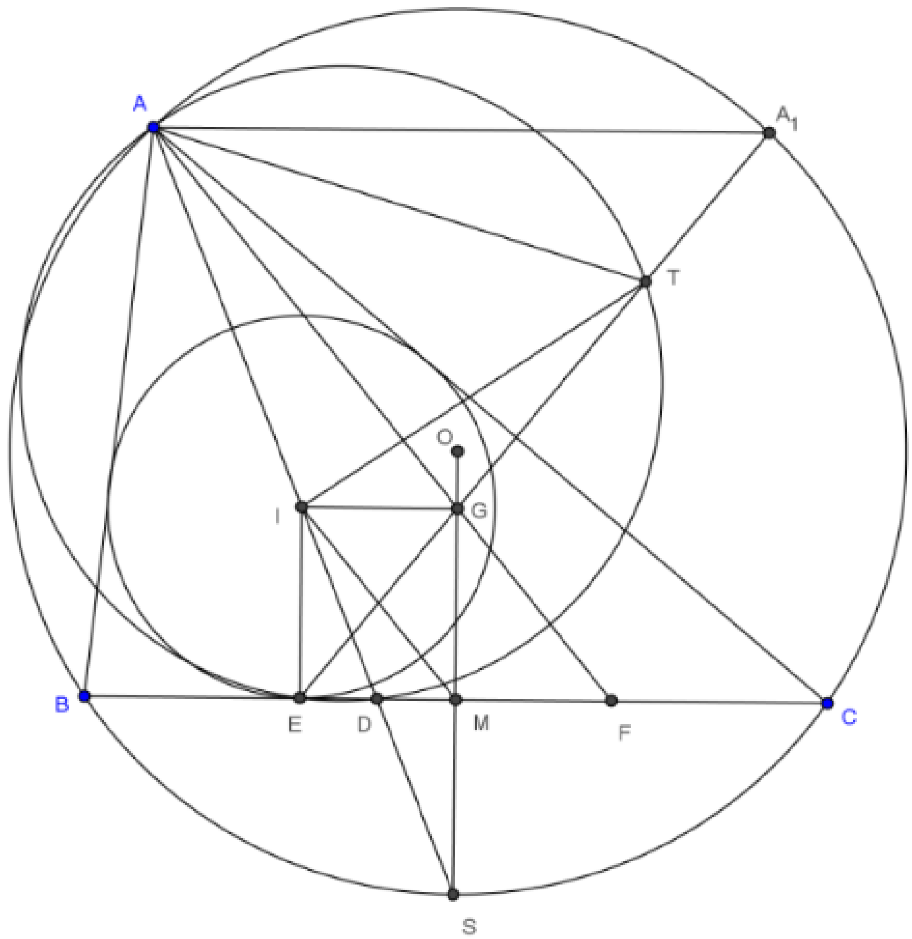

Figure 3.1: IRAN TST 2015 Day 3, P2

Problem 3.1.0.4: Generalization of Iran TST 2017 P5

Let ABC be triangle and the points P, Q lies on the side BC s.t B, C, P, Q are all different. The circumcircles of triangles ABP and ACQ intersect again at G. AG intersects BC at M. The circumcircle of triangle APQ intersects AB, AC again at E, F , respectively. EP and FQ intersect at T. The lines through M and parallel to AB, AC, intersect EP, FQ at X, Y, respectively. Prove that the circumcircles of triangle TXY and APQ are tangent to each other.

Problem 3.1.0.5: ARMO 2013 Grade 11 Day 2 P4

Let ω be the incircle of the triangle ABC and with center I. Let Γ be the circumcircle of the triangle AIB. Circles ω and Γ intersect at the point X and Y. Let Z be the intersection of the common tangents of the circles ω and Γ. Show that the circumcircle of the triangle XYZ is tangent to the circumcircle of the triangle ABC.

Problem 3.1.0.6: ISL 2016 G6

Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC < 90^\circ$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at E and F respectively, and meet each other at point P. Let M be the midpoint of AC and let ω be the circumcircle of triangle BPD. Segments BM and DM intersect ω again at X and Y respectively. Denote by Q the intersection point of lines XE and YF. Prove that $PQ \perp AC$.

Problem 3.1.0.7: AoPS

Let ABC be a triangle with incircle (I) and A-excircle (I_a). $(I), (I_a)$ are tangent to BC at D, P, respectively. Let $(I_1), (I_2)$ be the incircle of triangles APC, APB, respectively, $(J_1), (J_2)$ be the reflections of $(I_1), (I_2)$ wrt midpoints of AC, AB. Prove that AD is the radical axis of (J_1) and (J_2).

Problem 3.1.0.8: AoPS

Let ABC be a $A-$right-angled triangle and $MNPQ$ a square inscribed into it, with M, N onto BC in order $B - M - N - C$, and P, Q onto CA, AB respectively. Let $R = BP \cap QM, S = CQ \cap PN$. Prove that $AR = AS$ and RS is perpendicular to the $A-$inner angle bisector of $\triangle ABC$.

Problem 3.1.0.9: AoPS

P is an arbitrary point on the plane of $\triangle ABC$ and let $\triangle A'B'C'$ be the cevian triangle of P WRT $\triangle ABC$. The circles $\odot(ABB')$ and $\odot(ACC')$ meet at A, X. Similarly, define the points Y and Z WRT B and C. Prove that the lines AX, BY, CZ concur at the isogonal conjugate of the complement of P WRT $\triangle ABC$.

Problem 3.1.0.10: AoPS

Given are $\triangle ABC, L$ is Lemoine point,L_a, L_b, L_c are three Lemoine point of triangles LBC, LCA, LAB prove that AL_a, BL_b, CL_a are concurrent!
A question: What is the locus of point P such that AL_a, BL_b, CL_a are concurrent with L_a, L_b, L_c are three 'Lemoine points' of triangles PBC, PCA, PAB?

Problem 3.1.0.11: AoPS

Let ABC be a triangle inscribed circle (O). Let (O') be the circle wich is tangent to the circle (O) and the sides CA, AB at D and E, F, respectively. The line BC intersects the tangent line at A of (O), EF and AO' at T, S and L, respectively. The circle (O) intersects AS again at K. Prove that the circumcenter of triangle AKL lies on the circumcircle of triangle ADT.

Problem 3.1.0.12:

Let P and Q be isogonal conjugates of each other. Let $\triangle XYZ, \triangle KLM$ be the pedal triangles of P and Q wrt $\triangle ABC$. (X, K lie on BC; Y, L lie on CA; Z, M lie on AB) Prove that YM, ZL, PQ are concurrent.

Problem 3.1.0.13: 2nd Olympiad of Metropolises

Let $ABCDEF$ be a convex hexagon which has an inscribed circle and a circumscribed circle. Denote by ω_A, ω_B, ω_C, ω_D, ω_E, and ω_F the inscribed circles of the triangles FAB, ABC, BCD, CDE, DEF,and EFA,respectively. Let l_{AB} be the external common tangent of ω_A and ω_B other than the line AB; lines $l_{BC}, l_{CD}, l_{DE}, l_{EF}$, and l_{FA} are analogously defined. Let A_1 be the intersection point of the lines l_{FA} and l_{AB}; B_1 be the intersection point of the lines l_{AB} and l_{BC}; points C_1, D_1, E_1, and $F1$ are analogously defined. Suppose that $A_1B_1C_1D_1E_1F_1$ is a convex hexagon. Show that its diagonals A_1D_1, B_1E_1, and C_1F_1 meet at a single point.

3.2 Problems

Problem 3.2.0.1: IRAN 3rd Round 2016 P2 E

Let ABC be an arbitrary triangle. Let E, F be two points on AB, AC respectively such that their distance to the midpoint of BC is equal. Let P be the second intersection of the triangles ABC, AEF circumcircles . The tangents from E, F to the circumcircle of AEF intersect each other at K. Prove that : $\angle KPA = 90$

Problem 3.2.0.2: IRAN 2nd Round 2016 P6 E

Let ABC be a triangle and X be a point on its circumcircle. Q, P lie on a line BC such that $XQ \perp AC, XP \perp AB$. Let Y be the circumcenter of $\triangle XQP$. Prove that ABC is equilateral triangle if and if only Y moves on a circle when X varies on the circumcircle of ABC

Problem 3.2.0.3: AoPS E

Consider ABC with orthic triangle $A'B'C'$, let $AA' \cap B'C' = E$ and E' be reflection of E wrt BC. Let M be midpoint of BC and O be circumcenter of $E'B'C'$. Let M' be projection of O on BC and N be the intersection of a perpendicular to $B'C'$ through E with BC. Prove that $MM' = 1/4MN$.

Problem 3.2.0.4: IRAN 3rd Round 2010 D3, P5 M

In a triangle ABC, I is the incenter. D is the reflection of A to I. the incircle is tangent to BC at point E. DE cuts IG at P (G is centroid). M is the midpoint of BC. Prove that $AP||DM$ and $AP = 2DM$.

Problem 3.2.0.5: IRAN 3rd Round 2011 G5 M

Given triangle ABC, D is the foot of the external angle bisector of A, I its incenter and I_a its A-excenter. Perpendicular from I to DI_a intersects the circumcircle of triangle in A'. Define B' and C' similarly. Prove that AA', BB' and CC' are concurrent.

Problem 3.2.0.6: AoPS3 E

I is the incenter of ABC, $PI, QI \perp BC$, PA, QA intersect BC at DE. Prove: $IADE$ is on a circle.

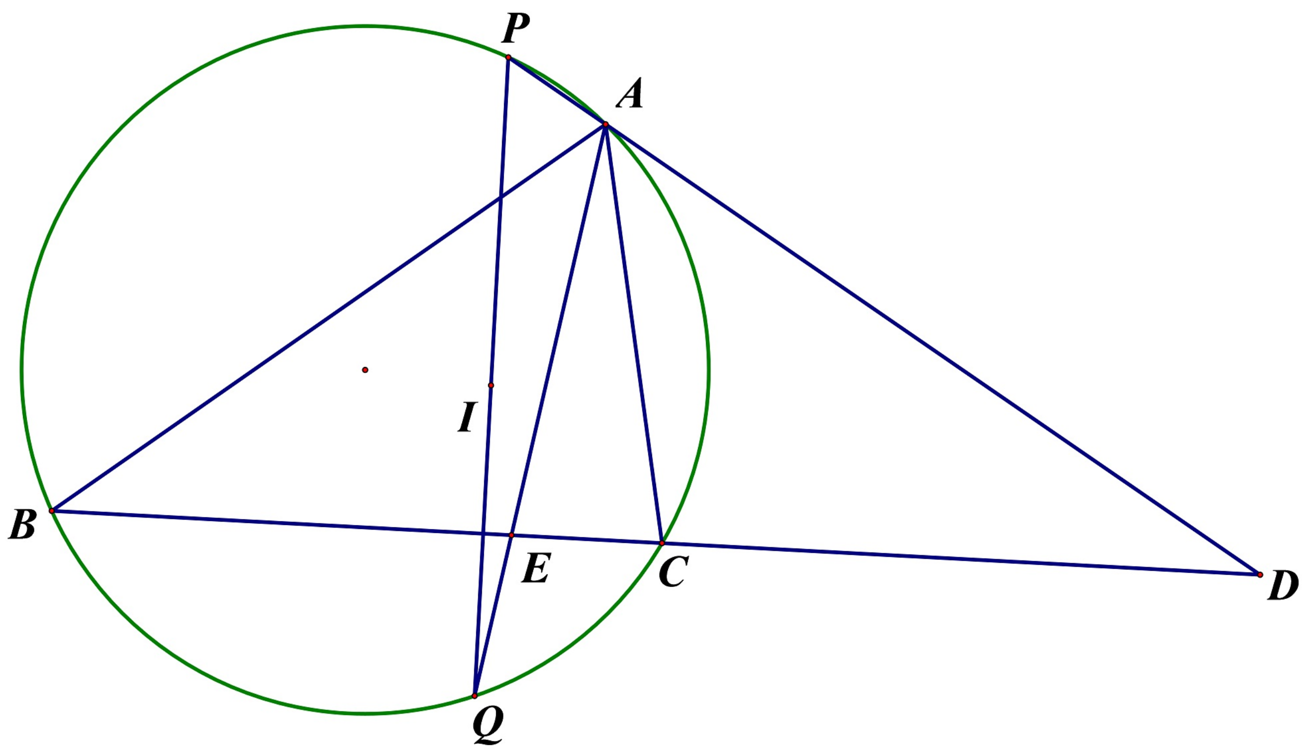

Figure 3.2: AoPS3

Problem 3.2.0.7: AoPS4 E

Given a triangle ABC, the incircle (I) touch BC, CA, AB at D, E, F respectively. Let AA_1, BB_1, CC_1 be $A, B, C - altitude$ respectively. Let N be the orthocenter of the triangle AEF. Prove that N is the incenter of AB_1C_1

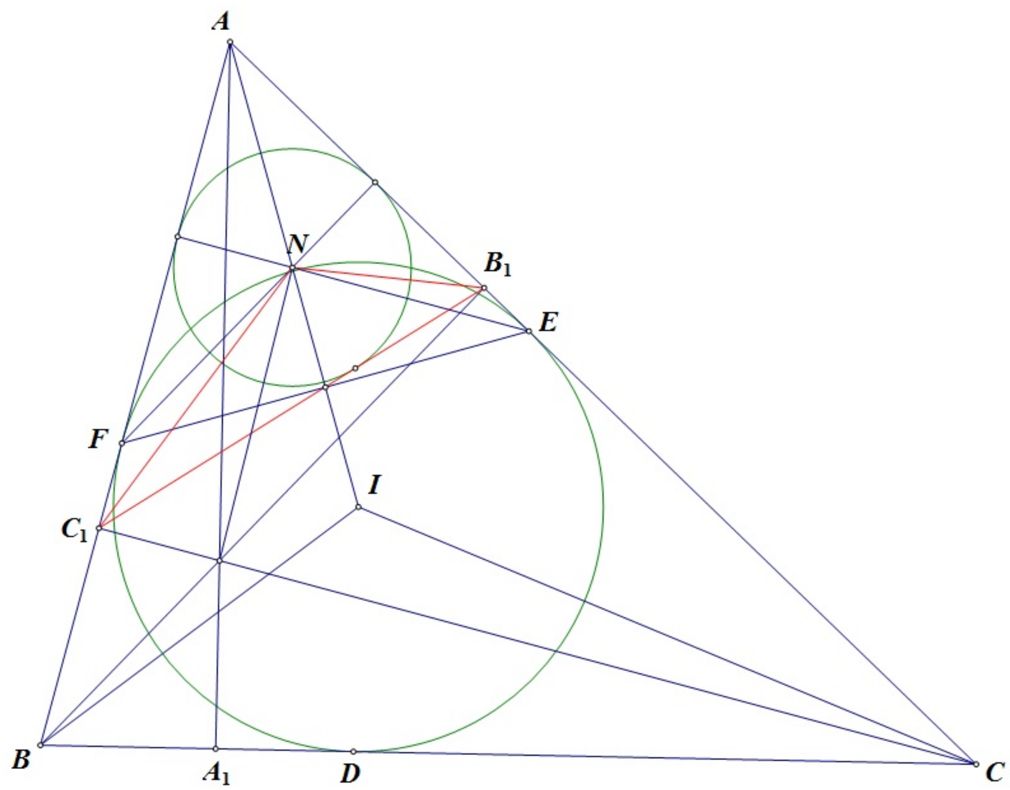

Figure 3.3: AoPS4

Problem 3.2.0.8: IRAN TST 2015 Day 2, P3 M

$ABCD$ is a circumscribed and inscribed quadrilateral. O is the circumcenter of the quadrilateral. E, F and S are the intersections of AB, CD; AD, BC and AC, BD respectively. E' and F' are points on AD and AB such that $\angle AEE' = \angle E'ED$ and $\angle AFF' = \angle F'FB$. X and Y are points on OE' and OF' such that $\frac{XA}{XD} = \frac{EA}{ED}$ and $\frac{YA}{YB} = \frac{FA}{FB}$. M is the midpoint of arc BD of (O) which contains A. Prove that the circumcircles of triangles OXY and OAM are coaxial with the circle with diameter OS.

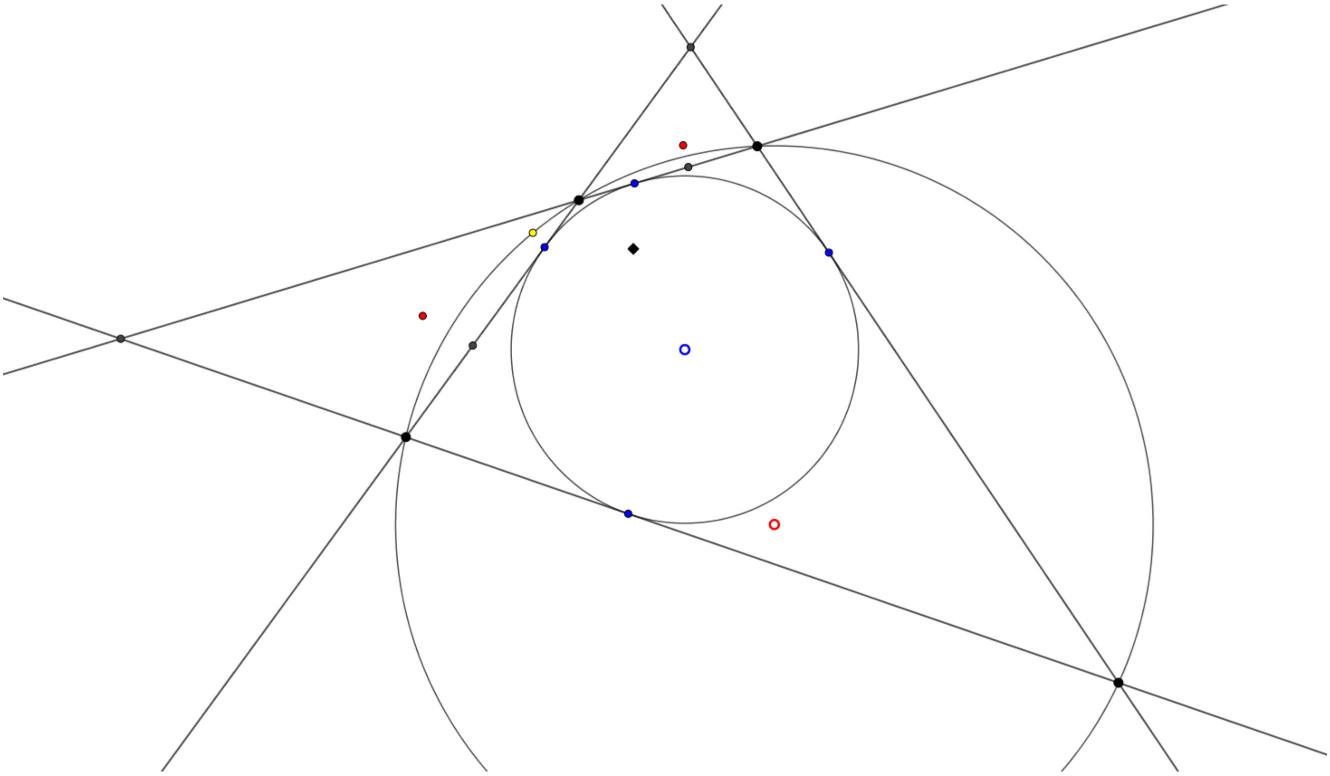

Figure 3.4: Actual Prob

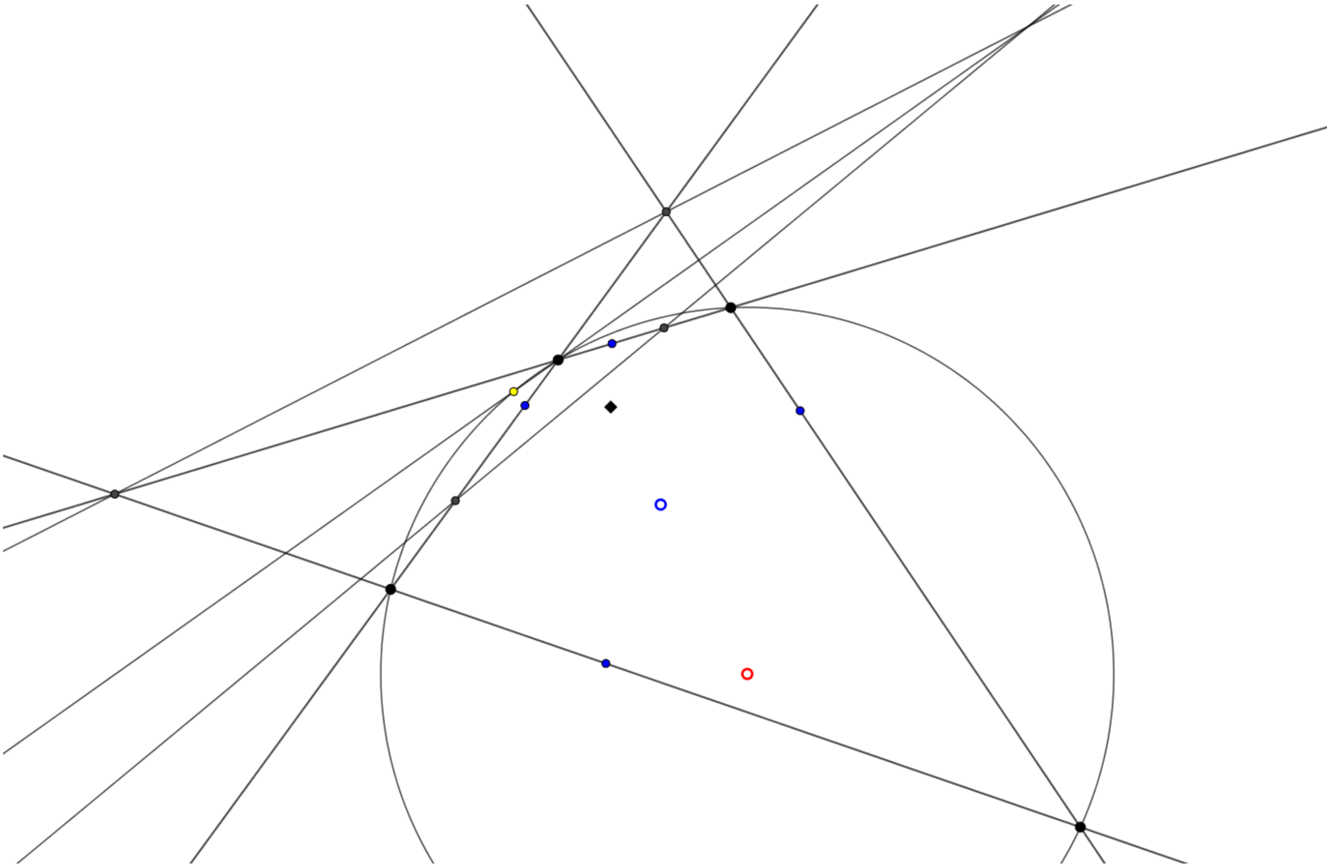

Figure 3.5: Inverted

Problem 3.2.0.9: USA TST 2017 P2 M

Let ABC be an acute scalene triangle with circumcenter O, and let T be on line BC such that $\angle TAO = 90^\circ$. The circle with diameter $\overline{AT}$ intersects the circumcircle of $\triangle BOC$ at two points A_1 and A_2, where $OA_1 < OA_2$. Points B_1, B_2, C_1, C_2 are defined analogously.

1. Prove that $\overline{AA_1}$, $\overline{BB_1}$, $\overline{CC_1}$ are concurrent.
2. Prove that $\overline{AA_2}$, $\overline{BB_2}$, $\overline{CC_2}$ are concurrent on the Euler line of triangle ABC.

Problem 3.2.0.10: AoPS2

Let ABC be a triangle with circumcenter O and altitude AH. AO meets BC at M and meets the circle (BOC) again at N. P is the midpoint of MN. K is the projection of P on line AH. Prove that the circle (K, KH) is tangent to the circle (BOC).

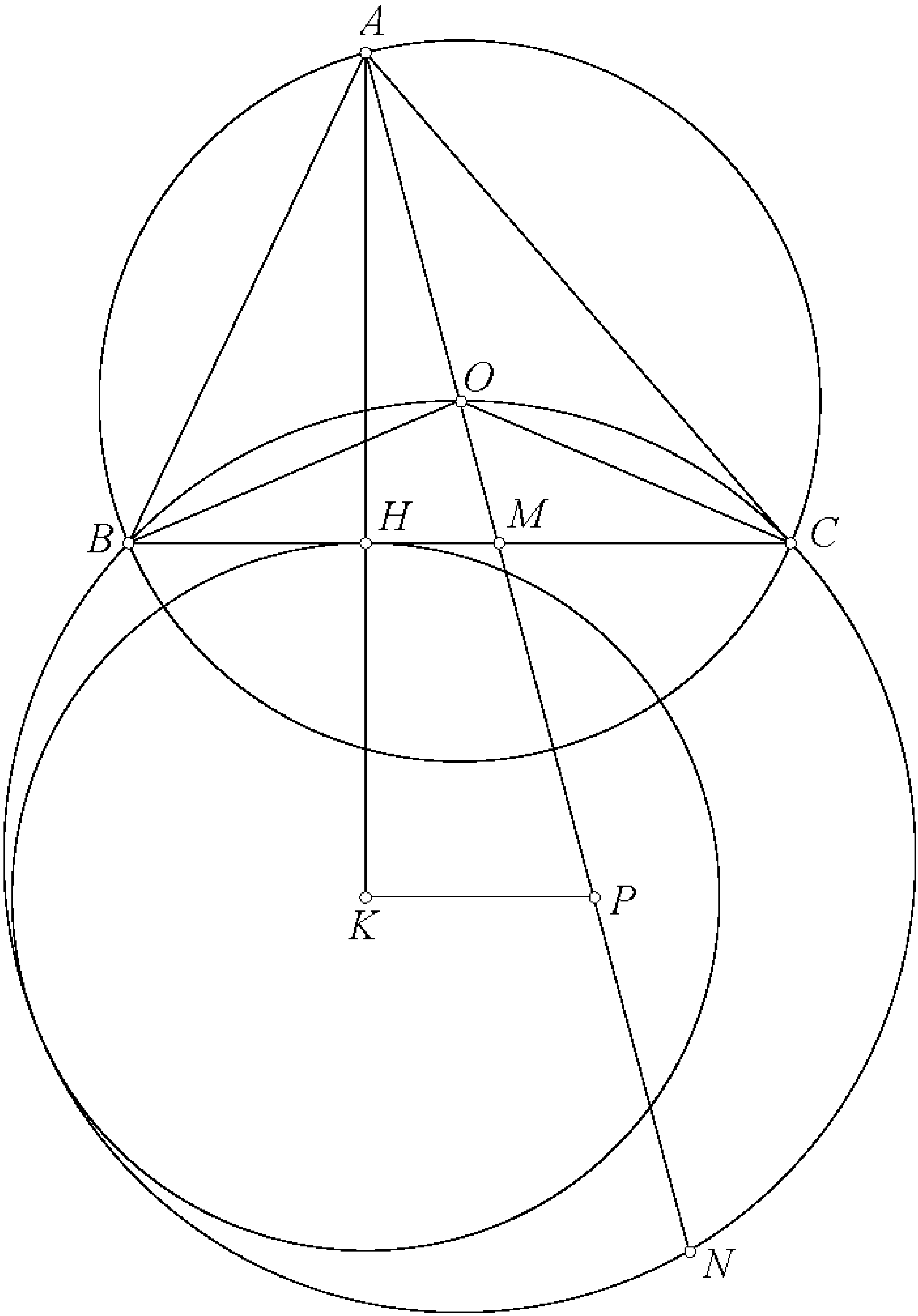

Figure 3.6: AoPS2

Idea Inversion all the way... □

Problem 3.2.0.11: AoPS5

Let ABC be a triangle inscribed in (O) and P be a point. Call P' be the isogonal conjugate point of P. Let A' be the second intersection of AP' and (O). Denote by M the intersection of BC and $A'P$. Prove that $P'M \parallel AP$.

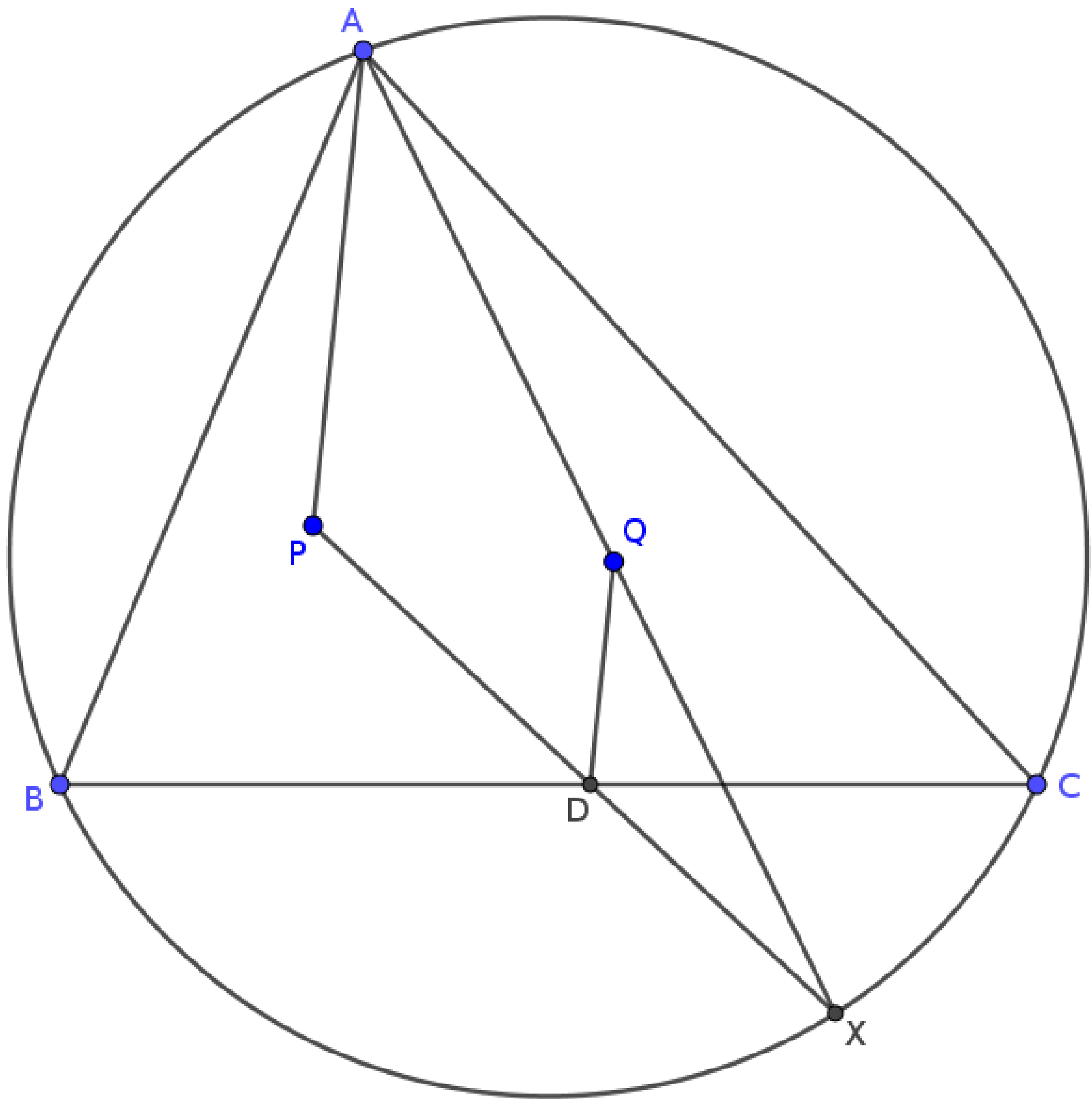

Figure 3.7: AoPS5

Problem 3.2.0.12: AoPS E

I is the incenter of a non-isosceles triangle $\triangle ABC$. If the incircle touches BC, CA, AB at A_1, B_1, C_1 respectively, prove that the circumcentres of the triangles $\triangle AIA_1$, $\triangle BIB_1$, $\triangle CIC_1$ are collinear.

Problem 3.2.0.13: AoPS M

Given $\triangle ABC$ and a point P inside. AP cuts BC at M. Let M', A' be the reflection of M, A in the perpendicular bisector of BC. $A'P$ cuts the perpendicular bisector of BC at N. Let Q be the isogonal conjugate of P in triangle ABC. Prove that $QM' \parallel AN$.

Problem 3.2.0.14: IRAN 3rd Round 2016 G6 E

Given triangle $\triangle ABC$ and let D, E, F be the foot of angle bisectors of A, B, C ,respectively. M, N lie on EF such that $AM = AN$. Let H be the foot of A-altitude on BC.
Points K, L lie on EF such that triangles $\triangle AKL, \triangle HMN$ are correspondingly similar (with the given order of vertices's) such that $AK \not\parallel HM$ and $AK \not\parallel HN$. Show that: $DK = DL$.

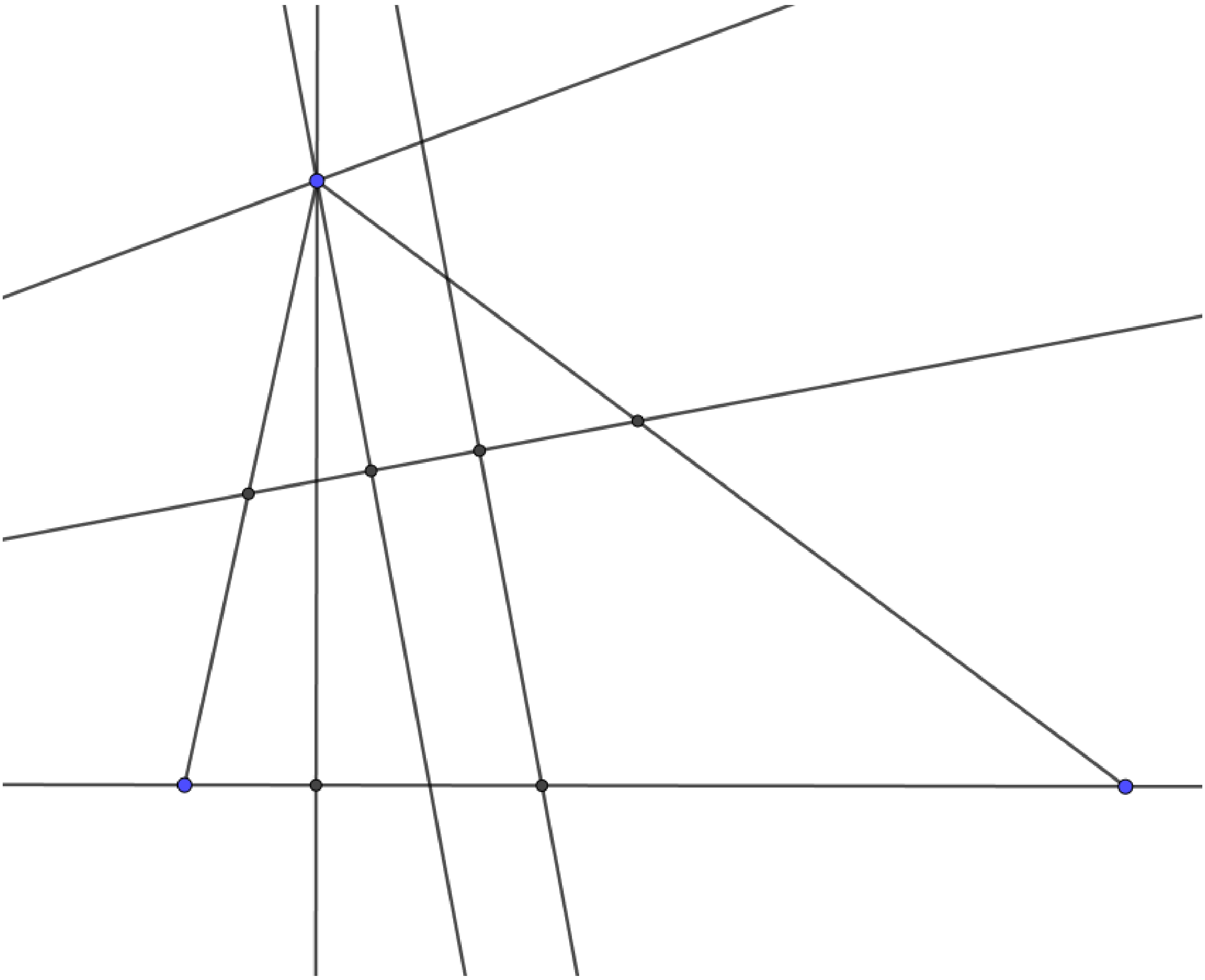

Figure 3.8: IRAN 3rd Round 2016 G6

Problem 3.2.0.15: Iran TST 2017 T3 P6 H

In triangle ABC let O and H be the circumcenter and the orthocenter. The point P is the reflection of A with respect to OH. Assume that P is not on the same side of BC as A. Points E, F lie on AB, AC respectively such that $BE = PC$, $CF = PB$. Let K be the intersection point of AP, OH. Prove that $\angle EKF = 90°$.

Spiral Similarity (points on AB, AC with some properties)

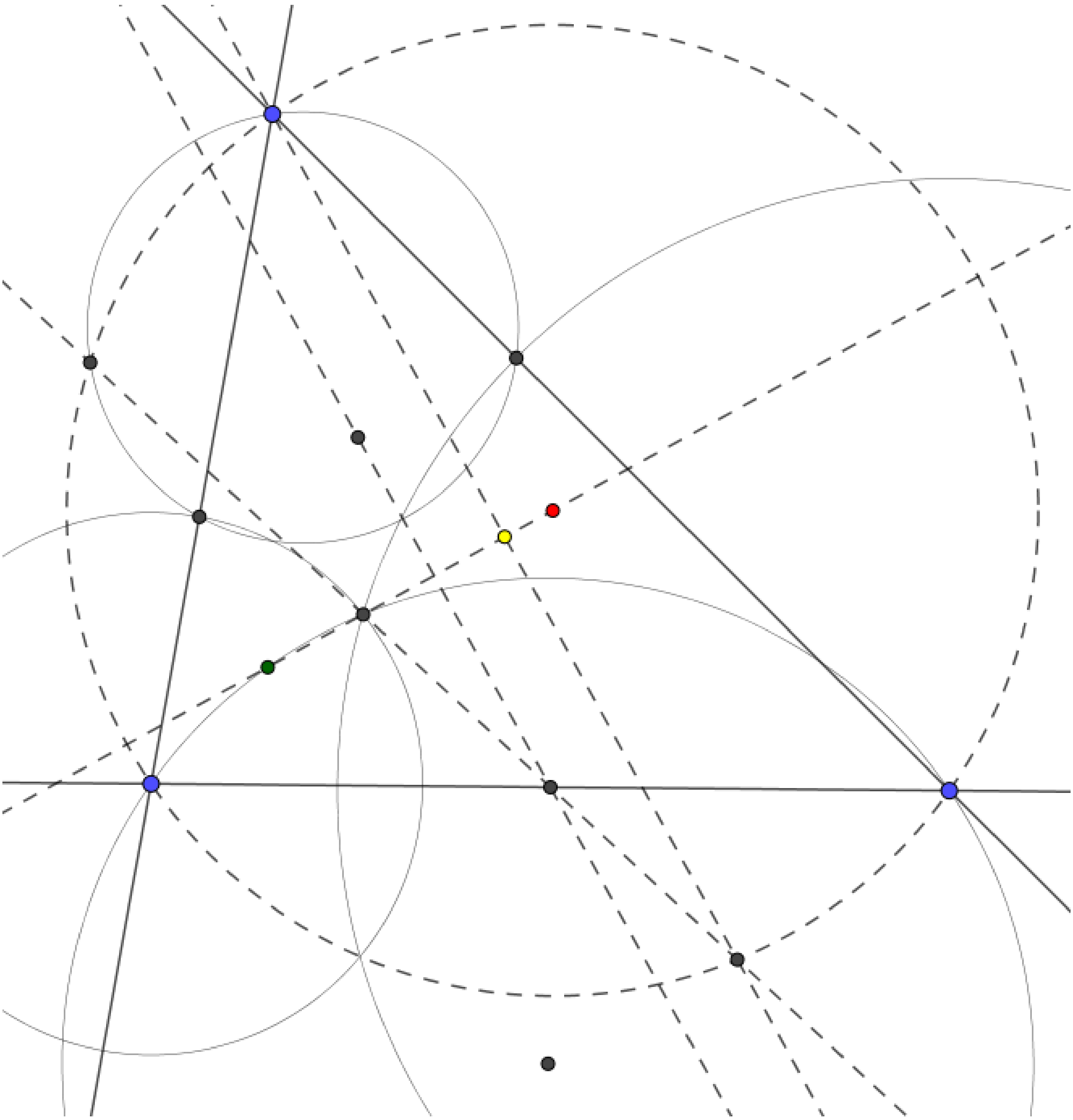

Figure 3.9: Iran TST 2017 T3 P6

Problem 3.2.0.16: IRAN 3rd Round 2010 D3, P6 M

In a triangle ABC, $\angle C = 45°$. AD is the altitude of the triangle. X is on AD such that $\angle XBC = 90 - \angle B$ (X is inside of the triangle). AD and CX cut the circumcircle of ABC in M and N respectively. Ff the tangent to $\odot ABC$ at M cuts AN at P, prove that P, B and O are collinear.

Cross-Ratio

Problem 3.2.0.17: Iran TST 2014 T1P6 M

I is the incenter of triangle ABC. perpendicular from I to AI meet AB and AC at B' and C' respectively. Suppose that B'' and C'' are points on half-line BC and CB such that $BB'' = BA$ and $CC'' = CA$. Suppose that the second intersection of circumcircles of $AB'B''$ and $AC'C''$ is T. Prove that the circumcenter of AIT is on the BC.

projective, inversion

Idea Too many collinearity, need to prove concurrency, what else can come into mind except projective approach. ☐

Idea Too many incenter related things, $\sqrt{bc}$-inversion :o ☐

Problem 3.2.0.18: APMO 2014 P5 M

Circles ω and Ω meet at points A and B. Let M be the midpoint of the arc AB of circle ω (M lies inside Ω). A chord MP of circle ω intersects Ω at Q (Q lies inside ω). Let ℓ_P be the tangent line to ω at P, and let ℓ_Q be the tangent line to Ω at Q. Prove that the circumcircle of the triangle formed by the lines ℓ_P, ℓ_Q and AB is tangent to Ω.

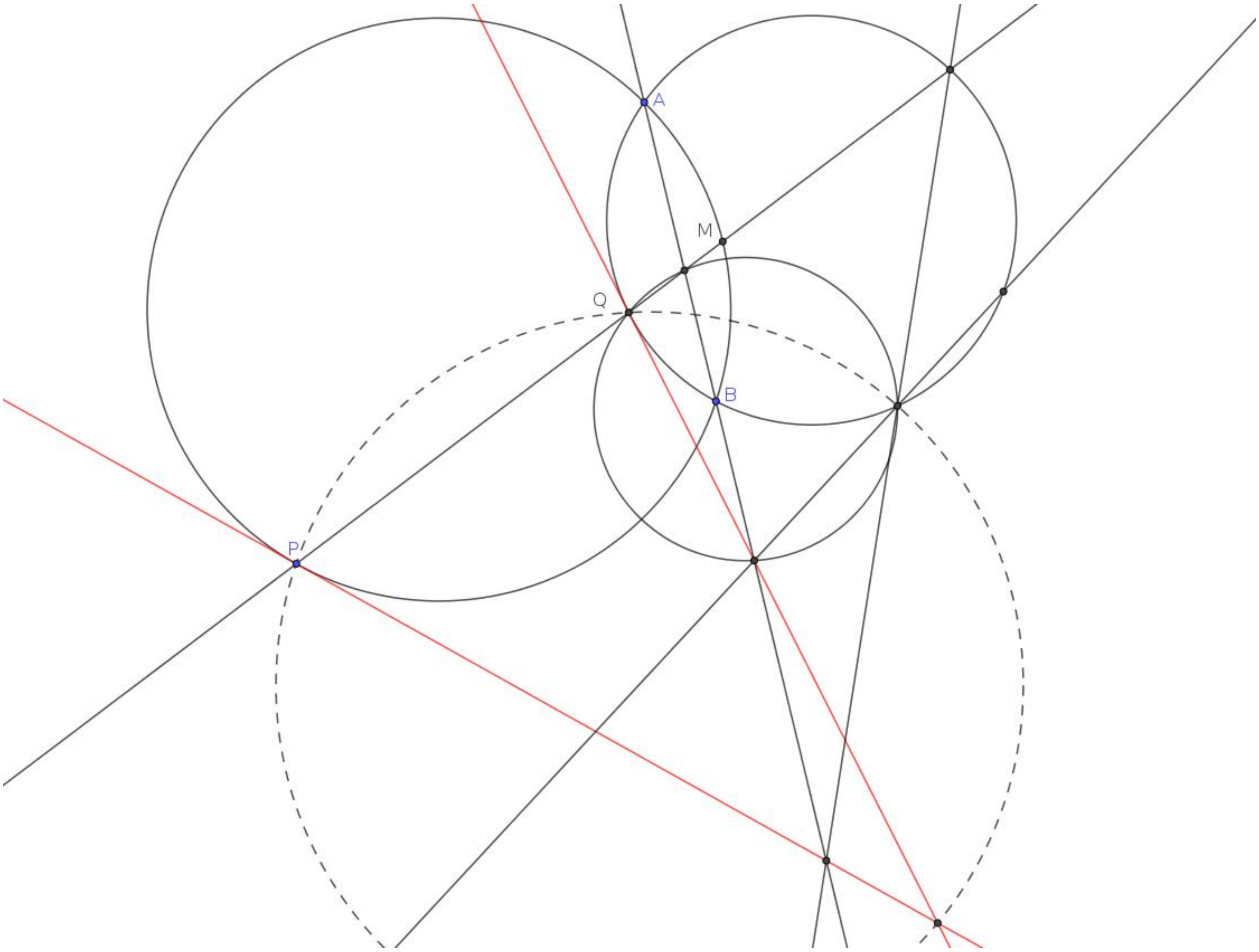

Figure 3.10: APMO 2014 P5

Problem 3.2.0.19: E

Let ABC be a triangle, D, E, F are the feet of the altitudes, $DF \cap BE \equiv P, DE \cap CF \equiv Q$. Prove that the perpendicular from A to PQ goes through the reflection of O on BC.

projective

Idea Projective approach. □

Problem 3.2.0.20: RMM 2018 P6 H

Fix a circle Γ, a line ℓ to tangent Γ, and another circle Ω disjoint from ℓ such that Γ and Ω lie on opposite sides of ℓ. The tangents to Γ from a variable point X on Ω meet ℓ at Y and Z. Prove that, as X varies over Ω, the circumcircle of XYZ is tangent to two fixed circles.

inversion

Idea Too many circles, plus tangency, what else other than inversion? After the inversion the problem turns into a pretty obvious work-around problem. □

Problem 3.2.0.21: AoPS6 H but Beautiful

Let O and I be the circumcenter and incenter of ΔABC. Draw circle ω so that $B, C \in \omega$ and ω touches (I) internally at P. AI intersects BC at X. Tangent at X to (I) which is different from BC, intersects tangent at P to (I) at S. $SA \cap (O) = T \neq A$. Prove that $\angle ATI = 90°$

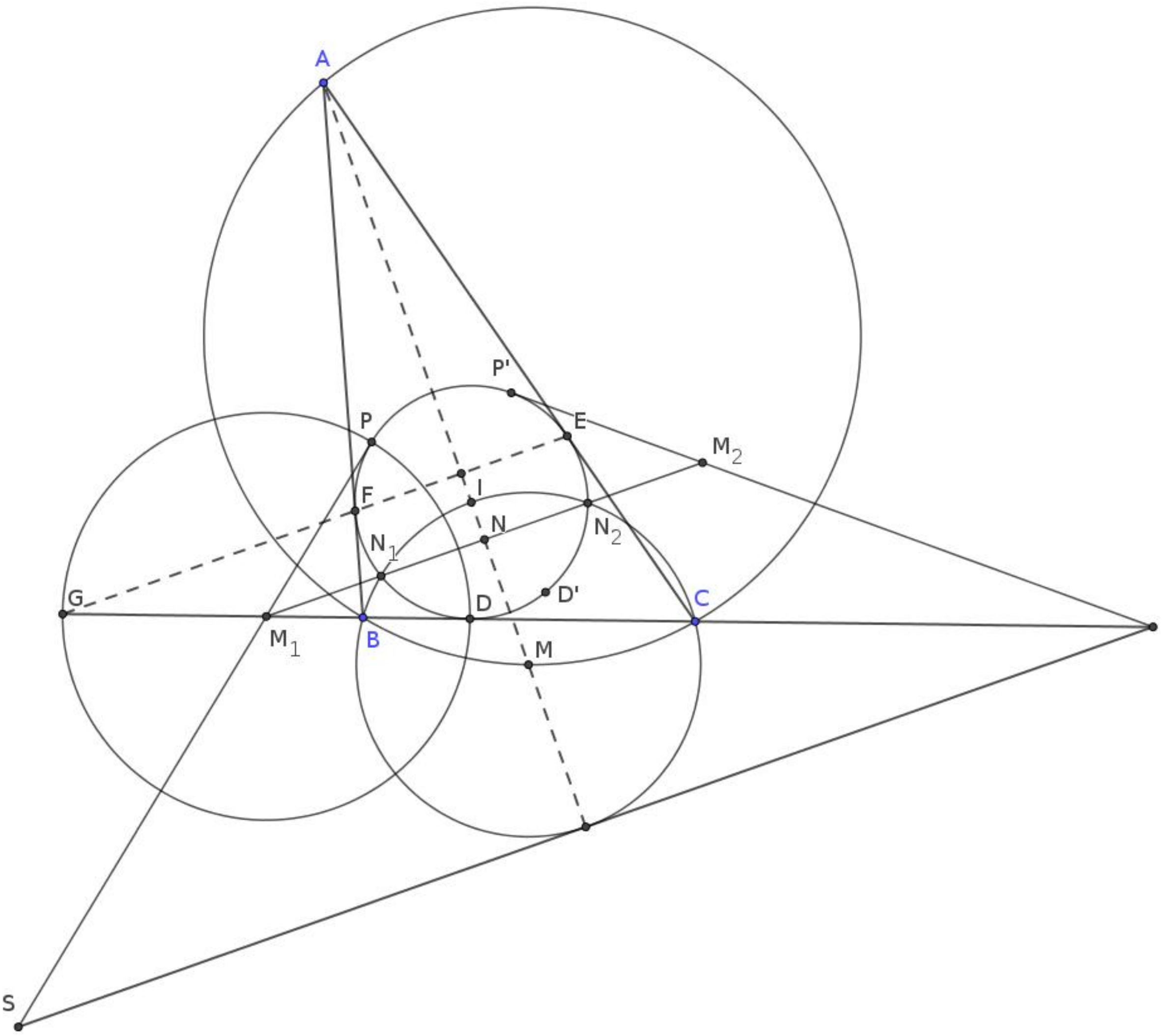

Figure 3.11: Solution 1

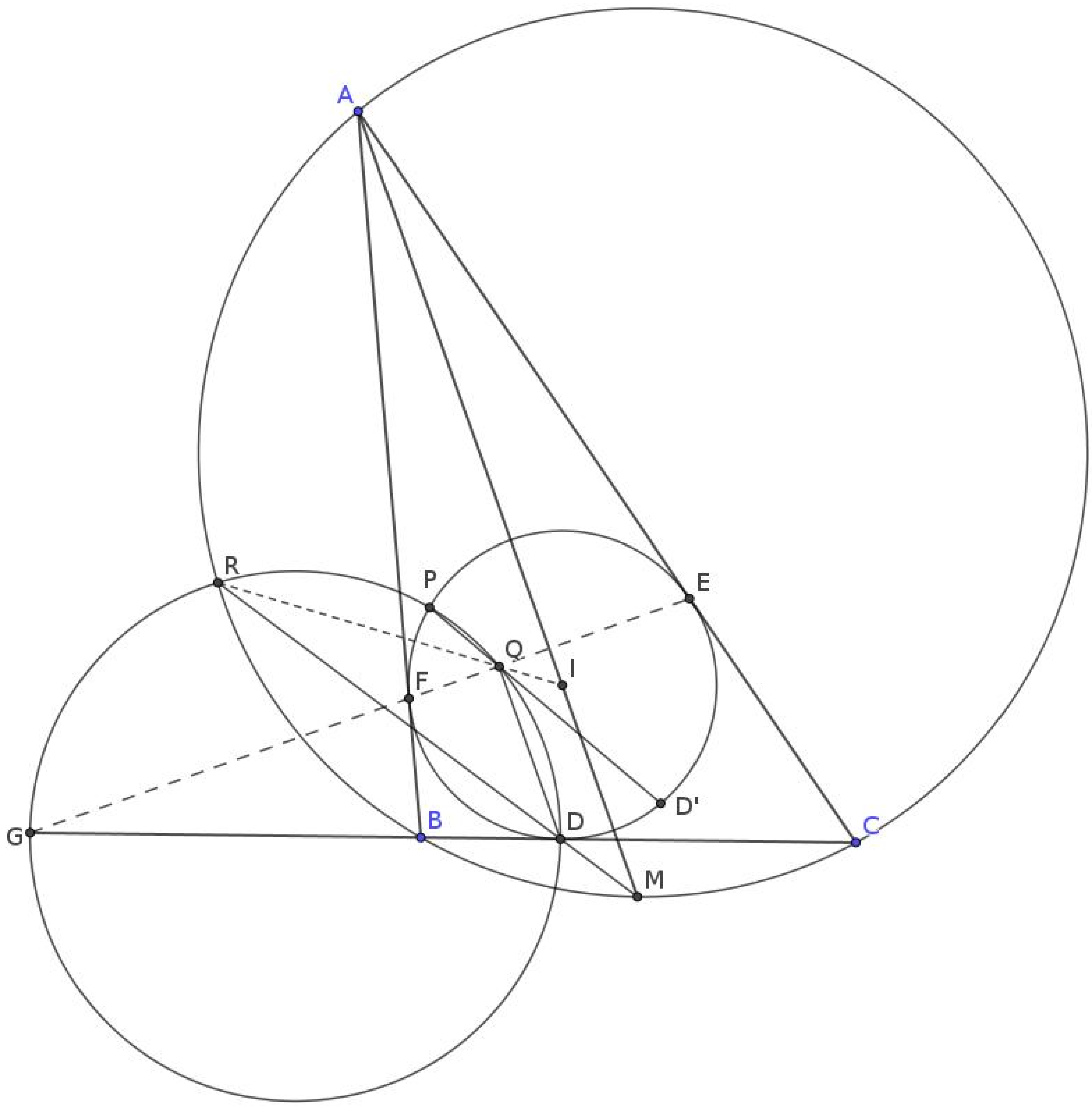

Figure 3.12: Solution 2

Problem 3.2.0.22: AoPS7 E

Let ABC be a triangle with incenter I and circumcircle Γ. Let the line through I perpendicular to AI meet AB at E and AC at F. Let the circumcircles of triangles AIB and AIC intersect the circumcircle of triangle AEF ω again at points M and N, and let ω intersect Γ again at Q. Prove that AQ, MN, and BC are concurrent.

Problem 3.2.0.23: AoPS E

Given a circle (O) with center O and A, B are 2 fixed points on (O). E lies on AB. C, D are on (O) and CD pass through E. P lies on the ray DA, Q lies on the ray DB such that E is the midpoint of PQ. Prove that the circle passing through C and touch PQ at E also pass through the midpoint of AB

Problem 3.2.0.24: WenWuGuangHua Mathematics Workshop E

O_B, O_C are the B and C mixtilinear centers respectively. (O_B) touches BC, AB at X_B, Y_B respectively, and $X_BY_B \cap O_BO_C$ at Z_B. Define X_C, Y_C, Z_C similarly. Prove that if $BZ_C \cap CZ_B = T$, then AT is the A-angle bisector.

Problem 3.2.0.25: All Russia 1999 P9.3 E

A triangle ABC is inscribed in a circle S. Let A_0 and C_0 be the midpoints of the arcs BC and AB on S, not containing the opposite vertex, respectively. The circle S_1 centered at A_0 is tangent to BC, and the circle S_2 centered at C_0 is tangent to AB. Prove that the incenter I of $\triangle ABC$ lies on a common tangent to S_1 and S_2.

Problem 3.2.0.26: All Russia 2000 P11.7 E

A quadrilateral $ABCD$ is circumscribed about a circle ω. The lines AB and CD meet at O. A circle ω_1 is tangent to side BC at K and to the extensions of sides AB and CD, and a circle ω_2 is tangent to side AD at L and to the extensions of sides AB and CD. Suppose that points O, K, L lie on a line. Prove that the midpoints of BC and AD and the center of ω also lie on a line.

Problem 3.2.0.27: All Russia 2000 P9.3 E

Let O be the center of the circumcircle ω of an acute-angle triangle ABC. A circle ω_1 with center K passes through A, O, C and intersects AB at M and BC at N. Point L is symmetric to K with respect to line NM. Prove that $BL \perp AC$.

Problem 3.2.0.28: WenWuGuangHua Mathematics Workshop M

1. AD, BE, CF are concurrent cevians. Angle bisectors of $\angle ADB$ and $\angle AEB$ meet at C_0. Again the angle bisectors of $\angle ADC$ and $\angle AFC$ meet at B_0. And bisectors of $\angle BEC$ and $\angle BFC$ meet at A_0. Prove that AA_0, BB_0, CC_0 are concurrent.

2. Angle bisectors of $\angle AEB$ and $\angle AFC$ meet at D_0, of $\angle BFC$ and BDA meet at E_0, and of $\angle CEB$ and $\angle CDA$ meet at F_0. Prove that DD_0, EE_0, FF_0 are concurrent.

Idea As this problem is purely made up with lines, we can do a projective transformation to simplify the problem. And as there are perpendicularity at D, E, F, we make D, E, F the feet of the altitudes of $\triangle ABC$. Then the angle bisector properties get replaced by simpler properties wrt DEF. $\square$

Problem 3.2.0.29: WenWuGuangHua Mathematics Workshop E

Generalization: Let AD, BE, CF be any cevians concurrent at T. $AD \cap EF = A'$, $BE \cap DF = B'$, $CF \cap DE = C'$, $B'A' \cap AC = X$, $B'A' \cap BC = Y$, $C'X \cap EF = Z$. Prove that T, Y, Z are collinear.

Problem 3.2.0.30: AoPS E

On circumcircle of triangle ABC, T and K are midpoints of arcs BC and BAC respectively . And E is foot of altitude from C on AB . Point P is on extension of AK such that PE is perpendicular to ET . Prove that $PC = CK$.

Problem 3.2.0.31: USJMO 2018 P3 E

Let $ABCD$ be a quadrilateral inscribed in circle ω with $\overline{AC} \perp \overline{BD}$. Let E and F be the reflections of D over lines BA and BC, respectively, and let P be the intersection of lines BD and EF. Suppose that the circumcircle of $\triangle EPD$ meets ω at D and Q, and the circumcircle of $\triangle FPD$ meets ω at D and R. Show that $EQ = FR$.

Problem 3.2.0.32: All Russia 2002 P11.6 M

The diagonals AC and BD of a cyclic quadrilateral $ABCD$ meet at O. The circumcircles of triangles AOB and COD intersect again at K. Point L is such that the triangles BLC and AKD are similar and equally oriented. Prove that if the quadrilateral $BLCK$ is convex, then it has an incircle.

Problem 3.2.0.33: WenWuGuangHua Mathematics Workshop M

Let O_B, O_C be the B, C mixtilinear excircles. O meet CA, CB at X_C, Y_C and O_B meet BA, BC at X_B, Y_B. Let I_C be the C-excircle. $I_C Y_B$ meet $O_B O_C$ at T. Prove that $BT \perp O_B O_C$

Idea From what we have to prove, we find two circles, from where we get another circle. This circle suggests that we try power of point. $\square$

Problem 3.2.0.34: Iran TST 2018 T1P3 M

In triangle ABC let M be the midpoint of BC. Let ω be a circle inside of ABC and is tangent to AB, AC at E, F, respectively. The tangents from M to ω meet ω at P, Q such that P and B lie on the same side of AM. Let $X \equiv PM \cap BF$ and $Y \equiv QM \cap CE$. If $2PM = BC$ prove that XY is tangent to ω.

Problem 3.2.0.35: Iran TST 2018 T1P4 E

Let ABC be a triangle ($\angle A \neq 90°$). BE, CF are the altitudes of the triangle. The bisector of $\angle A$ intersects EF, BC at M, N. Let P be a point such that $MP \perp EF$ and $NP \perp BC$. Prove that AP passes through the midpoint of BC.

Idea :'3 kala para na T_T $\square$

Problem 3.2.0.36: Iran TST 2018 T3P6 H

Consider quadrilateral $ABCD$ inscribed in circle ω. $AC \cap BD = P$. E, F lie on sides AB, CD, respectively such that $\angle APE = \angle DPF$. Circles ω_1, ω_2 are tangent to ω at X, Y respectively and also both tangent to the circumcircle of PEF at P. Prove that:

$$\frac{EX}{EY} = \frac{FX}{FY}$$

Idea fucking beautiful. □

Problem 3.2.0.37: ISL 2006 G6 E

Circles ω_1 and ω_2 with centres O_1 and O_2 are externally tangent at point D and internally tangent to a circle ω at points E and F respectively. Line t is the common tangent of ω_1 and ω_2 at D. Let AB be the diameter of ω perpendicular to t, so that A, E, O_1 are on the same side of t. Prove that lines AO_1, BO_2, EF and t are concurrent.

Problem 3.2.0.38: ISL 2006 G7 E

In a triangle ABC, let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB, respectively, and T_a, T_b, T_c be the midpoints of the arcs BC, CA, AB of the circumcircle of ABC, not containing the vertices's A, B, C, respectively. For $i \in a, b, c$, let w_i be the circle with M_iT_i as diameter. Let p_i be the common external common tangent to the circles w_j and w_k (for all $i, j, k = a, b, c$) such that w_i lies on the opposite side of p_i than w_j and w_k do.
Prove that the lines p_a, p_b, p_c form a triangle similar to ABC and find the ratio of similitude

Problem 3.2.0.39: ISL 2006 G9 H

Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC, respectively. The circumcircles of triangles $AB_1C_1,\ BC_1A_1,\ CA_1B_1$ intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2, respectively ($A_2 \neq A, B_2 \neq B, C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

Idea In this type of "Miquel's Point and the intersections of the circumcircles" related problems, it is useful to think about the second intersections of the lines joining the first intersections and the Miquel's Point with the main circle. □

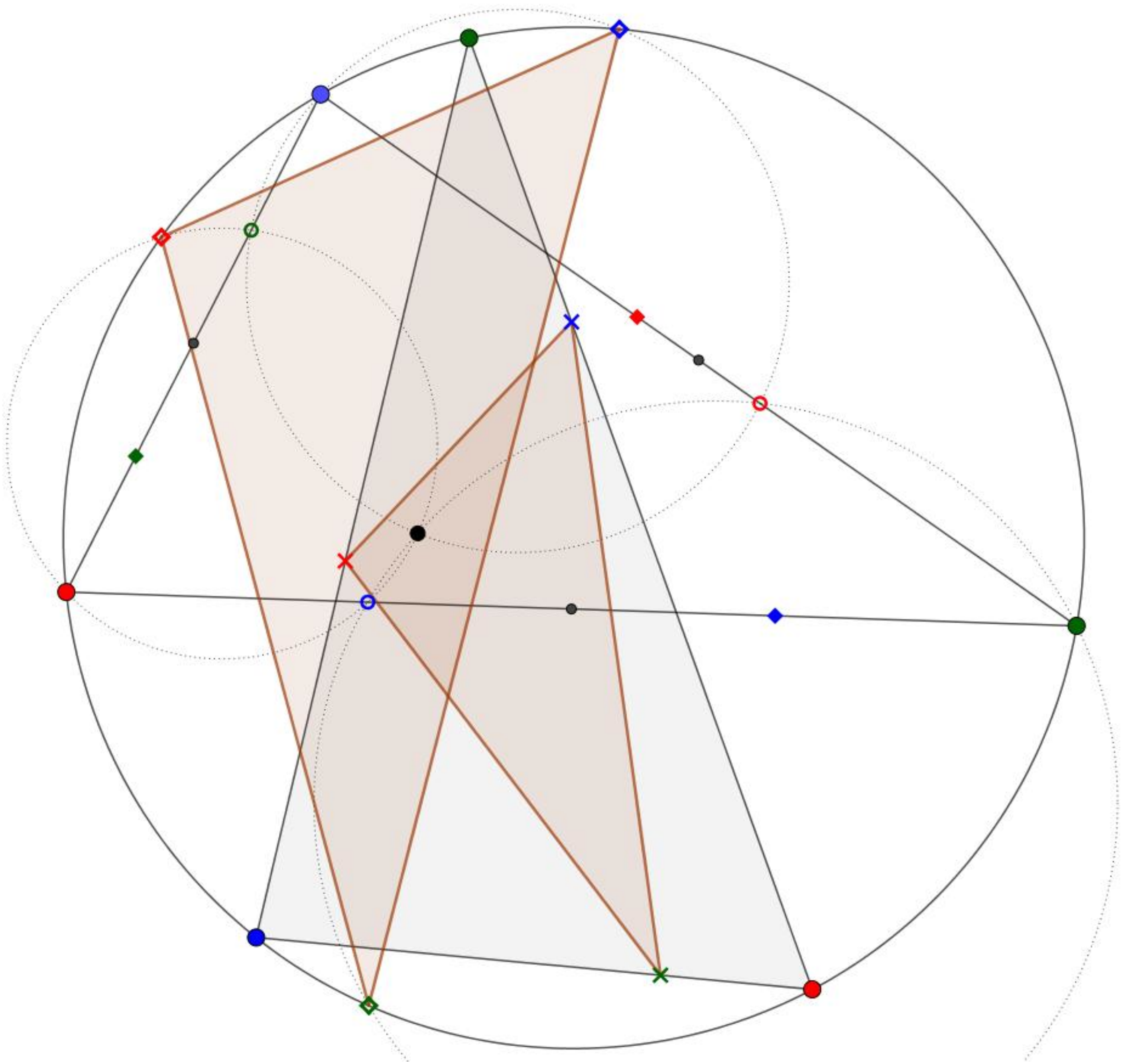

Figure 3.13: IMO Shortlist G9

Problem 3.2.0.40: Iran TST 2017 P5

In triangle ABC, arbitrary points P, Q lie on side BC such that $BP = CQ$ and P lies between B, Q. The circumcircle of triangle APQ intersects sides AB, AC at E, F respectively. The point T is the intersection of EP, FQ. Two lines passing through the midpoint of BC and parallel to AB and AC, intersect EP and FQ at points X, Y respectively. Prove that the circumcircle of triangle TXY and triangle APQ are tangent to each other.

Problem 3.2.0.41: E

Let X be the touchpoint of the incircle with BC and let AX meet $\cdot ABC$ at D. The tangents from D to the incircle meet $\cdot ABC$ at E, F. Prove that the tangent to the circumcircle at A, EF and BC are concurrent.

Problem 3.2.0.42: ISL 2012 G8 M

Let ABC be a triangle with circumcircle ω and ℓ a line without common points with ω. Denote by P the foot of the perpendicular from the center of ω to ℓ. The side-lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove that the circumcircles of the triangles AXP, BYP and CZP have a common point different from P or are mutually tangent at P.

Idea Using Cross ratio and Desergaus's Involution Theorem. □

Problem 3.2.0.43: E

Suppose an involution on a line l sending X, Y, Z to X', Y', Z'. Let l_x, l_y, l_z be three lines passing through X, Y, Z respectively. And let $X_0 = l_y \cap l_z,\ Y_0 = l_x \cap l_z,\ Z_0 = l_x \cap l_y$. Then X_0X', Y_0Y', Z_0Z' are concurrent.

Problem 3.2.0.44: USAMO 2018 P5 E

In convex cyclic quadrilateral $ABCD$, we know that lines AC and BD intersect at E, lines AB and CD intersect at F, and lines BC and DA intersect at G. Suppose that the circumcircle of $\triangle ABE$ intersects line CB at B and P, and the circumcircle of $\triangle ADE$ intersects line CD at D and Q, where C, B, P, G and C, Q, D, F are collinear in that order. Prove that if lines FP and GQ intersect at M, then $\angle MAC = 90^\circ$.

Problem 3.2.0.45: Japan MO 2017 P3 E

Let ABC be an acute-angled triangle with the circumcenter O. Let D, E and F be the feet of the altitudes from A, B and C, respectively, and let M be the midpoint of BC. AD and EF meet at X, AO and BC meet at Y, and let Z be the midpoint of XY. Prove that A, Z, M are collinear.

Problem 3.2.0.46: ISL 2002 G1 E

Let B be a point on a circle S_1, and let A be a point distinct from B on the tangent at B to S_1. Let C be a point not on S_1 such that the line segment AC meets S_1 at two distinct points. Let S_2 be the circle touching AC at C and touching S_1 at a point D on the opposite side of AC from B. Prove that the circumcenter of triangle BCD lies on the circumcircle of triangle ABC.

Problem 3.2.0.47: ISL 2002 G2 M

Let ABC be a triangle for which there exists an interior point F such that $\angle AFB = \angle BFC = \angle CFA$. Let the lines BF and CF meet the sides AC and AB at D and E respectively. Prove that

$$AB + AC \geq 4DE.$$

Idea Pari nai. □

Problem 3.2.0.48: ISL 2002 G3 E

The circle S has center O, and BC is a diameter of S. Let A be a point of S such that $\angle AOB < 120°$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and at F. Prove that I is the incenter of the triangle CEF.

Problem 3.2.0.49: ISL 2002 G4 E

Circles S_1 and S_2 intersect at points P and Q. Distinct points A_1 and B_1 (not at P or Q) are selected on S_1. The lines A_1P and B_1P meet S_2 again at A_2 and B_2 respectively, and the lines A_1B_1 and A_2B_2 meet at C. Prove that, as A_1 and B_1 vary, the circumcentres of triangles A_1A_2C all lie on one fixed circle.

Problem 3.2.0.50: ISL 2002 G7 E

The incircle Ω of the acute-angled triangle ABC is tangent to its side BC at a point K. Let AD be an altitude of triangle ABC, and let M be the midpoint of the segment AD. If N is the common point of the circle Ω and the line KM (distinct from K), then prove that the incircle Ω and the circumcircle of triangle BCN are tangent to each other at the point N.

Problem 3.2.0.51: Japan MO 2017 P3 E

Let ABC be an acute-angled triangle with the circumcenter O. Let D, E and F be the feet of the altitudes from A, B and C, respectively, and let M be the midpoint of BC. AD and EF meet at X, AO and BC meet at Y, and let Z be the midpoint of XY. Prove that A, Z, M are collinear.

Problem 3.2.0.52: India TST E

ABC triangle, D, E, F touchpoints, M midpoint of BC, K orthocenter of $\triangle AIC$, prove that $MI \perp KD$

Problem 3.2.0.53: ISL 2009 G3 E

Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelogram. Prove that $GR = GS$.

Idea Point Circle, distance same means Power same wrt point circles. □

Problem 3.2.0.54: ARO 2018 P11.6 E

Three diagonals of a regular n-gon prism intersect at an interior point O. Show that O is the center of the prism.
(The diagonal of the prism is a segment joining two vertices's not lying on the same face of the prism.)

Problem 3.2.0.55: ISL 2011 G4 EM

Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.

Problem 3.2.0.56: Constructing a forth circle tangent

Given 3 circle, construct another circle that is tangent to these three circles.

Idea A trick to remember: decreasing the radius's of some circles doesn't effect much. □

Problem 3.2.0.57: H

Let $ABCD$ be a convex quadrilateral, let $AD \cap BC = P$. Let O, O'; H, H' be the circumcentres and orthocenter of $\triangle PCD, \triangle PAB$. $\odot DOC$ is tangent to $\odot AD'B$, if and only if $\odot DHC$ is tangent to $\odot AH'B$

Chapter 4

Number Theory

4.1 Tricks

A general trick to remember: If the problem condition is completely or partially but crucially depended on some problem object, but the proof condition doesn't directly depend on that object, think of a way to include that object in the proof condition.

- permutation type problem
- do there exist...
- proving identities
- Dunno
- divisibility by primes and prime divisors stuff

1. Add. Everything. Up.
2. Infinitude of primes:
   (a) Eulerian infinitude trick
   (b) For large enough numbers, there is a larger prime divisor
   (c) Assuming contradiction, if there are any number co-prime to the product of the primes, then that must be 1.

4.1.1 Digit Sum or Product

When dealing with the sum of the digits or the product of them, to find the construction it is very important to consider 0 and 1's in the number.

4.1.2 Diophantine Equations

1. finding some solutions
2. trying modular cases
3. making some variables depended on other variables
4. putting constrains on variables which would make the problems easier
5. if there are infinitely many solutions, can you find a construction?
6. factorize (this is BIG)
7. In these problems, investigation, induction, recursion, constructions etc. are essentials

4.1.3 Sequences

4.1.4 NT Functions

4.1.5 Construction Problems

4.1.6 Sets satisfying certain properties

4.1.7 Other Small Techniques to Remember

1. $a - b$ stays invariant upon addition, just as $\frac{a}{b}$ stays invariant upon multiplication.

4.2 Lemmas

Lemma 4.2.0.1 Let $p \geq 5$ be a prime number. Prove that if $p \mid a^2 + ab + b^2$, then

$$p^3 \mid a + b^p - a^p - b^p$$

Lemma 4.2.0.2 If $b \geq 2$ and $b^n - 1 \mid a$ then there exist at least n non-zero digits in the representation of a in base b

Lemma 4.2.0.3 There are infinitely many primes p for every non-square integers a such that a is a non-quadratic residue $\pmod p$

Theorem 4.2.0.4: Wythoff Array

An infinite array of Positive integers such that every row of it forms Fibonacci-Type sequences.

Theorem 4.2.0.5: Frobenius Coin Problem

For any integers $a_1, a_2 \dots a_n$ such that $\gcd(a_1, a_2 \dots a_n) = 1$, there exists positive integers m such that for any integer $M \geq m$, there are non-negative integers $b_1, b_2 \dots b_n$ such that

$$\sum_{i=1}^{n} b_i a_i = M$$

If $n = 2$ then $m = a_1 a_2 - a_1 - a_2$.

If $n \geq 2$, then there doesn't exist an explicit formula, but if $\{a_i\}$ are in arithmetic progression, $(a_i = a_1 + (i-1)d)$ then

$$m = \left\lfloor \frac{a-2}{n-1} \right\rfloor a + (d-1)(a-1) - 1$$

Theorem 4.2.0.6: Beatty's Theorem

If a, b are two integers such that $\frac{1}{a} + \frac{1}{b} = 1$, then the two sets $\{\lfloor ia \rfloor\}$ and $\{\lfloor ib \rfloor\}$, where i are the positive integers, form a partition of the set of natural numbers.

Theorem 4.2.0.7: Cyclotomic Formulas

Lemma 4.2.0.8 Let x, y be co-prime. Then

$$\gcd(z, xy) = \gcd(z, x)\gcd(z, y) = \gcd(z \bmod x, x)\gcd(z \bmod y, y)$$

$$\implies \gcd(r(a, b), xy)) = \gcd(a, x)\gcd(b, y)$$

(here $r(a, b)$ denotes the smallest integer that satisfies $r(a, b) \equiv a \bmod x,\ r(a, b) \equiv b \bmod y$)

Lemma 4.2.0.9 Let $0 < a_1 < a_2 < \Delta\Delta\Delta < a_(mn + 1)$ be $mn + 1$ integers. Prove that you can select either $m + 1$ of them no one of which divides any other, or $n + 1$ of them each dividing the following one.

Theorem 4.2.0.10: Prime divisors of an integer polynomial

If $P(x) \in \mathbb{Z}[x]$, then the set of primes, $P = \{p : p \mid P(x)\}$ is infinite.

4.2.1 Quadratic Residue

Theorem 4.2.1.1:

There are exactly $\frac{p-1}{2}$ quadratic residue classes $\mod p$

Theorem 4.2.1.2:

Let p be an odd prime. Then,

1. The product of two quadratic residue is a quadratic residue.
2. The product of two quadratic non-residue is a quadratic residue.
3. The product of a qudratic residue and a quadratic non-residue is a quadratic non-residue

Definition 4.2.1.1 Legendre Symbol: We call $\left(\frac{a}{p}\right)$ the Legendre symbol for an integer a and a prime p where,

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p|a \\ 1 & \text{if } a \text{ is a qr of } p \\ -1 & \text{otherwise} \end{cases}$$

Theorem 4.2.1.3:

For an odd prime p and any two integers a, b, we have $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$

Theorem 4.2.1.4: Euler's Criterion

Let p be an odd prime. Then,

$$a^{\frac{p-1}{2}} \equiv \left(\frac{a}{p}\right) (mod\ p)$$

Theorem 4.2.1.5:

Let $(a, b) = 1$. Then every prime divisors of $a^2 + b^2$ is either 2 or a prime of the form $4k + 1$.

Theorem 4.2.1.6: Gauss's Criterion

Let p be a prime number and a be an integer coprime to p. Let $\mu(a)$ be the number of integers $x \in \{a, a*2, \dots a*\frac{p-1}{2}\}$ such that $x(mod\ p) > \frac{p}{2}$. Then

$$\left(\frac{a}{p}\right) = -1^{\mu(a)}$$

Theorem 4.2.1.7:

The smallest quadratic non-residue of an odd prime p is a prime which is less than $\sqrt{p}+1$

Theorem 4.2.1.8: Quadratic Residue Law

Using the usual Legendre Symbol, for two prime numbers p, q we have:

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$$

Definition 4.2.1.2 Jacobi Symbol: Let $a, n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$. We define Jacobi symbol as

$$\left(\frac{a}{n}\right) = \prod_{i=1}^{k} \left(\frac{a}{p_i}\right)^{\alpha_i}$$

Note 1: J

cobi symbol is not as accurate as Legendre symbol. $\left(\frac{a}{n}\right) = -1$ means that a is a quadratic non-residue of n, but $= 1$ doesnt necessarily mean that a is a quadratic residue of n.

Theorem 4.2.1.9:

Let $a, n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$, then a is a quadratic residue of n iff it is a quaadratic residue of every $p_i^{\alpha_i}$.

Theorem 4.2.1.10:

If an integer is a quadratic residue of every prime, then it is a square.

4.2.2 Modular Arithmatic Theorems and Useful Results

Theorem 4.2.2.1: Wolstenholme's Theorem

For all prime p the following relation is true:

$$p^2 \,\Big|\, 1 + \frac{1}{2} + \frac{1}{3} \cdots \frac{1}{p-1} = \sum_{i=1}^{p-1} \frac{1}{i}$$

Corollary 4.2.2.1.1

$$p \,\Big|\, 1 + \frac{1}{2^2} + \frac{1}{3^2} \cdots \frac{1}{(p-1)^2} = \sum_{i=1}^{p-1} \frac{1}{i^2}$$

Corollary 4.2.2.1.2

If $p > 3$ is a prime, then

$$\binom{2p}{p} \equiv 2 \ (mod\ p^3)$$

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$$\binom{2p-1}{p-1} \equiv 1 \ (mod\ p^3)$$

Lemma 4.2.2.2

$$\frac{1}{(p-i)!} \equiv (-1)^i (i-1)! \ (mod\ p)$$

$$\binom{p-1}{k} \equiv (-1)^k \ (mod\ p)$$

Problem 4.2.2.1: E

$$\binom{p^{n+1}}{p} \equiv p^n \ (mod\ p^{2n+3})$$

4.3 Problems

Problem 4.3.0.1: APMO 1999 P4 EM

Determine all pairs (a, b) of integers with the property that the numbers $a^2 + 4b$ and $b^2 + 4a$ are both perfect squares.

Idea Easy case work assuming positive or negative values for a, b. ☐

Problem 4.3.0.2: E

Let n be an odd integer, and let $S = \{x \mid 1 \leq x \leq n,\ (x, n) = (x + 1, n) = 1\}$. Prove that

$$\prod_{x \in S} x = 1\ (mod\ n)$$

Problem 4.3.0.3: All Squares E

Prove that there are infinitely many pairs of positive integers (x, y) satisfying that $x + y, x - y, xy + 1$ are all perfect squares.

Problem 4.3.0.4: USAMO 2007 P1 E

n be a positive integer. Define a sequence by setting $a_1 = n$ and, for each $k > 1$, letting a_k be the unique integer in the range $0 \leq a_k \leq k - 1$ for which $a_0 + a_1 \cdots + a_k$ is divisible by k. Prove that for any n the sequence a_i eventually becomes constant.

Problem 4.3.0.5: USAMO 1998 E

Prove that for each $n \geq 2$, there is a set S of n integers such that $(a - b)^2$ divides ab for every distinct $a, b \in S$.

Idea Induction comes to the rescue. Trying to find a way to get from n to $n + 1$, we see that we can *shift* the integers by any integer k. So after shifting, what stays the same, and what changes? ☐

Problem 4.3.0.6: Iran TST 2015, P4 M

Let n is a fixed natural number. Find the least k such that for every set A of k natural numbers, there exists a subset of A with an even number of elements which the sum of its members is divisible by n.

Idea Odd-Even, so lets first try for odd n's. It is quite easy.

So now, for evens, lets first try the simplest kind of evens. As we need a set with an even number of elements, this tells us to pair things up. We can try to partition A into pairs of e-e's and o-o's. This gives us our desired result. ☐

Problem 4.3.0.7: Iran TST 2015 P11 M

We call a permutation $(a_1, a_2 \dots a_n)$ of the set $\{1, 2 \dots n\}$ "good" if for any three natural numbers $i < j < k$,

$$n \nmid a_i + a_k - 2a_j$$

find all natural numbers $n \geq 3$ such that there exist a "good" permutation of the set $\{1, 2 \dots n\}$.

Idea Looking for "possibilities" for the first element, we get some more restrictions for the values of other terms. □

Problem 4.3.0.8: ISL 2004 N2 M

The function $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(n) = \sum_{k=1}^{n} \gcd(k, n)$, $n \in \mathbb{N}$.

1. Prove that $f(mn) = f(m)f(n)$ for every two relatively prime $m, n \in \mathbb{N}$.
2. Prove that for each $a \in \mathbb{N}$ the equation $f(x) = ax$ has a solution.
3. Find all $a \in \mathbb{N}$ such that the equation $f(x) = ax$ has a unique solution.

Idea Why not casually try to multiply $f(m)$ and $f(n)$?? And also find a formula for $n =$ prime power. □

Problem 4.3.0.9: Balkan MO 2017 P1 S

Find all ordered pairs of positive integers (x, y) such that: $x^3 + y^3 = x^2 + 42xy + y^2$.

Problem 4.3.0.10: Balkan MO 2017 P3 E

Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$n + f(m) \mid f(n) + nf(m)$$

for all $n, m \in \mathbb{N}$.

Idea Check sizes and bound for large n. □

Problem 4.3.0.11: Iran MO 3rd Round N3 E

Let $p > 5$ be a prime number and $A = \{b_1, b_2 \dots b_{\frac{p-1}{2}}\}$ be the set of all quadratic residues modulo p, excluding zero. Prove that there doesn't exist any natural a, c satisfying $\gcd(ac, p) = 1$ such that set $B = \{x \mid x = ay + c, y \in A\}$ and set A are disjoint modulo p.

Idea Sum it up. □

Idea For every integer a, b and prime p such that, $\gcd(a, p) = \gcd(b, p) = 1$, there exist (x, y) such that $x^2 \equiv ay^2 + c \pmod{p}$. □

Idea For a prime p, there exists an integer x such that x and $x+1$ both are quadratic residues $\pmod p$. $\square$

Problem 4.3.0.12: All Russia 2014 P9.5 E

Define $m(n)$ to be the greatest proper natural divisor of n. Find all $n \in \mathbb{N}$ such that $n + m(n)$ is a power of 10.

Problem 4.3.0.13: ISL 2000 N1 E

Determine all positive integers $n \geq 2$ that satisfy the following condition: for all a and b relatively prime to n we have $a \equiv b \pmod n$ iff $ab \equiv 1 \pmod n$.

Idea Don't forget the details. $\square$

Problem 4.3.0.14: ISL 2000 N3 M

Does there exist a positive integer n such that n has exactly 2000 prime divisors (not necessarily distinct) and $n \mid 2^n + 1$?

Idea Goriber Bondhu Induction. As the number 2000 seems so out of the place, we replace 2000 by k. Now suppose that for some k, the condition works. For simplicity let $k = p^i$ for some i, as it is quite clear that there is another prime q that divides $2^k + 1$, let $k' = kq$. So k' also satisfies the condition. So it is quite intuitive to think that for every x there exist some p and i for which $2^{p^i} + 1$ has x prime factors. So we search for such p. $\square$

Problem 4.3.0.15: USAMO 2001 P5 H

Let S be a set of integers (not necessarily positive) such that

1. There exist $a, b \in S$ with $\gcd(a, b) = \gcd(a-2, b-2) = 1$;
2. If x and y are elements of S (possibly equal), then $x^2 - y \in S$

Prove that S is the set of all integers.

Idea One possible intuition could be trying to make the problem statement a little bit more stable, like the term $x^2 - y$ is not so symmetric. So trying to make it a little bit more symmetric can come handy. $\square$

Idea If $c, x, y \in S$ then we can easily see that $A(x^2 - y^2) - c \in S$ for all $A \in \mathbb{Z}$. We take this a little too far and show that if $c, x, y, u, v \in S$, then $A(x^2 - y^2) + B(u^2 - v^2) - c \in S$ for all $A, B \in \mathbb{Z}$. So if we can find such x, y, u, v such that $\gcd(x^2 - y^2, u^2 - v^2) = 1$, we are almost done by Frobenius Coin Problem. So we start looking for integers that can be obtained from a, b. After some playing around we get the feeling (or maybe not) that we need one more pair. Again playing around for some time we find three pairs. FCP gives us an upper bound for all integers that are not in S. Easily we include them in S. $\square$

Problem 4.3.0.16: Vietnam TST 2017 P2 M

For each positive integer n, set $x_n = \binom{2n}{n}$

1. Prove that if $\frac{2017^k}{2} < n < 2017^k$ for some positive integer k then $2017 \mid x_n$.
2. Find all positive integer $h > 1$ such that there exist positive integers N, T such that the sequence (x_n) for $n > N$, is periodic $(bmodh)$ with period T.

Problem 4.3.0.17: Vietnam 2017 TST P6 H

For each integer $n > 0$, a permutation $(a_1, a_2 \ldots a_{2n})$ of $1, 2 \ldots 2n$ is called *beautiful* if for every $1 \leq i < j \leq 2n$, $a_i + a_{n+i} = 2n + 1$ and $a_i - a_{i+1} \not\equiv a_j - a_{j+1} \pmod{2n+1}$ (suppose that $a_i = a_{2n+i}$).

1. For $n = 6$, point out a *beautiful* permutation.
2. Prove that there exists a *beautiful* permutation for every n.

Idea Trial and Error. □

Problem 4.3.0.18: BrMO 2008 M

Find all sequences $a_{i}{}_{i=0}^{\infty}$ of rational numbers which follow the following conditions:

1. $a_n = 2a_{n-1}^2 - 1$ for all $n > 0$
2. $a_i = a_j$ for some $i, j > 0$, $i \neq j$

Idea Trial and Error. Don't forget that you only need the numbers to be rational. □

Problem 4.3.0.19: USA TST 2000 P4 E

Let n be a positive integer. Prove that

$$\sum_{i=0}^{n} \binom{n}{i}^{-1} = \frac{n+1}{2^{n+1}} \left(\sum_{i=0}^{n+1} \frac{2^i}{i} \right)$$

Idea *Positive Integer*, nuff said. □

Problem 4.3.0.20: USA TST 2000 P3 EH NOT TRIED

Let p be a prime number. For integers r, s such that $rs(r^2 - s^2)$ is not divisible by p, let $f(r, s)$ denote the number of integers $1 \leq n \leq p$ such that $\{\frac{rn}{p}\}$ and $\{\frac{sn}{p}\}$ are either both less than $\frac{1}{2}$ or both greater than $\frac{1}{2}$. Prove that there exists $N > 0$ such that for $p \geq N$ and all r, s,

$$\left\lceil \frac{(p-1)}{3} \right\rceil \leq f(r, s) \leq \left\lfloor \frac{2(p-1)}{3} \right\rfloor$$

Problem 4.3.0.21: China TST 2005 EH NOT TRIED

Let n be a positive integer and $f_n = 2^{2^n} + 1$. Prove that for all $n \geq 3$, there exists a prime factor of f_n which is larger than $2^{n+2}(n+1)$ [Stronger Version: $2^{n+4}(n+1)$].

Problem 4.3.0.22: IRAN TST 2009 P2 H NOT TRIED

Let a be a fixed natural number. Prove that the set of prime divisors of $2^{2^n} + a$ for $n = 1, 2, 3 \dots$ is infinite.

Problem 4.3.0.23: USAMO 2004 P2 E

Suppose $a_1, a_2 \dots a_n$ are integers whose greatest common divisor is 1. Let S be a set of integers with the following properties:

1. $a_i \in S$
2. For $i, j = 1, 2 \dots n$ (not necessarily distinct), $a_i - a_j \in S$.
3. For any integers $x, y \in S$, if $x + y \in S$, then $x - y \in S$.

Prove that S must be equal to the set of all integers.

Idea First we see that if $d = \gcd(x, y)$ and $x, y \in S$ then $d \in S$. So all we have to do is to find two x, y with $d = 1$. □

Problem 4.3.0.24: USAMO 2008 P1 E

Prove that for each positive integer n, there are pairwise relatively prime integers $k_0, k_1 \dots k_n$, all strictly greater than 1, such that $k_0 k_1 \dots k_{n-1}$ is the product of two consecutive integers.

Idea *Positive Integer n* nuff said. □

Problem 4.3.0.25: USAMO 2007 P5 M

Prove that for every nonnegative integer n, the number $7^{7^n} + 1$ is the product of at least $2n + 3$ (not necessarily distinct) primes.

Idea When you try to apply induction, always name the hypothesis. In this case name $7^{7^n} + 1 = a_n$. And try to relate a_n with a_{n+1}. □

Problem 4.3.0.26: IMO 1998 P3 E

For any positive integer n, let $\tau(n)$ denote the number of its positive divisors (including 1 and itself). Determine all positive integers m for which there exists a positive integer n such that $\frac{\tau(n^2)}{\tau(n)} = m$.

Idea Easy go with the flow. □

Problem 4.3.0.27: USA TST 2002 P2 M

Let $p > 5$ be a prime number. For any integer x, define

$$f_p(x) = \sum_{k=1}^{p-1} \frac{1}{(px+k)^2}$$

Prove that for any two integers x, y,

$$p^3 \mid f(x) - f(y)$$

Idea If there is $f(x) - f(y)$ for all x, y, then always make one of those equal to 0 or in other words make one side constant □

Idea All fractional sum problems should be solved by some expression manipulation. □

Idea Sometimes try adding things up. □

Problem 4.3.0.28: USAMO 2012 P4 E

Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ (where $\mathbb{Z}^+$ is the set of positive integers) such that $f(n!) = f(n)!$ for all positive integers n and such that $(m - n)$ divides $f(m) - f(n)$ for all distinct positive integers m, n.

Positive Integer n nuff said.

Problem 4.3.0.29: USAMO 2013 P5 M

Given positive integers m and n, prove that there is a positive integer c such that the numbers cm and cn have the same number of occurrences of each non-zero digit when written in base ten.

What if we make cm and cn have the same digits, occuring same number of time, when the digits are sorted? This will make the things a whole lot easier. Again for more simplicity, what if the arrangement of the digits in both of these numbers are "almost" the same? Like, if the digits are in blocks and if decomposed into such blocks, we get the same set for both of those problems? This idea of simplicity is more that enough to "Simplify" a problem. Call this strategy Simplify.

Problem 4.3.0.30: ISL 2015 N4 M

Suppose that $a_0, a_1, \cdots$ and $b_0, b_1, \cdots$ are two sequences of positive integers such that $a_0, b_0 \geq 2$ and

$$a_{n+1} = \gcd(a_n, b_n) + 1, \qquad b_{n+1} = \operatorname{lcm}(a_n, b_n) - 1.$$

Show that the sequence a_n is eventually periodic; in other words, there exist integers $N \geq 0$ and $t > 0$ such that $a_{n+t} = a_n$ for all $n \geq N$.

Like most NT probs, pure investigation. We see that the function a_n is mostly decreasing, but it is increasing as well. But the increase rate is not greater than the decrease rate. After some time playing around, we see that the value of a_n rises gradually and then suddenly drops. But the peak value of a_n doesn't seem to increase. Well, this is true, we prove that. After that, we are mostly done, we just show that eventually the least value of a_n becomes stable as well. We use the intuitions we get from working around.

Problem 4.3.0.31: ISL 2015 N3 E-M

Let m and n be positive integers such that $m > n$. Define $x_k = \frac{m+k}{n+k}$ for $k = 1, 2, \ldots, n+1$. Prove that if all the numbers $x_1, x_2, \ldots, x_{n+1}$ are integers, then $x_1 x_2 \ldots x_{n+1} - 1$ is divisible by an odd prime.

As there are powers of 2, we use the powers of 2.

Problem 4.3.0.32: USA TST 2018 P1 E

Let $n \geq 2$ be a positive integer, and let $\sigma(n)$ denote the sum of the positive divisors of n. Prove that the n^{th} smallest positive integer relatively prime to n is at least $\sigma(n)$, and determine for which n equality holds.

even-odd

Pretty starightforward, as the ques suggests, there are fewer than n coprimes in the interval $[1, \sigma(n)]$, we directly show this. [as consttructions don't seem to be trivial/ easy to get] Inclusion/Exclusion all the way. But remember, not checking floors can get you doomed.

Problem 4.3.0.33: APMO 2014 P3 M

Find all positive integers n such that for any integer k there exists an integer a for which $a^3 + a - k$ is divisible by n.

factorize, quadratic residue, complete residue class

Idea You have to show that the set $\{x \mid x \equiv a^3 + a \ (mod\ p)$ is equal to the set $\{1 \ldots p-1\}$. So some properties shared by a set as a whole must be satisfied by both of the sets. The quickest such properties that come into mind are the summation of the set and the product of the set. While the former doesnt help out much, the later seems promising. Where we get a nice relation that we have to satisfy:

$$\prod_{i=0}^{p-1} a^2 + 1 \equiv 1 \ (mod\ p)$$

One way of concluding from here is to use the quadratic residue ideas, or using the fact that $x^2 + 1 = (x+i)(x-i)$. The later requires some higher tricks tho. □

Idea The most natural way must be to show that for any prime p we will find two integers $p \nmid (a-b)$, and $p \mid a^2 + b^2 + ab + 1$, factorizing the later and getting two squares and a constant gives us our desired result. □

Problem 4.3.0.34: APMO 2014 P1 M

For a positive integer m denote by $S(m)$ and $P(m)$ the sum and product, respectively, of the digits of m. Show that for each positive integer n, there exist positive integers $a_1, a_2, \ldots, a_n$ satisfying the following conditions:

$$S(a_1) < S(a_2) < \cdots < S(a_n) \text{ and } S(a_i) = P(a_{i+1}) \quad (i = 1, 2, \ldots, n).$$

(We let $a_{n+1} = a_1$.)

Idea 1 is the only integer that increases the sum, but doesnt change the product. This may seem trivial, but on problems like this where both the sum and the product of the digits of a number are concerned, this tiny little fact can change everything. □

Problem 4.3.0.35: RMM 2018 P4 E

Let a, b, c, d be positive integers such that $ad \neq bc$ and $gcd(a, b, c, d) = 1$. Let S be the set of values attained by $\gcd(an + b, cn + d)$ as n runs through the positive integers. Show that S is the set of all positive divisors of some positive integer.

Problem 4.3.0.36: ISL 2011 N1 E

For any integer $d > 0$, let $f(d)$ be the smallest possible integer that has exactly d positive divisors (so for example we have $f(1) = 1, f(5) = 16$, and $f(6) = 12$). Prove that for every integer $k \geq 0$ the number $f\left(2^k\right)$ divides $f\left(2^{k+1}\right)$.

Idea Construct the function for 2^n. □

Problem 4.3.0.37: ISL 2004 N1 E

Let $\tau(n)$ denote the number of positive divisors of the positive integer n. Prove that there exist infinitely many positive integers a such that the equation $\tau(an) = n$ does not have a positive integer solution n.

Idea Infinitely many, divisor, what else should come to mind except prime powers... □

Problem 4.3.0.38: USAMO 2018 P4 E

Let p be a prime number and let $a_1, a_2 \ldots a_p$ be integers. Prove that there exists an integer k s.t. the $S = \{a_i + ik\}$ has at least $\frac{p}{2}$ elements modulo p

Idea As the only thing that is holding ourselves down is the equivalence of any two elements of S, we investigate it furthur. It is a good idea to represent by graphs. □

Problem 4.3.0.39: ISL 2009 N1 E

Let n be a positive integer and let $a_1, a_2, a_3, \ldots, a_k$ $(k \geq 2)$ be distinct integers in the set $1, 2, \ldots, n$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, 2, \ldots, k - 1$. Prove that n does not divide $a_k(a_1 - 1)$.

Problem 4.3.0.40: ISL 2009 N2 E

A positive integer N is called balanced, if $N = 1$ or if N can be written as a product of an even number of not necessarily distinct primes. Given positive integers a and b, consider the polynomial P defined by $P(x) = (x + a)(x + b)$.

1. Prove that there exist distinct positive integers a and b such that all the number $P(1)$, $P(2)$,..., $P(50)$ are balanced.
2. Prove that if $P(n)$ is balanced for all positive integers n, then $a = b$

Problem 4.3.0.41: USA TSTST 2015 P5 EM

Let $\varphi(n)$ denote the number of positive integers less than n that are relatively prime to n. Prove that there exists a positive integer m for which the equation $\varphi(n) = m$ has at least 2015 solutions in n.

Idea When does the equation has multiple solutions? Suppose $m = \prod_{i=1}^{t} p_i^{\alpha_i}(p_i - 1)$ then $\Phi(n) = m$ has multiple solutions if for some $p's$ in m, their product is one less from another prime. Which gives us necessary intuition to construct a m for which there are A LOT of solutions for the equation. □

Problem 4.3.0.42: Iran 2018 T1P1 M

Let $A_1, A_2, ..., A_k$ be the subsets of $\{1, 2, 3, ..., n\}$ such that for all $1 \leq i, j \leq k$:$A_i \cap A_j \neq \varnothing$. Prove that there are n distinct positive integers $x_1, x_2, ..., x_n$ such that for each $1 \leq j \leq k$:

$$lcm_{i \in A_j} \{x_i\} > lcm_{i \notin A_j} \{x_i\}$$

Idea Main part of the problem is to notice that the first $|A_i|$ columns of the matrix has 1 from all of the rows. Which triggers the idea of giving one prime to every row, and count x_i's with them. □

Problem 4.3.0.43: Iran TST 2018 T2P4 E

Call a positive integer "useful but not optimized " (!), if it can be written as a sum of distinct powers of 3 and powers of 5. Prove that there exist infinitely many positive integers which they are not "useful but not optimized".
e.g. 37 is a "useful but not optimized" number since $37 = (3^0 + 3^1 + 3^3) + (5^0 + 5^1)$

Problem 4.3.0.44: ISL 2014 N2 E

Determine all pairs (x, y) of positive integers such that

$$\sqrt[3]{7x^2 - 13xy + 7y^2} = |x - y| + 1.$$

Idea Always factor, before everything else. □

Problem 4.3.0.45: ISL 2014 N1 M

Let $n \geq 2$ be an integer, and let A_n be the set

$$A_n = \{2^n - 2^k \mid k \in \mathbb{Z},\, 0 \leq k < n\}.$$

Determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of A_n .

Idea In every problem, conjecture from smaller case, and check if the conjecture is true in bigger cases. □

Problem 4.3.0.46: ISL 2002 N1 E

What is the smallest positive integer t such that there exist integers $x_1, x_2, \ldots, x_t$ with

$$x_1^3 + x_2^3 + \ \ldots \ + x_t^3 = 2002^{2002}\,?$$

Idea $1000 + 1000 + 1 + 1$. □

Problem 4.3.0.47: ISL 2002 N2 M

Let $n \geq 2$ be a positive integer, with divisors $1 = d_1 < d_2 < \ \ldots < d_k = n$. Prove that $d_1 d_2 + d_2 d_3 + \ldots + d_{k-1} d_k$ is always less than n^2, and determine when it is a divisor of n^2.

Idea In problems with all of the divisors of n involved, it is a good choice to substitute $d_i = \frac{n}{d_{k-i+1}}$. That way, you get the exact same set, represented differently, with n involved. And $\sum_{i=1}^{n} \frac{1}{i*(i+1)} = \frac{n}{n+1}$ □

Problem 4.3.0.48: Japan MO 2017 P2, TST Mock 2018 M

Let N be a positive integer. There are positive integers $a_1, a_2, \cdots, a_N$ and all of them are not multiples of 2^{N+1}. For each integer $n \geq N+1$, set a_n as below:
If the remainder of a_k divided by 2^n is the smallest of the remainder of $a_1, \cdots, a_{n-1}$ divided by 2^n, set $a_n = 2a_k$. If there are several integers k which satisfy the above condition, put the biggest one.
Prove the existence of a positive integer M which satisfies $a_n = a_M$ for $n \geq M$.

Idea Things must go far... □

Problem 4.3.0.49: ISL 2002 N3 E

Let $p_1, p_2, \ldots, p_n$ be distinct primes greater than 3. Show that $2^{p_1 p_2 \cdots p_n} + 1$ has at least 4^n divisors.

Problem 4.3.0.50: Japan MO 2017 P1 E

Let a, b, c be positive integers. Prove that $lcm(a, b) \neq lcm(a + c, b + c)$.

Problem 4.3.0.51: ISL 2009 N3 M

Let f be a non-constant function from the set of positive integers into the set of positive integer, such that $a-b$ divides $f(a)-f(b)$ for all distinct positive integers a, b. Prove that there exist infinitely many primes p such that p divides $f(c)$ for some positive integer c.

Idea Notice if $f(1)=1$, we can easily prove the result, so assume that $f(1)=c$. Now see that, if we can somehow, create another function g from the domain and range of f with the same properties as f, and with $g(1)=1$, we will be done. So to do this, we need to perform some kind of division by c. □

Problem 4.3.0.52: ARO 2018 P9.1 E

Suppose $a_1, a_2, \dots$ is an infinite strictly increasing sequence of positive integers and $p_1, p_2, \dots$ is a sequence of distinct primes such that $p_n \mid a_n$ for all $n \geq 1$. It turned out that $a_n - a_k = p_n - p_k$ for all $n, k \geq 1$. Prove that the sequence $(a_n)_n$ consists only of prime numbers.

Problem 4.3.0.53: ARO 2018 P10.4 H

Initially, a positive integer is written on the blackboard. Every second, one adds to the number on the board the product of all its nonzero digits, writes down the results on the board, and erases the previous number. Prove that there exists a positive integer which will be added infinitely many times.

Proof. Using Bounding and the

Problem 4.3.0.54: APMO 2008 P4 E

Consider the function $f : \mathbb{N}_0 \to \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of all non-negative integers, defined by the following conditions :

$$f(0) = 0, \quad f(2n) = 2f(n) \text{ and } f(2n+1) = n + 2f(n) \quad \text{for all } n \geq 0$$

1. Determine the three sets $L = \{n|f(n) < f(n+1)\}$, $E = \{n|f(n) = f(n+1)\}$, and $G = \{n|f(n) > f(n+1)\}$.
2. For each $k \geq 0$, find a formula for $a_k = \max\{f(n) : 0 \leq n \leq 2^k\}$ in terms of k.

Problem 4.3.0.55: ISL 2009 N4 E

Find all positive integers n such that there exists a sequence of positive integers a_1, a_2,. . ., a_n satisfying:

$$a_{k+1} = \frac{a_k^2+1}{a_{k-1}+1} - 1$$

for every k with $2 \leq k \leq n-1$.

Idea Rewriting the condition, and doing some parity check. Then assuming the contrary and taking extreme case. □

Problem 4.3.0.56: ISL 2015 N2 E

Let a and b be positive integers such that $a! + b!$ divides $a!b!$. Prove that $3a \geq 2b + 2$.

Idea Size Chase □

Problem 4.3.0.57: USA TST 2019 P2 MH

Let $\mathbb{Z}/n\mathbb{Z}$ denote the set of integers considered modulo n (hence $\mathbb{Z}/n\mathbb{Z}$ has n elements). Find all positive integers n for which there exists a bijective function $g : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, such that the 101 functions

$$g(x), \quad g(x) + x, \quad g(x) + 2x, \quad \ldots, \quad g(x) + 100x$$

are all bijections on $\mathbb{Z}/n\mathbb{Z}$.

Idea A very nice problem. We get the motivation by trying the cases for 2, 3 replacing 101. In the case of 2, we just consider the sum $\sum g(x)$. We get that $2 \nmid n$. So in the case of 3, we conjecture that $3 \nmid n$. But we can't prove this similarly as before. Whats the most common 'sum-type' invariant after the normal sum? Sum of the Squares.

Now that we have proved that $(6, n) = 1$, most probably our conjecture is correct. So lets try for any k, we need to show that $(k!, n) = 1$. In the case of 3, we used the 2nd power sum. So probably to prove that $k \nmid n$ we need to take the $(k-1)$th power sum.

Now the real thing begins. In the case of 3, doesn't the modular sum equation looks something like the first finite difference? This rings a bell that whenever there is powers involved, we should consider using the derivatives. For $(k-1)$th power, the $(k-1)$th derivative that is.

Another thing here, in the case of 3, we did something like

$$\sum (g(x) + 2x)^2 - \sum (g(x) + x)^2 \equiv \sum (g(x) + x)^2 - \sum (g(x))^2 \equiv 0 \pmod{n}$$

We try something similar again. □