National Math Camp 2020 Exam 2

August 13, 2020

Problem 1. Two circles Γ_1 and Γ_2 meet at A and B. Let r be a line through B that meets Γ_1 at C and Γ_2 at D, such that B is between C and D. Let s be the line parallel to AD, which is tangent to Γ_1 in E and has the minimal distance from $AD.EA$ meets Γ_2 in F, and let t be the line through F which is tangent to Γ_2. Prove that:

a) $t \parallel AC$.

b) r, s and t are concurrent.

Problem 2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the following equation:

$$f(xy) = \begin{cases} f(x)f(y) & \text{if } f(x+y) \leq f(x)f(y) \\ f(x+y) & \text{otherwise} \end{cases}$$

Problem 3. Let n be a positive integer. Prove that there exist integers $b_1, b_2, \ldots, b_n$ such that for any integer m, the number

$$\left(\cdots\left(\left(\left(m^2+b_1\right)^2+b_2\right)^2+\cdots\right)^2+b_{n-1}\right)^2+b_n$$

is divisible by $2n-1$

Problem 4. 2015 positive integers are arranged on a circle. The difference between any two adjacent numbers equals their greatest common divisor. Determine the maximal value of N which divides the product of all 2015 numbers, regardless of their choice.