BCS Question Bank

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Contents

Chapter 1

Thoughts on PSolving, a note to thyself

1.1 Be DUMB, Keep it SIMPLE

Remember what Paul-lord said? Think wishfully, make dumb wishes. When first approaching the problem, you can do whatever you want. You can loosen some constraints, you can add some new. This works exceptionally well when you need to build an object from one given object, you can do whatever you want. Putting additional constraints decreases the number of test cases. Sometimes loosening some constraints help to give better observation of the problem.

Like in ISL 2016 N5, after deciding that we are going to build a pair (x_2, y_2) from the previous pair (x_1, y_1), we should look for the most innocent looking relation between these four variable. Now it's time to play around, try dumb things. Rewriting the equation, we want to use the fact that x_1, x_2 have to be on different sides of $\sqrt{a}$. How can we insert this constraint in our equation in the most simple and natural way? This is where we need to be dumb, and amature.

In 5.7 the trick is to keep things simple. Making the most natural assumptions. In construction problems, think of how the result can be achieved in the most natural way. Can we make some extra assumptions that might result in the immediate proof the result's existence?

Chapter 2

Combinatorics

2.1 Tricks

2.1.1 Bijection

Ideas for the bijection function:

- Induction
- Forming sets that are not already formed
- Building combinatorial models from the investigation of the problem conditions.
- Trying to define the later set by the former set.

1 `ISL 2002 C1,` Red-Blue Under $x + y < n$ and Bijection
2 `OC Chap2 P2,` Magic trick of hiding two digits
3 `ISL 2009 C3`
4 `USAMO 1996 P4,` Binary Strings NOT containing certain Combinations
5 `ISL 2008 C4,` Lamp States and Probability
6 `APMO 2017 P3,` Bijection Problem
7 `USAMO 2013 P2,` Around the circle on points with move or 1 or 2
8 `APMO 2008 P2`
9 `ISL 2006 C2`
10 `USA TST 2009 P1`
11 `ISL 2005 C3`
12 `ISL 2002 C2` Cover all black squares with L-tromino
13 `ISL 2002 C3` Full-Sequences

2.1.1.1 Hall's Marriage Lemma

1 `OC Chap2 P2`
2 `ARO 2005 P9.4`

2.1.2 Extremal Principal

2.1.2.1 Whole Extremal Cases

Exploring the extreme case as a whole

1 `ISL 2013 C1`
2 `Brazilian Olympic Revenge 2014`
3 `ISL 2009 C1`
4 `EGMO 2017 P2`
5 `APMO 2008 P2`
6 `Belarus 2001`
7 `ISL 2014 C3`
8 `USA TST 2017 P1`
9 `Polynomials and Roots problem`
10 `USAMO 2006 P2`
11 `ISL 2014 N3,` Cape Town Coin problem

2.1.2.2 Forget and Focus

Explore only one part of the problem at a time, choose the most crucial part of the problem and focus only on that.

1 `ARO 2018 P9.5`
2 `Polish OI`
3 `ARO 2008 P9.5`
4 `ISL 2001 C6`
5 `Swell Coloring`
6 `Indian Postal Coaching 2011`
7 `Romanian TST 2016 D1P2,` associating x_i with S_i

2.1.2.2.1 Swapping / Forget and Focus (2) Focusing on two neighboring elements in the extremal case.

1 `Polish OI`
2 `IOI 2007 P3`
3 `Problem`
4 `ARO 2014 P9.8`
5 `Problem,` Robot goes at a speed of i for i seconds in mode i
6 `Problem,` The same with speed $n - i$

2.1.2.2.2 Game Positions Considering Winning/Losing positions and describing the game with these definitions is an important game theory tactic.

1 `ISL 2004 C5`

2.1.2.3 Extreme Objects

Concentrating on the Extreme object only

1 `USAMO 2008 P4`

2 `ISL 2008 C1`

3 `IMO 2011 P2`

4 `Iran TST 2002 P3`

5 `Problem`

6 `ISL 2006 C4`

7 `ISL 2014 C1`

8 `ARO 1993 P10.4`

9 `Sunflower Lemma`

10 `ISL 1998 C4`

11 `USA TST 2009 P1`

12 `AoPS`

13 `APMO 2012 P2`

14 `Problem,` Confidence in solving a P6

15 `ISL 2001 C3,` 3-cliques with common points

16 `ISL 2007 C2,` dissecting a rectangle into n smaller rectangles, there exists a rectangle inside

2.1.3 Coloring

- If the nodes are connected in lattice point manner, then **Checkerboard** coloring is the most natural coloring technique. But if this coloring does not do any good, then there may be other alternatives and derivatives of checkerboard, like **Pseudo Checkerboard** or **Double Checkerboard**. The Pseudo Checkerboard's each row (or column) starts and end with the same color (If there are odd nodes in each row). In a Double Checkerboard, two consecutive nodes are of the same color. (You get the picture, don't you?)
- Checkerboard with $\frac{1}{2} \times \frac{1}{2}$ sized cells. Proof of the rectangle with integer side problem.
- Color with "Roots of Unity".
- A knight's move always changes the color of the cell.

1 `USAMO 2014 P1`
2 `USAMO 2008 P3`
3 `IMO 2018 P4`
4 `Codeforces 101954/G`

2.1.3.1 Plane divided by lines

In problems regarding the plane being divided by straight lines, color the plane with chessboard colors.

1 `EGMO 2017 P3`

2.1.4 Divide and Conquer

Divide the problem/grid/graph into smaller pieces and work through them separately and finally join them together. The main difference between this and induction/recursion is that we have to actually work in the smaller cases instead of assuming that they are true.

1 `USA TST 2011 P2,` Capacity $1, 2$ roads
2 `CodeForces 744B,` Finding the minimum number in the rows
3 `Problem,` Double binary search
4 `ISL 2005 C1,` Lamps in rooms
5 `IOI 2016 P5,` Bug changes the strings
6 `Iran TST 2007 P2,` x divides at most one other element in A

2.1.4.1 Induction

Cauchy Induction: $n \to 2n, n \to n-1$
Can be used in almost any kind of problems, often called '*goriber bondhu*'.

- In MO probs $2-3-5-6$ or SL $3+$ (often $1, 2$ as well) you can be sure that applying only induction isn't going to do any good. You'll need extra tools, and you might need to apply induction more than once.
  - Sometimes, in graph probs, apply indution on more than one node gives better results.
  - Often you can set up your induction in more than one way, and finding the right way makes the problem much simpler.
  - Sometimes trying to prove more by adding a stronger induction hypothesis makes it easier to carry out the induction.

Type 1: $n-1 \to n$

1 `ISL 2004 C2,` n circles intersect, colors
2 `ISL 1997 P4,` Silver matrix
3 `ARO 2018 P11.5,` an easy graph
4 `ISL 2006 C1,` lamps will eventually be off
5 `ISL 2002 C1,` bijection in $x + y = n$ and red-blue colors
6 `ISL 2012 C2`
7 `ARO 2013 P9.4`
8 `ISL 2005 C2`
9 `ISL 2013 C3`
10 `ISL 2005 C1`
11 `ISL 2016 C6,` the ferry problem

12 IMO SL 1985
13 ELMO 2017 P5
14 ISL 1990
15 USAMO 2017 P4
16 Jacob Tsimerman Induction
17 All Russia 2017 9.1
18 Iran TST 2008 D3P1
19 USA TST 2011 D3P2
20 Sunflower Lemma
21 ISL 1998 C4
22 Generalization of USAMO 1999 P1
23 USAMO 2005 P1, arranging divisors on a circle with no co-prime neighbors
24 USAMO 2006 P5, a frog jumps jumps of 2-powers
25 Romanian TST 2016 D1P2, associating x_i with S_i
26 American Mathematical Monthly, n subsets from $S = 1 \dots n-1$ and a weird relation
27 ISL 1991 P10, Color the graph by numbers such that any vertex is gcd 1
28 Problem, Circles 1 unit apart, gotta cover them up.
29 IOI 2018 P1, Prefixes of a string
30 German TST 2004 E7P3, A white graph to a black graph
31 ISL 2002 C5, An finite family of sets of size r has a intersecting set of size $r-1$
32 US Dec TST 2016, P1, k bijections, and cycles in those

Type 2: $k(k < n-1) \to n$

1 ARO 2014 P9.3
2 Mexican Regional 2014 P6
3 USA TST 2011 P2
4 ISL 2006 C2
5 APMO 1999 P2 $a_{i+j} \leq a_i + a_j$

2.1.4.2 Inductive/Recursive Relations

Building other solutions depending on already or easily tweakable solutions.

1 India TST 2013 Test 3, P1
2 ISL 2002 C1
3 ISL 2009 C3
4 USAMO 2013 P2
5 IMO 2011 P4
6 ARO 2014 P9.7, stable coin system with coins of value α^k
7 Saint Petersburg 2001

2.1.4.2.1 Catalan Numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$, this little number is associated with a lot of combinatorial setups. And has the recursion.

2.1.5 Count the Shit Up

2.1.5.1 Double Counting

Explicitly count the number of things, but Twice!

1 `ISL 2016 C3,` n-gon colored with 3 colors, exists isosceles triangle.
2 `Iran TST 2012 P4,` Path inside of a $m \times n$.
3 `ISL 2014 C1,` Dissecting a Rectangle wrt to some given points inside.
4 `Problem,` 10 person bookstore problem.
5 `USA TST 2005 P1,` Subsets of the set $\{1, 2, \ldots, mn\}$
6 `USAMO 2012 P2,` 4 colors on the circle, rotating.
7 `ISL 2004 C1,` A Cauchy type function based on a coloring of the integers.
8 `China MO 2018 P2,` $3n^2 - 3n + 1 + k$ points are red, k good boxes exist.
9 `Problem`
10 `ISL 2003 C3`

2.1.5.2 Generating Function

For a sequence $A = (a_0, a_1, a_2 \ldots)$, the generating function$E_A(x)$ for this sequence is of several types types:

1 $E_A(x) = a_0x^0 + a_1x^1 \cdots + a_ix^i \ldots$, Useful for usual recursive sequences.
2 $E_A(x) = x^{a_0} + x^{a_1} \cdots + x^{a_i} \ldots$, Useful for sum of any two elements from *any* two sequences.
3 $E_A(x) = \prod(1 + x^{a_i})$, Useful for sum of multiple numbers from one sequence.
4 $E_A(t, x) = \prod(t + x^{a_i})$, Useful for keeping track of how many numbers are being added to the sum.

1 `Problem,` two sequence's pairwise sum's tuples are the same, $n = 2^k$.
2 `Result by Erdos,` Partitioning the integers into arithmetic sequences.
3 `Problem on Catalan's Recursion`

2.1.5.2.1 Roots of Unity These things can be used in a lot of places, like coloring boards, coloring (dividing into modular classes) the integers, using as variables in generating functions etc.

1 `Result by Erdos`

2.1.6 Different Representation

Represent the problem or the problem objects differently, usually by binary strings, graphs or matrices.

2.1.6.1 Binary

Associate a binary string to elements, like for handling subsets, add a string to each element, representing if it is in a certain set or not.

1 `IOI Practice 2017`

2 `ISL 1988 P10`

2.1.6.2 Binary Query

Asking questions of the kind: if in the binary expansion of k , if the i th bit is 0 , add k to one kind of query and if it is 1 , then add it to another.

1 `CodeForces 744B`

2 `Problem`

2.1.6.3 Matrix Creation

When there is some sort of $a \times b$ always try to create a matrix.

1 `IMO 2017 P5`

2.1.6.4 Graph

Problems concerning sets and their relations, consider representing using graph, with some fixed mapping rules. Some times changing grid cells into vertices's also helps.

1 `USAMO 1986 P2`

2 `Mexican Regional 2014 P6`

3 `AoPS`

4 `ARO 2013 P9.5`

5 `ISL 2002 C6`

6 `Problem,` Mailman messes up

7 `timus 1862,` Sum of operations

8 `ARO 2007 P9.7,` Adding diagrams that cut even number of already drawn ones

9 `ISL 2002 C3` Full-Sequences

2.1.6.5 Changing the Target Term

Changing the term you have to achieve to a slightly more intuitive one, usually thinking about what values you can get more naturally from the given conditions, and to build a similar term from the original term.

1 `ARO 2014 P10.8,` Mutually intersecting k-gons, one point inside of a bunch of gons

2.1.7 Algorithms

2.1.7.1 Greedy Algorithm

1 `ISL 2014 N3,` Cape Town Coin problem

2 `China TST 2006`

3 `Timus 1578`

2.1.7.2 Constructive Algorithm

In these kinda problems, you have to prove using a construction. In other words, proof by *Existence*. The key is to add one object to the solution set one at a time depending on already added objects in the set and maintaining the problem conditions. Sometimes by adding additional constraints or prioritizing already given constraints.

1 `Bulgarian IMO TST 2004, D3P3`

2 `ARO 2018 10.3`

3 `CodeForces 960/C`

4 `ARO 2005 P10.3, P11.2`

5 `IOI 2007 P3`

6 `Problem`

7 `ARO 2018 P11.5`

8 `ARO 2013 P9.4`

9 `ARO 2014 P9.8`

10 `ISL 2016 C1`

11 `India IMO Camp 2017`

12 `ISL 2012 C2`

13 `CodeForces 989C`

14 `CodeForces 989B`

15 `ISL 2011 A5`

16 `Iran TST 2017 D1P1`

17 `ISL 2009 C2`

18 `Problem`

19 `Putnam 2017 A4`

20 `Serbia TST 2017 P4`

21 `ISL 2014 A1`

22 `ISL 2005 N2,` sequence that contains all of the integers

23 `Problem,` switch states of a row and column

24 `USAMO 2015 P4,` piles of stone on cells, mone on the corners of a rectangle

25 `ISL 2003 C4`

2.1.7.3 Element : Time

Adding an element of Time to give the static problem a dynamic view. In less formal words, if a problem environments seems to just *exist*, add a dynamic way to slowly visualize the environment to exist. One kind of constructive algorithm, but this algo doesn't build the answer or solution, instead it builds up the whole environment step by step.

1 `ARO 2016 P3`

2 `Brazilian Olympic Revenge 2014`

3 `ISL 2008 C1`

4 `USAMO 1999 P1`

5 `AoPS`

2.1.7.4 Gaming Tricks

2.1.7.4.1 Pairing and Copying Who said you can't cheat in a combinatorial game? Just follow your opponents movements, and copy them cleverly.

1 `ARO 2011 P11.6,` Take a number of stones off the heap of size n^2

2.1.7.4.2 Nim Equivalence

2.1.8 Invariance Rules of Thumb

1 Natural Sum
2 Alternating Sum
3 Sum of Squares
4 Product
5 Giving weight to each of the elements, in problems where usually no trivial invariants exists. Like weights of 2^i, $\frac{1}{i}$, roots of unity etc. depending on the problem's nature.

- ISL 2014 C2, 1 written on 2^m papers and and an addition operation
- ISL 2012 C1, Operation almost alike to swapping and sorting
- Indian TST 2004, Pebble makes a clone and moves up and right
- ISL 1998 C7, One lamp on each cell, switching one lamp switches neighbors
- APMO 2017 P1, $a - b + c - d + e = 29$
- AoPS
- ARO 2016 P1
- Serbia TST 2017 P2, $(x + y) \to (\frac{x}{2}, y + \frac{x}{2})$ or $(x + \frac{y}{2}, \frac{y}{2})$
- ISL 1994 C3, 3 bank accounts
- Codeforces 987E
- ISL 2007 C4, Dividing a sequence with almost equal sum, and getting another sequence from it.
- USAMO 2015 P4, piles of stone on cells, mone on the corners of a rectangle

2.1.8.1 Monotonicity with strict constraints

If regular monotonicity doesn't apply, then some monotonicity with special properties might work. Like keeping the sum *even*, *odd*, *square* etc.

1 USAMO 2013 P6, Replace x with the difference of the two neighboring numbers.
2 MEMO 2008, Team, P6, $a, b \to a + b, a + b$

2.1.9 Pigeonhole Principal

2.1.9.1 Alternating Chains Technique

In a Cyclic graph with n nodes, if your task is to color some of the nodes so that no two neighboring nodes will be colored, you can color at most $\lfloor \frac{n}{2} \rfloor$. And in a path, this value is $\lceil \frac{n}{2} \rceil$

1 ISL 1990 P3

2 USAMO 2008 P3

2.1.10 Other Useful Techniques and Philosophies

2.1.10.1 Include potentially important players in the game

If the problem condition is completely or partially but crucially depended on some problem object, but the proof condition doesn't directly depend on that object, think of a way to include that object in the proof condition.

1 `APMO 2017 P3` **2** `USAMO 2008 P5`

2.1.10.2 Finding the Tough Nut

Solving an easier version of the problem with some sort of constraints lose, to find out exactly what makes the problem so tough. This way we get valuable information about on what our main focus should be.

2.1.10.3 eChen trick

Subtract a constant from all the numbers to make the sum 0. This makes the numbers easier to handle.

1 `APMO 2017 P1` **2** `ARO 2013 P9.5`

2.1.10.4 Send objects to the infinity

If there are too many arbitrary objects in the problem, try making some of them vanish by sending them to the infinity.

2.1.10.5 n+1 = (n-i) + (i+1)

$n+1=(n-i)+(i+1)$ might prove to be useful when applying induction to $\binom{n}{k}$

2.1.10.6 Convex Hulls, and Sandwiching two points

You know what convex hull is. AND, One can draw two lines to separate two points - draw two parallel lines very close to the two points.

1 `ISL 2013 C2`

2 `Putnam 1979,` n red n blue, pair them

3 `USAMO 2005 P5,` n red n blue, at least two segments dividing them

4 `ILL 1985`

2.2 Binomial Identities

Theorem 2.2.1: Vandermonde's identity

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}$$

Theorem 2.2.2:

$$\binom{2n}{n} = \binom{p}{0}^2 \binom{p}{1}^2 \cdots + \binom{p}{p}^2$$

Problem 2.2.1: Somewhere **M**

Let n and m be positive integers with $n < m$. Prove that

$$\sum_{k=0}^{n} \frac{(-1)^{n-k}}{m-k}\binom{n}{k} = \frac{1}{(n+1)\binom{m}{n+1}}.$$

Idea Expressing the left side as the finite difference of a polynomial. □

Problem 2.2.2: USA TST 2010 P8 **M**

Let m, n be positive integers with $m \geq n$, and let S be the set of all n-term sequences of positive integers $(a_1, a_2, \ldots a_n)$ such that $a_1 + a_2 + \cdots + a_n = m$. Show that

$$\sum_{S} 1^{a_1} 2^{a_2} \cdots n^{a_n} = \binom{n}{n} n^m - \binom{n}{n-1}(n-1)^m + \cdots + (-1)^{n-2}\binom{n}{2}2^m + (-1)^{n-1}\binom{n}{1}$$

this problem is just f**king amazing, both the combinatorial and algebraic solutions are cute as hell

Idea First, the combi solution: Look at the right side, try to translate it to combinatorial model. If we had used the inclusion-exclusion method, the right side would be the number of ways to color a row of length m with n colors, with each color appearing at least once. Now our remaining job is to prove the same for the left side. □

Idea Now, the algebraic, GF solution: Now that we have made up our mind that we are going to use GF, let's try to apply GF to the more innocent looking term on the left side. We quickly get a polynomial for that. Now the right side, alternating sums of binomials, gives an experienced psolver the idea of finite differences. If we think of the polynomial as the n'th finite difference of a polynomial, we need the original polynomial to be a 2D poly. from here the idea follows and we get a nice polynomial for the right side. But how do we show that the two polynomials on the two sides are equal? We need to get a closed form for the right side poly. But to get this, we think inductively. □

Problem 2.2.3: Binom 1

$$\binom{2n}{0} - \binom{2n-1}{1} + \binom{2n-2}{2} - \ldots + (-1)^n \binom{n}{n} = ?$$

Idea Generating funtion □

Idea Combinaorial Model: Consider the number of ways to tile a $1 \times 2n$ rectangle with squares and dominoes. Let E be the set of ways to do it with an even number of dominoes, and let O be the same for odd. We want to find $|E| - |O|$. □

Problem 2.2.4: Binom 2

$$\binom{n}{0}^2 - \binom{n}{1}^2 + \binom{n}{2}^2 - \ldots + (-1)^n \binom{n}{n}^2 = ?$$

Idea Generating function □

Idea Consider the number of ways to paint the squares of a $2 \times n$ rectangle red and blue such that both rows have the same number of red squares. Let E be the set of ways with an even number of red squares in each row, and let O be the same for odd. We want to find $|E| - |O|$. $\square$

2.3 Sets

2.3.1 Lemmas

Lemma 2.3.1

Let S be a set with n elements, and let F be a family of subsets of S such that for any pair A, B in F, $A \cap B \neq \varnothing$. Then $|F| \leq 2^{n-1}$.

1 Iran TST 2008 D3P1

Theorem 2.3.2: Erdos Ko Rado theorem

Suppose that A is a family of distinct subsets of $\{1, 2 \ldots n\}$ such that each subset is of size r and each pair of subsets has a nonempty intersection, and suppose that $n \geq 2r$. Then the number of sets in A is less than or equal to the binomial coefficient

$$\binom{n-1}{r-1}$$

Lemma 2.3.3

Let S be a set with n elements, and let F be a family of subsets of S such that for any pair A, B in F, S is not contained by $A \cup B$. Then $|F| \leq 2^{n-1}$.

Lemma 2.3.4: Kleitman lemma

A set family F is said to be downwards closed if the following holds: if X is a set in F, then all subsets of X are also sets in F. Similarly, F is said to be upwards closed if whenever X is a set in F, all sets containing X are also sets in F. Let F_1 and F_2 be downwards closed families of subsets of $S = \{1, 2, ..., n\}$, and let F_3 be an upwards closed family of subsets of S. Then we have

$$|F_1 \cap F_2| \geq \frac{|F_1| \cdot |F_2|}{2^n} \tag{2.1}$$

$$|F_1 \cap F_3| \leq \frac{|F_1| \cdot |F_3|}{2^n} \tag{2.2}$$

Lemma 2.3.5

Let S be a set with n elements, and let F be a family of subsets of S such that for any pair A, B in F, $A \cap B \neq \varnothing$ and $A \cap B \neq S$. Then $|F| \leq 2^{n-2}$.

Idea Using the sets in lemma 1 and lemma 2, defining upwards and downwards sets like in Kleitman's Lemma. □

Lemma 2.3.6: The Sunflower Lemma

A sunflower with k petals and a core X is a family of sets $S_1, S_2, \ldots, S_k$ such that $S_i \cap S_j = X$ for each $i \neq j$. (The reason for the name is that the Venn diagram representation for such a family resembles a sunflower.) The sets $S_i \setminus X$ are known as petals and must be nonempty, though X can be empty. Show that if F is a family of sets of cardinality s, and $|F| > s!(k-1)^s$, then F contains a sunflower with k petals.

Idea Applying induction and considering the best case where $|X| = 0$ □

2.3.2 Extremal Set Theory

MIT 18.314 Lecture-8

Theorem 2.3.7: Mirsky Theorem

A set S with a chain of height h can't be partitioned into t anti-chains if $t < h$. In other words, the minimum number of sets in any anti-chain partition of S is equal to the maximum height of the chains in S. (And Vice Versa)

Theorem 2.3.8:

In any poset, the largest cardinality of an antichain is at most the smallest cardinality of a chain-decomposition of that poset.

Theorem 2.3.9: Dilworth's Theorem

Let P be a poset. Then there exist an antichain A and a chain decomposition $\mathcal{C}$ of P such that $|A| = |\mathcal{C}|$

Theorem 2.3.10: Erdos-Szekeres Theorem

Any sequence of $ab+1$ real numbers contains either a monotonically decreasing subsequence of length $a+1$ or a monotonically increasing subsequence of length $b+1$. The more useful case is when $a = b = n$.

Problem 2.3.1: **E**

Let $n \geq 1$ be an integer and let X be a set of n^2+1 positive integers such that in any subset of X with $n+1$ elements there exist two elements $x \neq y$ such that $x|y$. Prove that there exists a subset $\{x_1, x_2 \ldots x_{n+1} \in X$ such that $x_i|x_{i+1}$ for all $i = 1, 2, \ldots n$.

2.3.3 Problems

Problem 2.3.2: USA TST 2005 P1 **E**

Let n be an integer greater than 1. For a positive integer m , let $S_m = \{1, 2, \ldots, mn\}$. Suppose that there exists a $2n$ -element set T such that

1 each element of T is an m -element subset of S_m
2 each pair of elements of T shares at most one common element
3 each element of S_m is contained in exactly two elements of T

Determine the maximum possible value of m in terms of n.

Idea We use double counting to find the ans, after that the rest is easy. □

Problem 2.3.3: Iran TST 2008 D3P1 **E**

Let S be a set with n elements, and F be a family of subsets of S with 2^{n-1} elements, such that for each $A, B, C \in F$, $A \cap B \cap C$ is not empty. Prove that the intersection of all of the elements of F is not empty.

Idea Using Induction with this lemma. □

Problem 2.3.4: Romanian TST 2016 D1P2 **EM**

Let n be a positive integer, and let $S_1, S_2, \ldots S_n$ be a collection of finite non-empty sets such that

$$\sum_{1 \le i < j \le n} \frac{|S_i \cap S_j|}{|S_i||S_j|} < 1.$$

Prove that there exist pairwise distinct elements $x_1, x_2 \ldots x_n$ such that x_i is a member of S_i for each index i.

Idea The Inductive proof reduces the problem to this problem. □

Idea The other approach is to focus on the given weird condition, and interpolate it to something nice, like probabilistic condition. □

Problem 2.3.5: American Mathematical Monthly problem E2309 **EM**

If A_1, A_2,... A_n are n nonempty subsets of the set $\{1, 2, ..., n-1\}$, then prove that $\sum_{1\le i<j\le n} \frac{|A_i \cap A_j|}{|A_i|\cdot|A_j|} \ge 1$.

Problem 2.3.6: CGMO 2010 P1 **E**

Let n be an integer greater than two, and let $A_1, A_2, \cdots, A_{2n}$ be pairwise distinct subsets of $\{1, 2, , n\}$. Determine the maximum value of

$$\sum_{i=1}^{2n} \frac{|A_i \cap A_{i+1}|}{|A_i| \cdot |A_{i+1}|}$$

Where $A_{2n+1} = A_1$ and $|X|$ denote the number of elements in X.

Problem 2.3.7: ISL 2002 C5 **M**

Let $r \ge 2$ be a fixed positive integer, and let F be an infinite family of sets, each of size r, no two of which are disjoint. Prove that there exists a set of size $r-1$ that meets each set in F.

Generalization 2.3.7.1: HMMT 2016 Team Round

Fix positive integers $r > s$, and let $\mathcal{F}$ be an infinite family of sets, each of size r, no two of which share fewer than s elements. Prove that there exists a set of size $r-1$ that shares at least s elements with each set in F.

Idea First idea, if we take an arbitrary set, we can say that there exists infinitely many sets $\in \mathbb{F}$ which includes a fixed element from our test set. If we do this argument for $r-1$ times, we get a set X of $r-1$ elements, and an infinte family of sets that contains X completely. At this point the problem is trivial. □

Idea Since it's tricky to work with one family, why not introduce another family, like the second monk. This solution generalizes the problem as such. □

Problem 2.3.8: ISL 1988 P10 **M**

Let $N = \{1, 2 \ldots, n\}, n \ge 2$. A collection $F = \{A_1, \ldots, A_t\}$ of subsets $A_i \subseteq N$, $i = 1, \ldots, t$, is said to be separating, if for every pair $\{x, y\} \subseteq N$, there is a set $A_i \in F$ so that $A_i \cap \{x, y\}$ contains just one element. F is said to be covering, if every element of N is contained in at least one set $A_i \in F$. What is the smallest value $f(n)$ of t, so there is a set $F = \{A_1, \ldots, A_t\}$ which is simultaneously separating and covering

Idea Using Binary Representations for the elements as in or not in, we get an easy bijection. □

Problem 2.3.9: Iran TST 2013 D1P2 **E**

Find the maximum number of subsets from $\{1, ..., n\}$ such that for any two of them like A, B if $A \subset B$ then $|B - A| \geq 3$. (Here $|X|$ is the number of elements of the set X.)

2.4 Algorithmic

- Handout by Cody Johnson

2.4.1 Data Structures

Binary Heap Segment tree where, node `n` has children `2n+1, 2n+2` .

1 `timus 1862,` Sum of operations

Disjoint Set This data structure keeps track of connectivity by assigning a representative to a connected subset. And for any two nodes, `u, v` they are connected iff they have the same representative.

2.4.2 Minimal Spanning Tree

A minimum spanning tree or minimum weight spanning tree is a subset of the edges of a connected, edge-weighted (un)directed graph that connects all the vertices's together, without any cycles and with the minimum possible total edge weight.

Pics/minimum_spanning_tree.png

Figure 2.1: Minimal Spanning Tree

Lemma 2.4.1: MST Cut

For any cut of the graph, the lightest edge in that cut-set is in evey MST of the graph.

Kruskal's Algorithm

Kruskal's algorithm is a 'minimum-spanning-tree algorithm' which finds an edge of the least possible weight that connects any two trees in the forest. It is a greedy algorithm in graph theory as it finds a minimum spanning tree for a connected weighted graph adding increasing cost arcs at each step.

Idea To optimize this algorithm, Disjoint Set DS is used. □

Prim's Algorithm

Greedily build the tree by adding edges one by one. At one step we add the minimal cost edge that connects the tree to the vertices's that are not in the tree.

2.4.3 Shortest Path Problem

Finding the shortest path between two nodes in a weighted or unweighted graph.

Breadth-First Search

This algo runs from a node and ''levelizes'' the other nodes.

Dijkstra's Algorithm

It picks the unvisited vertex with the lowest distance, calculates the distance through it to each unvisited neighbor, and updates the neighbor's distance if smaller.

2.4.4 Other CP Tricks

Theorem 2.4.2: Swap Sort

In any swap sorting algorithm, the number of swaps needed has the same parity.

Convex Hull Trick

Given a lot of lines on the plane, and a lot of queries each asking for the smallest value for y among the lines for a given x, the optimal strategy is to sort the lines according to their slopes, and adding them to a stack, checking if they are relevant to the 'minimal' convex hull of those lines.

2.4.5 Problems

Problem 2.4.1: Iran TST 2018 P1 **E**

Let $A_1, A_2, ..., A_k$ be the subsets of $\{1, 2, 3, ..., n\}$ such that for all $1 \leq i, j \leq k$:$A_i \cap A_j \neq \varnothing$. Prove that there are n distinct positive integers $x_1, x_2, ..., x_n$ such that for each $1 \leq j \leq k$:

$$lcm_{i \in A_j} \{x_i\} > lcm_{i \notin A_j} \{x_i\}$$

Idea Apply induction on either k or n. $\square$

Problem 2.4.2: ISL 2016 C1 **TE**

The leader of an IMO team chooses positive integers n and k with $n > k$, and announces them to the deputy leader and a contestant. The leader then secretly tells the deputy leader an n -digit binary string, and the deputy leader writes down all n -digit binary strings which differ from the leader's in exactly k positions. (For example, if $n = 3$ and $k = 1$, and if the leader chooses 101 , the deputy leader would write down $001, 111$ and 100.) The contestant is allowed to look at the strings written by the deputy leader and guess the leader's string. What is the minimum number of guesses (in terms of n and k) needed to guarantee the correct answer?

Idea Small cases check. $\square$

Problem 2.4.3: ISL 2005 C2 **E**

Let $a_1, a_2, \ldots$ be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer n the numbers $a_1, a_2, \ldots, a_n$ leave n different remainders upon division by n. Prove that every integer occurs exactly once in the sequence $a_1, a_2, \ldots$.

Idea Constructing for the beginning. $\square$

Problem 2.4.4: IOI Practice 2017 **M**

C plays a game with A and B. There's a room with a table. First C goes in the room and puts 64 coins on the table in a row. Each coin is facing either heads or tails. Coins are identical to one another, but one of them is cursed. C decides to put that coin in position c. Then he calls in A and shows him the position of the cursed coin. Now he allows A to flip some coins if he wants (he can't move any coin to other positions). After that A and C leave the room and sends in B. If B can identify the cursed coin then C loses, otherwise C wins.
The rules of the game are explained to A and B beforehand, so they can discuss their strategy before entering the room. Find the minimum number k of coin flips required by A so that no matter what configuration of 64 coins C gives them and where he puts the cursed coin, A and B can win with A flipping at most k coins.
Find constructions for $k = 32, 8, 6, 3, 2, 1$

Idea XOR XOR XOR binary representation ☐

Problem 2.4.5: Codeforces 987E **E**

Petr likes to come up with problems about randomly generated data. This time problem is about random permutation. He decided to generate a random permutation this way: he takes identity permutation of numbers from 1 to n and then $3n$ times takes a random pair of different elements and swaps them. Alex envies Petr and tries to imitate him in all kind of things. Alex has also come up with a problem about random permutation. He generates a random permutation just like Petr but swaps elements $7n + 1$ times instead of $3n$ times. Because it is more random, OK?!

You somehow get a test from one of these problems and now you want to know from which one.

Idea This theorem kills this problem instantly. ☐

Problem 2.4.6: USAMO 2013 P6 **H**

At the vertices's of a regular hexagon are written six nonnegative integers whose sum is 2003^{2003}. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

Idea Firstly what comes into mind is to decrease the maximum value, but since this is a P6, there must be some mistakes. Surely, we can't follow this algo in the case $(k, k, 0, k, k, 0)$. But this time, the sum becomes even. So we have to slowly minimize the maximum, keeping the sum odd. And since only odd number on the board is the easiest to handle, we solve that case first, and the other cases can be easly handled with an additional algo. □

Problem 2.4.7: ISL 2012 C1 **E**

Several positive integers are written in a row. Iteratively, Alice chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y , and replaces the pair (x, y) by either $(y + 1, x)$ or $(x - 1, x)$. Prove that she can perform only finitely many such iterations.

Idea Easy invariant. □

Problem 2.4.8: AoPS **E**

There is a number from the set $\{1, -1\}$ written in each of the vertices's of a regular do-decagon (12 -gon). In a single turn we select 3 numbers going in the row and change their signs. In the beginning all numbers, except one are equal to 1. Can we transfer the only -1 into adjacent vertex after a finite number of turns?

Idea Algo+Proof $\implies$ Invariant. □

Problem 2.4.9: ISL 1994 C3 **EH**

Peter has three accounts in a bank, each with an integral number of dollars. He is only allowed to transfer money from one account to another so that the amount of money in the latter is doubled. Prove that Peter can always transfer all his money into two accounts. Can Peter always transfer all his money into one account?

Idea Since we want to decrease the minimum, and one of the most simple way is to consider Euclidean algorithm. So we sort the accounts, $A < B < C$, and write $B = qA + r$, and do some experiment to turn B into r. □

Problem 2.4.10: MEMO 2008, Team, P6 **M**

On a blackboard there are $n \geq 2, n \in \mathbb{Z}^+$ numbers. In each step we select two numbers from the blackboard and replace both of them by their sum. Determine all numbers n for which it is possible to yield n identical number after a finite number of steps.

Idea The pair thing rules out the case of odds. For evens, we make two identical sets, and focus on only one of the sets, with an additional move $x \to 2x$ available to use. Since we can now change the powers of 2 at out will at any time, we only focus on the greatest odd divisors. Our aim is to slowly decrease the largest odd divisor. $\square$

Problem 2.4.11: US Dec TST 2016, P1 **E**

Let $S = \{1, \dots, n\}$. Given a bijection $f : S \to S$ an orbit of f is a set of the form $\{x, f(x), f(f(x)), \dots\}$ for some $x \in S$. We denote by $c(f)$ the number of distinct orbits of f. For example, if $n = 3$ and $f(1) = 2$, $f(2) = 1$, $f(3) = 3$, the two orbits are $\{1, 2\}$ and $\{3\}$, hence $c(f) = 2$.

Given k bijections f_1, ..., f_k from S to itself, prove that

$$c(f_1) + \cdots + c(f_k) \leq n(k-1) + c(f)$$

where $f : S \to S$ is the composed function $f_1 \circ \cdots \circ f_k$.

Idea Induction reduces the problem to the case of $k = 2$. Then another inuction on $c(f_1)$ solves the problem. The later induction works on the basis of the fact that a "swap" in the bijection changes the number of cycles by 1 (either adds $+1$ or -1). $\square$

Problem 2.4.12: Cody Johnson **E**

Consider a set of 6 integers $S = \{a_1 \dots a_6\}$. At on step, you can add $+1$ or -1 to all of the 6 integers. Prove that you can make a finite number of moves so that after the moves, you have $a_1a_5a_6 = a_2a_4a_6 = a_3a_4a_5$

Problem 2.4.13: ISL 2014 A1 **E**

Let $a_0 < a_1 < a_2 \dots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + a_2 + \cdots + a_n}{n} \leq a_{n+1}.$$

Idea My idea was to construct the sequence with the assumtion that the condition is false. It leads to either all of the right ineq false or the condition being true. □

Idea The magical solution: defining $b_n = (a_n - a_{n-1}) + \cdots + (a_n - a_1)$ which eases the inequlity. □

Idea The beautiful solution: defining δ_i as $a_n = a_0 + \Delta_1 + \Delta_2 + ... + \Delta_n$ for all n, i. □

Idea Another idea is to first prove the existence and then to prove the uniqueness. □

Problem 2.4.14: ISL 2014 N3 **EM**

For each positive integer n, the Bank of Cape Town issues coins of denomination $\frac{1}{n}$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most most $99 + \frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.

Idea Notice that the sum of the geometric series $S = \dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} \dots$ is 1. And in another problem we partitioned the set of integers into subsets with each subset starting with an odd number k and every other elements of the subset being $2^i * k$. We do similarly in this problem, and partition the set of the coins in a similar way. Then we take the first 100 sets whose sum is less than 1 and insert the other left coins in these sets, with the condition that the sum of all of the coins is $99 + \dfrac{1}{2}$. Solu □

Idea Replacing 100 by n, we show that for all n the codition is valid. Assume otherwise. Take the minimal n for which the condition does not work. Ta-Da! We can show that if n does not work, so doesn't $n-1$. Solu □

Idea Or just be an EChen and prove the result for at most $k - \frac{k}{2k+1}$ with k groups. □

Idea Very similar to this problem ☐

Problem 2.4.15: China TST 2006 **E**

Given positive integer n, find the biggest real number C which satisfy the condition that if the sum of the reciprocals ($\frac{1}{n}$ is the reciprocal of n) of a set of integers (They can be the same.) that are greater than 1 is less than C, then we can divide the set of numbers into no more than n groups so that the sum of reciprocals of every group is less than 1.

Problem 2.4.16: **E**

In a $n * n$ grid, every cell is either black or white. A 'command' is a pair of integers, $i, j \leq n$, after which all of the cells in the i^{th} row and the j^{th} column (meaning a total of $2n-1$ cells) will switch the state. Our goal is to make every cell of the same state.

1. Prove that if it can be done, it can be done in less than $\frac{n^2}{2}$ commands.
2. Prove that it can always be done if n is even.
3. Prove or disprove for odd n.

Idea (a) is really easy, just take into account that flipping all cells result in the switch of all of the cells. And the question did not ask for an algorithm. ☐

Idea (b) is also easy, notice that we can pair the columns and then make them look like the same, with a compound command. A better algo is to take the original algo and to modify it like, take one cell, then do the original move on all cells in the row and column of this cell. ☐

Idea (c) uses Linear Algebra, which I dont know yet, or... use double counting to build the criteria of the fucntion f : states $\to$ subset of moves being bijective. ☐

Problem 2.4.17: OIM 1994, PSMiC **E**

In every square of an $n \times n$ board there is a lamp. Initially all the lamps are turned off. Touching a lamp changes the state of all the lamps in its row and its column (including the lamp that was touched). Prove that we can always turn on all the lamps and find the minimum number of lamps we have to touch to do this.

Problem 2.4.18: Timus 1578 **E**

The very last mammoth runs away from a group of primeval hunters. The hunters are fierce, hungry and are armed with bludgeons and stone axes. In order to escape from his pursuers, the mammoth tries to foul the trail. Its path is a polyline (not necessarily simple). Besides, all the pairs of adjacent segments of the polyline form acute angles (an angle of 0 degrees is also considered acute).

After the mammoth vanished, it turned out that it had made exactly N turns while running away. The points where the mammoth turned, as well as the points where the pursuit started and where the pursuit ended, are known. You are to determine one if the possible paths of the mammoth.

Problem 2.4.19: CodeForces 744B **E**

Given a hidden matrix of $n \times n$, $n \leq 1000$ where for every i , $M_{(i,i)=0}$, Luffy's task is to find the minimum value in the n rows, formarly spoken, he has to find values $\min_{j=1\ldots n,\ j\neq i} M_{(i,j)}$. To do this he can ask the computer questions of following types: In one question, Luffy picks up a set, $a_1, a_2 \ldots a_k$ with $a_i, k \leq n$. And gives the computer this set. The computer will respond with n integers. The i -th integer will contain the minimum value of $\min_{j=1\ldots k} M_{(i,a_j)}$. And on top of this, he can only ask 20 questions. Luffy being the stupid he is, doesn't even have clue how to do this, you have to help him solving this problem.

Idea If we draw the diagonal in the matrix, we see that we can fit boxes of $2^i \times 2^i$ in there depending on the i 's value. Now after we have decomposed the matrix into such boxes, we can choose several from them to ask a question. The trick is that for every row, there must be questions asked from each of the boxes this row covers and no question from here must contain the (i,i) cell. □

Idea The magical solution goes as following: For $i \leq 10$, for every $k = 1 \ldots n$, include k in the question if the i th bit of k 's binary form is 0. And then for the second round include k in the question if the i th bit of k 's binary form is 1. □

Problem 2.4.20: **E**

Alice wants to add an edge (u, v) in a graph. You want to know what this edge is. So, you can ask some questions to Alice. For each question, you will give Alice 2 non-empty disjoint sets S and T to Alice, and Alice will answer "true" iff u and v belongs to different sets. You can ask atmost $3 * \lceil log_2|V|\rceil$ questions to Alice. Describe a strategy to find the edge (u, v).

Idea First find one true answer in $\lceil log_2|V|\rceil$ questions, and then get the result out of these two sets in $2 * \lceil log_2|V|\rceil$ questions. $\square$

Idea The magical solution goes as following: In the i^{th} question, $S = x : i^{th}$ bit of x is 0, $T = x : i^{th}$ bit of x is 1 $\square$

Problem 2.4.21: USAMO 2015 P4 **E**

Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he finished piling his stones in some manner, he can then perform stone moves, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions $(i, k), (i, l), (j, k), (j, l)$ for some $1 \leq i, j, k, l \leq n$, such that $i < j$ and $k < l$. A stone move consists of either removing one stone from each of (i, k) and (j, l) and moving them to (i, l) and (j, k) respectively,j or removing one stone from each of (i, l) and (j, k) and moving them to (i, k) and (j, l) respectively.

Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves.

How many different non-equivalent ways can Steve pile the stones on the grid?

Idea Building an invariant, we see that only the sum of the columns is not sufficient. So to get more control, we take the row sums into account as well. $\square$

Problem 2.4.22: ISL 2003 C4 **E**

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be real numbers. Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be the matrix with entries

$$a_{ij} = \begin{cases} 1, & \text{if } x_i + y_j \geq 0; \\ 0, & \text{if } x_i + y_j < 0. \end{cases}$$

Suppose that B is an $n \times n$ matrix with entries 0, 1 such that the sum of the elements in each row and each column of B is equal to the corresponding sum for the matrix A. Prove that $A = B$.

Idea If done after this problem, this problem seems straigthforward. $\square$

Problem 2.4.23: India TST 2017 D1 P3 **E**

Let $n \geq 1$ be a positive integer. An $n \times n$ matrix is called *good* if each entry is a non-negative integer, the sum of entries in each row and each column is equal. A *permutation* matrix is an $n \times n$ matrix consisting of n ones and $n(n-1)$ zeroes such that each row and each column has exactly one non-zero entry.

Prove that any *good* matrix is a sum of finitely many *permutation* matrices.

Idea Same algo as above. Either distributing uniformly or gathering all in a diagonal ☐

Problem 2.4.24: Tournament of Towns 2015F S7 **MH**

N children no two of the same height stand in a line. The following two-step procedure is applied: first, the line is split into the least possible number of groups so that in each group all children are arranged from the left to the right in ascending order of their heights (a group may consist of a single child). Second, the order of children in each group is reversed, so now in each group the children stand in descending order of their heights. Prove that in result of applying this procedure $N-1$ times the children in the line would stand from the left to the right in descending order of their heights.

Idea It's obvious that we need to find some invariant or mono-variant. Now, an idea, we need to show that for any i, for it to be on it's rightful place, it doesn't need more than $N-1$ moves. How do we show that? Another idea, think about the bad bois on either of its sides. Now, observation, 'junctions' decrease with each move. Find the 'junctions'. ☐

Problem 2.4.25: Polish OI **E**

Given n jobs, indexed from $1, 2 \ldots n$. Given two sequences of reals, $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n$ where, $0 \leq a_i, b_i \leq 1$. If job i starts at time t , then the job takes $h_i(t) = a_i t + b_i$ time to finish. Order the jobs in a way such that the total time taken by all of the jobs is the minimum.

Idea Example of a problem which is solved by investigating two adjacent objects in the optimal arrangement. ☐

Problem 2.4.26: CodeForces 960/C **E**

Pikachu had an array with him. He wrote down all the non-empty subsequences of the array on paper. Note that an array of size n has $2^n - 1$ non-empty subsequences in it.

Pikachu being mischievous as he always is, removed all the subsequences in which

$$\text{Maximum element of the subsequence} - \text{Minimum element of subsequence} \geq d$$

Pikachu was finally left with X subsequences.

However, he lost the initial array he had, and now is in serious trouble. He still remembers the numbers X and d. He now wants you to construct any such array which will satisfy the above conditions. All the numbers in the final array should be positive integers less than 10^{18}.

Note the number of elements in the output array should not be more than 10^4. If no answer is possible, print -1.

Problem 2.4.27: ARO 2005 P10.3, P11.2 **M**

Given 2005 distinct numbers $a_1, a_2, \ldots, a_{2005}$. By one question, we may take three different indices $1 \leq i < j < k \leq 2005$ and find out the set of numbers $\{a_i, a_j, a_k\}$ (unordered, of course). Find the minimal number of questions, which are necessary to find out all numbers a_i.

Idea The key idea is to ask questions such that it is connected to multiple other questions, and each question uniquely finds out multiple elements together. One by itself immediately after the question has been asked, and one after the next question which is related to this one has been asked. As we find out three elements' values after one question, first, second, third, so, let us find first from the previous question, second from the current question, third from the next question. □

Problem 2.4.28: IOI 2007 P3 **M**

You are given two sets of integers $A = \{a_1, a_2 \ldots a_n\}$ and $B = \{b_1, b_2 \ldots b_n\}$ such that $a_i \geq b_i$. At move i you have to pick b_i distinct integers from the set $A_i = \{1, 2, \ldots a_i\}$. In total, $(b_1 + b_2 + \cdots + b_n)$ integers are selected, but not all of these are distinct. Suppose k distinct integers have been selected, with multiplicities $c_1, c_2, c_3 \ldots c_k$. Your score is defined as

$$\sum_{i=1}^{k} c_i(c_i - 1)$$

Give an efficient algorithm to select numbers in order to "minimize" your score.

Idea Some investigation shows that if $c_i > c_j + 1$ and $i > j$, then we can always minimize the score. and if $i < j$, then we can minimize the score only when $i, j \in A_k$ but i has been taken at move k , but j hasn't. So in the minimal state, either both i, j has been taken at move k , or $a_k < j$. So the idea is to take elements from A_i as large as possible, and then taking smaller values after wards if the c_i value of a big element gets more than that of a small element. In this algorithm, we see that we greedily manipulate c_i. So it is a good idea to greedily choose c_i 's from the very beginning. $\square$

Idea **Solution Algo:** at step i , take the set $\{c_1, c_2 \dots c_{a_i}\}$ and take the smallest b_i from this set, and add 1 to each of them (in other words, take their index numbers as the numbers to take). $\square$

Problem 2.4.29: **E**

Given n numbers $\{a_1, a_2, ..., a_n\}$, you have to select k of them such that no two consecutive numbers are selected and their sum is maximized.

Idea Notice that if a_i is the maximum value, and if a_i is not counted in the optimal solution, then both of a_{i-1}, a_{i+1} must be in the optimal solution, and $a_{i-1} + a_{i+1} > a_i$. And if a_i is counted in the optimal solution, then none of a_{i-1}, a_{i+1} can be counted in the optimal solution. So either way, we can remove these three and replace them by a single element to use induction. So remove a_{i-1}, a_{i+1} and replace a_i by $a_{i-1} + a_{i+1} - a_i$. $\square$

Problem 2.4.30: USAMO 2010 P2 **EM**

There are n students standing in a circle, one behind the other. The students have heights $h_1 < h_2 < \cdots < h_n$. If a student with height h_k is standing directly behind a student with height h_{k-2} or less, the two students are permitted to switch places. Prove that it is not possible to make more than $\binom{n}{3}$ such switches before reaching a position in which no further switches are possible.

Problem 2.4.31: Serbia TST 2015 P3 **H**

We have 2015 prisinoers. The king gives everyone a hat coloured in one of 5 colors. Everyone sees all hats expect his own. Now,the King orders them in a line(a prisioner can see all guys behind and in front of him). The king asks the prisinoers one by one does he know the color of his hat.If he answers NO,then he is killed.If he answers YES,then answers which color is his hat,if his answers is true,he goes to freedom,if not,he is killed.All the prisinors can hear did he answer YES or NO,but if he answered YES,they don't know what did he answered(he is killed in public).They can think of a strategy before the King comes,but after that they can't comunicate.What is the largest number of prisinors we can guarentee that can survive?

Problem 2.4.32: Taiwan TST 2015 R3D1P1 **E**

A plane has several seats on it, each with its own price, as shown below. $2n-2$ passengers wish to take this plane, but none of them wants to sit with any other passenger in the same column or row. The captain realize that, no matter how he arranges the passengers, the total money he can collect is the same. Proof this fact, and compute how much money the captain can collect.

`Pics/TaiwanTST2015R3D1P1.png`

Figure 2.2

Problem 2.4.33: ISL 2014 N1 **E**

Let $n \geq 2$ be an integer, and let A_n be the set

$$A_n = \{2^n - 2^k \mid k \in \mathbb{Z},\ 0 \leq k < n\}.$$

Determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of A_n .

Idea Inductive approach □

Problem 2.4.34: Codeforces 330D **E**

Idea Generalize the condition for a meet-up. □

2.4.5.1 Covering Area with Squares

- A nice blog post by ankogonit

Problem 2.4.35: Brazilian MO 2002, ARO 1979 **E**

Given a finite collection of squares with total area at least 4, prove that you can cover a unit square completely with these squares (with overlapping allowed, of course).

Idea Maybe motivated by the number 4 and how nice it would be if all the squares had 2's power side lengths, the idea is to shrink every square to a side with side of 2's power. □

Problem 2.4.36: ARO 1979's Sharper Version **E**

Given a finite collection of squares with total area at least 3, prove that you can cover a unit square completely with these squares (with overlapping allowed).

Idea The idea is to greedily cover the unit square by covering the lowest row uncovered. And then using boundings to prove that it is possible. □

Problem 2.4.37: **E**

Prove that a finite collection of squares of total area $\frac{1}{2}$ can be placed inside a unit square without overlap.

Idea The same idea as before. □

Problem 2.4.38: Tournament of Towns Spring 2012 S7 **EM**

We attempt to cover the plane with an infinite sequence of rectangles, overlapping allowed.

1. Is the task always possible if the area of the $n-$th rectangle is n^2 for each n?
2. Is the task always possible if each rectangle is a square, and for any number N, there exist squares with total area greater than N?

Idea Identical algo and proving technique as above. ☐

Idea Using the first problem in this subsection to find a better algo. ☐

Problem 2.4.39: ISL 2006 C6 **E**

A holey triangle is an upward equilateral triangle of side length n with n upward unit triangular holes cut out. A diamond is a $60° - 120°$ unit rhombus.

Prove that a holey triangle T can be tiled with diamonds if and only if the following condition holds: Every upward equilateral triangle of side length k in T contains at most k holes, for $1 \le k \le n$.

Idea Think of induction and how you can deal with that. ☐

Problem 2.4.40: Putnam 2002 A3 **E**

Let N be an integer greater than 1 and let T_n be the number of non empty subsets S of $\{1, 2, ..., n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

Idea Small cases check and an bijective approach suggests that the fact of the average being in or out of the set is important. ☐

Problem 2.4.41: USAMO 1998 **E**

Prove that for each $n \ge 2$, there is a set S of n integders such that $(a-b)^2$ divides ab for every distinct $a, b \in S$.

Idea Induction comes to the rescue. Trying to find a way to get from n to $n+1$, we see that we can *shift* the integers by any integer k. So after shifting, what stays the same, and what changes? ☐

2.5 Graph Theory

Stuck? Try These: Turning grids into graphs

- One common way to turn a grid into graphs is to create a bipartite graph between the columns and rows such that c_i and r_j are connected iff (i, j) is marked. This way we can find cycles alternating row and column.
- Creating a bipartite graph between all rows and columns and particular objects. This helps to prove matching.

Cut A cut is a partition of the vertices of a graph into two disjoint subsets. Any cut determines a cut-set, the set of edges that have one endpoint in each subset of the partition. These edges are said to cross the cut. In a connected graph, each cut-set determines a unique cut, and in some cases cuts are identified with their cut-sets rather than with their vertex partitions.

Pics/min_cut.png	Pics/max_cut.png
(a) Minimum Cut	(b) Maximum Cut

Lemma 2.5.1: Average of Degrees

In a graph G with n vertexes, let E be the set of all edges. Assign an integer f_i to every vertex v_i such that f_i equals to the everage degree of the neighbors of v_i. We have,

$$\sum_{i=1}^{n} f_i \geq 2|E|$$

Lemma 2.5.2: Bipartite Graph Criteria

Any graph having only even cycles are BIPARTITE.

1 AoPS
2 ISL 2004 C3
3 Problem
4 Problem

Theorem 2.5.3: Euler's Polyhedron Formula

- For any polyhedron with E, V, F edges, vertices's and faces resp. the following relation holds

$$V + F = E + 2$$

- In a planar graph with V vertices, E edges and C cycles, the following condition is always satisfied:

$$V - E + C = 1$$

Theorem 2.5.4: Prufer sequence

Consider a labeled tree T with vertices's $\{1, 2, ..., n\}$. At step i, remove the leaf with the smallest label and set the ith element of the *Prüfer sequence* to be the label of this leaf's neighbour. Prove that a Prüfer sequence of length $n - 2$ defines a Tree with length n.

Pics/prufer_code_example.png

Figure 2.3: A labeled tree with Prüfer sequence $4, 4, 4, 5$.

Lemma 2.5.5: Criteria of partitioning a graph into disconnected sub-graphs

If there exist no three vertices, u, v, w that $uv \in E(G)$ but $uw, vw \notin E(G)$, then the graph can be partitioned into equivalence classes based on their non-neighbors.

Theorem 2.5.6: Turan's theorem

Let G be any graph with n vertices, such that G is K_{r+1} -free. Then G is the "Turán's Graph" and is a complete r partite graph. And the number of edges in G is at most

$$\frac{r-1}{r} \cdot \frac{n^2}{2} = \left(1 - \frac{1}{r}\right) \cdot \frac{n^2}{2}$$

A special case of Turán's theorem for $n = 2$ is the **Mantel's Theorem**. It states that the maximal triangle free graph is a complete bipartite graph with at most $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges.

`Pics/turan_graph.png`

Figure 2.4: Turán's Graph

Proof. We need to prove that the maximal graph is the r partite one, and the rest will follow. We can directly try to prove that this graph is r colorable, but that is quite troublesome. Instead, we try to show that, we can partition the vertices of G into equivalence classes based on their non-neighbors. Since this is imply the former. So we need to prove that this holds for this graph.

The way it is done is quite interesting. We need to show that if the criteria doesn't hold in this graph, then this graph is not the maximal graph. How are we going to do that? We compare the degrees of u, w, and replace either u by w or w by u to get a graph with more edges and without the nasty situation.

Problem 2.5.1: **E**

155 birds $P_1, P_2, \ldots, P_{155}$ are sitting down no the boundary of a circle C. Two birds P_i, P_j are mutually visible if the angle at the center of their cord, $m(P_iP_j) \leq 10°$. Find the smallest number of mutually visible pairs of birds.

Problem 2.5.2: **E**

For a pair $A = (x_1, y_1)$ and $B = (x_2, y_2)$ of points on the coordinate plane, let $d(A, B) = |x_1 - x_2| + |y_1 - y_2|$. We call a pair (A, B) of unordered points harmonic if $1 < d(A, B) \leq 2$. Determine the maximum number of harminc pairs among 100 points in the plane.

Problem 2.5.3: German TST 2004 E7P3 **M**

We consider graphs with vertices colored black or white. "Switching" a vertex means: coloring it black if it was formerly white, and coloring it white if it was formerly black.

Consider a finite graph with all vertices colored white. Now, we can do the following operation: Switch a vertex and simultaneously switch all of its neighbours (i. e. all vertices connected to this vertex by an edge). Can we, just by performing this operation several times, obtain a graph with all vertices colored black?

Idea A classical example of creating complex moves from counter cases. □

Problem 2.5.4: ARO 2014 P9.8 **H**

In a country of n cities, an express train runs both ways between any two cities. For any train, ticket prices either direction are equal, but for any different routes these prices are different. Prove that the traveler can select the starting city, leave it and go on, successively, $n-1$ trains, such that each fare is smaller than that of the previous fare. (A traveler can enter the same city several times.)

Problem 2.5.5: Generalization **H**

Let A be a set of n points in the space. From the family of all segments with endpoints in A , q segments have been selected and colored yellow. Suppose that all yellow segments are of different length. Prove that there exists a polygonal line composed of m yellow segments, where $m \geq \frac{2q}{n}$, arranged in order of increasing length.

Idea Make one person go to every node. Then let the two people on the two sides of the most expensive edge swap their position. This ensures that every edge was used exactly 2 times. Using PHP, we have the desired result. Another solution is by this □

Problem 2.5.6: **E**

Given a bipartite graph, prove that the minimum number of colors required to color the edges of the graph such that no node is adjacent to 2 edges of same color is the maximum degree of the graph.

Problem 2.5.7: **E**

For every bipartite graph prove that it's edges can be bicolored so that each node is adjacent to atmost $\lceil \frac{deg}{2} \rceil$ edges of any color.

Idea Using the main property of a bipartite graph. □

Idea After finding the cycle solution, to optimize it, we recall that we can find a Eulerian Path (if it exists) in $O(V+E)$. Now we want to make the graph have a Eulerian path, so we add a vertice to both sides of the graph, and join them with odd vertices from the other side. □

Problem 2.5.8: ISL 2005 C1 **E**

A house has an even number of lamps distributed among its rooms in such a way that there are at least three lamps in every room. Each lamp shares a switch with exactly one other lamp, not necessarily from the same room. Each change in the switch shared by two lamps changes their states simultaneously. Prove that for every initial state of the lamps there exists a sequence of changes in some of the switches at the end of which each room contains lamps which are on as well as lamps which are off.

Problem 2.5.9: ISL 2013 C3 **M**

A crazy physicist discovered a new kind of particle which he called an i -mon, after some of them mysteriously appeared in his lab. Some pairs of i -mons in the lab can be entangled, and each i -mon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.

1. If some i -mon is entangled with an odd number of other i -mons in the lab, then the physicist can destroy it.

2. At any moment, he may double the whole family of i -mons in the lab by creating a copy I' of each i -mon I. During this procedure, the two copies I' and J' become entangled if and only if the original i -mons I and J are entangled, and each copy I' becomes entangled with its original i -mon I ; no other entanglements occur or disappear at this moment.

Idea Prove that the physicist may apply a sequence of much operations resulting in a family of i -mons, no two of which are entangled.
As there are an integer number of i -mons, it is quite natural to use induction. We try to find an algorithm to reduce the number of particles.
Another way to do this is to consider the chromatic number of the graph. If we can show that this number reduces after some move, then we are done by induction. □

Problem 2.5.10: ISL 2005 C2 **E**

A forest consists of rooted (i. e. oriented) trees. Each vertex of the forest is either a leaf or has two successors. A vertex v is called an extended successor of a vertex u if there is a chain of vertices's $u_0 = u, u_1, u_2 \ldots u_{t-1}, u_t = v$ with $t > 0$ such that the vertex u_{i+1} is a successor of the vertex u_i for every integer i with $0 \leq i \leq t-1$.

Let k be a nonnegative integer. A vertex is called dynastic if it has two successors and each of these successors has at least k extended successors.

Prove that if the forest has n vertices, then there are at most $\frac{n}{k+2}$ dynastic vertices.

Idea Trying to apply induction, we realize the bound is very loosy. That's why when we try to add in the inductive step, the value becomes larger than the bound. To stop that overflow, we tighten the bound. □

Idea The second and dummy approach is to first doing some smaller cases, finding small infos, taking the root, seeing that the bound doesnt work, but it would work if one of the successors of the root would have exactly or less than $2k+3$ successors. As we can't always guarantee that, we look for such a vertex with $2k+3$ successors. We do some work with it and by induction its done. □

Problem 2.5.11: All Russia 2017 9.1 **E**

In a country some cities are connected by oneway flights (There are no more then one flight between two cities). City A called "available" for city B , if there is flight from B to A , maybe with some transfers. It is known, that for every 2 cities P and Q exist city R , such that P and Q are available from R. Prove, that exist city A , such that every city is available for A.

Problem 2.5.12: Jacob Tsimerman Induction **E**

There are 2010 ninjas in the village of Konoha (what? Ninjas are cool.) Certain ninjas are friends, but it is known that there do not exist 3 ninjas such that they are all pairwise friends. Find the maximum possible number of pairs of friends.(If ninja A is friends with ninja B , then ninja B is also friends with ninja A.)

Problem 2.5.13: USAMO 2017 P4 **M**

Let P_1 , P_2 , . . . , P_{2n} be $2n$ distinct points on the unit circle $x^2 + y^2 = 1$, other than $(1, 0)$. Each point is colored either red or blue, with exactly n red points and n blue points. Let R_1 , R_2 , . . . , R_n be any ordering of the red points. Let B_1 be the nearest blue point to R_1 traveling counterclockwise around the circle starting from R_1. Then let B_2 be the nearest of the remaining blue points to R_2 travelling counterclockwise around the circle from R_2 , and so on, until we have labeled all of the blue points $B_1, \ldots, B_n$. Show that the number of counterclockwise arcs of the form $R_i \to B_i$ that contain the point $(1, 0)$ is independent of the way we chose the ordering $R_1, \ldots, R_n$ of the red points.

Idea As the ques is saying that the condition is true for every positive integer n , can't we try induction? Which part of the condition makes the problem challenging? Obviously the circle condition and the $(1, 0)$ point. So Lets remove a "problematic" pair R_t, B_t. We encounter some problems, but we can pull it out of there. □

Problem 2.5.14: Swell coloring **E**

Let K_n denote the complete graph on n vertices, that is, the graph with n vertices's such that every pair of vertices's is connected by an edge. A swell coloring of K_n is an assignment of a color to each of the edges such that the edges of any triangle are either all of distinct colors or all the same color. Further, more than one color must be used in total (otherwise trivially if all edges are the same color we would have a swell coloring). Show that if K_n can be swell colored with k colors, then $k \geq \sqrt{n} + 1$.

Idea Concentrate on only one vertex. □

Problem 2.5.15: Belarus 2001 **MH**

Given n people, any two are either friends or enemies, and friendship and enmity are mutual. I want to distribute hats to them, in such a way that any two friends possess a hat of the same color but no two enemies possess a hat of the same color. Each person can receive multiple hats. What is the minimum number of colors required to always guarantee that I can do this?

Idea In this problem, finding the worst case is a big help, because once the answer is guessed, the things become really clear. □

Problem 2.5.16: USA TST 2011 D3P2 **M**

Let $n \geq 1$ be an integer, and let S be a set of integer pairs (a, b) with $1 \leq a < b \leq 2^n$. Assume $|S| > n \cdot 2^{n+1}$. Prove that there exists four integers $a < b < c < d$ such that S contains all three pairs (a, c), (b, d) and (a, d).

Idea Using Induction to the first and last half of the set S shows us the hardest part of the problem. Then ordering the left and right elements with some sort of hierarchy is all the work left to do. □

Problem 2.5.17: ISL 2016 C6 **H**

There are $n \geq 3$ islands in a city. Initially, the ferry company offers some routes between some pairs of islands so that it is impossible to divide the islands into two groups such that no two islands in different groups are connected by a ferry route.

After each year, the ferry company will close a ferry route between some two islands X and Y. At the same time, in order to maintain its service, the company will open new routes according to the following rule: for any island which is connected to a ferry route to exactly one of X and Y , a new route between this island and the other of X and Y is added.

Suppose at any moment, if we partition all islands into two nonempty groups in any way, then it is known that the ferry company will close a certain route connecting two islands from the two groups after some years. Prove that after some years there will be an island which is connected to all other islands by ferry routes.

Idea It is only natural to use induction on this kinda problems. After some trying, we see that if we remove 1 node, We get to nowhere, but if we remove 2 nodes, we get something interesting. So now focus on those two nodes and the rest of the nodes separately. Its not hard from there. □

Idea As it seems, the separation of the graph was the main observation. We can call this trick Bringing Order in the Chaos. □

Problem 2.5.18: ELMO 2017 P5 **M (8/10)**

The edges of K_{2017} are each labeled with $1, 2$ or 3 such that any triangle has sum of labels at least 5. Determine the minimum possible average of all labels. (Here K_{2017} is defined as the complete graph on 2017 vertices's, with an edge between every pair of vertices's.)

Idea A starting idea to get the ans: if we discard of all the 2-edges, we see that in any triangle, one edge has to be a 3-edge. So... Turan-kinda... □

Idea After getting the ans, and thinking about approaching inductively, if we remove only one vertex, there will be pairs to consider. But if we remove two vertices, we will only need to consider single vertices after the removal of these two vertices.

Now which pair of vertices are the best choice to remove? Before doing that, lets first think how much change will we get in the sum after we remove two vertices. Since we have the ans, we do quick maffs:

$$m(4m+1) - (m-1)(4m-3) = 8m - 3 = 4 \times (2m-1) + 1$$

Doesn't this indicate that we remove a 1-edge, so the other edges coming out of the two vertices will sum up to be at least $4 * (2m-1)$. □

Idea The solution by bern is very pretty. What he probably had thought was:

If we pick a vertex, say u, and take an 1-edge from this vertex to another vertex v, we see that there are at least as many 3-edges in u than there are 1-edges in v. Now if to get a more accurate value of $d_3(u)$ (defined naturally), we need to take the maximum of the values $d_1(v)$ for all v's connected to u.

Now we need to evaluate the number of 3 edges from the d_1 values. Can we put a bound on this sum? We have **this lemma**, does this help? Turns out that it does.

What left is to sum it all up to see if we can get the ans. □

Problem 2.5.19: ARO 2013 P9.5 **M (8/10)**

$2n$ real numbers with a positive sum are aligned in a circle. For each of the numbers, we can see there are two sets of n numbers such that this number is on the end. Prove that at least one of the numbers has a positive sum for both of these two sets.

Idea Since there is nothing specfic about the sum, we may safely assume that it is 0, because (1) probably it works, and (2) it makes things more convenient. How we do that? we decrease every number by the average.

Now, Consider every block of n consecutive blockes of numbers. When are two blocks connected? When they share the same end. What if we consider them as vertices, and this "connectivity" as edges? We see that cycles pop out.

And we make use of the fact that our sum is 0. So signs are sure to bet flipped at the opposite side, and there are odd and even -ness in cycles that we can use. □

Problem 2.5.20: USA TST 2011 P2 **HM (9/10)**

In the nation of Onewaynia, certain pairs of cities are connected by roads. Every road connects exactly two cities (roads are allowed to cross each other, e.g., via bridges). Some roads have a traffic capacity of 1 unit and other roads have a traffic capacity of 2 units. However, on every road, traffic is only allowed to travel in one direction. It is known that for every city, the sum of the capacities of the roads connected to it is always odd. The transportation minister needs to assign a direction to every road. Prove that he can do it in such a way that for every city, the difference between the sum of the capacities of roads entering the city and the sum of the capacities of roads leaving the city is always exactly one.

Idea As there are two types of subgraph, 1 -type and 2 -type. By some work-arounds, we see that we have to work distinctly in both types of graphs. Firstly, if we work in type-1 , we see after making a path from node x, y , the degrees of x, y will be $\{1, -1\}$ and the degrees of other nodes on the path will be the same. After that, we make every nodes have degree either $\{1, -1\}$. So after this operation we remove the 1 -edges. Now, when dealing with the type- 2 sub-graph. Start over from zero, we see that when making a path between nodes x, y the degree of those two changes parity, and other nodes on the path stays the same. So select two odd nodes.... □

Idea Dealing with two different kind of edges simultaneously is messy, so we work with graph 1 and graph 2 differently. Now on both graphs, we can remove cycles. And in graph 2 , we see that we can remove any big paths if there is a edge 1 joining the two endpoints. Since if the new graph works then the previous graph works too. [Several cases to show here] And if there is no edge joining the two endpoints, replace the path by joining the two endpoints by a edge 2.

Now there are only edge 1 s, and lone edge 2 s. Now dividing the graph 1 into paths of edge 1 , and dealing with several small cases, we are done. □

Problem 2.5.21: Iran TST 2009 P6 **E-M (9/10)**

We have a closed path that goes from one vertex to another neighboring vertex, on the vertices of a $n \times n$ square which pass throgugh each vertex exactly once. Prove that we have two adjacent vertices such that if we cut the path at these two points then the length of each open paths is at least $n^2/4$.

Idea Draw a path, doesn't it look like a snake? Now can we relate the area of the tiled path with its perimeter? If we could do that, we would be able to replace two neighboring vertices by an edge inside the path, which seems to make the problem simpler. □

Problem 2.5.22: OC Chap2 P2 **M (6/10)**

Arutyun and Amayak perform a magic trick as follows. A spectator writes down on a board a sequence of N (decimal) digits. Amayak covers two adjacent digits by a black disc. Then Arutyun comes and says both closed digits (and their order). For which minimal N can this trick always work? NOTE: Arutyun and Amayak have a strategy determined beforehand.

Idea We have to actually find a bijection between all of the combinations the spectator can create, and all of the combinations that Arutyun might see when he comes back. Which tells us to use "Perfect Matching" tricks. □

Idea Existential proof: for this trick to always work, they have to make a bijection from a set of N digits with two covered, to an unique set of N digits. Consider a bijection from the set of $0-9$ strings with length N to the set of $0-9$ strings with length N with 2 adjacent digits unknown. There exist a bijection iff the two sets satisfy Hall's Marriage Theorem. By double counting we get the value of N from here. □

Problem 2.5.23: ARO 2005 P9.4 **M (7/10)**

100 people from 50 countries, two from each countries, stay on a circle. Prove that one may partition them onto 2 groups in such way that neither no two countrymen, nor three consecutive people on a circle, are in the same group.

Variant: There are 100 people from 25 countries sitting around a circular table. Prove that they can be separated into four classes, so that no two countrymen are in the same class, nor any two people sitting adjacent in the circle.

Idea Thinking of the most natural way of eliminating the consecutive condition – pair two consecutive verices. □

Problem 2.5.24: Romanian TST 2012 P4 **E (7/10)**

Prove that a finite simple planar graph has an orientation so that every vertex has out-degree at most 3.

Problem 2.5.25: USA TST 2006 P1 **E-M (8/10)**

A communications network consisting of some terminals is called a 3-connector if among any three terminals, some two of them can directly communicate with each other. A communications network contains a windmill with n blades if there exist n pairs of terminals $\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_n, y_n\}$ such that each x_i can directly communicate with the corresponding y_i and there is a hub terminal that can directly communicate with each of the $2n$ terminals $x_1, y_1, \ldots, x_n, y_n$. Determine the minimum value of $f(n)$, in terms of n, such that a 3 -connector with $f(n)$ terminals always contains a windmill with n blades.

Idea Windmills won't be there if among any $2n+1$ vertices, there were one vertex that were not connected to any of the other $2n$ vertices. So that means that we are dealing Turan-kinda config here. So we can make several 'compact' graphs that are mutually disconnected, and each have at most $2n$ verices. Guessing from this, the ans is probably of some form $k * 2n + 1$. Now we have another condition to consider, 3-connector. Lets see, if we had 3 disconnected componets, the resulting graph wouldn't be a 3-connector. Done... □

Problem 2.5.26: **E**

Graph G on n vertices has the property that the degree of every vertex is greater than 2. Prove that for every $0 < k < n$, there is a simple path with lenght at least n/k or, k cycles, such that every cycle has at least one node which none of the other cycles has, and its lenght is not divisible by 3.

Problem 2.5.27: Simurgh 2019 P3 **E**

Call a graph *symmetric*, if one can put its vertices on the plane such that it becomes symmetric wrt a line (which doesn't pass through any vertex). Find the minimum value of k such that (the edges of) every graph on 100 vertices, can be decomposed into k symmetric subgraph.

Problem 2.5.28: RMM 2019 P3 **H**

Given any positive real number ε, prove that, for all but finitely many positive integers v, any graph on v vertices with at least $(1+\varepsilon)v$ edges has two distinct simple cycles of equal lengths. (Recall that the notion of a simple cycle does not allow repetition of vertices in a cycle.)

Idea We want to bound the number of edges. If we take a spanning tree of this graph, we see that whenever we add one more edge to this tree, we create new cycles. So we want to bound the number of 'new-addable-edges' with the fact that there doens't exist two cycles of the same lenght.
What is the best choice for a spanning tree? Of course a extreme one, like DFS tree. In this tree, we are forcing leftover edges to be between two consecutive levels only.
Now we first bound the sum of the lenghts of the cycle with the number of leftover edges. Then we use double count to describe the lenghts in a different way, and bound that wrt to the number of cycles. Combining these two, we get our desired bound. □

Problem 2.5.29: China TST 2015 T1D2P3 **E**

There are some players in a Ping Pong tournament, where every 2 players play with each other at most once. Given:

1. Each player wins at least a players, and loses to at least b players. $(a, b \geq 1)$
2. For any two players A, B, there exist some players $P_1, ..., P_k$ $(k \geq 2)$ (where $P_1 = A, P_k = B$), such that P_i wins P_{i+1} $(i = 1, 2..., k-1)$

Prove that there exist $a+b+1$ distinct players $Q_1, ...Q_{a+b+1}$, such that Q_i wins Q_{i+1} $(i = 1, ..., a+b)$

Idea Take the longest path, and do obvious works on it. □

Problem 2.5.30: ARO 2005 P10.8 **E**

A white plane is partitioned onto cells (in a usual way). A finite number of cells are coloured black. Each black cell has an even (0, 2 or 4) adjacent (by the side) white cells. Prove that one may colour each white cell in green or red such that every black cell will have equal number of red and green adjacent cells.

Idea Join the white cells and prove that that graph is bipartite. □

Problem 2.5.31: ARO 1999 P9.8 **M**

There are 2000 components in a circuit, every two of which were initially joined by a wire. The hooligans Vasya and Petya cut the wires one after another. Vasya, who starts, cuts one wire on his turn, while Petya cuts one or three. The hooligan who cuts the last wire from some component loses. Who has the winning strategy?

Problem 2.5.32: ISL 2001 C3 **E**

Define a k-clique to be a set of k people such that every pair of them are acquainted with each other. At a certain party, every pair of 3-cliques has at least one person in common, and there are no 5-cliques. Prove that there are two or fewer people at the party whose departure leaves no 3-clique remaining.

Idea Casework with the point where most of the triangles are joined. □

2.6 Game Theory

- Zawad's Game Theory Pset

2.6.1 Games

Nimbers: Nimbers are simply 'Nim values' which are assigned to a game configuration - these values are written as $0, *1, *2, *3\dots$ We shall first describe how to obtain the Nim values for the game Squaring the Number. First, the Nim value of $n = 0$ is assigned 0, since it is a state in which neither player has a valid move. We then recursively adopt the following rule for each n : find all the possible moves from n and pick the smallest Nim value which does not occur among all these possible moves.

Theorem 2.6.1: Sprague-Grund Theorem

The *Sprague–Grundy theorem* states that every impartial game under the normal play convention is equivalent to a nimber.

Game 2.6.2: Chip Firing Game

Let $G = (V, E)$ be a graph without any loops or multiedges. Let a number of s_i chips be stacked on vertex i. The game follows with the player choosing a vertex i, taking d_i chips from it $(s_i - d_i > 0)$, and sending one chip to each of the neighbors of the vertex where d_i is the degree of i. The Problem of this game is to determine when the game will be infinte.

- If N is the total number of edges in G, and S is the total number of chips, then

Game 2.6.3: Cutting a stack in half

Given a number of stacks, at his/her move, a player can choose a stack with even number stones, and divide it in two stacks with the same number of stones.

Game 2.6.4: Cutting a stack in several

Given a number of stacks, at his/her move, a player can choose a stack, and divide it in several stacks with the same number of stones.

2.6.2 Problems

Problem 2.6.1: USAMO 2008 P5 **M**

Three non-negative real numbers r_1 , r_2 , r_3 are written on a blackboard. These numbers have the property that there exist integers a_1 , a_2 , a_3 , not all zero, satisfying $a_1r_1 + a_2r_2 + a_3r_3 = 0$. We are permitted to perform the following operation: find two numbers x , y on the blackboard with $x \leq y$, then erase y and write $y - x$ in its place. Prove that after a finite number of such operations, we can end up with at least one 0 on the blackboard.

Idea When can't get info out of the reals, try the integers. Observe the integers, and check if they have any invariant. Rule of thumb of finding an invariant. $\square$

Problem 2.6.2: USAMO 2014 P1 **E**

Let k be a positive integer. Two players A and B play a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with A moving first. In his move, A may choose two adjacent hexagons in the grid which are empty and place a counter in both of them. In his move, B may choose any counter on the board and remove it. If at any time there are k consecutive grid cells in a line all of which contain a counter, A wins. Find the minimum value of k for which A cannot win in a finite number of moves, or prove that no such minimum value exists.

Idea Trying to block A. We see that if we could alternately color the points black and white, we could've found some strategy for B. But the triangle grid doesn't seem very friendly. How can we color the triangles? And don't forget the details idiot. $\square$

Problem 2.6.3: Indian TST 2004 **M**

The game of pebbles is played as followed: Initially there is one pebble at $(0, 0)$. In a move one can remove the pebble at (i, j) and put one pebble each at $(i + 1, j)$ and $(i, j + 1)$, given that both $(i + 1, j)$ and $(i, j + 1)$ were empty. Prove that at any point in the game, there will be a pebble at some lattice point (a, b) with $a + b \leq 3$.

Idea Two from one, means if the weight is reduced by half in the second level, then the sum would be the same. $\square$

Problem 2.6.4: ISL 1998 C7 **H**

A solitaire game is played on an $m \times n$ rectangular board, using mn markers which are white on one side and black on the other. Initially, each square of the board contains a marker with its white side up, except for one corner square, which contains a marker with its black side up. In each move, one may take away one marker with its black side up, but must then turn over all markers which are in squares having an edge in common with the square of the removed marker. Determine all pairs (m, n) of positive integers such that all markers can be removed from the board.

Idea If we remove one marker, then this cell becomes useless. So the neighbors to this cell will act like they are not connected to this cell. Now if a cell is connected to w white cells, and b black cells, then the resulting board state will have $b - w$ more cells. Now only this info doesn't build up an invariant. Notice that as we are doing moves, we are reducing neighborhood relations as well, in other words, neighborhood relations decrease by $b + w$. So if we consider the sum $W + E$ where W is the number of all white cells, and E is the number of all neighborhood relations, we get an invariant on this value. □

Problem 2.6.5: ARO 1999 P10.1 **E**

There are three empty jugs on a table. Winnie the pooh, Rabbit, and Piglet put walnuts in the jugs one by one. They play successively, with the initial determined by a draw. Thereby Winnie the pooh plays either in the first or second jug, Rabbit in the second or third, and Piglet in the first or third. The player after whose move there are exactly 1999 walnuts loses the games. Show that Winnie the pooh and Piglet can cooperate so as to make Rabbit lose.

Problem 2.6.6: USAMO 2004, P4 **E**

Alice and Bob play a game on a 6 by 6 grid. On his or her turn, a player chooses a rational number not yet appearing in the grid and writes it in an empty square of the grid. Alice goes first and then the players alternate. When all squares have numbers written in them, in each row, the square with the greatest number in that row is colored black. Alice wins if she can then draw a line from the top of the grid to the bottom of the grid that stays in black squares, and Bob wins if she can't. (If two squares share a vertex, Alice can draw a line from one to the other that stays in those two squares.) Find, with proof, a winning strategy for one of the players.

Problem 2.6.7: ARO 1999 P10.1 **E**

There are three empty jugs on a table. Winnie the pooh, Rabbit, and Piglet put walnuts in the jugs one by one. They play successively, with the initial determined by a draw. Thereby Winnie the pooh plays either in the first or second jug, Rabbit in the second or third, and Piglet in the first or third. The player after whose move there are exactly 1999 walnuts loses the games. Show that Winnie the pooh and Piglet can cooperate so as to make Rabbit lose.

Problem 2.6.8: RMM 2019 P1 **E**

Amy and Bob play the game. At the beginning, Amy writes down a positive integer on the board. Then the players take moves in turn, Bob moves first. On any move of his, Bob replaces the number n on the blackboard with a number of the form $n - a^2$, where a is a positive integer. On any move of hers, Amy replaces the number n on the blackboard with a number of the form n^k, where k is a positive integer. Bob wins if the number on the board becomes zero. Can Amy prevent Bob's win?

Idea Decent. □

Problem 2.6.9: ISL 2015 C4 **M**

Let n be a positive integer. Two players A and B play a game in which they take turns choosing positive integers $k \leq n$. The rules of the game are:

1. A player cannot choose a number that has been chosen by either player on any previous turn.
2. A player cannot choose a number consecutive to any of those the player has already chosen on any previous turn.
3. The game is a draw if all numbers have been chosen; otherwise the player who cannot choose a number anymore loses the game.

The player A takes the first turn. Determine the outcome of the game, assuming that both players use optimal strategies.

Idea Look at the simplest things, first produce data like a good boy, and then see what A has to do to win, or at least draw that he can't because B is an asshole. □

Problem 2.6.10: ISL 2012 C4 **EM**

Players A and B play a game with $N \geq 2012$ coins and 2012 boxes arranged around a circle. Initially A distributes the coins among the boxes so that there is at least 1 coin in each box. Then the two of them make moves in the order $B, A, B, A, \ldots$ by the following rules:

1. On every move of his B passes 1 coin from every box to an adjacent box.
2. On every move of hers A chooses several coins that were not involved in B's previous move and are in different boxes. She passes every coin to and adjacent box.

Player A's goal is to ensure at least 1 coin in each box after every move of hers, regardless of how B plays and how many moves are made. Find the least N that enables her to succeed.

Idea Investigate B's move, see how and where he can make 0's $\square$

2.7 Combinatorial Geometry

- Combinatorial Geometry - Maria Monk (MOP 2010)

Stuck? Try These

- Consider the convex hull made up of the points.
- Consider the extreme points: smallest or highest x or y coordinate.
- Find the triangle (quadrilateral, pentagon, etc.) with the vertices being the points from your set S, so that the area of the triangle is minimal/maximal.

Theorem 2.7.1: Helly's Theorem

Let $X_1, ..., X_n$ be a finite collection of convex subsets of $\mathbb{R}^d$, with $n > d$. If the intersection of every $d+1$ of these sets is nonempty, then the whole collection has a nonempty intersection; that is,

$$\bigcap_{j=1}^{n} X_j \neq \varnothing$$

Problem 2.7.1: ARO 2013 P9.4 **E**

N lines lie on a plane, no two of which are parallel and no three of which are concurrent. Prove that there exists a non-self-intersecting broken line $A_1A_2A_3\dots A_N$ with N parts, such that on each of the N lines lies exactly one of the N segments of the line.

Problem 2.7.2: EGMO 2017 P3 **M**

There are 2017 lines in the plane such that no three of them go through the same point. Turbo the snail sits on a point on exactly one of the lines and starts sliding along the lines in the following fashion: she moves on a given line until she reaches an intersection of two lines. At the intersection, she follows her journey on the other line turning left or right, alternating her choice at each intersection point she reaches. She can only change direction at an intersection point. Can there exist a line segment through which she passes in both directions during her journey?

Idea The condition that tells us to go either right or left, seems very non-rigorous. So to rigorize this condition, instead of using right or left condition in the direction, we consider what's on our right and left. (INTUITION) After some experiment we see (not all of us) that if we color the plane with two colors in a way where every neighboring regions have different colors, we find some interesting stuff. (CREATIVITY) With this we are done.
Color the Plane □

Problem 2.7.3: ISL 2006 C2 **TE**

Let P be a regular 2006 -gon. A diagonal is called good if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P. The sides of P are also called good.
Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

Idea The straight way, induction. □

Idea The intuitive way, bijection. There are at most n good triangles, there are $2n$ edges, so a mapping that takes a two edges to a single good triangle must exist. Finding it is not that hard. □

Problem 2.7.4: ARO 2014 P9.3 **E**

In a convex n -gon, several diagonals are drawn. Among these diagonals, a diagonal is called good if it intersects exactly one other diagonal drawn (in the interior of the n -gon). Find the maximum number of good diagonals.

Idea There can be two cases, two good diagonals intersecting each other, and no two good diagonals intersecting each other. In the first case, we just use induction, and in the later, all of the good diagonals create a "triangulation" of the polygon, which gives us the numbers. □

Problem 2.7.5: ISL 2013 C2, IMO 2013 P2 **E**

A configuration of 4027 points in the plane is called Colombian if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is good for a Colombian configuration if the following two conditions are satisfied:

1. No line passes through any point of the configuration.
2. No region contains points of both colors.

Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

Idea Obviously a n00b would think about induction. The only problem occurs when the convex hull completely consists of red points. In this case, after some investigation, we should get the sandwiching two points idea. □

Idea Another way of inductive approach is like this, as the problem condition says that no region contains points of both colors, which means if we connect any two red and blue points, some line must bisect this segment. Now it is known that there is non intersecting partition of the points in to red-blue segments. So suppose in such a partition, we draw bisectors of each segments. Now there will be some holes in this proof. We see that to fill this holes, we have to focus on two red points with their respective blue partners, and draw the two bisectors in a way that separates the two red points form the blue points. So to remove further holes, we get the sandwiching idea. □

Problem 2.7.6: Putnam 1979 **E**

Let A be a set of $2n$ points in the plane, no three of which are collinear, n of them are colored red and the other blue. Prove that there are n line segments, no two with a point in common, such that the endpoints of each segment are points of A having different colors.

Idea Strong induction, and a way to divide the points into two sets with the same number of red and blue points. Travel through the points around a certain point and keep track of the number of red and blue points. □

Problem 2.7.7: USAMO 2005 P5 **E**

Let n be an integer greater than 1. Suppose $2n$ points are given in the plane, no three of which are collinear. Suppose n of the given $2n$ points are colored blue and the other n colored red. A line in the plane is called a balancing line if it passes through one blue and one red point and, for each side of the line, the number of blue points on that side is equal to the number of red points on the same side. Prove that there exist at least two balancing lines.

Idea Using the same idea as in this problem. □

Problem 2.7.8: ILL 1985 **E**

Let A and B be two finite disjoint sets of points in the plane such that no three distinct points in $A \cup B$ are collinear. Assume that at least one of the sets A, B contains at least five points. Show that there exists a triangle all of whose vertices's are contained in A or in B that does not contain in its interior any point from the other set.

Idea Concentrating on one of the sets five points such that there is no other points of the same set inside the hull of those five points. □

Problem 2.7.9: APMO 1999 P5 **M**

Let S be a set of $2n+1$ points in the plane such that no three are collinear and no four concyclic. A circle will be called "Good" if it has 3 points of S on its circumference, $n-1$ points in its interior and $n-1$ points in its exterior. Prove that the number of good circles has the same parity as n.

Idea When thinking about induction, got a feeling that double counting with the number of good circles going through pairs of points might be useful, because a good circle will be counted three times, if we can show that every pair has odd number of good circles, we are done. So, take a pair. Now we need to 'sort' the points somehow. See that, we can't sort the points in a trivial way with numbers, so moving to angles. Now setting conditions for a point inside of a circle in terms of angles, we see amazing patter, and an easy way to calculate the number of good circle of that pair of points. □

Problem 2.7.10: ISL 2014 C1 **E**

Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. The rectangle R is to be dissected into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect R into at least $n+1$ smaller rectangles.

Idea Work with the largest continuous segments, and their endpoints. □

Problem 2.7.11: ISL 2007 C2 **EM**

A rectangle D is partitioned in several (≥ 2) rectangles with sides parallel to those of D. Given that any line parallel to one of the sides of D, and having common points with the interior of D, also has common interior points with the interior of at least one rectangle of the partition; prove that there is at least one rectangle of the partition having no common points with D's boundary.

Idea There existing such a rectangle means that there is a rectangular region inside of the original rectangle. So what if we walked along the segments, and cut a smaller rectangle from the inside of the rectangle? Like the way in the game.

`Pics/2007c2.png`

Figure 2.5

□

Idea Starting from one corner, and taking the opposite corner of the rectangle containing that corner, we use infinite decent to reach a contradiction. □

Idea Using ISL 2014 C1 as a lemma. □

Idea Take one side of the square. Take a "sandwiched" rectangle touching that side. If no such rectangle exists, then it's just a special case that can be dealt with ease. □

Problem 2.7.12: ISL 2003 C2 **E**

Let $D_1, D_2 \ldots, D_n$ be closed discs in the plane. (A closed disc is the region limited by a circle, taken jointly with this circle.) Suppose that every point in the plane is contained in at most 2003 discs D_i. Prove that there exists a disc D_k which intersects at most $7 \cdot 2003 - 1 = 14020$ other discs D_i.

Idea Just go with the natural idea. □

Problem 2.7.13: ISL 2003 C3 **E**

Let $n \geq 5$ be a given integer. Determine the greatest integer k for which there exists a polygon with n vertices (convex or not, with non-selfintersecting boundary) having k internal right angles.

Idea double count □

Problem 2.7.14: Tournament of Towns 2015S S4 **A convex$N-$gon with equal sides is located inside a circle. Each side is extended in both directions up to the intersection with the circle so that it contains two new segments outside the polygon. Prove that one can paint some of these new $2N$ segments in red and the rest in blue so that the sum of lengths of all the red segments would be the same as for the blue ones.**

Idea Just use what's the most natural, POP, on one vertex point. □

Problem 2.7.15: USAMO 2007 P2 **E**

A square grid on the Euclidean plane consists of all points (m, n), where m and n are integers. Is it possible to cover all grid points by an infinite family of discs with non-overlapping interiors if each disc in the family has radius at least 5?

Problem 2.7.16: MEMO 2015 T4 **EM**

Let N be a positive integer. In each of the N^2 unit squares of an $N \times N$ board, one of the two diagonals is drawn. The drawn diagonals divide the $N \times N$ board into K regions. For each N, determine the smallest and the largest possible values of K.

`Pics/MEMO2015T4.png`

Figure 2.6

Idea An Algorithmic Approach: Consider each diagonal as 0 or 1, prove that the maximum configuration is the one with alternating $0, 1$s and the minimum one is the one with all 0s. □

Idea A Counting Approach: Just count and bound with the minimum areas of the regions. □

Problem 2.7.17: **EM**

In every cells of a $m \times n$ grid, one of the two diagonals are drawn. Prove that there exist a path on these diagonals from left to right or from up to bottom of the grid.

Idea First remove the cycles, then take the largest path from left to right, and use induction. □

Problem 2.7.18: Math Price for Girls 2017 P4 **E**

A lattice point is a point in the plane whose two coordinates are both integers. A lattice line is a line in the plane that contains at least two lattice points. Is it possible to color every lattice point red or blue such that every lattice line contains exactly 2017 red lattice points? Prove that your answer is correct.

Idea Transfinite induction. □

Problem 2.7.19: China TST 2016 T3P2 **E**

In the coordinate plane the points with both coordinates being rational numbers are called rational points. For any positive integer n, is there a way to use n colours to colour all rational points, every point is coloured one colour, such that any line segment with both endpoints being rational points contains the rational points of every colour?

Idea Transfinite induction □

Problem 2.7.20: IGO 2018 A3 **E**

Find all possible values of integer $n > 3$ such that there is a convex n-gon in which, each diagonal is the perpendicular bisector of at least one other diagonal.

Idea Taking maximum terminal triangle. □

Problem 2.7.21: Lithuania ?? **E**

Prove that in every polygon there is a diagonal that cuts off a triangle and lies completely within the polygon.

Problem 2.7.22: Romanian TST 2008 T1P4 **E**

Prove that there exists a set S of $n-2$ points inside a convex polygon P with n sides, such that any triangle determined by 3 vertices of P contains exactly one point from S inside or on the boundaries.

Idea Checking small cases inductively quickly shows a construction. □

Problem 2.7.23: Iran TST ?? **E**

In an isosceles right-angled triangle shaped billiards table, a ball starts moving from one of the vertices adjacent to hypotenuse. When it reaches to one side then it will reflect its path. Prove that if we reach to a vertex then it is not the vertex at initial position

Problem 2.7.24: APMO 2018 P4 **EM**

Let ABC be an equilateral triangle. From the vertex A we draw a ray towards the interior of the triangle such that the ray reaches one of the sides of the triangle. When the ray reaches a side, it then bounces off following the law of reflection, that is, if it arrives with a directed angle α, it leaves with a directed angle $180^\circ - \alpha$. After n bounces, the ray returns to A without ever landing on any of the other two vertices. Find all possible values of n.

Idea Reflect the whole board when just reflecting the ball doesn't seem to be helping. GLOBAL □

Problem 2.7.25: ISL 2007 C5 **EM**

In the Cartesian coordinate plane define the strips $S_n = \{(x, y) | n \leq x < n + 1\}$, $n \in \mathbb{Z}$ and color each strip black or white. Prove that any rectangle which is not a square can be placed in the plane so that its vertices have the same color.

Idea Proceed step by step. See what happens if the parity of a, b are different. Then the case with two coprimes. In this case, we want to tilt the rectangle to some extent where the desired result is achieved. We just need to show that this is possible. A bit of wishful thinking and a bit of algebra does the rest. □

Problem 2.7.26: APMO 2018 P3 **M**

A collection of n squares on the plane is called tri-connected if the following criteria are satisfied:

1. All the squares are congruent.
2. If two squares have a point P in common, then P is a vertex of each of the squares.
3. Each square touches exactly three other squares.

How many positive integers n are there with $2018 \leq n \leq 3018$, such that there exists a collection of n squares that is tri-connected?

Idea Play around to find that $6k$ for $k > 4$ is good. Then play around a little bit more for a different construction. Another construction for $6k$ gives rise to a construction for $10k$. Which integers can be written as a sum of $6k$ and $10k$? □

Problem 2.7.27: Iran 2005 **E**

A simple polygon is one where the perimeter of the polygon does not intersect itself (but is not necessarily convex). Prove that a simple polygon P contains a diagonal which is completely inside P such that the diagonal divides the perimeter into two parts both containing at least $\frac{n}{3} - 1$ vertices. (Do not count the vertices which are endpoints of the diagonal.)

Idea Triangulate. □

Problem 2.7.28: ISL 2008 C3 **E**

In the coordinate plane consider the set S of all points with integer coordinates. For a positive integer k, two distinct points a, $B \in S$ will be called k-friends if there is a point $C \in S$ such that the area of the triangle ABC is equal to k. A set $T \subset S$ will be called k-clique if every two points in T are k-friends. Find the least positive integer k for which there exits a k-clique with more than 200 elements.

Idea When does $ax + by = c$ have integer solution? Fix one point as origin and check other points friendliness with other points. □

Problem 2.7.29: ISL 2015 C2 **E**

We say that a finite set $\mathcal{S}$ of points in the plane is balanced if, for any two different points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is centre-free if for any three different points A, B and C in $\mathcal{S}$, there is no points P in $\mathcal{S}$ such that $PA = PB = PC$.

1. Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.
2. Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of n points.

Idea Simple, think about circles, then think about "center-free" in a graph theoritic manner. $\square$

2.7.1 Chessboard Pieces

Lemma 2.7.2

What is the maximum number of knights that can be placed on a chessboard such that no two knights attack each other?

Idea A knight's move always changes the color of the cell. □

Problem 2.7.30: IMO 2018 P4 **E/H**

A site is any point (x, y) in the plane such that x and y are both positive integers less than or equal to 20.

Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones with Amy going first. On her turn, Amy places a new red stone on an unoccupied site such that the distance between any two sites occupied by red stones is not equal to $\sqrt{5}$. On his turn, Ben places a new blue stone on any unoccupied site. (A site occupied by a blue stone is allowed to be at any distance from any other occupied site.) They stop as soon as a player cannot place a stone.

Find the greatest K such that Amy can ensure that she places at least K red stones, no matter how Ben places his blue stones.

Idea Using the maximum knight problem as a lemma. □

Problem 2.7.31: **E**

How many rooks can be placed on an $n \times n$ board such that each rook attacks at most one other rook?

Idea Use graphs with one set of degrees being rows, and the other set of degrees being columns. □

Problem 2.7.32:

How many queens can be placed on an $n \times n$ board such that no queen attacks another queen?

Problem 2.7.33: Serbia National D2P2 **HM**

How many queens can be placed on an $n \times n$ board such that each queen attacks at most one other queen?

Problem 2.7.34: BdMO 2019 P10 **E**

Define a new chess piece named warrior. it can either go three steps forward and one step to the side, or t2wo steps forward and two steps to the side in some orientation. In a 2020×2020 chessboard, prove that the mazimum number of warriors so that none of them attack each other is leass htan or equal to $\frac{2}{5}$ of the number of cells.

Idea Color and partition □

Problem 2.7.35: RMM 2019 P4 **EM**

Prove that for every positive integer n there exists a (not necessarily convex) polygon with no three collinear vertices, which admits exactly n diffferent triangulations.
(A triangulation is a dissection of the polygon into triangles by interior diagonals which have no common interior points with each other nor with the sides of the polygon)

Idea

Pics/RMM2019P4.png

Figure 2.7: Fixes

□

Problem 2.7.36: China TST 2015 T1D2P1 **EM**

Prove that : For each integer $n \geq 3$, there exists the positive integers $a_1 < a_2 < \cdots < a_n$, such that for $i = 1, 2, \cdots, n-2$, With a_i, a_{i+1}, a_{i+2} may be formed as a triangle side length , and the area of the triangle is a positive integer.

Idea First of all we dont need to limit us to integers, we can work with rationals. We want to build a_4 from a_1, a_2, a_3. with $a_4 > a_3$ while keeping the area rational i.e. keeping the height and base rational. $\square$

Problem 2.7.37: Codeforces 1158D **E**

You are given n points on the plane, and a sequence S of length $n-2$ consisting of L and R. You need to generate a sequence of the points $a_1, a_2 \ldots a_n$ such that

- the polyline $a_1 a_2 \ldots a_n$ is not self intersecting.
- the directed segment $a_{i+1}a_{i+2}$ is on the left side of the the directed segment $a_i a_{i+1}$ if $S_i = L$, and on the right side if $S_i = R$.

2.8 Sequences

2.8.1 Lemmas

Theorem 2.8.1: Van der Waerden's Theorem

For any given positive integers$r$ and k, there is some number N such that if the integers $\{1, 2, ..., N\}$ are colored, each with one of r different colors, then there are at least k integers in arithmetic progression all of the same color.

2.8.2 Generating Function Lemmas

Theorem 2.8.2: Catalan Recursion

The infinite series defined as following:

$$a_0 = a_1 = 1,\ a_n = \prod_{i=0}^{n} a_i a_{n-i+1} = a_0 a_{n-1} + a_1 a_{n-2} \cdots + a_{n-1} a_0$$

has the general term $a_n = C_n = \frac{1}{n+1}\binom{2n}{n}$

1 `Problem to do with generating function`

2.8.3 Sequence Problems

Sequences - Alexander Remorov

Problem 2.8.1: ISL 1990 **E**

Assume that the set of all positive integers is decomposed into r (disjoint) subsets $A_1 \cup A_2 \cup \cdots \cup A_r = \mathbb{N}$. Prove that one of them, say A_i, has the following property: There exists a positive m such that for any k one can find numbers $a_1, a_2, \ldots, a_k$ in A_i with $0 < a_{j+1} - a_j \leq m$, $(1 \leq j \leq k-1)$.

Problem 2.8.2: Result by Erdos, Dividing the integers into arithmetic progressions **E**

Let $d_1, d_2, \ldots, d_k$ be differences of k arithmetic progressions that partition $\mathbb{N}$. Show that $d_i = d_j$ for some i, j.

Problem 2.8.3: APMO 1999 P1 **E**

Find the smallest positive integer n with the following property: there does not exist an arithmetic progression of 1999 real numbers containing exactly n integers.

Problem 2.8.4: APMO 1999 P2 **E**

Let $a_1, a_2, \ldots$ be a sequence of real numbers satisfying $a_{i+j} \leq a_i + a_j$ for all $i, j = 1, 2, \ldots$. Prove that

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \geq a_n$$

for each positive integer n.

Problem 2.8.5: ISL 1994 A1 **E**

Let $a_0 = 1994$ and $a_{n+1} = \frac{a_n^2}{a_n+1}$ for each nonnegative integer n. Prove that $1994 - n$ is the greatest integer less than or equal to a_n, $0 \leq n \leq 998$

Idea Take the differences. □

Problem 2.8.6: ISL 2007 C4 **M**

Let $A_0 = (a_1, \dots, a_n)$ be a finite sequence of real numbers. For each $k \geq 0$, from the sequence $A_k = (x_1, \dots, x_k)$ we construct a new sequence A_{k+1} in the following way.

1. We choose a partition $\{1, \dots, n\} = I \cup J$, where I and J are two disjoint sets, such that the expression
$$\left| \sum_{i \in I} x_i - \sum_{j \in J} x_j \right|$$
attains the smallest value. (We allow I or J to be empty; in this case the corresponding sum is 0.) If there are several such partitions, one is chosen arbitrarily.

2. We set $A_{k+1} = (y_1, \dots, y_n)$ where $y_i = x_i + 1$ if $i \in I$, and $y_i = x_i - 1$ if $i \in J$.

Prove that for some k, the sequence A_k contains an element x such that $|x| \geq \frac{n}{2}$.

Idea Suppose the contrary. Now, since A_i can only attain finite values, So $A_i = A_j$ for some i, j. Now, we are taking about changes here, so we need to think of some invariants. Firstly the sum, it's not much of an help, because it doesn't give us much control. So kinda sum-ish invariant with a bit more control is the sum of squares. We combine these two ideas. □

Problem 2.8.7: ISL 2009 A6 **EM**

Suppose that $s_1, s_2, s_3, \dots$ is a strictly increasing sequence of positive integers such that the sub-sequences
$$s_{s_1},\ s_{s_2},\ s_{s_3},\ \dots \qquad \text{and} \qquad s_{s_1+1},\ s_{s_2+1},\ s_{s_3+1},\ \dots$$
are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \dots$ is itself an arithmetic progression.

Idea First notice that the two arithmetic sequences has the same common difference. Then notice that the diffrences of the original sequence is bounded. Another advice, give everything names. After naming the smallest difference and the largest differnce, we get two different inequalities, from where we deduce that the difference is constant. □

Problem 2.8.8: ISL 2013 N2 **E**

A

ssume that k and n are two positive integers. Prove that there exist positive integers $m_1, \ldots, m_k$ such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right) \cdots \left(1 + \frac{1}{m_k}\right).$$

Idea Just induct, and think wishfully. □

Problem 2.8.9: USAMO 2007 P1 **E**

n be a positive integer. Define a sequence by setting $a_1 = n$ and, for each $k > 1$, letting a_k be the unique integer in the range $0 \leq a_k \leq k-1$ for which $a_0 + a_1 \cdots + a_k$ is divisible by k. Prove that for any n the sequence a_i eventually becomes constant.

Idea Investigate and done. □

2.9 Exploring Configurations

Problems where there is some kind of a configuration is given, the question usually asks to proof or find some specific properties of the configuration.

2.9.1 Problems

Problem 2.9.1: APMO 2017 P3 **H**

Let $A(n)$ denote the number of sequences $a_1 \geq a_2 \geq \cdots \geq a_k$ of positive integers for which $\sum_{i=1}^{k} a_k = n$ and each $a_i + 1$ is a power of two. Let $B(n)$ denote the number of sequences $b_1 \geq b_2 \geq \cdots \geq b_k$ of positive integers for which $\sum_{i=1}^{k} b_k = n$ and each inequality $b_j \geq 2b_{j+1}$ holds $(j = 1, 2 \ldots m-1)$. Prove that $|A(n)| = |B(n)|$ for every positive integer.

Idea A sequence of the first type can be rewritten as:

$$n = x_1 + 3x_2 + 7x_3 \cdots + (2^i - 1)x_i + \ldots (2^k - 1)x_k$$

Where x_i are non-negative integers. This motivates us to find a way to represent b_i as sums of $(2^i - 1)x_i$. Then since $b_j \geq 2b_{j+1}$, we write: $b_i = 2b_{i-1} + x_i$ with x_i being non-negative integers. $\square$

Problem 2.9.2: ISL 2008 C4 **M**

Let n and k be positive integers with $k \geq n$ and $k - n$ an even number. Let $2n$ lamps labeled $1, 2 \ldots 2n$ be given, each of which can be either on or off. Initially, all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on).

Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off.

Let M be number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off, but where none of the lamps $n+1$ through $2n$ is ever switched on.

Determine $\frac{N}{M}$.

Idea These type of problems most of the time have bijection or algo solutions. Think of a way to perform bijection from the set $S\{M\} \rightarrow S\{N\}$. Find an algorithm to get a sequence of the first type from a sequence of the second type. □

Problem 2.9.3: USAMO 1996 P4 **E**

An n -term sequence $(x_1, x_2, \ldots, x_n)$ in which each term is either 0 or 1 is called a binary sequence of length n. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n.

Idea These type of problems cries for a nice bijection. That is a way to get from $a \rightarrow b$ and vice versa. What if there is no $0,0,1,1$? Or what if there is no $0,1,0$? What is an one way bijection? □

Problem 2.9.4: APMO 2017 P1 **M**

We call a 5 -tuple of integers arrangeable if its elements can be labeled a, b, c, d, e in some order so that $a-b+c-d+e=29$. Determine all 2017 -tuples of integers $n_1, n_2, n_3 \ldots n_{2017}$ such that if we place them in a circle in clockwise order, then any 5 -tuple of numbers in consecutive positions on the circle is arrangeable.

Idea EChen trick. □

Problem 2.9.5: ISL 2004 C1 **E**

There are 10001 students at an university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Find all possible values of k so that the following conditions are satisfied:

1 Each pair of students are in exactly one club.
2 For each student and each society, the student is in exactly one club of the society.
3 Each club has an odd number of students. In addition, a club with $2m+1$ students (m is a positive integer) is in exactly m societies.

Idea Just Double-Counting. □

Problem 2.9.6: ISL 2002 C1 **E**

Let n be a positive integer. Each point (x, y) in the plane, where x and y are non-negative integers with $x + y < n$, is coloured red or blue, subject to the following condition: if a point (x, y) is red, then so are all points (x', y') with $x' \leq x$ and $y' \leq y$. Let A be the number of ways to choose n blue points with distinct x -coordinates, and let B be the number of ways to choose n blue points with distinct y -coordinates. Prove that $A = B$.

Problem 2.9.7: USAMO 2012 P2 **M**

A circle is divided into 432 congruent arcs by 432 points. The points are colored in four colors such that some 108 points are colored Red, some 108 points are colored Green, some 108 points are colored Blue, and the remaining 108 points are colored Yellow. Prove that one can choose three points of each color in such a way that the four triangles formed by the chosen points of the same color are congruent.

Idea Double counting saves the day :) The trick is to rotate ;) □

Problem 2.9.8: European Mathematics Cup 2018 P1 **E**

Call a partition of n a set $a_1, \dots a_k$ with $a_1 \leq a_2 \cdots \leq a_k$ and $a_1 + a_2 \cdots + a_k =$. A partition of a positive integer is 'even' if all of its elements are even numbers. Similarly, a partition is 'odd' if all of its elements are odd. Determine all positive integers n such that the number of even partitions of n is equal to the number of odd partitions of n.

Idea Bijection. □

Problem 2.9.9: APMO 2008 P2 **EM**

Students in a class form groups each of which contains exactly three members such that any two distinct groups have at most one member in common. Prove that, when the class size is 46 , there is a set of 10 students in which no group is properly contained.

Idea Taking the maximum set that follows the "in which no group is properly contained" rule. Now the elements that are *not* in this set, we can connect this element to only one of the pairs from the set. Now defining a bijection, and counting the elements, we are done. □

Problem 2.9.10: IMO SL 1985 **M**

A set of 1985 points is distributed around the circumference of a circle and each of the points is marked with 1 or -1. A point is called "good" if the partial sums that can be formed by starting at that point and proceeding around the circle for any distance in either direction are all strictly positive. Show that if the number of points marked with -1 is less than 662 , there must be at least one good point.

Idea First thing to notice, the number $3 * 661 + 2 = 1985$. And these numbers are completely random. So what if we try to replace 1985 by n ? Will the condition still hold? □

Problem 2.9.11: IMO 2011 P4 **E**

Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \cdots, 2^{n-1}$. We are to place each of the n weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed. Determine the number of ways in which this can be done.

Idea Writing the whole process as a sum, we see that only 2^0 is the odd term here, if we remove that we can divide by 2 to get a recursive formula. □

Idea Calculating wrt to the last placed weight. □

Idea Getting recursive formula considering the position of 2^{n-1}. □

Problem 2.9.12: USAMO 2008 P3 **H**

Let n be a positive integer. Denote by S_n the set of points (x, y) with integer coordinates such that

$$|x| + \left|y + \frac{1}{2}\right| < n.$$

A path is a sequence of distinct points $(x_1, y_1), (x_2, y_2), \ldots, (x_\ell, y_\ell)$ in S_n such that, for $i = 2, \ldots, \ell$, the distance between (x_i, y_i) and (x_{i-1}, y_{i-1}) is 1 (in other words, the points (x_i, y_i) and (x_{i-1}, y_{i-1}) are neighbors in the lattice of points with integer coordinates). Prove that the points in S_n cannot be partitioned into fewer than n paths (a partition of S_n into m paths is a set $\mathcal{P}$ of m nonempty paths such that each point in S_n appears in exactly one of the m paths in $\mathcal{P}$).

Idea Graph + Partition, coloring is just natural. Again, the edges join two neighbor lattice points, so checkerboard coloring. But checkerboard doesn't do much good. So the next thing we try is to apply some derivations of it, pseudo!!! Well, overkill. □

Idea For all n, induction is very natural. The optimal partition (the most beautiful one) and the longest path in it, say P , gives us a way to perform induction. As always, we suppose a partition with $n-1$ paths. As there are a lot of partitions, we need to choose a certain partition, say $\mathbb{M}$. Again as our goal is to include P in $\mathbb{M}$. So suppose that the set with all the points in P is A. And further more, suppose that in $\mathbb{M}$ there is a path Q with $|Q \cap A|$ being maximal among all other partitions of the points. Some some easy case work shows that we must have $P \in \mathbb{M}$. □

Problem 2.9.13: USAMO 2013 P2 **H**

For a positive integer $n \geq 3$ plot n equally spaced points around a circle. Label one of them A , and place a marker at A. One may move the marker forward in a clockwise direction to either the next point or the point after that. Hence there are a total of $2n$ distinct moves available; two from each point. Let a_n count the number of ways to advance around the circle exactly twice, beginning and ending at A , without repeating a move. Prove that $a_{n-1} + a_n = 2^n$ for all $n \geq 4$.

Idea Problems where there are multiple possible value of a function regardless of the current position, one of dealing with these is to assigning labels of these possible values to each points of the function, and this will give a combinatorial model and a way to deal it with bijection. □

Idea First investigate the problem condition, $a_n + a_{n-1} = 2^n$, now, 2^n means the number of differently coloring every point black or white, and the left side is the number of such paths for n and $n-1$. Which means we should try to color the points and see what happens. □

Proof. EChen's solution: In this problem, the main obstacle seems to be the circle condition. And on top of that, on can land on the starting point. So things are pretty messed up here. What we want to do is to make things a little bit more easy to deal with. So our best option is to change the problem so that we get the similar problem with a different explanation. So we change the condition circle with matrix, 2 round with 2 rows. n points with n entries in each rows. What we get now is the same problem, just a bit easier to deal with. We call this Tweak The Problem strategy.

Problem 2.9.14: **E**

10 persons went to a bookstore. It is known that: Every person has bought 3 kinds on books and for every 2 persons, there is at least one kind of books which they both have bought. Let m_i be the number of the persons who bought the i^{th} kind of books and $M = \max\{m_i\}$ Find the smallest possible value of M.

Problem 2.9.15: ISL 2002 C3 **EM**

Let n be a positive integer. A sequence of n positive integers (not necessarily distinct) is called full if it satisfies the following condition: for each positive integer $k \geq 2$, if the number k appears in the sequence then so does the number $k-1$, and moreover the first occurrence of $k-1$ comes before the last occurrence of k. For each n, how many full sequences are there?

Proof. After guessing the ans, the first thing that I did was to draw a level based graph. Suppose that a full sequence has k different entries. Then the top level contains the positions of k in the sequence sorted from left to right. The next level contains the positions of $k-1$ in the sequence sorted so, and so on till the last level. What I noticed is that if we draw arrows pointing from a larger integer to a smaller integer, the only arrows (or more like relations between entries of the sequence) we need to worry about are the arrows pointing left to right in each levels, and the arrows from the last entry of level i to the first entry of level $i+1$. After this, if we try with a smaller case, we see that this leads to a bijection from the set of sequences of length n with n different integers to the set of full-sequences of length n.

Idea Another bijection approach is as followed, in a full-sequence, on first run, go from right to left, placing integers starting with 1 onwards on the 1's in the sequence. on the second run continue counting and placing integers on the 2's and so on. □

Idea Another idea is to prove $a_n = na_{n-1}$. To do this, remove the rightmost 1 and do some casework. $\square$

Problem 2.9.16: ISL 1994 C2 **M**

In a certain city, age is reckoned in terms of real numbers rather than integers. Every two citizens x and x' either know each other or do not know each other. Moreover, if they do not, then there exists a chain of citizens $x = x_0, x_1, \ldots, x_n = x'$ for some integer $n \geq 2$ such that x_{i-1} and x_i know each other. In a census, all male citizens declare their ages, and there is at least one male citizen. Each female citizen provides only the information that her age is the average of the ages of all the citizens she knows. Prove that this is enough to determine uniquely the ages of all the female citizens.

Idea Describing the problem using matrix and vector spaces, the problem reduces to well known theorems of linear algebra. $\square$

Problem 2.9.17: ISL 2003 C1 **E**

Let A be a 101-element subset of the set $S = \{1, 2, \ldots, 1000000\}$. Prove that there exist numbers t_1, $t_2, \ldots, t_{100}$ in S such that the sets

$$A_j = \{x + t_j \mid x \in A\}, \qquad j = 1, 2, \ldots, 100$$

are pairwise disjoint.

Idea just count... $\square$

Problem 2.9.18: USAMO 2006 P2 **E**

For a given positive integer k find, in terms of k, the minimum value of N for which there is a set of $2k + 1$ distinct positive integers that has sum greater than N but every subset of size k has sum at most $\frac{N}{2}$.

Idea Compactness is the optimal decision. $\square$

Problem 2.9.19: MOP Problem **E**

Prove that for any positive integer c, there exists an integer n such that n has more 1's in its binary expansion than $n^2 + c$ does.

Idea For $x = 2^a - 1$, x and x^2 have the same number of $1's$. So does $x = 2^a - 2^b$. But what if increase the number of $1's$ in this x by substracting 1? Let $x = 2^a - 2^b - 1$. This might work if we can choose nice a, b's □

Problem 2.9.20: EGMO 2015 P2 **EM**

A domino is a 2×1 or 1×2 tile. Determine in how many ways exactly n^2 dominoes can be placed without overlapping on a $2n \times 2n$ chessboard so that every 2×2 square contains at least two uncovered unit squares which lie in the same row or column.

Idea Notice how each of the four kind of dominoes needs to be in a group. So if we separated them into blocks, inverstigation shows that there can only be 4 blocks and each strictly attached to the sides. The reason why this is happening is pretty obvious. Now those blocks create two paths between the two opposite vertices of the square. This gives our desired bijection. □

Problem 2.9.21: USA TST 2006 P5 **TE**

Let n be a given integer with n greater than 7 , and let $\mathcal{P}$ be a convex polygon with n sides. Any set of $n-3$ diagonals of $\mathcal{P}$ that do not intersect in the interior of the polygon determine a triangulation of $\mathcal{P}$ into $n-2$ triangles. A triangle in the triangulation of $\mathcal{P}$ is an interior triangle if all of its sides are diagonals of $\mathcal{P}$. Express, in terms of n, the number of triangulations of $\mathcal{P}$ with exactly two interior triangles, in closed form.

Idea Just mindless calculation... □

Problem 2.9.22: ISL 2010 C3 **E**

2500 chess kings have to be placed on a 100×100 chessboard so that

1. no king can capture any other one (i.e. no two kings are placed in two squares sharing a common vertex);

2. each row and each column contains exactly 25 kings.

Find the number of such arrangements. (Two arrangements differing by rotation or symmetry are supposed to be different.)

Idea In a 2×2 box, one can place only one king. So we divide the board in that way, and explore... □

Problem 2.9.23: ISL 2015 C1 **E**

In Lineland there are $n \geq 1$ towns, arranged along a road running from left to right. Each town has a left bulldozer (put to the left of the town and facing left) and a right bulldozer (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a left and right bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, bulldozers are quite unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

Let A and B be two towns, with B to the right of A. We say that town A can sweep town B away if the right bulldozer of A can move over to B pushing off all bulldozers it meets. Similarly town B can sweep town A away if the left bulldozer of B can move over to A pushing off all bulldozers of all towns on its way.

Prove that there is exactly one town that cannot be swept away by any other one.

Idea Focus on the heaviest bulldozer. □

2.9.1.1 Conway's Soldiers

Problem 2.9.24: ISL 1993 C5 **MH**

On an infinite chessboard, a solitaire game is played as follows: at the start, we have n^2 pieces occupying a square of side n. The only allowed move is to jump over an occupied square to an unoccupied one, and the piece which has been jumped over is removed. For which n can the game end with only one piece remaining on the board?

Idea We want to find an invariant. So we need to find a weight for each of the cells such that any two consecutive cells' values equals to the values of the two cells on the two sides. Some mind bashing gives the idea of mod 3. And a construction for the other n's can be easily generated after some casework. □

Problem 2.9.25: ARO 1999 P4 **M**

A frog is placed on each cell of a $n \times n$ square inside an infinite chessboard (so initially there are a total of $n \times n$ frogs). Each move consists of a frog A jumping over a frog B adjacent to it with A landing in the next cell and B disappearing (adjacent means two cells sharing a side). Prove that at least $\left[\frac{n^2}{3}\right]$ moves are needed to reach a configuration where no more moves are possible.

Idea In the final stage, no two neighboring cells are occupied. Could we double count the number of frogs with this information? What about the number of frogs in the original $n \times n$ board? Another small information needed for this is that we need 2 moves to "empty" a 2×2 board. □

2.9.1.2 Triominos

Problem 2.9.26: ARO 2011 P10.8 **M**

A 2010×2010 board is divided into corner-shaped figures of three cells. Prove that it is possible to mark one cell in each figure such that each row and each column will have the same number of marked cells.

Idea First we will mark the corner pieces of the triominos. Then shift the mark to either of the legs. Our objective is to show that we can always do this. First if we only focus on the rows, we can easily show that this can be done using some counting. To show that we can do the same for columns as well, we create a graph from columns and rows to triominos which should be operated on, and using Hall's Marriage we pove the result. □

Problem 2.9.27: St. Petersburg 2000 **E**

On an infinite checkerboard are placed 111 non-overlapping corners, L-shaped figures made of 3 unit squares. Suppose that for any corner, the 2×2 square containing it is entirely covered by the corners. Prove that one can remove some number between 1 and 110 of the corners so that the property will be preserved.

2.9.1.3 Dominos

Problem 2.9.28: **E**

An $m \times n$ rectangular grid is covered by dominoes. Prove that the vertices of the grid can be coloured using three colours so that any two vertices a distance 1 apart are colored with different colours if and only if their segment lies on the boundary of a domino.

Idea Create a graph with the midpoints of the dominos. $\square$

2.9.2 Coloring Problems

Problem 2.9.29: EGMO 2017 P2 **M**

Find the smallest positive integer k for which there exists a colouring of the positive integers $\mathbb{Z}_{>0}$ with k colours and a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ with the following two properties:

1. For all positive integers m, n of the same colour, $f(m+n) = f(m) + f(n)$.
2. There are positive integers m, n such that $f(m+n) \neq f(m) + f(n)$.

In a colouring of $\mathbb{Z}_{>0}$ with k colours, every integer is coloured in exactly one of the k colours. In both (i) and (ii) the positive integers m, n are not necessarily distinct.

Idea Firstly a modular coloring shows that $1 < k \leq 2$. For $k = 2$ we do some trivial case works. □

Problem 2.9.30: ISL 2002 C2 **E**

For n an odd positive integer, the unit squares of an $n \times n$ chessboard are coloured alternately black and white, with the four corners coloured black. A it tromino is an L-shape formed by three connected unit squares. For which values of n is it possible to cover all the black squares with non-overlapping trominos? When it is possible, what is the minimum number of trominos needed?

Idea First find the first ans and a configuration that works. Then guess the second ans, and see from where that might come from, usually these anses come from some special set of problems, where bijection is applicable. □

Problem 2.9.31: Codeforces 101954/G **E/H**

Two Knights are given on a chessboard, one black one white. Which player has a winning possibility?

Idea A knight's move always changes the color of the cell. □

Problem 2.9.32: ARO 1993 P10.4 **M**

Thirty people sit at a round table. Each of them is either smart or dumb. Each of them is asked: "Is your neighbor to the right smart or dumb?" A smart person always answers correctly, while a dumb person can answer both correctly and incorrectly. It is known that the number of dumb people does not exceed F. What is the largest possible value of F such that knowing what the answers of the people are, you can point at at least one person, knowing he is smart?

Idea We see that the strings of truth only exist either when all people are dumb or the last one is the truthful one. Now we take the longest such string, and this sting has to be of the second kind. To prove this, we use bounding with the given constraint. □

Problem 2.9.33: Tournament of Towns 2015S S6 **E**

An Emperor invited 2015 wizards to a festival. Each of the wizards knows who of them is good and who is evil, however the Emperor doesn't know this. A good wizard always tells the truth, while an evil wizard can tell the truth or lie at any moment. The Emperor gives each wizard a card with a single question, maybe different for different wizards, and after that listens to the answers of all wizards which are either "yes" or "no". Having listened to all the answers, the Emperor expels a single wizard through a magic door which shows if this wizard is good or evil. Then the Emperor makes new cards with questions and repeats the procedure with the remaining wizards, and so on. The Emperor may stop at any moment, and after this the Emperor may expel or not expel a wizard. Prove that the Emperor can expel all the evil wizards having expelled at most one good wizard.

Idea There is only one problem with the cyclic arrangement, that is what if all the answers are 'yes'? We get rid of this problem by trying small case with $n = 3$ and trying the most simple way to connect this strategy to any n. Simplicity is the key. □

Problem 2.9.34: ISL 2007 C1 **E**

Let $n > 1$ be an integer. Find all sequences $a_1, a_2, \ldots a_{n^2+n}$ satisfying the following conditions:

1. $a_i \in \{0, 1\}$ for all $1 \leq i \leq n^2 + n$
2. for all $0 \leq i \leq n^2 - n$

$$a_{i+1} + a_{i+2} + \ldots + a_{i+n} < a_{i+n+1} + a_{i+n+2} + \ldots + a_{i+2n}$$

Idea $n+1$ blocks of n, each strictly greater than the previous one. means the sums of the blocks have to be $0, 1, \dots n$. construction's easy from examples of $2, 3$. □

Problem 2.9.35: ISL 2016 C2 **E**

Find all positive integers n for which all positive divisors of n can be put into the cells of a rectangular table under the following constraints: each cell contains a distinct divisor;
the sums of all rows are equal; and
the sums of all columns are equal.

Idea Check the sizes. □

Problem 2.9.36: ISL 2007 C3 **E**

Find all positive integers n for which the numbers in the set $S = \{1, 2, \dots, n\}$ can be colored red and blue, with the following condition being satisfied: The set $S \times S \times S$ contains exactly 2007 ordered triples (x, y, z) such that:

1. the numbers x, y, z are of the same color
2. the number $x + y + z$ is divisible by n.

Idea Trying out small cases, noticing patter. It doesn't matter 'which' numbers are red, but 'how' many numbers are red. □

Problem 2.9.37: ISL 2014 C4 **EM**

Construct a tetromino by attaching two 2×1 dominoes along their longer sides such that the midpoint of the longer side of one domino is a corner of the other domino. This construction yields two kinds of tetrominoes with opposite orientations. Let us call them S- and Z-tetrominoes, respectively.

Assume that a lattice polygon P can be tiled with S-tetrominoes. Prove that no matter how we tile P using only S- and Z-tetrominoes, we always use an even number of Z-tetrominoes.

Idea So after we are determined to do coloring, it is not very hard to come up with a coloring. Start from stracth type coloring. Color one square at a time, this might take several tries.

`Pics/ISL2014C4.pdf`

Figure 2.8

□

2.10 Double Counting and Other Algebraic Methods

Problem 2.10.1: ISL 2012 C3 **M**

In a 999×999 square table some cells are white and the remaining ones are red. Let T be the number of triples (C_1, C_2, C_3) of cells, the first two in the same row and the last two in the same column, with C_1, C_3 white and C_2 red. Find the maximum value T can attain.

Idea Expicitly count the value, and bound it using ineq... □

2.10.1 Probabilistic Methods

Problem 2.10.2: ISL 2006 C3 **M/E**

Let S be a finite set of points in the plane such that no three of them are on a line. For each convex polygon P whose vertices are in S, let $a(P)$ be the number of vertices of P, and let $b(P)$ be the number of points of S which are outside P. A line segment, a point, and the empty set are considered as convex polygons of 2, 1, and 0 vertices respectively. Prove that for every real number x

$$\sum_{P} x^{a(P)}(1-x)^{b(P)} = 1,$$

where the sum is taken over all convex polygons with vertices in S.

Idea This can be done by strong induction and double counting. Counting for every possibile subsets with every points inside of the convex hull and at least one point on the convex hull outside of the set. □

Idea The beautiful solution on the other hand uses probability. Color each point black or white, then translate the condition in terms of probability. :D □

RMM 2019 P3

Idea □

2.11 Problems

Problem 2.11.1: ISL 2009 C1 **E**

Consider 2009 cards, each having one gold side and one black side, lying on parallel on a long table. Initially all cards show their gold sides. Two player, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those which showed gold now show black and vice versa. The last player who can make a legal move wins.

1. Does the game necessarily end?

2. Does there exist a winning strategy for the starting player?

Idea Simplicity is the key. □

Problem 2.11.2: ISL 2009 C3 **H**

Let n be a positive integer. Given a sequence ε_1 , . . . , ε_{n-1} with $\varepsilon_i = 0$ or $\varepsilon_i = 1$ for each $i = 1$, . . . , $n - 1$, the sequences a_0 , . . . , a_n and b_0 , . . . , b_n are constructed by the following rules:

$$a_0 = b_0 = 1, \quad a_1 = b_1 = 7,$$

$$a_{i+1} = \begin{cases} 2a_{i-1} + 3a_i, & \text{if } \varepsilon_i = 0, \\ 3a_{i-1} + a_i, & \text{if } \varepsilon_i = 1, \end{cases} \quad \text{for each } i = 1, \dots, n-1,$$

$$b_{i+1} = \begin{cases} 2b_{i-1} + 3b_i, & \text{if } \varepsilon_{n-i} = 0, \\ 3b_{i-1} + b_i, & \text{if } \varepsilon_{n-i} = 1, \end{cases} \quad \text{for each } i = 1, \dots, n-1.$$

Prove that $a_n = b_n$.

Idea Got the idea, will try later. □

Problem 2.11.3: ARO 2018 P11.5 **E**

On the table, there're 1000 cards arranged on a circle. On each card, a positive integer was written so that all 1000 numbers are distinct. First, Vasya selects one of the card, remove it from the circle, and do the following operation: If on the last card taken out was written positive integer k , count the k^{th} clockwise card not removed, from that position, then remove it and repeat the operation. This continues until only one card left on the table. Is it possible that, initially, there's a card A such that, no matter what other card Vasya selects as first card, the one that left is always card A ?

Problem 2.11.4: ARO 2017 P9.1 **E**

In country some cities are connected by oneway flights (There are no more then one flight between two cities). City A called "available" for city B , if there is flight from B to A , maybe with some transfers. It is known, that for every 2 cities P and Q exist city R , such that P and Q are available from R. Prove, that exist city A , such that every city is available for A.

Problem 2.11.5: ARO 2018 10.3 **E**

A positive integer k is given. Initially, N cells are marked on an infinite checkered plane. We say that the cross of a cell A is the set of all cells lying in the same row or in the same column as A. By a turn, it is allowed to mark an unmarked cell A if the cross of A contains at least k marked cells. It appears that every cell can be marked in a sequence of such turns. Determine the smallest possible value of N.

Idea First find the construction. □

Problem 2.11.6: ARO 2018 P9.5 **E**

On the circle, 99 points are marked, dividing this circle into 99 equal arcs. Petya and Vasya play the game, taking turns. Petya goes first; on his first move, he paints in red or blue any marked point. Then each player can paint on his own turn, in red or blue, any uncolored marked point adjacent to the already painted one. Vasya wins, if after painting all points there is an equilateral triangle, all three vertices's of which are colored in the same color. Could Petya prevent him?

Idea Think of what Petya must do to prevent immediate losing. □

Problem 2.11.7: ISL 2004 C2 **E**

Let n and k be positive integers. There are given n circles in the plane. Every two of them intersect at two distinct points, and all points of intersection they determine are pairwise distinct (i. e. no three circles have a common point). No three circles have a point in common. Each intersection point must be colored with one of n distinct colors so that each color is used at least once and exactly k distinct colors occur on each circle. Find all values of $n \geq 2$ and k for which such a coloring is possible.

Problem 2.11.8: ISL 2004 C3 **E**

The following operation is allowed on a finite graph: Choose an arbitrary cycle of length 4 (if there is any), choose an arbitrary edge in that cycle, and delete it from the graph. For a fixed integer $n \geq 4$, find the least number of edges of a graph that can be obtained by repeated applications of this operation from the complete graph on n vertices's (where each pair of vertices's are joined by an edge).

Idea Walk backwards. or the same thing with Bipartite Graphs. □

Problem 2.11.9: Iran TST 2012 P4 **E**

Consider $m+1$ horizontal and $n+1$ vertical lines ($m,n \geq 4$) in the plane forming an $m \times n$ table. Cosider a closed path on the segments of this table such that it does not intersect itself and also it passes through all $(m-1)(n-1)$ interior vertices's (each vertex is an intersection point of two lines) and it doesn't pass through any of outer vertices. Suppose A is the number of vertices's such that the path passes through them straight forward, B number of the table squares that only their two opposite sides are used in the path, and C number of the table squares that none of their sides is used in the path. Prove that $A = B - C + m + n - 1$.

Problem 2.11.10: AoPS **E**

Given $2n+1$ irrational numbers, prove that one can pick n from them s.t. no two of the choosen n sum up to a rational number.

Idea Use a graph theory representation. □

Problem 2.11.11: Bulgarian IMO TST 2004, Day 3, Problem 3 **H**

Prove that among any $2n+1$ irrational numbers there are $n+1$ numbers such that the sum of any k of them is irrational, for all $k \in \{1, 2, 3, \ldots, n+1\}$.

Idea We first create a set B such that any linear combination of the elements in it are irrational. Then for convenience, we add 1 to it, so that now the sum equals to 0 of any linear combinations. An algorithm for building it comes into our mind, which leaves some other original elements, which we then later add to the final solution set A along with the elements in the set B except 1. □

Problem 2.11.12: ISL 1997 P4 **E**

An $n \times n$ matrix whose entries come from the set $S = \{1, 2, ..., 2n-1\}$ is called a "silver matrix" if, for each $i = 1, 2, ..., n$, the i -th row and the i -th column together contain all elements of S. Show that:

1 there is no silver matrix for $n = 1997$;
2 silver matrices exist for infinitely many values of n.

Idea Proving that for odd n 's isn't hard. Then A small try-around with $n = 2, 4$, we see a pattern that leads to a construction for 2^n □

Problem 2.11.13: **E**

A rectangle is completely partitioned into smaller rectangles such that each smaller rectangles has at least one integral side. Prove that the original rectangle also has at least one integral side.

Idea Try a special grid system with $.5 \times .5$ boxes. □

Idea Consider the number of corners in the rectangle. □

Problem 2.11.14: ISL 2004 C5 **M**

A and B play a game, given an integer N, A writes down 1 first, then every player sees the last number written and if it is n then in his turn he writes $n+1$ or $2n$, but his number cannot be bigger than N. The player who writes N wins. For which values of N does B win?

Idea Trying with smaller cases, it's easy. Using most important game theory trick. □

Problem 2.11.15: ISL 2006 C1 **E**

We have $n \geq 2$ lamps $L_1, L_2 \ldots L_n$ in a row, each of them being either on or off. Every second we simultaneously modify the state of each lamp as follows: if the lamp L_i and its neighbors (only one neighbor for $i = 1$ or $i = n$, two neighbors for other i) are in the same state, then L_i is switched off; otherwise, L_i is switched on. Initially all the lamps are off except the leftmost one which is on.

1 Prove that there are infinitely many integers n for which all the lamps will eventually be off.
2 Prove that there are infinitely many integers n for which the lamps will never be all off

Problem 2.11.16: ISL 2006 C4 **M**

A cake has the form of an $n \times n$ square composed of n^2 unit squares. Strawberries lie on some of the unit squares so that each row or column contains exactly one strawberry; call this arrangement $\mathbb{A}$.
Let $\mathbb{B}$ be another such arrangement. Suppose that every grid rectangle with one vertex at the top left corner of the cake contains no fewer strawberries of arrangement $\mathbb{B}$ than of arrangement $\mathbb{A}$. Prove that arrangement $\mathbb{B}$ can be obtained from $\mathbb{A}$ by performing a number of switches, defined as follows:
A switch consists in selecting a grid rectangle with only two strawberries, situated at its top right corner and bottom left corner, and moving these two strawberries to the other two corners of that rectangle.

Idea When the first approach fails, don't throw that idea yet. Stick to it, as it is most probably the closest to a correct solution. Taking the smallest rectangle with 0 's equal to 1 's, we see that we can 'shrink' the rectangle. Which leads to a solution instantly. □

Problem 2.11.17: ISL 2014 C2 **E**

We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b , then we erase these numbers and write the number $a+b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m.

Idea When you know that the problem can be solved using invariants, go through all of the possible invariants (from the rules of thumb). Don't give up on one so quickly. And product and sum are actually more close than you think. Because if you are told to prove some bound on the sum, then product can come very handy. After all there is AM-GM to connect sum and product. □

Problem 2.11.18: ISL 2016 C3 **E**

Let n be a positive integer relatively prime to 6. We paint the vertices's of a regular n -gon with three colours so that there is an odd number of vertices's of each colour. Show that there exists an isosceles triangle whose three vertices's are of different colours.

Idea Double Count with the number of points of each colors. □

Problem 2.11.19: Iran TST 2002 P3 **E**

A "2-line" is the area between two parallel lines. Length of "2-line" is distance of two parallel lines. We have covered unit circle with some "2-lines". Prove sum of lengths of "2-lines" is at least 2.

Idea Consider the "2-line" of the largest length. □

Problem 2.11.20: ARO 2008 P9.5 **E**

The distance between two cells of an infinite chessboard is defined as the minimum number to moves needed for a king to move from one to the other. On the board are chosen three cells on pairwise distances equal to 100. How many cells are there that are at the distance 50 from each of the three cells?

Problem 2.11.21: USAMO 1986 P2 **E**

During a certain lecture, each of five mathematicians fell asleep exactly twice. For each pair of mathematicians, there was some moment when both were asleep simultaneously. Prove that, at some moment, three of them were sleeping simultaneously.

Problem 2.11.22: Mexican Regional 2014 P6 **E**

Let $A = n \times n$ be a $\{0, 1\}$ matrix, where each row is different. Prove that you can remove a column such that the resulting $n \times (n-1)$ matrix has n different rows.

Idea Try to represent the sets in a nicer way, with graph. or. Induction on the number of columns deleted and the number or different rows being there. □

Problem 2.11.23: IMO 2017 P5 **M (H)**

An integer $N \geq 2$ is given. A collection of $N(N+1)$ soccer players, no two of whom are of the same height, stand in a row. Show that Sir Alex can always remove $N(N-1)$ players from this row leaving a new row of $2N$ players in which the following N conditions hold:
(1) no one stands between the two tallest players,
(2) no one stands between the third and fourth tallest players,
$\vdots$
(N) no one stands between the two shortest players.

Idea $N(N+1)$, rows, removing ... these things just begs for to be arranged in a systematic order. As arranging thing in a matrix is the simplest way, we arrange the bad-bois in a $N \cdot (N+1)$ matrix. Now finding the algorithm is not very hard. □

Problem 2.11.24: ISL 1990 P3 **E**

Let $n \geq 3$ and consider a set E of $2n-1$ distinct points on a circle. Suppose that exactly k of these points are to be colored black. Such a coloring is good if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly n points from E. Find the smallest value of k so that every such coloring of k points of E is good.

Idea Creating a graph and using Alternating Chains Technique □

Problem 2.11.25: USAMO 1999 P1 **E**

Some checkers placed on an $n \times n$ checkerboard satisfy the following conditions:

1 every square that does not contain a checker shares a side with one that does;
2 given any pair of squares that contain checkers, there is a sequence of squares containing checkers, starting and ending with the given squares, such that every two consecutive squares of the sequence share a side.

Prove that at least $(n^2 - 2)/3$ checkers have been placed on the board.

Idea As the problem simply seems to exist, we can't count how much contribution a checker cntaining square contributes to the whole board. So we place one at a time and see the changes. □

Generalization 2.11.25.1: USAMO 1999 P1 generalization

Find the smallest positive integer m such that if m squares of an $n \times n$ board are colored, then there will exist 3 colored squares whose centers form a right triangle with sides parallel to the edges of the board.

Problem 2.11.26: ISL 2013 C1 **E**

Let n be an positive integer. Find the smallest integer k with the following property; Given any real numbers $a_1, \cdots, a_d$ such that $a_1 + a_2 + \cdots + a_d = n$ and $0 \leq a_i \leq 1$ for $i = 1, 2, \cdots, d$, it is possible to partition these numbers into k groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.

Idea Think about the worst case where d is the minimum and the ans is d , it would only be possible if each $a_i > \frac{1}{2}$ but this can't be true, so, the ans is $2n - 1$. Now the ques should become obvious. □

Problem 2.11.27: Brazilian Olympic Revenge 2014 **M**

Let n a positive integer. In a $2n \times 2n$ board, $1 \times n$ and $n \times 1$ pieces are arranged without overlap. Call an arrangement maximal if it is impossible to put a new piece in the board without overlapping the previous ones. Find the least k such that there is a maximal arrangement that uses k pieces.

Idea Intuition gives that there is at least one n -mino in each row. But we can easily guess that there is no maximal arrangement with $2n$ minos. Suppose in a maximal arrangement, there are no vertical n -mino, that means there are more than $2n+1$ n-minos. So suppose that there is at least one vertical suppose that it lies in a column i between 1 and n. Then we have that there is at least one n -mino in each column in between 1 and i. If there is one in between 1 and $2n$, say j , then there is one in each of the columns on the right side of it. Then we count horizontal n -minos, we show that $2n+1$ is the answer. □

Problem 2.11.28: ISL 2008 C1 **E**

In the plane we consider rectangles whose sides are parallel to the coordinate axes and have positive length. Such a rectangle will be called a box. Two boxes intersect if they have a common point in their interior or on their boundary. Find the largest n for which there exist n boxes $B_1, B_2 \dots B_n$ such that B_i and B_j intersect if and only if $i \not\equiv j \pm 1 \pmod{n}$.

Idea Instead of focusing on building the boxes from only one side (i.e. starting with $1, 2 \dots$, we should include n in our investigation, and follow from both direction, (i.e. $1, 2 \dots$ and $\dots, n-1, n$). □

Problem 2.11.29: USAMO 2008 P4 **E**

Let $\mathcal{P}$ be a convex polygon with n sides, $n \geq 3$. Any set of $n-3$ diagonals of $\mathcal{P}$ that do not intersect in the interior of the polygon determine a triangulation of $\mathcal{P}$ into $n-2$ triangles. If $\mathcal{P}$ is regular and there is a triangulation of $\mathcal{P}$ consisting of only isosceles triangles, find all the possible values of n.

Idea It's not hard after getting the ans. □

Problem 2.11.30: ARO 2016 P3 **M**

We have a sheet of paper, divided into 100×100 unit squares. In some squares, we put right-angled isosceles triangles with $leg = 1$ (Every triangle lies in one unit square and is half of this square). Every unit grid segment (boundary too) is under one leg of a triangle. Find maximal number of unit squares, that don't contains any triangles.

Idea What is the minimum number of triangles you can use in a row? Create a good row one at a time □

Problem 2.11.31: India TST 2013 Test 3, P1 **E**

For a positive integer n , a *Sum-Friendly Odd Partition* of n is a sequence $(a_1, a_2 \dots a_k)$ of odd positive integers with $a_1 \le a_2 \le \dots \le a_k$ and $a_1 + a_2 + \dots + a_k = n$ such that for all positive integers $m \le n$, m can be uniquely written as a subsum $m = a_{i_1} + a_{i_2} + \dots + a_{i_r}$. (Two subsums $a_{i_1} + a_{i_2} + \dots + a_{i_r}$ and $a_{j_1} + a_{j_2} + \dots + a_{j_s}$ with $i_1 < i_2 < \dots < i_r$ and $j_1 < j_2 < \dots < j_s$ are considered the same if $r = s$ and $a_{i_l} = a_{j_l}$ for $1 \le l \le r$.) For example, $(1, 1, 3, 3)$ is a *sum-friendly odd partition* of 8. Find the number of sum-friendly odd partitions of 9999.

Idea Firstly we explore one SFOP at a time. Which gives us a way to tell what a_{i+1} is going to be by looking at $a_1 \dots a_i$. □

Problem 2.11.32: IMO 2011 P2 **H**

Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A windmill is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q , takes over as the new pivot, and the line now rotates clockwise about Q , until it next meets a point of $\mathcal{S}$. This process continues indefinitely.
Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

Idea Some workaround gives us the idea that the starting line has to be kinda "in between" the points. Formal words could be: the line should divide the set of points in two sets so that the two sets have equal number of points. Once we take a such line, we see that after every move we get a new line which has similar properties of the first line. □

Idea So moral of the story is that if you get some vague idea that something has to satisfy something-ish, remove the -ish part, and try with a formal assumption. □

Problem 2.11.33: IOI 2016 P5 **M**

A computer bug has a permutation P of length $2^k = N$ that changes any string added to a DS according to the permutation, i.e. it makes $S[i] = S[P[i]]$. Your task it to find the permutation in the following ways:

1 You can add at most $n \log_2 n$ N bit binary strings to the DS.
2 You can ask at most $n \log_2 n$, in the form of N bit binary strings. The answer will be "true" if the string exists in the DS after the Bug had changed the strings and "no" otherwise.

Idea Typical Divide and Conquer approach. You want to do the same thing for $N = \frac{N}{2}$, and to do so you need to tell exactly what the first $\frac{N}{2}$ terms of the permutation are. To do this, you can use at most N questions. This is easy, you first add strings with only one bit present in the first $\frac{N}{2}$ positions, and then ask N questions with only one bit in every N positions. This maps the first $\frac{N}{2}$ numbers of the permutation to a set of $\frac{N}{2}$ integers. And we can proceed by induction now. $\square$

Problem 2.11.34: ISL 2001 C6 **M**

For a positive integer n define a sequence of zeros and ones to be balanced if it contains n zeros and n ones. Two balanced sequences a and b are neighbors if you can move one of the $2n$ symbols of a to another position to form b. For instance, when $n = 4$, the balanced sequences 01101001 and 00110101 are neighbors because the third (or fourth) zero in the first sequence can be moved to the first or second position to form the second sequence. Prove that there is a set S of at most $\frac{1}{n+1}\binom{2n}{n}$ balanced sequences such that every balanced sequence is equal to or is a neighbor of at least one sequence in S.

Problem 2.11.35: ISL 1998 C4 **M**

Let $U = \{1, 2, \ldots, n\}$, where $n \geq 3$. A subset S of U is said to be split by an arrangement of the elements of U if an element not in S occurs in the arrangement somewhere between two elements of S. For example, 13542 splits $\{1, 2, 3\}$ but not $\{3, 4, 5\}$. Prove that for any $n - 2$ subsets of U, each containing at least 2 and at most $n - 1$ elements, there is an arrangement of the elements of U which splits all of them.

Idea If we try to apply induction, we see that the sets with 2 and $n-1$ elements create problems, so we handle them first. $\square$

Problem 2.11.36: USA TST 2009 P1 **M**

Let m and n be positive integers. Mr. Fat has a set S containing every rectangular tile with integer side lengths and area of a power of 2. Mr. Fat also has a rectangle R with dimensions $2^m \times 2^n$ and a 1×1 square removed from one of the corners. Mr. Fat wants to choose $m + n$ rectangles from S, with respective areas $2^0, 2^1, \ldots, 2^{m+n-1}$, and then tile R with the chosen rectangles. Prove that this can be done in at most $(m + n)!$ ways.

Idea The fact that this can be done in $(m + n)!$ asks for a bijective proof. Now an intuition gives us that we have to sort the tiles wrt the missing square in some way. Now since the numbers $\square$

Problem 2.11.37: ARO 2016 P1 **E**

There are 30 teams in **NBA** and every team play 82 games in the year. Bosses of **NBA** want to divide all teams on Western and Eastern Conferences (not necessarily equally), such that the number of games between teams from different conferences is half of the number of all games. Can they do it?

Idea You want to divide something. Check the parity. □

Problem 2.11.38: AoPS **M**

Each edge of a polyhedron is oriented with an arrow such that at each vertex, there is at least one arrow leaving the vertex and at least one arrow entering the vertex. Prove that there exists a face on the polyhedron such that the edges on its boundary form a directed cycle.

Idea The trick which is used to prove Euler's Polyhedron theorem. □

Problem 2.11.39: ISL 2014 C3 **M**

Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is peaceful if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 unit squares.

Idea Guessing the "Correct" ans is the challenge, think of the worst case you can produce. □

Problem 2.11.40: APMO 2012 P2 **E**

Into each box of a $n \times n$ square grid, a real number greater than or equal to 0 and less than or equal to 1 is inserted. Consider splitting the grid into 2 non-empty rectangles consisting of boxes of the grid by drawing a line parallel either to the horizontal or the vertical side of the grid. Suppose that for at least one of the resulting rectangles the sum of the numbers in the boxes within the rectangle is less than or equal to 1, no matter how the grid is split into 2 such rectangles. Determine the maximum possible value for the sum of all the $n \times n$ numbers inserted into the boxes. Find the ans for k-dimension grids too.

Idea As the maximal rectangle defines other smaller rectangles in it, we take that. □

Problem 2.11.41: Indian Postal Coaching 2011 **M**

Consider 2011^2 points arranged in the form of a 2011×2011 grid. What is the maximum number of points that can be chosen among them so that no four of them form the vertices's of either an isosceles trapezium or a rectangle whose parallel sides are parallel to the grid lines?

Idea Since we need to maintain the relation of perpendicular bisectors, we focus on perp bisectors and the points on one line only and then count. □

Problem 2.11.42: ISL 2010 C2 **M**

On some planet, there are 2^N countries $(N \geq 4)$. Each country has a flag N units wide and one unit high composed of N fields of size 1×1, each field being either yellow or blue. No two countries have the same flag. We say that a set of N flags is diverse if these flags can be arranged into an $N \times N$ square so that all N fields on its main diagonal will have the same color. Determine the smallest positive integer M such that among any M distinct flags, there exist N flags forming a diverse set.

Idea Using induction we see that if we have found the value of M for $N-1$, then possibly the value for M_N is twice as large than M_{N-1}. With some further calculation, we see that if we have $2 * M_{N-1} - 1 = M_N$, then we can pick half of them and apply induction and still be left with a 'lot' of flags to choose the Nth element of the diverse set.

After that the only work left is to proof for $N = 4$. Which is easy casework. □

Idea Another way to prove the ans, is to prove the bound for any non-diverse set. In this case, we use hall's marriage to prove the contradiction. □

Problem 2.11.43: Iran TST 2007 P2 **E**

Let A be the largest subset of $\{1, \ldots, n\}$ such that for each $x \in A$, x divides at most one other element in A. Prove that

$$\frac{2n}{3} \leq |A| \leq \left\lceil \frac{3n}{4} \right\rceil .$$

Idea Partition the set optimally. □

Problem 2.11.44: India IMO Camp 2017 **H**

Find all positive integers n s.t. the set $\{1, 2, \dots, 3n\}$ can be partitioned into n triplets (a_i, b_i, c_i) such that $a_i + b_i = c_i$ for all $1 \le i \le n$.

Problem 2.11.45: ISL 2012 C2 **TE**

Let $n \ge 1$ be an integer. What is the maximum number of disjoint pairs of elements of the set $\{1, 2, \dots, n\}$ such that the sums of the different pairs are different integers not exceeding n ?

Idea As Usual, first find the ans. Using double counting is quite natural. Working with small cases easily gives a construction. □

Problem 2.11.46: CodeForces 989C **E**

Problem 2.11.47: CodeForces 989B **E**

Problem 2.11.48: ISL 2011 A5 **MH**

Prove that for every positive integer n , the set $\{2, 3, \dots, 3n+1\}$ can be partitioned into n triples in such a way that the numbers from each triple are the lengths of the sides of some obtuse triangle.

Idea What is the best way to choose the side lengths of an obtuse triangle? Obviously by maintaining some strict rules to get the third side from the first two sides and making the rules invariant. One way of doing this is to take $(a, b, a + b - 1)$.
After that, some (literally this is the hardest part of the problem) experiment to find a construction. First, we try to partition the set into tuples of our desired form, but we soon realize that that can't be done so easily. So we try a little bit of different approach and make one tuple different from the others. Luckily this approach gives us a nice construction. □

Problem 2.11.49: Iran TST 2017 D1P1 **TE**

In the country of Sugarland, there are 13 students in the IMO team selection camp. 6 team selection tests were taken and the results have came out. Assume that no students have the same score on the same test. To select the IMO team, the national committee of math Olympiad have decided to choose a permutation of these 6 tests and starting from the first test, the person with the highest score between the remaining students will become a member of the team. The committee is having a session to choose the permutation.
Is it possible that all 13 students have a chance of being a team member?

Idea If a student is in x^{th} place in a test t_y , and he has a chance to get into the team iff the $1^th, 2^th \dots x-1^th$ persons in test t_y are already in the team. So $x \leq 5$. Make a $6 \cdot 6$ grid with place $\cdot$ test. WHY?? Because it makes the best sense among other possible choices of the grid. A little bit of work produces a configuration where every student has a chance to get into the team. $\square$

Problem 2.11.50: ISL 2009 C2 **M**

For any integer $n \geq 2$, let $N(n)$ be the maximum number of triples (a_i, b_i, c_i) , $i = 1, 2 \dots, N(n)$, consisting of nonnegative integers a_i , b_i and c_i such that the following two conditions are satisfied:

1 $a_i + b_i + c_i = n$ for all $i = 1, \dots, N(n)$,
2 If $i \neq j$ then $a_i \neq a_j$, $b_i \neq b_j$ and $c_i \neq c_j$

Determine $N(n)$ for all $n \geq 2$.

Idea Find an upper bound. It's easy. Then with some experiment, we see that this upper bound is achievable. So our next task is to find a construction. As it is related to 3 , we first try with $n = 3k$. Some experiment and experience gives us a construction. $\square$

Problem 2.11.51: **M**

Let n be an integer. What is the maximum number of disjoint pairs of elements of the set $\{1, 2, \dots, n\}$ such that the sums of the different pairs are different integers not exceeding n ?

Problem 2.11.52: ISL 2002 C6 **H**

Let n be an even positive integer. Show that there is a permutation $(x_1, x_2 \dots x_n)$ of $(1, 2 \dots n)$ such that for every $1 \leq i \leq n$, the number x_{i+1} is one of the numbers $2x_i, 2x_i - 1, 2x_i - n, 2x_i - n - 1$. Hereby, we use the cyclic subscript convention, so that x_{n+1} means x_1.

Some experiments show that our graph has more than 2 incoming and outgoing degree in all vertexes expect the first and last vertexes. So our lemma won't work yet. To make use of our lemma we take a graph with half of the vertexes of our original graph and make each vertex v_{2k} represent two integers: $(2k-1, 2k)$. Simple argument shows that this graph has an Euler Circuit, and surprisingly this itself is sufficient, as we can follow this circuit to get every integers in the interval $[1, n]$.

Problem 2.11.53: USA TST 2017 P1 **E**

In a sports league, each team uses a set of at most t signature colors. A set S of teams is color-identifiable if one can assign each team in S one of their signature colors, such that no team in S is assigned any signature color of a different team in S.

For all positive integers n and t, determine the maximum integer $g(n,t)$ such that: In any sports league with exactly n distinct colors present over all teams, one can always find a color-identifiable set of size at least $g(n,t)$.

Idea First, guess the answer, then try taking the minimal set. □

Problem 2.11.54: Putnam 2017 A4 **E**

$2N$ students take a quiz in which the possible scores are $0, 1 \dots 10$. It is given that each of these scores appeared at least once, and the average of their scores is 7.4. Prove that the students can be divided into two sets of N student with both sets having an average score of 7.4.

Idea We take a set $S_1 = \{0, 1 \dots 10\}$. Basically we have to partition the set of $2N$ into two equal sets with equal sum. So we pair S , and other leftovers and see what happens. □

Problem 2.11.55: ISL 2005 C3 **MH**

Consider a $m \times n$ rectangular board consisting of mn unit squares. Two of its unit squares are called adjacent if they have a common edge, and a path is a sequence of unit squares in which any two consecutive squares are adjacent. Two paths are called non-intersecting if they don't share any common squares.

Each unit square of the rectangular board can be colored black or white. We speak of a coloring of the board if all its mn unit squares are colored.

Let N be the number of colorings of the board such that there exists at least one black path from the left edge of the board to its right edge. Let M be the number of colorings of the board for which there exist at least two non-intersecting black paths from the left edge of the board to its right edge.

Prove that $N^2 \geq M \times 2^{mn}$.

Idea Bijective relation problem, the condition has $\times$, means we find a combinatorial model for the R.H.S. which is a pair of boards satisfying conditions. We want to show a surjection from this model to the model on the L.H.S. □

Problem 2.11.56: Result by Erdos **MH**

Given two *different* sequence of integers $(a_1, a_2 \dots a_n), (b_1, b_2, \dots b_n)$ such that two $\frac{n(n-1)}{2}$-tuples

$$a_1 + a_2, a_1 + a_3 \dots a_{n-1}a_n \quad \text{and} \quad b_1 + b_2, b_1 + b_3 \dots b_{n-1}b_n$$

are equal upto permutation. Prove that $n = 2^k$ for some k.

Problem 2.11.57: A reformulation of Catalan's Numbers **MH**

Let $n \geq 3$ students all have different heights. In how many ways can they be arranged such that the heights of any three of them are not from left to right in the order: medium, tall, short?

Idea The proof uses derivatives to construct a polynomial similar to a **Maclaurin Series**. □

Problem 2.11.58: **E**

There are n cubic polynomials with three distinct real roots each. Call them $P_1(x), P_2(x), \dots, P_n(x)$. Furthermore for any two polynomials P_i, P_j, $P_i(x)P_j(x) = 0$ has exactly 5 distinct real roots. Let S be the set of roots of the equation

$$P_1(x)P_2(x)\dots P_n(x) = 0$$

. Prove that

1 If for each a, b there is exactly one $i \in \{1, \dots n\}$ such that $P_i(a) = P_i(b) = 0$, then $n = 7$.
2 If $n > 7$, $|S| = 2n + 1$.

Problem 2.11.59: Serbia TST 2017 P2 **E**

Initially a pair (x, y) is written on the board, such that exactly one of it's coordinates is odd. On such a pair we perform an operation to get pair $(\frac{x}{2}, y + \frac{x}{2})$ if $2|x$ and $(x + \frac{y}{2}, \frac{y}{2})$ if $2|y$. Prove that for every odd $n > 1$ there is a even positive integer $b < n$ such that starting from the pair (n, b) we will get the pair (b, n) after finitely many operations.

Idea Finding a construction through investigation and realizing that the infos and operations on x only defines the changes are enough for this problem. □

Problem 2.11.60: Serbia TST 2017 P4 **E**

We have an $n \times n$ square divided into unit squares. Each side of unit square is called unit segment. Some isosceles right triangles of hypotenuse 2 are put on the square so all their vertices's are also vertices's of unit squares. For which n it is possible that every unit segment belongs to exactly one triangle (unit segment belongs to a triangle even if it's on the border of the triangle)?

Idea Finding n is even, seeing 4 fails... □

Problem 2.11.61: China MO 2018 P2 **M**

Let n and k be positive integers and let

$$T = \{(x, y, z) \in \mathbb{N}^3 \mid 1 \le x, y, z \le n\}$$

be the length n lattice cube. Suppose that $3n^2 - 3n + 1 + k$ points of T are colored red such that if P and Q are red points and PQ is parallel to one of the coordinate axes, then the whole line segment PQ consists of only red points.

Prove that there exists at least k unit cubes of length 1, all of whose vertices's are colored red.

Idea The inductive solution is tedious, and since we have to count the number of "good" boxes, we can try double counting. Explicitly counting all the "good" boxes. □

Problem 2.11.62: China MO 2018 P5 **MH**

Let $n \geq 3$ be an odd number and suppose that each square in a $n \times n$ chessboard is colored either black or white. Two squares are considered adjacent if they are of the same color and share a common vertex and two squares a, b are considered connected if there exists a sequence of squares $c_1, \ldots, c_k$ with $c_1 = a, c_k = b$ such that c_i, c_{i+1} are adjacent for $i = 1, 2, \ldots, k-1$.

Find the maximal number M such that there exists a coloring admitting M pairwise disconnected squares.

Idea It's not hard to get the ans, now that the answer is guesses, and we have tried to prove with induction and couldn't find anything good, we try double counting. We notice that all the connected components in the $n \times n$ are planar graphs. Now we use Euler's theorem on Planar Graphs to find a value of M wrt to other values, and we double count the other values. □

Problem 2.11.63: USAMO 2006 P2 **E**

For a given positive integer k find, in terms of k, the minimum value of N for which there is a set of $2k+1$ distinct positive integers that has sum greater than N but every subset of size k has sum at most $\frac{N}{2}$.

Idea The best or simple looking set is the set of consecutive integers. So if there are some 'holes', we can fill them up to some extent, this opens two sub-cases. □

Problem 2.11.64: USAMO 2005 P1 **E**

Determine all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.

Problem 2.11.65: USAMO 2005 P5 **E**

A mathematical frog jumps along the number line. The frog starts at 1, and jumps according to the following rule: if the frog is at integer n, then it can jump either to $n+1$ or to $n+2^{m_n+1}$ where 2^{m_n} is the largest power of 2 that is a factor of n. Show that if $k \geq 2$ is a positive integer and i is a nonnegative integer, then the minimum number of jumps needed to reach $2^i k$ is greater than the minimum number of jumps needed to reach 2^i.

Idea The main idea is to notice that the operation only uses powers of 2. And it depends on only the power of 2 in the integers, and in the sequence of 2-powers, the operation is very nice. □

Problem 2.11.66: ISL 1991 P10 **E**

Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2, \ldots, k$ in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is equal to 1.

Problem 2.11.67: **E**

A robot has n modes, and programmed as such: in mode i the robot will go at a speed of $i\text{ms}^{-1}$ for i seconds. At the beginning of its journey, you have to give it a permutation of $\{1, 2, \ldots n\}$. What is the maximum distance you can make the robot go?

Problem 2.11.68: **E**

A slight variation of the previous problem, in this case, the problem goes at a speed of $(n-1)\text{ms}^{-1}$ for i seconds in mode i.

Problem 2.11.69: **E**

m people each ordered n books but because Ittihad was the mailman, he messed up. Everyone got n books but not necessarily the one they wanted you need to fix this. To go to a house from another house it takes one hour. You can carry one book with you during any trip (at most one). You know who has which books and all books are different (i,e, $n * m$ different books). Prove that you can always finish the job in $m * (n + \frac{1}{2})$ hours

Idea Thinking about the penultimate step, when we have to go to a house empty handed. Thinking in this way gives us a way to pair the houses up, and since pairing... □

Idea Another way to do this is to convert it to a multi-graph. Now go to a house and return with a book means removing two edges from that vertex. We play around with it for sometime □

Problem 2.11.70: **E**

There are n campers in a camp and they will try to solve a IMO P6 but everyone has a confidence threshold (they will solve the problem by group solving). For example Laxem has threshold 5. I.e. if he's in the group, the group needs to contain at least 5 people (him included). A group is 'confident' when everyone of the team is confident. Now MM wants to make a list of possible "perfect confident" groups. I.e. groups that are confident but adding anyone else will destroy the confidence. How long can his list be?

Problem 2.11.71: timus 1862 **ME**

Problem 2.11.72: ARO 2014 P9.7 **E**

In a country, mathematicians chose an $\alpha > 2$ and issued coins in denominations of 1 ruble, as well as α^k rubles for each positive integer k. α was chosen so that the value of each coins, except the smallest, was irrational. Is it possible that any natural number of rubles can be formed with at most 6 of each denomination of coins?

Problem 2.11.73: Saint Petersburg 2001 **MH**

The number n is written on a board. A and B take turns, each turn consisting of replacing the number n on the board with $n-1$ or $\lfloor \frac{n+1}{2} \rfloor$. The player who writes the number 1 wins. Who has the winning strategy?

Idea Recursively building the losing positions. □

Problem 2.11.74: ARO 2011 P11.6 **E**

There are more than n^2 stones on the table. Peter and Vasya play a game, Peter starts. Each turn, a player can take any prime number less than n stones, or any multiple of n stones, or 1 stone. Prove that Peter always can take the last stone (regardless of Vasya's strategy).

Problem 2.11.75: ARO 2007 P9.7 **E**

Two players by turns draw diagonals in a regular $(2n+1)$-gon $(n > 1)$. It is forbidden to draw a diagonal, which was already drawn, or intersects an odd number of already drawn diagonals. The player, who has no legal move, loses. Who has a winning strategy?

Idea Turning the diagonals as vertices, and connection being intersections, we get a graph to play the game on. We then count the degrees. □

Problem 2.11.76: **E**

After tiling a 6×6 box with dominoes, prove that a line parallel to the sides of the box can be drawn that this line doesn't cut any dominoes.

Idea Double count how many lines "cut" a domino, and domino number. □

Problem 2.11.77: **E**

There are 100 points on the plane. You have to cover them with discs, so that any two disks are at a distance of 1. Prove that you can do this in such a way that the total diameter of the disks is < 100.

Idea As the number 100 is very random, we suspect that is true for all values. So we can use induction □

Problem 2.11.78: ARO 2014 P10.8 **M**

Given are n pairwise intersecting convex k-gons on the plane. Any of them can be transferred to any other by a homothety with a positive coefficient. Prove that there is a point in a plane belonging to at least $1 + \frac{n-1}{2k}$ of these k-gons.

Idea The most natural such point should be a vertex of a polygon. And these kinda problems use PHP more often, so we will have to divide by k somewhere. Again to find the polygon to use the PHP we will have to divide by n also. So we want to have nk in the denominator. We change the term to achieve this and Ta-Da! we get a fine term to work with. □

Problem 2.11.79: IOI 2018 P1 **E**

Problem 2.11.80: USAMO 2005 P1 **E**

Determine all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.

Problem 2.11.81: USAMO 2005 P4 **E**

Legs L_1, L_2, L_3, L_4 of a square table each have length n, where n is a positive integer. For how many ordered 4-tuples (k_1, k_2, k_3, k_4) of nonnegative integers can we cut a piece of length k_i from the end of leg L_i $(i = 1, 2, 3, 4)$ and still have a stable table?
(The table is stable if it can be placed so that all four of the leg ends touch the floor. Note that a cut leg of length 0 is permitted.)

Problem 2.11.82: USAMO 2006 P2 **M**

For a given positive integer k find, in terms of k, the minimum value of N for which there is a set of $2k + 1$ distinct positive integers that has sum greater than N but every subset of size k has sum at most $\frac{N}{2}$.

Problem 2.11.83: USAMO 2006 P5 **M**

A mathematical frog jumps along the number line. The frog starts at 1, and jumps according to the following rule: if the frog is at integer n, then it can jump either to $n + 1$ or to $n + 2^{m_n+1}$ where 2^{m_n} is the largest power of 2 that is a factor of n. Show that if $k \geq 2$ is a positive integer and i is a nonnegative integer, then the minimum number of jumps needed to reach $2^i k$ is greater than the minimum number of jumps needed to reach 2^i.

Problem 2.11.84: USAMO 2009 P2 **EM**

Let n be a positive integer. Determine the size of the largest subset of $\{-n, -n+1, \ldots, n-1, n\}$ which does not contain three elements a, b, c (not necessarily distinct) satisfying $a + b + c = 0$.

Chapter 3

Algebra

3.1 Functional Equations

Can't Start? Try These

1 GUESS THE POSSIBLE SOLUTIONS.

2 SUBSTITUTION

a Try EVERY possible substitutions, and write them in a list, dont think during this time.

b Now think what these results give you.

c Find values of $f(0), f(1), f(2), f(-x)$ etc.

d Tweak the function a little bit, do substitution again.

e Assume some other functions according to the solutions, substitute them to make the fe easier to get info out of.

3 PROPERTIES OF THE FUNCTION

a Try proving INJECTIVITY, SURJECTIVITY etc.

b Look for Injectivity or Surjectivity of $f(x) - f(y)$.

4 Assume for the sake of contradiction that the value of the function is greater or smaller than the estimated value at some point.

5 Sometimes consider the difference of two values of f.

Stuck? Try These

1 Proving that $f(x) - x$ is injective might come handy in some cases.

2 If you're *NOT* able to make one side of the equation equal to 0, try to make it equal to any real or some particular real. (pco 169 P11)

3 Sometimes in integer functions, divisibility of the type $f(1)^{k-1} \mid f(x)^k$ helps.

4 Durr… I want things to cancel.

Lemma 3.1.1

Inverse Funtion

An inverse function $f^{-1}(x)$ is the reflection of the function $f(x)$ wrt to the line $y = x$.

3.1.1 Problems

Problem 3.1.1: USA TST 2018 P2 **H**

Find all functions $f : \mathbb{Z}^2 \to [0, 1]$ such that for any integers x and y,

$$f(x, y) = \frac{f(x - 1, y) + f(x, y - 1)}{2}.$$

Idea We know that the function has to be a constant function. So it is a intuitive idea considering the difference of two values of the function. Again as we wish to show that this difference is 0, we have to use either equality of limit. As equality is quite ambiguous in this problem, we approach with limits. We see that $f(x, y)$ can be written as a term depending on the values of the 3rd quarter of the plane with (x, y) as its origin. With infinite values in our hand, we try bounding. $\square$

Problem 3.1.2: EGMO 2012 P3 **E**

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(yf(x + y) + f(x)) = 4x + 2yf(x + y)$$

for all $x, y \in \mathbb{R}$.

Problem 3.1.3: pco 169 P11 **M**

Find all $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x, y the following holds:

$$f(x)^2 + 2yf(x) + f(y) = f(y + f(x))$$

Problem 3.1.4: IMO 1994 P5 **MH**

Let S be the set of all real numbers strictly greater than -1. Find all functions $f : S \to S$ satisfying the two conditions:

1. $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all x, y in S;
2. $\frac{f(x)}{x}$ is strictly increasing on each of the two intervals $-1 < x < 0$ and $0 < x$.

Problem 3.1.5: ISL 1994 A4 **H**

Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{R}^+$ the subset of all positive ones. Let α and β be given elements in $\mathbb{R}$, not necessarily distinct. Find all functions $f : \mathbb{R}^+ \mapsto \mathbb{R}$ such that:

$$f(x)f(y) = y^\alpha f\left(\frac{x}{2}\right) + x^\beta f\left(\frac{y}{2}\right) \forall x, y \in \mathbb{R}^+.$$

Problem 3.1.6: IMO 2017 P2 **H**

Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for any real numbers x and y,

$$f(f(x)f(y)) + f(x+y) = f(xy).$$

Problem 3.1.7: ISL 2008 A1 **E**

Find all functions $f : (0, \infty) \mapsto (0, \infty)$ (so f is a function from the positive real numbers) such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers w, x, y, z, satisfying $wx = yz$.

Problem 3.1.8: pco 169 P15 **EH**

Find all $a \in \mathbb{R}$ for which there exists a non-constant function $f : (0, 1] \to \mathbb{R}$ such that

$$a + f(x + y - xy) + f(x)f(y) \leq f(x) + f(y)$$

for all $x, y \in (0, 1]$

Problem 3.1.9: pco 168 P18 **E**

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

for all $x, y \in \mathbb{R}$

Problem 3.1.10: ISL 2011 A3 **M**

Determine all pairs (f, g) of functions from the set of real numbers to itself that satisfy

$$g(f(x+y)) = f(x) + (2x+y)g(y)$$

for all real numbers x and y.

Problem 3.1.11: ISL 2005 A2 **M**

We denote by $\mathbb{R}^+$ the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ which have the property:

$$f(x)f(y) = 2f(x + yf(x))$$

for all positive real numbers x and y.

Idea As the function has to be constant, we first show that the funtion is either greater or smaller than the estimated solution. Then we show contradiction when the function is not a constant function. This is actually a general idea which can be applied to any fe: suppose $x' = f(x)$ the estimated solution, then assume for the sake of contradiction that for some x, $f(x) < x'$ or $f(x) > x'$. □

Problem 3.1.12: ISL 2005 A4 **M**

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) + f(x)f(y) = f(xy) + 2xy + 1$ for all real numbers x and y.

Idea Substitution. □

Problem 3.1.13: Iran TST T2P1 **E**

Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the following conditions:

1. $x + f(y + f(x)) = y + f(x + f(y)) \quad \forall x, y \in \mathbb{R}$
2. The set $I = \left\{ \frac{f(x)-f(y)}{x-y} \mid x, y \in \mathbb{R}, x \neq y \right\}$ is an interval.

Problem 3.1.14: 169 P20 **E**

Let a be a real number and let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying: $f(0) = \frac{1}{2}$ and

$$f(x+y) = f(x)f(a-y) + f(y)f(a-x)$$

$\forall x, y \in \mathbb{R}$. Prove that f is constant

Problem 3.1.15: Vietnam 1991 **E**

Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which

$$\frac{1}{2}f(xy) + \frac{1}{2}f(xz) - f(x)f(yz) \geq \frac{1}{4}$$

Idea Just substitute. □

Problem 3.1.16: **M**

Suppose that f and g are two functions defined on the set of positive integers and taking positive integer values. Suppose also that the equations $f(g(n)) = f(n) + 1$ and $g(f(n)) = g(n) + 1$ hold for all positive integer n. Prove that $f(n) = g(n)$ for all positive integer n.

Idea Durrr... I want things to cancel... Hint: You want to show $f(n) - g(n) = 0$. □

Problem 3.1.17: ISL 2002 A1 **EM**

Find all functions f from the reals to the reals such that

$$f\left(f(x) + y\right) = 2x + f\left(f(y) - x\right)$$

for all real x, y.

Idea On one of our substitution, we see that there is surjectivity in the equation. So trying to show injectivity is the most intuitive move after that. Again, we have x on the outside, so we need to make x, a once and b once. but we have $f(y) - x$ which we need to eleminate, keeping y constant. We can make it either a or b since we already have $f(a) = f(b)$. And again we can take whatever value we want for $f(y)$. □

Problem 3.1.18: ISL 2001 A1 **EM**

Let T denote the set of all ordered triples (p, q, r) of nonnegative integers. Find all functions $f : T \to \mathbb{R}$ satisfying

$$f(p,q,r) = \begin{cases} 0 & \text{if } pqr = 0, \\ 1 + \frac{1}{6}(f(p+1,q-1,r) + f(p-1,q+1,r) & \\ +f(p-1,q,r+1) + f(p+1,q,r-1) & \\ +f(p,q+1,r-1) + f(p,q-1,r+1)) & \text{otherwise} \end{cases}$$

for all nonnegative integers p, q, r.

Idea First let us guess the ans. For all points on the 3 sides, our function gives 0. We get $f(1,1,1) = 1$. We get $f(1,1,2) = f(1,2,1) = f(2,1,1) = \frac{3}{2}$. We get $f(1,1,3) = \frac{9}{5}$. We get $f(1,2,2) = \frac{12}{5}$. Now, since for $pqr = 0$, we have $f = 0$, we need the expression pqr on the numerator. And we kinda guess that the denominator is $p+q+r$. From here the guess is obvious.
Now proving that this solution is the only solution. Let the solution be g. Define, $h := f - g$. Our aim is to prove that $h = 0$ for all inputs. □

Problem 3.1.19: RMM 2019 P5 **M**

Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x + yf(x)) + f(xy) = f(x) + f(2019y),$$

for all real numbers x and y.

Idea After getting $f(yf(0)) = f(y2019)$, one should think of proving that either f is constant, all zero except 0, or linear. How to do this? □

Problem 3.1.20: APMO 2015 P2 **E**

Let $S = \{2, 3, 4, \ldots\}$ denote the set of integers that are greater than or equal to 2. Does there exist a function $f : S \to S$ such that

$$f(a)f(b) = f(a^2b^2) \text{ for all } a, b \in S \text{ with } a \neq b?$$

Idea Try to break the symmetry, add another variable. □

Problem 3.1.21: ISL 2015 A2 **E**

Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$.

Idea It just flows. $\square$

Problem 3.1.22: ISL 2015 A4 **M**

Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$

for all real numbers x and y.

Idea When you don't know any heavy techniques, just plug in simple values into the function, and write down all of the equations in a list. $\square$

3.2 FE cantonmathguy Seclected Problems

1. Determine all functions $f : \mathbb{Q} \to \mathbb{Q}$ satisfying $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{Q}$.

2. Let $a_1, a_2, \dots$ be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer n the numbers $a_1, a_2, \dots, a_n$ leave n different remainders upon division by n. Prove that every integer occurs exactly once in the sequence $a_1, a_2, \dots$.

3. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying
$$f(x)f(y) = f(x + y) + xy$$
for all real x and y.

4. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a + b + c = 0$, the following equality holds:
$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

5. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying
$$f(x) + f(y) = f(x + y) \quad \text{and} \quad f(xy) = f(x)f(y)$$
for all $x, y \in \mathbb{R}$.

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$, the following equality holds
$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$
where $\lfloor a \rfloor$ is the greatest integer not greater than a.

7. ★ Let k be a real number. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying
$$|f(x) - f(y)| \le k(x - y)^2$$
for all real x and y.

8. Let $f : \mathbb{N} \to \mathbb{N}$ be a function, and suppose that positive integers k and c satisfy
$$f^k(n) = n + c$$
for all $n \in \mathbb{N}$, where f^k denotes f applied k times. Show that $k \mid c$.

9. Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying
$$f(f(f(n))) + f(f(n)) + f(n) = 3n$$
for every positive integer n.

10. Let S be the set of integers greater than 1. Find all functions $f : S \to S$ such that (i) $f(n) \mid n$ for all $n \in S$, (ii) $f(a) \ge f(b)$ for all $a, b \in S$ with $a \mid b$.

11. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all pairs of real numbers x and y.

12. ★ Let T denote the set of all ordered triples (p, q, r) of nonnegative integers. Find all functions $f : T \to \mathbb{R}$ satisfying

$$f(p,q,r) = \begin{cases} 0 & \text{if } pqr = 0, \\ 1 + \frac{1}{6}(f(p+1,q-1,r) + f(p-1,q+1,r) & \\ +f(p-1,q,r+1) + f(p+1,q,r-1) & \\ +f(p,q+1,r-1) + f(p,q-1,r+1)) & \text{otherwise} \end{cases}$$

for all nonnegative integers p, q, r.

13. Determine all strictly increasing functions $f : \mathbb{N} \to \mathbb{N}$ satisfying $nf(f(n)) = f(n)^2$ for all positive integers n.

14. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$.

15. Find all real-valued functions f defined on pairs of real numbers, having the following property: for all real numbers a, b, c, the median of $f(a, b), f(b, c), f(c, a)$ equals the median of a, b, c.

16. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that, for all positive integer n, we have $f(f(n)) < f(n + 1)$.

17. Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that, for any $w, x, y, z \in N$,

$$f(f(f(z)))f(wxf(yf(z))) = z^2 f(xf(y))f(w).$$

Show that $f(n!) \geq n!$ for every positive integer n.

18. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(n!) = f(n)!$ for all positive integers n and such that $m - n$ divides $f(m) - f(n)$ for all distinct positive integers m, n.

19. Find all functions f from the reals to the reals such that

$$(f(a) + f(b))(f(c) + f(d)) = f(ac + bd) + f(ad - bc)$$

for all real a, b, c, d.

20. Determine all functions f defined on the natural numbers that take values among the natural numbers for which

$$(f(n))^p \equiv n \pmod{f(p)}$$

for all $n \in \mathbb{N}$ and all prime numbers p.

21. Let $n \geq 4$ be an integer, and define $[n] = \{1, 2, \ldots, n\}$. Find all functions $W : [n]^2 \to \mathbb{R}$ such that for every partition $[n] = A \cup B \cup C$ into disjoint sets,
$$\sum_{a\in A}\sum_{b\in B}\sum_{c\in C} W(a,b)W(b,c) = |A||B||C|.$$

22. ★ Find all infinite sequences $a_1, a_2, \ldots$ of positive integers satisfying the following properties: (a) $a_1 < a_2 < a_3 < \cdots$, (b) there are no positive integers i, j, k, not necessarily distinct, such that $a_i + a_j = a_k$, (c) there are infinitely many k such that $a_k = 2k - 1$.

23. Show that there exists a bijective function $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that for all $m, n \in \mathbb{N}_0$,
$$f(3mn + m + n) = 4f(m)f(n) + f(m) + f(n)$$

24. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying
$$f(f(m) + n) + f(m) = f(n) + f(3m) + 2014$$
for all integers m and n.

25. Let $n \geq 3$ be a given positive integer. We wish to label each side and each diagonal of a regular n-gon $P_1 \ldots P_n$ with a positive integer less than or equal to r so that:

    a) every integer between 1 and r occurs as a label;
    b) in each triangle $P_iP_jP_k$ two of the labels are equal and greater than the third.

    Given these conditions:

    a) Determine the largest positive integer r for which this can be done.
    b) For that value of r, how many such labellings are there?

26. ★ Suppose that f and g are two functions defined on the set of positive integers and taking positive integer values. Suppose also that the equations $f(g(n)) = f(n) + 1$ and $g(f(n)) = g(n) + 1$ hold for all positive integer n. Prove that $f(n) = g(n)$ for all positive integer n.

27. Find all the functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ satisfying the relation

28. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation
$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$
for all real numbers x and y.

29. Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the sub-sequences
$$s_{s_1}, s_{s_2}, s_{s_3}, \ldots \qquad \text{and} \qquad s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \ldots$$
are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

30. Find all functions f from $\mathbb{N}_0$ to itself such that

$$f(m + f(n)) = f(f(m)) + f(n)$$

for all $m, n \in \mathbb{N}_0$.

31. ★ Consider a function $f : \mathbb{N} \to \mathbb{N}$. For any $m, n \in \mathbb{N}$ we write $f^n(m) = \underbrace{f(f(\ldots f}_{n}(m)\ldots))$.

Suppose that f has the following two properties:

a) if $m, n \in \mathbb{N}$, then $\frac{f^n(m)-m}{n} \in \mathbb{N}$;
b) The set $\mathbb{N} \setminus \{f(n) \mid n \in \mathbb{N}\}$ is finite.

Prove that the sequence $f(1) - 1, f(2) - 2, f(3) - 3, \ldots$ is periodic.

32. Let $\mathbb{N}$ be the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ that satisfy the equation

$$f^{abc-a}(abc) + f^{abc-b}(abc) + f^{abc-c}(abc) = a + b + c$$

for all $a, b, c \geq 2$.

33. Let $2\mathbb{Z} + 1$ denote the set of odd integers. Find all functions $f : \mathbb{Z} \to 2\mathbb{Z} + 1$ satisfying

$$f(x + f(x) + y) + f(x - f(x) - y) = f(x + y) + f(x - y)$$

for every $x, y \in \mathbb{Z}$.

3.3 Polynomials

3.3.1 Techniques to remember

Stuck? Try These

1 A polynomial with odd degree always has at least one real root.

2 If a polynomial with even degree has a negative value on its graph, then it has at least one real root.

3 Roots of unity divide a polynomial in parts like congruence classes.

4 MODULUS SIGN: use Triangle Inequality.

5 They say, In Poly, chase ROOTS.

6 $x^n f(\frac{1}{x})$ has the same coefficients as $f(x)$, but in opposite order.

Theorem 3.3.1: Lagrange Interpolation Theorem

Given n real numbers, there exist a polynomial with at most $n-1$ degree such that the graph of the polynomial goes through all of the points.

Theorem 3.3.2: Finite Differences

This is the descrete form of derivatives. The first finite difference of a function f is defined as $g(x) := f(x+1) - f(x)$.
$n+1$th finite difference of a n degree polynomial: For any polynomial $P(x)$ of degree at most n the following equation holds:

$$\sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} P(i) = 0$$

Remark: This can be used

1. to reduce the degree of a polynomial, manipulate the coefficients etc.
2. to solve recurrences, where the recurrence equation is a bit messy and contains a lot of previous values. Like this recurrence is quite messy to solve as it is, but if we take the first

finite difference here, it becomes easy:

$$f(x) = \frac{1}{3}f(x+1) + \frac{2}{3}f(x-1) + 1$$

It's like solving for the first derivative and then finding the original function.

□

3.3.2 General Problems

Problem 3.3.1: USA TST 2014 P4 **E**

Let n be a positive even integer, and let $c_1, c_2, \ldots, c_{n-1}$ be real numbers satisfying

$$\sum_{i=1}^{n-1} |c_i - 1| < 1.$$

Prove that

$$2x^n - c_{n-1}x^{n-1} + c_{n-2}x^{n-2} - \cdots - c_1x^1 + 2$$

has no real roots.

Idea A polynomial has no real root means that the polynomial completely lies in either of the two sides of the x-axis. So in this case, we have to prove that $P(x) > 0$. So we gotta try to bound. Again, to make $|c_i - 1|$ a little bit more approachable, we assign $b_i = c_i - 1$ and write $P(x)$ in terms of b_i. Now, how to bring the modulus sign in our polynomial? Oh, we have triangle ineq for those kinda work :0 □

Problem 3.3.2: USAMO 2002 P3 **H**

Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

Idea (i) If we have $n+1$ points, we have an unique polynomial through them.
(ii) If we have one positive value of a polynomial and one negative value, then there exists a real root between that two values. □

Problem 3.3.3: China TST 1995 P5 **M**

A and B play the following game with a polynomial of degree at least 4:

$$x^{2n} + \_\, x^{2n-1} + \_\, x^{2n-2} + \ldots + \_\, x + 1 = 0$$

A and B take turns to fill in one of the blanks with a real number until all the blanks are filled up. If the resulting polynomial has no real roots, A wins. Otherwise, B wins. If A begins, which player has a winning strategy?

Idea Not always (actually in very few cases) the first move decides the winning strategy. In this case, if B could make the last move, he would definitely win. But as he can't, consider the final two moves. Again "Waves". □

Problem 3.3.4: Zhao Polynomials **E**

A set of n numbers are considered to be k-cool if $a_1 + a_{k+1} \cdots = a_2 + a_{k+2} \cdots = \cdots = a_k + a_{2k} \ldots$. Suppose a set of 50 numbers are $3, 5, 7, 11, 13, 17$-cool. Prove that every element of that set is 0.

Idea Equivalence class :0 roots of unity :0 :0 :0 □

Problem 3.3.5: All Russian Olympiad 2016, Day2, Grade 11, P5 **E**

Let n be a positive integer and let $k_0, k_1, \ldots, k_{2n}$ be nonzero integers such that $k_0 + k_1 + \cdots + k_{2n} \neq 0$. Is it always possible to find a permutation $(a_0, a_1, \ldots, a_{2n})$ of $(k_0, k_1, \ldots, k_{2n})$ so that the equation

$$a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_0 = 0$$

has no integer roots?

Idea The degree is $2n$, and we have to find a zero, so proving/disproving the existence of negative value of $P(x)$ is enough. If all the values of $P(x)$ are to be positive, the leading coefficient must be very big... □

Problem 3.3.6: Zhao Poly **E**

Let $f(x)$ be a monic polynomial with degree n with distinct zeroes $x_1, x_2, ..., x_n$. Let $g(x)$ be any monic polynomial of degree $n-1$. Show that

$$\sum_{j=1}^{n} \frac{g(x_j)}{f'(x_j)} = 1$$

where $f'(x_i) = \prod_{j \neq i} (x_i - x_j)$

Idea Lagrange's Interpolation □

Problem 3.3.7: ARO 2018 P11.1 **E**

The polynomial $P(x)$ is such that the polynomials $P(P(x))$ and $P(P(P(x)))$ are strictly monotone on the whole real axis. Prove that $P(x)$ is also strictly monotone on the whole real axis.

Problem 3.3.8: Serbia 2018 P4 **E**

Prove that there exists a uniqe $P(x)$ polynomial with real coefficients such that

$$xy - x - y \mid (x+y)^{1000} - P(x) - P(y)$$

for all real x, y.

Idea Substitution. □

3.3.3 Root Hunting

Problem 3.3.9: Putnam 2017 A2 **E**

Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

3.3.4 NT Polynomials

Problem 3.3.10: Iran TST 2009 P4 **E**

Find all polynomials f with integer coefficient such that, for every prime p and natural numbers u and v with the condition:

$$p \mid uv - 1$$

we always have

$$p \mid f(u)f(v) - 1$$

Idea Notice that we can disregard v by considering it $\frac{1}{u}$, and the condition won't be affected, because primes allow multiplicative inverses. After this observation the problem is almost solved. $\square$

Problem 3.3.11: Iran TST 2004 P6 **E**

p is a polynomial with integer coefficients and for every natural n we have $p(n) > n$. x_k is a sequence that: $x_1 = 1, x_{i+1} = p(x_i)$ for every N one of x_i is divisible by N. Prove that $p(x) = x + 1$

Idea Notice that $\{x_i\}$ becomes periodic mod any prime. Now, we start by showing that $P(1) = 2$. We have, $P(x)$ has to be even. If it is > 2 then what happens? what if we take $N = P(1) - 1$? $\square$

Problem 3.3.12: ISL 2006 N4 **E**

Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\dots P(P(x))\dots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

Idea Suppose that there are more than n fixed points. So at least one of them can't be a fixed point of P. Use that. Follow. $\square$

3.3.5 Irreducibility

- Summer Camp 2015 Handout
- yufei Zhao's Handout

Theorem 3.3.3: Gauss's Lemma

Let $P(x) \in \mathbb{Z}$. P is irreducible over $\mathbb{Q}$ iff it is irreducible over $\mathbb{Z}$.

Stuck? Try These: What can be showed to prove Irreducibility

- Writing $f = g \cdot h$ and equating coefficients
- If the polynomial involves some prime, it's often useful to try factoring modulo that prime
- If the last coefficient is a prime, then there are some obvious bounds on the roots
- If there are bounds on the coefficients, then try root bounding

Lemma 3.3.4: Bounds On Roots

- P is a monic polynomial. Suppose $P(0) \neq 0$ and at most one complex root of P has absolute value at least 1. Then P is irreducible.

- P is a monic polynomial. Suppose that $|P(0)|$ is prime, and all complex roots of P have absolute value greater than 1. Then P is irreducible.

- Let $P(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$ such that

$$|b_n| > |b_{n-1}| + |b_{n-2}| + \cdots + |b_1| + |b_0|$$

Then every root α of P is strictly inside of the unit circle, i.e. $|\alpha| < 1$.

i.e. If the first coefficient of the polynomial is very large, then all of the roots lie inside the unit circle.

- Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a real polynomial. Such that,

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0$$

Then any complex z of $P(x)$ satisfies $|z| \leq 1$

i.e. If the coefficients form a decreasing sequence then all of the roots lie on or inside the unit circle.

- Perron's Criterion: Let $P(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \ldots a_1 x + a_0$ be a polynomial over integers such that

$$|a_{n-1}| > 1 + |a_{n-2}| + |a_{n-3}| + \cdots + |a_1| + |a_0|$$

Then $P(x)$ is irreducible.

-

Proof. Proof of Perron's Criterion: Can we bound the roots? We have two relations with bounded roots and irreducibility, we might be able to use them. In this case it seems that the first one is more usable.

First we show that there is no root $|\alpha| = 1$ such that $P(\alpha) = 0$.

Then we show that if one root has absolute value more than 1, then all other roots have absolute value less than 1. We do this by introducing $Q(x)$ such that,

$$P(x) = (x - \alpha)Q(x)$$

And then compare the coefficients.

Theorem 3.3.5: Perron's Criterion's Generalization (Dominating Term)

Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a complex polynomial, such that its a_k term is dominant, that is,

$$|a_k| > |a_0| + |a_1| + \ldots |a_{k-1}| + |a_{k+1}| + \cdots + |a_n| s$$

for some $0 \le k \le n$. Then exactly k roots of P lies strictly inside of the unit circle, and the other $n - k$ roots of P lies strictly outside of the unit circle.

Proof. Its proof uses Rouché's theorem.

Theorem 3.3.6: Rouché's Theorem

Let f, g be analytic functions on and inside a simple closed curve $\mathcal{C}$. Suppose that $|f(z)| > |g(z)|$ for all points z on $\mathcal{C}$. Then f and $f + g$ have the same number of zeroes (counting multiplicities) interior to $\mathcal{C}$

Problem 3.3.13: **H**

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} \cdots + a_1 x + a_0$ be a polynomial over integers. Where, a_0 is a prime, and

$$|a_0| > |a_n| + |a_{n-1}| + \cdots + |a_1|$$

Prove that $P(x)$ is irreducible.

Idea Bound the roots. □

Problem 3.3.14: ISL 2005 A1 **E**

Find all pairs of integers a, b for which there exists a polynomial $P(x) \in \mathbb{Z}[X]$ such that product $(x^2 + ax + b) \cdot P(x)$ is a polynomial of a form

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$$

where each of $c_0, c_1, \ldots, c_{n-1}$ is equal to 1 or -1.

Idea The idea of bounding the roots using the coefficients. □

Problem 3.3.15: **EM**

Let $P(x)$ be a polynomial with real coefficients, and $P(x) \geq 0$ for all $x \in \mathbb{R}$. Prove that there exists two polynomials $R, S \in \mathbb{Q}$ such that

$$P(x) = R(x)^2 + Q(x)^2$$

Problem 3.3.16: Romanian TST 2006 P2 **M**

Let p a prime number, $p \geq 5$. Find the number of polynomials of the form

$$x^p + px^k + px^l + 1, \quad k > l, \quad k, l \in \{1, 2, \cdots, p-1\},$$

which are irreducible in $\mathbb{Z}[X]$.

Idea Taking mod p, we have that $x^p + 1 \equiv (x+1)^p (\mod p)$. Now we can try equating terms or plug in some values to check for equality. □

Problem 3.3.17: Romanian TST 2003 P5 **M**

Let $f \in \mathbb{Z}[X]$ be an irreducible polynomial over the ring of integer polynomials, such that $|f(0)|$ is not a perfect square. Prove that if the leading coefficient of f is 1 (the coefficient of the term having the highest degree in f) then $f(X^2)$ is also irreducible in the ring of integer polynomials.

Idea If $f(x^2) = g(x)h(x)$, plugging $-x$ gives us $g(x)h(x) = g(-x)h(-x)$. So we should look at the common roots of $h(x)$ and $h(-x)$. And it is straightforward from here. □

Idea □

3.3.6 Fourier Transformation

Lemma 3.3.7: Dealing with binomial terms with a common factor

Let n, k be two integers, and let z be a kth root of unity other than 1. Then,

$$\binom{n}{0} + \binom{n}{k} + \binom{n}{2k} + \cdots = \frac{(1+z^1)^n + (1+z^2)^n + \cdots + (1+z^k)^n}{k}$$

Proof. For j not divisible by k,

$$z^j \left(\sum_{i=1}^{k} z^i \right) = 0$$

3.4 Inequalities

- Olympiad Inequalities - Thomas J. Mildorf
- $A < B$ - Keran Kedlaya
- Convexity - Po Shen Loh

3.4.1 Theorems

'Mean' Inequalities

Four different 'Mean'ies we have Given n "positive" numbers $a_1, a_2 \dots a_n$, we have

- Arithmetic Mean,
$$\text{AM} = \frac{\sum a_i}{n} = \frac{a_1 + a_2 + \cdots + a_n}{n}$$
- Geometric Mean,
$$\text{GM} = \sqrt[n]{\prod a_i} = \sqrt[n]{a_1 a_2 \dots a_n}$$
- Quadratic Mean or Root - Mean Square,
$$\text{QM} = \sqrt{\frac{\sum a_i^2}{n}} = \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}$$
- Harmonic Mean,
$$\text{HM} = \frac{n}{\sum \frac{1}{a_i}} = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

Theorem 3.4.1: Triangle Inequality

For any complex numbers $a_1, a_2 \dots a_n$ the following holds:

$$|a_1 + a_2 \cdots + a_k| \leq |a_1| + |a_2| + \cdots + |a_k|$$

§1 One Mean Ineq – QM-AM-GM-HM

Given n positive real numbers $x_1, x_2, \ldots x_n$, the following relation holds:

$$\boxed{\sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}} \geq \frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n} \geq \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Theorem 3.4.2: Weighted AM-GM

If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are nonnegative real numbers (the "weights") which sum to 1, then

$$\boxed{\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n \geq a_1^{\lambda_1} a_2^{\lambda_2} \ldots a_n^{\lambda_n}}$$

Equality holds if and only if $a_i = a_j$ for all integers i, j such that $\lambda_i \neq 0$ and $\lambda_j \neq 0$. We obtain the unweighted form of AM-GM by setting

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = \frac{1}{n}$$

Theorem 3.4.3: Cauchy-Schward Inequality

For any real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$,

$$\boxed{\left(a_1^2 + a_2^2 \ldots a_n^2\right)\left(b_1^2 + b_2^2 \ldots b_n^2\right) \geq (a_1 b_1 + a_2 b_2 \ldots a_n b_n)^2}$$

with equality when there exist constants μ, λ not both zero such that for all $1 \leq i \leq n$, $\mu a_i = \lambda b_i$.

The inequality sometimes appears in the following form.

Theorem 3.4.4: Cauchy-Schwarz Inequality Complex form

Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be complex numbers. Then

$$\left(|a_1^2| + |a_2^2| \ldots |a_n^2|\right)\left(|b_1^2| + |b_2^2| \ldots |b_n^2|\right) \geq |a_1 b_1 + a_2 b_2 \ldots a_n b_n|^2$$

Theorem 3.4.5: Titu's Lemma

For positive reals $a_1, a_2 \ldots a_n$ and $b_1, b_2 \ldots b_n$ the following holds:

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{(b_1 + b_2 + \cdots + b_n)}$$

Theorem 3.4.6: Holder's Inequality

If $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, \ldots, z_1, z_2, \ldots, z_n$ are nonnegative real numbers and $\lambda_a, \lambda_b, \ldots, \lambda_z$ are nonnegative reals with sum of 1, then

$$a_1^{\lambda_a} b_1^{\lambda_b} \ldots z_1^{\lambda_z} + \cdots + a_n^{\lambda_a} b_n^{\lambda_b} \ldots z_n^{\lambda_z} \leq (a_1 + \cdots + a_n)^{\lambda_a} (b_1 + \cdots + b_n)^{\lambda_b} \ldots (z_1 + \cdots + z_n)^{\lambda_z}$$

.

3.4.2 Tricks

Some tricks to try

1 Replace trigonometric functions by reals, and translate the problem

2 Smoothing, replace two variable while keeping something invariant, to make the inequality sharper.

3 Convexity, differentiate to check convexity, if the second derivative is positive on some interval, then the function is convex on that interval except probably at the end-points, and concave otherwise.

An example is $\ln \dfrac{1-x}{x}$. It is convex in $(0, \dfrac{1}{2}]$ and concave in $[\dfrac{1}{2}, 1)$

4 If there is product, and if the problem is 'ad-hoc'y, then apply $AM - GM$ and ln to see if there is something to play with.

Homogeneous Expression Expression $F(a_1, a_2 \dots a_n)$ is said to be homogeneous of degree k if and only if there exists real k such that for every $t > 0$ we have

$$t^k F(a_1, a_2 \dots a_n) = F(ta_1, ta_2 \dots ta_n)$$

If an expression is homogeneous, then the following can be assumed (one at a time):

$$\sum_{i=1}^{n} a_i = 1 \tag{3.1}$$

$$\prod_{i=1}^{n} a_i = 1 \tag{3.2}$$

$$a_1 = 1 \text{ or for some } i,\ a_i = 1 \tag{3.3}$$

$$\sum_{i=1}^{n} a_i^2 = 1 \tag{3.4}$$

$$\sum_{Cyc} a_i a_{i+1} = 1 \tag{3.5}$$

Substitutions

- For the condition $abc = 1$, set
$$a = \frac{x}{y},\ b = \frac{y}{z},\ c = \frac{z}{x}$$

- $$xyx = x + y + z + 2 \implies \frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1$$
implies the existence of a, b, c such that
$$x = \frac{b+c}{a},\ y = \frac{a+c}{b},\ z = \frac{b+a}{c}$$
.

- $2xyz + xy + yz + zx = 1$ is just the inverse of the previouss condition.

- $$x^2 + y^2 + z^2 = xyz + 4 \text{ and } |x|,\ |y|,\ |z| \geq 2$$
implies the existence of a, b, c such that
$$abc = 1 \text{ and } x = a + \frac{1}{a},\ y = b + \frac{1}{b},\ z = c + \frac{1}{c}$$
In fact even if only $\max(|x|, |y|, |z|) > 2$ is given, the result still holds.

Lemma 3.4.7

The following inequality holds for every positive integer n

$$2\sqrt{n+1} - 2\sqrt{n} < \sqrt{\frac{1}{n}} < 2\sqrt{n} - 2\sqrt{n-1}$$

Lemma 3.4.8

Given 4 posititve real numbers $a < b < c < d$. Call the score of a permutation a_1, a_2, a_3, a_4 of the four given reals be equal to the real

$$|\frac{a_1}{a_2} - \frac{a_3}{a_4}|$$

Prove that the minimum the score can get is equal to

$$|\frac{a}{c} - \frac{b}{d}|$$

3.4.3 Problems

Problem 3.4.1: APMO 1991 P3 **E**

Let $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$. Show that

$$\frac{a_1^2}{a_1+b_1} + \frac{a_2^2}{a_2+b_2} + \cdots + \frac{a_n^2}{a_n+b_n} \geq \frac{a_1+a_2+\cdots+a_n}{2}$$

Problem 3.4.2: ISL 2009 A2 **(Kori nai)**

Let a, b, c be positive real numbers such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = a + b + c$. Prove that:

$$\frac{1}{(2a+b+c)^2} + \frac{1}{(a+2b+c)^2} + \frac{1}{(a+b+2c)^2} \leq \frac{3}{16}.$$

Problem 3.4.3: ARO 2018 P11.2 **M**

Let $n \geq 2$ and $x_1, x_2, \ldots, x_n$ positive real numbers. Prove that

$$\frac{1+x_1^2}{1+x_1x_2} + \frac{1+x_2^2}{1+x_2x_3} + \ldots + \frac{1+x_n^2}{1+x_nx_1} \geq n$$

Idea The inequality says sum is greater, so if the product is greater, then we are done by AM-GM (**??**). □

Problem 3.4.4: Turkey TST 2017 P5 **M**

For all positive real numbers a, b, c with $a + b + c = 3$, show that

$$a^3b + b^3c + c^3a + 9 \geq 4(ab + bc + ca)$$

Idea Always try the most simple ineq possible, AM-GM □

Problem 3.4.5: IMO 2012 P2 **Varies**

Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1+a_2)^2(1+a_3)^3 \cdots (1+a_n)^n > n^n.$$

Idea The main idea is to look for the ans of the ques, $(1+a_k)^k >=?$. We have k^{th} power. So if we can get a k term sum inside of the brackets, we can get a clean term for ? from AM-GM. And 1 seems like it's crying to be partitioned. So we write the term as $\left(a_k + \frac{1}{k-1} + \cdots + \frac{1}{k-1}\right)$ □

Idea *Looks at the $a_2 a_3 \ldots a_n = 1$ condition*
Hey, we have a substitution for this one, why not try it out...
darn it, i still have to do the partition thing to cancel out the powers > (□

Problem 3.4.6: ISL 1998 A1 **can't judge now**

Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n < 1$. Prove that

$$\frac{a_1 a_2 \cdots a_n \left[1 - (a_1 + a_2 + \cdots + a_n)\right]}{(a_1 + a_2 + \cdots + a_n)(1-a_1)(1-a_2)\cdots(1-a_n)} \leq \frac{1}{n^{n+1}}.$$

Idea Simplifying and making it symmetric, we get to the inequality

$$\prod_{i=1} n\frac{1-a_i}{a_i} \geq n^{n+1}$$

Now approaching similarly as this problem, we get to the solution. □

Problem 3.4.7: ISL 2001 A1 **E**

Let a, b, c be positive real numbers so that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Idea Substitute. □

Problem 3.4.8: ISL 1999 A1 **EM**

Let $n \geq 2$ be a fixed integer. Find the least constant C such the inequality

$$\sum_{i<j} x_i x_j \left(x_i^2 + x_j^2\right) \leq C \left(\sum_i x_i\right)^4$$

holds for any $x_1, \ldots, x_n \geq 0$ (the sum on the left consists of $\binom{n}{2}$ summands). For this constant C, characterize the instances of equality.

Idea Follow the ineq sign and remember AM-GM. □

Problem 3.4.9: ISL 2017 A1 **E**

Let $a_1, a_2, \ldots a_n, k$, and M be positive integers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = k \quad \text{and} \quad a_1 a_2 \cdots a_n = M$$

If $M > 1$, prove that the polynomial

$$P(x) = M(x+1)^k - (x+a_1)(x+a_2)\cdots(x+a_n)$$

has no positive roots.

Idea The same idea used in this and this, spreading an expression to perform AM-GM on it. □

Problem 3.4.10: ISL 2016 A1 **M**

Let a, b, c be positive real numbers such that $\min(ab, bc, ca) \geq 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1.$$

Idea Try the simpler version with two variables first. Now you can use this discovery with a little bit of cleverness to solve the problem. The clever part is to notice that 4 variable ineq is more solvable than a 3 variable one. □

Problem 3.4.11: ISL 2016 A2 **M**

Find the smallest constant $C > 0$ for which the following statement holds: among any five positive real numbers a_1, a_2, a_3, a_4, a_5 (not necessarily distinct), one can always choose distinct subscripts i, j, k, l such that

$$\left| \frac{a_i}{a_j} - \frac{a_k}{a_l} \right| \leq C.$$

Idea Simplify the problem to get the ans first. Think about what is the smallest such value for any given 4 positive reals. □

Problem 3.4.12: ISL 2004 A1 **E**

Let $n \geq 3$ be an integer. Let t_1, t_2, ..., t_n be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n) \left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n} \right).$$

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

Idea Easy solution by induction. For a more elegant solution, write the right side as sum of paired factors. Finding when the inequality breaks and relating it to the end statement. □

Problem 3.4.13: ISL 1996 A2 **EM**

Let $a_1 \geq a_2 \geq \ldots \geq a_n$ be real numbers such that for all integers $k > 0$,

$$a_1^k + a_2^k + \ldots + a_n^k \geq 0.$$

Let $p = \max\{|a_1|, \ldots, |a_n|\}$. Prove that $p = a_1$ and that

$$(x - a_1) \cdot (x - a_2) \cdots (x - a_n) \leq x^n - a_1^n$$

for all $x > a_1$.

Idea After the first part, apply AM-GM on the whole left side, this not gonna work, since we can't bound $\sum a_i$ wrt a_1. So what if we divide both side by $(x - a_1)$ and then apply AM-GM? □

3.4.3.1 Smoothing And Convexity

Some usual tricks

1. Bring x, y closer, keeping $x + y$ constant.
2. If we need to smoothen up the value xy, then take ln on both side.
3. Work with different variables.

Theorem 3.4.9: Convexity

1. The function is convex in interval I iff for all $a, b \in I$ and for all $t < 1$,

$$tf(a) + (1-t)f(b) \geq f(ta + (1-t)b)$$

Which if put in words, means that the line segment joining $(a, f(a))$ and $(b, f(b))$ lies completely above the graph of the function.

2. The function is convex in interval I if f' is increasing in I or f'' is positive in I.

Theorem 3.4.10: Jensen's Inequality

Let $x_1, \ldots, x_n \in \mathbb{R}$ and let $\alpha_1, \ldots, \alpha_n \geq 0$ satisfy $\alpha_1 + \cdots + \alpha_n = 1$.

If f is a Convex Function, we have:

$$\alpha_1 f(x_1) + \alpha_2 f(x_2) \cdots + a_n f(x_n) \geq f(\alpha_1 x_1 + \alpha_2 x_2 \cdots + \alpha_n x_n)$$

If f is a Concave Function, we have:

$$\alpha_1 f(x_1) + \alpha_2 f(x_2) \cdots + a_n f(x_n) \leq f(\alpha_1 x_1 + \alpha_2 x_2 \cdots + \alpha_n x_n)$$

Theorem 3.4.11: Popoviciu's inequality

Let f be a convex function on and interval $I \in \mathbb{R}$. Then for any numbers $x, y, z \in I$,

$$f(x) + f(y) + f(z) \ + 3f\left(\frac{x+y+z}{3}\right) \ \geq \ 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{y+z}{2}\right) + 2f\left(\frac{z+x}{2}\right)$$

Problem 3.4.14: USAMO 1998 P3 **H**

Let $a_0, a_1, \cdots, a_n$ be numbers from the interval $(0, \pi/2)$ such that

$$\tan(a_0 - \frac{\pi}{4}) + \tan(a_1 - \frac{\pi}{4}) + \cdots + \tan(a_n - \frac{\pi}{4}) \geq n - 1.$$

Prove that

$$\tan a_0 \tan a_1 \cdots \tan a_n \geq n^{n+1}.$$

Idea Get rid of the tan's. AM-GM, Jensen doesn't works, so try **smoothing**. The conventional smoothing trick fails at one case, but works for all other cases. Means we have to deal with that case specially. □

Problem 3.4.15: USAMO 1974 P2 **E**

Prove that if a, b, and c are positive real numbers, then

$$a^a b^b c^c \geq (abc)^{(a+b+c)/3}.$$

Idea ln □

Problem 3.4.16: India 1995 **E**

Let $x_1, x_2, ..., x_n > 0$ be real numbers such that $x_1 + x_2 + x_3 + ... + x_n = 1$. Prove the inequality

$$\frac{x_1}{\sqrt{1 - x_1}} + \frac{x_2}{\sqrt{1 - x_2}} + ... + \frac{x_n}{\sqrt{1 - x_n}} \geq \sqrt{\frac{n}{n-1}}$$

Idea easy smoothing □

Problem 3.4.17: Vietnam 1998 **E**

$x_1, x_2 ... x_n$ are real numbers such that

$$\frac{1}{x_1 + 1998} + \cdots + \frac{1}{x_n + 1998} = \frac{1}{1998}$$

Prove that

$$\frac{\sqrt[n]{x_1 \ldots x_n}}{n-1} \geq 1998$$

Idea Translate the given expression in a nicer way with new variables... □

Problem 3.4.18: IMO 1974 P2 **M**

The variables a, b, c, d, traverse, independently from each other, the set of positive real values. What are the values which the expression

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

takes?

Idea $\dfrac{x}{y+c} \leq \dfrac{x}{y} \leq \dfrac{x}{y-c}$ □

Problem 3.4.19: Bulgaria 1995 **E**

Given n real number $x_1, x_2, ..., x_n \in [0, 1]$. Prove the following inequality

$$(x_1 + x_2 + \cdots + x_n) - (x_1x_2 + x_2x_3 + \cdots + x_nx_1) \leq \left\lfloor \frac{n}{2} \right\rfloor$$

3.5 Calculus

Lemma 3.5.1

$$(f(x) + g(x))' = f'(x) + g'(x)$$

$$(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

$$(\frac{f(x)}{g(x)})' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

$$(f(g(x)))' = f'(g(x)) + g'(x)$$

Lemma 3.5.2: The Derivative of an Odd function is always an Even function, and vice versa.

3.6 Ad-Hocs

Problem 3.6.1: USAMO 2013 P4 **E**

Find all real numbers $x, y, z \geq 1$ satisfying

$$\min(\sqrt{x+xyz}, \sqrt{y+xyz}, \sqrt{z+xyz}) = \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

The trick used here can be called Direct Action which is applicable for a lot of problems. You just start by replacing the variables by some "easier to deal with" variables, and do bashing.

Again, even if a polynomial/function cannot be factorized, it can be semi-factorized or can be written as a closed form. Always try to keep the equations as much earthly as possible.

Problem 3.6.2: ISL 2011 A2 **M**

Determine all sequences $(x_1, x_2, \ldots, x_{2011})$ of positive integers, such that for every positive integer n there exists an integer a with

$$\sum_{j=1}^{2011} jx_j^n = a^{n+1} + 1$$

In such problems where trying with smaller cases doesn't help, try playing around with the condition as much as you can. If an equation is given, try adding something, subtracting something, dividing by something or multiplying by something etc. Try tweaking the conditions a little bit. Valueable advises taught by this problem :D

Problem 3.6.3: ISL 2004 A2 **E**

Let a_0, a_1, a_2, ... be an infinite sequence of real numbers satisfying the equation $a_n = |a_{n+1} - a_{n+2}|$ for all $n \geq 0$, where a_0 and a_1 are two different positive reals.
Can this sequence a_0, a_1, $a_2 \ldots$ be bounded?

We can show by some sort of inductional process, that after some time, a_i increases by either a_0 or a_1. So the general term is $a_i = \alpha a_0 + \beta a_1$ for $\alpha, \beta \in \mathbb{N}$, where α and β increases without bound.

Problem 3.6.4: ISL 2014 A1 **E**

Let $z_0, z_1, z_2, \ldots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$z_n < \frac{z_0 + z_1 + \cdots + z_n}{n} \leq z_{n+1}$$

.

Problem 3.6.5: ISL 2014 A2 **E**

Define the function $f : (0, 1) \to (0, 1)$ by

$$f(x) = x^2, \text{ for } x \geq \frac{1}{2} \text{ and } x + \frac{1}{2}, \text{ for } x < \frac{1}{2}$$

Let a and b be two real numbers such that $0 < a < b < 1$. We define the sequences an and bn by $a_0 = a, b_0 = b$, and $a_n = f(a_{n-1}), b_n = f(b_{n-1})$ for $n > 0$. Show that there exists a positive integer n such that

$$(a_n - a_{n-1})(b_n - b_{n-1}) < 0$$

.

Problem 3.6.6: ARO 2018 P10.1 **E**

Determine the number of real roots of the equation

$$|x| + |x + 1| + \cdots + |x + 2018| = x^2 + 2018x - 2019$$

Problem 3.6.7: European Mathematics Cup 2018 P3 **E**

Find all $k > 1$ such that there exists a set S such that,

1. There exists $N > 0$ such that, if $x \in S, then x < N$.
2. If $a, b \in S$, and $a > b$, then $k(a - b) \in S$

Idea Find some constraints such as, $k(a - b) \not> a$, S has a smallest element. These two combined with a sequence of decreasing elements of S is enough to solve this problem. □

Problem 3.6.8: APMO 2018 P2 **EM**

Let $f(x)$ and $g(x)$ be given by

$$f(x) = \frac{1}{x} + \frac{1}{x-2} + \frac{1}{x-4} + \cdots + \frac{1}{x-2018}$$

$$g(x) = \frac{1}{x-1} + \frac{1}{x-3} + \frac{1}{x-5} + \cdots + \frac{1}{x-2017}$$

Prove that $|f(x) - g(x)| > 2$ for any non-integer real number x satisfying $0 < x < 2018$.

Idea Substract, manipulate, see that for $\epsilon < 1$, for $x = 2k + \epsilon$ it's true, if it's true for $x = 2k + 1 + \epsilon$. Then for $x = 2k + 1 + \epsilon$, sustitute the value to find a common term appearing in all of those equations. So if that term were to be greater than 2, we would be done. How do we test that? Take the first derivative to find the minima. ☐

Problem 3.6.9: ISL 2015 A1 **E**

Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \geq \frac{ka_k}{a_k^2 + (k-1)}$$

for every positive integer k. Prove that $a_1 + a_2 + \ldots + a_n \geq n$ for every $n \geq 2$.

Idea Simplify the inequality. And then sum it up. ☐

Problem 3.6.10: ISL 2015 A3 **M**

Let n be a fixed positive integer. Find the maximum possible value of

$$\sum_{1 \leq r < s \leq 2n} (s - r - n)x_r x_s,$$

where $-1 \leq x_i \leq 1$ for all $i = 1, \cdots, 2n$.

Idea The expression is weird, and beautiful. Now if we write the expression as a single variable function, we see that $x_i \in 1, -1$. Now, there is $x_i x_j$ in the expression. So we need to multiply two expressions. Again, see that $s - r - n$ can be rewritten as $-(n - s) - (r)$. Now, how do we get an expression like $x_i x_j$ which can be found in squares, with a coefficient $(n - s)$ and r? By summing it up r times, simple. ☐

3.7 Tricks and Lemmas

Theorem 3.7.1: Minkowski's theorem

Any convex set in $\mathbb{R}^n$, which is symmetric with respect to the origin and with volume greater than $2^n d(L)$ contains a non-zero lattice point.

3.7.1 Ad Hocs

1. $x^2 + 1 = (x+i)(x-i)$ [USAMO 2014 P1]
2. Add. Everything. Up.
3. Send SHIT to the infinity.

Chapter 4

Geometry

4.1 Orthocenter–Circumcircle–NinePoint Circle

Problem 4.1.1: Balkan MO 2017 P3 **S**

Consider an acute-angled triangle ABC with $AB < AC$ and let ω be its circumscribed circle. Let t_B and t_C be the tangents to the circle ω at points B and C, respectively, and let L be their intersection. The straight line passing through the point B and parallel to AC intersects t_C in point D. The straight line passing through the point C and parallel to AB intersects t_B in point E. The circumcircle of the triangle BDC intersects AC in T, where T is located between A and C. The circumcircle of the triangle BEC intersects the line AB (or its extension) in S, where B is located between S and A. Prove that ST, AL, and BC are concurrent.

Problem 4.1.2: USAMO 2014 P5 **E**

Let ABC be a triangle with orthocenter H and let P be the second intersection of $\odot AHC$ with the internal bisector of $\angle BAC$. Let X be the circumcenter of triangle APB and Y the orthocenter of triangle APC. Prove that the length of segment XY is equal to the circumradius of triangle ABC.

Idea No length conditions given, yet we need to prove that two lengths are equal. Parallelogram !!! □

Problem 4.1.3: **E**

Let DEF be the orthic triangle, and let $EF \cap BC = P$. Let the tangent at A to $\odot ABC$ meet BC at Q. Let T be the reflection of Q over P. Let K be the orthogonal projection of H on AM. Prove that $\angle OKT = 90$.

Problem 4.1.4: buratinogigle's proposed problems for Arab Saudi team 2015 **E**

Let ABC be a triangle with orthocenter H. P is a point. (K) is the circle with diameter AP. (K) cuts CA, AB again at E, F. PH cuts (K) again at G. Tangent line at E, F of (K) intersect at T. M is midpoint of BC. L is the point on MG such that $AL \parallel MT$. Prove that $LA \perp LH$.

Pics/SATST2015proposed_by_bura/derakynay1134-1.png

Figure 4.1

Problem 4.1.5: buratinogigle's proposed problems for Arab Saudi team 2015 E

Let ABC be a triangle inscribed circle (O). P lies on (O). The line passes through P and parallel to BC cuts CA at E. K is circumcenter of triangle PCE and L is nine point center of triangle PBC. Prove that the line passes through L and parallel to PK, always passes through a fixed point when P moves.

Pics/SATST2015proposed_by_bura/derakynay1134-2.png

Figure 4.2

Problem 4.1.6: buratinogigle's proposed problems for Arab Saudi team 2015 E

Let ABC be acute triangle inscribed circle (O), altitude AH, H lies on BC. P is a point that lies on bisector $\angle BAC$ and P is inside triangle ABC. Circle diameter AP cuts (O) again at G. L is projection of P on AH. Assume that GL bisects HP. Prove that P is incenter of ABC.

`Pics/SATST2015proposed_by_bura/derakynay1134-4.png`

Figure 4.3

Problem 4.1.7: buratinogigle's proposed problems for Arab Saudi team 2015 E

Let ABC be an acute triangle inscribed circle (O). M lies on small arc $\overline{BC}$. P lies on AM. Circle diameter MP cuts (O) again at N. MO cuts circle diameter MP again at Q. AN cuts circle diameter MP again at R. Prove that $\angle PRA = \angle PQA$.

Pics/SATST2015proposed_by_bura/derakynay1134-5.png

Figure 4.4

Problem 4.1.8: buratinogigle's proposed problems for Arab Saudi team 2015 E

Let ABC be right triangle with hypotenuse BC, bisector BE, E lies on CA. Assume that circumcircle of triangle BCE cuts segment AB again at F. K is projection of A on BC. L lies on segment AB such that $BL = BK$. Prove that $\frac{AL}{AF} = \sqrt{\frac{BK}{BC}}$.

Pics/SATST2015proposed_by_bura/derakynay1134-6.png

Figure 4.5

Problem 4.1.9: buratinogigle's proposed problems for Arab Saudi team 2015 E

Let ABC be acute triangle inscribed circle (O). AD is diameter of (O). M, N lie on BC such that $OM \parallel AB$, $ON \parallel AC$. DM, DN cut (O) again at P, Q. Prove that $BC = DP = DQ$.

Pics/SATST2015proposed_by_bura/derakynay1134-7.png

Figure 4.6

Problem 4.1.10: **E**

Let $\triangle ABC$ be a triangle. F, G be arbitrary points on AB, AC. Take D, E midpoint of BF, CG. Show that the center of nine-point circle of $\triangle ABC$, $\triangle ADE$, $\triangle AFG$ are collinear.

Problem 4.1.11: IGO 2017 Advance P3

Let O be the circumcenter of $\triangle ABC$. Line CO intersects the altitude through A at point K. Let P, M be the midpoints of AK, AC respectively. If PO intersects BC at Y , and the circumcircle of $\triangle BCM$ meets AB at X, prove that $BXOY$ is cyclic

Idea There is no easily measurable angles, in this case use projective geometry. And since we still don't have any easy angles, we look for the second way of concyclicity, POP □

Problem 4.1.12: Turkey TST 2018 P4 **E**

In a non-isosceles acute triangle ABC, D is the midpoint of BC. The points E and F lie on AC and AB, respectively, and the circumcircles of CDE and AEF intersect in P on AD. The angle bisector from P in triangle EFP intersects EF in Q. Prove that the tangent line to the circumcircle of AQP at A is perpendicular to BC.

Idea Inverting around A. □

Problem 4.1.13: IRAN 3rd Round 2016 P1 **E**

Let ABC be an arbitrary triangle, P is the intersection point of the altitude from C and the tangent line from A to the circumcircle. The bisector of angle A intersects BC at D. PD intersects AB at K, if H is the orthocenter then prove : $HK \perp AD$

Idea Finding a set of Collinear points. □

Problem 4.1.14: IGO 2017 Advance P4

Three circles W_1, W_2 and W_3 touches a line l at A, B, C respectively (B lies between A and C). W_2 touches W_1 and W_3. Let l_2 be the other common external tangent of W_1 and W_3. l_2 cuts W_2 at X, Y. Perpendicular to l at B intersects W_2 again at K. Prove that KX and KY are tangent to the circle with diameter AC.

Idea Finding a Orthocenter Figure in these circle simplifies the problem a lot. □

Problem 4.1.15: 2017 IGO Advanced P2 **E**

We have six pairwise non-intersecting circles that the radius of each is at least one (no circle lies in the interior of any other circle). Prove that the radius of any circle intersecting all the six circles, is at least one.

Problem 4.1.16: ARO 2018 P10.2 **E**

Let $\triangle ABC$ be an acute-angled triangle with $AB < AC$. Let M and N be the midpoints of AB and AC, respectively; let AD be an altitude in this triangle. A point K is chosen on the segment MN so that $BK = CK$. The ray KD meets the circumcircle Ω of ABC at Q. Prove that C, N, K, Q are concyclic.

Problem 4.1.17: ARO 2014 P9.4 **E**

Let M be the midpoint of the side AC of acute-angled triangle ABC with $AB > BC$. Let Ω be the circumcircle of ABC. The tangents to Ω at the points A and C meet at P, and BP and AC intersect at S. Let AD be the altitude of the triangle ABP and ω the circumcircle of the triangle CSD. Suppose ω and Ω intersect at $K \neq C$. Prove that $\angle CKM = 90°$.

Problem 4.1.18: APMO 1999 P3 **E**

Let Γ_1 and Γ_2 be two circles intersecting at P and Q. The common tangent, closer to P, of Γ_1 and Γ_2 touches Γ_1 at A and Γ_2 at B. The tangent of Γ_1 at P meets Γ_2 at C, which is different from P, and the extension of AP meets BC at R. Prove that the circumcircle of triangle PQR is tangent to BP and BR.

Problem 4.1.19: Simurgh 2019 P2 **E**

Let ABC be an isosceles triangle, $AB = AC$. Suppoe that Q is a point such that $AQ = AB$, $AQ||BC$. Let P be the foot of perpendicular line from Q to BC. Prove that the circle with diameter PQ is tangent to the circumcircle of ABC.

Problem 4.1.20: European Mathematics Cup 2018 P2 **E**

Later

Problem 4.1.21: RMM 2019 P2 **E**

Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. Let E be the midpoint of AC. Denote by ω and Ω the circumcircles of the triangles ABE and CDE, respectively. Let P be the crossing point of the tangent to ω at A with the tangent to Ω at D. Prove that PE is tangent to Ω.

Problem 4.1.22: IGO 2018 A5 **E**

$ABCD$ is a cyclic quadrilateral. A circle passing through A, B is tangent to segment CD at point E. Another circle passing through C, D is tangent to AB at point F. Point G is the intersection point of AE, DF, and point H is the intersection point of BE, CF. Prove that the incenters of triangles AGF, BHF, CHE, DGE lie on a circle.

Pics/IGO2018A5.pdf

Figure 4.7

Problem 4.1.23: ISL 2011 G8 **EM**

Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.

Idea Find the translated triangle circumscribed in $\odot ABC$. Once you find the properties of this triangle and the relations between this and the common touch point, the problem becomes obvious. $\square$

Problem 4.1.24: ELMO 2019 P3 **EM**

Let ABC be a triangle such that $\angle CAB > \angle ABC$, and let I be its incentre. Let D be the point on segment BC such that $\angle CAD = \angle ABC$. Let ω be the circle tangent to AC at A and passing through I. Let X be the second point of intersection of ω and the circumcircle of ABC. Prove that the angle bisectors of $\angle DAB$ and $\angle CXB$ intersect at a point on line BC.

Idea Keep finding new things in the diagram... $\square$

Problem 4.1.25: ISL 2014 G5 **M**

Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90°$. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and

$$\angle CHS - \angle CSB = 90^\circ, \quad \angle THC - \angle DTC = 90^\circ.$$

Prove that line BD is tangent to the circumcircle of triangle TSH.

Idea First construct using nice circles, then prove the center is on AH using angle bisector theorem.

Pics/ISL2014G5.pdf

Figure 4.8: Construction

Pics/ISL2014G5_lemma1.pdf

Figure 4.9: Lemma

□

Problem 4.1.26: ISL 2014 G7 **M**

Let ABC be a triangle with circumcircle Ω and incentre I. Let the line passing through I and perpendicular to CI intersect the segment BC and the arc BC (not containing A) of Ω at points U and V, respectively. Let the line passing through U and parallel to AI intersect AV at X, and let the line passing through V and parallel to AI intersect AB at Y. Let W and Z be the midpoints of AX and BC, respectively. Prove that if the points I, X, and Y are collinear, then the points I, W, and Z are also collinear.

Idea Draw a nice diagram, and use the parallel property to find circles.

Pics/ISL2014G7.pdf

Figure 4.10: ISL 2014 G7

□

Problem 4.1.27: ISL 2015 G6 **E**

Let ABC be an acute triangle with $AB > AC$. Let Γ be its cirumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = 90°$ and let K be the point on Γ such that $\angle HKQ = 90°$. Assume that the points A, B, C, K and Q are all different and lie on Γ in this order.

Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

Idea Draw the tangent line, and find angles.

`Pics/ISL2015G6.pdf`

Figure 4.11: ISL 2015 G6

□

Problem 4.1.28: ISL 2015 G5 **M**

Let ABC be a triangle with $CA \neq CB$. Let D, F, and G be the midpoints of the sides AB, AC, and BC respectively. A circle Γ passing through C and tangent to AB at D meets the segments AF and BG at H and I, respectively. The points H' and I' are symmetric to H and I about F and G, respectively. The line $H'I'$ meets CD and FG at Q and M, respectively. The line CM meets Γ again at P. Prove that $CQ = QP$.

Idea Don't depend on the figure too much, find facts using facts, not figure.

`Pics/ISL2015G5.pdf`

Figure 4.12: ISL 2015 G5

$\square$

4.1.1 The line parallel to BC

Let the line parallel to BC through O meet AB, AC at D, E. Let K be the midpoint of AH, M be the midpoint of BC. F be the feet of A-altitude on BC and let H' be the reflection of H on F. Let O' be the circumcenter of KBC.

Lemma 4.1.1

$\angle DKC = \angle EKB = 90^\circ$

Pics/linethroughO_1.png

Figure 4.13

Lemma 4.1.2

CD, BE, OH', AM, KO' are concurrent. (by lemma)

Pics/linethroughO_2.png

Figure 4.14

Problem 4.1.29: InfinityDots MO Problem 3 **EM**

Let $\triangle ABC$ be an acute triangle with circumcenter O and orthocenter H. The line through O parallel to BC intersect AB at D and AC at E. X is the midpoint of AH. Prove that the circumcircles of $\triangle BDX$ and $\triangle CEX$ intersect again at a point on line AO.

Idea Just using lemma to get another pair of circle where we can apply radical axis arguments. $\square$

Idea Noticing that the resulting point is the isogonal conjugate of a well defined point, □

Lemma 4.1.3

Let P, Q be on AB, AC resp. such that $PQ \parallel BC$. And let A' be such that $A' \in \odot ABC, AA' \parallel BC$. Let $CP \cap BQ = X$, and let the perpendicular bisector of BC meet PQ at Y. Prove that A', X, Y are collinear.

Idea No angles... Do Lengths... □

Pics/concurrency_in_prallel_lines_with_BC.png

Figure 4.15

Problem 4.1.30: ARO 2018 P11.4 **M**

$P \in AB$, $Q \in AC$, $PQ \parallel BC$, $BQ \cap CP = X$. A' is the reflection of A wrt BC. $A'X \cap \odot APQ = Y$. Prove that $\odot BYC$ is tangent to $\odot APQ$.

Idea Of co it can be solved using angle chase, lemma makes it almost trivial. □

Problem 4.1.31: buratinogigle **EM**

Let (O) be a circle and E, F are two points inside (O). $(K), (L)$ are two circles passing though E, F and tangent internally to (O) at A, D, respectively. AE, AF cut (O) again at B, C, respectively. BF cuts CE at G. Prove that reflection of A though EF lies on line DG.

Rephrasing the problem as such: In the setup of this lemma, let $A'X \cap \odot ABC = Z$, then $\odot PQZ$ is tangent to $\odot ABC$.

Idea Simple angle chase. □

Idea Another solution to this is by taking D as a phantom point. □

Idea Another solution is with cross ratios □

4.1.2 Simson Line and Stuffs

Lemma 4.1.4: Simson Line Parallel

Let P be a point on the circumcircle, let P' be the reflection of P on BC and let $PP' \cap \Omega = D$, and let l_p be the Simson line of P. Prove that $l_p \parallel AD \parallel HP'$.

Pics/SimsonLineLemma1.png

Figure 4.16: The dotted lines are parallel

Lemma 4.1.5: Simson Line Angle

Given triangle ABC and its circumcircle (O). Let E, F be two arbitrary points on (O). Then the angle between the Simson lines of two points E and F is half the measure of the arc EF.

Pics/SimsonLineLemma2.png

Figure 4.17

4.1.3 Euler Line

Theorem 4.1.6: Perspectivity Line with Orthic triangle is perpendicular to Euler line

Let DEF be the orthic triangle. Then $BC \cap EF, CA \cap FD, AB \cap ED$ are collinear, and the line is perpendicular to the Euler line. In in fact this line is the radical axis of the Circumcircle and the NinePoint circle

Lemma 4.1.7

DEF is orthic triangle of ABC, XYZ is the orthic triangle of DEF. Prove that the perspective point of ABC and XYZ lies on the Euler line of ABC

Idea Thinking the stuff wrt to the incircle and using cross ratio. □

4.2 Cevian and Circumcevian Triangles

4.2.1 Circumcevian Triangle

Theorem 4.2.1: Hagge's circles

Let P be a point on the plane of $\triangle ABC$, let Ω be the circumcircle. Let A_1, B_1, C_1 be the intersections of AP, BP, CP with Ω for the second time. Let A_2, B_2, C_2 be the reflections of A_1, B_1, C_1 wrt BC, CA, AB. Prove that H, A_2, B_2, C_2 lie on a circle. This circle is called the **P-Hagge's Circle**.

Idea Either using the dual of Hagge's Circle, or using the reflection points of A, B, C wrt the isogonal conjugate of P. And using Lemma 1.1 to finish. □

Pics/P-HaggeCircle.png

Figure 4.18: P-Hagge Circle

Corollary 4.2.1.1

$\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2$.

Idea Straightforward use of Lemma 1.2. □

Corollary 4.2.1.2

If AH, BH, CH meet $\odot A_2B_2C_2H$ at A_3, B_3, C_3, then A_2A_3, B_2B_3, C_2B_3 meet at P.

Idea Simple angle chase and similarity transformation. □

Corollary 4.2.1.3

If I is the incenter of $\odot A_2B_2C_2$, K is the reflection of H over I, AK, BK, CK meet $\odot A_2B_2C_2$ at A_4, B_4, C_4, then A_4A_3, B_4B_3, C_4B_3 are concurrent.

Idea Simple angle chasing and trig-ceva. □

Problem 4.2.1: China TST D2P2, Dual of the Hagge's Circle theorem **M**

Let ω be the circumcircle of $\triangle ABC$. P is an interior point of $\triangle ABC$. A_1, B_1, C_1 are the intersections of AP, BP, CP respectively and A_2, B_2, C_2 are the symmetrical points of A_1, B_1, C_1 with respect to the midpoints of side BC, CA, AB. Show that the circumcircle of $\triangle A_2B_2C_2$ passes through the orthocenter of $\triangle ABC$. Further proof that if this circle's center is O_1, then $HOPO_1$ is a parallelogram.

Idea Construct Parallelograms. You have to prove two angles are equal. Reflection the smaller trig wrt one of the midpoints. □

Problem 4.2.2: China TST 2011, Quiz 2, D2, P1 **E**

Let AA', BB', CC' be three diameters of the circumcircle of an acute triangle ABC. Let P be an arbitrary point in the interior of $\triangle ABC$, and let D, E, F be the orthogonal projection of P on BC, CA, AB, respectively. Let X be the point such that D is the midpoint of $A'X$, let Y be the point such that E is the midpoint of $B'Y$, and similarly let Z be the point such that F is the midpoint of $C'Z$. Prove that triangle XYZ is similar to triangle ABC.

Idea A straightforward application of Lemma 6.1 using the O-Hagge's Circle. □

4.2.2 Cevian Triangle

Lemma 4.2.2: Isogonal Conjugate Lemma

Let a circle ω meet the sides of triangle ABC at $A_1, A_2;\ B_1, B_2;\ C_1, C_2$. Let P_1, P_2 be the miquel points of ABC wrt $A_1B_1C_1, A_2B_2C_2$ resp. Then P_1, P_2 are isogonal conjugates.

`Pics/IsoConjuCirclesLemma1.png`

Figure 4.19: The two round points are isogonal conjugates.

Theorem 4.2.3: Terquem's Cevian Theorem

Let a circle ω meet the sides of triangle ABC at $A_1, A_2;\ B_1, B_2;\ C_1, C_2$. If AA_1, BB_1, CC_1 are concurrent, then so are AA_2, BB_2, CC_2

.

Theorem 4.2.4: Mannheim's Theorem

Let ABC be a triangle, and let L, M, N be points on BC, CA, AB respectively. Let A', B', C' be points on $(AMN), (BNL), (CLM)$, and denote $K \equiv AA' \cap BB'$. Then if $K \in CC'$, A', B', C', K are concyclic.

Pics/MannheimTheorem.png

Figure 4.20: Mannheim's Theorem

Theorem 4.2.5: Mannheim's Theorem's Converse

Let ABC be a triangle, and let L, M, N be points on BC, CA, AB respectively. Let A', B', C' be points on $(AMN), (BNL), (CLM)$, and denote $K \equiv AA' \cap BB'$. Then if A', B', C', K are concyclic, $C' \in CK$.

Theorem 4.2.6: Brocard Points

Brocard Points are points inside a triangle such that

$$\angle PAB = \angle PBC = \angle PCA = \omega$$

and

$$\angle QCB = \angle QBA = \angle QAC = \omega.$$

Pics/BrocardPoints.png

Figure 4.21: Brocard Points

Problem 4.2.3: Rioplatense Olympiad 2013 Problem 6 **E**

Let ABC be an acute-angled scalene triangle, with centroid G and orthocenter H. The circle with diameter AH cuts the circumcircle of BHC at A', distinct from H. Analogously define B', C'. Prove that A', B', C', G are concyclic.

Problem 4.2.4: Iran 3rd Round Training 2016 **E**

ABC is an acute triangle and H, O are its orthocenter and circumcenter respectively. If AO, BO, CO intersect BH, CH, AH at X, Y, Z respectively,then prove that H, X, Y, Z lie on a circle

Idea Using Brocard Point □

Idea Using Mannheim's Theorem □

Theorem 4.2.7: Jacobi's Theorem

Suppose that D, E, F are points such that AE, AF are isogonal wrt $\angle BAC$. Similarly with D, E, F. Then AD, BE, CF are concurrent.

4.3 Incenter–Excenter Lemma stuff

Let $\triangle ABC$ be an ordinary triangle, I is its incenter, D, E, F are the touch points of the incenter with BC, CA, AB and D', E', F' are the reflections of D, E, F wrt I.

Let the I_a, I_b, I_c excircles touch BC, CA, AB at D_1, E_1, F_1.

Let M_a, M_b, M_c be the midpoints of the smaller arcs BC, CA, AB, and M_A, M_B, M_C be the midpoints of the major arcs BC, CA, AB. M are the midpoint of BC.

Let (I_a) touch BC, CA, AB at D_A, E_A, F_A. So, $D_A \equiv D_1$.

Let A' be the antipode of A wrt $\odot ABC$.

Call EF, 'A-tangent line', and DE, DF similarly. And call $E_A F_A$ 'A_A-tangent line.

Lemma 4.3.1

Let $AD \cap \odot(I) = G, AD' \cap \odot(I) = H$. Let the line through D' parallel to BC meet AB, AC at B', C'. Then AM, EF, GH, DD', BC', CB' are concurrent.

Pics/concurrent_lines_in_incenter.png

Figure 4.22: The lines are concurrent.

Lemma 4.3.2

AI, B, B_A-tangent lines and C-mid-line are concurrent. And, if the concurrency point is X, then $CS \perp AI$

Pics/InExLemma4.png

Figure 4.23: AI, B, I_A-tangent lines and C-mid-line are concurrent.

Corollary 4.3.2.1

Let triangle ABC, incircle (I), the A - excircle (I_a) touches BC at M. IM intersects (I_a) at the second point X. Similarly, we get Y, Z. Prove that AX, BY, CZ are concurrent.

Corollary 4.3.2.2

Extension, by buratinogigle: Triangle ABC and XYZ are homothetic with center I is incenter of ABC. Excircles touches BC, CA, AB at D, E, F. XD, YE, FZ meets excircles again at U, V, W. Prove that AU, BV, CW are concurrent.

Theorem 4.3.3: Paul Yui Theorem

B-tangent line, C_A-tangent line, and AH are concurrent.

Lemma 4.3.4

Let ω_a be the circle that goes through B, C and is tangent to (I) at X. Then XD', EF, BC are concurrent and X, D, I_a are collinear. The same properties is found if the roles of incenter and excenter are swapped.

`Pics/InExLemma3.png`

Figure 4.24: Circle through BC tangent to incircle

Idea Pole-Polar. ISL 2002 G7 □

Lemma 4.3.5

$A'I, \odot ABC, \odot AEIF$ are concurrent at Y_A. And Y_A, D, M_a are collinear.

Lemma 4.3.6

$DD_H \perp EF$, then D_H, I, A' are collinear.

Lemma 4.3.7

Let X be any point on BC, and let I_1, I_2 be incenters of $\triangle ABX, \triangle ACX$. Then $\square XI_1I_2D$ are cyclic. And the other common tangent of $\odot I_1$ and $\odot I_2$ goes through D.

Lemma 4.3.8

$$M_AE_1 = M_AF_1$$
$$M_BF_1 = M_BD_1$$
$$M_CD_1 = M_CE_1$$

Pics/excenter_touchpoint_bigarc-midpoints.png

Figure 4.25: Excenter Toucpoints are equidistance from the Bigger Arc-midpoint

Lemma 4.3.9: Incircle Touchpoint and Cevian

Let a cevian be AX and let I_1, I_2 be the incirlces of $\triangle ABX, \triangle ACX$. Then D, I_1, I_2, X are concyclic.

`Pics/incircle_touchpoint_and_cevian.pdf`

Figure 4.26

`Isodynamic Points` Let ABC be a triangle, and let the angle bisectors of $\angle A$ meet BC at X, Y. Call ω_a the circumcircle of $\triangle AXY$. Define ω_b, ω_c similarly. The first and second isodynamic points are the points where the three circles $\omega_a, \omega_b, \omega_c$ meet. These two points satisfy the following relations:

1. $$PA \sin A = PB \sin B = PC \sin C$$

2. They are the isogonal congugates of the Fermat Points, and they lie on the 'Brocard Axis'

Problem 4.3.1: **E**

Prove that the pedal triangles of the isodynamic points are equilateral triangles.

Problem 4.3.2: Vietnamese TST 2018 P6.a **M**

Triangle ABC circumscribed (O) has A-excircle (I_A) that touches AB, BC, AC at F_A, D_1, E_A, resp. M is the midpoint of BC. Circle with diameter MI_A cuts D_1E_A, D_1F_A at K, H. Prove that (BD_1K), (CD_1H) have an intersecting point on (I_A).

Idea Inverting around point D, and using harmonic bundle and harmonic pencils. □

Problem 4.3.3: Vietnamese TST 2018 P6.a Modified Version **M**

Let G, H, B', C' be defined the same way in Lemma 3.2. Prove that F lies on the radical axis of $\odot D'GI, D'C'H$. By extension prove that B lies on the radical axis of $\odot D'B'I, D'C'H$

Pics/c6t45786f6h1618685_a_1.png

Figure 4.27: Vietnamese TST 2018 P6.a

Idea With a little bit of work we come to the point where we have to show that

$$\frac{FG}{GD} \times \cot\frac{C}{2} = \frac{HF}{HE}$$

We can do this by applying Ptolemy's theorem on the two harmonic quads $GFDE, D'FHE$. □

Generalization 4.3.3.1: Vietnamese TST 2018 P6.a Generalization

Let ABC be a triangle. The points D, E, F are on the lines BC, CA, AB respectively. The circles (AEF), (CFD), (CDE) have a common point M. A circle (K) passes through P, D meet DE, DF again at Q, R respectively. Prove that the circles (DBQ), (DCR) and (DEF) are coaxial.

Idea Inversion around point D.

`Pics/VNTST2018P6a_Gene.pdf`

Figure 4.28: Vietnamese TST 2018 P6.a Generalization

□

Theorem 4.3.10: Poncelet's Porism

Poncelet's porism (sometimes referred to as Poncelet's closure theorem) states that whenever a polygon is inscribed in one conic section and circumscribes another one, the polygon must be part of an infinite family of polygons that are all inscribed in and circumscribe the same two conics.

Problem 4.3.4: IMO 2013 P3 **M**

Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.

Idea Straightforward use of this lemma □

Problem 4.3.5: buratinogigle's proposed probs for Arab Saudi team 2015 **E**

Let ABC be acute triangle with $AB < AC$ inscribed circle (O). Bisector of $\angle BAC$ cuts (O) again at D. E is reflection of B through AD. DE cuts BC at F. Let (K) be circumcircle of triangle BEF. BD, EA cut (K) again at M, N, reps. Prove that $\angle BMN = \angle KFM$.

Pics/SATST2015proposed_by_bura/derakynay1134-8.png

Figure 4.29

Problem 4.3.6: USAMO 1999 P6 **E**

Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles.

Problem 4.3.7: Serbia 2018 P1 **E**

Let $\triangle ABC$ be a triangle with incenter I. Points P and Q are chosen on segments BI and CI such that $2\angle PAQ = \angle BAC$. If D is the touch point of incircle and side BC prove that $\angle PDQ = 90$.

Idea Straightforward Trig application. □

Problem 4.3.8: Iran TST T2P5 **E**

Let ω be the circumcircle of isosceles triangle ABC $(AB = AC)$. Points P and Q lie on ω and BC respectively such that $AP = AQ$ .AP and BC intersect at R. Prove that the tangents from B and C to the incircle of $\triangle AQR$ (different from BC) are concurrent on ω.

Problem 4.3.9: **M**

Let a point P inside of $\triangle ABC$ be such that the following condition is satisfied

$$\frac{AP + BP}{AB} = \frac{BP + CP}{BC} = \frac{CP + AP}{CA}$$

Lines AP, BP, CP intesect the circumcirle again at A', B', C'. Prove that ABC and A', B', C' have the same incircle.

Idea After finiding the point P, we get a lot of ideas.

Pics/itti_same_incircle.pdf

Figure 4.30: two lines are parallel

□

Problem 4.3.10: Iran TST 2018 P3 **EM**

In triangle ABC let M be the midpoint of BC. Let ω be a circle inside of ABC and is tangent to AB, AC at E, F, respectively. The tangents from M to ω meet ω at P, Q such that P and B lie on the same side of AM. Let $X \equiv PM \cap BF$ and $Y \equiv QM \cap CE$. If $2PM = BC$ prove that XY is tangent to ω.

Idea Work backwards □

Problem 4.3.11: Iran TST 2018 P4 **E**

Let ABC be a triangle ($\angle A \neq 90°$). BE, CF are the altitudes of the triangle. The bisector of $\angle A$ intersects EF, BC at M, N. Let P be a point such that $MP \perp EF$ and $NP \perp BC$. Prove that AP passes through the midpoint of BC.

Problem 4.3.12: APMO 2018 P1 **E**

Let H be the orthocenter of the triangle ABC. Let M and N be the midpoints of the sides AB and AC, respectively. Assume that H lies inside the quadrilateral $BMNC$ and that the circumcircles of triangles BMH and CNH are tangent to each other. The line through H parallel to BC intersects the circumcircles of the triangles BMH and CNH in the points K and L, respectively. Let F be the intersection point of MK and NL and let J be the incenter of triangle MHN. Prove that $FJ = FA$.

Problem 4.3.13: ISL 2006 G6 **E**

Circles w_1 and w_2 with centres O_1 and O_2 are externally tangent at point D and internally tangent to a circle w at points E and F respectively. Line t is the common tangent of w_1 and w_2 at D. Let AB be the diameter of w perpendicular to t, so that A, E, O_1 are on the same side of t. Prove that lines AO_1, BO_2, EF and t are concurrent.

Idea This □

Lemma 4.3.11: Tangential Quadrilateral Incenters

Let $ABCD$ be a tangential quatrilateral. Let I_1, I_2 be the incenters of $\triangle ABD, \triangle BCD$. Then $(I_1), (I_2)$ is tangent to BD at the same point.

`Pics/tangential_quad_incenters.png`

Figure 4.31

Problem 4.3.14: Four Incenters in a Tangential Quadrilateral **E**

Let $ABCD$ be a quadrilateral. Denote by X the point of intersection of the lines AC and BD. Let I_1, I_2, I_3, I_4 be the centers of the incircles of the triangles XAB, XBC, XCD, XDA, respectively. Prove that the quadrilateral $I_1I_2I_3I_4$ has a circumscribed circle if and only if the quadrilateral $ABCD$ has an inscribed circle.

Idea There is a lot going on in this figure, firstly, the J_1, J_2 and M, then K, then $\angle I_4ME = \angle I_3ME$. Connecting them with the lemma.

`Pics/tangential_quad_four_incenters.pdf`

Figure 4.32

$\square$

Problem 4.3.15: Geodip **E**

Let G be the centeroid. Dilate $\odot I$ from G with constant -2 to get I'. Then I' is tangent to the circumcircle.

`Pics/nice_prob_by_geodip.pdf`

Figure 4.33

Theorem 4.3.12: Fuhrmann Circle

Let X', Y', Z' be the midpoints of the arcs not containing A, B, C of $\odot ABC$. Let X, Y, Z be the reflections of these points on the sides. Then $\odot XYZ$ is called the **Fuhrmann Circle**. The orthocenter H and the nagel point N lies on this circle, and HN is a diameter of this circle.
Furthermore, AH, BH, CH cut the circle for the second time at a distance $2r$ from the vertices.

```
Pics/fuhrmann_circle.pdf
```

Figure 4.34: Fuhrmann Circle

Problem 4.3.16: Iran TST 2008 P12 **E**

In the acute-angled triangle ABC, D is the intersection of the altitude passing through A with BC and I_a is the excenter of the triangle with respect to A. K is a point on the extension of AB from B, for which $\angle AKI_a = 90^\circ + \frac{3}{4}\angle C$. I_aK intersects the extension of AD at L. Prove that DI_a bisects the angle $\angle AI_aB$ iff $AL = 2R$. (R is the circumradius of ABC)

Idea □

4.4 Conjugates

4.4.1 Isogonal Conjugate

Theorem 4.4.1: Isogonal Lemma

Let AP, AS and AQ, AR be two pairs of isogonal lines with respect to $\angle BAC$. Let $PR \cap QS = X$ and $PQ \cap RS = Y$. Then AX, AY are isogonal line with respect to $\angle BAC$

Pics/Isogonal_Lemma.pdf

Figure 4.35

Problem 4.4.1: India Postals 2015 Set 2 **E**

Let $ABCD$ be a convex quadrilateral. In the triangle ABC let I and J be the incenter and the excenter opposite the vertex A, respectively. In the triangle ACD let K and L be the incenter and the excenter opposite the vertex A, respectively. Show that the lines IL and JK, and the bisector of the angle BCD are concurrent.

Idea Using this lemma □

Lemma 4.4.2

Let ω_1, ω_2 be two circles such that ω_1 passes through A, B and is tangent to AC at A. ω_2 is defined similarly by swapping B with C. $\omega_1 \cap \omega_2 = X$.

Let γ_1, γ_2 be two circles such that γ_1 passes through A, B and is tangent to BC at B. γ_2 is defined similarly by swapping B with C. $\gamma_1 \cap \gamma_2 = Y$.

Then X, Y are isogonal conjugates wrt $\triangle ABC$.

Lemma 4.4.3: Isogonality in quadrilateral

For a point X, its isogonal conjugate wrt a quadrilateral $ABCD$ exists iff

$$\measuredangle BXA + \measuredangle DXC = 180^\circ$$

`Pics/isogonal_in_quad.pdf`

Figure 4.36: Isogonality in quadrilateral

Idea Draw the cirles, looks for similarity. □

4.4.1.1 Symmedians

In $\triangle ABC$, let T_a, T_b, T_c be the meet points of the tangents at A, B, C. Let $\triangle N_a N_b N_c$ be the cevian triangle of AT_a, BT_b, CT_c. Let S be the symmedian point of $\triangle ABC$. Let M_a, M_b, M_c be the midpoints of BC, CA, AB.

Pics/Symmedian_Point.png

Figure 4.37: The Symmedian Point

Lemma 4.4.4: Most Important Symmedian Property

Let the circles tangent to AC, AB at A and passes through B, C respectively meet at T' for the second time. Let $AT_a \cap \odot ABC = A'$. Let the tangents to $\odot ABC$ at A, A' meet BC at T. Prove that, $A, T', T_a,$ and T, T', O are collinear.

Pics/Symmedian_Lemma_1.png

Figure 4.38

Problem 4.4.2: USAMO 2008 P2 **E**

Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.

Idea Phantom point generated from symmedian properties □

Idea Taking the isogonal conjugate of F and showing that it lies on AM □

Idea Using Isogonal Lemma intelligently by taking the reflections of B, C over D, F □

Problem 4.4.3: IRAN TST 2015 Day 3, P3 **M**

AH is the altitude of triangle ABC and H' is the reflection of H trough the midpoint of BC. If the tangent lines to the circumcircle of ABC at B and C, intersect each other at X and the perpendicular line to XH' at H', intersects AB and AC at Y and Z respectively, prove that $\angle ZXC = \angle YXB$.

Problem 4.4.4: Two Symmedian Points **E**

Let E, F be the feet of B, C-altitudes. Let K, K_A be the symmedian points of $\triangle ABC, \triangle AEF$. Prove that $KK_A \perp BC, KK_A \cap BC = P$ and $KK_A = KP$

Pics/Two_Symmedian_Points.png

Figure 4.39: $KK_A \perp BC$

4.4.2 Isotonic Conjugate

Theorem 4.4.5: Isotonic Lemma

Let M be the midpoint of BC, and PQ such that Q is the reflection of P on M. Two points Q, R on AP, AQ, $BQ \cap CR = X$, $BR \cap CQ = Y$. Then AX, AY are isotonic wrt BC.

Pics/Isotonic_Lemma.png

Figure 4.40

Problem 4.4.5: IGO 2014 S5 **M**

Two points P and Q lying on side BC of triangle ABC and their distance from the midpoint of BC are equal.The perpendiculars from P and Q to BC intersect AC and AB at E and F,respectively.M is point of intersection PF and EQ.If H_1 and H_2 be the orthocenters of triangles BFP and CEQ, respectively, prove that $AM \perp H_1H_2$.

Idea We first show that the slope of H_1H_2 is fixed, and then show that AM is fixed where we use isotonic lemma, and finally show that these two lines are perpendicular. □

4.4.3 Reflection

Lemma 4.4.6: Homothety and Reflection

Let two oppositely oriented congruent triangles be $\triangle ABC, \triangle DEF$. Prove that the midpoints of AD, BE, CF are collinear.

Pics/homothety+reflection.png

Figure 4.41: Oppositely oriented congruent triangles

Problem 4.4.6: Autumn Tournament, 2012	**E**
Let two oppositely oriented equilateral triangles be $\triangle ABC, \triangle DEF$. What is the least possible value of $\max(AD, BE, CF)$?	

4.5 Mixtilinear–Curvilinear–Normal In-Excricles

Let $\triangle ABC$ be an ordinary triangle, I is its incenter, D is the touch points of the incenter with BC. Let ω be the mixtilinear incircle. Let it touch CA, AB at E, F. Furthermore, let $\omega \cap \odot ABC \equiv T$. Let M_a, M_b, M_c be the midpoints of the smaller arcs BC, CA, AB, and M_A, M_B, M_C be the midpoints of the major arcs BC, CA, AB.

Pics/Mixt1.png

Figure 4.42: Mixtilinear Incircle: Construction

`Pics/Mixt2.png`

Figure 4.43: Mixtilinear Incircle: Circlicity Lemmas

Idea List of small proofs

1. E, I, F are collinear. Consider the circle $TI'EC$ and do some angle chasing.
2. T, I, M_A are collinear. Consider the circle $TIEC$ and apply reim's theorem.

$\square$

Lemma 4.5.1

$\frac{TM_c}{M_cA} = \frac{TM_b}{M_bA}$, in other words, the bundle $(A, T; M_b, M_c)$ is harmonic. And TA is a symmedian of $\triangle TM_cM_b$.

Lemma 4.5.2

Let X be a variable point on the arc AB, and let O_1 and O_2 be the incenters of the triangles CAX and CBX. Then X, O_1, O_2 and T lie on a circle.

Idea Using similarity and this lemma. □

Pics/incenters_cyclic_with_mixtilinear_touchpoint.png

Figure 4.44: The two incenters are cyclic with T, X

Problem 4.5.1: ISL 1999 G8 **M**

Given a triangle ABC. The points A, B, C divide the circumcircle Ω of the triangle ABC into three arcs BC, CA, AB. Let X be a variable point on the arc AB, and let O_1 and O_2 be the incenters of the triangles CAX and CBX. Prove that the circumcircle of the triangle XO_1O_2 intersects the circle Ω in a fixed point.

Idea This is actually this lemma. □

Problem 4.5.2: AoPS1 **H**

Let $ABCD$ be a quadrilateral inscribed in a circle, such that the inradius of $\triangle ABC$ and ACD are the same. Let T be the touchpoint of A-mixtilinear incircle of the triangle ABD with $\odot ABCD$. Let I_1, I_2 be the incenters of the triangles ABC, ACD respectively. Show that I_1I_2 and the tangents of A, T wrt $\odot ABCD$ are concurrent.

Idea The main problem is how to relate the two mixtilinear touchpoints to the two incenters. But with our mixtilinear lemma, we can do that easily. □

Pics/c6t48f6h1508805.png

Figure 4.45

Problem 4.5.3: Generalization of Mixtilinear Incirlce **E**

Consider triangle ABC and let M, N are midpoints of arcs AB, AC. Let E, F on AB, AC such that $EF \parallel MN$. Let EM, FN meet (ABC) second time at P, Q. Consider two intersection points E', F' of $(EFPQ)$ with AB, AC different from E, F. Then $EF' \cap E'F$ is the incenter of ABC.

Problem 4.5.4:

Let the B-mixtilinear and C-mixtilinear circles touch BC at X_B, X_C respectively. Then X_B, X_C, T, M_a lie on a circle

Problem 4.5.5: Taiwan TST 2014 T3P3 **EH**

Let M be any point on the circumcircle of $\triangle ABC$. Suppose the tangents from M to the incircle meet BC at two points X_1 and X_2. Prove that T, M, X_1, X_2 lie on a circle.

Problem 4.5.6: Archer - EChen M1P3 **E**

Let the incenter touch BC at D. Let $AI \cap BC = E,\ AI \cap \odot ABC = F$. Let $\odot DEF \cap \odot ABC = X,\ \odot DEF \cap \odot(I_a) = S_1, S_2$. Prove that AX goes through either S_1 or S_2.

`Pics/Archer_EChen_M1P3.pdf`

Figure 4.46

`Curvilinear Incircle` Let $ABCD$ be a cyclic quadrilateral. AC meets BD at X. We call the circle that touches AX, BX and the circumcircle from the inside a curvilinear incircle.

Lemma 4.5.3

Let the previously defined curvilinear incircle touch AX, BX at P, Q resp. And let the incircle of $\triangle ABD$ be I. Then P, Q, I are collinear.

`Pics/curvi_lem_1.png`

Figure 4.47

Idea Notice that this is similar to the circles $TIEC$ and $TIFB$ in the mixtilinear circle figures. □

Theorem 4.5.4: Sawayama and Thebault's theorem

Through the vertex A of a triangle ABC, a straight line AD is drawn, cutting the side BC at D. I is the center of the incircle of triangle ABC. Let P be the center of the circle which touches DC, DA at E, F, and the circumcircle of ABC, and let Q be the center of a further circle which touches DB, DA in G, H and the circumcircle of ABC. Then P, I and Q are collinear

4.6 Complete Quadrilateral + Spiral Similarity

Lemma 4.6.1

Three lines, l_a, l_b, l_c, origin at point P. Two circles ω_1, ω_2 passing through P meet the lines at A_1, B_1, C_1; A_2, B_2, C_2 resp. Let A_3 be the reflection of A_2 on A_1. Define B_3, C_3 similarly. Then $PA_3B_3C_3$ are concyclic.

Pics/SpiralSimiLemma1.png

Figure 4.48: Spiral Similarity Lemma 1: the Blue points have been reflected wrt to the Red points to get the Green points

Problem 4.6.1: ISL 2009 G4 **E**

Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G and H.

Idea This problem generalizes to this □

Problem 4.6.2: All Russian 2014 Grade 10 Day 1 P4 **E**

Given a triangle ABC with $AB > BC$, let Ω be the circumcircle. Let M, N lie on the sides AB, BC respectively, such that $AM = CN$. Let K be the intersection of MN and AC. Let P be the incenter of the triangle AMK and Q be the K-excenter of the triangle CNK. If R is midpoint of the arc ABC of Ω then prove that $RP = RQ$.

Lemma 4.6.2

Let E and F be the intersections of opposite sides of a convex quadrilateral $ABCD$. The two diagonals meet at P. Let M be the foot of the perpendicular from P to EF. Show that $\angle BMC = \angle AMD$. And PM is the bisector of angles $\angle AMC, \angle BMD$.

Pics/Complete_quad_anglebisector.png

Figure 4.49

Theorem 4.6.3: Newton-Gauss Line

Among the points A, B, C, D no three are collinear. The lines AB and CD intersect at E, and BC and DA intersect at F. Prove that either the circles with diameters AC, BD, EF pass through two common points, or no two of them have any common point.
The previous can be stated differently: The midpoints of AC, BD, EF are collinear and this line is called "Newton-Gauss Line".

Idea Either by E.R.I.Q. Lemma, Length Chase, or configurations like Varignon Parallelogram □

Problem 4.6.3: **M**

Let $ABCD$ incribed (O) and a point so-called M .Call $X,\ Y,\ Z,\ T,\ U,\ V$ are the projection of M onto $AB,\ BC,\ CD,\ DA,\ CA,\ BD$ respectively. Call $I,\ J,\ H$ the midpoints of $XZ,\ UV,\ YT$ respectively. Prove that $N,\ P,\ Q$ are collinear.

Idea Divide the problem in cases, and prove the easiest case first. □

Lemma 4.6.4

In a cyclic quadrilateral $ABCD$, $AC \cap BD = P$, $AD \cap BC = Q$, $AB \cap CD = R$. S, T are the midpoints of PQ, PR. And a point X is on ST. Prove that the power of X wrt $ABCD$ is XP^2.

Idea Using polar argument wrt P □

Pics/power_of_point_on_a_midline_of_a_self-polar_triangle.png

Figure 4.50

Problem 4.6.4: USA TST 2000 P2 **E-M**

Let $ABCD$ be a cyclic quadrilateral and let E and F be the feet of perpendiculars from the intersection of diagonals AC and BD to AB and CD, respectively. Prove that EF is perpendicular to the line through the midpoints of AD and BC.

Pics/USATST2000P2.png

Figure 4.51: USA TST 2000 P2

Idea First solution is using some properties of the complete quad and angle bash the angle $\measuredangle(MN, EF)$ □

Idea Second solution is to notice the two brow triangles and proving them congruent. □

Problem 4.6.5: **E**

Let 2 equal circle $(O_1), (O_2)$ meet each other at P, Q. O be the midpoint of PQ. 2 line through P meet the circles at A, B, C, D, $(A, C \in (O_1);\ B, D \in (O_2))$. M, N be midpoint of AD, BC. Prove that M, N, O are collinear.

Problem 4.6.6: AoPS

In $\triangle ADE$ a circle with center O, passes through A, D meets AE, ED respectively at B, C, $BD \cap AC = G$, line OG meets $\odot ADE$ at P. Prove that $\triangle PBD, \triangle PAC$ has the same incenter (preferably without using inversion).

Problem 4.6.7: Archer - EChen M1P2 **E**

Let a circle ω centered at A meet BC at D, E, such that B, D, E, C all lie on BC in that order. Let ω meet $\odot ABC$ at F, G such that A, F, B, C, G lie on the circle in that order. Let $\odot BFD \cap AB = K$, $\odot CGE \cap AC = L$. Prove that FK, GL, AO are concurrent.

Pics/Archer_EChen_M1P2.pdf

Figure 4.52

Problem 4.6.8: Sharygin 2012 P22 **E**

A circle ω with center I is inscribed into a segment of the disk, formed by an arc and a chord AB. Point M is the midpoint of this arc AB, and point N is the midpoint of the complementary arc. The tangents from N touch ω in points C and D. The opposite sidelines AC and BD of quadrilateral $ABCD$ meet in point X, and the diagonals of $ABCD$ meet in point Y. Prove that points X, Y, I and M are collinear.

Pics/sharygin/2012_22.png

Figure 4.53

Idea La Hire □

Problem 4.6.9: Sharygin 2012 P21 **E**

Two perpendicular lines pass through the orthocenter of an acute-angled triangle. The sidelines of the triangle cut on each of these lines two segments: one lying inside the triangle and another one lying outside it. Prove that the product of two internal segments is equal to the product of two external segments.

Pics/sharygin/2012_21.png

Figure 4.54

Idea Spiral Similarity □

Problem 4.6.10: Iran TST 2004 P4 **E**

Let M, M' be two conjugates point in triangle ABC (in the sense that $\angle MAB = \angle M'AC, \dots$). Let P, Q, R, P', Q', R' be foots of perpendiculars from M and M' to BC, CA, AB. Let $E = QR \cap Q'R'$, $F = RP \cap R'P'$ and $G = PQ \cap P'Q'$. Prove that the lines AG, BF, CE are parallel.

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Pics/irantst2004P4.pdf
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Figure 4.55: The points are collinear, by Zhao Lemmas

Problem 4.6.11: Iran TST 2018 D2P6 **EM**

Consider quadrilateral $ABCD$ inscribed in circle ω. $P \equiv AC \cap BD$. E, F lie on sides AB, CD respectively such that $A\hat{P}E = D\hat{P}F$. Circles ω_1, ω_2 are tangent to ω at X, Y respectively and also both tangent to the circumcircle of $\triangle PEF$ at P. Prove that:

$$\frac{EX}{EY} = \frac{FX}{FY}$$

4.7 Purely Projective and Inversion problems

TelvCohl's $\sqrt{bc}$ inversion problem collection

Theorem 4.7.1: Pascal's Theorem for Octagons: A special case

Let $ABCDA'B'C'D'$ be a octagon inscribed in a conic section. If the points:

$$AD \cap BC,\ AC' \cap BB',\ AD' \cap CA',\ BD' \cap DA',\ DB' \cap CC'$$

are collinear, then so are the points

$$A'D' \cap B'C',\ A'C \cap B'B,\ A'D \cap C'A,\ B'D \cap D'A,\ D'B \cap C'C$$

Pics/PG1_OctagonalPascal.pdf

Figure 4.56: If the small Red points are collinear, then the Blue ones are too.

Theorem 4.7.2: Inscribed Conic in Pascal's theorem

$A_1A_2A_3A_4A_5A_6$ be a hexagon inscribed in a conic section. Then the hexagon formed by

$$A_1A_3 \cap A_2A_6,\ A_2A_4 \cap A_1A_3,\ A_2A_4 \cap A_3A_5,\ A_3A_5 \cap A_4A_6,\ A_5A_1 \cap A_4A_6,\ A_1A_5 \cap A_2A_6$$

has an inscribed conic section.

Problem 4.7.1: **E**

Let $ABCD$ have an incircle (I). Let (I) meet AB, BC, CD, DA at M, N, P, Q. Let K, L be the circumcenters of AMN, APQ. $KL \cap BD = R,\ AI \cap MQ = J$. Prove that $RA = RJ$.

Problem 4.7.2: **E**

Let the A mixtilinear incircle (O) of $\triangle ABC$ meet $\odot ABC, AC, AB$ at P, E, F. Let M be the BC arc midpoint. Let $\mathcal{H}$ be the conic that goes through E, F, O, P, M meet $\odot ABC$ at X, Y. Prove that AA, XY, EF are concurrent.

Problem 4.7.3: Iran 3rd Round G4 **EM**

Let ABC be a triangle with incenter I. Let K be the midpoint of AI and $BI \cap \odot(\triangle ABC) = M, CI \cap \odot(\triangle ABC) = N$. points P, Q lie on AM, AN respectively such that $\angle ABK = \angle PBC, \angle ACK = \angle QCB$. Prove that P, Q, I are collinear.

Idea Since we are dealing with collinearity, and usually we use harmonic bundle in these cases to show collinearity. But in this problem, there is no harmonic bundle. So we use cross ratio... □

Generalization 4.7.3.1: Iran 3rd Round G4 Generalized version

Let ABC be a triangle inscribed in circle (O) and P, Q are two isogonal conjugate points. PB, PC cut (O) again at M, N. QA cuts MN at K. L is isogonal conjugate of K. LB, LC cut AM, AN at S, T, resp. Prove that S, Q, T are collinear.

Lemma 4.7.3

Too long, can't explain, loot at the figure. The dotted lines go through that concurrency point.

Pics/concurrent_segments_almost_pascal.pdf

Figure 4.57: Everything concurrs

Problem 4.7.4: Sharygin Olympiad 2010 **M**

The circumcircle of $\triangle ABC$ is drawn, and three points A', B', C' on its sides BC, CA, AB are chosen. Then the original triangle is erased. Prove that the original triangle can be constructed iff AA', BB', CC' are concurrent.

Pics/reconstruct_ABC.png

Figure 4.58: A lot of things are going on here, look closely

4.7.1 Involution

An involution $f : \mathcal{P} \to \mathcal{P}$ is a function such that for all $A \in \mathcal{P}$, $f(f(A)) = A$ and reserves cross ratio.

Theorem 4.7.4: Involution on A Line

An involution on a line l is an inversion around some point on l.

Theorem 4.7.5: Involution on A Conic

For any involution on $\mathcal{C}$ the segments joining the reciprocal points, passes through a fixed point.

Idea First project the conic to a circle, then invert the circle to a line, and then using this, we prove the rest. □

Theorem 4.7.6: Desargues' Involution Theorem

Let $ABCD$ be a quadrilateral, let a conic $\mathcal{C}$ pass through A, B, C, D. And let a line l intersect $(AB, CD), (AD, BC), (AC, BD), \mathcal{C}$ at $(X_1, X_2), (Y_1, Y_2), (Z_1, Z_2), (W_1, W_2)$. Then $(X_1, X_2), (Y_1, Y_2), (Z_1, Z_2), (W_1, W_2)$ are reciprocal pairs of some involution on l.

Pics/desargues_invo_theorem.png

Figure 4.59

Idea Fix two pairs, and with those two, prove for other pairs using cross ration and pencils. □

Theorem 4.7.7: Desargues' Involution Theorem 2 Points

Let A, B, be two points on a conic $\mathcal{C}$, let a line l meet $AB, \mathcal{C}$ and the tangents at A, B to $\mathcal{C}$ at $X, (W_1, W_2), (Y_1, Y_2)$. Then $(X, X), (W_1, W_2), (Y_1, Y_2)$ are reciprocal pairs of an involution on l.

Pics/desargues_invo_theorem_2points.png

Figure 4.60

4.7.2 Inversion Properties

Lemma 4.7.8

WRT a circle ω with center O the polar of a point A can be constructed as the radical axis of ω and the circle with diameter OA.

Lemma 4.7.9

$\measuredangle(a, b) = \measuredangle AOB$

4.8 Parallelogram Stuff

Theorem 4.8.1: Maximality of the Area of a Cyclic Quadrilateral

Among all quadrilaterals with given side lengths, the cyclic one has maximal area.

`Pics/quad-same_sides-diff_area.png`

Figure 4.61: The cyclic quad has the maximal area

Problem 4.8.1: IOM 2017 P1 **E**

Let $ABCD$ be a parallelogram in which angle at B is obtuse and $AD > AB$. Points K and L on AC such that $\angle ADL = \angle KBA$(the points A, K, C, L are all different, with K between A and L). The line BK intersects the circumcircle ω of ABC at points B and E, and the line EL intersects ω at points E and F. Prove that $BF \parallel AC$.

Simplify: Make the diagram easier to draw.

Problem 4.8.2: USA TST 2006 P6 **E**

Let ABC be a triangle. Triangles PAB and QAC are constructed outside of triangle ABC such that $AP = AB$ and $AQ = AC$ and $\angle BAP = \angle CAQ$. Segments BQ and CP meet at R. Let O be the circumcenter of triangle BCR. Prove that $AO \perp PQ$.

Pics/USATST2006P6.pdf

Figure 4.62: USA TST 2006 P6, That is a prallelogram

4.9 Length Relations

Lemma 4.9.1: E.R.I.Q. (Equal Ration in Quadrilateral) Lemma

Let $A_1, B_1, C_1;\ A_2, B_2, C_2$ be two sets of collinear points such that

$$\frac{A_1B_1}{B_1C_1} = \frac{A_2B_2}{B_2C_2} = k$$

. Let points A, B, C be on A_1A_2, B_1B_2, C_1, C_2 such that:

$$\frac{A_1A}{A_2A} = \frac{B_1B}{B_2B} = \frac{C_1C}{C_2C}$$

Then we have,

$$A, B, C \text{ are collinear and, } \frac{AB}{BC} = k$$

`Pics/ERIQ_Lemma.png`

Figure 4.63: E.R.I.Q. Lemma

Idea A great use of this problem is in proving some midpoints collinear. Line in Newton-Gauss Line and some other such problems (1, 2, 3) where it is asked to proved that some midpoints are collinear. □

Lemma 4.9.2: Steiner's Isogonal Cevian Lemma

In $\triangle ABC$, AA_1, AA_2 are two isogonal cevians, with $A_1, A_2 \in BC$. Then we have

$$\frac{BA_1}{A_1C} \times \frac{BA_2}{A_2C} = \frac{BA^2}{AC^2}$$

Theorem 4.9.3:

Let P_1, P_2 be two isogonal conjugates wrt $\triangle ABC$. Then if the Pedal triangle of P_1 is homological wrt to $\triangle ABC$ then so is the Pedal triangle of P_2.

Theorem 4.9.4: Erdos-Mordell Theorem (Forum Geometricorum Volume 1 (2001) 7-8)

If from a point O inside a given $\triangle ABC$ perpendiculars OD, OE, OF are drawn to its sides, then $OA + OB + OC \geq 2(OD + OE + OF)$. Equality holds if and only if $\triangle ABC$ is equilateral.

Apparently nothing is needed except "Ptolemy's Theorem". Think of a way to connect OA with OE, OF and the sides of the triangle. As it is the most natural to use AB, AC, we have to deal with BE, CF too. And dealing with lengths is the easiest when we have similar triangles. So we do some construction.

Problem 4.9.1: ISL 2011 G7 **E**

Let $ABCDEF$ be a convex hexagon all of whose sides are tangent to a circle ω with centre O. Suppose that the circumcircle of triangle ACE is concentric with ω. Let J be the foot of the perpendicular from B to CD. Suppose that the perpendicular from B to DF intersects the line EO at a point K. Let L be the foot of the perpendicular from K to DE. Prove that $DJ = DL$.

Idea There are a LOT of equal lenths, equal angles, and we have a perpendicularity lemma working as well. Why don't we try cosine :0 □

4.10 Pedal Triangles

Let P be an arbitrary point, let $\triangle A_1B_1C_1$ be its pedal triangle wrt $\triangle ABC$. Let A', B', C' and A_0, B_0, C_0 be the feet of the altitudes and the midpoints of $\triangle ABC$.

$$B_1C_1 \cap B_0C_0 = A_2,\ C_1A_1 \cap C_0A_0 = B_2,\ A_1B_1 \cap A_0B_0 = C_2$$
$$B'C' \cap B_0C_0 = A_3,\ C'A' \cap C_0A_0 = B_3,\ A'B' \cap A_0B_0 = C_3$$

Theorem 4.10.1: Fontene's First Theorem

A_1A_2, B_1B_2, C_1C_2 are concurrent at the intersection of $\odot A_1B_1C_1$ and $\odot A_0B_0C_0$

Lemma 4.10.2

$A'A_3, B'B_3, C'C_3$ and A_0A_3, B_0B_3, C_0C_3 concur at the nine point circle of $\triangle ABC$.

Theorem 4.10.3: Fontene's Second Theorem

Let the concurrency point in the first theorem be Q. Then, if the line OP is fixed and P moves along that line, Q will stay fixed.

The previous result leads to another beautiful result:

Lemma 4.10.4

Suppose a varying point P is chosen on the Euler Line of $\triangle ABC$. Then the pedal circle of P wrt $\triangle ABC$ intersects the 9p circle at a fixed point which is the Euler Reflection Point of the median triangle.

4.11 Pending Problems

Problem 4.11.1:

In $\triangle ABC$, I is the incenter, D is the touch point of the incenter with BC. $AD \cap \odot ABC \equiv X$. The tangents line from X to $\odot I$ meet $\odot ABC$ at Y, Z. Prove that YZ, BC and the tangent at A to $\odot ABC$ concur.

Problem 4.11.2: IRAN TST 2017 Day 1, P3 **M**

In triangle ABC let I_a be the A-excenter. Let ω be an arbitrary circle that passes through A, I_a and intersects the extensions of sides AB, AC (extended from B, C) at X, Y respectively. Let S, T be points on segments I_aB, I_aC respectively such that $\angle AXI_a = \angle BTI_a$ and $\angle AYI_a = \angle CSI_a$. Lines BT, CS intersect at K. Lines KI_a, TS intersect at Z. Prove that X, Y, Z are collinear.

Problem 4.11.3: IRAN TST 2015 Day 3, P2

In triangle ABC(with incenter I) let the line parallel to BC from A intersect circumcircle of $\triangle ABC$ at A_1 let $AI \cap BC = D$ and E is tangency point of incircle with BC let $EA_1 \cap \odot(\triangle ADE) = T$ prove that $AI = TI$.

Pics/ITST2015D3P2.png

Figure 4.64: IRAN TST 2015 Day 3, P2

Problem 4.11.4: Generalization of Iran TST 2017 P5

Let ABC be triangle and the points P, Q lies on the side BC s.t B, C, P, Q are all different. The circumcircles of triangles ABP and ACQ intersect again at G. AG intersects BC at M. The circumcircle of triangle APQ intersects AB, AC again at E, F , respectively. EP and FQ intersect at T. The lines through M and parallel to AB, AC, intersect EP, FQ at X, Y, respectively. Prove that the circumcircles of triangle TXY and APQ are tangent to each other.

Problem 4.11.5: ARMO 2013 Grade 11 Day 2 P4

Let ω be the incircle of the triangle ABC and with center I. Let Γ be the circumcircle of the triangle AIB. Circles ω and Γ intersect at the point X and Y. Let Z be the intersection of the common tangents of the circles ω and Γ. Show that the circumcircle of the triangle XYZ is tangent to the circumcircle of the triangle ABC.

Problem 4.11.6: AoPS

Let ABC be a triangle with incircle (I) and A-excircle (I_a). $(I),(I_a)$ are tangent to BC at D,P, respectively. Let $(I_1),(I_2)$ be the incircle of triangles APC,APB, respectively, $(J_1),(J_2)$ be the reflections of $(I_1),(I_2)$ wrt midpoints of AC,AB. Prove that AD is the radical axis of (J_1) and (J_2).

Problem 4.11.7: AoPS

Let ABC be a $A-$right-angled triangle and $MNPQ$ a square inscribed into it, with M,N onto BC in order $B-M-N-C$, and P,Q onto CA,AB respectively. Let $R = BP \cap QM, S = CQ \cap PN$. Prove that $AR = AS$ and RS is perpendicular to the $A-$inner angle bisector of $\triangle ABC$.

Problem 4.11.8: AoPS

P is an arbitrary point on the plane of $\triangle ABC$ and let $\triangle A'B'C'$ be the cevian triangle of P WRT $\triangle ABC$. The circles $\odot(ABB')$ and $\odot(ACC')$ meet at A,X. Similarly, define the points Y and Z WRT B and C. Prove that the lines AX,BY,CZ concur at the isogonal conjugate of the complement of P WRT $\triangle ABC$.

Problem 4.11.9: AoPS

Given are $\triangle ABC, L$ is Lemoine point,L_a, L_b, L_c are three Lemoine point of triangles LBC, LCA, LAB prove that AL_a, BL_b, CL_a are concurrent!
A question: What is the locus of point P such that AL_a, BL_b, CL_a are concurrent with L_a, L_b, L_c are three 'Lemoine points' of triangles PBC, PCA, PAB?

Problem 4.11.10: AoPS

Let ABC be a triangle inscribed circle (O). Let (O') be the circle wich is tangent to the circle (O) and the sides CA, AB at D and E,F, respectively. The line BC intersects the tangent line at A of (O), EF and AO' at T,S and L, respectively. The circle (O) intersects AS again at K. Prove that the circumcenter of triangle AKL lies on the circumcircle of triangle ADT.

Problem 4.11.11:

Let P and Q be isogonal conjugates of each other. Let $\triangle XYZ, \triangle KLM$ be the pedal triangles of P and Q wrt $\triangle ABC$. (X,K lie on BC; Y,L lie on CA; Z,M lie on AB) Prove that YM, ZL, PQ are concurrent.

Problem 4.11.12: 2nd Olympiad of Metropolises

Let $ABCDEF$ be a convex hexagon which has an inscribed circle and a circumscribed circle. Denote by ω_A, ω_B, ω_C, ω_D, ω_E, and ω_F the inscribed circles of the triangles FAB, ABC, BCD, CDE, DEF,and EFA,respectively. Let l_{AB} be the external common tangent of ω_A and ω_B other than the line AB; lines $l_{BC}, l_{CD}, l_{DE}, l_{EF}$, and l_{FA} are analogously defined. Let A_1 be the intersection point of the lines l_{FA} and l_{AB}; B_1 be the intersection point of the lines l_{AB} and l_{BC}; points C_1, D_1, E_1, and $F1$ are analogously defined. Suppose that $A_1B_1C_1D_1E_1F_1$ is a convex hexagon. Show that its diagonals A_1D_1, B_1E_1, and C_1F_1 meet at a single point.

Problem 4.11.13: ISL 2016 G6

Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC < 90^\circ$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at E and F respectively, and meet each other at point P. Let M be the midpoint of AC and let ω be the circumcircle of triangle BPD. Segments BM and DM intersect ω again at X and Y respectively. Denote by Q the intersection point of lines XE and YF. Prove that $PQ \perp AC$.

4.12 Problems

Problem 4.12.1: IRAN 3rd Round 2016 P2 **E**

Let ABC be an arbitrary triangle. Let E, F be two points on AB, AC respectively such that their distance to the midpoint of BC is equal. Let P be the second intersection of the triangles ABC, AEF circumcircles . The tangents from E, F to the circumcircle of AEF intersect each other at K. Prove that : $\angle KPA = 90$

Problem 4.12.2: IRAN 2nd Round 2016 P6 **E**

Let ABC be a triangle and X be a point on its circumcircle. Q, P lie on a line BC such that $XQ \perp AC, XP \perp AB$. Let Y be the circumcenter of $\triangle XQP$. Prove that ABC is equilateral triangle if and only if Y moves on a circle when X varies on the circumcircle of ABC

Problem 4.12.3: AoPS **E**

Consider ABC with orthic triangle $A'B'C'$, let $AA' \cap B'C' = E$ and E' be reflection of E wrt BC. Let M be midpoint of BC and O be circumcenter of $E'B'C'$. Let M' be projection of O on BC and N be the intersection of a perpendicular to $B'C'$ through E with BC. Prove that $MM' = 1/4MN$.

Problem 4.12.4: IRAN 3rd Round 2010 D3, P5 **M**

In a triangle ABC, I is the incenter. D is the reflection of A to I. the incircle is tangent to BC at point E. DE cuts IG at P (G is centroid). M is the midpoint of BC. Prove that $AP||DM$ and $AP = 2DM$.

Problem 4.12.5: IRAN 3rd Round 2011 G5 **M**

Given triangle ABC, D is the foot of the external angle bisector of A, I its incenter and I_a its A-excenter. Perpendicular from I to DI_a intersects the circumcircle of triangle in A'. Define B' and C' similarly. Prove that AA', BB' and CC' are concurrent.

Problem 4.12.6: AoPS3 **E**

I is the incenter of ABC, $PI, QI \perp BC$, PA, QA intersect BC at DE. Prove: $IADE$ is on a circle.

Pics/AoPS3.png

Figure 4.65: AoPS3

Problem 4.12.7: AoPS4 **E**

Given a triangle ABC, the incircle (I) touch BC, CA, AB at D, E, F respectively. Let AA_1, BB_1, CC_1 be $A, B, C - altitude$ respectively. Let N be the orthocenter of the triangle AEF. Prove that N is the incenter of AB_1C_1

Pics/AoPS4.png

Figure 4.66: AoPS4

Problem 4.12.8: IRAN TST 2015 Day 2, P3 **M**

$ABCD$ is a circumscribed and inscribed quadrilateral. O is the circumcenter of the quadrilateral. E, F and S are the intersections of AB, CD; AD, BC and AC, BD respectively. E' and F' are points on AD and AB such that $\angle AEE' = \angle E'ED$ and $\angle AFF' = \angle F'FB$. X and Y are points on OE' and OF' such that $\frac{XA}{XD} = \frac{EA}{ED}$ and $\frac{YA}{YB} = \frac{FA}{FB}$. M is the midpoint of arc BD of (O) which contains A. Prove that the circumcircles of triangles OXY and OAM are coaxial with the circle with diameter OS.

Pics/ITST2015D3P3i.png

Figure 4.67: Actual Prob

Pics/ITST2015D3P3ii.png

Figure 4.68: Inverted

Problem 4.12.9: USA TST 2017 P2 **M**

Let ABC be an acute scalene triangle with circumcenter O, and let T be on line BC such that $\angle TAO = 90^\circ$. The circle with diameter $\overline{AT}$ intersects the circumcircle of $\triangle BOC$ at two points A_1 and A_2, where $OA_1 < OA_2$. Points B_1, B_2, C_1, C_2 are defined analogously.

1. Prove that $\overline{AA_1}$, $\overline{BB_1}$, $\overline{CC_1}$ are concurrent.

2. Prove that $\overline{AA_2}$, $\overline{BB_2}$, $\overline{CC_2}$ are concurrent on the Euler line of triangle ABC.

Problem 4.12.10: AoPS2

Let ABC be a triangle with circumcenter O and altitude AH. AO meets BC at M and meets the circle (BOC) again at N. P is the midpoint of MN. K is the projection of P on line AH. Prove that the circle (K, KH) is tangent to the circle (BOC).

Pics/AoPS2.png

Figure 4.69: AoPS2

Idea Inversion all the way... □

Problem 4.12.11: AoPS5

Let ABC be a triangle inscribed in (O) and P be a point. Call P' be the isogonal conjugate point of P. Let A' be the second intersection of AP' and (O). Denote by M the intersection of BC and $A'P$. Prove that $P'M \parallel AP$.

`Pics/AoPS5.png`

Figure 4.70: AoPS5

Problem 4.12.12: AoPS **E**

I is the incenter of a non-isosceles triangle $\triangle ABC$. If the incircle touches BC, CA, AB at A_1, B_1, C_1 respectively, prove that the circumcentres of the triangles $\triangle AIA_1$, $\triangle BIB_1$, $\triangle CIC_1$ are collinear.

Problem 4.12.13: AoPS **M**

Given $\triangle ABC$ and a point P inside. AP cuts BC at M. Let M', A' be the reflection of M, A in the perpendicular bisector of BC. $A'P$ cuts the perpendicular bisector of BC at N. Let Q be the isogonal conjugate of P in triangle ABC. Prove that $QM' \parallel AN$.

Problem 4.12.14: IRAN 3rd Round 2016 G6 **E**

Given triangle $\triangle ABC$ and let D, E, F be the foot of angle bisectors of A, B, C ,respectively. M, N lie on EF such that $AM = AN$. Let H be the foot of A-altitude on BC.
Points K, L lie on EF such that triangles $\triangle AKL, \triangle HMN$ are correspondingly similar (with the given order of vertices's) such that $AK \nparallel HM$ and $AK \nparallel HN$. Show that: $DK = DL$.

Pics/IRAN3rd2016G6.png

Figure 4.71: IRAN 3rd Round 2016 G6

Problem 4.12.15: Iran TST 2017 T3 P6 **H**

In triangle ABC let O and H be the circumcenter and the orthocenter. The point P is the reflection of A with respect to OH. Assume that P is not on the same side of BC as A. Points E, F lie on AB, AC respectively such that $BE = PC$, $CF = PB$. Let K be the intersection point of AP, OH. Prove that $\angle EKF = 90°$.

Spiral Similarity (points on AB, AC with some properties)

Pics/ITST2017T3P6.pdf

Figure 4.72: Iran TST 2017 T3 P6

Problem 4.12.16: IRAN 3rd Round 2010 D3, P6 **M**

In a triangle ABC, $\angle C = 45°$. AD is the altitude of the triangle. X is on AD such that $\angle XBC = 90 - \angle B$ (X is inside of the triangle). AD and CX cut the circumcircle of ABC in M and N respectively. Ff the tangent to $\odot ABC$ at M cuts AN at P, prove that P, B and O are collinear.

Cross-Ratio

Problem 4.12.17: Iran TST 2014 T1P6 **M**

I is the incenter of triangle ABC. perpendicular from I to AI meet AB and AC at B' and C' respectively. Suppose that B'' and C'' are points on half-line BC and CB such that $BB'' = BA$ and $CC'' = CA$. Suppose that the second intersection of circumcircles of $AB'B''$ and $AC'C''$ is T. Prove that the circumcenter of AIT is on the BC.

projective, inversion

Idea Too many collinearity, need to prove concurrency, what else can come into mind except projective approach. □

Idea Too many incenter related things, $\sqrt{bc}$-inversion :o □

Problem 4.12.18: APMO 2014 P5 **M**

Circles ω and Ω meet at points A and B. Let M be the midpoint of the arc AB of circle ω (M lies inside Ω). A chord MP of circle ω intersects Ω at Q (Q lies inside ω). Let ℓ_P be the tangent line to ω at P, and let ℓ_Q be the tangent line to Ω at Q. Prove that the circumcircle of the triangle formed by the lines ℓ_P, ℓ_Q and AB is tangent to Ω.

Pics/APMO2014P5.png

Figure 4.73: APMO 2014 P5

Problem 4.12.19: **E**

Let ABC be a triangle, D, E, F are the feet of the altitudes, $DF \cap BE \equiv P, DE \cap CF \equiv Q$. Prove that the perpendicular from A to PQ goes through the reflection of O on BC.

projective

Idea Projective approach. □

Problem 4.12.20: RMM 2018 P6 **H**

Fix a circle Γ, a line ℓ to tangent Γ, and another circle Ω disjoint from ℓ such that Γ and Ω lie on opposite sides of ℓ. The tangents to Γ from a variable point X on Ω meet ℓ at Y and Z. Prove that, as X varies over Ω, the circumcircle of XYZ is tangent to two fixed circles.

inversion

Idea Too many circles, plus tangency, what else other than inversion? After the inversion the problem turns into a pretty obvious work-around problem. □

Problem 4.12.21: AoPS6 **H but Beautiful**

Let O and I be the circumcenter and incenter of ΔABC. Draw circle ω so that $B, C \in \omega$ and ω touches (I) internally at P. AI intersects BC at X. Tangent at X to (I) which is different from BC, intersects tangent at P to (I) at S. $SA \cap (O) = T \neq A$. Prove that $\angle ATI = 90°$

Pics/AoPS6_1.pdf

Figure 4.74: Solution 1

Pics/AoPS6_2.pdf

Figure 4.75: Solution 2

Problem 4.12.22: AoPS7 **E**

Let ABC be a triangle with incenter I and circumcircle Γ. Let the line through I perpendicular to AI meet AB at E and AC at F. Let the circumcircles of triangles AIB and AIC intersect the circumcircle of triangle AEF ω again at points M and N, and let ω intersect Γ again at Q. Prove that AQ, MN, and BC are concurrent.

Problem 4.12.23: AoPS **E**

Given a circle (O) with center O and A, B are 2 fixed points on (O). E lies on AB. C, D are on (O) and CD pass through E. P lies on the ray DA, Q lies on the ray DB such that E is the midpoint of PQ. Prove that the circle passing through C and touch PQ at E also pass through the midpoint of AB

Problem 4.12.24: WenWuGuangHua Mathematics Workshop **E**

O_B, O_C are the B and C mixtilinear centers respectively. (O_B) touches BC, AB at X_B, Y_B respectively, and $X_BY_B \cap O_BO_C$ at Z_B. Define X_C, Y_C, Z_C similarly. Prove that if $BZ_C \cap CZ_B = T$, then AT is the A-angle bisector.

Problem 4.12.25: All Russia 1999 P9.3 **E**

A triangle ABC is inscribed in a circle S. Let A_0 and C_0 be the midpoints of the arcs BC and AB on S, not containing the opposite vertex, respectively. The circle S_1 centered at A_0 is tangent to BC, and the circle S_2 centered at C_0 is tangent to AB. Prove that the incenter I of $\triangle ABC$ lies on a common tangent to S_1 and S_2.

Problem 4.12.26: All Russia 2000 P11.7 **E**

A quadrilateral $ABCD$ is circumscribed about a circle ω. The lines AB and CD meet at O. A circle ω_1 is tangent to side BC at K and to the extensions of sides AB and CD, and a circle ω_2 is tangent to side AD at L and to the extensions of sides AB and CD. Suppose that points O, K, L lie on a line. Prove that the midpoints of BC and AD and the center of ω also lie on a line.

Problem 4.12.27: All Russia 2000 P9.3 **E**

Let O be the center of the circumcircle ω of an acute-angle triangle ABC. A circle ω_1 with center K passes through A, O, C and intersects AB at M and BC at N. Point L is symmetric to K with respect to line NM. Prove that $BL \perp AC$.

Problem 4.12.28: WenWuGuangHua Mathematics Workshop **M**

1. AD, BE, CF are concurrent cevians. Angle bisectors of $\angle ADB$ and $\angle AEB$ meet at C_0. Again the angle bisectors of $\angle ADC$ and $\angle AFC$ meet at B_0. And bisectors of $\angle BEC$ and $\angle BFC$ meet at A_0. Prove that AA_0, BB_0, CC_0 are concurrent.
2. Angle bisectors of $\angle AEB$ and $\angle AFC$ meet at D_0, of $\angle BFC$ and BDA meet at E_0, and of $\angle CEB$ and $\angle CDA$ meet at F_0. Prove that DD_0, EE_0, FF_0 are concurrent.

Idea As this problem is purely made up with lines, we can do a projective transformation to simplify the problem. And as there are perpendicularity at D, E, F, we make D, E, F the feet of the altitudes of $\triangle ABC$. Then the angle bisector properties get replaced by simpler properties wrt DEF. □

Problem 4.12.29: WenWuGuangHua Mathematics Workshop **E**

Generalization: Let AD, BE, CF be any cevians concurrent at T. $AD \cap EF = A'$, $BE \cap DF = B'$, $CF \cap DE = C'$, $B'A' \cap AC = X$, $B'A' \cap BC = Y$, $C'X \cap EF = Z$. Prove that T, Y, Z are collinear.

Problem 4.12.30: AoPS **E**

On circumcircle of triangle ABC, T and K are midpoints of arcs BC and BAC respectively . And E is foot of altitude from C on AB . Point P is on extension of AK such that PE is perpendicular to ET . Prove that $PC = CK$.

Problem 4.12.31: USJMO 2018 P3 **E**

Let $ABCD$ be a quadrilateral inscribed in circle ω with $\overline{AC} \perp \overline{BD}$. Let E and F be the reflections of D over lines BA and BC, respectively, and let P be the intersection of lines BD and EF. Suppose that the circumcircle of $\triangle EPD$ meets ω at D and Q, and the circumcircle of $\triangle FPD$ meets ω at D and R. Show that $EQ = FR$.

Problem 4.12.32: All Russia 2002 P11.6 **M**

The diagonals AC and BD of a cyclic quadrilateral $ABCD$ meet at O. The circumcircles of triangles AOB and COD intersect again at K. Point L is such that the triangles BLC and AKD are similar and equally oriented. Prove that if the quadrilateral $BLCK$ is convex, then it has an incircle.

Problem 4.12.33: WenWuGuangHua Mathematics Workshop **M**

Let O_B, O_C be the B, C mixtilinear excircles. O meet CA, CB at X_C, Y_C and O_B meet BA, BC at X_B, Y_B. Let I_C be the C-excircle. $I_C Y_B$ meet $O_B O_C$ at T. Prove that $BT \perp O_B O_C$

Idea From what we have to prove, we find two circles, from where we get another circle. This circle suggests that we try power of point. □

Problem 4.12.34: Iran TST 2018 T1P3 **M**

In triangle ABC let M be the midpoint of BC. Let ω be a circle inside of ABC and is tangent to AB, AC at E, F, respectively. The tangents from M to ω meet ω at P, Q such that P and B lie on the same side of AM. Let $X \equiv PM \cap BF$ and $Y \equiv QM \cap CE$. If $2PM = BC$ prove that XY is tangent to ω.

Problem 4.12.35: Iran TST 2018 T1P4 **E**

Let ABC be a triangle ($\angle A \neq 90°$). BE, CF are the altitudes of the triangle. The bisector of $\angle A$ intersects EF, BC at M, N. Let P be a point such that $MP \perp EF$ and $NP \perp BC$. Prove that AP passes through the midpoint of BC.

Idea :'3 kala para na T_T □

Problem 4.12.36: Iran TST 2018 T3P6 **H**

Consider quadrilateral $ABCD$ inscribed in circle ω. $AC \cap BD = P$. E, F lie on sides AB, CD, respectively such that $\angle APE = \angle DPF$. Circles ω_1, ω_2 are tangent to ω at X, Y respectively and also both tangent to the circumcircle of PEF at P. Prove that:

$$\frac{EX}{EY} = \frac{FX}{FY}$$

Idea fucking beautiful. □

Problem 4.12.37: ISL 2006 G6 **E**

Circles ω_1 and ω_2 with centres O_1 and O_2 are externally tangent at point D and internally tangent to a circle ω at points E and F respectively. Line t is the common tangent of ω_1 and ω_2 at D. Let AB be the diameter of ω perpendicular to t, so that A, E, O_1 are on the same side of t. Prove that lines AO_1, BO_2, EF and t are concurrent.

Problem 4.12.38: ISL 2006 G7 **E**

In a triangle ABC, let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB, respectively, and T_a, T_b, T_c be the midpoints of the arcs BC, CA, AB of the circumcircle of ABC, not containing the vertices's A, B, C, respectively. For $i \in a, b, c$, let w_i be the circle with M_iT_i as diameter. Let p_i be the common external common tangent to the circles w_j and w_k (for all $i, j, k = a, b, c$) such that w_i lies on the opposite side of p_i than w_j and w_k do.
Prove that the lines p_a, p_b, p_c form a triangle similar to ABC and find the ratio of similitude

Problem 4.12.39: ISL 2006 G9 **H**

Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC, respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2, respectively ($A_2 \neq A, B_2 \neq B, C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

Idea In this type of "Miquel's Point and the intersections of the circumcircles" related problems, it is useful to think about the second intersections of the lines joining the first intersections and the Miquel's Point with the main circle. $\square$

Pics/ISL2006G9.pdf

Figure 4.76: IMO Shortlist G9

Problem 4.12.40: Iran TST 2017 P5

In triangle ABC, arbitrary points P, Q lie on side BC such that $BP = CQ$ and P lies between B, Q. The circumcircle of triangle APQ intersects sides AB, AC at E, F respectively. The point T is the intersection of EP, FQ. Two lines passing through the midpoint of BC and parallel to AB and AC, intersect EP and FQ at points X, Y respectively. Prove that the circumcircle of triangle TXY and triangle APQ are tangent to each other.

Problem 4.12.41: **E**

Let X be the touchpoint of the incircle with BC and let AX meet $\cdot ABC$ at D. The tangents from D to the incircle meet $\cdot ABC$ at E, F. Prove that the tangent to the circumcircle at A, EF and BC are concurrent.

Problem 4.12.42: ISL 2012 G8 **M**

Let ABC be a triangle with circumcircle ω and ℓ a line without common points with ω. Denote by P the foot of the perpendicular from the center of ω to ℓ. The side-lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove that the circumcircles of the triangles AXP, BYP and CZP have a common point different from P or are mutually tangent at P.

Idea Using Cross ratio and Desergaus's Involution Theorem. □

Problem 4.12.43: **E**

Suppose an involution on a line l sending X, Y, Z to X', Y', Z'. Let l_x, l_y, l_z be three lines passing through X, Y, Z respectively. And let $X_0 = l_y \cap l_z$, $Y_0 = l_x \cap l_z$, $Z_0 = l_x \cap l_y$. Then X_0X', Y_0Y', Z_0Z' are concurrent.

Problem 4.12.44: USAMO 2018 P5 **E**

In convex cyclic quadrilateral $ABCD$, we know that lines AC and BD intersect at E, lines AB and CD intersect at F, and lines BC and DA intersect at G. Suppose that the circumcircle of $\triangle ABE$ intersects line CB at B and P, and the circumcircle of $\triangle ADE$ intersects line CD at D and Q, where C, B, P, G and C, Q, D, F are collinear in that order. Prove that if lines FP and GQ intersect at M, then $\angle MAC = 90°$.

Problem 4.12.45: Japan MO 2017 P3 **E**

Let ABC be an acute-angled triangle with the circumcenter O. Let D, E and F be the feet of the altitudes from A, B and C, respectively, and let M be the midpoint of BC. AD and EF meet at X, AO and BC meet at Y, and let Z be the midpoint of XY. Prove that A, Z, M are collinear.

Problem 4.12.46: ISL 2002 G1 **E**

Let B be a point on a circle S_1, and let A be a point distinct from B on the tangent at B to S_1. Let C be a point not on S_1 such that the line segment AC meets S_1 at two distinct points. Let S_2 be the circle touching AC at C and touching S_1 at a point D on the opposite side of AC from B. Prove that the circumcenter of triangle BCD lies on the circumcircle of triangle ABC.

Problem 4.12.47: ISL 2002 G2 **M**

Let ABC be a triangle for which there exists an interior point F such that $\angle AFB = \angle BFC = \angle CFA$. Let the lines BF and CF meet the sides AC and AB at D and E respectively. Prove that

$$AB + AC \geq 4DE.$$

Idea Pari nai. □

Problem 4.12.48: ISL 2002 G3 **E**

The circle S has center O, and BC is a diameter of S. Let A be a point of S such that $\angle AOB < 120°$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and at F. Prove that I is the incenter of the triangle CEF.

Problem 4.12.49: ISL 2002 G4 **E**

Circles S_1 and S_2 intersect at points P and Q. Distinct points A_1 and B_1 (not at P or Q) are selected on S_1. The lines A_1P and B_1P meet S_2 again at A_2 and B_2 respectively, and the lines A_1B_1 and A_2B_2 meet at C. Prove that, as A_1 and B_1 vary, the circumcentres of triangles A_1A_2C all lie on one fixed circle.

Problem 4.12.50: ISL 2002 G7 **E**

The incircle Ω of the acute-angled triangle ABC is tangent to its side BC at a point K. Let AD be an altitude of triangle ABC, and let M be the midpoint of the segment AD. If N is the common point of the circle Ω and the line KM (distinct from K), then prove that the incircle Ω and the circumcircle of triangle BCN are tangent to each other at the point N.

Problem 4.12.51: Japan MO 2017 P3 **E**

Let ABC be an acute-angled triangle with the circumcenter O. Let D, E and F be the feet of the altitudes from A, B and C, respectively, and let M be the midpoint of BC. AD and EF meet at X, AO and BC meet at Y, and let Z be the midpoint of XY. Prove that A, Z, M are collinear.

Problem 4.12.52: India TST **E**

ABC triangle, D, E, F touchpoints, M midpoint of BC, K orthocenter of $\triangle AIC$, prove that $MI \perp KD$

Problem 4.12.53: ISL 2009 G3 **E**

Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelogram. Prove that $GR = GS$.

Idea Point Circle, distance same means Power same wrt point circles. $\square$

Problem 4.12.54: ARO 2018 P11.6 **E**

Three diagonals of a regular n-gon prism intersect at an interior point O. Show that O is the center of the prism.
(The diagonal of the prism is a segment joining two vertices's not lying on the same face of the prism.)

Problem 4.12.55: ISL 2011 G4 **EM**

Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.

Problem 4.12.56: **Constructing a forth circle tangent**

Given 3 circle, construct another circle that is tangent to these three circles.

Idea A trick to remember: decreasing the radius's of some circles doesn't effect much. □

Problem 4.12.57: **H**

Let $ABCD$ be a convex quadrilateral, let $AD \cap BC = P$. Let O, O'; H, H' be the circumcentres and orthocenter of $\triangle PCD, \triangle PAB$. $\odot DOC$ is tangent to $\odot AD'B$, if and only if $\odot DHC$ is tangent to $\odot AH'B$

4.13 Research Stuffs for later

Problem 4.13.1: AoPS

Let ABC be a triangle with incenter I. L_a, L_b, L_c are symmedian points of triangles IBC, ICA, IAB. Let X, Y, Z be the reflections of I through L_a, L_b, L_c.

- Prove that AX, BY, CZ and OI are concurrent.
- Let I_a, I_b, I_c be the excenters of ABC. Prove that I_aX, I_bY, I_cZ are concurrent at a point P and isogonal conjugate of P with respect to triangle $I_aI_bI_c$ lies on Euler line of ABC.

Problem 4.13.2: buratinogigle Tough P1

Let ABC be a triangle inscribed in circle (O) with A-excircle (J). Circle passing through A, B touches (J) at M. Circle passing through A, C touches (J) at N. BM cuts CN at P. Prove that AP passes through tangent point of A-mixtilinear incircle with (O).

Chapter 5

Number Theory

5.1 Tricks

A general trick to remember: If the problem condition is completely or partially but crucially depended on some problem object, but the proof condition doesn't directly depend on that object, think of a way to include that object in the proof condition.

- permutation type problem
- do there exist...
- proving identities
- Dunno
- divisibility by primes and prime divisors stuff

1. Add. Everything. Up.
2. Infinitude of primes:
   a) Eulerian infinitude trick
   b) For large enough numbers, there is a larger prime divisor
   c) Assuming contradiction, if there are any number co-prime to the product of the primes, then that must be 1.

5.1.1 Digit Sum or Product

When dealing with the sum of the digits or the product of them, to find the construction it is very important to consider 0 and 1's in the number.

5.1.2 Diophantine Equations

1. finding some solutions
2. trying modular cases
3. making some variables depended on other variables
4. putting constrains on variables which would make the problems easier
5. if there are infinitely many solutions, can you find a construction?

6. factorize (this is BIG)
7. In these problems, investigation, induction, recursion, constructions etc. are essentials

5.1.3 Sequences

5.1.4 NT Functions

5.1.5 Construction Problems

5.1.6 Sets satisfying certain properties

5.1.7 Other Small Techniques to Remember

1. $a - b$ stays invariant upon addition, just as $\frac{a}{b}$ stays invariant upon multiplication.

5.2 Lemmas

Lemma 5.2.1

Let $p \geq 5$ be a prime number. Prove that if $p \mid a^2 + ab + b^2$, then

$$p^3 \mid a + b^p - a^p - b^p$$

Lemma 5.2.2

If $b \geq 2$ and $b^n - 1 \mid a$ then there exist at least n non-zero digits in the representation of a in base b

Lemma 5.2.3

There are infinitely many primes p for every non-square integers a such that a is a non-quadratic residue $\pmod{p}$

Theorem 5.2.4: Wythoff Array

An infinite array of Positive integers such that every row of it forms Fibonacci-Type sequences.

Theorem 5.2.5: Frobenius Coin Problem

For any integers $a_1, a_2 \dots a_n$ such that $\gcd(a_1, a_2 \dots a_n) = 1$, there exists positive integers m such that for any integer $M \geq m$, there are non-negative integers $b_1, b_2 \dots b_n$ such that

$$\sum_{i=1}^{n} b_i a_i = M$$

If $n = 2$ then $m = a_1 a_2 - a_1 - a_2$.

If $n \geq 2$, then there doesn't exist an explicit formula, but if $\{a_i\}$ are in arithmetic progression, $(a_i = a_1 + (i-1)d)$ then

$$m = \left\lfloor \frac{a-2}{n-1} \right\rfloor a + (d-1)(a-1) - 1$$

Theorem 5.2.6: Beatty's Theorem

If a, b are two integers such that $\frac{1}{a} + \frac{1}{b} = 1$, then the two sets $\{\lfloor ia \rfloor\}$ and $\{\lfloor ib \rfloor\}$, where i are the positive integers, form a partition of the set of natural numbers.

Theorem 5.2.7: Cyclotomic Formulas

Lemma 5.2.8

Let x, y be co-prime. Then

$$\gcd(z, xy) = \gcd(z, x)\gcd(z, y) = \gcd(z \bmod x, x)\gcd(z \bmod y, y)$$

$$\implies \gcd(r(a, b), xy)) = \gcd(a, x)\gcd(b, y)$$

(here $r(a, b)$ denotes the smallest integer that satisfies $r(a, b) \equiv a \bmod x,\ r(a, b) \equiv b \bmod y$)

Lemma 5.2.9

Let $0 < a_1 < a_2 < \Delta\Delta\Delta < a_(mn + 1)$ be $mn + 1$ integers. Prove that you can select either $m + 1$ of them no one of which divides any other, or $n + 1$ of them each dividing the following one.

Theorem 5.2.10: Prime divisors of an integer polynomial

If $P(x) \in \mathbb{Z}[x]$, then the set of primes, $P = \{p : p \mid P(x)\}$ is infinite.

5.2.1 Quadratic Residue

Theorem 5.2.11:

There are exactly $\frac{p-1}{2}$ quadratic residue classes $\mod p$

Theorem 5.2.12:

Let p be an odd prime. Then,

1. The product of two quadratic residue is a quadratic residue.
2. The product of two quadratic non-residue is a quadratic residue.
3. The product of a qudratic residue and a quadratic non-residue is a quadratic non-residue

Legendre Symbol We call $\left(\frac{a}{p}\right)$ the Legendre symbol for an integer a and a prime p where,

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p|a \\ 1 & \text{if } a \text{ is a qr of } p \\ -1 & \text{otherwise} \end{cases}$$

Theorem 5.2.13:

For an odd prime p and any two integers a, b, we have $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$

Theorem 5.2.14: Euler's Criterion

Let p be an odd prime. Then,

$$a^{\frac{p-1}{2}} \equiv \left(\frac{a}{p}\right) (mod\ p)$$

Theorem 5.2.15:

Let $(a, b) = 1$. Then every prime divisors of $a^2 + b^2$ is either 2 or a prime of the form $4k + 1$.

Theorem 5.2.16: Gauss's Criterion

Let p be a prime number and a be an integer coprime to p. Let $\mu(a)$ be the number of integers $x \in \{a, a * 2, \dots a * \frac{p-1}{2}\}$ such that $x(mod\ p) > \frac{p}{2}$. Then

$$\left(\frac{a}{p}\right) = -1^{\mu(a)}$$

Theorem 5.2.17:

The smallest quadratic non-residue of an odd prime p is a prime which is less than $\sqrt{p} + 1$

Theorem 5.2.18: Quadratic Residue Law

Using the usual Legendre Symbol, for two prime numbers p, q we have:

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$$

Jacobi Symbol Let $a, n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$. We define Jacobi symbol as

$$\left(\frac{a}{n}\right) = \prod_{i=1}^{k} \left(\frac{a}{p_i}\right)^{\alpha_i}$$

Note 1: J

cobi symbol is not as accurate as Legendre symbol. $\left(\frac{a}{n}\right) = -1$ means that a is a quadratic non-residue of n, but $= 1$ doesnt necessarily mean that a is a quadratic residue of n.

Theorem 5.2.19:

Let $a, n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$, then a is a quadratic residue of n iff it is a quaadratic residue of every $p_i^{\alpha_i}$.

Theorem 5.2.20:

If an integer is a quadratic residue of every prime, then it is a square.

5.2.2 Modular Arithmatic Theorems and Useful Results

Theorem 5.2.21: Wolstenholme's Theorem

For all prime p the following relation is true:

$$p^2 \,\Big|\, 1 + \frac{1}{2} + \frac{1}{3} \dots \frac{1}{p-1} = \sum_{i=1}^{p-1} \frac{1}{i}$$

Corollary 5.2.21.1

$$p \,\Big|\, 1 + \frac{1}{2^2} + \frac{1}{3^2} \dots \frac{1}{(p-1)^2} = \sum_{i=1}^{p-1} \frac{1}{i^2}$$

Corollary 5.2.21.2

If $p > 3$ is a prime, then

$$\binom{2p}{p} \equiv 2 \ (mod\ p^3)$$

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$$\binom{2p-1}{p-1} \equiv 1 \ (mod\ p^3)$$

Lemma 5.2.22

$$\frac{1}{(p-i)!} \equiv (-1)^i (i-1)! \ (mod\ p)$$

$$\binom{p-1}{k} \equiv (-1)^k \ (mod\ p)$$

Problem 5.2.1: **E**

$$\binom{p^{n+1}}{p} \equiv p^n \ (mod\ p^{2n+3})$$

5.3 Orders Modulo a Prime and Related Stuffs

- Order Modulo a Prime - Evan Chen
- Zsigmondy's Theorem's Proof, has some useful lemmas
- Another source of the proof of Zsigmondy's Theorem

Möbius Function The Möbius function μ maps the natural numbers to the set $-1, 0, 1$. μ can be defined in multiple ways:

- Let ζ_i be the primitive roots of n^{th} cyclotomic polynomial, then $\mu(n) = \sum \zeta_i$
- $$\mu(n) = \begin{cases} 1 & \text{if } n \text{ is square free and has an even number of primes} \\ -1 & \text{if } n \text{ is square free and has an odd number of primes} \\ 0 & \text{if } n \text{ has a prime square divisor} \end{cases}$$

5.3.1 Cyclotomic Polynomials

Cyclotomic Polynomials $\Phi_n(x)$ is the polynomial with the primitive roots of $x^n - 1 = 0$.

Lemma 5.3.1: Product of Cyclotomic Polynomials

For any n we have,
$$x^n - 1 = \prod_{d|n} \Phi_d(x)$$

Lemma 5.3.2: Prime Divisors of Cyclotomic Polynomials

If $a \in \mathbb{Z}$ such that $\Phi_n(a) \neq 0$ and for some prime p,

$$\Phi_n(a) \equiv 0 \pmod{p}$$

Then either

- $p \equiv 1 \pmod{n}$, or
- $p|n$

Lemma 5.3.3

If p does not divide m, then

$$\Phi_{pm}(x)\Phi_m(x) = \Phi_m(x^p)$$

5.4 Primes

Problem 5.4.1: ISL 2013 N5 **M**

Fix an integer $k > 2$. Two players, called Ana and Banana, play the following game of numbers. Initially, some integer $n \geq k$ gets written on the blackboard. Then they take moves in turn, with Ana beginning. A player making a move erases the number m just written on the blackboard and replaces it by some number m' with $k \leq m' < m$ that is coprime to m. The first player who cannot move anymore loses.

An integer $n \geq k$ is called good if Banana has a winning strategy when the initial number is n, and bad otherwise.

Consider two integers $n, n' \geq k$ with the property that each prime number $p \leq k$ divides n if and only if it divides n'. Prove that either both n and n' are good or both are bad.

Idea Every idea that naturally follows lead to a solution, so after getting the idea of working on a single equivalence class is enough, we face the problem that "big primes" cause the trouble. So can we get rid of them by making some minimal number that arent "contaminated" by big primes? □

Problem 5.4.2: ISL 2014 N4 **EM**

Let $n > 1$ be a given integer. Prove that infinitely many terms of the sequence $(a_k)_{k\geq 1}$, defined by

$$a_k = \left\lfloor \frac{n^k}{k} \right\rfloor,$$

are odd. (For a real number x, $\lfloor x \rfloor$ denotes the largest integer not exceeding x.)

Idea First we take a prime, doesnt work, then we take two primes, one being 2 (Since we need it in the bottom), but that doesn't work either. Then take n instead of 2, because we want the 2's in the numerator vanish. Surprisingly this works. □

5.5 NT Functions and Polynomials

Problem 5.5.1: ISL 2013 N1 **E**

Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m and n.

Idea Go with the flow. □

Problem 5.5.2: ISL 2010 N5 **M**

Find all functions $g : \mathbb{N} \to \mathbb{N}$ such that

$$(g(m) + n)(g(n) + m)$$

is a perfect square for all $m, n \in \mathbb{N}$.

Idea Playig around with some primes give us that for every "big" primes, we need to have every residue class present in the range of g. Now with this fact, we can prove the injectivity as well. Now we want to show that $g(n+1) = g(n) + 1$. How to show that? We can show that by saying that no prime p exits such that $p|g(n+1) - g(n)$, in other words, again the residue classes. □

5.6 Diophantine Equations

Problem 5.6.1: APMO 1999 P4 **EM**

Determine all pairs (a, b) of integers with the property that the numbers $a^2 + 4b$ and $b^2 + 4a$ are both perfect squares.

Idea Easy case work assuming positive or negative values for a, b. $\square$

Problem 5.6.2: All Squares **E**

Prove that there are infinitely many pairs of positive integers (x, y) satisfying that $x + y, x - y, xy + 1$ are all perfect squares.

Idea Just work it out. $\square$

Problem 5.6.3: ISL 2010 N3 **E**

Find the smallest number n such that there exist polynomials $f_1, f_2, \ldots, f_n$ with rational coefficients satisfying

$$x^2 + 7 = f_1(x)^2 + f_2(x)^2 + \ldots + f_n(x)^2 .$$

Idea Find the obvious answer, which is very small, so we can probably case work it out. We find the case for 3 a bit challenging. But we plan to show that $7a^2$ can't be written as a sum of 3 squares. Which numbers can be written as the sum of 3 squares? Investigate... $\square$

Problem 5.6.4: ISL 2014 N5 **E**

Find all triples (p, x, y) consisting of a prime number p and two positive integers x and y such that $x^{p-1} + y$ and $x + y^{p-1}$ are both powers of p.

Idea The old school trick, replacement. Assume $x < y$ and replace y, and then compare the power of p. $\square$

5.7 Problems

Problem 5.7.1: **E**

Let n be an odd integer, and let $S = \{x \mid 1 \leq x \leq n,\ (x,n) = (x+1,n) = 1\}$. Prove that

$$\prod_{x \in S} x = 1\ (mod\ n)$$

Problem 5.7.2: Iran TST 2015, P4 **M**

Let n is a fixed natural number. Find the least k such that for every set A of k natural numbers, there exists a subset of A with an even number of elements which the sum of its members is divisible by n.

Idea Odd-Even, so lets first try for odd n's. It is quite easy.

So now, for evens, lets first try the simplest kind of evens. As we need a set with an even number of elements, this tells us to pair things up. We can try to partition A into pairs of e-e's and o-o's. This gives us our desired result. □

Problem 5.7.3: Iran TST 2015 P11 **M**

We call a permutation $(a_1, a_2 \ldots a_n)$ of the set $\{1, 2 \ldots n\}$ "good" if for any three natural numbers $i < j < k$,

$$n \nmid a_i + a_k - 2a_j$$

find all natural numbers $n \geq 3$ such that there exist a "good" permutation of the set $\{1, 2 \ldots n\}$.

Idea Looking for "possibilities" for the first element, we get some more restrictions for the values of other terms. □

Problem 5.7.4: ISL 2004 N2 **M**

The function $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(n) = \sum_{k=1}^{n} \gcd(k, n)$, $n \in \mathbb{N}$.

1. Prove that $f(mn) = f(m)f(n)$ for every two relatively prime $m, n \in \mathbb{N}$.
2. Prove that for each $a \in \mathbb{N}$ the equation $f(x) = ax$ has a solution.
3. Find all $a \in \mathbb{N}$ such that the equation $f(x) = ax$ has a unique solution.

Idea Why not casually try to multiply $f(m)$ and $f(n)$?? And also find a formula for $n =$ prime power. $\square$

Problem 5.7.5: Balkan MO 2017 P1 **S**

Find all ordered pairs of positive integers (x, y) such that: $x^3 + y^3 = x^2 + 42xy + y^2$.

Problem 5.7.6: Balkan MO 2017 P3 **E**

Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$n + f(m) \mid f(n) + nf(m)$$

for all $n, m \in \mathbb{N}$.

Idea Check sizes and bound for large n. $\square$

Problem 5.7.7: Iran MO 3rd Round N3 **E**

Let $p > 5$ be a prime number and $A = \{b_1, b_2 \dots b_{\frac{p-1}{2}}\}$ be the set of all quadratic residues modulo p, excluding zero. Prove that there doesn't exist any natural a, c satisfying $\gcd(ac, p) = 1$ such that set $B = \{x \mid x = ay + c, y \in A\}$ and set A are disjoint modulo p.

Idea Sum it up. $\square$

Idea For every integer a, b and prime p such that, $\gcd(a, p) = \gcd(b, p) = 1$, there exist (x, y) such that $x^2 \equiv ay^2 + c \pmod p$. $\square$

Idea For a prime p, there exists an integer x such that x and $x + 1$ both are quadratic residues $\pmod p$. $\square$

Problem 5.7.8: All Russia 2014 P9.5 **E**

Define $m(n)$ to be the greatest proper natural divisor of n. Find all $n \in \mathbb{N}$ such that $n + m(n)$ is a power of 10.

Problem 5.7.9: ISL 2000 N1 **E**

Determine all positive integers $n \geq 2$ that satisfy the following condition: for all a and b relatively prime to n we have $a \equiv b \pmod{n}$ iff $ab \equiv 1 \pmod{n}$.

Idea Don't forget the details. □

Problem 5.7.10: ISL 2000 N3 **M**

Does there exist a positive integer n such that n has exactly 2000 prime divisors (not necessarily distinct) and $n \mid 2^n + 1$?

Idea Goriber Bondhu Induction. As the number 2000 seems so out of the place, we replace 2000 by k. Now suppose that for some k, the condition works. For simplicity let $k = p^i$ for some i, as it is quite clear that there is another prime q that divides $2^k + 1$, let $k' = kq$. So k' also satisfies the condition. So it is quite intuitive to think that for every x there exist some p and i for which $2^{p^i} + 1$ has x prime factors. So we search for such p. □

Problem 5.7.11: USAMO 2001 P5 **H**

Let S be a set of integers (not necessarily positive) such that

1. There exist $a, b \in S$ with $\gcd(a, b) = \gcd(a - 2, b - 2) = 1$;
2. If x and y are elements of S (possibly equal), then $x^2 - y \in S$

Prove that S is the set of all integers.

Idea One possible intuition could be trying to make the problem statement a little bit more stable, like the term $x^2 - y$ is not so symmetric. So trying to make it a little bit more symmetric can come handy. □

Idea If $c, x, y \in S$ then we can easily see that $A(x^2 - y^2) - c \in S$ for all $A \in \mathbb{Z}$. We take this a little too far and show that if $c, x, y, u, v \in S$, then $A(x^2 - y^2) + B(u^2 - v^2) - c \in S$ for all $A, B \in \mathbb{Z}$. So if we can find such x, y, u, v such that $\gcd(x^2 - y^2, u^2 - v^2) = 1$, we are almost done by Frobenius Coin Problem. So we start looking for integers that can be obtained from a, b. After some playing around we get the feeling (or maybe not) that we need one more pair. Again playing around for some time we find three pairs. FCP gives us an upper bound for all integers that are not in S. Easily we include them in S. □

Problem 5.7.12: Vietnam TST 2017 P2 **M**

For each positive integer n, set $x_n = \binom{2n}{n}$

1. Prove that if $\frac{2017^k}{2} < n < 2017^k$ for some positive integer k then $2017 \mid x_n$.
2. Find all positive integer $h > 1$ such that there exist positive integers N, T such that the sequence (x_n) for $n > N$, is periodic $(bmodh)$ with period T.

Problem 5.7.13: Vietnam 2017 TST P6 **H**

For each integer $n > 0$, a permutation $(a_1, a_2 \dots a_{2n})$ of $1, 2 \dots 2n$ is called *beautiful* if for every $1 \leq i < j \leq 2n$, $a_i + a_{n+i} = 2n + 1$ and $a_i - a_{i+1} \not\equiv a_j - a_{j+1} \pmod{2n+1}$ (suppose that $a_i = a_{2n+i}$).

1. For $n = 6$, point out a *beautiful* permutation.
2. Prove that there exists a *beautiful* permutation for every n.

Idea Trial and Error. □

Problem 5.7.14: BrMO 2008 **M**

Find all sequences $a_{i}{}_{i=0}^{\infty}$ of rational numbers which follow the following conditions:

1. $a_n = 2a_{n-1}^2 - 1$ for all $n > 0$
2. $a_i = a_j$ for some $i, j > 0$, $i \neq j$

Idea Trial and Error. Don't forget that you only need the numbers to be rational. □

Problem 5.7.15: USA TST 2000 P4 **E**

Let n be a positive integer. Prove that

$$\sum_{i=0}^{n}\binom{n}{i}^{-1}=\frac{n+1}{2^{n+1}}\left(\sum_{i=0}^{n+1}\frac{2^i}{i}\right)$$

Idea *Positive Integer*, nuff said. □

Problem 5.7.16: USA TST 2000 P3 **EH NOT TRIED**

Let p be a prime number. For integers r, s such that $rs(r^2-s^2)$ is not divisible by p, let $f(r,s)$ denote the number of integers $1 \le n \le p$ such that $\{\frac{rn}{p}\}$ and $\{\frac{sn}{p}\}$ are either both less than $\frac{1}{2}$ or both greater than $\frac{1}{2}$. Prove that there exists $N>0$ such that for $p \ge N$ and all r, s,

$$\left\lceil\frac{(p-1)}{3}\right\rceil \le f(r,s) \le \left\lfloor\frac{2(p-1)}{3}\right\rfloor$$

Problem 5.7.17: China TST 2005 **EH NOT TRIED**

Let n be a positive integer and $f_n = 2^{2^n}+1$. Prove that for all $n \ge 3$, there exists a prime factor of f_n which is larger than $2^{n+2}(n+1)$ [Stronger Version: $2^{n+4}(n+1)$].

Problem 5.7.18: IRAN TST 2009 P2 **H NOT TRIED**

Let a be a fixed natural number. Prove that the set of prime divisors of $2^{2^n}+a$ for $n=1,2,3\ldots$ is infinite.

Problem 5.7.19: USAMO 2004 P2 **E**

Suppose $a_1, a_2 \ldots a_n$ are integers whose greatest common divisor is 1. Let S be a set of integers with the following properties:

1. $a_i \in S$
2. For $i, j = 1, 2 \ldots n$ (not necessarily distinct), $a_i - a_j \in S$.
3. For any integers $x, y \in S$, if $x + y \in S$, then $x - y \in S$.

Prove that S must be equal to the set of all integers.

Idea First we see that if $d = \gcd(x, y)$ and $x, y \in S$ then $d \in S$. So all we have to do is to find two x, y with $d = 1$. $\square$

Problem 5.7.20: USAMO 2008 P1 **E**

Prove that for each positive integer n, there are pairwise relatively prime integers $k_0, k_1 \ldots k_n$, all strictly greater than 1, such that $k_0 k_1 \ldots k_{n-1}$ is the product of two consecutive integers.

Idea *Positive Integer n* nuff said. $\square$

Problem 5.7.21: USAMO 2007 P5 **M**

Prove that for every nonnegative integer n, the number $7^{7^n} + 1$ is the product of at least $2n + 3$ (not necessarily distinct) primes.

Idea When you try to apply induction, always name the hypothesis. In this case name $7^{7^n} + 1 = a_n$. And try to relate a_n with a_{n+1}. $\square$

Problem 5.7.22: IMO 1998 P3 **E**

For any positive integer n, let $\tau(n)$ denote the number of its positive divisors (including 1 and itself). Determine all positive integers m for which there exists a positive integer n such that $\frac{\tau(n^2)}{\tau(n)} = m$.

Idea Easy go with the flow. ☐

Problem 5.7.23: USA TST 2002 P2 **M**

Let $p > 5$ be a prime number. For any integer x, define

$$f_p(x) = \sum_{k=1}^{p-1} \frac{1}{(px+k)^2}$$

Prove that for any two integers x, y,

$$p^3 \mid f(x) - f(y)$$

Idea If there is $f(x) - f(y)$ for all x, y, then always make one of those equal to 0 or in other words make one side constant ☐

Idea All fractional sum problems should be solved by some expression manipulation. ☐

Idea Sometimes try adding things up. ☐

Problem 5.7.24: USAMO 2012 P4 **E**

Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ (where $\mathbb{Z}^+$ is the set of positive integers) such that $f(n!) = f(n)!$ for all positive integers n and such that $(m-n)$ divides $f(m) - f(n)$ for all distinct positive integers m, n.

Positive Integer n nuff said.

Problem 5.7.25: USAMO 2013 P5 **M**

Given positive integers m and n, prove that there is a positive integer c such that the numbers cm and cn have the same number of occurrences of each non-zero digit when written in base ten.

What if we make cm and cn have the same digits, occuring same number of time, when the digits are sorted? This will make the things a whole lot easier. Again for more simplicity, what if the arrangement of the digits in both of these numbers are "almost" the same? Like, if the digits are in blocks and if decomposed into such blocks, we get the same set for both of those problems? This idea of simplicity is more that enough to "Simplify" a problem. Call this strategry Simplify.

Problem 5.7.26: ISL 2015 N4 **M**

Suppose that $a_0, a_1, \cdots$ and $b_0, b_1, \cdots$ are two sequences of positive integers such that $a_0, b_0 \geq 2$ and

$$a_{n+1} = \gcd(a_n, b_n) + 1, \qquad b_{n+1} = \operatorname{lcm}(a_n, b_n) - 1.$$

Show that the sequence a_n is eventually periodic; in other words, there exist integers $N \geq 0$ and $t > 0$ such that $a_{n+t} = a_n$ for all $n \geq N$.

Like most NT probs, pure investigation. We see that the function a_n is mostly decreasing, but it is increasing as well. But the increase rate is not greater than the decrease rate. After some time playing around, we see that the value of a_n rises gradually and then suddenly drops. But the peak value of a_n doesn't seem to increase. Well, this is true, we prove that. After that, we are mostly done, we just show that eventually the least value of a_n becomes stable as well. We use the intuitions we get from working around.

Problem 5.7.27: ISL 2015 N3 **E-M**

Let m and n be positive integers such that $m > n$. Define $x_k = \frac{m+k}{n+k}$ for $k = 1, 2, \ldots, n+1$. Prove that if all the numbers $x_1, x_2, \ldots, x_{n+1}$ are integers, then $x_1 x_2 \ldots x_{n+1} - 1$ is divisible by an odd prime.

As there are powers of 2, we use the powers of 2.

Problem 5.7.28: USA TST 2018 P1 **E**

Let $n \geq 2$ be a positive integer, and let $\sigma(n)$ denote the sum of the positive divisors of n. Prove that the n^{th} smallest positive integer relatively prime to n is at least $\sigma(n)$, and determine for which n equality holds.

even-odd

Pretty starightforward, as the ques suggests, there are fewer than n coprimes in the interval $[1, \sigma(n)]$, we directly show this. [as consttructions don't seem to be trivial/ easy to get] Inclusion/Exclusion all the way. But remember, not checking floors can get you doomed.

Problem 5.7.29: APMO 2014 P3 **M**

Find all positive integers n such that for any integer k there exists an integer a for which a^3+a-k is divisible by n.

factorize, quadratic residue, complete residue class

Idea You have to show that the set $\{x \mid x \equiv a^3+a \ (mod\ p)$ is equal to the set $\{1 \ldots p-1\}$. So some properties shared by a set as a whole must be satisfied by both of the sets. The quickest such properties that come into mind are the summation of the set and the product of the set. While the former doesnt help out much, the later seems promising. Where we get a nice relation that we have to satisfy:

$$\prod_{i=0}^{p-1} a^2 + 1 \equiv 1 \ (mod\ p)$$

One way of concluding from here is to use the quadratic residue ideas, or using the fact that $x^2 + 1 = (x + i)(x - i)$. The later requires some higher tricks tho. □

Idea The most natural way must be to show that for any prime p we will find two integers $p \nmid (a - b)$, and $p \mid a^2 + b^2 + ab + 1$, factorizing the later and getting two squares and a constant gives us our desired result. □

Problem 5.7.30: APMO 2014 P1 **M**

For a positive integer m denote by $S(m)$ and $P(m)$ the sum and product, respectively, of the digits of m. Show that for each positive integer n, there exist positive integers $a_1, a_2, \ldots, a_n$ satisfying the following conditions:

$$S(a_1) < S(a_2) < \cdots < S(a_n) \text{ and } S(a_i) = P(a_{i+1}) \quad (i = 1, 2, \ldots, n).$$

(We let $a_{n+1} = a_1$.)

Idea 1 is the only integer that increases the sum, but doesnt change the product. This may seem trivial, but on problems like this where both the sum and the product of the digits of a number are concerned, this tiny little fact can change everything. □

Problem 5.7.31: RMM 2018 P4 **E**

Let a, b, c, d be positive integers such that $ad \neq bc$ and $gcd(a, b, c, d) = 1$. Let S be the set of values attained by $\gcd(an + b, cn + d)$ as n runs through the positive integers. Show that S is the set of all positive divisors of some positive integer.

Problem 5.7.32: ISL 2011 N1 **E**

For any integer $d > 0$, let $f(d)$ be the smallest possible integer that has exactly d positive divisors (so for example we have $f(1) = 1, f(5) = 16$, and $f(6) = 12$). Prove that for every integer $k \geq 0$ the number $f\left(2^k\right)$ divides $f\left(2^{k+1}\right)$.

Idea Construct the function for 2^n. $\square$

Problem 5.7.33: ISL 2004 N1 **E**

Let $\tau(n)$ denote the number of positive divisors of the positive integer n. Prove that there exist infinitely many positive integers a such that the equation $\tau(an) = n$ does not have a positive integer solution n.

Idea Infinitely many, divisor, what else should come to mind except prime powers... $\square$

Problem 5.7.34: USAMO 2018 P4 **E**

Let p be a prime number and let $a_1, a_2 \dots a_p$ be integers. Prove that there exists an integer k s.t. the $S = \{a_i + ik\}$ has at least $\frac{p}{2}$ elements modulo p

Idea As the only thing that is holding ourselves down is the equivalence of any two elements of S, we investigate it furthur. It is a good idea to represent by graphs. $\square$

Problem 5.7.35: ISL 2009 N1 **E**

Let n be a positive integer and let $a_1, a_2, a_3, \dots, a_k$ $(k \geq 2)$ be distinct integers in the set $1, 2, \dots, n$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, 2, \dots, k - 1$. Prove that n does not divide $a_k(a_1 - 1)$.

Problem 5.7.36: ISL 2009 N2 **E**

A positive integer N is called balanced, if $N = 1$ or if N can be written as a product of an even number of not necessarily distinct primes. Given positive integers a and b, consider the polynomial P defined by $P(x) = (x + a)(x + b)$.

1. Prove that there exist distinct positive integers a and b such that all the number $P(1)$, $P(2)$,..., $P(50)$ are balanced.

2. Prove that if $P(n)$ is balanced for all positive integers n, then $a = b$

Problem 5.7.37: USA TSTST 2015 P5 **EM**

Let $\varphi(n)$ denote the number of positive integers less than n that are relatively prime to n. Prove that there exists a positive integer m for which the equation $\varphi(n) = m$ has at least 2015 solutions in n.

Idea When does the equation has multiple solutions? Suppose $m = \prod_{i=1}^{t} p_i^{\alpha_i}(p_i - 1)$ then $\Phi(n) = m$ has multiple solutions if for some $p's$ in m, their product is one less from another prime. Which gives us necessary intuition to construct a m for which there are A LOT of solutions for the equation. $\square$

Problem 5.7.38: Iran 2018 T1P1 **M**

Let $A_1, A_2, ..., A_k$ be the subsets of $\{1, 2, 3, ..., n\}$ such that for all $1 \leq i, j \leq k$:$A_i \cap A_j \neq \varnothing$. Prove that there are n distinct positive integers $x_1, x_2, ..., x_n$ such that for each $1 \leq j \leq k$:

$$lcm_{i \in A_j} \{x_i\} > lcm_{i \notin A_j} \{x_i\}$$

Idea Main part of the problem is to notice that the first $|A_i|$ columns of the matrix has 1 from all of the rows. Which triggers the idea of giving one prime to every row, and count x_i's with them. $\square$

Problem 5.7.39: Iran TST 2018 T2P4 **E**

Call a positive integer "useful but not optimized " (!), if it can be written as a sum of distinct powers of 3 and powers of 5. Prove that there exist infinitely many positive integers which they are not "useful but not optimized".
e.g. 37 is a "useful but not optimized" number since $37 = (3^0 + 3^1 + 3^3) + (5^0 + 5^1)$

Problem 5.7.40: ISL 2014 N2 **E**

Determine all pairs (x, y) of positive integers such that

$$\sqrt[3]{7x^2 - 13xy + 7y^2} = |x - y| + 1.$$

Idea Always factor, before everything else. $\square$

Problem 5.7.41: ISL 2014 N1 **M**

Let $n \geq 2$ be an integer, and let A_n be the set

$$A_n = \{2^n - 2^k \mid k \in \mathbb{Z},\, 0 \leq k < n\}.$$

Determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of A_n .

Idea In every problem, conjecture from smaller case, and check if the conjecture is true in bigger cases. □

Problem 5.7.42: ISL 2002 N1 **E**

What is the smallest positive integer t such that there exist integers $x_1, x_2, \ldots, x_t$ with

$$x_1^3 + x_2^3 + \ldots + x_t^3 = 2002^{2002}\,?$$

Idea $1000 + 1000 + 1 + 1$. □

Problem 5.7.43: ISL 2002 N2 **M**

Let $n \geq 2$ be a positive integer, with divisors $1 = d_1 < d_2 < \ldots < d_k = n$. Prove that $d_1d_2 + d_2d_3 + \ldots + d_{k-1}d_k$ is always less than n^2, and determine when it is a divisor of n^2.

Idea In problems with all of the divisors of n involved, it is a good choice to substitute $d_i = \frac{n}{d_{k-i+1}}$. That way, you get the exact same set, represented differently, with n involved. And $\sum_{i=1}^{n} \frac{1}{i*(i+1)} = \frac{n}{n+1}$ □

Problem 5.7.44: Japan MO 2017 P2, TST Mock 2018 **M**

Let N be a positive integer. There are positive integers $a_1, a_2, \cdots, a_N$ and all of them are not multiples of 2^{N+1}. For each integer $n \geq N+1$, set a_n as below:
If the remainder of a_k divided by 2^n is the smallest of the remainder of $a_1, \cdots, a_{n-1}$ divided by 2^n, set $a_n = 2a_k$. If there are several integers k which satisfy the above condition, put the biggest one.
Prove the existence of a positive integer M which satisfies $a_n = a_M$ for $n \geq M$.

Idea Things must go far... □

Problem 5.7.45: ISL 2002 N3 **E**

Let $p_1, p_2, \dots, p_n$ be distinct primes greater than 3. Show that $2^{p_1 p_2 \cdots p_n} + 1$ has at least 4^n divisors.

Problem 5.7.46: Japan MO 2017 P1 **E**

Let a, b, c be positive integers. Prove that $lcm(a, b) \neq lcm(a + c, b + c)$.

Problem 5.7.47: ISL 2009 N3 **M**

Let f be a non-constant function from the set of positive integers into the set of positive integer, such that $a - b$ divides $f(a) - f(b)$ for all distinct positive integers a, b. Prove that there exist infinitely many primes p such that p divides $f(c)$ for some positive integer c.

Idea Notice if $f(1) = 1$, we can easily prove the result, so assume that $f(1) = c$. Now see that, if we can somehow, create another function g from the domain and range of f with the same properties as f, and with $g(1) = 1$, we will be done. So to do this, we need to perform some kind of division by c. □

Problem 5.7.48: ARO 2018 P9.1 **E**

Suppose $a_1, a_2, \dots$ is an infinite strictly increasing sequence of positive integers and $p_1, p_2, \dots$ is a sequence of distinct primes such that $p_n \mid a_n$ for all $n \geq 1$. It turned out that $a_n - a_k = p_n - p_k$ for all $n, k \geq 1$. Prove that the sequence $(a_n)_n$ consists only of prime numbers.

Problem 5.7.49: ARO 2018 P10.4 **H**

Initially, a positive integer is written on the blackboard. Every second, one adds to the number on the board the product of all its nonzero digits, writes down the results on the board, and erases the previous number. Prove that there exists a positive integer which will be added infinitely many times.

Proof. Using Bounding and the

Problem 5.7.50: APMO 2008 P4 **E**

Consider the function $f : \mathbb{N}_0 \to \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of all non-negative integers, defined by the following conditions :

$$f(0) = 0, \quad f(2n) = 2f(n) \quad \text{and} \quad f(2n+1) = n + 2f(n) \quad \text{for all } n \geq 0$$

1. Determine the three sets $L = \{n | f(n) < f(n+1)\}$, $E = \{n | f(n) = f(n+1)\}$, and $G = \{n | f(n) > f(n+1)\}$.
2. For each $k \geq 0$, find a formula for $a_k = \max\{f(n) : 0 \leq n \leq 2^k\}$ in terms of k.

Problem 5.7.51: ISL 2009 N4 **E**

Find all positive integers n such that there exists a sequence of positive integers a_1, a_2,…, a_n satisfying:

$$a_{k+1} = \frac{a_k^2 + 1}{a_{k-1} + 1} - 1$$

for every k with $2 \leq k \leq n - 1$.

Idea Rewriting the condition, and doing some parity check. Then assuming the contrary and taking extreme case. □

Problem 5.7.52: ISL 2015 N2 **E**

Let a and b be positive integers such that $a! + b!$ divides $a!b!$. Prove that $3a \geq 2b + 2$.

Idea Size Chase □

Problem 5.7.53: USA TST 2019 P2 **MH**

Let $\mathbb{Z}/n\mathbb{Z}$ denote the set of integers considered modulo n (hence $\mathbb{Z}/n\mathbb{Z}$ has n elements). Find all positive integers n for which there exists a bijective function $g : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, such that the 101 functions

$$g(x), \quad g(x) + x, \quad g(x) + 2x, \quad \ldots, \quad g(x) + 100x$$

are all bijections on $\mathbb{Z}/n\mathbb{Z}$.

Idea A very nice problem. We get the motivation by trying the cases for 2, 3 replacing 101. In the case of 2, we just consider the sum $\sum g(x)$. We get that $2 \nmid n$. So in the case of 3, we conjecture that $3 \nmid n$. But we can't prove this similarly as before. Whats the most common 'sum-type' invariant after the normal sum? Sum of the Squares.

Now that we have proved that $(6, n) = 1$, most probably our conjecture is correct. So lets try for any k, we need to show that $(k!, n) = 1$. In the case of 3, we used the 2nd power sum. So probably to prove that $k \nmid n$ we need to take the $(k-1)$th power sum.

Now the real thing begins. In the case of 3, doesn't the modular sum equation looks something like the first finite difference? This rings a bell that whenever there is powers involved, we should consider using the derivatives. For $(k-1)$th power, the $(k-1)$th derivative that is.

Another thing here, in the case of 3, we did something like

$$\sum (g(x) + 2x)^2 - \sum (g(x) + x)^2 \equiv \sum (g(x) + x)^2 - \sum (g(x))^2 \equiv 0 \pmod{n}$$

We try something similar again. □

Problem 5.7.54: Tuymaada 2016, P5 **E**

The ratio of prime numbers p and q does not exceed 2 ($p \neq q$). Prove that there are two consecutive positive integers such that the largest prime divisor of one of them is p and that of the other is q.

Problem 5.7.55: ISL 2016 N5 **M**

Let a be a positive integer which is not a perfect square, and consider the equation

$$k = \frac{x^2 - a}{x^2 - y^2}.$$

Let A be the set of positive integers k for which the equation admits a solution in $\mathbb{Z}^2$ with $x > \sqrt{a}$, and let B be the set of positive integers for which the equation admits a solution in $\mathbb{Z}^2$ with $0 \leq x < \sqrt{a}$. Show that $A = B$.

Idea Building x_2, y_2 from x_1, y_1 in the most simple and dumb way. □

Problem 5.7.56: Simurgh 2019 P1 **E**

Prove that there exists a 10×10 table of 'different' positive integers such that, if we define r_i, s_i be te product of the elements of the ith row and ith column respectively, then $r_1, r_2 \ldots r_{10}$ and $s_1, s_2 \ldots s_{10}$ form a non-constant arithmetic progression.

Idea We want to keep things simple. The simpliest arithmetic progression is the $a, 2a, 3a \ldots$ one. Again, we have $r_1 r_2 \ldots r_{10} = s_1 s_2 \ldots s_{10}$, we can wish that $r_i = s_i$. With these two assumptions, we can hope that we will find a table with the two sequences being a constant arithmetic progression. □

Problem 5.7.57: APMO 2017 P4 **EM**

Call a rational number r powerful if r can be expressed in the form $\frac{p^k}{q}$ for some relatively prime positive integers p, q and some integer $k > 1$. Let a, b, c be positive rational numbers such that $abc = 1$. Suppose there exist positive integers x, y, z such that $a^x + b^y + c^z$ is an integer. Prove that a, b, c are all powerful.

Problem 5.7.58: USA TST 2010 P9 **H**

Determine whether or not there exists a positive integer k such that $p = 6k + 1$ is a prime and

$$\binom{3k}{k} \equiv 1 \pmod p$$

Idea $\binom{3k}{k} \equiv 1 \implies \binom{3k}{2k} \equiv 1$ which implies,

$$\binom{3k}{0} + \binom{3k}{k} + \binom{3k}{2k} + \binom{3k}{3k} \equiv 4(\mod p)$$

Which gives the idea to find how the following term works in mod p

$$\sum_{i=0}^{\infty} \binom{n}{ki} \text{ for any arbitrary } k$$

From **??** we know a nice way of representing it with the k^{th} roots of unity. Roots of unity are primitive roots mod prime. □

Chapter 6

PSets

Chapter 7

IMO Shortlist Algebra Problems A1-3 from year 1998-2017

7.1 A1

Problem 7.1.1:

Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n < 1$. Prove that

$$\frac{a_1 a_2 \cdots a_n \left[1 - (a_1 + a_2 + \cdots + a_n)\right]}{(a_1 + a_2 + \cdots + a_n)(1 - a_1)(1 - a_2) \cdots (1 - a_n)} \leq \frac{1}{n^{n+1}}.$$

Problem 7.1.2:

Let $n \geq 2$ be a fixed integer. Find the least constant C such the inequality

$$\sum_{i<j} x_i x_j \left(x_i^2 + x_j^2\right) \leq C \left(\sum_i x_i\right)^4$$

holds for any $x_1, \ldots, x_n \geq 0$ (the sum on the left consists of $\binom{n}{2}$ summands). For this constant C, characterize the instances of equality.

Problem 7.1.3:

Let a, b, c be positive real numbers so that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Problem 7.1.4:

Let T denote the set of all ordered triples (p, q, r) of nonnegative integers. Find all functions $f : T \to \mathbb{R}$ satisfying

$$f(p,q,r) = \begin{cases} 0 & \text{if } pqr = 0, \\ 1 + \frac{1}{6}\{f(p+1, q-1, r) + f(p-1, q+1, r) & \\ +f(p-1, q, r+1) + f(p+1, q, r-1) & \\ +f(p, q+1, r-1) + f(p, q-1, r+1)\} & \text{otherwise} \end{cases}$$

for all nonnegative integers p, q, r.

Problem 7.1.5:

Find all functions f from the reals to the reals such that

$$f(f(x)+y) = 2x + f(f(y)-x)$$

for all real x, y.

Problem 7.1.6:

Let a_{ij} $i = 1, 2, 3$; $j = 1, 2, 3$ be real numbers such that a_{ij} is positive for $i = j$ and negative for $i \neq j$.
Prove the existence of positive real numbers c_1, c_2, c_3 such that the numbers

$$a_{11}c_1 + a_{12}c_2 + a_{13}c_3, \qquad a_{21}c_1 + a_{22}c_2 + a_{23}c_3, \qquad a_{31}c_1 + a_{32}c_2 + a_{33}c_3$$

are either all negative, all positive, or all zero.

Problem 7.1.7:

Let $n \geq 3$ be an integer. Let t_1, t_2, ..., t_n be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n}\right).$$

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

Problem 7.1.8:

Find all pairs of integers a, b for which there exists a polynomial $P(x) \in \mathbb{Z}[X]$ such that product $(x^2 + ax + b) \cdot P(x)$ is a polynomial of a form

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$$

Problem 7.1.9:

A sequence of real numbers $a_0,\ a_1,\ a_2, \ldots$ is defined by the formula

$$a_{i+1} = \lfloor a_i \rfloor \cdot \langle a_i \rangle \qquad \text{for} \quad i \geq 0;$$

here a_0 is an arbitrary real number, $\lfloor a_i \rfloor$ denotes the greatest integer not exceeding a_i, and $\langle a_i \rangle = a_i - \lfloor a_i \rfloor$. Prove that $a_i = a_{i+2}$ for i sufficiently large.

Problem 7.1.10:

Real numbers a_1, a_2, ..., a_n are given. For each i, $(1 \le i \le n)$, define

$$d_i = \max\{a_j \mid 1 \le j \le i\} - \min\{a_j \mid i \le j \le n\}$$

and let $d = \max\{d_i \mid 1 \le i \le n\}$.
(a) Prove that, for any real numbers $x_1 \le x_2 \le \cdots \le x_n$,

$$\max\{|x_i - a_i| \mid 1 \le i \le n\} \ge \frac{d}{2}. \qquad (*)$$

(b) Show that there are real numbers $x_1 \le x_2 \le \cdots \le x_n$ such that the equality holds in (*).

Problem 7.1.11:

Find all functions $f : (0, \infty) \mapsto (0, \infty)$ (so f is a function from the positive real numbers) such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers w, x, y, z, satisfying $wx = yz$.

Problem 7.1.12:

Find the largest possible integer k, such that the following statement is true: Let 2009 arbitrary non-degenerated triangles be given. In every triangle the three sides are coloured, such that one is blue, one is red and one is white. Now, for every colour separately, let us sort the lengths of the sides. We obtain

$$\begin{array}{rll} & b_1 \le b_2 \le \ldots \le b_{2009} & \text{the lengths of the blue sides} \\ & r_1 \le r_2 \le \ldots \le r_{2009} & \text{the lengths of the red sides} \\ \text{and} & w_1 \le w_2 \le \ldots \le w_{2009} & \text{the lengths of the white sides} \end{array}$$

Then there exist k indices j such that we can form a non-degenerated triangle with side lengths b_j, r_j, w_j.

Problem 7.1.13:

Find all function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following equality holds

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

where $\lfloor a \rfloor$ is greatest integer not greater than a.

Problem 7.1.14:

Given any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by s_A. Let n_A denote the number of pairs (i, j) with $1 \leq i < j \leq 4$ for which $a_i + a_j$ divides s_A. Find all sets A of four distinct positive integers which achieve the largest possible value of n_A.

Problem 7.1.15:

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a + b + c = 0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

(Here $\mathbb{Z}$ denotes the set of integers.)

Problem 7.1.16:

Let n be a positive integer and let $a_1, \ldots, a_{n-1}$ be arbitrary real numbers. Define the sequences $u_0, \ldots, u_n$ and $v_0, \ldots, v_n$ inductively by $u_0 = u_1 = v_0 = v_1 = 1$, and $u_{k+1} = u_k + a_k u_{k-1}$, $v_{k+1} = v_k + a_{n-k} v_{k-1}$ for $k = 1, \ldots, n-1$.
Prove that $u_n = v_n$.

Problem 7.1.17:

Let $a_0 < a_1 < a_2 \ldots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + a_2 + \cdots + a_n}{n} \leq a_{n+1}.$$

Problem 7.1.18:

Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \geq \frac{k a_k}{a_k^2 + (k-1)}$$

for every positive integer k. Prove that $a_1 + a_2 + \ldots + a_n \geq n$ for every $n \geq 2$.

Problem 7.1.19:

Let a, b, c be positive real numbers such that $\min(ab, bc, ca) \geq 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1.$$

Problem 7.1.20:

Let $a_1, a_2, \ldots a_n, k$, and M be positive integers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = k \quad \text{and} \quad a_1 a_2 \cdots a_n = M.$$

If $M > 1$, prove that the polynomial

$$P(x) = M(x+1)^k - (x+a_1)(x+a_2)\cdots(x+a_n)$$

has no positive roots.

7.2 A2

Problem 7.2.1:

Let $r_1, r_2, \ldots, r_n$ be real numbers greater than or equal to 1. Prove that

$$\frac{1}{r_1+1}+\frac{1}{r_2+1}+\cdots+\frac{1}{r_n+1} \geq \frac{n}{\sqrt[n]{r_1 r_2 \cdots r_n}+1}.$$

Problem 7.2.2:

The numbers from 1 to n^2 are randomly arranged in the cells of a $n \times n$ square ($n \geq 2$). For any pair of numbers situated on the same row or on the same column the ratio of the greater number to the smaller number is calculated. Let us call the characteristic of the arrangement the smallest of these $n^2(n-1)$ fractions. What is the highest possible value of the characteristic ?

Problem 7.2.3:

Let a, b, c be positive integers satisfying the conditions $b > 2a$ and $c > 2b$. Show that there exists a real number λ with the property that all the three numbers $\lambda a, \lambda b, \lambda c$ have their fractional parts lying in the interval $\left(\frac{1}{3}, \frac{2}{3}\right]$.

Problem 7.2.4:

Let $a_0, a_1, a_2, \ldots$ be an arbitrary infinite sequence of positive numbers. Show that the inequality $1 + a_n > a_{n-1}\sqrt[n]{2}$ holds for infinitely many positive integers n.

Problem 7.2.5:

Let $a_1, a_2, \ldots$ be an infinite sequence of real numbers, for which there exists a real number c with $0 \leq a_i \leq c$ for all i, such that

$$|a_i - a_j| \geq \frac{1}{i+j} \quad \text{for all } i,\ j \text{ with } i \neq j.$$

Prove that $c \geq 1$.

Problem 7.2.6:

Find all nondecreasing functions $f : \mathbb{R} \to \mathbb{R}$ such that (i) $f(0) = 0, f(1) = 1$; (ii) $f(a) + f(b) = f(a)f(b) + f(a+b-ab)$ for all real numbers a, b such that $a < 1 < b$.

Problem 7.2.7:

Let $a_0,\ a_1,\ a_2,\ \ldots$ be an infinite sequence of real numbers satisfying the equation $a_n = |a_{n+1} - a_{n+2}|$ for all $n \geq 0$, where a_0 and a_1 are two different positive reals.
Can this sequence $a_0,\ a_1,\ a_2,\ \ldots$ be bounded?

Problem 7.2.8:

We denote by $\mathbb{R}^+$ the set of all positive real numbers.
Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ which have the property:

$$f(x)f(y) = 2f(x + yf(x))$$

for all positive real numbers x and y.

Problem 7.2.9:

The sequence of real numbers $a_0, a_1, a_2, \ldots$ is defined recursively by

$$a_0 = -1, \qquad \sum_{k=0}^{n} \frac{a_{n-k}}{k+1} = 0 \quad \text{for} \quad n \geq 1.$$

Show that $a_n > 0$ for all $n \geq 1$.

Problem 7.2.10:

Consider those functions $f : \mathbb{N} \mapsto \mathbb{N}$ which satisfy the condition

$$f(m+n) \geq f(m) + f(f(n)) - 1$$

for all $m, n \in \mathbb{N}$. Find all possible values of $f(2007)$.

Problem 7.2.11:

(a) Prove that

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$$

for all real numbers x, y, z, each different from 1, and satisfying $xyz = 1$.
(b) Prove that equality holds above for infinitely many triples of rational numbers x, y, z, each different from 1, and satisfying $xyz = 1$.

Problem 7.2.12:

Let a, b, c be positive real numbers such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=a+b+c$. Prove that:

$$\frac{1}{(2a+b+c)^2}+\frac{1}{(a+2b+c)^2}+\frac{1}{(a+b+2c)^2}\le\frac{3}{16}.$$

Problem 7.2.13:

Let the real numbers a, b, c, d satisfy the relations $a+b+c+d=6$ and $a^2+b^2+c^2+d^2=12$. Prove that

$$36\le 4\left(a^3+b^3+c^3+d^3\right)-\left(a^4+b^4+c^4+d^4\right)\le 48.$$

Problem 7.2.14:

Determine all sequences $(x_1, x_2, \ldots, x_{2011})$ of positive integers, such that for every positive integer n there exists an integer a with

$$\sum_{j=1}^{2011} j x_j^n = a^{n+1}+1$$

Problem 7.2.15:

Let $\mathbb{Z}$ and $\mathbb{Q}$ be the sets of integers and rationals respectively. a) Does there exist a partition of $\mathbb{Z}$ into three non-empty subsets A, B, C such that the sets $A+B, B+C, C+A$ are disjoint? b) Does there exist a partition of $\mathbb{Q}$ into three non-empty subsets A, B, C such that the sets $A+B, B+C, C+A$ are disjoint?
Here $X+Y$ denotes the set $\{x+y : x\in X, y\in Y\}$, for $X, Y\subseteq\mathbb{Z}$ and for $X, Y\subseteq\mathbb{Q}$.

Problem 7.2.16:

Prove that in any set of 2000 distinct real numbers there exist two pairs $a>b$ and $c>d$ with $a\neq c$ or $b\neq d$, such that

$$\left|\frac{a-b}{c-d}-1\right|<\frac{1}{100000}.$$

Problem 7.2.17:

Define the function $f : (0, 1) \to (0, 1)$ by

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{if } x < \frac{1}{2} \\ x^2 & \text{if } x \geq \frac{1}{2} \end{cases}$$

Let a and b be two real numbers such that $0 < a < b < 1$. We define the sequences a_n and b_n by $a_0 = a, b_0 = b$, and $a_n = f(a_{n-1})$, $b_n = f(b_{n-1})$ for $n > 0$. Show that there exists a positive integer n such that

$$(a_n - a_{n-1})(b_n - b_{n-1}) < 0.$$

Problem 7.2.18:

Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$.

Problem 7.2.19:

Find the smallest constant $C > 0$ for which the following statement holds: among any five positive real numbers a_1, a_2, a_3, a_4, a_5 (not necessarily distinct), one can always choose distinct subscripts i, j, k, l such that

$$\left| \frac{a_i}{a_j} - \frac{a_k}{a_l} \right| \leq C.$$

Problem 7.2.20:

Let q be a real number. Gugu has a napkin with ten distinct real numbers written on it, and he writes the following three lines of real numbers on the blackboard:
In the first line, Gugu writes down every number of the form $a - b$, where a and b are two (not necessarily distinct) numbers on his napkin. In the second line, Gugu writes down every number of the form qab, where a and b are two (not necessarily distinct) numbers from the first line. In the third line, Gugu writes down every number of the form $a^2 + b^2 - c^2 - d^2$, where a, b, c, d are four (not necessarily distinct) numbers from the first line.
Determine all values of q such that, regardless of the numbers on Gugu's napkin, every number in the second line is also a number in the third line.

7.3 A3

Problem 7.3.1:

Let x, y and z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

Problem 7.3.2:

A game is played by n girls ($n \geq 2$), everybody having a ball. Each of the $\binom{n}{2}$ pairs of players, is an arbitrary order, exchange the balls they have at the moment. The game is called nice nice if at the end nobody has her own ball and it is called tiresome if at the end everybody has her initial ball. Determine the values of n for which there exists a nice game and those for which there exists a tiresome game.

Problem 7.3.3:

Find all pairs of functions $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(x + g(y)\right) = xf(y) - yf(x) + g(x) \quad \text{for all } x, y \in \mathbb{R}.$$

Problem 7.3.4:

Let $x_1, x_2, \ldots, x_n$ be arbitrary real numbers. Prove the inequality

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+\cdots+x_n^2} < \sqrt{n}.$$

Problem 7.3.5:

Let P be a cubic polynomial given by $P(x) = ax^3 + bx^2 + cx + d$, where a, b, c, d are integers and $a \neq 0$. Suppose that $xP(x) = yP(y)$ for infinitely many pairs x, y of integers with $x \neq y$. Prove that the equation $P(x) = 0$ has an integer root.

Problem 7.3.6:

Consider pairs of the sequences of positive real numbers

$$a_1 \geq a_2 \geq a_3 \geq \cdots, \qquad b_1 \geq b_2 \geq b_3 \geq \cdots$$

and the sums

$$A_n = a_1 + \cdots + a_n, \quad B_n = b_1 + \cdots + b_n; \qquad n = 1, 2, \ldots.$$

For any pair define $c_n = \min\{a_i, b_i\}$ and $C_n = c_1 + \cdots + c_n$, $n = 1, 2, \ldots$.
(1) Does there exist a pair $(a_i)_{i\geq 1}$, $(b_i)_{i\geq 1}$ such that the sequences $(A_n)_{n\geq 1}$ and $(B_n)_{n\geq 1}$ are unbounded while the sequence $(C_n)_{n\geq 1}$ is bounded?
(2) Does the answer to question (1) change by assuming additionally that $b_i = 1/i$, $i = 1, 2, \ldots$? Justify your answer.

Problem 7.3.7:

Does there exist a function $s\colon \mathbb{Q} \to \{-1, 1\}$ such that if x and y are distinct rational numbers satisfying $xy = 1$ or $x + y \in \{0, 1\}$, then $s(x)s(y) = -1$? Justify your answer.

Problem 7.3.8:

Four real numbers p, q, r, s satisfy $p + q + r + s = 9$ and $p^2 + q^2 + r^2 + s^2 = 21$. Prove that there exists a permutation (a, b, c, d) of (p, q, r, s) such that $ab - cd \geq 2$.

Problem 7.3.9:

The sequence $c_0, c_1, \ldots, c_n, \ldots$ is defined by $c_0 = 1, c_1 = 0$, and $c_{n+2} = c_{n+1} + c_n$ for $n \geq 0$. Consider the set S of ordered pairs (x, y) for which there is a finite set J of positive integers such that $x = \sum_{j\in J} c_j$, $y = \sum_{j\in J} c_{j-1}$. Prove that there exist real numbers α, β, and M with the following property: An ordered pair of nonnegative integers (x, y) satisfies the inequality

$$m < \alpha x + \beta y < M$$

if and only if $(x, y) \in S$.

Problem 7.3.10:

Let n be a positive integer, and let x and y be a positive real number such that $x^n + y^n = 1$. Prove that

$$\left(\sum_{k=1}^{n} \frac{1 + x^{2k}}{1 + x^{4k}}\right) \cdot \left(\sum_{k=1}^{n} \frac{1 + y^{2k}}{1 + y^{4k}}\right) < \frac{1}{(1 - x) \cdot (1 - y)}.$$

Problem 7.3.11:

Let $S \subseteq \mathbb{R}$ be a set of real numbers. We say that a pair (f, g) of functions from S into S is a Spanish Couple on S, if they satisfy the following conditions: (i) Both functions are strictly increasing, i.e. $f(x) < f(y)$ and $g(x) < g(y)$ for all x, $y \in S$ with $x < y$;
(ii) The inequality $f(g(g(x))) < g(f(x))$ holds for all $x \in S$.
Decide whether there exists a Spanish Couple
on the set $S = \mathbb{N}$ of positive integers; on the set $S = \{a - \frac{1}{b} : a, b \in \mathbb{N}\}$

Problem 7.3.12:

Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths

$$a, f(b) \text{ and } f(b + f(a) - 1).$$

(A triangle is non-degenerate if its vertices are not collinear.)

Problem 7.3.13:

Let $x_1, \ldots, x_{100}$ be nonnegative real numbers such that $x_i + x_{i+1} + x_{i+2} \leq 1$ for all $i = 1, \ldots, 100$ (we put $x_{101} = x_1, x_{102} = x_2$). Find the maximal possible value of the sum $S = \sum_{i=1}^{100} x_i x_{i+2}$.

Problem 7.3.14:

Determine all pairs (f, g) of functions from the set of real numbers to itself that satisfy

$$g(f(x + y)) = f(x) + (2x + y)g(y)$$

for all real numbers x and y.

Problem 7.3.15:

Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n > n^n.$$

Problem 7.3.16:

Let $\mathbb{Q}_{>0}$ be the set of all positive rational numbers. Let $f : \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the following three conditions:
(i) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x)f(y) \geq f(xy)$; (ii) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x + y) \geq f(x) + f(y)$; (iii) there exists a rational number $a > 1$ such that $f(a) = a$.
Prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

Problem 7.3.17:

For a sequence $x_1, x_2, \ldots, x_n$ of real numbers, we define its *price* as

$$\max_{1 \le i \le n} |x_1 + \cdots + x_i|.$$

Given n real numbers, Dave and George want to arrange them into a sequence with a low price. Diligent Dave checks all possible ways and finds the minimum possible price D. Greedy George, on the other hand, chooses x_1 such that $|x_1|$ is as small as possible; among the remaining numbers, he chooses x_2 such that $|x_1 + x_2|$ is as small as possible, and so on. Thus, in the i-th step he chooses x_i among the remaining numbers so as to minimise the value of $|x_1 + x_2 + \cdots x_i|$. In each step, if several numbers provide the same value, George chooses one at random. Finally he gets a sequence with price G.
Find the least possible constant c such that for every positive integer n, for every collection of n real numbers, and for every possible sequence that George might obtain, the resulting values satisfy the inequality $G \le cD$.

Problem 7.3.18:

Let n be a fixed positive integer. Find the maximum possible value of

$$\sum_{1 \le r < s \le 2n} (s - r - n) x_r x_s,$$

where $-1 \le x_i \le 1$ for all $i = 1, \cdots, 2n$.

Problem 7.3.19:

Find all positive integers n such that the following statement holds: Suppose real numbers a_1, a_2, $\ldots$, a_n, b_1, b_2, $\ldots$, b_n satisfy $|a_k| + |b_k| = 1$ for all $k = 1, \ldots, n$. Then there exists ε_1, ε_2, $\ldots$, ε_n, each of which is either -1 or 1, such that

$$\left|\sum_{i=1}^{n} \varepsilon_i a_i\right| + \left|\sum_{i=1}^{n} \varepsilon_i b_i\right| \le 1.$$

Problem 7.3.20:

Let S be a finite set, and let $\mathcal{A}$ be the set of all functions from S to S. Let f be an element of $\mathcal{A}$, and let $T = f(S)$ be the image of S under f. Suppose that $f \circ g \circ f \ne g \circ f \circ g$ for every g in $\mathcal{A}$ with $g \ne f$. Show that $f(T) = T$.

Chapter 8

IMO Shortlist Geometry Problems from year 2000-2017

8.1 G1

Problem 8.1.1:

In the plane we are given two circles intersecting at X and Y. Prove that there exist four points with the following property:
(P) For every circle touching the two given circles at A and B, and meeting the line XY at C and D, each of the lines AC, AD, BC, BD passes through one of these points.

Problem 8.1.2:

Let A_1 be the center of the square inscribed in acute triangle ABC with two vertices of the square on side BC. Thus one of the two remaining vertices of the square is on side AB and the other is on AC. Points $B_1,\ C_1$ are defined in a similar way for inscribed squares with two vertices on sides AC and AB, respectively. Prove that lines $AA_1,\ BB_1,\ CC_1$ are concurrent.

Problem 8.1.3:

Let B be a point on a circle S_1, and let A be a point distinct from B on the tangent at B to S_1. Let C be a point not on S_1 such that the line segment AC meets S_1 at two distinct points. Let S_2 be the circle touching AC at C and touching S_1 at a point D on the opposite side of AC from B. Prove that the circumcentre of triangle BCD lies on the circumcircle of triangle ABC.

Problem 8.1.4:

Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

Problem 8.1.5:

Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.

Problem 8.1.6:

Given a triangle ABC satisfying $AC + BC = 3 \cdot AB$. The incircle of triangle ABC has center I and touches the sides BC and CA at the points D and E, respectively. Let K and L be the reflections of the points D and E with respect to I. Prove that the points A, B, K, L lie on one circle.

Problem 8.1.7:

Let ABC be triangle with incenter I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

Problem 8.1.8:

In triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.

Problem 8.1.9:

Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1 and C_2.
Prove that the six points A_1, A_2, B_1, B_2, C_1 and C_2 are concyclic.

Problem 8.1.10:

Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E , respectively. Let K be the incentre of triangle ADC. Suppose that $\angle BEK = 45°$. Find all possible values of $\angle CAB$.

Problem 8.1.11:

Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

Problem 8.1.12:

Let ABC be an acute triangle. Let ω be a circle whose centre L lies on the side BC. Suppose that ω is tangent to AB at B' and AC at C'. Suppose also that the circumcentre O of triangle ABC lies on the shorter arc $B'C'$ of ω. Prove that the circumcircle of ABC and ω meet at two points.

Problem 8.1.13:

Given triangle ABC the point J is the centre of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.
(The excircle of ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C.)

Problem 8.1.14:

Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 is the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of triangle CWM, and let Y be the point such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

Problem 8.1.15:

Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be the points on AP and AQ, respectively, such that P is the midpoint of AM and Q is the midpoint of AN. Prove that the intersection of BM and CN is on the circumference of triangle ABC.

Problem 8.1.16:

Let ABC be an acute triangle with orthocenter H. Let G be the point such that the quadrilateral $ABGH$ is a parallelogram. Let I be the point on the line GH such that AC bisects HI. Suppose that the line AC intersects the circumcircle of the triangle GCI at C and J. Prove that $IJ = AH$.

Problem 8.1.17:

Triangle BCF has a right angle at B. Let A be the point on line CF such that $FA = FB$ and F lies between A and C. Point D is chosen so that $DA = DC$ and AC is the bisector of $\angle DAB$. Point E is chosen so that $EA = ED$ and AD is the bisector of $\angle EAC$. Let M be the midpoint of CF. Let X be the point such that $AMXE$ is a parallelogram. Prove that BD, FX and ME are concurrent.

Problem 8.1.18:

Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$, $\angle EAB = \angle BCD$, and $\angle EDC = \angle CBA$. Prove that the perpendicular line from E to BC and the line segments AC and BD are concurrent.

8.2 G2

Problem 8.2.1:

Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

Problem 8.2.2:

Consider an acute-angled triangle ABC. Let P be the foot of the altitude of triangle ABC issuing from the vertex A, and let O be the circumcenter of triangle ABC. Assume that $\angle C \geq \angle B + 30°$. Prove that $\angle A + \angle COP < 90°$.

Problem 8.2.3:

Let ABC be a triangle for which there exists an interior point F such that $\angle AFB = \angle BFC = \angle CFA$. Let the lines BF and CF meet the sides AC and AB at D and E respectively. Prove that

$$AB + AC \geq 4DE.$$

Problem 8.2.4:

Three distinct points A, B, and C are fixed on a line in this order. Let Γ be a circle passing through A and C whose center does not lie on the line AC. Denote by P the intersection of the tangents to Γ at A and C. Suppose Γ meets the segment PB at Q. Prove that the intersection of the bisector of $\angle AQC$ and the line AC does not depend on the choice of Γ.

Problem 8.2.5:

Let Γ be a circle and let d be a line such that Γ and d have no common points. Further, let AB be a diameter of the circle Γ; assume that this diameter AB is perpendicular to the line d, and the point B is nearer to the line d than the point A. Let C be an arbitrary point on the circle Γ, different from the points A and B. Let D be the point of intersection of the lines AC and d. One of the two tangents from the point D to the circle Γ touches this circle Γ at a point E; hereby, we assume that the points B and E lie in the same halfplane with respect to the line AC. Denote by F the point of intersection of the lines BE and d. Let the line AF intersect the circle Γ at a point G, different from A.
Prove that the reflection of the point G in the line AB lies on the line CF.

Problem 8.2.6:

Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths.
Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

Problem 8.2.7:

Let ABC be a trapezoid with parallel sides $AB > CD$. Points K and L lie on the line segments AB and CD, respectively, so that $AK/KB = DL/LC$. Suppose that there are points P and Q on the line segment KL satisfying

$$\angle APB = \angle BCD \qquad \text{and} \qquad \angle CQD = \angle ABC.$$

Prove that the points P, Q, B and C are concyclic.

Problem 8.2.8:

Denote by M midpoint of side BC in an isosceles triangle $\triangle ABC$ with $AC = AB$. Take a point X on a smaller arc MA of circumcircle of triangle $\triangle ABM$. Denote by T point inside of angle BMA such that $\angle TMX = 90$ and $TX = BX$.
Prove that $\angle MTB - \angle CTM$ does not depend on choice of X.

Problem 8.2.9:

Given trapezoid $ABCD$ with parallel sides AB and CD, assume that there exist points E on line BC outside segment BC, and F inside segment AD such that $\angle DAE = \angle CBF$. Denote by I the point of intersection of CD and EF, and by J the point of intersection of AB and EF. Let K be the midpoint of segment EF, assume it does not lie on line AB. Prove that I belongs to the circumcircle of ABK if and only if K belongs to the circumcircle of CDJ.

Problem 8.2.10:

Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ. respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

Problem 8.2.11:

Let P be a point interior to triangle ABC (with $CA \neq CB$). The lines AP, BP and CP meet again its circumcircle Γ at K, L, respectively M. The tangent line at C to Γ meets the line AB at S. Show that from $SC = SP$ follows $MK = ML$.

Problem 8.2.12:

Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let O_1 and r_1 be the circumcentre and the circumradius of the triangle $A_2A_3A_4$. Define O_2, O_3, O_4 and r_2, r_3, r_4 in a similar way. Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

Proposed by Alexey Gladkich, Israel

Problem 8.2.13:

Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that D, H, F, G are concyclic.

Problem 8.2.14:

Let ω be the circumcircle of a triangle ABC. Denote by M and N the midpoints of the sides AB and AC, respectively, and denote by T the midpoint of the arc BC of ω not containing A. The circumcircles of the triangles AMT and ANT intersect the perpendicular bisectors of AC and AB at points X and Y, respectively; assume that X and Y lie inside the triangle ABC. The lines MN and XY intersect at K. Prove that $KA = KT$.

Problem 8.2.15:

Let ABC be a triangle. The points K, L, and M lie on the segments BC, CA, and AB, respectively, such that the lines AK, BL, and CM intersect in a common point. Prove that it is possible to choose two of the triangles ALM, BMK, and CKL whose inradii sum up to at least the inradius of the triangle ABC.

Problem 8.2.16:

Triangle ABC has circumcircle Ω and circumcenter O. A circle Γ with center A intersects the segment BC at points D and E, such that B, D, E, and C are all different and lie on line BC in this order. Let F and G be the points of intersection of Γ and Ω, such that A, F, B, C, and G lie on Ω in this order. Let K be the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle CGE and the segment CA.
Suppose that the lines FK and GL are different and intersect at the point X. Prove that X lies on the line AO.

Problem 8.2.17:

Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of $\overline{BC}$. The points D, E, F are selected on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

Problem 8.2.18:

Let R and S be different points on a circle Ω such that RS is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen on the shorter arc RS of Ω so that the circumcircle Γ of triangle JST intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ that is closer to R. Line AJ meets Ω again at K. Prove that the line KT is tangent to Γ.

8.3 G3

Problem 8.3.1:

Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Show that there exist points D, E, and F on sides BC, CA, and AB respectively such that

$$OD + DH = OE + EH = OF + FH$$

and the lines AD, BE, and CF are concurrent.

Problem 8.3.2:

Let ABC be a triangle with centroid G. Determine, with proof, the position of the point P in the plane of ABC such that $AP{\cdot}AG + BP{\cdot}BG + CP{\cdot}CG$ is a minimum, and express this minimum value in terms of the side lengths of ABC.

Problem 8.3.3:

The circle S has centre O, and BC is a diameter of S. Let A be a point of S such that $\angle AOB < 120°$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and at F. Prove that I is the incentre of the triangle CEF.

Problem 8.3.4:

Let ABC be a triangle and let P be a point in its interior. Denote by D, E, F the feet of the perpendiculars from P to the lines BC, CA, AB, respectively. Suppose that

$$AP^2 + PD^2 = BP^2 + PE^2 = CP^2 + PF^2.$$

Denote by I_A, I_B, I_C the excenters of the triangle ABC. Prove that P is the circumcenter of the triangle $I_AI_BI_C$.

Problem 8.3.5:

Let O be the circumcenter of an acute-angled triangle ABC with $\angle B < \angle C$. The line AO meets the side BC at D. The circumcenters of the triangles ABD and ACD are E and F, respectively. Extend the sides BA and CA beyond A, and choose on the respective extensions points G and H such that $AG = AC$ and $AH = AB$. Prove that the quadrilateral $EFGH$ is a rectangle if and only if $\angle ACB - \angle ABC = 60°$.

Problem 8.3.6:

Let $ABCD$ be a parallelogram. A variable line g through the vertex A intersects the rays BC and DC at the points X and Y, respectively. Let K and L be the A-excenters of the triangles ABX and ADY. Show that the angle $\measuredangle KCL$ is independent of the line g.

Problem 8.3.7:

Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \qquad \text{and} \qquad \angle ABC = \angle ACD = \angle ADE.$$

The diagonals BD and CE meet at P. Prove that the line AP bisects the side CD.

Problem 8.3.8:

The diagonals of a trapezoid $ABCD$ intersect at point P. Point Q lies between the parallel lines BC and AD such that $\angle AQD = \angle CQB$, and line CD separates points P and Q. Prove that $\angle BQP = \angle DAQ$.

Problem 8.3.9:

Let $ABCD$ be a convex quadrilateral and let P and Q be points in $ABCD$ such that $PQDA$ and $QPBC$ are cyclic quadrilaterals. Suppose that there exists a point E on the line segment PQ such that $\angle PAE = \angle QDE$ and $\angle PBE = \angle QCE$. Show that the quadrilateral $ABCD$ is cyclic.

Problem 8.3.10:

Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelogram. Prove that $GR = GS$.

Problem 8.3.11:

Let $A_1A_2 \ldots A_n$ be a convex polygon. Point P inside this polygon is chosen so that its projections $P_1, \ldots, P_n$ onto lines $A_1A_2, \ldots, A_nA_1$ respectively lie on the sides of the polygon. Prove that for arbitrary points $X_1, \ldots, X_n$ on sides $A_1A_2, \ldots, A_nA_1$ respectively,

$$\max\left\{\frac{X_1X_2}{P_1P_2}, \ldots, \frac{X_nX_1}{P_nP_1}\right\} \geq 1.$$

Problem 8.3.12:

Let $ABCD$ be a convex quadrilateral whose sides AD and BC are not parallel. Suppose that the circles with diameters AB and CD meet at points E and F inside the quadrilateral. Let ω_E be the circle through the feet of the perpendiculars from E to the lines AB, BC and CD. Let ω_F be the circle through the feet of the perpendiculars from F to the lines CD, DA and AB. Prove that the midpoint of the segment EF lies on the line through the two intersections of ω_E and ω_F.

Problem 8.3.13:

In an acute triangle ABC the points D, E and F are the feet of the altitudes through A, B and C respectively. The incenters of the triangles AEF and BDF are I_1 and I_2 respectively; the circumcenters of the triangles ACI_1 and BCI_2 are O_1 and O_2 respectively. Prove that I_1I_2 and O_1O_2 are parallel.

Problem 8.3.14:

In a triangle ABC, let D and E be the feet of the angle bisectors of angles A and B, respectively. A rhombus is inscribed into the quadrilateral $AEDB$ (all vertices of the rhombus lie on different sides of $AEDB$). Let φ be the non-obtuse angle of the rhombus. Prove that $\varphi \le \max\{\angle BAC, \angle ABC\}$.

Problem 8.3.15:

Let Ω and O be the circumcircle and the circumcentre of an acute-angled triangle ABC with $AB > BC$. The angle bisector of $\angle ABC$ intersects Ω at $M \neq B$. Let Γ be the circle with diameter BM. The angle bisectors of $\angle AOB$ and $\angle BOC$ intersect Γ at points P and Q, respectively. The point R is chosen on the line PQ so that $BR = MR$. Prove that $BR \parallel AC$. (Here we always assume that an angle bisector is a ray.)

Problem 8.3.16:

Let ABC be a triangle with $\angle C = 90°$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

Problem 8.3.17:

Let $B = (-1, 0)$ and $C = (1, 0)$ be fixed points on the coordinate plane. A nonempty, bounded subset S of the plane is said to be nice if
(i) there is a point T in S such that for every point Q in S, the segment TQ lies entirely in S; and
(ii) for any triangle $P_1P_2P_3$, there exists a unique point A in S and a permutation σ of the indices $\{1, 2, 3\}$ for which triangles ABC and $P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$ are similar.
Prove that there exist two distinct nice subsets S and S' of the set $\{(x, y) : x \geq 0, y \geq 0\}$ such that if $A \in S$ and $A' \in S'$ are the unique choices of points in (ii), then the product $BA \cdot BA'$ is a constant independent of the triangle $P_1P_2P_3$.

Problem 8.3.18:

Let O be the circumcenter of an acute triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of triangle PQH lies on a median of triangle ABC.

8.4 G4

Problem 8.4.1:

Let $A_1A_2\dots A_n$ be a convex polygon, $n \geq 4$. Prove that $A_1A_2\dots A_n$ is cyclic if and only if to each vertex A_j one can assign a pair (b_j, c_j) of real numbers, $j = 1, 2, \dots, n$, so that $A_iA_j = b_jc_i - b_ic_j$ for all i, j with $1 \leq i < j \leq n$.

Problem 8.4.2:

Let M be a point in the interior of triangle ABC. Let A' lie on BC with MA' perpendicular to BC. Define B' on CA and C' on AB similarly. Define

$$p(M) = \frac{MA' \cdot MB' \cdot MC'}{MA \cdot MB \cdot MC}.$$

Determine, with proof, the location of M such that $p(M)$ is maximal. Let $\mu(ABC)$ denote this maximum value. For which triangles ABC is the value of $\mu(ABC)$ maximal?

Problem 8.4.3:

Circles S_1 and S_2 intersect at points P and Q. Distinct points A_1 and B_1 (not at P or Q) are selected on S_1. The lines A_1P and B_1P meet S_2 again at A_2 and B_2 respectively, and the lines A_1B_1 and A_2B_2 meet at C. Prove that, as A_1 and B_1 vary, the circumcentres of triangles A_1A_2C all lie on one fixed circle.

Problem 8.4.4:

Let Γ_1, Γ_2, Γ_3, Γ_4 be distinct circles such that Γ_1, Γ_3 are externally tangent at P, and Γ_2, Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2; Γ_2 and Γ_3; Γ_3 and Γ_4; Γ_4 and Γ_1 meet at A, B, C, D, respectively, and that all these points are different from P. Prove that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

Problem 8.4.5:

In a convex quadrilateral $ABCD$, the diagonal BD bisects neither the angle ABC nor the angle CDA. The point P lies inside $ABCD$ and satisfies

$$\angle PBC = \angle DBA \quad \text{and} \quad \angle PDC = \angle BDA.$$

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

Problem 8.4.6:

Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and BC not parallel with DA. Let two variable points E and F lie of the sides BC and DA, respectively and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R.
Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.

Problem 8.4.7:

A point D is chosen on the side AC of a triangle ABC with $\angle C < \angle A < 90^\circ$ in such a way that $BD = BA$. The incircle of ABC is tangent to AB and AC at points K and L, respectively. Let J be the incenter of triangle BCD. Prove that the line KL intersects the line segment AJ at its midpoint.

Problem 8.4.8:

Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB.

Problem 8.4.9:

In an acute triangle ABC segments BE and CF are altitudes. Two circles passing through the point A anf F and tangent to the line BC at the points P and Q so that B lies between C and Q. Prove that lines PE and QF intersect on the circumcircle of triangle AEF.

Problem 8.4.10:

Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G and H.

Problem 8.4.11:

Given a triangle ABC, with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D. Let E be a point on the arc BDC, and F a point on the segment BC, such that $\angle BAF = \angle CAE < \dfrac{1}{2}\angle BAC$. If G is the midpoint of IF, prove that the meeting point of the lines EI and DG lies on Γ.

Problem 8.4.12:

Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.

Problem 8.4.13:

Let ABC be a triangle with $AB \neq AC$ and circumcenter O. The bisector of $\angle BAC$ intersects BC at D. Let E be the reflection of D with respect to the midpoint of BC. The lines through D and E perpendicular to BC intersect the lines AO and AD at X and Y respectively. Prove that the quadrilateral $BXCY$ is cyclic.

Problem 8.4.14:

Let ABC be a triangle with $\angle B > \angle C$. Let P and Q be two different points on line AC such that $\angle PBA = \angle QBA = \angle ACB$ and A is located between P and C. Suppose that there exists an interior point D of segment BQ for which $PD = PB$. Let the ray AD intersect the circle ABC at $R \neq A$. Prove that $QB = QR$.

Problem 8.4.15:

Consider a fixed circle Γ with three fixed points A, B, and C on it. Also, let us fix a real number $\lambda \in (0, 1)$. For a variable point $P \notin \{A, B, C\}$ on Γ, let M be the point on the segment CP such that $CM = \lambda \cdot CP$. Let Q be the second point of intersection of the circumcircles of the triangles AMP and BMC. Prove that as P varies, the point Q lies on a fixed circle.

Problem 8.4.16:

Let ABC be an acute triangle and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{BT}{BM}$.

Problem 8.4.17:

Let ABC be a triangle with $AB = AC \neq BC$ and let I be its incentre. The line BI meets AC at D, and the line through D perpendicular to AC meets AI at E. Prove that the reflection of I in AC lies on the circumcircle of triangle BDE.

Problem 8.4.18:

In triangle ABC, let ω be the excircle opposite to A. Let D, E and F be the points where ω is tangent to BC, CA, and AB, respectively. The circle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that the circle MPQ is tangent to ω.

8.5 G5

Problem 8.5.1:

The tangents at B and A to the circumcircle of an acute angled triangle ABC meet the tangent at C at T and U respectively. AT meets BC at P, and Q is the midpoint of AP; BU meets CA at R, and S is the midpoint of BR. Prove that $\angle ABQ = \angle BAS$. Determine, in terms of ratios of side lengths, the triangles for which this angle is a maximum.

Problem 8.5.2:

Let ABC be an acute triangle. Let DAC, EAB, and FBC be isosceles triangles exterior to ABC, with $DA = DC, EA = EB$, and $FB = FC$, such that

$$\angle ADC = 2\angle BAC, \quad \angle BEA = 2\angle ABC, \quad \angle CFB = 2\angle ACB.$$

Let D' be the intersection of lines DB and EF, let E' be the intersection of EC and DF, and let F' be the intersection of FA and DE. Find, with proof, the value of the sum

$$\frac{DB}{DD'} + \frac{EC}{EE'} + \frac{FA}{FF'}.$$

Problem 8.5.3:

For any set S of five points in the plane, no three of which are collinear, let $M(S)$ and $m(S)$ denote the greatest and smallest areas, respectively, of triangles determined by three points from S. What is the minimum possible value of $M(S)/m(S)$?

Problem 8.5.4:

Let ABC be an isosceles triangle with $AC = BC$, whose incentre is I. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC. The lines through P parallel to CA and CB meet AB at D and E, respectively. The line through P parallel to AB meets CA and CB at F and G, respectively. Prove that the lines DF and EG intersect on the circumcircle of the triangle ABC.

Problem 8.5.5:

Let $A_1A_2A_3 \dots A_n$ be a regular n-gon. Let B_1 and B_n be the midpoints of its sides A_1A_2 and $A_{n-1}A_n$. Also, for every $i \in \{2, 3, 4, \dots, n-1\}$. Let S be the point of intersection of the lines A_1A_{i+1} and A_nA_i, and let B_i be the point of intersection of the angle bisector bisector of the angle $\measuredangle A_iSA_{i+1}$ with the segment A_iA_{i+1}.
Prove that $\sum_{i=1}^{n-1} \measuredangle A_1B_iA_n = 180^\circ$.

Problem 8.5.6:

Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocenter of triangle ABC, and let M be the midpoint of the side BC. Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the common chord of the circumscribed circles of triangle $\triangle ABC$ and triangle $\triangle ADE$.

Problem 8.5.7:

In triangle ABC, let J be the center of the excircle tangent to side BC at A_1 and to the extensions of the sides AC and AB at B_1 and C_1 respectively. Suppose that the lines A_1B_1 and AB are perpendicular and intersect at D. Let E be the foot of the perpendicular from C_1 to line DJ. Determine the angles $\angle BEA_1$ and $\angle AEB_1$.

Problem 8.5.8:

Let ABC be a fixed triangle, and let A_1, B_1, C_1 be the midpoints of sides BC, CA, AB, respectively. Let P be a variable point on the circumcircle. Let lines PA_1, PB_1, PC_1 meet the circumcircle again at A', B', C', respectively. Assume that the points A, B, C, A', B', C' are distinct, and lines AA', BB', CC' form a triangle. Prove that the area of this triangle does not depend on P.

Problem 8.5.9:

Let k and n be integers with $0 \leq k \leq n-2$. Consider a set L of n lines in the plane such that no two of them are parallel and no three have a common point. Denote by I the set of intersections of lines in L. Let O be a point in the plane not lying on any line of L. A point $X \in I$ is colored red if the open line segment OX intersects at most k lines in L. Prove that I contains at least $\frac{1}{2}(k+1)(k+2)$ red points.

Problem 8.5.10:

Let P be a polygon that is convex and symmetric to some point O. Prove that for some parallelogram R satisfying $P \subset R$ we have

$$\frac{|R|}{|P|} \leq \sqrt{2}$$

where $|R|$ and $|P|$ denote the area of the sets R and P, respectively.

Problem 8.5.11:

Let $ABCDE$ be a convex pentagon such that $BC \parallel AE$, $AB = BC + AE$, and $\angle ABC = \angle CDE$. Let M be the midpoint of CE, and let O be the circumcenter of triangle BCD. Given that $\angle DMO = 90^\circ$, prove that $2\angle BDA = \angle CDE$.

Problem 8.5.12:

Let ABC be a triangle with incentre I and circumcircle ω. Let D and E be the second intersection points of ω with AI and BI, respectively. The chord DE meets AC at a point F, and BC at a point G. Let P be the intersection point of the line through F parallel to AD and the line through G parallel to BE. Suppose that the tangents to ω at A and B meet at a point K. Prove that the three lines AE, BD and KP are either parallel or concurrent.

Problem 8.5.13:

Let ABC be a triangle with $\angle BCA = 90^\circ$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let M be the point of intersection of AL and BK.
Show that $MK = ML$.

Problem 8.5.14:

Let $ABCDEF$ be a convex hexagon with $AB = DE$, $BC = EF$, $CD = FA$, and $\angle A - \angle D = \angle C - \angle F = \angle E - \angle B$. Prove that the diagonals AD, BE, and CF are concurrent.

Problem 8.5.15:

Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90^\circ$. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and

$$\angle CHS - \angle CSB = 90^\circ, \quad \angle THC - \angle DTC = 90^\circ.$$

Prove that line BD is tangent to the circumcircle of triangle TSH.

Problem 8.5.16:

Let ABC be a triangle with $CA \neq CB$. Let D, F, and G be the midpoints of the sides AB, AC, and BC respectively. A circle Γ passing through C and tangent to AB at D meets the segments AF and BG at H and I, respectively. The points H' and I' are symmetric to H and I about F and G, respectively. The line $H'I'$ meets CD and FG at Q and M, respectively. The line CM meets Γ again at P. Prove that $CQ = QP$.

Problem 8.5.17:

Let D be the foot of perpendicular from A to the Euler line (the line passing through the circumcentre and the orthocentre) of an acute scalene triangle ABC. A circle ω with centre S passes through A and D, and it intersects sides AB and AC at X and Y respectively. Let P be the foot of altitude from A to BC, and let M be the midpoint of BC. Prove that the circumcentre of triangle XSY is equidistant from P and M.

Problem 8.5.18:

Let $ABCC_1B_1A_1$ be a convex hexagon such that $AB = BC$, and suppose that the line segments AA_1, BB_1, and CC_1 have the same perpendicular bisector. Let the diagonals AC_1 and A_1C meet at D, and denote by ω the circle ABC. Let ω intersect the circle A_1BC_1 again at $E \neq B$. Prove that the lines BB_1 and DE intersect on ω.

8.6 G6

Problem 8.6.1:

Let $ABCD$ be a convex quadrilateral. The perpendicular bisectors of its sides AB and CD meet at Y. Denote by X a point inside the quadrilateral $ABCD$ such that $\measuredangle ADX = \measuredangle BCX < 90^\circ$ and $\measuredangle DAX = \measuredangle CBX < 90^\circ$. Show that $\measuredangle AYB = 2 \cdot \measuredangle ADX$.

Problem 8.6.2:

Let ABC be a triangle and P an exterior point in the plane of the triangle. Suppose the lines AP, BP, CP meet the sides BC, CA, AB (or extensions thereof) in D, E, F, respectively. Suppose further that the areas of triangles PBD, PCE, PAF are all equal. Prove that each of these areas is equal to the area of triangle ABC itself.

Problem 8.6.3:

Let $n \geq 3$ be a positive integer. Let $C_1, C_2, C_3, \ldots, C_n$ be unit circles in the plane, with centres $O_1, O_2, O_3, \ldots, O_n$ respectively. If no line meets more than two of the circles, prove that

$$\sum_{1 \leq i < j \leq n} \frac{1}{O_i O_j} \leq \frac{(n-1)\pi}{4}.$$

Problem 8.6.4:

Each pair of opposite sides of a convex hexagon has the following property: the distance between their midpoints is equal to $\dfrac{\sqrt{3}}{2}$ times the sum of their lengths. Prove that all the angles of the hexagon are equal.

Problem 8.6.5:

Let P be a convex polygon. Prove that there exists a convex hexagon that is contained in P and whose area is at least $\frac{3}{4}$ of the area of the polygon P.
Alternative version. Let P be a convex polygon with $n \geq 6$ vertices. Prove that there exists a convex hexagon with
a) vertices on the sides of the polygon (or) b) vertices among the vertices of the polygon
such that the area of the hexagon is at least $\frac{3}{4}$ of the area of the polygon.

Problem 8.6.6:

Let ABC be a triangle, and M the midpoint of its side BC. Let γ be the incircle of triangle ABC. The median AM of triangle ABC intersects the incircle γ at two points K and L. Let the lines passing through K and L, parallel to BC, intersect the incircle γ again in two points X and Y. Let the lines AX and AY intersect BC again at the points P and Q. Prove that $BP = CQ$.

Problem 8.6.7:

Circles w_1 and w_2 with centres O_1 and O_2 are externally tangent at point D and internally tangent to a circle w at points E and F respectively. Line t is the common tangent of w_1 and w_2 at D. Let AB be the diameter of w perpendicular to t, so that A, E, O_1 are on the same side of t. Prove that lines AO_1, BO_2, EF and t are concurrent.

Problem 8.6.8:

Determine the smallest positive real number k with the following property. Let $ABCD$ be a convex quadrilateral, and let points A_1, B_1, C_1, and D_1 lie on sides AB, BC, CD, and DA, respectively. Consider the areas of triangles AA_1D_1, BB_1A_1, CC_1B_1 and DD_1C_1; let S be the sum of the two smallest ones, and let S_1 be the area of quadrilateral $A_1B_1C_1D_1$. Then we always have $kS_1 \geq S$.

Problem 8.6.9:

There is given a convex quadrilateral $ABCD$. Prove that there exists a point P inside the quadrilateral such that

$$\angle PAB + \angle PDC = \angle PBC + \angle PAD = \angle PCD + \angle PBA = \angle PDA + \angle PCB = 90^\circ$$

if and only if the diagonals AC and BD are perpendicular.

Problem 8.6.10:

Let the sides AD and BC of the quadrilateral $ABCD$ (such that AB is not parallel to CD) intersect at point P. Points O_1 and O_2 are circumcenters and points H_1 and H_2 are orthocenters of triangles ABP and CDP, respectively. Denote the midpoints of segments O_1H_1 and O_2H_2 by E_1 and E_2, respectively. Prove that the perpendicular from E_1 on CD, the perpendicular from E_2 on AB and the lines H_1H_2 are concurrent.

Problem 8.6.11:

The vertices X, Y, Z of an equilateral triangle XYZ lie respectively on the sides BC, CA, AB of an acute-angled triangle ABC. Prove that the incenter of triangle ABC lies inside triangle XYZ.

Problem 8.6.12:

Let ABC be a triangle with $AB = AC$ and let D be the midpoint of AC. The angle bisector of $\angle BAC$ intersects the circle through D, B and C at the point E inside the triangle ABC. The line BD intersects the circle through A, E and B in two points B and F. The lines AF and BE meet at a point I, and the lines CI and BD meet at a point K. Show that I is the incentre of triangle KAB.

Idea Draw the circle centered at D with radius $DA = DC$ □

Problem 8.6.13:

Let ABC be a triangle with circumcenter O and incenter I. The points D, E and F on the sides BC, CA and AB respectively are such that $BD + BF = CA$ and $CD + CE = AB$. The circumcircles of the triangles BFD and CDE intersect at $P \neq D$. Prove that $OP = OI$.

Problem 8.6.14:

Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.

Problem 8.6.15:

Let ABC be a fixed acute-angled triangle. Consider some points E and F lying on the sides AC and AB, respectively, and let M be the midpoint of EF . Let the perpendicular bisector of EF intersect the line BC at K, and let the perpendicular bisector of MK intersect the lines AC and AB at S and T , respectively. We call the pair (E, F) *interesting*, if the quadrilateral $KSAT$ is cyclic. Suppose that the pairs (E_1, F_1) and (E_2, F_2) are interesting. Prove that $\frac{E_1E_2}{AB} = \frac{F_1F_2}{AC}$

Problem 8.6.16:

Let ABC be an acute triangle with $AB > AC$. Let Γ be its cirumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = 90°$ and let K be the point on Γ such that $\angle HKQ = 90°$. Assume that the points A, B, C, K and Q are all different and lie on Γ in this order.
Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

Problem 8.6.17:

Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC < 90°$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at E and F respectively, and meet each other at point P. Let M be the midpoint of AC and let ω be the circumcircle of triangle BPD. Segments BM and DM intersect ω again at X and Y respectively. Denote by Q the intersection point of lines XE and YF. Prove that $PQ \perp AC$.

Problem 8.6.18:

Let $n \geq 3$ be an integer. Two regular n-gons $\mathcal{A}$ and $\mathcal{B}$ are given in the plane. Prove that the vertices of $\mathcal{A}$ that lie inside $\mathcal{B}$ or on its boundary are consecutive.
(That is, prove that there exists a line separating those vertices of $\mathcal{A}$ that lie inside $\mathcal{B}$ or on its boundary from the other vertices of $\mathcal{A}$.)

8.7 G7

Problem 8.7.1:

Ten gangsters are standing on a flat surface, and the distances between them are all distinct. At twelve o'clock, when the church bells start chiming, each of them fatally shoots the one among the other nine gangsters who is the nearest. At least how many gangsters will be killed?

Problem 8.7.2:

Let O be an interior point of acute triangle ABC. Let A_1 lie on BC with OA_1 perpendicular to BC. Define B_1 on CA and C_1 on AB similarly. Prove that O is the circumcenter of ABC if and only if the perimeter of $A_1B_1C_1$ is not less than any one of the perimeters of AB_1C_1, BC_1A_1, and CA_1B_1.

Problem 8.7.3:

The incircle Ω of the acute-angled triangle ABC is tangent to its side BC at a point K. Let AD be an altitude of triangle ABC, and let M be the midpoint of the segment AD. If N is the common point of the circle Ω and the line KM (distinct from K), then prove that the incircle Ω and the circumcircle of triangle BCN are tangent to each other at the point N.

Problem 8.7.4:

Let ABC be a triangle with semiperimeter s and inradius r. The semicircles with diameters BC, CA, AB are drawn on the outside of the triangle ABC. The circle tangent to all of these three semicircles has radius t. Prove that

$$\frac{s}{2} < t \leq \frac{s}{2} + \left(1 - \frac{\sqrt{3}}{2}\right) r.$$

Alternative formulation. In a triangle ABC, construct circles with diameters BC, CA, and AB, respectively. Construct a circle w externally tangent to these three circles. Let the radius of this circle w be t. Prove: $\frac{s}{2} < t \leq \frac{s}{2} + \frac{1}{2}\left(2 - \sqrt{3}\right) r$, where r is the inradius and s is the semiperimeter of triangle ABC.

Problem 8.7.5:

For a given triangle ABC, let X be a variable point on the line BC such that C lies between B and X and the incircles of the triangles ABX and ACX intersect at two distinct points P and Q. Prove that the line PQ passes through a point independent of X.

Problem 8.7.6:

In an acute triangle ABC, let D, E, F be the feet of the perpendiculars from the points A, B, C to the lines BC, CA, AB, respectively, and let P, Q, R be the feet of the perpendiculars from the points A, B, C to the lines EF, FD, DE, respectively.
Prove that $p(ABC)\,p(PQR) \geq (p(DEF))^2$, where $p(T)$ denotes the perimeter of triangle T .

Problem 8.7.7:

In a triangle ABC, let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB, respectively, and T_a, T_b, T_c be the midpoints of the arcs BC, CA, AB of the circumcircle of ABC, not containing the vertices A, B, C, respectively. For $i \in \{a, b, c\}$, let w_i be the circle with M_iT_i as diameter. Let p_i be the common external common tangent to the circles w_j and w_k (for all $\{i, j, k\} = \{a, b, c\}$) such that w_i lies on the opposite side of p_i than w_j and w_k do. Prove that the lines p_a, p_b, p_c form a triangle similar to ABC and find the ratio of similitude.

Problem 8.7.8:

Given an acute triangle ABC with $\angle B > \angle C$. Point I is the incenter, and R the circumradius. Point D is the foot of the altitude from vertex A. Point K lies on line AD such that $AK = 2R$, and D separates A and K. Lines DI and KI meet sides AC and BC at E, F respectively. Let $IE = IF$.
Prove that $\angle B \leq 3\angle C$.

Problem 8.7.9:

Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents to ω_1 and ω_2 intersect on ω.

Problem 8.7.10:

Let ABC be a triangle with incenter I and let X, Y and Z be the incenters of the triangles BIC, CIA and AIB, respectively. Let the triangle XYZ be equilateral. Prove that ABC is equilateral too.

Problem 8.7.11:

Three circular arcs γ_1, γ_2, and γ_3 connect the points A and C. These arcs lie in the same half-plane defined by line AC in such a way that arc γ_2 lies between the arcs γ_1 and γ_3. Point B lies on the segment AC. Let h_1, h_2, and h_3 be three rays starting at B, lying in the same half-plane, h_2 being between h_1 and h_3. For $i, j = 1, 2, 3$, denote by V_{ij} the point of intersection of h_i and γ_j (see the Figure below). Denote by $\widehat{V_{ij}V_{kj}}\widehat{V_{kl}V_{il}}$ the curved quadrilateral, whose sides are the segments $V_{ij}V_{il}$, $V_{kj}V_{kl}$ and arcs $V_{ij}V_{kj}$ and $V_{il}V_{kl}$. We say that this quadrilateral is *circumscribed* if there exists a circle touching these two segments and two arcs. Prove that if the curved quadrilaterals $\widehat{V_{11}V_{21}}\widehat{V_{22}V_{12}}, \widehat{V_{12}V_{22}}\widehat{V_{23}V_{13}}, \widehat{V_{21}V_{31}}\widehat{V_{32}V_{22}}$ are circumscribed, then the curved quadrilateral $\widehat{V_{22}V_{32}}\widehat{V_{33}V_{23}}$ is circumscribed, too.

Problem 8.7.12:

Let $ABCDEF$ be a convex hexagon all of whose sides are tangent to a circle ω with centre O. Suppose that the circumcircle of triangle ACE is concentric with ω. Let J be the foot of the perpendicular from B to CD. Suppose that the perpendicular from B to DF intersects the line EO at a point K. Let L be the foot of the perpendicular from K to DE. Prove that $DJ = DL$.

Problem 8.7.13:

Let $ABCD$ be a convex quadrilateral with non-parallel sides BC and AD. Assume that there is a point E on the side BC such that the quadrilaterals $ABED$ and $AECD$ are circumscribed. Prove that there is a point F on the side AD such that the quadrilaterals $ABCF$ and $BCDF$ are circumscribed if and only if AB is parallel to CD.

Problem 8.7.14:

Let ABC be a triangle with circumcircle Ω and incentre I. Let the line passing through I and perpendicular to CI intersect the segment BC and the arc BC (not containing A) of Ω at points U and V , respectively. Let the line passing through U and parallel to AI intersect AV at X, and let the line passing through V and parallel to AI intersect AB at Y . Let W and Z be the midpoints of AX and BC, respectively. Prove that if the points I, X, and Y are collinear, then the points I, W, and Z are also collinear.

Problem 8.7.15:

Let $ABCD$ be a convex quadrilateral, and let P, Q, R, and S be points on the sides AB, BC, CD, and DA, respectively. Let the line segment PR and QS meet at O. Suppose that each of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$ has an incircle. Prove that the lines AC, PQ, and RS are either concurrent or parallel to each other.

Problem 8.7.16:

Let I be the incentre of a non-equilateral triangle ABC, I_A be the A-excentre, I'_A be the reflection of I_A in BC, and l_A be the reflection of line AI'_A in AI. Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B.
Prove that P lies on line OI where O is the circumcentre of triangle ABC. Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y. Show that $\angle XIY = 120^\circ$.

Problem 8.7.17:

A convex quadrilateral $ABCD$ has an inscribed circle with center I. Let I_a, I_b, I_c and I_d be the incenters of the triangles DAB, ABC, BCD and CDA, respectively. Suppose that the common external tangents of the circles AI_bI_d and CI_bI_d meet at X, and the common external tangents of the circles BI_aI_c and DI_aI_c meet at Y. Prove that $\angle XIY = 90^\circ$.

8.8 G8

Problem 8.8.1:

Let AH_1, BH_2, CH_3 be the altitudes of an acute angled triangle ABC. Its incircle touches the sides BC, AC and AB at T_1, T_2 and T_3 respectively. Consider the symmetric images of the lines H_1H_2, H_2H_3 and H_3H_1 with respect to the lines T_1T_2, T_2T_3 and T_3T_1. Prove that these images form a triangle whose vertices lie on the incircle of ABC.

Problem 8.8.2:

Let ABC be a triangle with $\angle BAC = 60^\circ$. Let AP bisect $\angle BAC$ and let BQ bisect $\angle ABC$, with P on BC and Q on AC. If $AB + BP = AQ + QB$, what are the angles of the triangle?

Problem 8.8.3:

Let two circles S_1 and S_2 meet at the points A and B. A line through A meets S_1 again at C and S_2 again at D. Let M, N, K be three points on the line segments CD, BC, BD respectively, with MN parallel to BD and MK parallel to BC. Let E and F be points on those arcs BC of S_1 and BD of S_2 respectively that do not contain A. Given that EN is perpendicular to BC and FK is perpendicular to BD prove that $\angle EMF = 90^\circ$.

Problem 8.8.4:

Given a cyclic quadrilateral $ABCD$, let M be the midpoint of the side CD, and let N be a point on the circumcircle of triangle ABM. Assume that the point N is different from the point M and satisfies $\frac{AN}{BN} = \frac{AM}{BM}$. Prove that the points E, F, N are collinear, where $E = AC \cap BD$ and $F = BC \cap DA$.

Problem 8.8.5:

Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P. Show that the sum of the areas assigned to the sides of P is at least twice the area of P.

Problem 8.8.6:

Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2 respectively ($A_2 \neq A, B_2 \neq B, C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

Problem 8.8.7:

Let $ABCD$ be a convex quadrilateral. A circle passing through the points A and D and a circle passing through the points B and C are externally tangent at a point P inside the quadrilateral. Suppose that

$$\angle PAB + \angle PDC \leq 90^\circ \quad \text{and} \quad \angle PBA + \angle PCD \leq 90^\circ.$$

Prove that $AB + CD \geq BC + AD$.

Problem 8.8.8:

Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incenter. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. Let lines AC and BD meet at E, and let lines AK and BL meet at F. Prove that points E, I, and F are collinear.

Problem 8.8.9:

Let $ABCD$ be a circumscribed quadrilateral. Let g be a line through A which meets the segment BC in M and the line CD in N. Denote by I_1, I_2 and I_3 the incenters of $\triangle ABM$, $\triangle MNC$ and $\triangle NDA$, respectively. Prove that the orthocenter of $\triangle I_1I_2I_3$ lies on g.

Problem 8.8.10:

Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.

Problem 8.8.11:

Let ABC be a triangle with circumcircle ω and ℓ a line without common points with ω. Denote by P the foot of the perpendicular from the center of ω to ℓ. The side-lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove that the circumcircles of the triangles AXP, BYP and CZP have a common point different from P or are mutually tangent at P.

Problem 8.8.12:

A triangulation of a convex polygon Π is a partitioning of Π into triangles by diagonals having no common points other than the vertices of the polygon. We say that a triangulation is a Thaiangulation if all triangles in it have the same area.
Prove that any two different Thaiangulations of a convex polygon Π differ by exactly two triangles. (In other words, prove that it is possible to replace one pair of triangles in the first Thaiangulation with a different pair of triangles so as to obtain the second Thaiangulation.)

Problem 8.8.13:

Let A_1, B_1 and C_1 be points on sides BC, CA and AB of an acute triangle ABC respectively, such that AA_1, BB_1 and CC_1 are the internal angle bisectors of triangle ABC. Let I be the incentre of triangle ABC, and H be the orthocentre of triangle $A_1B_1C_1$. Show that

$$AH + BH + CH \geq AI + BI + CI.$$

Problem 8.8.14:

There are 2017 mutually external circles drawn on a blackboard, such that no two are tangent and no three share a common tangent. A tangent segment is a line segment that is a common tangent to two circles, starting at one tangent point and ending at the other one. Luciano is drawing tangent segments on the blackboard, one at a time, so that no tangent segment intersects any other circles or previously drawn tangent segments. Luciano keeps drawing tangent segments until no more can be drawn.

8.9 National Camp 2019 Problem Set

Problem 8.9.1: **E**

Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

Problem 8.9.2:

Alice and Bob play a game. There is a threshold n, Initially, the game starts with the number 1. In a move the player replaced the number m with either $m+1$ or $2m$. Assuming optimal play, count the number of threshold under 2019 where Alice wins.

Problem 8.9.3:

For an integer n, let $a_1, a_2, \dots a_{\phi(n)}$ be the integers less than and coprime to n. Determine all possible values of

$$\prod_{i=1}^{\phi(n)} a_i \pmod{n}$$

and for which values of n they appear.

Problem 8.9.4:

Let O and N be the circumcenter and nine-point-center of $\triangle ABC$ respectively. Let I_b and I_c be the B and C excenters of $\triangle ABC$ respectively. Prove that

$$\angle I_b O I_c = 180^\circ - \frac{1}{2}\angle I_b N I_c$$

Problem 8.9.5: Putnam 2001 A5 **E**

Prove that there are unique positive integers a, n such that

$$a^{n+1} - (a+1)^n = 2001$$

Problem 8.9.6:

Find all polynomials $P(x)$ with real coefficients such that for all real numbers $x+y+z=0$, then the determinant of the following matrix is 0

$$\begin{bmatrix} 1 & x & P(x) \\ 1 & y & P(y) \\ 1 & z & P(z) \end{bmatrix}$$

Problem 8.9.7: IMO 1971 P5 **M**

Prove that for every positive integer m we can find a finite set S of points in the plane, such that given any point A of S, there are exactly m points in S at unit distance from A.

Problem 8.9.8: ISL 1971 **M**

Consider a sequence of polynomials $P_0(x), P_1(x), P_2(x), \ldots, P_n(x), \ldots,$ where $P_0(x) = 2, P_1(x) = x$ and for every $n \geq 1$ the following equality holds:

$$P_{n+1}(x) + P_{n-1}(x) = xP_n(x).$$

Prove that there exist three real numbers a, b, c such that for all $n \geq 1$,

$$(x^2 - 4)[P_n^2(x) - 4] = [aP_{n+1}(x) + bP_n(x) + cP_{n-1}(x)]^2.$$

Problem 8.9.9: IMO 1974 P3

Prove that for any n natural, the number

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^3 k$$

cannot be divided by 5.

Problem 8.9.10: Putnam 1999 B6 **M**

Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that $\gcd(s, n) = 1$ or $\gcd(s, n) = s$. Show that there exist $s, t \in S$ such that $\gcd(s, t)$ is prime.

Problem 8.9.11: Sharygin 2009 P4 **E**

Let P and Q be the common points of two circles. The ray with origin Q reflects from the first circle in points A_1, A_2,... according to the rule "the angle of incidence is equal to the angle of reflection". Another ray with origin Q reflects from the second circle in the points B_1, B_2,... in the same manner. Points A_1, B_1 and P occurred to be collinear. Prove that all lines A_iB_i pass through P.

Problem 8.9.12: Greece **M**

Find all surjective functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $m, n \in \mathbb{N}$

$$m|n \iff f(m)|f(n)$$

Problem 8.9.13: USA TST 2010 P9 **H**

Determine whether or not there exists a positive integer k such that $p = 6k + 1$ is a prime and

$$\binom{3k}{k} \equiv 1 \pmod p$$

Problem 8.9.14: USA TST 2011 P6 **H**

A polynomial $P(x)$ is called nice if $P(0) = 1$ and the nonzero coefficients of $P(x)$ alternate between 1 and -1 when written in order. Suppose that $P(x)$ is nice, and let m and n be two relatively prime positive integers. Show that

$$Q(x) = P(x^n) \cdot \frac{(x^{mn} - 1)(x - 1)}{(x^m - 1)(x^n - 1)}$$

is nice as well.

Problem 8.9.15:

Find all pairs of positive integers (m, n) such that $mn - 1$ divides $(n^2 - n + 1)^2$.

Problem 8.9.16:

In isosceles $\triangle ABC$, $AB = AC$, points D, E, F lie on segments BC, AC, AB such that $DE \parallel AB$, $DF \parallel AC$. The circumcircle of $\triangle ABC$ ω_1 and the circumcircle of $\triangle AEF$ ω_2 intersect at A, G. Let DE meet ω_2 at $K \neq E$. Points L, M lie on ω_1, ω_2 respectively such that $LG \perp KG, MG \perp CG$. Let P, Q be the circumcenters of $\triangle DGL$ and $\triangle DGM$ respectively. Prove that A, G, P, Q are concyclic.

Pics/nat_pset_16.pdf

Figure 8.1

Problem 8.9.17:

Given a polynomial $f(x)$ with rational coefficients, of degree $d \geq 2$, we define the sequence of sets $f^0(\mathbb{Q}), f^1(\mathbb{Q}), \ldots$ as $f^0(\mathbb{Q}) = \mathbb{Q}$, $f^{n+1}(\mathbb{Q}) = f(f^n(\mathbb{Q}))$ for $n \geq 0$. (Given a set S, we write $f(S)$ for the set $\{f(x) \mid x \in S\}$). Let $f^\omega(\mathbb{Q}) = \bigcap_{n=0}^{\infty} f^n(\mathbb{Q})$ be the set of numbers that are in all of the sets $f^n(\mathbb{Q})$, $n \geq 0$. Prove that $f^\omega(\mathbb{Q})$ is a finite set.

Problem 8.9.18:

Define the polymonial sequence $\{f_n(x)\}_{n\geq 1}$ with $f_1(x) = 1$,

$$f_{2n}(x) = x f_n(x), \ f_{2n+1}(x) = f_n(x) + f_{n+1}(x), \ n \geq 1.$$

Look for all the rational number a which is a root of certain $f_n(x)$.

Problem 8.9.19: Sharygin 2013 Final Round 10.8

Two fixed circles are given on the plane, one of them lies inside the other one. From a point C moving arbitrarily on the external circle, draw two chords CA, CB of the larger circle such that they tangent to the smaller one. Find the locus of the incenter of triangle ABC.

Problem 8.9.20: IMC 2014 P3

Let n be a positive integer. Show that there are positive real numbers $a_0, a_1, \ldots, a_n$ such that for each choice of signs the polynomial

$$\pm a_n x^n \pm a_{n-1}x^{n-1} \pm \cdots \pm a_1 x \pm a_0$$

has n distinct real roots.

Problem 8.9.21: Dunno

Suppose that the real numbers $a_0, a_1, ..., a_n$ and x, with $0 < x < 1$, satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + ... + \frac{a_n}{1-x^{n+1}} = 0$$

Prove that there exist a real number y with $0 < y < 1$ such that

$$a_0 + a_1 y + a_2 y^2 + ... + a_n y^n = 0$$

Problem 8.9.22: RMM 2013 P6

A token is placed at each vertex of a regular $2n$-gon. A move consists in choosing an edge of the $2n$-gon and swapping the two tokens placed at the endpoints of that edge. After a finite number of moves have been performed, it turns out that every two tokens have been swapped exactly once. Prove that some edge has never been chosen.

Problem 8.9.23: Sharygin 2018 Final Round 8.4

Find all sets of six points in the plane, no three collinear, such that if we partition the set into two sets, then the obtained triangles are congruent.

Chapter 9

Database

9.1 Problems

Combo Somewhere
USA TST 2010 P8
Binom 1
Binom 2
USA TST 2005 P1
Iran TST 2008 D3P1
Romanian TST 2016 D1P2
American Mathematical Monthly problem E2309
CGMO 2010 P1
ISL 2002 C5
ISL 1988 P10
Iran TST 2013 D1P2
Iran TST 2018 P1
ISL 2016 C1
ISL 2005 C2
IOI Practice 2017
Codeforces 987E
USAMO 2013 P6
ISL 2012 C1
AoPS
ISL 1994 C3
MEMO 2008, Team, P6
US Dec TST 2016, P1
Cody Johnson
ISL 2014 A1
ISL 2014 N3
China TST 2006
OIM 1994, PSMiC
Timus 1578
CodeForces 744B
USAMO 2015 P4
ISL 2003 C4
India TST 2017 D1 P3
Tournament of Towns 2015F S7
Polish OI
CodeForces 960/C
ARO 2005 P10.3, P11.2
IOI 2007 P3
USAMO 2010 P2
Serbia TST 2015 P3
Taiwan TST 2015 R3D1P1
ISL 2014 N1
Codeforces 330D
Brazilian MO 2002, ARO 1979
ARO 1979's Sharper Version
Tournament of Towns Spring 2012 S7
ISL 2006 C6
Putnam 2002 A3
USAMO 1998
German TST 2004 E7P3
ARO 2014 P9.8
Generalization
ISL 2005 C1
ISL 2013 C3
ISL 2005 C2
All Russia 2017 9.1
Jacob Tsimerman Induction
USAMO 2017 P4
Swell coloring
Belarus 2001
USA TST 2011 D3P2
ISL 2016 C6
ELMO 2017 P5
ARO 2013 P9.5
USA TST 2011 P2
Iran TST 2009 P6
OC Chap2 P2
ARO 2005 P9.4
Romanian TST 2012 P4
USA TST 2006 P1
Simurgh 2019 P3
RMM 2019 P3
China TST 2015 T1D2P3
ARO 2005 P10.8
ARO 1999 P9.8
ISL 2001 C3

USAMO 2008 P5
USAMO 2014 P1
Indian TST 2004
ISL 1998 C7
ARO 1999 P10.1
USAMO 2004, P4
ARO 1999 P10.1
RMM 2019 P1
ISL 2015 C4
ISL 2012 C4
ARO 2013 P9.4
EGMO 2017 P3
ISL 2006 C2
ARO 2014 P9.3
ISL 2013 C2, IMO 2013 P2
Putnam 1979
USAMO 2005 P5
ILL 1985
APMO 1999 P5
ISL 2014 C1
ISL 2007 C2
ISL 2003 C2
ISL 2003 C3
Tournament of Towns 2015S S4
USAMO 2007 P2
MEMO 2015 T4
Math Price for Girls 2017 P4
China TST 2016 T3P2
IGO 2018 A3
Lithuania ??
Romanian TST 2008 T1P4
Iran TST ??
APMO 2018 P4
ISL 2007 C5
APMO 2018 P3
Iran 2005
ISL 2008 C3
ISL 2015 C2
IMO 2018 P4
Serbia National D2P2
BdMO 2019 P10
RMM 2019 P4
China TST 2015 T1D2P1
Codeforces 1158D
ISL 1990
Result by Erdos, Dividing the integers into arithmetic progressions
APMO 1999 P1
APMO 1999 P2
ISL 1994 A1
ISL 2007 C4
ISL 2009 A6
ISL 2013 N2
USAMO 2007 P1
APMO 2017 P3
ISL 2008 C4
USAMO 1996 P4
APMO 2017 P1
ISL 2004 C1
ISL 2002 C1
USAMO 2012 P2
European Mathematics Cup 2018 P1
APMO 2008 P2
IMO SL 1985
IMO 2011 P4
USAMO 2008 P3
USAMO 2013 P2
ISL 2002 C3
ISL 1994 C2
ISL 2003 C1
USAMO 2006 P2
MOP Problem
EGMO 2015 P2
USA TST 2006 P5
ISL 2010 C3
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ISL 1993 C5
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Iran TST 2018 T1P4
Iran TST 2018 T3P6
ISL 2006 G6
ISL 2006 G7
ISL 2006 G9
Iran TST 2017 P5
ISL 2012 G8
USAMO 2018 P5
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ISL 2002 G1
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India TST
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ISL 2009 N1
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ISL 2002 N2
Japan MO 2017 P2, TST Mock 2018
ISL 2002 N3
Japan MO 2017 P1
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APMO 2008 P4
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9.2 Theorems

Combo Vandermonde's identity
Erdos Ko Rado theorem
Kleitman lemma
The Sunflower Lemma
Mirsky Theorem
Dilworth's Theorem
Erdos-Szekeres Theorem
MST Cut
Swap Sort
Average of Degrees
Bipartite Graph Criteria
Euler's Polyhedron Formula
Prufer sequence
Criteria of partitioning a graph into disconnected sub-graphs
Turan's theorem
Sprague-Grund Theorem
Helly's Theorem
Van der Waerden's Theorem
Catalan Recursion
Alg
Lagrange Interpolation Theorem
Finite Differences
Gauss's Lemma
Bounds On Roots
Perron's Criterion's Generalization (Dominating Term)
Dealing with binomial terms with a common factor
Triangle Inequality
Weighted AM-GM
Cauchy-Schward Inequality
Cauchy-Schwarz Inequality Complex form
Titu's Lemma
Holder's Inequality
Convexity
Jensen's Inequality
Popoviciu's inequality
The Derivative of an Odd function is always an Even function, and vice versa.
Minkowski's theorem
Geo
Simson Line Parallel
Simson Line Angle
Perspectivity Line with Orthic triangle is perpendicular to Euler line
Hagge's circles
Isogonal Conjugate Lemma
Terquem's Cevian Theorem
Mannheim's Theorem
Mannheim's Theorem's Converse
Brocard Points
Jacobi's Theorem
Paul Yui Theorem
Incircle Touchpoint and Cevian
Poncelet's Porism
Tangential Quadrilateral Incenters
Fuhrmann Circle
Isogonal Lemma
Isogonality in quadrilateral
Most Important Symmedian Property
Isotonic Lemma
Homothety and Reflection
Sawayama and Thebault's theorem
Newton-Gauss Line
Pascal's Theorem for Octagons: A special case
Inscribed Conic in Pascal's theorem
Involution on A Line
Involution on A Conic
Desargues' Involution Theorem
Desargues' Involution Theorem 2 Points
Maximality of the Area of a Cyclic Quadrilateral
E.R.I.Q. (Equal Ration in Quadrilateral) Lemma
Steiner's Isogonal Cevian Lemma
Erdos-Mordell Theorem (Forum Geometricorum Volume 1 (2001) 7-8)
Fontene's First Theorem
Fontene's Second Theorem
NT
Wythoff Array
Frobenius Coin Problem
Beatty's Theorem
Cyclotomic Formulas
Prime divisors of an integer polynomial
Euler's Criterion
Gauss's Criterion
Quadratic Residue Law
Wolstenholme's Theorem
Product of Cyclotomic Polynomials
Prime Divisors of Cyclotomic Polynomials