GEOMETRY

IDEAS & LEMMAS

LEMMAS

Isogonality:

1. Let P, Q be points in the plane of triangle ΔABC such that AP, AQ are symmetric wrt the bisector of $\angle BAC$. Let $\cap CP \equiv X, BP \cap CQ \equiv Y$. Then we have AX, AY symmetric wrt $\angle BAC$.

Incircle:

Let I be the incenter of ΔABC, and let D, E, F be the touch points of the incircle with the sides BC, CA, AB resp.

Let P, Q, R be the midpoints of the arcs BC, CA, AB resp. Let H, O, ω_n, ω be the orthocenter, circumcenter, nine-point circle and circumcircle of ΔABC.

Let L, M, N be the midpoints of BC, CA, AB resp.

Let I_A be the A-excircle of ΔABC, and let D_A, E_A, F_A be the touch points of the A-excircle with the sides BC, CA, AB resp.

Let A' be the antipode of A.

1. $DF, D_A F_A, AI, LM$ are concurrent at X_{A_B}, and $CX \perp AI$. [Angle Chasing]
2. $DF, D_A E_A, AH$ are concurrent. [Again Angle Chasing] {PAUL YUI TEOREM}
3. $A' I, \omega, \odot AEIF$ are concurrent at Y_A. [Straightforward]
4. Y_A, D, P are collinear. [Inversion centered at P]
5. Let the altitude from D to EF meet EF at D_H. Then D_H, I, A' are collinear. [Spiral similarity centered at Y_A]
6. Let X be a point on ΔABC, let the tangents from X to $\odot I$ meet $\odot ABC$ at Y, Z. Then YZ is tangent to $\odot I$. [Just invert wrt $\odot I$]
7. In ΔABC, I is the incenter. Let the incircle touch BC at H. Let D be any point on BC. Let P, Q be the incenters of $\Delta ABD, \Delta ACD$. Then $\square HDPQ$ is cyclic. The common internal tangent to $\odot P, \odot Q$ except AD passes through H.
8.

An IRAN TST 2012 G3

Problem:

In ΔABC, O is the circumcircle. Points $A^{'}, B^{'}, C'$ are on lines BC, CA, AB resp. such that $(A, B^{'}, C^{'}, O), (B, C^{'}, A^{'}, O), (C, A^{'}, B^{'}, O)$ lie on a circle. Define (X, XY) to be the circle centered at X, with radius XY. Let l_a be the radical axis of $(B^{'}, B^{'}C)$ and $(C^{'}, C^{'}B)$. Define l_b and l_c similarly. PROVE that the orthocenter of the triangle formed by l_a, l_b, l_c is the orthocenter of ΔABC.

Let $l_c \cap l_b \equiv A_0, l_a \cap l_b \equiv C_0, l_a \cap l_c \equiv B_0$.

Things in the Figure:

1. $(B^{'}, B^{'}C)$, $(C^{'}, C^{'}B)$, $\odot ABC$ and $(B^{'}, B^{'}C)$, $(C^{'}, C^{'}B)$, BC are concurrent.
2. $l_a \parallel OA^{'}, l_b \parallel OB^{'}, l_c \parallel OC^{'}$.
3. Let $D^{'}$ be the intersection point of A-altitude with $\odot ABC$. Then l_a passes through $D^{'}$.

PROBLEMS

1. USAMO 2014 P5: Use angle chase to prove that $XY \vee ¿ AH$, some intuition.

2. In ΔABC, H is the orthocenter, and AD, BE are arbitrary cevians. Let ω_1, ω_2 denote the circles with diameters AD, BE resp. HD, HE meet ω_1, ω_2 again at F, G. DE meet ω_1, ω_2 again at P_1, P_2. FG meet ω_1, ω_2 again at Q_1, Q_2. P_1H, P_2H meet ω_1, ω_2 at R_1, R_2 and Q_1H, Q_2H meet ω_1, ω_2 at S_1, S_2. $P_1Q_1 \cap P_2Q_2 = X$ and $R_1S_1 \cap R_2S_2 = Y$. Prove that X, Y, H are collinear.

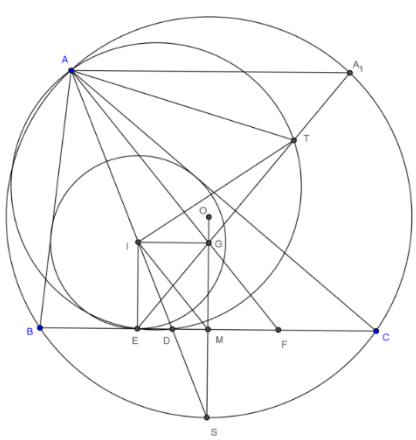
   Radical Axis and POP.

3. Iran TST 2017 P3: Thought of using Miquel point, got some equal lengths, but alas :(Should have worked backward.

4. Iran TST 2015 P18: Isogonal Conjugate.

5. Iran TST 2015 P6: Cool Geo. Radical Axis, Angle chase, POP, Projective.

6. Look at the fig: (Iran TST 2015 P8) $\Im \vee ¿ AF$



7. Iran TST 2017 P5 (Hard Version): Take the second intersection point of AM and the circumcircle of ΔAPQ. One way from there is to take A' such that $AA' \vee ¿ BC$ (length bash :p) (Pascal), or just chase angles.

8. Balkan MO 2017 P3:_S_ Consider an acute-angled triangle ABC with $AB < AC$ and let ω be its circumscribed circle. Let t_B and t_C be the tangents to the circle ω at points B and C, respectively, and let L be their intersection. The straight line passing through the point B and parallel to AC intersects t_C in point D. The straight line passing through the point C and parallel to AB intersects t_B in point E. The circumcircle of the triangle BDC intersects AC in T, where T is located between A and C. The circumcircle of the triangle BEC intersects the line AB at S, where B is located between S and A. Prove that AL, ST and BC are concurrent.

9. All Russia 2014 P10.4: Given a triangle ABC with $AB > BC$, let ω be the circumcircle. Let M, N lie on the sides AB, BC respectively, such that $AM = CN$. Let K be the intersection of MN and AC

. Let P be the incentre of the triangle AMK and Q be the K-excentre of the triangle CNK. If R is midpoint of the arc ABC of ω then prove that $RP=RQ$.

Hello Spiry my old friend...

10. All Russia 2013 P11.8:_H_ Let ω be the incircle of the triangle ABC and with centre I. Let γ be the circumcircle of the triangle AIB. Circles ω and γ intersect at the point X and Y. Let Z be the intersection of the common tangents of the circles ω and γ. Show that the circumcircle of the triangle XYZ is tangent to the circumcircle of the triangle ABC.

Inversion, POP, Radical Axis. Another way to approach: Angle chase, Homothety, Isogonality, Length chase.

11. China TST 2011 Quiz 2 D2.P1: Let AA', BB', CC' be three diameters of the circumcircle of an acute triangle ABC. Let P be an arbitrary point in the interior of ΔABC, and let D, E, F be the orthogonal projection of P on BC, CA, AB respectively. Let X be the point such that D is the midpoint of $A'X$. Define Y, Z similarly. Prove that ΔXYZ is similar to ΔABC.

Too many reflections, does this ring a bell??

12. IMO Shortlist 2016 G6:_H_ Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC < 90°$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at E and F respectively, and meet each other at point P. Let M be the midpoint of AC and let ω be the circumcircle of triangle BPD. Segments BM and DM intersect ω again at X and Y respectively. Denote by Q the intersection point of lines XE and YF. Prove that $PQ \perp AC$.

13. USA TST 2000 P2: _E_ Let $ABCD$ be a cyclic quadrilateral and let E and F be the feet of perpendiculars from the intersection of diagonals AC and BD to AB and CD, respectively. Prove that EF is perpendicular to the line through the midpoints of AD and BC.

14. Erdős-Mordell Theorem: If from a point O inside a given triangle ABC perpendiculars OD, OE, OF are drawn to its sides, then $OA+OB+OC \geq 2(OD+OE+OF)$. Equality holds if and only if triangle ABC is equilateral.

Forum Geometricorum Volume 1 (2001) 7–8 (Hoojo Lee).

Apparently nothing is needed except Ptolemy's Theorem. Think of a way to connect OA with OE, OF and the sides of the triangle. As it is the most natural to use AB, AC, we have to deal with BE, CF too. And dealing with lengths is the easiest when we have similar triangles. So we do some construction.

- TelvCohl's $\sqrt{bc}$ inversion problem collection.

15. IRAN 3rd 2016 G1.P1:_E_ Let ABC be an arbitrary triangle, P is the intersection point of the altitude from C and the tangent line from A to the circumcircle. The bisector of angle A intersects BC at . PD intersects AB at K, if H is the orthocenter then prove : $HK \perp AD$

Draw a good figure moron.

16. IRAN 3rd 2016 G1.P2:_E_ Let ABC be an arbitrary triangle. Let E, F be two points on AB, AC respectively such that their distance to the midpoint of BC is equal. Let P be the second intersection of the triangles

$\odot ABC, \odot AEF$. The tangents from E, F to $\odot AEF$ intersect each other at K. Prove that : $\angle KPA = 90°$.

17. IRAN 2nd 2016 P6:_E_ Let ABC be a triangle and X be a point on its circumcircle. Q, P lie on a line BC such that $XQ \perp AC, XP \perp AB$. Let Y be the circumcenter of ΔXQP. Prove that ΔABC is equilateral triangle if and if only Y moves on a circle when X varies on $\odot ABC$.

18. AoPS:_M_ Let ABC be a triangle with incircle (I) and A-excircle (I_a). $(I),(I_a)$ are tangent to BC at D, P respectively. Let $(I_1),(I_2)$ be the incircle of triangles APC, APB respectively, $(J_1),(J_2)$ be the reflections of $(I_1),(I_2)$ wrt midpoints of AC, AB.Prove that AD is the radical axis of (J_1) and $(J¿¿2)¿$.

19. AoPS: _M_ Let ABC be a $A-¿$right-angled triangle and $MNPQ$ a square inscribed into it, with M, N onto BC in order $B-M-N-C$, and P, Q onto CA, AB respectively.
Let $R = BP \cap QM, S = CQ \cap PN$. Prove that $AR = AS$ and RS is perpendicular to the $A-¿$inner angle bisector of ΔABC.

⪢ Let U be the point on the A symmedian such that BU, CU are tangents on $\odot ABC$. After inverting across A, U goes to a point U' such that $\odot AU'B', \odot AU'C'$ are tangent to $B'C'$.

20. AoPS: Given are $\Delta ABC, L$ is the Lemoine point, L_a, L_b, L_c are the three Lemoine points of triangles $\Delta LBC, \Delta LAC, \Delta LAB$ Prove that AL_a, BL_b, CL_c are concurrent.

A question: What is the locus of P such that AL_a, BL_b, CL_c are concurrent with L_a, L_b, L_c are three Lemoine point of triangles $\Delta PBC, \Delta PAC, \Delta PAB$.

21. AoPS: P is an arbitrary point on the plane of ΔABC and let $\Delta A'B'C'$ be the cevian triangle of P WRT ΔABC. The circles $\odot ABB'$ and $\odot ACC'$ meet at A, X Similarly, define the points Y and Z WRT B and C. Prove that the lines AX, BY, CZ concur at the isogonal conjugate of the complement of P WRT ΔABC.

⪢ Compliment of point P wrt ΔABC: Reflect P over the midpoints of the sides of ΔABC and get P_a, P_b, P_c. The compliment of point P is the concurrency point of AP_a, BP_b, CP_c.

22. AoPS:_E_ Consider ΔABC with orthic triangle $\Delta A'B'C'$, let $AA' \cap B'C' = E$ and E' be reflection of E wrt BC. Let M be midpoint of BC and O be circumcenter of $\Delta E'B'C'$. Let M' be projection of O on BC and N be the intersection of a perpendicular to $B'C'$ through E with BC. Prove that $MM' = \frac{1}{4}MN$.

23. AoPS: _M_ Let ABC be a triangle inscribed circle (O). Let (O') be the circle wich is tangent to the circle $(O)$and the sides CA, AB at $D$and E, F, respectively. The line BC intersects the tangent line at A of (O), EF and AO' at T, S and L, respectively. The circle (O) intersects AS again at K. Prove that the circumcenter of triangle AKL lies on the circumcircle of triangle ADT.

24. IRAN 3rd 2010 D3.P5:_M_ In a ΔABC, I is the incenter. D is the reflection of A to I. the incircle is tangent to BC at point E. DE cuts IG at P (G is centroid). M is the midpoint of BC. Prove that a) $AP \vee ¿ DM$ b) $AP = 2DM$

It's easy after discovering this: 6

25. IRAN 3rd 2010 D3.P6:_M_ In a triangle ABC, $\angle C = 45$. AD is the altitude of the triangle. X is on AD such that $\angle XBC = 90 - \angle B$ (X is in the triangle). AD and CX cut the circumcircle of ABC in M and N respectively. if tangent to circumcircle of ABC at M cuts AN at P, prove that P, B and O are collinear.

26. IRAN 3rd 2011 G5:_M_ Given ΔABC, D is the foot of the external angle bisector of A, I its incenter and I_a its A-excenter. Perpendicular from I to DI_a intersects the circumcircle of triangle in A'. Define B' and C' similarly. Prove that AA', BB' and CC' are concurrent.

27. AoPS:_E_ I is the incenter of a non-isosceles triangle ΔABC .If the incircle touches BC, CA, AB at A', B', C' respectively, prove that the circumcenter of the triangles $\Delta AIA', \Delta BIB', \Delta CIC'$ are collinear.

28. AoPS:_M_ Given ΔABC and a point P inside. AP cuts BC at M. Let M', P' be the reflection of M, P in the perpendicular bisector of BC. Let Q be the isogonal conjugate of P in ΔABC. Prove that $QM' \parallel AP'$.

Come on Spiry my ol' friend.

29. AoPS:_M_ Let ABC be a triangle inscribed in (O) and P be a point. Call Q be the isogonal conjugate point of P. Let S be the second intersection of AQ and (O). Denote by M the intersection of BC and SP. Prove that $QM \parallel AP$.

MORAL OF THE STORY: If the problem has isogonal conjugate and some intersections with the circumcircle, bring in the reflection of the original point P wrt the perp bisector of BC.

30. _H_ Let P and Q be isogonal conjugates of each other. Let $\Delta XYZ, \Delta KLM$ be the pedal triangles of P and Q wrt ΔABC. (X, K lie on BC; Y, L lie on CA; Z, M lie on AB) Prove that YM, ZL, PQ are concurrent.

31. 2nd Olympiad of Metropolises:_H_ Let $ABCDEF$ be a convex hexagon which has an inscribed circle and a circumscribed circle. Denote by $\omega_A, \omega_B, \omega_C, \omega_D, \omega_E$, and ω_F the inscribed circles of the triangles $FAB, ABC, BCD, CDE, \stackrel{\text{def}}{=} ¿$,and EFA,respectively. Let l_{AB} be the external common tangent of ω_A and ω_B other than the line AB; lines $l_{BC}, l_{CD}, l_{DE}, l_{EF}$, and l_{FA} are analogously defined. Let A_1 be the intersection point of thelines l_{FA} and l_{AB}; B_1 be the intersection point of the lines l_{AB} and l_{BC}; points C_1, D_1, E_1, and $F1$ are analogously defined. Suppose that $A_1B_1C_1D_1E_1F_1$ is a convex hexagon. Show that its diagonals A_1D_1, B_1E_1, and C_1F_1 meet at a single point.

Finding Stuffs.

32. AoPS:_M_ Let ΔABC, circumcenter O and altitude AH. AO meets BC at M and meets the circle $\odot BOC$ again at N. P is the midpoint of MN. K is the projection of P on line AH. Prove that the circle $\odot\left(K, KH\right)$ is tangent to the circle $\odot BOC$.

33. AoPS:_E_ I is the incenter of ΔABC. The line perpendicular to BC passing through I cuts $\odot ABC$ at P, Q. $AP, AQ \cap BC \equiv X, Y$. Prove that $\square AIXY$ is cyclic.

34. AoPS:_E_ In ΔABC, I is the incenter, D, E, F are the touchpoints of the incircle. A', B', C' are the feet of the perpendiculars. If X is the orthocenter of ΔAEF, then prove that X is the incenter of $\Delta AB'C'$.

35. USA TST 2017 P2:_M_ Let ABC be an acute scalene triangle with circumcenter O, and let T be on line BC such that $\angle TAO = 90°$. The circle with diameter AT intersects the circumcircle of ΔBOC at two points A_1 and A_2, where $OA_1 < OA_2$. Points B_1, B_2, C_1, C_2 are defined analogously.

- Prove that AA_1, BB_1, CC_1 are concurrent.

- Prove that AA_2, BB_2, CC_2 are concurrent on the Euler line of ΔABC.

PENDING

1. IOM 2017 P1: Let $ABCD$ be a parallelogram in which angle at B is obtuse and $AD > AB$. Points K and L on AC such that $\angle ADL = \angle KBA$ (the points A, K, C, L are all different, with K between A and L). The line BK intersects the circumcircle ω of ABC at points B and E, and the line EL intersects ω at points E and F. Prove that $BF \parallel AC$.
2. AoPS: Let $ABCD$ be a quadrilateral inscribed in a circle, such that the inradius of triangle ABC and ACD are the same. Let T be the A-mixtillinear incircle of the triangle ABD.Let I_1, I_2 be the incenters of the triangles ABC, ACD respectively. Show that $I_1 I_2$ the tangent of A, T wrt $\odot ABCD$ are concurrent.
3. AoPS: Let ABC be a triangle with circumcenter O and incenter I. Let K_A be the symmedian point of ΔBIC and define K_B, K_C similarly. Let X be the midpoint of AI and define Y, Z similarly. Show that $X K_A, Y K_B, Z K_C, OI$ concur.
4. IMO SL 2012 G8: Let ABC be a triangle with circumcircle ω and l a line without common points with ω. Denote by P the foot of the perpendicular from the center of ω to l. The side-lines BC, CA, AB intersect l at the points X, Y, Z different from P. Prove that the circumcircles of the triangles AXP, BYP and CZP have a common point different from P or are mutually tangent at P.
5. IGO Advanced P3: In a triangle ABC with circumcenter O, the line CO cuts the altitude though A at K. Let P, M be the midpoints of AK, AC respectively. PO cuts BC at S. $\odot BMC$ cuts AB again at T.Prove that $BTOS$ is cyclic .
6. IGO Advanced P4: Three circles W_1, W_2 & W_3 touches a line L at A, B, C respectively (B lies between A & C). W_2 touches W_1 & W_3. Let L_2 be the other common external tangent of W_1 & W_3. L_2 cuts W_2 at X, Y. Perpendicular to L at B intersects W_2 again at K. Prove that KX and KY are tangent to the circle with diameter AC.
7. AoPS
8. IMO SL 2012 G8:_EEH_ Let ABC be a triangle with circumcircle ω and l a line without common points with ω. Denote by P the foot of the perpendicular from the center of ω to l. The side-lines BC, CA, AB intersect l at the points X, Y, Z different from P. Prove that the circumcircles of the triangles AXP, BYP and CZP have a common point different from P or are mutually tangent at P.
9. buratinogigle Tough Prob1:_H_ Let ABC be a triangle inscribed in circle (O) with A-excircle (J). Circle passing through A, B touches (J) at M. Circle passing through A, C touches (J) at N. BM cuts CN at P. Prove that APpasses through tangent point of A-mixtilinear incircle with (O).
10. FUCKIT: In ΔABC, I is the incenter, D is the touch point of the incenter with BC. $AD \cap \odot ABC \equiv X$. The tangents line from X to $\odot I$ meet $\odot ABC$ at Y, Z. Prove that YZ, BC and the tangent at A to $\odot ABC$ concur.