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0.1 Conjugates

0.1.1 Isogonal Conjugate

Theorem 0.1.1 (Isogonal Line Lemma) — Let AP, AQ are isogonal lines with respect to $\angle BAC$. Let $BP \cap CQ = F$ and $BQ \cap CP = E$. Then AE, AF are isogonal lines with respect to $\angle BAC$.

Proof.

$$\begin{aligned} A(B,F;P,X) &= (B,F;P,X) = C(B,Q;E,X) \\ &= (B,Q;E,X) = (X,E;Q,B) \end{aligned}$$

So if we define a projective transformation that swaps isogonal lines wrt $\angle BAC$, we see AE, AF are conjugates of each other.

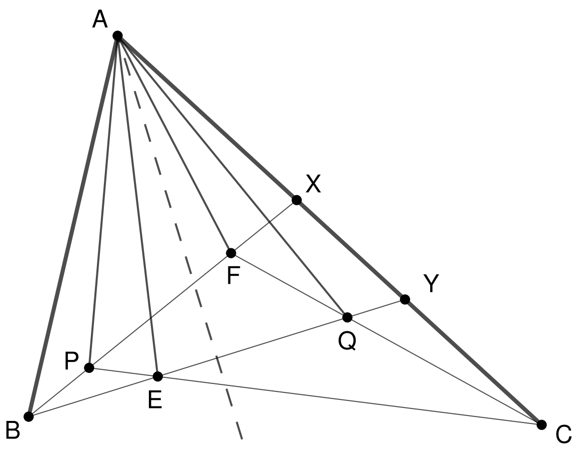

Figure 0.1

Problem 0.1.1 (India Postals 2015 Set 2). Let $ABCD$ be a convex quadrilateral. In the triangle ABC let I and J be the incenter and the excenter opposite the vertex A, respectively. In the triangle ACD let K and L be the incenter and the excenter opposite the vertex A, respectively. Show that the lines IL and JK, and the bisector of the angle BCD are concurrent.

Solution. Using Theorem 0.1.1

Lemma 0.1.2 — Let ω_1, ω_2 be two circles such that ω_1 passes through A, B and is tangent to AC at A. ω_2 is defined similarly by swapping B with C. $\omega_1 \cap \omega_2 = X$.
Let γ_1, γ_2 be two circles such that γ_1 passes through A, B and is tangent to BC at B. γ_2 is defined similarly by swapping B with C. $\gamma_1 \cap \gamma_2 = Y$.
Then X, Y are isogonal conjugates wrt $\triangle ABC$.

Lemma 0.1.3 (Isogonality in quadrilateral) — For a point X, its isogonal conjugate wrt a quadrilateral $ABCD$ exists iff

$$\measuredangle BXA + \measuredangle DXC = 180^\circ$$

Solution. Draw the cirles, look for similarity.

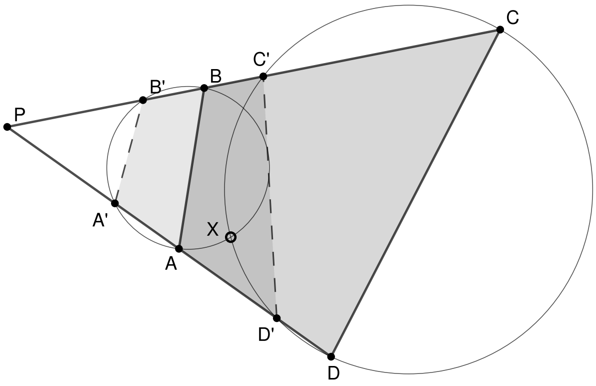

Figure 0.2: Isogonality in quadrilateral

Lemma 0.1.4 (Ratio) — Given a $\triangle ABC$ with isogonal conjugate P, Q. Let AP, AQ cut the circumcircle of $\triangle ABC$ again at U, V, respectively and let $D \equiv AP \cap BC$. Then

$$\frac{AQ}{QV} = \frac{PD}{DU}$$

Proof. By using cross ratio:

$$\begin{aligned}(A,F;Q,V) &= C(A,F;Q,V)\\ &= C(D,A;P,V*) = (D,A;P,V*)\\ &= (A,D;V*,P)\end{aligned}$$

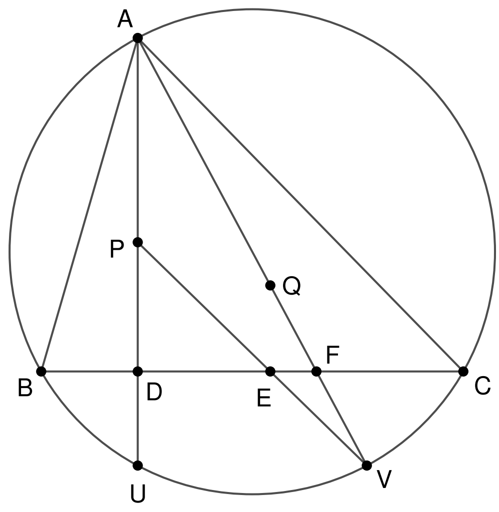

Figure 0.3

0.1.1.1 Symmedians

Definition (Symmedians)— In $\triangle ABC$, let T_a, T_b, T_c be the meet points of the tangents at A, B, C. Let $\triangle N_a N_b N_c$ be the cevian triangle of AT_a, BT_b, CT_c. Let S be the symmedian point of $\triangle ABC$. Let M_a, M_b, M_c be the midpoints of BC, CA, AB.

> **Lemma 0.1.5** (Most Important Symmedian Property) — Let the circles tangent to AC, AB at A and passes through B, C respectively meet at T' for the second time. Let $AT_a \cap \odot ABC = A'$. Let the tangents to $\odot ABC$ at A, A' meet BC at T. Prove that, A, T', T_a, and T, T', O are collinear.

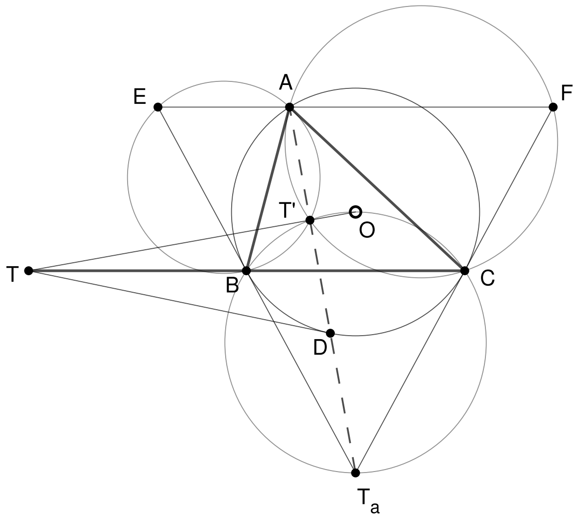

Figure 0.4: T' is quite special!

Problem 0.1.2 (USAMO 2008 P2). Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of BC, CA, and AB, respectively. Let the perpendicular bisectors of AB and AC intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.

> **Solution** [Phantom Point]. First assume $F \in BD$, and $F = T'$ (Where T' comes from Lemma 0.1.5, and prove that $F \in CE$.)

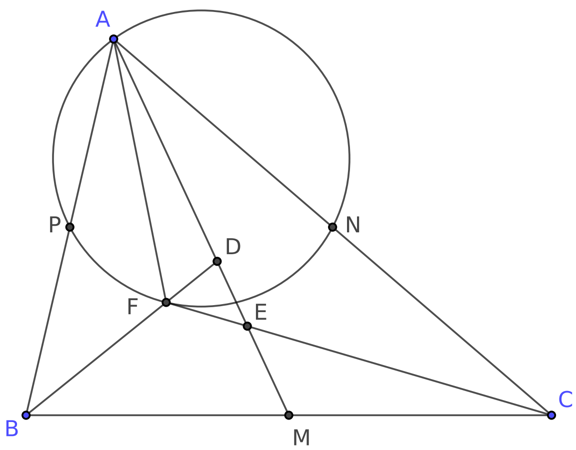

Figure 0.5: USAMO 2008 P2

> **Solution** [Isogonal Conjugate]. Construct the isogonal conjugate of F, which is the intersection of the circles touching BC and passing through A, B and A, C.

> **Solution.** Using Theorem 0.1.1 by taking the reflections of B, C over D, F

Problem 0.1.3 (IRAN TST 2015 Day 3, P3). AH is the altitude of triangle ABC and H' is the reflection of H trough the midpoint of BC. If the tangent lines to the circumcircle of ABC at B and C, intersect each other at X and the perpendicular line to XH' at H', intersects AB and AC at Y and Z respectively, prove that $\angle ZXC = \angle YXB$.

Problem 0.1.4 (`Two Symmedian Points`). Let E,F be the feet of B,C-altitudes. Let K,K_A be the symmedian points of $\triangle ABC, \triangle AEF$. Prove that $KK_A \perp BC, KK_A \cap BC = P$ and $KK_A = KP$

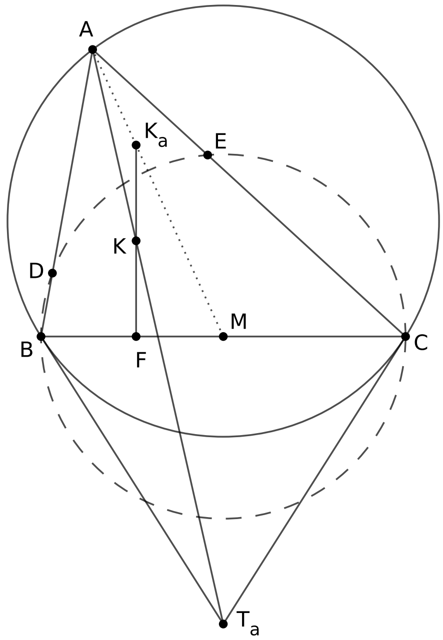

Figure 0.6: $KK_A \perp BC$

0.1.2 Isotonic Conjugate

Theorem 0.1.6 (Isotonic Lemma) — Let M be the midpoint of BC, and PQ such that Q is the reflection of P on M. Two points Q,R on AP,AQ, $BQ \cap CR = X$, $BR \cap CQ = Y$. Then AX,AY are isotonic wrt BC.

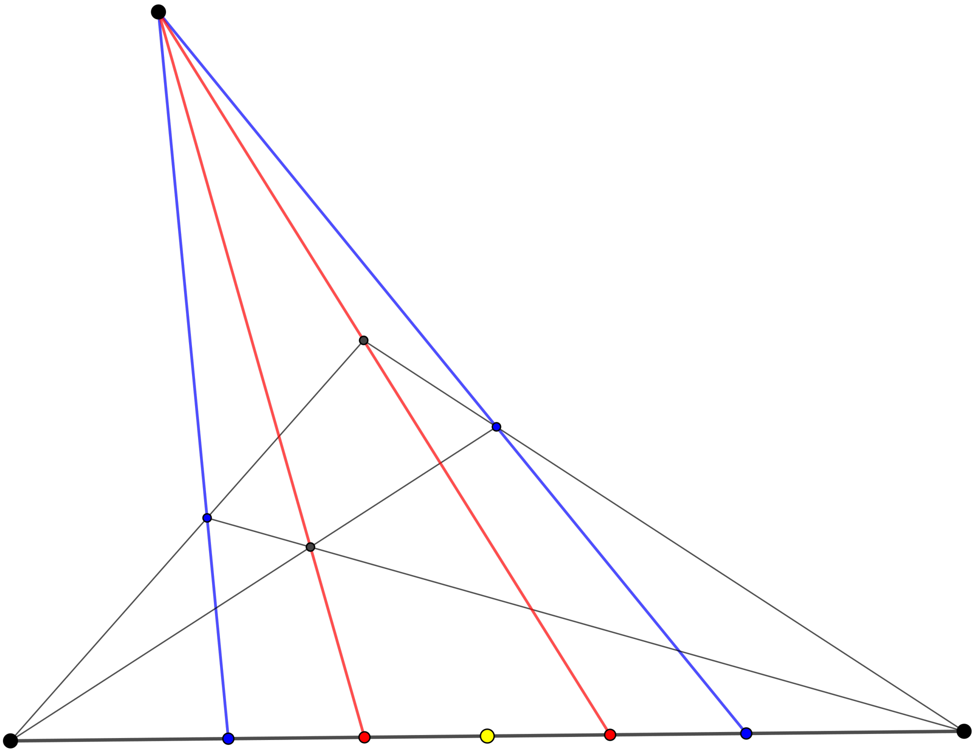

Figure 0.7

Problem 0.1.5 (IGO 2014 S5). Two points P and Q lying on side BC of triangle ABC and their distance from the midpoint of BC are equal.The perpendiculars from P and Q to BC intersect AC and AB at E and F,respectively.M is point of intersection PF and

EQ.If H_1 and H_2 be the orthocenters of triangles BFP and CEQ, respectively, prove that $AM \perp H_1H_2$.

> **Solution.** We first show that the slope of H_1H_2 is fixed, and then show that AM is fixed where we use isotonic lemma, and finally show that these two lines are perpendicular.

0.1.3 Reflection

Lemma 0.1.7 (Homothety and Reflection) — Let two oppositely oriented congruent triangles be $\triangle ABC, \triangle DEF$. Prove that the midpoints of AD, BE, CF are collinear.

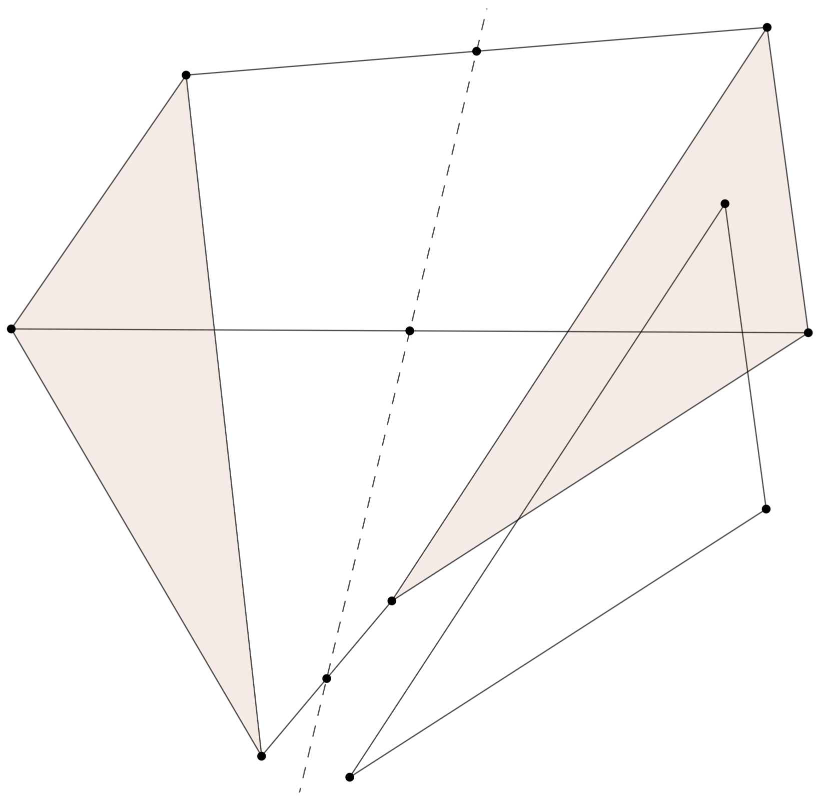

Figure 0.8: Oppositely oriented congruent triangles

Problem 0.1.6 (`Autumn Tournament, 2012`). Let two oppositely oriented equilateral triangles be $\triangle ABC, \triangle DEF$. What is the least possible value of $\max(AD, BE, CF)$?