I



GEOMETRY

IDEAS & LEMMAS



## LEMMAS

Isogonality:

1. Let be points in the plane of triangle such that are symmetric wrt the bisector of . Let . Then we have symmetric wrt .

Incircle:

Let be the incenter of, and let be the touch points of the incircle with the sides resp.

Let be the midpoints of the arcs resp. Let be the orthocenter, circumcenter, nine-point circle and circumcircle of

Let be the midpoints of resp.

Let be the A-excircle of, and let be the touch points of the A-excircle with the sides resp.

Let be the antipode of .

1. are concurrent at, and . [Angle Chasing]
2. are concurrent. [Again Angle Chasing] {PAUL YUI TEOREM}
3. are concurrent at . [Straightforward]
4. are collinear. [Inversion centered at ]
5. Let the altitude from to meet at. Then are collinear. [Spiral similarity centered at ]
6. Let be a point on let the tangents from to meet at. Then is tangent to . [Just invert wrt ]
7. In, is the incenter. Let the incircle touch at. Let be any point on. Let be the incenters of. Then is cyclic. The common internal tangent to except passes through.

An IRAN TST 2012 G3

Problem:

In , is the circumcircle. Points are on lines resp. such that lie on a circle. Define to be the circle centered at , with radius . Let be the radical axis of and . Define and similarly. PROVE that the orthocenter of the triangle formed by is the orthocenter of .

Let .

Things in the Figure:

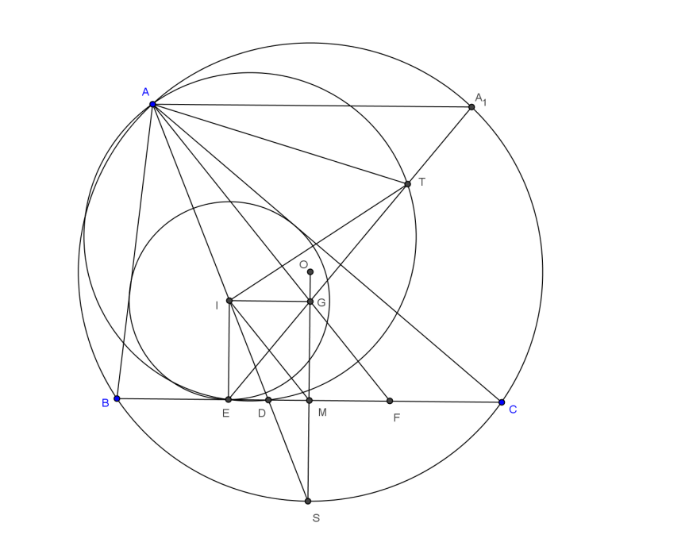
1. , , and , , are concurrent.
2. .
3. Let be the intersection point of A-altitude with . Then passes through .

## PROBLEMS

1. USAMO 2014 P5: Use angle chase to prove that , some intuition.
2. In , is the orthocenter, and are arbitrary cevians. Let denote the circles with diameters resp. meet again at . meet again at meet again at . meet at and meet at . and . Prove that are collinear.

Radical Axis and POP.

1. Iran TST 2017 P3: Thought of using Miquel point, got some equal lengths, but alas :( Should have worked backward.
2. Iran TST 2015 P18: Isogonal Conjugate.
3. [Iran TST 2015 P6](https://artofproblemsolving.com/community/c6h1087613p4817114): Cool Geo. Radical Axis, Angle chase, POP, Projective.
4. Look at the fig: (Iran TST 2015 P8)



1. [Iran TST 2017 P5](https://artofproblemsolving.com/community/c6h1423742p8012536) (Hard Version): Take the second intersection point of and the circumcircle of . One way from there is to take such that (length bash :p) (Pascal), or just chase angles.
2. [Balkan MO 2017 P3](https://artofproblemsolving.com/community/c6h1441717p8209926):\_S\_ Consider an acute-angled triangle  with  and let  be its circumscribed circle. Let  and  be the tangents to the circle  at points  and , respectively, and let  be their intersection. The straight line passing through the point  and parallel to  intersects  in point . The straight line passing through the point  and parallel to  intersects  in point . The circumcircle of the triangle  intersects  in , where  is located between  and . The circumcircle of the triangle  intersects the line  at , where  is located between  and . Prove that  and  are concurrent.
3. [All Russia 2014 P10.4](https://artofproblemsolving.com/community/c6h587990p3480801): Given a triangle  with , let  be the circumcircle. Let  lie on the sides  respectively, such that . Let be the intersection of  and . Let  be the incentre of the triangle  and  be the -excentre of the triangle . If  is midpoint of the arc  of  then prove that .

Hello Spiry my old friend…

1. [All Russia 2013 P11.8](https://artofproblemsolving.com/community/c6h535325p3070851):\_H\_ Let  be the incircle of the triangle  and with centre . Let  be the circumcircle of the triangle . Circles  and intersect at the point  and . Let  be the intersection of the common tangents of the circles  and . Show that the circumcircle of the triangle  is tangent to the circumcircle of the triangle .

Inversion, POP, Radical Axis. Another way to approach: Angle chase, Homothety, Isogonality, Length chase.

1. [China TST 2011 Quiz 2 D2.P1](https://artofproblemsolving.com/community/c6h407514p2276387): Let be three diameters of the circumcircle of an acute triangle. Let be an arbitrary point in the interior of, and let be the orthogonal projection of on respectively. Let be the point such that is the midpoint of . Define similarly. Prove that is similar to .

Too many reflections, does this ring a bell??

1. [IMO Shortlist 2016 G6](https://artofproblemsolving.com/community/c6h1480719p8639316):\_H\_ Let be a convex quadrilateral with . The internal angle bisectors of and meet at and respectively, and meet each other at point . Let be the midpoint of and let be the circumcircle of triangle . Segments and intersect again at and respectively. Denote by the intersection point of lines and . Prove that .
2. [USA TST 2000 P2](https://artofproblemsolving.com/community/c6h326960p1752047): \_E\_ Let be a cyclic quadrilateral and let and be the feet of perpendiculars from the intersection of diagonals and to and , respectively. Prove that is perpendicular to the line through the midpoints of and .
3. Erdős-Mordell Theorem: If from a point inside a given triangle perpendiculars are drawn to its sides, then . Equality holds if and only if triangle is equilateral.

Forum Geometricorum Volume 1 (2001) 7–8 (Hoojo Lee).

Apparently nothing is needed except Ptolemy’s Theorem. Think of a way to connect with and the sides of the triangle. As it is the most natural to use , we have to deal with too. And dealing with lengths is the easiest when we have similar triangles. So we do some construction.

[TelvCohl’s inversion problem collection.](http://artofproblemsolving.com/community/c5h1101224p5015914)

1. [IRAN 3](https://artofproblemsolving.com/community/c6h1289432p6815893)[rd](https://artofproblemsolving.com/community/c6h1289432p6815893) [2016 G1.P1](https://artofproblemsolving.com/community/c6h1289432p6815893):\_E\_ Let be an arbitrary triangle, is the intersection point of the altitude from and the tangent line from to the circumcircle. The bisector of angle intersects at . intersects at , if is the orthocenter then prove :

Draw a good figure moron.

1. [IRAN 3rd 2016 G1.P2](https://artofproblemsolving.com/community/c6h1291293p6832329):\_E\_ Let be an arbitrary triangle. Let be two points on respectively such that their distance to the midpoint of is equal. Let be the second intersection of the triangles . The tangents from to intersect each other at . Prove that : .
2. [IRAN 2nd 2016 P6](https://artofproblemsolving.com/community/c6h1434843p8120660):\_E\_ Let be a triangle and be a point on its circumcircle. lie on a line such that . Let be the circumcenter of . Prove that is equilateral triangle if and if only moves on a circle when varies on .
3. [AoPS](https://artofproblemsolving.com/community/c6h1490457p8788075):\_M\_ Let  be a triangle with incircle  and -excircle .  are tangent to  at  respectively. Let  be the incircle of triangles  respectively,  be the reflections of  wrt midpoints of .Prove that  is the radical axis of  and .
4. [AoPS](https://artofproblemsolving.com/community/c6h1491448p8781804): \_M\_ Let  be a right-angled triangle and  a square inscribed into it, with  onto  in order , and  onto  respectively. Let . Prove that  and  is perpendicular to the inner angle bisector of .

Let be the point on the symmedian such that are tangents on . After inverting across , goes to a point such that are tangent to .

1. [AoPS](https://artofproblemsolving.com/community/c6h108111p8724592): Given are  is the Lemoine point, are the three Lemoine points of triangles Prove that  are concurrent.

*A question: What is the locus of  such that  are concurrent with  are three Lemoine point of triangles .*

1. [AoPS](https://artofproblemsolving.com/community/c6h282732):  is an arbitrary point on the plane of  and let  be the cevian triangle of  WRT . The circles  and  meet at  Similarly, define the points  and  WRT  and . Prove that the lines  concur at the isogonal conjugate of the complement of  WRT .

Compliment of point wrt : Reflect over the midpoints of the sides of and get . The compliment of point is the concurrency point of .

1. [AoPS](https://artofproblemsolving.com/community/c6h1460735p8471885):\_E\_ Consider  with orthic triangle , let  and  be reflection of  wrt . Let  be midpoint of  and  be circumcenter of . Let  be projection of  on  and  be the intersection of a perpendicular to  through  with . Prove that .
2. [AoPS](https://artofproblemsolving.com/community/c6h1432386p8088082): \_M\_ Let  be a triangle inscribed circle . Let  be the circle wich is tangent to the circle and the sides  at and , respectively. The line  intersects the tangent line at  of ,  and  at  and , respectively. The circle  intersects  again at . Prove that the circumcenter of triangle  lies on the circumcircle of triangle .
3. [IRAN 3rd 2010 D3.P5](https://artofproblemsolving.com/community/c6h360730p1973873):\_M\_ In a , is the incenter. is the reflection of to . the incircle is tangent to at point . cuts at ( is centroid). is the midpoint of . Prove that a) b)

It’s easy after discovering this: 6

1. [IRAN 3rd 2010 D3.P6](https://artofproblemsolving.com/community/c3497):\_M\_ In a triangle . AD is the altitude of the triangle. is on such that ( is in the triangle). and cut the circumcircle of in and respectively. if tangent to circumcircle of at cuts at , prove that and are collinear.
2. [IRAN 3rd 2011 G5](https://artofproblemsolving.com/community/c6h429226p2428694):\_M\_ Given , is the foot of the external angle bisector of , its incenter and its -excenter. Perpendicular from to intersects the circumcircle of triangle in . Define and similarly. Prove that and are concurrent.
3. [AoPS](https://artofproblemsolving.com/community/c6h283185):\_E\_  is the incenter of a non-isosceles triangle  .If the incircle touches  at  respectively, prove that the circumcenter of the triangles  are collinear.
4. [AoPS](https://artofproblemsolving.com/community/c6h1501095p8898504):\_M\_ Given  and a point  inside.  cuts  at . Let  be the reflection of  in the perpendicular bisector of . Let  be the isogonal conjugate of  in . Prove that .

Come on Spiry my ol’ friend.

1. [AoPS](https://artofproblemsolving.com/community/c6h46718):\_M\_ Let  be a triangle inscribed in  and  be a point. Call  be the isogonal conjugate point of . Let  be the second intersection of  and . Denote by  the intersection of  and . Prove that .

MORAL OF THE STORY: If the problem has isogonal conjugate and some intersections with the circumcircle, bring in the reflection of the original point wrt the perp bisector of .

1. \_H\_ Let and be isogonal conjugates of each other. Let be the pedal triangles of and wrt . ( lie on ; lie on ; lie on ) Prove that are concurrent.
2. 2nd Olympiad of Metropolises:\_H\_ Let be a convex hexagon which has an inscribed circle and a circumscribed circle. Denote by , and the inscribed circles of the triangles ,and ,respectively. Let be the external common tangent of and other than the line ; lines , and are analogously defined. Let be the intersection point of thelines and ; be the intersection point of the lines and ; points , and are analogously defined. Suppose that is a convex hexagon. Show that its diagonals , and meet at a single point.

Finding Stuffs.

1. [AoPS](https://artofproblemsolving.com/community/c6t48f6h1522766_tangent_circles):\_M\_ Let , circumcenter  and altitude .  meets  at  and meets the circle  again at .  is the midpoint of .  is the projection of  on line . Prove that the circle  is tangent to the circle .
2. [AoPS](https://artofproblemsolving.com/community/c6t48f6h1519616_geometry):\_E\_ is the incenter of . The line perpendicular to passing through cuts at . . Prove that is cyclic.
3. [AoPS](https://artofproblemsolving.com/community/c6t48f6h1523774_nice_property):\_E\_ In , is the incenter, are the touchpoints of the incircle. are the feet of the perpendiculars. If is the orthocenter of , then prove that is the incenter of .
4. [USA TST 2017 P2](https://artofproblemsolving.com/community/c6h1352164p7389108):\_M\_ Let be an acute scalene triangle with circumcenter , and let be on line such that . The circle with diameter intersects the circumcircle of at two points and , where . Points are deﬁned analogously.

- Prove that are concurrent.

- Prove that are concurrent on the Euler line of .

## PENDING

1. [IOM 2017 P1](https://artofproblemsolving.com/community/c6h1508225_simple_parallelogram_geo): Let  be a parallelogram in which angle at  is obtuse and . Points  and  on  such that  (the points  are all different, with  between  and ). The line  intersects the circumcircle  of  at points  and , and the line  intersects  at points  and . Prove that .
2. [AoPS](https://artofproblemsolving.com/community/c6t48f6h1508805_nice_but_not_hard): Let  be a quadrilateral inscribed in a circle, such that the inradius of triangle  and  are the same. Let  be the -mixtillinear incircle of the triangle .Let  be the incenters of the triangles  respectively. Show that  the tangent of  wrt  are concurrent.
3. [AoPS](https://artofproblemsolving.com/community/c6h1499703p8903397): Let  be a triangle with circumcenter  and incenter . Let  be the symmedian point of  and define  similarly. Let  be the midpoint of  and define  similarly. Show that  concur.
4. IMO SL 2012 G8: Let be a triangle with circumcircle and a line without common points with . Denote by the foot of the perpendicular from the center of to . The side-lines intersect at the points diﬀerent from . Prove that the circumcircles of the triangles and have a common point diﬀerent from or are mutually tangent at .
5. IGO Advanced P3: In a triangle with circumcenter , the line cuts the altitude though at . Let be the midpoints of respectively. cuts at . cuts again at .Prove that is cyclic .
6. IGO Advanced P4: Three circles , & touches a line at respectively ( lies between & ). touches & . Let be the other common external tangent of & . cuts at. Perpendicular to at intersects again at . Prove that and are tangent to the circle with diameter .
7. [AoPS](https://artofproblemsolving.com/community/c6h1502855p8912529)
8. IMO SL 2012 G8:\_EEH\_ Let be a triangle with circumcircle and a line without common points with . Denote by the foot of the perpendicular from the center of to . The side-lines intersect at the points diﬀerent from . Prove that the circumcircles of the triangles , and have a common point diﬀerent from or are mutually tangent at .
9. [buratinogigle Tough Prob1](https://artofproblemsolving.com/community/c6h1180223):\_H\_ Let  be a triangle inscribed in circle  with -excircle . Circle passing through  touches at . Circle passing through  touches  at .  cuts  at . Prove that passes through tangent point of -mixtilinear incircle with .
10. FUCKIT: In , is the incenter, is the touch point of the incenter with . . The tangents line from to meet at . Prove that and the tangent at to concur.