BCS Question Bank

M Ahsan Al Mahir

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About

Hello, I'm M Ahsan Al Mahir, a math olympiad contestant from Bangladesh. I have been with math olympiads since 2016. And this is my journal of problem solving that I have been keeping since 2017. At the moment of compiling, this has 680 randomly ordered problems, 210 theorems and lemmas, and 174 figures, mostly geometric, drawn in geogebra.

My motivation for keeping the problems I encountered is the same as that of an adventurer. When I got serious about math olympiad in 2017, I was really bad at combinatorics. It was a comletely wild topic, I couldn't seem to find any idea on how to approach any combi problem wahtsoever. So what I decided was to keep a list of general tricks that I would look through whenever I would try a combi problem. As time went by, that list grew longer, and so I had to be serious about keeping it organized.

And that's how this journal came to existence. I have tried my best to organize combi problems into the categories I found most intuitive, but that wasn't really rigorous. So expect to find many misplaced problems. I also added anything I found interesting related to olympiad math in this journal. But I didn't follow through with most of those topics, so you might find a few really fancy start of a topic, that never made its way to the second page.

Also there are a LOT of spelling and grammatical mistakes, as most of the entries I made here was right after solving (or in most cases, after failing to solve them and reading the solution), I never went back to proofread the comments I left when I added the problems. So expect a lot of nonsense talking and typos. Apologies for those :3

How to use this file

If you are reading this electronically (which I assume is the case), you can use the table of contents to move around the file. There are also separate indices for problems, theorems, definitions and strategies at the end of the file. You can use them to navigate as well.

Each section and subsection starts with essential theorems and lemmas. I have also added lists of important handouts at the beginning of each section. So check them out if you find the topics interesting. There are also some boxes titled "Stuck? Try These" at the beginning of the sections, that contain "rules of thumb" ideas to keep in mind while approaching a problem related to that section.

Most of the problems, theorems and lemmas have links to the AoPS page, Wiki page or whatever source I learned them from, linked to their titles. But there might be some cases where I missed to link the sources, or couldn't find any sources. If you notice something like that, please let me know.

There aren't that many full solutions in this journal, but I listed at least some hints for most of the problems (though I can't vouch for their usefulness).

I intended this journal to be just a list of tricks when I began working on it. But over the years, this file has grown in size and has become massive. But don't mistake it for a book or anything. The things are all over the place, and not nearly as helpful as an actual book. But there are interesting things hidden beneath the unorganized texts, and there are a lot of problems at one place. So it is advised to use it as an extra large problem set rather than a book.

On “familiarity” or, How to avoid “going down the Math Rabbit Hole”?

An excerpt from the math.stackexchange post of the same title.

Anyone trying to learn mathematics on his/her own has had the experience of “going down the Math Rabbit Hole.”

For example, suppose you come across the novel term vector space, and want to learn more about it. You look up various definitions, and they all refer to something called a field. So now you’re off to learn what a field is, but it’s the same story all over again: all the definitions you find refer to something called a group. Off to learn about what a group is. Ad infinitum. That’s what I’m calling here “to go down the Math Rabbit Hole.”

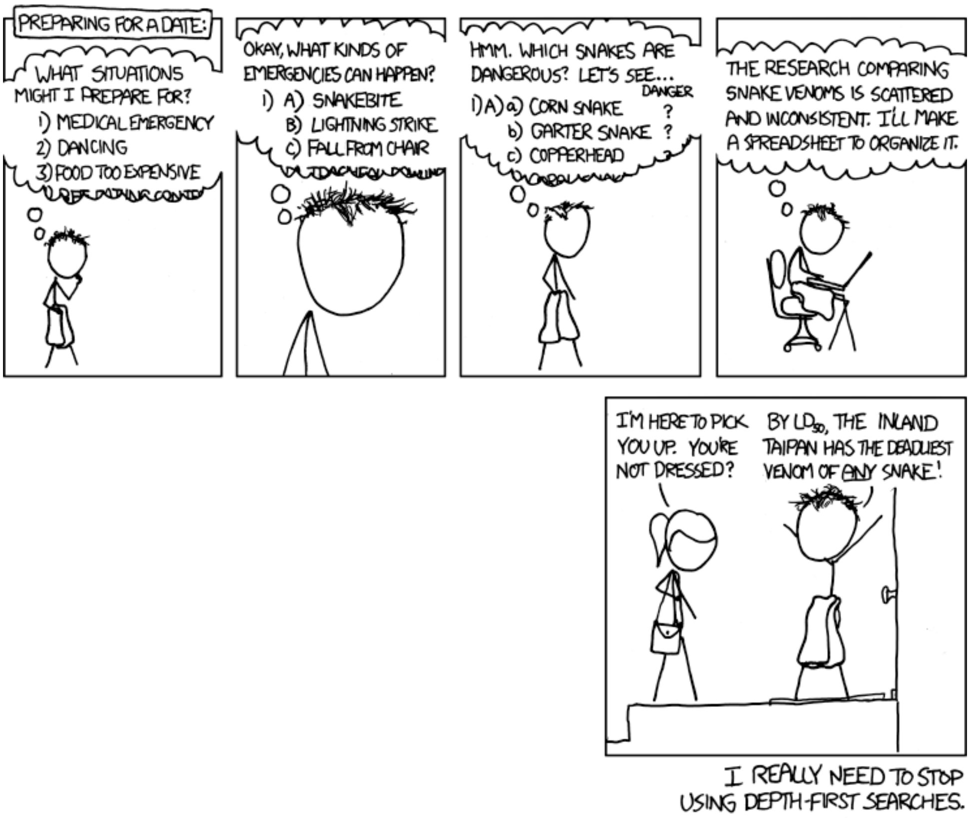
Imagine some nice, helpful fellow came along, and made a big graph of every math concept ever, where each concept is one node and related concepts are connected by edges. Now you can take a copy of this graph, and color every node green based on whether you “know” that concept (unknowns can be grey).

How to define “know"? In this case, when somebody mentions that concept while talking about something, do you immediately feel confused and get the urge to look the concept up? If no, then you know it (funnily enough, you may be deluding yourself into thinking you know something that you completely misunderstand, and it would be classed as “knowing” based on this rule - but that’s fine and I’ll explain why in a bit). For purposes of determining whether you “know” it, try to assume that the particular thing the person is talking about isn’t some intricate argument that hinges on obscure details of the concept or bizarre interpretations - it’s just mentioned matter-of-factly, as a tangential remark.

When you are studying a topic, you are basically picking one grey node and trying to color it green. But you may discover that to do this, you must color some adjacent grey nodes first. So the moment you discover a prerequisite node, you go to color it right away, and put your original topic on hold. But this node also has prerequisites, so you put it on hold, and... What you are doing is known as a depth first search. It’s natural for it to feel like a rabbit hole - you are trying to go as deep as possible. The hope is that sooner or later you will run into a wall of greens, which is when your long, arduous search will have born fruit, and you will get to feel that unique rush of climbing back up the stack with your little jewel of recursion terminating return value.

Then you get back to coloring your original node and find out about the other prerequisite, so now you can do it all over again.

DFS is suited for some applications, but it is bad for others. If your goal is to color the whole graph (ie. learn all of math), any strategy will have you visit the same number of nodes, so it doesn’t matter as much. But if you are not seriously attempting to learn everything right now, DFS is not the best choice.



https://xkcd.com/761/

So, the solution to your problem is straightforward - use a more appropriate search algorithm!

Immediately obvious is breadth-first search. This means, when reading an article (or page, or book chapter), don't rush off to look up every new term as soon as you see it. Circle it or make a note of it on a separate paper, but force yourself to finish your text even if its completely incomprehensible to you without knowing the new term. You will now have a list of prerequisite nodes, and can deal with them in a more organized manner.

Compared to your DFS, this already makes it much easier to avoid straying too far from your original area of interest. It also has another benefit which is not common in actual graph problems: Often in math, and in general, understanding is cooperative. If you have a concept A which has prerequisite concept B and C, you may find that B is very difficult to understand (it leads down a deep rabbit hole), but only if you don't yet know the very easy topic C, which if you do, make B very easy to "get" because you quickly figure out the salient and relevant points (or it may be turn out that knowing either B or C is sufficient to learn A). In this case, you really don't want to have a learning strategy which will not make sure you do C before B!

BFS not only allows you to exploit cooperativities, but it also allows you to manage your time better. After your first pass, let's say you ended up with a list of 30 topics you need to learn first. They won't all be equally hard. Maybe 10 will take you 5 minutes of skimming wikipedia to figure out. Maybe another 10 are so simple, that the first Google Image diagram explains everything. Then there will be 1 or 2 which will take days or even months of work. You don't want to get tripped up on the big ones while you have the small ones to take care of. After all, it may turn out that the big topic is not essential, but the small topic is. If that's the case, you would feel very silly if you tried to tackle the big topic first! But if the small one proves useless, you haven't really lost much energy or time.

Once you're doing BFS, you might as well benefit from the other, very nice and clever twists on it, such as Dijkstra or A*. When you have the list of topics, can you order them by how promising they seem? Chances are you can, and chances are, your intuition will be right. Another thing to do - since ultimately, your aim is to link up with some green nodes, why not try to prioritize topics which seem like they would be getting closer to things you do know? The beauty of A* is that these heuristics don't even have to be very correct - even "wrong" or "unrealistic" heuristics may end up making your search faster.

Contents

Chapter 1

Thoughts on PSolving, a note to thyself

1.1 Be DUMB, Keep it SIMPLE

Remember what Paul Zeitz said? Think wishfully, make dumb wishes. When first approaching the problem, you can do whatever you want. You can loosen some constraints, you can add some new. This works exceptionally well when you need to build an object from one given object, you can do whatever you want. Putting additional constraints decreases the number of test cases. Sometimes loosening some constraints help to give better observation of the problem.

Like in ISL 2016 N5, after deciding that we are going to build a pair (x_2, y_2) from the previous pair (x_1, y_1), we should look for the most innocent looking relation between these four variable. Now it's time to play around, try dumb things. Rewriting the equation, we want to use the fact that x_1, x_2 have to be on different sides of $\sqrt{a}$. How can we insert this constraint in our equation in the most simple and natural way? This is where we need to be dumb, and amature.

In Problem 5.9 the trick is to keep things simple. Making the most natural assumptions. In construction problems, think of how the result can be achieved in the most natural way. Can we make some extra assumptions that might result in the immediate proof the result's existence?

What to do in a contest

1. Avoid immediate fixation on one approach.
2. Try to be wishful. Try some smaller cases first. But don't spend too much time behind finding patterns. **Keep a balance between working out examples and conjecturing outcomes.**
3. During investigating, focusing on only approach won't do good. During the investigation phase, try out all the approaches at once, and then go with the most promising ones. But mass-create examples and approaches at first.
4. **First attacks:**
   a) First study the problem carefully. Don't miss any details. That means in a geo problem, naming the points and writing the corrent condition and conclusions, and in combi/alg/nt problems, checking positive vs negative or finite vs infinite conditions.
   b) Try to classify the problems with previous encounters. Maybe remember some popular ways to approach it.
   c) Which arguments seem the most plausible? What kind of solution there might be? Will it use some construction? Maybe we will need to show contradiction assuming otherwise, or we might need to use projective geo?
   d) Repeat at least twice.
   e) Specially for geo and graphs, try to find an easier construction? Maybe we can think of graphs instead of grids, or maybe we can interpret in another way?
5. Now examine the problem closely:
   a) **Get your hands dirty**: write down possible solutions for functional equations, or workout some smaller cases.
   b) **Penultimate Steps**: Try to think of the situations, where the conclusion would follow trivially from. Write them down.
   c) **Wishful thinking, Making it easier**: Try to simplify the problems using stricter assumtions, or maybe loosing some constraints. Play out with the conditions.

1.2 Problems to try before some fixed day

Problem 1.2.1 (Thue's Note). Let S be a set of all positive integers which can be represented as $a^2 + 5b^2$ for some coprime integers a, b. Let p be a prime number such that $p = 4n + 3$ for some integer n. Show that if for some positive integer k the numberk p is in S, then $2p$ is in S as well.

Konig's theorem, Dilworth theorem, Max flow Min cut

Mock 1

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Problem 1.2.2. Let I, G be the incenter and the centeroid of $\triangle ABC$. Prove that $IG \perp BC$ iff either $AB = AC$ or $AB + AC = 3BC$.

Solution. Let D, H be the foot of I, H, M the midpoint of BC. WLOG assume that $b > c$.

We have $HD = 2MD$ since G lies on ID. And we also have $2MD = b - c$. Which means

$$\begin{aligned} BH &= BD - HD \\ &= \frac{c+a-b}{2} - (b-c) \\ &= \frac{3(c-b)+a}{2} \\ \therefore \cos B &= \frac{3(c-b)+a}{2c} \end{aligned}$$

By cosine rule, we get

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos B \\ &= a^2 + c^2 + 3ab - 3ac - a^2 \\ \implies b^2 - c^2 &= 3a(b-c) \\ (b-c)(b+c) &= 3a(b-c) \end{aligned}$$

Which immediately implies that either $b = c$ or $b + c = 3a$.

Problem 1.2.3. Prove that for every positive integer n, there is a positive integer a_n for which

$$\left(1+\sqrt{5}\right)^n = \sqrt{a_n} + \sqrt{a_n + 4^n}$$

Also prove that $5 \cdot 4^{2009}$ divides a_{2010}, and find the quotient.

Solution. We define x_n, y_n by,

$$(1+\sqrt{5})^n = x_n + y_n\sqrt{5}$$

Where the recursion relation is given by the matrix

$$\begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix}$$

We prove by induction that

$$a_n = \begin{cases} x_n^2 & n \text{ is odd} \\ 5y_n^2 & n \text{ is even} \end{cases}$$

That is,

$$x_n - 5y_n = (-4)^n$$

The base cases for $n = 1, 2$ are easily proved by hand. So suppose for some n, we have

$$x_n - 5y^n = (-4)^n$$

We have by the recursion relation,

$$\begin{aligned} x_{n+1} &= x_n + 5y_n \\ \implies x_{n+1}^2 &= x_n^2 + 25y_n^2 + 10x_ny_n \end{aligned}$$

$$\begin{aligned} \text{and } y_{n+1} &= x_n + y_n \\ \implies 5y_{n+1}^2 &= 5x_n^2 + 10x_ny_n + 5y_n^2 \end{aligned}$$

$$\begin{aligned} \therefore x_{n+1}^2 - 5y_{n+1}^2 &= 4(5y_n^2 - x_n^2) \\ &= (-4)^{n+1} \end{aligned}$$

Now we find a closed form for y_{2n}. Note that

$$\begin{bmatrix} x_{2n} \\ y_{2n} \end{bmatrix} = 2^{n-1} \begin{bmatrix} 3 & 5 \\ 1 & 3 \end{bmatrix}^{n-1} \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

Which gives us that $2^{n-1} | y_{2n}$, and so, $5 \cdot 4^{2009} | a_{2010} = 5y_{2010}^2$.

By eigenvalue decomposition, we get

$$\begin{bmatrix} 3 & 5 \\ 1 & 3 \end{bmatrix} = \frac{1}{2\sqrt{5}} \begin{bmatrix} \sqrt{5} & -\sqrt{5} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3+\sqrt{5} & 0 \\ 0 & 3-\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{5} \\ -1 & \sqrt{5} \end{bmatrix}$$

So, we get

$$\begin{bmatrix} x_{2n} \\ y_{2n} \end{bmatrix} = 2^{n-1} \times \frac{1}{\left(2\sqrt{5}\right)^{n-1}} \begin{bmatrix} \sqrt{5} & -\sqrt{5} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(3+\sqrt{5}\right)^{n-1} & 0 \\ 0 & \left(3-\sqrt{5}\right)^{n-1} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{5} \\ -1 & \sqrt{5} \end{bmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

The gives us the result of y_{2n},

$$y_{2n} = 2^{n-1} \frac{1}{2\sqrt{5}} \left(6\left(3+\sqrt{5}\right)^{n-1} - 6\left(3-\sqrt{5}\right)^{n-1} + 2\sqrt{5}\left(3+\sqrt{5}\right)^{n-1} + 2\sqrt{5}\left(3-\sqrt{5}\right)^{n-1} \right)$$

Simplifying it we get

$$y_{2n} = 2^n \times \frac{\left(3+\sqrt{5}\right)^n - \left(3-\sqrt{5}\right)^n}{2\sqrt{5}}$$

Chapter 2

Combinatorics

2.1 Binomial Identities

Theorem 2.1.1 (Vandermonde's identity) — For every positive integer m, n, k

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}$$

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 \cdots + \binom{n}{n}^2$$

Problem 2.1.1 (Somewhere). Let n and m be positive integers with $n < m$. Prove that

$$\sum_{k=0}^{n} (-1)^{n-k} \frac{1}{m-k}\binom{n}{k} = \frac{1}{(n+1)\binom{m}{n+1}}$$

Solution [MellowMelon]. The left side is the nth finite difference of the unique nth degree polynomial satisfying $P(x) = \frac{1}{m-x}$ for $x = 0, 1, \ldots, n$. This means that if a_n is the x^n coefficient of P, then the left side is equal to $n! \cdot a_n$. Setting this equal to the right side, it suffices to show that

$$a_n = \frac{1}{(n+1)!\binom{m}{n+1}} = \frac{1}{m(m-1)\cdots(m-n)}.$$

Let $Q(x) = (x-m)P(x) + 1$. Then Q is degree $n+1$, has roots $0, 1, \ldots, n$, and has leading coefficient a_n. So

$$Q(x) = a_n x(x-1)(x-2)\cdots(x-n).$$

We also know $Q(m) = 1$. Plugging $x = m$ into the above equation and isolating a_n gives us exactly what we need to show.

Remark. The idea to express the left hand side with finite differences is pretty simple. The crux idea is to finding a Q that works. Since we have to show $a_n(m)(m-1)\ldots(m-n) = 1$, which looks like a polynomial with leading coefficient a_n with roots $0, 1, 2, \ldots n$ and $Q(m) = 1$. How can we get that?

The takeaway is, if the problem requires you to find a polynomial that almost looks like the one given in the problem, try to modify the given polynomial in some way first.

In this particular case was to modify P bu using the fact $(x-m)P(x) = -1$.

Solution [Hydroxide]. Let P be the unique polynomial of degree at most n such that $P(0) = P(1) = \cdots = P(n) = 1$. Obviously $P(x) = 1$, but we can also find P from Lagrange interpolation. This gives us the equality of polynomials

$$\sum_{k=0}^{n} \frac{(x-0)\cdots(x-(k-1))(x-(k+1))\cdots(x-n)}{(k-0)\cdots(k-(k-1))(k-(k+1))\cdots(k-n)} = 1.$$

Plugging in $x = m$ and doing a ton of rearranging gives the desired result.

Remark. Both solutions can easily be extended to prove the more general result that if $0 \le p < n < m$ are integers, then

$$\sum_{k=0}^{n} (-1)^{n-k} \frac{1}{m-k} \binom{n}{k} k^p = \frac{m^p}{(n+1)\binom{m}{n+1}}$$

Problem 2.1.2 (USA TST 2010 P8). Let m, n be positive integers with $m \ge n$, and let S be the set of all n-term sequences of positive integers $(a_1, a_2, \ldots a_n)$ such that $a_1 + a_2 + \cdots + a_n = m$. Show that

$$\sum_S 1^{a_1} 2^{a_2} \cdots n^{a_n} = \binom{n}{n} n^m - \binom{n}{n-1} (n-1)^m + \cdots + (-1)^{n-2} \binom{n}{2} 2^m + (-1)^{n-1} \binom{n}{1}$$

Solution [Combinatorial, MellowMelon]. Look at the right side, try to translate it to combinatorial model. If we had used the inclusion-exclusion method, the right side would be the number of ways to color m balls with n colors, with each color appearing at least once. Now our remaining job is to prove the same for the left side.

Trying to interpret the left hand side, we first notice that we need to partition the balls in n parts with sizes $a_1, a_2, \ldots a_n$ where the first partition will have 1 color to choose from, the second one will have 2 and so on. We need to refine this idea for using it.

Solution [Generating Function, MellowMelon]. The first step of a gf solution is to decide on a gf solution. Now we can think of the left hand side as a function of m, namely b_m. So we can make a generating function with it.

We let $F_n(x) = b_0 + b_1 x + b_2 x^2 + \cdots$ and so we get:

$$\begin{aligned} F_n(x) &= (x + x^2 + \cdots)(2x + 4x^2 + \cdots)\cdots(nx + n^2x^2 + \cdots), \\ &= \frac{x}{1-x}\frac{2x}{1-2x}\cdots\frac{nx}{1-nx}, \\ &= \frac{n!x^n}{(1-x)(1-2x)\cdots(1-nx)}. \end{aligned}$$

Now we need a generating function for the right hand side as well. The right hand side looks the nth finite difference of the function ${}_m g_0(x) = x^m$, that is

$${}_m g_n(0) = \binom{n}{n} n^m - \binom{n}{n-1}(n-1)^m + \cdots + (-1)^{n-2}\binom{n}{2}2^m + (-1)^{n-1}\binom{n}{1}$$

Now we can make a generating function with this:

$$G(y) = {}_0 g_n(0) y^0 + {}_1 g_n(0) y^1 \ldots$$

But this isn't very useful as we are getting the nth finite differences on the right side, where we want to work with the original polynomials.

So what we do is, we unravel the ${}_m g_n(0)$ as original polynomials and get:

$$G(x,y) = x^0 y^0 + x^1 y^1 + x^2 + y^2 \ldots$$

Where our desired generating function for n is:

$$G_n(0,y) = {}_0 g_n(0) y^0 + {}_1 g_n(0) y^1 + \ldots$$

And we want to show that

$$G_n(0,y) = \frac{n!x^n}{(1-x)(1-2x)\cdots(1-nx)}$$

Now we notice that we can actually use induction on n since we definded

$$G_{n+1}(x,y) = G_n(x+1,y) - G_n(x,y)$$

And so we are done.

Lemma 2.1.2 —

$$G(x) = \sum_{n=0}^{\infty} \binom{n+k}{k} x^n = \frac{1}{(1-x)^{k+1}}$$

Proof. We have the above relation because $\binom{n+k}{k}$ can be interpret as the number of ways write n as the sum of $k+1$ numbers. And the generating function for that is:

$$G(x) = (1 + x + x^2 \dots)^{k+1}$$

Problem 2.1.3 (Binom1). Solve for positive integer n

$$\binom{2n}{0} - \binom{2n-1}{1} + \binom{2n-2}{2} - \dots + (-1)^n \binom{n}{n} =?$$

Solution [Generating funtion, Tintarn]. Denote this sum a_n. Denote $F(x) = \sum_{n=0}^{\infty} a_n x^{2n}$.

$$\begin{aligned} F(x) &= \sum_{k=0}^{\infty} (-1)^k \sum_{n=0}^{\infty} \binom{2n-k}{k} x^{2n} \\ &= \sum_{k=0}^{\infty} (-x^2)^k \sum_{n=0}^{\infty} \binom{2n+k}{k} x^{2n}. \end{aligned}$$

Now, using the well-known power series

$$\boxed{G(x) = \sum_{n=0}^{\infty} \binom{n+k}{k} x^n = \frac{1}{(1-x)^{k+1}}}$$

we find

$$\sum_{n=0}^{\infty} \binom{2n+k}{k} x^{2n} = \frac{G(x) + G(-x)}{2}$$

and hence

$$\begin{aligned} F(x) &= \frac{1}{2(1-x)} \sum_{k=0}^{\infty} \left(\frac{x^2}{x-1} \right)^k + \frac{1}{2(1+x)} \sum_{k=0}^{\infty} \left(\frac{-x^2}{x+1} \right)^k \\ &= \frac{1}{2(x^2 - x + 1)} + \frac{1}{2(x^2 + x + 1)} \\ &= \frac{x^2 + 1}{x^4 + x^2 + 1} \\ &= \frac{1 - x^4}{1 - x^6} \\ &= (1 - x^4)(1 + x^6 + x^{12} + x^{18} + \dots) \\ &= 1 - x^4 + x^6 - x^{10} + x^{12} - x^{16} + \dots. \end{aligned}$$

and hence $\boxed{a_{3k} = 1, a_{3k+1} = 0, a_{3k+2} = -1}$.

Solution [Combinaorial Model, MellowMelon]**.** Consider the number of ways to tile a $1 \times 2n$ rectangle with squares and dominoes. Let E be the set of ways to do it with an even number of dominoes, and let O be the same for odd. We want to find $|E| - |O|$.

Trying some basic cases, we see that the only values we get are $-1, 0, 1$. So we want to find a pairing between the two sets that is almost a bijection, but doesn't work for only one element. And that bijection is:

Consider the first consecutive pair of squares or domino which either comes at the very start or comes right after another domino. Swap the pair of squares to a domino or vice versa.

Problem 2.1.4 (Binom 2).

$$\binom{n}{0}^2 - \binom{n}{1}^2 + \binom{n}{2}^2 - \cdots + (-1)^n \binom{n}{n}^2 =?$$

Solution [Generating function, TinTarn]**.** Denote

$$f(m,n) = \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} \binom{n}{m-k}$$

We are looking for $f(n,n)$. We have

$$\begin{aligned}
\sum_{m=0}^{\infty} f(m,n) x^m &= \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} \sum_{m=0}^{\infty} \binom{n}{m-k} x^m \\
&= \left[\sum_{k=0}^{\infty} (-x)^k \binom{n}{k} \right] \left[\sum_{m=0}^{\infty} x^m \binom{n}{m} \right] \\
&= (1-x)^n (1+x)^n \\
&= (1-x^2)^n \\
&= \sum_{t=0}^{\infty} (-1)^t \binom{n}{t} x^{2t}.
\end{aligned}$$

Hence we see that for odd m we have $f(m,n) = 0$ and hence for odd n we have $\boxed{f(n,n) = 0}$.

Now, for $n = 2k$ even, we find $\boxed{f(2k,2k) = (-1)^k \binom{2k}{k}}$.

Solution [Combinatorial, MellowMelon]**.** Consider the number of ways to paint the squares of a $2 \times n$ rectangle red and blue such that both rows have the same number of red squares. Let E be the set of ways with an even number of red squares in each row, and let O be the same for odd. We want to find $|E| - |O|$.

Pair the selections as follows: take the first column with both squares the same color, and flip both colors. This maps elements in E to O and vice versa and is invertible, so except cases where the pairing is undefined, E and O have the same number of elements.

2.2 Sets

2.2.1 Lemmas

Lemma 2.2.1 — Let S be a set with n elements, and let F be a family of subsets of S such that for any pair A, B in F, $A \cap B \neq \varnothing$. Then $|F| \leq 2^{n-1}$.

Theorem 2.2.2 (Erdos Ko Rado theorem) — Suppose that A is a family of distinct subsets of $\{1, 2 \ldots n\}$ such that each subset is of size r and each pair of subsets has a nonempty intersection, and suppose that $n \geq 2r$. Then the number of sets in A is less than or equal to the binomial coefficient

$$\binom{n-1}{r-1}$$

Lemma 2.2.3 — Let S be a set with n elements, and let F be a family of subsets of S such that for any pair A, B in F, S is not contained by $A \cup B$. Then $|F| \leq 2^{n-1}$.

Lemma 2.2.4 (Kleitman lemma) — A set family F is said to be downwards closed if the following holds: if X is a set in F, then all subsets of X are also sets in F. Similarly, F is said to be upwards closed if whenever X is a set in F, all sets containing X are also sets in F. Let F_1 and F_2 be downwards closed families of subsets of $S = \{1, 2, ..., n\}$, and let F_3 be an upwards closed family of subsets of S. Then we have

$$|F_1 \cap F_2| \geq \frac{|F_1| \cdot |F_2|}{2^n} \tag{2.1}$$

$$|F_1 \cap F_3| \leq \frac{|F_1| \cdot |F_3|}{2^n} \tag{2.2}$$

Lemma 2.2.5 — Let S be a set with n elements, and let F be a family of subsets of S such that for any pair A, B in F, $A \cap B \neq \varnothing$ and $A \cap B \neq S$. Then $|F| \leq 2^{n-2}$.

Solution. Using the sets in Lemma 2.2.1 and Lemma 2.2.1, defining upwards and downwards sets like in Equation 2.2.1.

Theorem 2.2.6 (The Sunflower Lemma) — A **sunflower** with k petals and a core X is a family of sets $S_1, S_2, \ldots, S_k$ such that $S_i \cap S_j = X$ for each $i \neq j$. (The reason for the name is that the Venn diagram representation for such a family resembles a sunflower.)

The sets $S_i \setminus X$ are known as petals and must be nonempty, though X can be empty.

Show that if F is a family of sets of cardinality s, and $|F| > s!(k-1)^s$, then F contains a sunflower with k petals.

Solution. Applying induction and considering the best case where $|X| = 0$

2.2.2 Extremal Set Theory

- MIT 18.314 Lecture-8

Theorem 2.2.7 (Mirsky Theorem) — A set S with a chain of height h cant be partitioned into t anti-chains if $t < h$. In other words, the minimum number of sets in any anti-chain partition of S is equal to the maximum height of the chains in S. (And Vice Versa)

Theorem 2.2.8 — In any poset, the largest cardinality of an antichain is at most the smallest cardinality of a chain-decomposition of that poset.

Theorem 2.2.9 (Dilworth's Theorem) — Let P be a poset. Then there exist an antichain A and a chain decomposition $\mathcal{C}$ of P such that $|A| = |\mathcal{C}|$

Theorem 2.2.10 (Erdos-Szekeres Theorem) — Any sequence of $ab+1$ real numbers contains either a monotonically decreasing subsequence of length $a+1$ or a monotonically increasing subsequence of length $b+1$. The more useful case is when $a = b = n$.

Problem 2.2.1. Let $n \geq 1$ be an integer and let X be a set of n^2+1 positive integers such that in any subset of X with $n+1$ elements there exist two elements $x \neq y$ such that $x|y$. Prove that there exists a subset $\{x_1, x_2 \dots x_{n+1} \in X$ such that $x_i | x_{i+1}$ for all $i = 1, 2, \dots n$.

Definition (Sperner Family)— A family of sets in which none of the sets is a strict subset of another is called a **Sperner family**, that is, an antichain of sets.

Theorem 2.2.11 (LYM Inequality) — Let U be a set of n elements and $\mathcal{S}$ is a Sperner family of subsets of U. If a_k is the number of k element subsets of $\mathcal{S}$, then

$$\sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}} \leq 1$$

Proof. Rewrite the inequality by

$$\sum_{k=0}^{n} a_k k!(n-k)! \leq n!$$

The left hand side equals to

$$\sum_{S \in \mathcal{S}} |S|!\,(n-|S|)!$$

For a $S \in \mathcal{S}$, $|S|!\,(n-|S|)!$ is the number of permutations of $[n]$ for which the first $|S|$ terms are the elements of S. Since the sets of $\mathcal{S}$ doesn't include each other, all of these permutations are different. And so if we count them together, we will get at most all the permutatios of $[n]$, which gives us our desired result.

Theorem 2.2.12 (Sperner's Theorem) — Sperner's theorem bounds the number of sets in any Sperner family. For any Sperner Family $\mathcal{S}$, if the union of those sets is a set of n elements, then

$$|\mathcal{S}| < \binom{n}{\lfloor n/2 \rfloor}$$

Where equality holds iff $\mathcal{S}$ is consisted of all subsets of $[n]$ with $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$ elements.

Proof. Let a_k be the number of k element subsets of $\mathcal{S}$. Then since $\binom{n}{\lfloor n/2 \rfloor} \leq \binom{n}{k}$, we have

$$\frac{a_k}{\binom{n}{\lfloor n/2 \rfloor}} \leq \frac{a_k}{\binom{n}{k}}$$

And since the sets doesn't contain each other, we can sum them up and by Theorem 2.2.11, we get

$$\sum_{k=0}^{n} \frac{a_k}{\binom{n}{\lfloor n/2 \rfloor}} \leq \sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}} \leq 1$$

2.2.3 Problems

Problem 2.2.2 (USA TST 2005 P1). Let n be an integer greater than 1. For a positive integer m , let $S_m = \{1, 2, \ldots, mn\}$. Suppose that there exists a $2n$ -element set T such that

1. each element of T is an m -element subset of S_m
2. each pair of elements of T shares at most one common element
3. each element of S_m is contained in exactly two elements of T

Determine the maximum possible value of m in terms of n.

Solution. We use double counting to find the ans, after that the rest is easy.

Problem 2.2.3 (Iran TST 2008 D3P1). Let S be a set with n elements, and F be a family of subsets of S with 2^{n-1} elements, such that for each $A, B, C \in F$, $A \cap B \cap C$ is not empty. Prove that the intersection of all of the elements of F is not empty.

Solution. Using Induction with this lemma.

Problem 2.2.4 (Romanian TST 2016 D1P2). Let n be a positive integer, and let $S_1, S_2, \ldots S_n$ be a collection of finite non-empty sets such that

$$\sum_{1 \le i < j \le n} \frac{|S_i \cap S_j|}{|S_i||S_j|} < 1$$

Prove that there exist pairwise distinct elements $x_1, x_2 \ldots x_n$ such that x_i is a member of S_i for each index i.

Solution. The Inductive proof reduces the problem to American Mathematical Monthly problem E2309

Solution. The other approach is to focus on the given weird condition, and interpolate it to something nice, like probabilistic condition.

Problem 2.2.5 (American Mathematical Monthly E2309). If A_1, A_2,... A_n are n nonempty subsets of the set $\{1, 2, ..., n-1\}$, then prove that

$$\sum_{1\le i<j\le n}\frac{|A_i\cap A_j|}{|A_i|\cdot|A_j|}\ge 1$$

Problem 2.2.6 (CGMO 2010 P1)**.** Let n be an integer greater than two, and let $A_1, A_2, \cdots, A_{2n}$ be pairwise distinct subsets of $\{1, 2, \ldots n\}$. Determine the maximum value of

$$\sum_{i=1}^{2n}\frac{|A_i\cap A_{i+1}|}{|A_i|\cdot|A_{i+1}|}$$

Where $A_{2n+1} = A_1$ and $|X|$ denote the number of elements in X.

Problem 2.2.7 (ISL 2002 C5)**.** Let $r \ge 2$ be a fixed positive integer, and let F be an infinite family of sets, each of size r, no two of which are disjoint. Prove that there exists a set of size $r-1$ that meets each set in F.

HMMT 2016 Team Round: Fix positive integers $r > s$, and let $\mathcal{F}$ be an infinite family of sets, each of size r, no two of which share fewer than s elements. Prove that there exists a set of size $r-1$ that shares at least s elements with each set in F.

> **Solution** [Focus on a set]**.** If we take an arbitrary set, we can say that there exists infinitely many sets $\in \mathbb{F}$ which includes a fixed element from our test set. If we do this argument for $r-1$ times, we get a set X of $r-1$ elements, and an infinte family of sets that contains X completely. At this point the problem is trivial.

> **Solution** [Adding Elements]**.** Since it's tricky to work with one family, why not introduce another family, like the second monk. This solution generalizes the problem as such.

Problem 2.2.8 (ISL 1988 P10)**.** Let $N = \{1, 2\ldots, n\}, n \ge 2$.

A collection $F = \{A_1, \ldots, A_t\}$ of subsets $A_i \subseteq N$, $i = 1, \ldots, t$, is said to be **separating**, if for every pair $\{x, y\} \subseteq N$, there is a set $A_i \in F$ so that $A_i \cap \{x, y\}$ contains just one element.

F is said to be **covering**, if every element of N is contained in at least one set $A_i \in F$.

What is the smallest value $f(n)$ of t, so there is a set $F = \{A_1, \ldots, A_t\}$ which is simultaneously separating and covering.

> **Solution** [Binary Representation]**.** Using Binary Representations for the elements as in or not in, we get an easy bijection.

Problem 2.2.9 (Iran TST 2013 D1P2)**.** Find the maximum number of subsets from $\{1, ..., n\}$ such that for any two of them like A, B if $A \subset B$ then $|B - A| \geq 3$. (Here $|X|$ is the number of elements of the set X.)

Solution. By partitioning the maximum set of subsets into groups which contain the number n and which don't and Induction on n we can show that the maximum number of subset is

$$\frac{2^n - (-1)^n}{3}$$

.

Problem 2.2.10 (Putnam 2005 B4)**.** For positive integers m and n, let $f(m, n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \leq m$. Show that $f(m, n) = f(n, m)$.

Solution. Try to show Bisection between the result and choosing m or n objects from $m + n$ objects or show that the result is $\binom{m+n}{n}$.

2.2.4 Hamming Distance

- Blogpost by dgrozev

Definition (Hamming Distance)— The Hamming distance between two equal-length strings of symbols is the number of positions at which the corresponding symbols are different.

Occasionally we might be able to use the idea of Hamming distance to "partition" a set into sets of equivalence classes. For example consider the following problem:

Problem 2.2.11. Suppose we have some real numbers $(x_n)\,, n = 1, 2, \ldots, N$ and consider the sums

$$\sum_{i=1}^{N} b_i x_i, \quad b_i \in \{0, 1\}$$

We want to estimate how many among those sums hit a fixed interval Δ with some length.

Solution. Suppose a sum $\sum_{i=1}^{N} b_i x_i$ is inside Δ.

Then changing any bit $b_i, i = 1, 2, \ldots, N$ we leave Δ if this interval is smaller than the largest $|x_i|\,, i \in [1 \ldots N]$.

So, for any N -tuple of bits $b = (b_1, b_2, \ldots, b_N)$ with corresponding sum in Δ, all the other tuples of bits that differ from b in exactly one bit have sums outside Δ.

In fact the set of those bits is the unit sphere around b with respect to the Hamming distance. So no other set on this sphere has the sum in Δ.

Now, we take the set B of all tuples $b = (b_1, b_2, \ldots, b_N)$ with $\sum_{i=1}^{N} b_i x_i \in \Delta$. Then for any $b \in B$ the unit sphere

$$S(b) := \{b' : b' \text{differs from } b \text{ in exactly one bit}\}$$

consists of tuples not in B. If it happens for $b \in B$, $S(b)$ are disjoint, we can estimate the number of elements in B as

$$|B| \leq \frac{2^N}{N+1}$$

Problem 2.2.12 (CIIM 2019 P3). Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of non-zero real numbers. For every positive integer m, we define:

$$X_m = \left\{ A \subseteq \{0,1,...,m-1\} \,\middle|\, \left|\sum_{a\in A} x_a\right| > \frac{1}{m} \right\}$$

Prove that:

$$\lim_{n\to\infty} \frac{|X_n|}{2^n} = 1$$

Remark. The idea roughly speaking is as follows.

For sufficiently large m we map each A having small sum $\sum_{a\in A} x_a$ to a family $F(A)$ of many other sets with big sums. If those families do not intersect, we obtain that the number of A's with small sums is a small portion of the all subsets.

The vague idea is to mark some finite set of distinct elements x_n and for a fixed set A we may add or remove any marked element, depending if it's in A or not. Thus we obtain another sets, but those sets cannot have small sums if the marked elements are appropriately taken.

Solution [dgrozev]. To apply this idea we consider two cases.

$\boxed{\textbf{1.}}$ There are infinitely many distinct numbers among $(x_n)_{n\geq 0}$.

Let us fix some $N \in \mathbb{N}$ and I be a set of N non negative integers (indices) such $x_i, i \in I$ are all distinct. Denote by ε any positive real number less than $\min\{|x_i - x_j|,\ |x_i|\}, i,j \in I$.

We take any m large enough such that $2/m < \varepsilon$ and $m > \max\{i \in I\}$. Let A is any subset of $\{0,1,\ldots,m\}$ satisfying

$$\left|\sum_{a\in A} x_a\right| \leq \frac{1}{m} \qquad (1)$$

For any $i \in I$ we construct the set A_i as

$$A_i := \begin{cases} A \cup i & \text{if } i \notin A \\ A \setminus \{i\} & \text{if } i \in A \end{cases}$$

It can be easily seen

$$\left|\sum_{a\in A_i} x_a\right| > \frac{1}{m},\ \forall i \in I$$

By $F(A)$ we denote the family $\{A_i : i \in I\}$. For any two subsets A, B of $\{0,1,\ldots,m\}$ satisfying (1), the corresponding families $F(A), F(B)$ do not meet.

Indeed, if $X \in F(A)$ and $X \in F(B)$, then X differs from A, B only in one element of I. That is, A differs from B only in two elements that are in I. Hence,

$$\left|\sum_{a\in A} x_a - \sum_{a\in B} x_a\right| > 2/m$$

because of the choice of m. It means A and B cannot both satisfy (1).

To recap. Let $\mathcal{A}$ be the family of all subsets A of $\{0, 1, \dots m-1\}$ satisfying (1). Then for any $A \in \mathcal{A}$ all sets in $F(A)$ are not in $\mathcal{A}$. Further, $F(A), A \in \mathcal{A}$ are disjoint. Since $|F(A)| = N, \forall A \in \mathcal{A}$, we have

$$|\mathcal{A}| \leq \frac{2^m}{N+1}$$

and henceh

$$\frac{|X_m|}{2^m} \geq \frac{N}{N+1}$$

Since N could be chosen arbitrary large, the result follows.

$\boxed{\textbf{2.}}$ Only finite values of $(x_n)_{n\geq 0}$ are distinct.

So, suppose $x_n = a$ for infinitely many $n \in \mathbb{N}$. Let I be a finite set of such indices with $|I| = N$ and $m \in \mathbb{N}$ satisfies $2/m < |a|$ and $m \geq \max\{i : i \in I\}$. For any fixed subset $A \subset [0..m] \setminus I$ there exists at most one $k \in \mathbb{N}$ with $\left|\sum_{i\in A} x_i + ka\right| \leq \frac{1}{m}$. Hence the number of subsets $A' \subset I$ with $\left|\sum_{i\in A\cup A'} x_i + ka\right| \leq \frac{1}{m}$ are at most $\binom{N}{\lfloor N/2 \rfloor}$.

It means the number of sets $A \subset [0..m]$ satisfying (1) are at most $2^{m-N}\binom{N}{\lfloor N/2 \rfloor}$, thus

$$\frac{|X_m|}{2^m} \geq 1 - \frac{\lfloor N/2 \rfloor}{2^N}$$

which proves the desired result.

Solution [Sperner's theorem, IMD2]. We call a subset "bad" if it doesn't satisfy the inequality.

Now fix an integer N. Let m be an integer such that

$$\frac{2}{m} < \min\{|x_1|, |x_2| \dots |x_N|\}$$

WLOG assume that there are as many positives in the first N terms as there are negatives. Let P be the positive elements subset of $\{x_1, x_2 \dots x_N\}$.

Define $S = \{x_1, x_2, \dots x_m\} - P$. Then every subset of the first m terms can be written as A, B, where $A \subseteq P, B \subseteq S$ and the subset is $A \cup B$.

Using this notation, notice that if $A \subset A' \subseteq P$, then both A, B and A', B can't be bad, because of our choice of m. So for a fixed $B \subseteq S$, the largest family of subsets of P is a *Sperner Family* of P. Which by Sperner's theorem has size at most

$$t_p = \binom{|P|}{\lfloor |P|/2 \rfloor}$$

So the probability of a subset of $\{x_1, x_2, \ldots x_m\}$ being bad is $\frac{t_p}{2^p}$, which tends to 0 for larger choice of N.

2.3 Algorithmic

- Handout by Cody Johnson

Stuck? Try These: When to use algorithms

Rules of thumb to applying algorithms:

$\boxed{1}$ A procedure or operation is given. An algorithm would be a helpful organization of a set of allowed steps

$\boxed{2}$ You seek a construction. An algorithm is a powerful method to construct a configuration satisfying some given conditions

$\boxed{3}$ A complex object or configuration is given that you want to reduce.

$\boxed{4}$ You are looking to represent all positive integers n in some given form. Often you are able to represent the items greedily.

$\boxed{5}$ You are packing anything. Often you are able to pack the items greedily.

$\boxed{6}$ You are looking for a winning strategy. A winning strategy is, by definition, an algorithm to ensure a player of winning a game.

2.3.1 Some CP techniques

Theorem 2.3.1 (Swap Sort) — In any swap sorting algorithm, the number of swaps needed has the same parity.

Algorithm (Convex Hull Trick)— Given a lot of lines on the plane, and a lot of queries each asking for the smallest value for y among the lines for a given x, the optimal strategy is to sort the lines according to their slopes, and adding them to a stack, checking if they are relevant to the 'minimal' convex hull of those lines.

Notice that the line y=4 will never be the lowest one, regardless of the x-value. Of the remaining three lines, each one is the minimum in a single contiguous interval (possibly having plus or minus infinity as one bound), which are colored green in the figure.

Thus, if we remove "irrelevant" lines such as $y = 4$ in this example and sort the remaining lines by slope, we obtain a collection of N intervals (where N is the number of lines remaining), in each of which one of the lines is the minimal one. If we can determine the endpoints of these intervals, it becomes a simple matter to use binary search to answer each query.

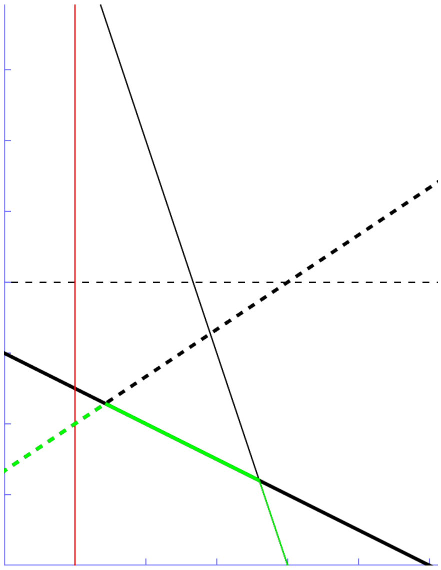


Figure 2.3.1: Convex Hull Trick

2.3.1.1 Minimal Spanning Tree

Definition (Minimum Spanning Tree)— A minimum spanning tree or minimum weight spanning tree is a subset of the edges of a connected, edge-weighted (un)directed graph that connects all the vertices's together, without any cycles and with the minimum possible total edge weight.

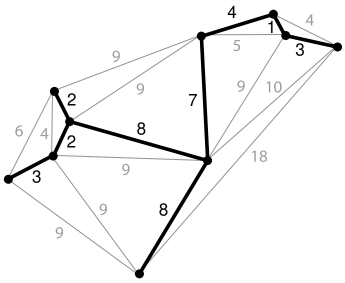


Figure 2.3.2: Minimal Spanning Tree

> **Lemma 2.3.2** (MST Cut) — For any cut of the graph, the lightest edge in that cut-set is in evey MST of the graph.

> **Proof.** Assume that there is an MST T that does not contain e. Adding e to T will produce a cycle, that crosses the cut once at e and crosses back at another edge e'. Deleting e' we get a spanning tree $T/e' \cup e$ of strictly smaller weight than T. This contradicts the assumption that T was a MST.

> **Lemma 2.3.3** (MST Cycle) — For any cycle C in the graph, the heaviest edge in the cycle cannot be in any MST of the graph.

Proof. Assume the contrary, i.e. that e belongs to an MST T_1. Then deleting e will break T_1 into two subtrees with the two ends of e in different subtrees. The remainder of C reconnects the subtrees, hence there is an edge f of C with ends in different subtrees, i.e., it reconnects the subtrees into a tree T_2 with weight less than that of T_1, because the weight of f is less than the weight of e.

Algorithm (Kruskal's Algorithm)— Kruskal's algorithm is a 'minimum-spanning-tree algorithm' which finds an edge of the least possible weight that connects any two trees in the forest. It is a greedy algorithm in graph theory as it finds a minimum spanning tree for a connected weighted graph adding increasing cost arcs at each step.

Solution. To optimize this algorithm, Disjoint Set DS is used.

Algorithm (Prim's Algorithm)— Greedily build the tree by adding edges one by one. At one step we add the minimal cost edge that connects the tree to the vertices's that are not in the tree.

2.3.1.2 Shortest Path Problem

Definition (Shortest Path Problem)— Finding the shortest path between two nodes in a weighted or unweighted graph.

Algorithm (Breadth-First Search)— This algo runs from a node and "levelizes" the other nodes.

Algorithm (Dijkstra's Algorithm)— It picks the unvisited vertex with the lowest distance, calculates the distance through it to each unvisited neighbor, and updates the neighbor's distance if smaller.

2.3.2 Parity of Permutation

- Sign of a permutation - Keith Conrad

Definition (Permutation as Composition of Transpositions)— Any cycle in S_n is a product of transpositions: the identity (1) is $(12)(12)$, and a k -cycle with $k \geq 2$ can be written as

$$\left(i_1 i_2 \cdots i_k\right)=\left(i_1 i_k\right)\left(i_1 i_{k-1}\right) \cdots\left(i_1 i_3\right)\left(i_1 i_2\right)$$

Or

$$\left(i_1 i_2 \cdots i_k\right)=\left(i_{k-1} i_k\right) \cdots\left(i_2 i_k\right)\left(i_1 i_k\right)$$

For example, a 3 -cycle (abc) can be written as

$$(abc)=(bc)(ac)=(ac)(ab)$$

since any permutation in S_n is a product of cycles and any cycle is a product of transpositions, any permutation in S_n is a product of transpositions.

Theorem 2.3.4 (Parity of Transpostion Composition) — If the permutation σ is written as composition of two sets of tanspositions:

$$\sigma=\tau_1 \tau_2 \ldots \tau_r=\tau_1' \tau_2' \ldots \tau_{r'}'$$

Then $r \equiv r' \pmod 2$

Proof. We can write the identity permutation as the compostion of those transpositions:

$$(1)=\tau_1 \tau_2 \ldots \tau_r \tau_{r'}' \ldots \tau_2' \tau_1'$$

We are done if we can prove that (1) can never be composed of an odd number of transpositions. Write (1) as:

$$(1)=\left(a_1, b_1\right)\left(a_1, b_2\right) \ldots\left(a_k, b_k\right)$$

We prove by induction that k is even. Firstly notice that there must be another $i \neq 1$ such that $a_i=a_1$. Since

$$(ab)(cd)=(cd)(ab)$$

We can safely assume that $a_2=a_1$. Now if $b_1=b_2$, we can just remove $(a_1, b_1)(a_2, b_2)$ and by induction hold our claim. So suppose $b_1 \neq b_2$. Since

$$(ab)(ac)=(bc)(ab)$$

We can replace $\left(a_1, b_1\right)\left(a_2, b_2\right)$ by $\left(b_1, b_2\right)\left(a_1, b_1\right)$. So,

$$(1)=\left(b_1, b_2\right)\left(a_1, b_1\right) \ldots\left(a_k, b_k\right)$$

But then there must be another $i \neq 1,2$ such that $a_i=a_1$. And since this can't continue infinitely, we will eventually need to reduce k by 2. Which completes the induction.

Definition (Inversions of a Permutation)— Inversions of a permutation σ are pairs (x, y) such that

$$x < y \text{ and } \sigma(x) > \sigma(y)$$

Definition (Parity of a Permutation)— Let m be the number of inversions of a permutation σ, and let $\sigma = \tau_1\tau_2\ldots\tau_r$ be a transposition composition of σ. Then the pairity of the permutation σ is defined by:

$$\operatorname{sgn}(\sigma) = (-1)^m = (-1)^r$$

Where the permutation is called *odd* if $\operatorname{sgn}(\sigma) = -1$ and *even* otherwise.

Parity of permutaions follow properties of integers under addition: even+even is even, odd+odd is even, even+odd is odd. Also the inverse of the polynomial has the same parity.

The parity can also be defined with polynomials. Let

$$P(x_1, x_2, \ldots, x_n) = \prod_{i<j}(x_i - x_j)$$

Now for a given permutation σ of the numbers $\{1, \ldots, n\}$, we can also define

$$\operatorname{sgn}(\sigma) = \frac{P\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right)}{P(x_1, \ldots, x_n)}$$

Theorem 2.3.5 (Definitions of parity are equivalent) — The parity of the number of transpositions in the decomposition of σ is the same as the parity of the number of inversions.

Proof. Suppose that σ has r inverses and its decomposition into transpositions has m elements, that is

$$\sigma = \tau_1\tau_2\ldots\tau_m$$

If we let $\sigma = a_1a_2\ldots a_n$, and $a_k = 1$, then if we exchange a_{k-1}, a_k, that increases m by 1, and reduces r by 1. We can do this till σ becomes the identity, and the parity of $m-r$ would not change. But in that case, as we have proved in Theorem 2.3.4, $m-r$ is even.

Corollary 2.3.6 — The pairity of σ is also the pairity of the number of *even cycles* of σ.

Definition (Permutations as Matrices)— We can translate permutations using matrices. For $\sigma \in S_n$, let $T_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ by permuting the columns of I_n by σ. For example, (123) translates to

$$T_{(123)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Then we have,

$$\det(T_\sigma) = \operatorname{sgn}(\sigma)$$

The mapping $\sigma \to T_\sigma$ is multiplicative, which explains why the signs of permutations are multiplicative.

2.3.3 Fast Fourier Transform

Let $A(x) = a_0 + a_1x + \ldots a_{n-1}x^{n-1}$ be a polinomial and let $\omega^n = 1$ be a nth root of unity. We use **FFT** to multiply two polynomials in $n \log n$ time.

Definition (DFT Matrix)— Let ω be a nth root of unity. The **DFT Matrix** is a $n \times n$ matrix given by:

$$W = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega & \omega^2 & \ldots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \ldots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \ldots & \omega^{(n-1)^2} \end{pmatrix}$$

And its inverse is given by:

$$W^{-1} = \frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \ldots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \ldots & \omega^{-2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \ldots & \omega^{-(n-1)^2} \end{pmatrix}$$

Definition (Discrete Fourier Transform)— The **Discrete Fourier transform** (DFT) of the polynomial $A(x)$ (or equivalently the vector of coefficients $(a_0, a_1, \ldots, a_{n-1})$ is defined as the values of the polynomial at the points $x = \omega_n^k$, i.e. it is the vector:

$$\mathrm{DFT}\left(a_0, a_1, \ldots, a_{n-1}\right) = \left(y_0, y_1, \ldots, y_{n-1}\right) = \left(A\left(\omega_n^0\right), A\left(\omega_n^1\right), \ldots, A\left(\omega_n^{n-1}\right)\right)$$

In other words, we can write $\mathrm{DFT}(A)$ as:

$$\mathrm{DFT}(A) = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega & \omega^2 & \ldots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \ldots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \ldots & \omega^{(n-1)^2} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Definition (Inverse Discrete Fourier Transform)— The **Inverse Discrete Fourier Transform** of values of the polynomial $(y_0, y_1, \ldots, y_{n-1})$ are the coefficients of the polynomial $(a_0, a_1, \ldots, a_{n-1})$

$$\text{InverseDFT}\ \left(y_0, y_1, \ldots, y_{n-1}\right) = \left(a_0, a_1, \ldots, a_{n-1}\right)$$

Thus, if a direct DFT computes the values of the polynomial at the points at the n -th roots, the inverse DFT can restore the coefficients of the polynomial using those values.

Algorithm (Multiplication of two polynomials)— Say we have A, B two polynomials, and we want to compute $(A \cdot B)(x)$. Then we first component-wise multiply $\mathrm{DFT}(A)$ and $\mathrm{DFT}(B)$, and then retrieve the coefficients of $A \cdot B$ by applying the Inverse DFT.

Algorithm (Fast Fourier Transform)— We want to compute the DFT(A) in $n \log n$ time. Suppose n is a power of 2, $\omega^n = 1$, and $A(x) = a_0 x^0 + a_1 x^1 + \cdots + a_{n-1} x^{n-1}$. We use divide and conquer by considering:

$$\begin{aligned} A_0(x) &= a_0 x^0 + a_2 x^1 + \cdots + a_{n-2} x^{\frac{n}{2}-1} \\ A_1(x) &= a_1 x^0 + a_3 x^1 + \cdots + a_{n-1} x^{\frac{n}{2}-1} \\ \text{and } A(x) &= A_0(x^2) + x A_1(x^2) \end{aligned}$$

After we have computed DFT$(A_0) = (y_i^0)_{i=0}^{\frac{n}{2}-1}$ and DFT$(A_0) = (y_i^1)_{i=0}^{\frac{n}{2}-1}$, we can compute $(y_i)_{i=0}^{n-1}$ by:

$$y_k = \begin{cases} y_k^0 + \omega^k y_k^1, & \text{for } k < \dfrac{n}{2} \\ y_k^0 - \omega^k y_k^1, & \text{for } k \geq \dfrac{n}{2} \end{cases}$$

Algorithm (Inverse FFT)— Since by definition we know $\mathrm{DFT}(A) = W \times A$, we have:

2.3.4 Problems

Problem 2.3.1 (Iran TST 2018 P1). Let $A_1, A_2, ..., A_k$ be the subsets of $\{1, 2, 3, ..., n\}$ such that for all

$$1 \leq i, j \leq k, \quad A_i \cap A_j \neq \varnothing$$

Prove that there are n distinct positive integers $x_1, x_2, ..., x_n$ such that for each $1 \leq j \leq k$:

$$\underset{i \in A_j}{\operatorname{lcm}} \{x_i\} > \underset{i \notin A_j}{\operatorname{lcm}} \{x_i\}$$

> **Solution** [elegance, a1267ab]. Pick k distinct primes $p_1, \ldots, p_k$. Let x_i be the product of all p_j where $i \in A_j$. By assumption, for any j, m, some element of A_j is also in A_m, so
>
> $$\operatorname{lcm}_{i \in A_j} \{x_i\} = p_1 \cdots p_k$$
>
> because every prime is represented at least once. On the other hand, A_j^c has no element in common with A_j, so
>
> $$\operatorname{lcm}_{i \notin A_j} \{x_i\} \leq \frac{p_1 \cdots p_k}{p_j}.$$
>
> Hence this works.

Remark. This solution probably came from the wish to keep all the lcm's of the left side equal, and making the right side, in some way, obviously smaller. Also we know that when dealing with lcm or $\gcd$, it is wise to think about primes.

> **Solution** [induction]. Apply induction on either k or n. We can either build another subset of $[n]$ with the help of some primes, or we can extend one more element, and build a new x_n, which actually gives similar results to the previous solution.

Problem 2.3.2 (ISL 2016 C1). The leader of an IMO team chooses positive integers n and k with $n > k$, and announces them to the deputy leader and a contestant. The leader then secretly tells the deputy leader an n-digit binary string, and the deputy leader writes down all n-digit binary strings which differ from the leaders in exactly k positions.

For example, if $n = 3$ and $k = 1$, and if the leader chooses 101, the deputy leader would write down $001, 111$ and 100.

The contestant is allowed to look at the strings written by the deputy leader and guess the leaders string. What is the minimum number of guesses (in terms of n and k) needed to guarantee the correct answer?

> **Solution** [dgrozev, blogpost]. Think about the problems in terms of "Hamming" distance. Then we need to find the center of a sphere of radius k. And what's the nicest thing with center's and points on the sphere? "Diameters". So we try to find the points

on this sphere with the greatest distance, and we hope to find something about the center from the information.

Problem 2.3.3 (ISL 2005 C2). Let $a_1, a_2, \dots$ be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer n the numbers $a_1, a_2, \dots, a_n$ leave n different remainders upon division by n. Prove that every integer occurs exactly once in the sequence $a_1, a_2, \dots$.

Solution. Constructing the initials.

Problem 2.3.4 (IOI Practice 2017). C plays a game with A and B. There's a room with a table. First C goes in the room and puts 64 coins on the table in a row. Each coin is facing either heads or tails. Coins are identical to one another, but one of them is cursed. C decides to put that coin in position c. Then he calls in A and shows him the position of the cursed coin. Now he allows A to flip some coins if he wants (he can't move any coin to other positions). After that A and C leave the room and sends in B. If B can identify the cursed coin then C loses, otherwise C wins.

The rules of the game are explained to A and B beforehand, so they can discuss their strategy before entering the room. Find the minimum number k of coin flips required by A so that no matter what configuration of 64 coins C gives them and where he puts the cursed coin, A and B can win with A flipping at most k coins.

Find constructions for $k = 32, 8, 6, 3, 2, 1$

Solution. XOR XOR XOR binary representation

Problem 2.3.5 (Codeforces 987E). Petr likes to come up with problems about randomly generated data. This time problem is about random permutation. He decided to generate a random permutation this way: he takes identity permutation of numbers from 1 to n and then $3n$ times takes a random pair of different elements and swaps them. Alex envies Petr and tries to imitate him in all kind of things. Alex has also come up with a problem about random permutation. He generates a random permutation just like Petr but swaps elements $7n + 1$ times instead of $3n$ times. Because it is more random, OK?!

You somehow get a test from one of these problems and now you want to know from which one.

Solution. This theorem kills this problem instantly.

Problem 2.3.6 (USAMO 2013 P6). At the vertices's of a regular hexagon are written six nonnegative integers whose sum is 2003^{2003}. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

> **Solution.** Firstly what comes into mind is to decrease the maximum value, but since this is a P6, there must be some mistakes. Surely, we can't follow this algo in the case $(k,k,0,k,k,0)$. But this time, the sum becomes even. So we have to slowly minimize the maximum, keeping the sum odd. And since only odd number on the board is the easiest to handle, we solve that case first, and the other cases can be easly handled with an additional algo.

Problem 2.3.7 (ISL 2012 C1). Several positive integers are written in a row. Iteratively, Alice chooses two adjacent numbers x and y such that $x>y$ and x is to the left of y , and replaces the pair (x,y) by either $(y+1,x)$ or $(x-1,x)$. Prove that she can perform only finitely many such iterations.

> **Solution.** Easy invariant.

Problem 2.3.8 (AoPS). There is a number from the set $\{1,-1\}$ written in each of the vertices's of a regular do-decagon (12 -gon). In a single turn we select 3 numbers going in the row and change their signs. In the beginning all numbers, except one are equal to 1. Can we transfer the only -1 into adjacent vertex after a finite number of turns?

> **Solution.** Algo+Proof $\implies$ Invariant.

Problem 2.3.9 (ISL 1994 C3). Peter has three accounts in a bank, each with an integral number of dollars. He is only allowed to transfer money from one account to another so that the amount of money in the latter is doubled. Prove that Peter can always transfer all his money into two accounts. Can Peter always transfer all his money into one account?

> **Solution.** Since we want to decrease the minimum, and one of the most simple way is to consider Euclidean algorithm. So we sort the accounts, $A<B<C$, and write $B=qA+r$, and do some experiment to turn B into r.

Problem 2.3.10 (MEMO 2008, Team, P6). On a blackboard there are $n\geq 2, n\in\mathbb{Z}^+$ numbers. In each step we select two numbers from the blackboard and replace both of them by their sum. Determine all numbers n for which it is possible to yield n identical number after a finite number of steps.

Solution. The pair thing rules out the case of odds. For evens, we make two identical sets, and focus on only one of the sets, with an additional move $x \to 2x$ available to use. Since we can now change the powers of 2 at out will at any time, we only focus on the greatest odd divisors. Our aim is to slowly decrease the largest odd divisor.

Problem 2.3.11 (USA Dec TST 2016, P1). Let $S = \{1, \dots, n\}$. Given a bijection $f : S \to S$ an orbit of f is a set of the form $\{x, f(x), f(f(x)), \dots\}$ for some $x \in S$. We denote by $c(f)$ the number of distinct orbits of f. For example, if $n = 3$ and $f(1) = 2$, $f(2) = 1$, $f(3) = 3$, the two orbits are $\{1, 2\}$ and $\{3\}$, hence $c(f) = 2$.

Given k bijections f_1, ..., f_k from S to itself, prove that

$$c(f_1) + \dots + c(f_k) \leq n(k-1) + c(f)$$

where $f : S \to S$ is the composed function $f_1 \circ \dots \circ f_k$.

Solution. Induction reduces the problem to the case of $k = 2$. Then another inuction on $c(f_1)$ solves the problem. The later induction works on the basis of the fact that a "swap" in the bijection changes the number of cycles by 1 (either adds $+1$ or -1).

Problem 2.3.12 (Cody Johnson). Consider a set of 6 integers $S = \{a_1 \dots a_6\}$. At on step, you can add $+1$ or -1 to all of the 6 integers. Prove that you can make a finite number of moves so that after the moves, you have $a_1a_5a_6 = a_2a_4a_6 = a_3a_4a_5$

Problem 2.3.13 (ISL 2014 A1). Let $a_0 < a_1 < a_2 \dots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + a_2 + \dots + a_n}{n} \leq a_{n+1}.$$

Solution. My idea was to construct the sequence with the assumtion that the condition is false. It leads to either all of the right ineq false or the condition being true.

Solution. The magical solution: defining $b_n = (a_n - a_{n-1}) + \dots + (a_n - a_1)$ which eases the inequlity.

Solution. The beautiful solution: defining δ_i as $a_n = a_0 + \Delta_1 + \Delta_2 + ... + \Delta_n$ for all n, i.

Solution. Another idea is to first prove the existence and then to prove the uniqueness.

Problem 2.3.14 (ISL 2014 N3). For each positive integer n, the Bank of Cape Town issues coins of denomination $\frac{1}{n}$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most most $99 + \frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.

Solution. Notice that the sum of the geometric series $S = \dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} \dots$ is 1. And in another problem we partitioned the set of integers into subsets with each subset starting with an odd number k and every other elements of the subset being $2^i * k$. We do similarly in this problem, and partition the set of the coins in a similar way. Then we take the first 100 sets whose sum is less than 1 and insert the other left coins in these sets, with the condition that the sum of all of the coins is $99 + \dfrac{1}{2}$. Solu

Solution. Replacing 100 by n, we show that for all n the codition is valid. Assume otherwise. Take the minimal n for which the condition does not work. Ta-Da! We can show that if n does not work, so doesn't $n - 1$. Solu

Solution. Or just be an EChen and prove the result for at most $k - \dfrac{k}{2k+1}$ with k groups.

Solution. Very similar to this problem

Problem 2.3.15 (China TST 2006). Given positive integer n, find the biggest real number C which satisfy the condition that if the sum of the reciprocals ($\frac{1}{n}$ is the reciprocal of n) of a set of integers (They can be the same.) that are greater than 1 is less than C, then we can divide the set of numbers into no more than n groups so that the sum of reciprocals of every group is less than 1.

Problem 2.3.16. In a $n * n$ grid, every cell is either black or white. A 'command' is a pair of integers, $i, j \leq n$, after which all of the cells in the i^{th} row and the j^{th} column (meaning a total of $2n - 1$ cells) will switch the state. Our goal is to make every cell of the same state.

1. Prove that if it can be done, it can be done in less than $\frac{n^2}{2}$ commands.
2. Prove that it can always be done if n is even.
3. Prove or disprove for odd n.

Solution. (a) is really easy, just take into account that flipping all cells result in the switch of all of the cells. And the question did not ask for an algorithm.

Solution. (b) is also easy, notice that we can pair the columns and then make them look like the same, with a compound command. A better algo is to take the original algo and to modify it like, take one cell, then do the original move on all cells in the row and column of this cell.

Solution. (c) uses Linear Algebra, which I dont know yet, or... use double counting to build the criteria of the fucntion f : states $\to$ subset of moves being bijective.

Problem 2.3.17 (OIM 1994). In every square of an $n \times n$ board there is a lamp. Initially all the lamps are turned off. Touching a lamp changes the state of all the lamps in its row and its column (including the lamp that was touched). Prove that we can always turn on all the lamps and find the minimum number of lamps we have to touch to do this.

Problem 2.3.18 (AtCoder GC043 B). Given is a sequence of N digits $a_1, a_2 \dots a_N$, where each element is $1, 2$, or 3. Let $x_{i,j}$ defined as follows:

- $x_{1,j} := a_j \quad (1 \leq j \leq N)$
- $x_{i,j} := |x_{i-1,j} - x_{i-1,j+1}| \quad (2 \leq i \leq N$ and $1 \leq j \leq N + 1 - i)$

Find $x_{N,1}$.

Solution. Since $|x - y| \equiv x + y (\bmod 2)$, we can determine the parity of $X_{N,1}$ using binomial coefficient. Which in turn we can get in $O(n)$ with bitwise operator. Now we have to distinguish between $0, 2$. For 2, all of the rows starting with the second one should have only 2 and 0. Where we can apply the same algorithm as before and find whether the final digit is 2 or 0.

Problem 2.3.19 (Timus 1578). The very last mammoth runs away from a group of primeval hunters. The hunters are fierce, hungry and are armed with bludgeons and stone axes. In order to escape from his pursuers, the mammoth tries to foul the trail. Its path is a polyline (not necessarily simple). Besides, all the pairs of adjacent segments of the polyline form acute angles (an angle of 0 degrees is also considered acute).

After the mammoth vanished, it turned out that it had made exactly N turns while running away. The points where the mammoth turned, as well as the points where the pursuit started and where the pursuit ended, are known. You are to determine one if the possible paths of the mammoth.

Problem 2.3.20 (Codeforces 744B). Given a hidden matrix of $n \times n$, $n \leq 1000$ where for every i , $M_{(i,i)=0}$, Luffy's task is to find the minimum value in the n rows, formarly spoken, he has to find values $\min_{j=1\ldots n,\ j\neq i} M_{(i,j)}$. To do this he can ask the computer questions of following types: In one question, Luffy picks up a set, $a_1, a_2 \ldots a_k$ with $a_i, k \leq n$. And gives the computer this set. The computer will respond with n integers. The i -th integer will contain the minimum value of $\min_{j=1\ldots k} M_{(i,\ a_j)}$. And on top of this, he can only ask 20 questions. Luffy being the stupid he is, doesn't even have any clue how to do this, you have to help him solving this problem.

Solution. If we draw the diagonal in the matrix, we see that we can fit boxes of $2^i \times 2^i$ in there depending on the i 's value. Now after we have decomposed the matrix into such boxes, we can choose several from them to ask a question. The trick is that for every row, there must be questions asked from each of the boxes this row covers and no question from here must contain the (i,i) cell.

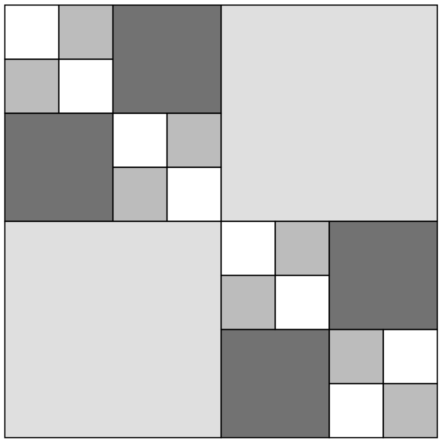


Figure 2.3.3: haha

Solution [magical solution]**.** For $i \leq 10$, for every $k = 1 \ldots n$, include k in the question if the ith bit of k's binary form is 0. And then for the second round include k in the question if the i th bit of k's binary form is 1.

Problem 2.3.21. Alice wants to add an edge (u, v) in a graph. You want to know what this edge is. So, you can ask some questions to Alice. For each question, you will give Alice 2 non-empty disjoint sets S and T to Alice, and Alice will answer "true" iff u and v belongs to different sets. You can ask atmost $3 * \lceil log_2|V| \rceil$ questions to Alice. Describe a strategy to find the edge (u, v).

Solution. First find one true answer in $\lceil log_2|V| \rceil$ questions, and then get the result out of these two sets in $2 * \lceil log_2|V| \rceil$ questions.

Solution [magical solution]**.** In the i^{th} question, $S = x : i^{th}$ bit of x is 0, $T = x : i^{th}$ bit of x is 1

Problem 2.3.22 (USAMO 2015 P4). Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he finished piling his stones in some manner, he can then perform stone moves, defined as follows.

Consider any four grid squares, which are corners of a rectangle, i.e. in positions $(i,k),(i,l)$, $(j,k),(j,l)$ for some $1 \le i,j,k,l \le n$, such that $i<j$ and $k<l$. A stone move consists of either

- ☐ removing one stone from each of (i,k) and (j,l) and moving them to (i,l) and (j,k) respectively, or
- ☐ removing one stone from each of (i,l) and (j,k) and moving them to (i,k) and (j,l) respectively.

Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves. How many different non-equivalent ways can Steve pile the stones on the grid?

> **Solution.** Building an invariant, we see that only the sum of the columns is not sufficient. So to get more control, we take the row sums into account as well.

Problem 2.3.23 (ISL 2003 C4). Let $x_1,\dots,x_n$ and $y_1,\dots,y_n$ be real numbers. Let $A=(a_{ij})_{1\le i,j\le n}$ be the matrix with entries

$$a_{ij}=\begin{cases}1, & \text{if } x_i+y_j\ge 0;\\ 0, & \text{if } x_i+y_j<0.\end{cases}$$

Suppose that B is an $n\times n$ matrix with entries 0, 1 such that the sum of the elements in each row and each column of B is equal to the corresponding sum for the matrix A. Prove that $A=B$.

> **Solution.** If done after Problem 2.3.22 problem, this problem seems straigthforward.

Problem 2.3.24 (India TST 2017 D1 P3). Let $n\ge 1$ be a positive integer. An $n\times n$ matrix is called *good* if each entry is a non-negative integer, the sum of entries in each row and each column is equal. A *permutation* matrix is an $n\times n$ matrix consisting of n ones and $n(n-1)$ zeroes such that each row and each column has exactly one non-zero entry.

Prove that any *good* matrix is a sum of finitely many *permutation* matrices.

> **Solution.** Same algo as Problem 2.3.22. Either distributing uniformly or gathering all in a diagonal

Problem 2.3.25 (Tournament of Towns 2015F S7). N children no two of the same height stand in a line. The following two-step procedure is applied:

- ☐ First, the line is split into the least possible number of groups so that in each group all children are arranged from the left to the right in ascending order of their heights (a group may consist of a single child).
- ☐ Second, the order of children in each group is reversed, so now in each group the children stand in descending order of their heights.

Prove that in result of applying this procedure $N-1$ times the children in the line would stand from the left to the right in descending order of their heights.

> **Solution.** It's obvious that we need to find some invariant or mono-variant. Now, an idea, we need to show that for any i, for it to be on it's rightful place, it doesn't need more than $N-1$ moves. How do we show that? Another idea, think about the bad bois on either of its sides. Now, observation, "junctions" decrease with each move. Find the "junctions".

Problem 2.3.26 (`Polish OI`). Given n jobs, indexed from $1, 2 \ldots n$. Given two sequences of reals, $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n$ where, $0 \leq a_i, b_i \leq 1$. If job i starts at time t , then the job takes $h_i(t) = a_i t + b_i$ time to finish. Order the jobs in a way such that the total time taken by all of the jobs is the minimum.

> **Solution.** We want to know about the optimal case, and how it behaves. Also if we think about the optimal case as a whole, we will have a lot of things to sort through. What we do instead is think about two consecutive jobs and check what properties they have. We then exploit what we found to build a strategy.

Problem 2.3.27 (`Codeforces 960C`). Pikachu had an array with him. He wrote down all the non-empty subsequences of the array on paper. Note that an array of size n has $2^n - 1$ non-empty subsequences in it.

Pikachu being mischievous as he always is, removed all the subsequences in which

$$\text{Maximum element of the subsequence} - \text{Minimum element of subsequence} \geq d$$

Pikachu was finally left with X subsequences.

However, he lost the initial array he had, and now is in serious trouble. He still remembers the numbers X and d. He now wants you to construct any such array which will satisfy the above conditions. All the numbers in the final array should be positive integers less than 10^{18}.

Note the number of elements in the output array should not be more than 10^4. If no answer is possible, print -1.

Problem 2.3.28 (`ARO 2005 P10.3, P11.2`). Given 2005 distinct numbers $a_1, a_2, \ldots, a_{2005}$. By one question, we may take three different indices $1 \leq i < j < k \leq 2005$ and find out the set of numbers $\{a_i,\ a_j,\ a_k\}$ (unordered, of course). Find the minimal number of questions, which are necessary to find out all numbers a_i.

> **Solution.** We want to maximize the amount of information each query gives us. So we need to tie our questions with our previous questions to find as many integers as possible. This leads us to the greedy approach where we circle around the positions.

Problem 2.3.29 (IOI 2007 P3). You are given two sets of integers $A = \{a_1, a_2 \dots a_n\}$ and $B = \{b_1, b_2 \dots b_n\}$ such that $a_i \geq b_i$. At move i you have to pick b_i distinct integers from the set $A_i = \{1, 2, \dots a_i\}$. In total, $(b_1 + b_2 + \cdots + b_n)$ integers are selected, but not all of these are distinct.

Suppose k distinct integers have been selected, with multiplicities $c_1, c_2, c_3 \dots c_k$. Your score is defined as

$$\sum_{i=1}^{k} c_i(c_i - 1)$$

Give an efficient algorithm to select numbers in order to "minimize" your score.

> **Solution.** Some investigation shows that if $c_i > c_j + 1$ and $i > j$, then we can always minimize the score. and if $i < j$, then we can minimize the score only when $i, j \in A_k$ but i has been taken at move k , but j hasn't.
>
> So in the minimal state, either both i, j has been taken at move k , or $a_k < j$. So the idea is to take elements from A_i as large as possible, and then taking smaller values after wards if the c_i value of a big element gets more than that of a small element. In this algorithm, we see that we greedily manipulate c_i. So it is a good idea to greedily choose c_i 's from the very beginning.
>
> At step i, take the set $\{c_1, c_2 \dots c_{a_i}\}$ and take the smallest b_i from this set, and add 1 to each of them (in other words, take their index numbers as the numbers to take).

Problem 2.3.30. Given n numbers $\{a_1, a_2, ..., a_n\}$ in arbitrary order, you have to select k of them such that no two consecutive numbers are selected and their sum is maximized.

> **Solution.** Notice that if a_i is the maximum value, and if a_i is not counted in the optimal solution, then both of a_{i-1}, a_{i+1} must be in the optimal solution, and $a_{i-1} + a_{i+1} > a_i$. And if a_i is counted in the optimal solution, then none of a_{i-1}, a_{i+1} can be counted in the optimal solution. So either way, we can remove these three and replace them by a single element to use induction. So remove a_{i-1}, a_{i+1} and replace a_i by $a_{i-1} + a_{i+1} - a_i$.

Problem 2.3.31 (USAMO 2010 P2). There are n students standing in a circle, one behind the other. The students have heights $h_1 < h_2 < \cdots < h_n$. If a student with height h_k is standing directly behind a student with height h_{k-2} or less, the two students are permitted to switch places. Prove that it is not possible to make more than $\binom{n}{3}$ such switches before reaching a position in which no further switches are possible.

> **Solution.** Instead of just thinking about the switches randomly, we try to count the number a_k of times when the student k switched his position. Trying out some smaller cases with this, we discover the upper bound of each a_k.

Problem 2.3.32 (Serbia TST 2015 P3). We have 2015 prisinoers. The king gives everyone a hat colored in one of 5 colors. Everyone sees others hats expect his own.

Now, the King orders them in a line (a prisioner can see all guys behind and in front of him). The King asks the prisinoers one by one if he knows the color of his hat.

If he answers NO, then he is killed. If he answers YES, then he is asked the color of his hat. If his answer is true, he is freed, if not, he is killed. All the prisinors can hear his YES-NO answer, but if he answered YES, they don't know what he answered (he is killed in public).

They can think of a strategy before the King comes, but after that they can't communicate. What is the largest number of prisoners they can guarentee to survive?

> **Solution** [MellowMelon]**.** Since everyone can see other's caps and hear other's answers, so we can try global approach. Usually in these "encryption" problems, we want to use some bits to express an integer that has all the information needed.
>
> In this case, if everyone knows the sum of others cap numbers, then they can easily find their's. They don't even need to know the sum, they can do it with the sum modulo 5. This idea gives us the answer 2012, because we need at least 3 bits to express all 5 cases.
>
> But wait, what about the YES answer and then getting killed or being freed? We didn't use that at all! It turns out we can use those extra informations to save one more person!
>
> Let A be $\#2$'s hat mod 5 and let B be the sum of $\#3$ through $\#2015$ mod 5. The goal is for everyone to know B after $\#1$ and $\#2$ take their turns.
>
> If $A + B \equiv 0, 1 \pmod 4$ or $(A, B) = (2, 2)$ or $(A, B) = (0, 4)$, then $\#1$ says no. Otherwise, $\#1$ says yes and guesses whatever. Let $X = 0$ if $\#1$ said no and $X = 1$ if $\#1$ said yes.
>
> If $\#2$ sees $B = 0$, (s)he says no.
> If $\#2$ sees $B = 1$, (s)he says yes and guesses hat color $3X$ mod 5.
> If $\#2$ sees $B = 2$, (s)he says no.
> If $\#2$ sees $B = 3$, (s)he says yes and guesses hat color $3 + 3X$ mod 5.

> If $\#2$ sees $B = 4$, (s)he says yes and guesses hat color $2 + 2X$ mod 5.
>
> If you draw out the five by five grid with all values of A, B and all results of the above, you'll see that the results of $\#1$'s and $\#2$'s guesses uniquely identify B in all cases.

Problem 2.3.33 (ISL 2014 N1). Let $n \geq 2$ be an integer, and let A_n be the set

$$A_n = \{2^n - 2^k \mid k \in \mathbb{Z},\, 0 \leq k < n\}.$$

Determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of A_n .

> **Solution.** We know similar problem from chicken mcnugget theorem, which motivates us to try out some smaller cases. And those cases further motivates us to formulate a conjecture. After the answer has been found, it is not hard to prove that, again by recursively trying to build the sum.

> **Solution** [getrektm9, motivation for the answer]**.** My idea for guessing the answer was observing modulo 2^{n-1}. I wasn't trying to find the smallest numeber that we can represent as sum of elements from the set with the condition that it is congruent to modulo 1 while dividing with 2^{n-1}. It turns out that the smallest number that we can't represent is exatcly $(n-2)\cdot 2^n + 1$.
>
> The reason for observing modulo 2^{n-1} is that the smallest number in the set is 2^{n-1} and later showed that every other residue bigger than 1 can be obtained with the condition that its smallest sum is less than when residue is 1.

Problem 2.3.34 (Taiwan TST 2015 R3D1P1). A plane has several seats on it, each with its own price, as shown below. $2n - 2$ passengers wish to take this plane, but none of them wants to sit with any other passenger in the same column or row. The captain realize that, no matter how he arranges the passengers, the total money he can collect is the same. Proof this fact, and compute how much money the captain can collect.

					n	n					
				n	n-1	n-1	n				
			n	n-1	n-2	n-2	n-1	n			
		$\iddots$	$\iddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$		
	n	...		...	3	3	...		...	n	
n	n-1	...		3	2	2	3		...	n-1	n
n-1	...	...		2	1	1	2		...	...	n-1
...	...		2	1			1	2		...	...
$\vdots$	$\vdots$									$\vdots$	$\vdots$
3	2	1							1	2	3
2	1									1	2
1											1

Table 2.1: Problem 2.3.34, there are $2n-2$ columns and $2n-2$ rows.

Solution [induction]. This table looks really appropriate for a nice induction approach. We want to prove that the total is n^2-1, which tells us how much change we will get by going from n to $n-1$. And the seats that will guarantee that this change will occure are the one in the first row, and the two seats in the last and first columns. But we need to do some work to get them nicely in position.

We use the usual grid movement trick: $(a,b),(c,d) \leftrightarrow (a,d),(c,b)$.

Solution [MellowMelon]. The price of the seats in this arrangement are almost like the pricing of the lattice points by $x+y$. And if we give the rows and columns their own prices and change the pricing system like so, we get an easy invariant.

Problem 2.3.35 (`Codeforces 330D`). Biridian Forest

Solution. Generalize the condition for a meet-up.

Problem 2.3.36 (`Codeforces 1270F`). Let's call a binary string s awesome, if it has at least 1 symbol `1` and length of the string is divisible by the number of `1` in it. In particular, `1`, `1010`, `111` are awesome, but `0`, `110`, `01010` aren't.

You are given a binary string s of size $\leq 2\times 10^5$. Count the number of its awesome substrings.

Solution. The constraint tells us the algorithm should be of complexity $O(n\sqrt{n})$. Playing with the funtion $f(i)$ and the divisibility condition gives us some ground to work with.

Since we need $\leq c \times \sqrt{n}$ queries around the full string, we know we need to use something similar to the prime sieve trick.

Problem 2.3.37 (IMO 1986 P3). To each vertex of a regular pentagon an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively, and $y < 0$, then the following operation is allowed: x, y, z are replaced by $x + y, -y, z + y$ respectively.

Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

Solution [invariant]**.** The move changes a lot except the sum. And if we wanted it to terminate at some point, we know there might be an invariant that decreases monotonically. What is the closest form of invariance to sum? Sum of squares. But sum of squares doesn't tell us much either. But we know sometimes sum of squares of differences bear good results. So we try some of the different functions we get from there.

And voila, if we let

$$f(a_1, a_2, a_3, a_4, a_5) = \sum_{i=1}^{5} (a_i - a_{i+2})^2$$

we get a value that decreases!

Solution. Notice how starting from one negative number and applying the move to the negative number created to its one side and keeping on doing this, we can see the numbers move to the other direction. We use this idea to come up with a compound move to increase the least number.

Problem 2.3.38 (Codeforces 1379E). Let G be a tree with n vertices that has a root, and every vertex has either 2 children or no child. A vertex is said to be "good" if the two subtrees of it, say X, Y, satisfy either $V(X) \geq 2V(Y)$ or $V(Y) \geq 2V(X)$ (here $V(X)$ is the number of vertices of tree X). Find all k such that there is a tree with n vertices with k good vertices, and give a construction to build the tree.

Solution. The first thing we notice is the for even n, no such tree exists. Now, binary trees are really nice, because we can build them recursively: Delete the root, and build the two subtrees by recursion. Trying out for some initial values, we get the solution pattern that if $S(n)$ is the set of possible values for k, then

$$S(n) = \begin{cases} \left\{1, 2, \dots \dfrac{n-3}{2}\right\} & \text{if } n \neq 2^t - 1 \\ \left\{0, 2, \dots \dfrac{n-3}{2}\right\} & \text{if } n = 2^t - 1 \end{cases}$$

Which isn't hard to proof by induction.

Problem 2.3.39 (ISL 2010 C4). Each of the six boxes B_1, B_2, B_3, B_4, B_5, B_6 initially contains one coin. The following operations are allowed

1. Choose a non-empty box B_j, $1 \leq j \leq 5$, remove one coin from B_j and add two coins to B_{j+1};
2. Choose a non-empty box B_k, $1 \leq k \leq 4$, remove one coin from B_k and swap the contents (maybe empty) of the boxes B_{k+1} and B_{k+2}.

Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes B_1, B_2, B_3, B_4, B_5 become empty, while box B_6 contains exactly $2010^{2010^{2010}}$ coins.

Solution. The given number is absurdly big, with no apparent reason right now, so we decide to just look for a way to maximize a box as much as we can.

After playing around with this idea, we soon notice that we can make powers of 2, with some compound moves. Which in turn provides the construction for nested powers of twos. So it might not be impossible to get a large enough nested power of two that is greater than our large number.

But what about "exactly" condition? The only move that allows us to reduce is the second move, which requires two boxes at the end, So we need to make the big number appear somewhere in B_1, B_2, B_3 or B_4.

2.3.5 Algorithm Analysis

Problem 2.3.40 (GQMO 2020 P4). Prove that, for all sufficiently large integers n, there exists n numbers $a_1, a_2, \ldots, a_n$ satisfying the following three conditions:

Each number a_i is equal to either $-1, 0$ or 1. At least $\frac{2n}{5}$ of the numbers $a_1, a_2, \ldots, a_n$ are non-zero. The sum $\frac{a_1}{1} + \frac{a_2}{2} + \cdots + \frac{a_n}{n}$ is 0.

Note: Results with 2/5 replaced by a constant c *will be awarded points depending on the value of* c

2.3.6 Covering Area with Squares

- A nice blog post by ankogonit

Problem 2.3.41 (Brazilian MO 2002, ARO 1979). Given a finite collection of squares with total area at least 4, prove that you can cover a unit square completely with these squares (with overlapping allowed, of course).

Solution. Maybe motivated by the number 4 and how nice it would be if all the squares had 2's power side lengths, the idea is to shrink every square to a side with side of 2's power.

Problem 2.3.42 (ARO 1979's Sharper Version). Given a finite collection of squares with total area at least 3, prove that you can cover a unit square completely with these squares (with overlapping allowed).

Solution. The idea is to greedily cover the unit square by covering the lowest row uncovered. And then using boundings to prove that it is possible.

Problem 2.3.43. Prove that a finite collection of squares of total area $\frac{1}{2}$ can be placed inside a unit square without overlap.

Solution. The same idea as before.

Problem 2.3.44 (Tournament of Towns Spring 2012 S7). We attempt to cover the plane with an infinite sequence of rectangles, overlapping allowed.

1. Is the task always possible if the area of the $n-$th rectangle is n^2 for each n?
2. Is the task always possible if each rectangle is a square, and for any number N, there exist squares with total area greater than N?

Solution. Identical algo and proving technique as above.

Solution. Using the first problem in this subsection to find a better algo.

Problem 2.3.45 (ISL 2006 C6). A holey triangle is an upward equilateral triangle of side length n with n upward unit triangular holes cut out. A diamond is a $60° - 120°$ unit rhombus.

Prove that a holey triangle T can be tiled with diamonds if and only if the following condition holds: Every upward equilateral triangle of side length k in T contains at most k holes, for $1 \le k \le n$.

> **Solution.** Think of induction and how you can deal with that.

Problem 2.3.46 (Putnam 2002 A3). Let N be an integer greater than 1 and let T_n be the number of non empty subsets S of $\{1, 2, ..., n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

> **Solution.** Try to show an bijection between the sets with average n which has k elements (Here k is an even integer) and the sets with average n but with number of elements $k + 1$. This implies that the number of such sets is even.

Problem 2.3.47 (USAMO 1998). Prove that for each $n \ge 2$, there is a set S of n integders such that $(a - b)^2$ divides ab for every distinct $a, b \in S$.

> **Solution.** Induction comes to the rescue. Trying to find a way to get from n to $n + 1$, we see that we can *shift* the integers by any integer k. So after shifting, what stays the same, and what changes?

2.4 Graph Theory

Stuck? Try These: Turning grids into graphs

- One common way to turn a grid into graphs is to create a bipartite graph between the columns and rows such that c_i and r_j are connected iff (i, j) is marked. This way we can find cycles alternating row and column.
- Creating a bipartite graph between all rows and columns and particular objects. This helps to prove matching.

Lemma 2.4.1 (Bipartite Graph) — Any graph having only even cycles are *Bipartite*.

1. AoPS
2. ISL 2004 C3
3. Problem
4. Problem

Theorem 2.4.2 (Euler's Polyhedron Formula) — For any polyhedron with E, V, F edges, vertices's and faces resp. the following relation holds

$$V + F = E + 2$$

In a planar graph with V vertices, E edges and C cycles, the following condition is always satisfied:

$$V + C = E + 1$$

Lemma 2.4.3 (Criteria of partitioning a graph into disconnected sub-graphs) — If there exist no three vertices, u, v, w that $uv \in E(G)$ also $uw, vw \in E(G)$, the graph can be partitioned into equivalence classes based on their non-neighbors.

Problem 2.4.1 (China TST 2015 T1 D2 P3). There are some players in a Ping Pong tournament, where every 2 players play with each other at most once. Given:

1. Each player wins against at least a players, and loses to at least b players. $(a, b \geq 1)$
2. For any two players A, B, there exist some players $P_1, ..., P_k$ $(k \geq 2)$ (where $P_1 = A$,$P_k = B$), such that P_i wins against P_{i+1} $(i = 1, 2..., k - 1)$

Prove that there exist $a + b + 1$ distinct players $Q_1, ...Q_{a+b+1}$, such that Q_i wins against Q_{i+1} $(i = 1, ..., a + b)$.

Remark. Typical largest path, some workaround with given constraints problem.

Solution. Take the largest path starting from a_1 to a_n.

Definition— Assume that $n \le a + b$. Since this is the largest path, a edges coming out of a_n are all in $S = \{a_1, a_2, \dots a_n\}$, and b edges going in a_1 are all in S. Let S_1 be the set of vertices that a_n wins against, and S_2 be the set of vertices that a_1 loses against. Moreover, let a_l be the smallest element of S_1, and a_{n-k} be the largest element of S_2 (smallest means leftmost in the part, and largest means rightmost).

Let $S' = \{a_l, a_{l+1}, \dots a_{n-k-1}, a_{n-k}\}$. Since $n \le a + b$, we have $S' \ne \varnothing$.

We also define:

$$S_1' = \{a_i \mid a_i \text{ defeats } a_{i+1} \text{ where } a_{i+1} \in S_1 \cap S'\}$$

$$S_2' = \{a_i \mid a_{i-1} \text{ defeats } a_i \text{ where } a_{i-1} \in S_2 \cap S'\}$$

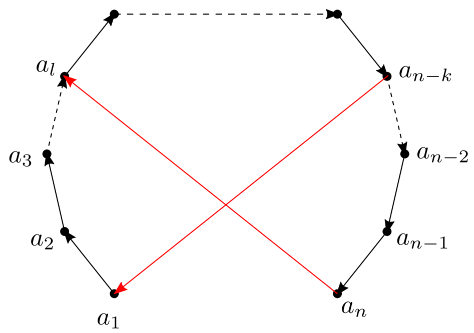


Figure 2.4.1

Now, note that for any $x \in S_2'$, $y \in S_1'$, there is a path between x, y with n vertices. So for all $x \in S_2'$, there does not exist a vertex outside of S that defeats x. And for all $y \in S_1'$, there doesn't exist a vertex outside of S that loses to y, because of the maximality of n.

We show that, $S_1' \cap S_2' \ne \varnothing$. Then there would exist a vertex that doesn't have any edge outside of S, meaning it has at least $a + b$ games inside S, proving the result.

We have, $S_1', S_2' \subset S' \cup \{a_{l-1}, a_{n-k+1}\}$. We have,

$$|S_1'| \ge a - k + 2 \quad [\because a_n \text{ has at most } k-2 \text{ vertices in } \{a_{n-k+1}, \dots a_n\}]$$
$$|S_2'| \ge b - l + 3$$

But $|S'| = n - (l-1) - (k) + 2 \le a + b - l - k + 3 < |S_1'| + |S_2'|$. So $S_1' \cap S_2' \ne \varnothing$, and we are done.

Problem 2.4.2 (ARO 2005 P10.8). A white plane is partitioned in to cells (in a usual way). A finite number of cells are coloured black. Each black cell has an even (0, 2 or 4) number of adjacent (by the side) white cells. Prove that one may colour each white cell in green or red such that every black cell will have equal number of red and green adjacent cells.

Solution. First we join the white cells like this:

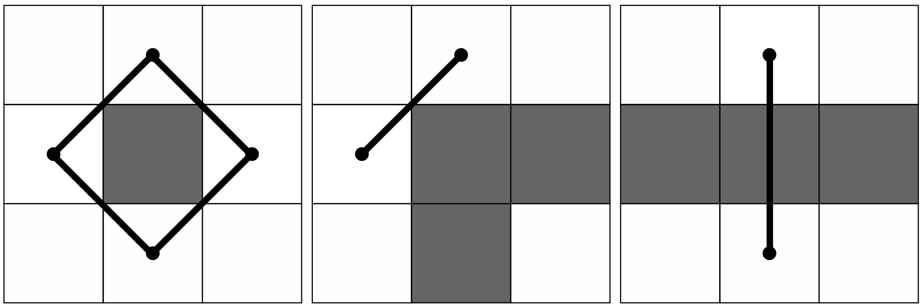


Figure 2.4.2

Now, notice that the plane have been divided into some cycles (a black cell that has no adjacent black cells is a cycle itself). So we can color the plane blue and yellow in a way that no region has the same color as its neighbors. We can do this because at any junction, there are an even number of regions connected because of the problem condition.

Now we focus on our graph that we created connected the white cells. Take any cycle on it. If we have a "slanted" edge, then both of the nodes are inside a region of either blue or yellow. But in a "straight" edge, the two nodes are in different colored region.

We know that there are an even number of slanted edges, which is trivial to prove (using the fact that any cycle on a grid system has even number of nodes, and on these cycles, most of the edges (the straight ones) have even lenght, but only the slanted one has odd lenghts on the sides). It is also easy to see that there are an even number of straight edges, because of going in and out of the regions of a fixed color.

So our cycle has an even number of nodes and thus bipartite. We can color the graph with two colors, so that along each edge, the two nodes are of different color.

Remark. There is a simple coloring using this solution. After we color the regions of the plane with blue and yellow, we number each column with integers. Then on the odd numbered columns, we color all the white cells that are in yellow region green and blue region red. And on the even numbered columns, we do the opposite. It is easy to check that this coloring works using the graph we created before.

Problem 2.4.3 (AtCoder GC033 C). Takahashi and Aoki will play a game on a tree. The tree has N vertices numbered 1 to N, and the i-th of the $N-1$ edges connects Vertex a_i and Vertex b_i

At the beginning of the game, each vertex contains a coin. Starting from Takahashi, he and Aoki will alternately perform the following operation:

- Choose a vertex v that contains one or more coins, and remove all the coins from v.
- Then, move each coin remaining on the tree to the vertex that is nearest to v among the adjacent vertices of the coin's current vertex.

The player who becomes unable to play, loses the game. That is, the player who takes his turn when there is no coin remaining on the tree, loses the game. Determine the winner of the game when both players play optimally.

> **Solution.** First transform the game by removing the idea of coins, and replacing it with deleting vertices. Now, notice that the longest path in this tree (i.e. the diameter) strictly decreases by 1 or 2 each turn depending on the move. So it's just a basic predetermined game.

Problem 2.4.4 (ARO 1999 P9.8). There are 2000 components in a circuit, every two of which were initially joined by a wire. The hooligans Vasya and Petya cut the wires one after another. Vasya, who starts, cuts one wire on his turn, while Petya cuts one or three. The hooligan who cuts the last wire from some component loses. Who has the winning strategy?

> **Solution** [Copycat]. The P-Hooligan Petya has a winning strategy, for he can be follow the old cunning trick of never losing. How does he do it?
> He starts by secretly partitioning the vertices in two 1000 degree subsets. He calls them $A = \{a_1, a_2 \dots a_{1000}\}$ and $B = \{b_1, b_2 \dots b_{1000}\}$. He then connects a_i with b_i with an edge with an invisible marker that only he can see.
> Now the game begins. Petya copies Vasyas moves following these rules:
>
> 1. If Vasya removes an edge $a_i - a_j$, where $i \neq j$, then Patya removes the edges $a_i - b_j,\ a_j - b_i$ and $b_i - b_j$.
> 2. If Vasya removes $a_j - b_i$, where $i \neq j$, then Patya removes the other three edges from the above rule.
> 3. If Vasya removes a_i–b_i, then Patya looks for another b_j, such that a_i–b_j exists. Then by the symmetry so far maintanined, a_j–b_i and a_j–b_j exist too. And Patya can remove a_j–b_j, and swap the names of b_j and b_i.
>
>   But if he can't, then that would mean after Vasya's move a_i would become isolated, and Patya would win.
>
> It is easy to see that the above moves are possible since Patya is always maintaining symmetry between A, B. So he can't move means Vasya has already disconnected one of the vertices.

Remark. The case with 4 vertices and 6 vertices give an idea to copy the opponent's moves.

Problem 2.4.5 (ISL 2001 C3). Define a k-clique to be a set of k people such that every pair of them are acquainted with each other. At a certain party, every pair of 3-cliques has at least one person in common, and there are no 5-cliques. Prove that there are two or fewer people at the party whose departure leaves no 3-clique remaining.

> **Solution.** Casework with the point where most of the triangles are joined.

Problem 2.4.6 (ARO 2017 P9.1). In a country some cities are connected by one-way flights (there is at most one flight between two cities). City A is called "available" from city B, if there is a flight from B to A, maybe with some transits. It is known, that for every 2 cities P and Q, there exists a city R, such that P and Q are both available from R. Prove, that exist city A, such that every city is available from A.

> **Solution.** Basic induction exercise.

Problem 2.4.7 (Tournament of Towns 2009 S6). Anna and Ben decided to visit a country with 2009 islands. Some pairs of islands are connected by boats which run both ways. Anna and Ben are playing during the trip:

Anna chooses the first island on which they arrive by plane. Then Ben chooses the next island which they could visit. Thereafter, the two take turns choosing an island which they have not yet visited. When they arrive at an island which is connected only to islands they had already visited, whoever's turn to choose next would be the loser. Prove that Anna could always win, regardless of the way Ben played and regardless of the way the islands were connected.

> **Solution.** The copycat idea, as always in games. How can Anna make sure she gets a move after Ben? Maybe she can "pair" vertices and follow them along. What happens if she tries this strategy?

Problem 2.4.8 (USA TST 2014 P5). Find the maximum number E such that the following holds: there is an edge-colored graph with 60 vertices and E edges, with each edge colored either red or blue, such that in that coloring, there is no monochromatic cycles of length 3 and no monochromatic cycles of length 5.

2.4.1 Counting in Graph

Lemma 2.4.4 (Average of Degrees) — In a graph G with n vertexes, let E be the set of all edges. Assign an integer f_i to every vertex v_i such that f_i equals to the everage degree of the neighbors of v_i. We have,

$$\sum_{i=1}^{n} f_i \geq 2|E|$$

Lemma 2.4.5 — In a graph G with n vertexes, let E be the set of all edges. Assign an integer g_i to every vertex v_i such that g_i equals to the maximum degree among its neighbors. We have,

$$\sum_{i=1}^{n} g_i \geq 2|E|$$

Problem 2.4.9 (USA TST 2014 P3). Let n be an even positive integer, and let G be an n-vertex graph with exactly $\frac{n^2}{4}$ edges, where there are no loops or multiple edges (each unordered pair of distinct vertices is joined by either 0 or 1 edge). An unordered pair of distinct vertices $\{x, y\}$ is said to be amicable if they have a common neighbor (there is a vertex z such that xz and yz are both edges). Prove that G has at least $2\binom{n/2}{2}$ pairs of vertices which are amicable.

Solution. Define friendship in a different way, bounding below, keeping in mind the equality case. Then using the previous lemma.

Theorem 2.4.6 (Turan's theorem) — Let G be any graph with n vertices, such that G is K_{r+1} -free. Then G is the "Turán's Graph" and is a complete r partite graph. And the number of edges in G is at most

$$\frac{r-1}{r} \cdot \frac{n^2}{2} = \left(1 - \frac{1}{r}\right) \cdot \frac{n^2}{2}$$

A special case of Turán's theorem for $n = 2$ is the **Mantel's Theorem**. It states that the maximal triangle free graph is a complete bipartite graph with at most $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges.

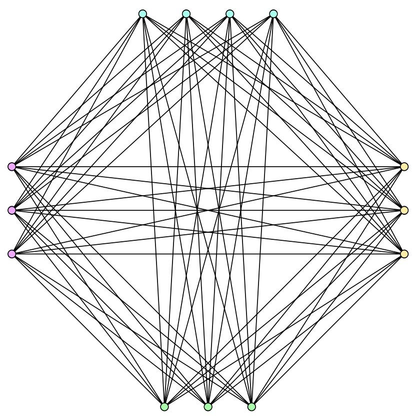


Figure 2.4.3: Turán's Graph

Proof. We need to prove that the maximal graph is the r partite one, and the rest will follow. We can directly try to prove that this graph is r colorable, but that is quite troublesome. Instead, we try to show that, we can partition the vertices of G into equivalence classes based on their non-neighbors. Since this is imply the former. So we need to prove that this holds for this graph.

The way it is done is quite interesting. We need to show that if the criteria doesn't hold in this graph, then this graph is not the maximal graph. How are we going to do that? We compare the degrees of u, w, and replace either u by w or w by u to get a graph with more edges and without the nasty situation.

Problem 2.4.10. 155 birds $P_1, P_2, \ldots, P_{155}$ are sitting down no the boundary of a circle C. Two birds P_i, P_j are mutually visible if the angle at the center of their cord, $m(P_iP_j) \leq 10°$. Find the smallest number of mutually visible pairs of birds.

Problem 2.4.11. For a pair $A = (x_1, y_1)$ and $B = (x_2, y_2)$ of points on the coordinate plane, let $d(A, B) = |x_1 - x_2| + |y_1 - y_2|$. We call a pair (A, B) of unordered points harmonic if $1 < d(A, B) \leq 2$. Determine the maximum number of harminc pairs among 100 points in the plane.

Problem 2.4.12 (`Swell coloring`)**.** Let K_n denote the complete graph on n vertices, that is, the graph with n vertices's such that every pair of vertices's is connected by an edge. A swell coloring of K_n is an assignment of a color to each of the edges such that the edges of any triangle are either all of distinct colors or all the same color. Further, more than one color must be used in total (otherwise trivially if all edges are the same color we would have a swell coloring). Show that if K_n can be swell colored with k colors, then $k \geq \sqrt{n} + 1$.

Solution. Concentrate on only one vertex.

Problem 2.4.13 (`Belarus 2001`)**.** Given n people, any two are either friends or enemies, and friendship and enmity are mutual. I want to distribute hats to them, in such a way that any two friends possess a hat of the same color but no two enemies possess a hat of the same color. Each person can receive multiple hats. What is the minimum number of colors required to always guarantee that I can do this?

Solution. In this problem, finding the worst case is a big help, because once the answer is guessed, the things become really clear.

Problem 2.4.14 (`ELMO 2017 P5`)**.** The edges of K_{2017} are each labeled with $1, 2$ or 3 such that any triangle has sum of labels at least 5. Determine the minimum possible average of all labels. (Here K_{2017} is defined as the complete graph on 2017 vertices's, with an edge between every pair of vertices's.)

Solution. A starting idea to get the ans: if we discard of all the 2-edges, we see that in any triangle, one edge has to be a 3-edge. So... Turan-kinda...

Solution. After getting the ans, and thinking about approaching inductively, if we remove only one vertex, there will be pairs to consider. But if we remove two vertices, we will only need to consider single vertices after the removal of these two vertices.

Now which pair of vertices are the best choice to remove? Before doing that, lets first think how much change will we get in the sum after we remove two vertices. Since we have the ans, we do quick maffs:

$$m(4m+1)-(m-1)(4m-3)=8m-3=4\times(2m-1)+1$$

Doesn't this indicate that we remove a 1-edge, so the other edges coming out of the two vertices will sum up to be at least $4*(2m-1)$.

Solution. The solution by bern is very pretty. What he probably had thought was:

If we pick a vertex, say u, and take an 1-edge from this vertex to another vertex v, we see that there are at least as many 3-edges in u than there are 1-edges in v. Now if to get a more accurate value of $d_3(u)$ (defined naturally), we need to take the maximum of the values $d_1(v)$ for all v's connected to u.

Now we need to evaluate the number of 3 edges from the d_1 values. Can we put a bound on this sum? We have **this lemma**, does this help? Turns out that it does.

What left is to sum it all up to see if we can get the ans.

Problem 2.4.15 (`ARO 2005 P9.4`). 100 people from 50 countries, two from each countries, stay on a circle. Prove that one may partition them onto 2 groups in such way that neither no two countrymen, nor three consecutive people on a circle, are in the same group.

Variant: There are 100 people from 25 countries sitting around a circular table. Prove that they can be separated into four classes, so that no two countrymen are in the same class, nor any two people sitting adjacent in the circle.

Solution. Thinking of the most natural way of eliminating the consecutive condition – pair two consecutive verices.

Problem 2.4.16 (Romanian TST 2012 P4)**.** Prove that a finite simple planar graph has an orientation so that every vertex has out-degree at most 3.

Problem 2.4.17 (USA TST 2006 P1)**.** A communications network consisting of some terminals is called a 3-connector if among any three terminals, some two of them can directly communicate with each other. A communications network contains a windmill with n blades if there exist n pairs of terminals $\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_n, y_n\}$ such that each x_i can directly communicate with the corresponding y_i and there is a hub terminal that can directly communicate with each of the $2n$ terminals $x_1, y_1, \ldots, x_n, y_n$. Determine the minimum value of $f(n)$, in terms of n, such that a 3 -connector with $f(n)$ terminals always contains a windmill with n blades.

> **Solution.** Windmills won't be there if among any $2n+1$ vertices, there were one vertex that were not connected to any of the other $2n$ vertices. So that means that we are dealing Turan-kinda config here. So we can make several 'compact' graphs that are mutually disconnected, and each have at most $2n$ verices. Guessing from this, the ans is probably of some form $k * 2n + 1$. Now we have another condition to consider, 3-connector. Lets see, if we had 3 disconnected componets, the resulting graph wouldn't be a 3-connector. Done...

Problem 2.4.18. Graph G on n vertices has the property that the degree of every vertex is greater than 2. Prove that for every $0 < k < n$, there is a simple path with lenght at least n/k or, k cycles, such that every cycle has at least one node which none of the other cycles has, and its lenght is not divisible by 3.

Problem 2.4.19 (ISL 2005 C4)**.** Let $n \geq 3$ be a fixed integer. Each side and each diagonal of a regular n-gon is labelled with a number from the set $\{1;\ 2;\ \ldots;\ r\}$ in a way such that the following two conditions are fulfilled:

- Each number from the set $\{1, 2, \ldots r\}$ occurs at least once as a label.
- In each triangle formed by three vertices of the n-gon, two of the sides are labelled with the same number, and this number is greater than the label of the third side.

1. Find the maximal r for which such a labelling is possible.
2. For this maximal value of r, how many such labellings are there?

> **Solution** [Extremal]**.** Take the edges labeled with r, and delete them. Study what is left. For the second part, formulate a recursive function, and try out small cases to find pattern.

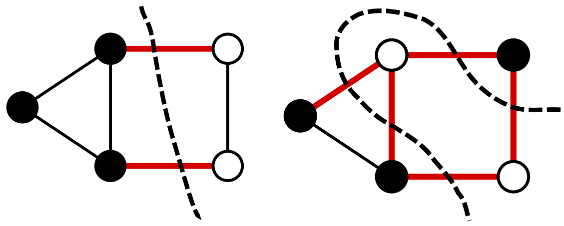
Problem 2.4.20 (St Petersburg 2020 P11.7)**.** N oligarchs built a country with N cities with each one of them owning one city. In addition, each oligarch built some roads such that the maximal amount of roads an oligarch can build between two cities is 1 (note that there can be more than 1 road going through two cities, but they would belong to different oligarchs).

A total of d roads were built. Some oligarchs wanted to create a corporation by combining their cities and roads so that from any city of the corporation you can go to any city of the corporation using only corporation roads (roads can go to other cities outside corporation) but it turned out that no group of less than N oligarchs can create a corporation. What is the maximal amount that d can have?

> **Solution.** At first I thought about "cuts" where we can only have roads owned by one oligarch, but it proved to be really complex to work with. So I thought about constructing the best solution. Trying it out for $3, 4$ immediately gave the idea to construct optimally. Now on forward to proving it.
>
> The proof is roughly as followed. We will show that if we remove the oligarch indexed N, then we need to remove at most $\binom{N}{2}$ roads. Since there is no road owned by N that connects to city N, the roads owned by N forms a forest of graphs with the other cities.
>
> We show that for every edge in that forest, there is one less road leaving city N. Which we do by induction. We take the set $\{1, 2, \ldots N-1\}$, one of these cities has no road with N. WLOG, it is 1. Then inductively we can assume that city i can have at most $i-1$ roads with N.
>
> Now for each i, starting with $N-1$, and ending at 1, we show that in reality, i can have at most $i-1-V_N(i)$ where $V_N(i)$ is the number of roads owned by N leaving i. It works inductively, and so after it, we can just remove N, and assume our inductive hypothesis. Which gives us our answer of
>
> $$\boxed{\binom{N}{3}}$$

2.4.2 Algorithms in Graph

Definition (Cut)— A cut is a partition of the vertices of a graph into two disjoint subsets. Any cut determines a cut-set, the set of edges that have one endpoint in each subset of the partition. These edges are said to cross the cut. In a connected graph, each cut-set determines a unique cut, and in some cases cuts are identified with their cut-sets rather than with their vertex partitions.



(a) Minimum Cut (b) Maximum Cut

Theorem 2.4.7 (Prufer sequence) — Consider a labeled tree T with vertices's $\{1, 2, ..., n\}$. At step i, remove the leaf with the smallest label and set the ith element of the *Prüfer sequence* to be the label of this leaf's neighbour. Prove that a Prüfer sequence of length $n-2$ defines a Tree with length n.

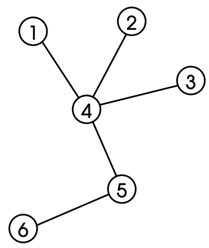


Figure 2.4.4: $4, 4, 4, 5$

Problem 2.4.21 (German TST 2004 E7P3). We consider graphs with vertices colored black or white. "Switching" a vertex means: coloring it black if it was formerly white, and coloring it white if it was formerly black.

Consider a finite graph with all vertices colored white. Now, we can do the following operation: Switch a vertex and simultaneously switch all of its neighbours (i.e. all vertices connected to this vertex by an edge). Can we, just by performing this operation several times, obtain a graph with all vertices colored black?

Solution. A classical example of creating complex moves from counter cases.

Problem 2.4.22 (ARO 2014 P9.8). In a country of n cities, an express train runs both ways between any two cities. For any train, ticket prices either direction are equal, but for any different routes these prices are different. Prove that the traveler can select the starting city, leave it and go on, successively, $n-1$ trains, such that each fare is smaller than that of the previous fare. (A traveler can enter the same city several times.)

Problem 2.4.23 (Generalization of ARO 2014 P9.8). Let A be a set of n points in the space. From the family of all segments with endpoints in A , q segments have been selected and colored yellow. Suppose that all yellow segments are of different length. Prove that there exists a polygonal line composed of m yellow segments, where $m \geq \frac{2q}{n}$, arranged in order of increasing length.

Solution. There are no local information given that we can use to consturct a way, so our only bet is to try some global approach. We want at least one path of length more than $n-1$, and there are a total of $\frac{n(n-1)}{2}$ edges. So either we want to partition the edges in $\frac{n}{2}$ paths, or we count each edge twice and then partition then in n paths, giving us our desired result.

We figure out the second approach works. So we want to find n paths that begin at each of the n vertices and cover all the edges twice. This idea soon lead us to the magical solution:

Let one traveller go to each of the n cities. Then we select the costliest train and swap the two travellers on its two ends. We keep doing this until all the trains have been used.

This works because every traveller follows a path of gradually decreasing train cost, and every edge is used exactly twice.

Solution [ninjaturtle]**.** Consider the direct graph where each edge has two directions ($2q$ edges). A directed edge e is called *terminal* if there isn't an edge f with larger value such that the end vertex of e is the same as the start vertex of f.

At step i, we remove all terminal edges in the remaining graph. If we can show that each step removes at most n direct edges, then $2q/n$ steps are required to remove all edges, implying there exists a polygonal line of increasing length of length at least $2q/n$.

Let $I_i(a), O_i(a)$ be the in-degree and out-degree of a before step i. Assume we are at the time before step k. Put $m = O_k(a)$, and let e be the outgoing edge of a with the largest value v. Then $v \geq m$. After step k, only incoming edges of a with value $\geq m$ will have possibly been removed. Thus $I_{k+1}(a) \geq m-1 = O_k(a) - 1$. Sum it over all a

$$\#\text{edges before step k+1} = \sum_a I_{k+1}(a) \geq \sum_a O_k(a) - n = \#\text{edges before step k}$$

and we have shown the number of edges after each step decreases by at most n.

Problem 2.4.24. Given a bipartite graph, prove that the minimum number of colors required to color the edges of the graph such that no node is adjacent to 2 edges of same color is the maximum degree of the graph.

Problem 2.4.25. For every bipartite graph prove that it's edges can be bicolored so that each node is adjacent to atmost $\left\lceil \frac{deg}{2} \right\rceil$ edges of any color.

Solution. Using the main property of a bipartite graph.

Solution. After finding the cycle solution, to optimize it, we recall that we can find a Eulerian Path (if it exists) in $O(V+E)$. Now we want to make the graph have a Eulerian path, so we add a vertice to both sides of the graph, and join them with odd vertices from the other side.

Problem 2.4.26 (Turkey National MO 2002 P3)**.** Graph Airlines (GA) operates flights between some of the cities of the Republic of Graphia. There are at least three GA flights from each city, and it is possible to travel from any city in Graphia to any city in Graphia using GA flights. GA decides to discontinue some of its flights. Show that this can be done in such a way that it is still possible to travel between any two cities using GA flights, yet at least $2/9$ of the cities have only one flight.

Simplified: In a connected graph, every vertex has degree at least 3. Prove that some edges can be deleted to turn that graph into a tree with at least $\frac{2}{9}$ leaves.

Better Approximation: We can actually achieve $\frac{1}{4}$ with careful construction.

Solution [dgrozev]**.** First we construct a spanning tree T that maximizes the number of leaves, then we bound the number. We define $G(V, E), n = |V|$ with the usual notations.

Let $f : V \to V$. We initialize the tree by selecting a vertex v by random, and adding it to T. We inductively add the vertices according to the following priority checks:

$\boxed{1.}$ If there is a vertex $v \notin T$ that is connected to $u \in T$ such that u is not a leaf, then add v, uv to T.

$\boxed{2.}$ If $u \in T$ is a leaf and there are two v_1, v_2 not in T that are connected to u, add v_1, v_2 and uv_1, uv_2 to T.

$\boxed{3.}$ If $u \in T$ is a leaf, and there is a v which has two neighbors outside of T, then add v, uv to T, and let $f(u) = v$.

$\boxed{4.}$ If $u \in T$ is a leaf, there is a $v \notin T$ which is connected to at most one vertex outside T, and connected to $u' \in T$, then add v, uv to T and let $f(u) = u'$.

This algorithm will add all the vertices to the tree. We now need to bound the number of leaves.

Let n_1, n_2, n_3 be the set of vertices that have $1, 2$ and more than 3 neighbors respectively. Since f is a injection from the set n_2 to either n_1 or n_3, we have since $n = n_1 + n_2 + n_3$,

$$n_2 \leq n_1 + n_3, \quad n_2 \leq \frac{n}{2}$$

Bounding the number of edges gives us:

$$\begin{aligned} 2n - 2 &\geq n_1 + 2n_2 + 3n_3 \\ &= n_1 + 3(n - n_1) - n_2 \\ &\geq 3n - 2n_1 - \frac{n}{2} \\ \implies n_1 &\geq \frac{n}{4} + 1 \end{aligned}$$

Solution [Paper]. The construction is the same as before. But we define a different cost fuction f to bound our leaves count. Let $D(T)$ be the number of leaves in T which doesn't have a neighbor outside of T. Let $L(T)$ be the number of all leaves, and $V(T)$ is the number of vertices of T. Then consider

$$\boxed{f(T) = 3L(T) + D(T) - V(T)}$$

We show that $f(T)$ is non decreasing in our construction. If it is, we will get by setting $f(T_0)$ for a one vertex and its neighbor tree T_0,

$$f(T) \geq f(T_0) \geq 3 \times 3 + 0 - 4 = 5$$

And since $D(T') = L(T')$ for a spanning tree T',

$$L(T') \geq \frac{N + 5}{4}$$

We now check that for all of your steps in construction, $f(T)$ is non decreasing.

Problem 2.4.27 (ISL 2005 C1). A house has an even number of lamps distributed among its rooms in such a way that there are at least three lamps in every room. Each lamp shares a switch with exactly one other lamp, not necessarily from the same room. Each change in the switch shared by two lamps changes their states simultaneously. Prove that for every initial state of the lamps there exists a sequence of changes in some of the switches at the end of which each room contains lamps which are on as well as lamps which are off.

Problem 2.4.28 (ISL 2013 C3). A crazy physicist discovered a new kind of particle which he called an i -mon, after some of them mysteriously appeared in his lab. Some pairs of i -mons in the lab can be entangled, and each i -mon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.

1. If some i -mon is entangled with an odd number of other i -mons in the lab, then the physicist can destroy it.
2. At any moment, he may double the whole family of i -mons in the lab by creating a copy I' of each i -mon I. During this procedure, the two copies I' and J' become entangled if and only if the original i -mons I and J are entangled, and each copy I' becomes entangled with its original i -mon I ; no other entanglements occur or disappear at this moment.

Prove that the physicist may apply a sequence of much operations resulting in a family of i -mons, no two of which are entangled.

Solution. As there are an integer number of i -mons, it is quite natural to use induction. We try to find an algorithm to reduce the number of particles.
Another way to do this is to consider the chromatic number of the graph. If we can show that this number reduces after some move, then we are done by induction.

Problem 2.4.29 (ISL 2005 C2). A forest consists of rooted (i. e. oriented) trees. Each vertex of the forest is either a leaf or has two successors. A vertex v is called an extended successor of a vertex u if there is a chain of vertices's $u_0 = u, u_1, u_2 \dots u_{t-1}, u_t = v$ with $t > 0$ such that the vertex u_{i+1} is a successor of the vertex u_i for every integer i with $0 \le i \le t-1$.

Let k be a nonnegative integer. A vertex is called dynastic if it has two successors and each of these successors has at least k extended successors.

Prove that if the forest has n vertices, then there are at most $\frac{n}{k+2}$ dynastic vertices.

Solution. Trying to apply induction, we realize the bound is very loosy. That's why when we try to add in the inductive step, the value becomes larger than the bound. To stop that overflow, we tighten the bound.

Solution. The second and dummy approach is to first doing some smaller cases, finding small infos, taking the root, seeing that the bound doesnt work, but it would work if one of the successors of the root would have exactly or less than $2k+3$ successors. As we can't always guarantee that, we look for such a vertex with $2k+3$ successors. We do some work with it and by induction its done.

Problem 2.4.30 (All Russia 2017 9.1). In a country some cities are connected by oneway flights (There are no more then one flight between two cities). City A called "available" for city B , if there is flight from B to A , maybe with some transfers. It is known, that for every 2 cities P and Q exist city R , such that P and Q are available from R. Prove, that exist city A , such that every city is available for A.

Problem 2.4.31 (Jacob Tsimerman Induction). There are 2010 ninjas in the village of Konoha (what? Ninjas are cool.) Certain ninjas are friends, but it is known that there do not exist 3 ninjas such that they are all pairwise friends. Find the maximum possible number of pairs of friends.(If ninja A is friends with ninja B , then ninja B is also friends with ninja A.)

Problem 2.4.32 (USA TST 2011 D3P2). Let $n \geq 1$ be an integer, and let S be a set of integer pairs (a, b) with $1 \leq a < b \leq 2^n$. Assume $|S| > n \cdot 2^{n+1}$. Prove that there exists four integers $a < b < c < d$ such that S contains all three pairs (a, c), (b, d) and (a, d).

> **Solution.** Using Induction to the first and last half of the set S shows us the hardest part of the problem. Then ordering the left and right elements with some sort of hierarchy is all the work left to do.

Problem 2.4.33 (ISL 2016 C6). There are $n \geq 3$ islands in a city. Initially, the ferry company offers some routes between some pairs of islands so that it is impossible to divide the islands into two groups such that no two islands in different groups are connected by a ferry route.

After each year, the ferry company will close a ferry route between some two islands X and Y. At the same time, in order to maintain its service, the company will open new routes according to the following rule: for any island which is connected to a ferry route to exactly one of X and Y, a new route between this island and the other of X and Y is added.

Suppose at any moment, if we partition all islands into two nonempty groups in any way, then it is known that the ferry company will close a certain route connecting two islands from the two groups after some years. Prove that after some years there will be an island which is connected to all other islands by ferry routes.

> **Solution.** It is only natural to use induction on this kinda problems. After some trying, we see that if we remove 1 node, We get to nowhere, but if we remove 2 nodes, we get something interesting. So now focus on those two nodes and the rest of the nodes separately. Its not hard from there.

Solution. As it seems, the separation of the graph was the main observation. We can call this trick Bringing Order in the Chaos.

Problem 2.4.34 (ARO 2013 P9.5). $2n$ real numbers with a positive sum are aligned in a circle. For each of the numbers, we can see there are two sets of n numbers such that this number is on the end. Prove that at least one of the numbers has a positive sum for both of these two sets.

Solution. Since there is nothing specfic about the sum, we may safely assume that it is 0, because (1) probably it works, and (2) it makes things more convenient. How we do that? we decrease every number by the average.

Now, Consider every block of n consecutive blockes of numbers. When are two blocks connected? When they share the same end. What if we consider them as vertices, and this "connectivity" as edges? We see that cycles pop out.

And we make use of the fact that our sum is 0. So signs are sure to bet flipped at the opposite side, and there are odd and even -ness in cycles that we can use.

Problem 2.4.35 (USA TST 2011 P2). In the nation of Onewaynia, certain pairs of cities are connected by roads. Every road connects exactly two cities (roads are allowed to cross each other, e.g., via bridges). Some roads have a traffic capacity of 1 unit and other roads have a traffic capacity of 2 units. However, on every road, traffic is only allowed to travel in one direction. It is known that for every city, the sum of the capacities of the roads connected to it is always odd. The transportation minister needs to assign a direction to every road. Prove that he can do it in such a way that for every city, the difference between the sum of the capacities of roads entering the city and the sum of the capacities of roads leaving the city is always exactly one.

Solution. As there are two types of subgraph, 1 -type and 2 -type. By some work-arounds, we see that we have to work distinctly in both types of graphs. Firstly, if we work in type- 1 , we see after making a path from node x, y , the degrees of x, y will be $\{1, -1\}$ and the degrees of other nodes on the path will be the same. After that, we make every nodes have degree either $\{1, -1\}$. So after this operation we remove the 1 -edges. Now, when dealing with the type- 2 sub-graph. Start over from zero, we see that when making a path between nodes x, y the degree of those two changes parity, and other nodes on the path stays the same. So select two odd nodes....

Solution. Dealing with two different kind of edges simultaneously is messy, so we work with graph 1 and graph 2 differently. Now on both graphs, we can remove cycles. And in graph 2 , we see that we can remove any big paths if there is a edge 1 joining the two endpoints. Since if the new graph works then the previous graph works too. [Several cases to show here] And if there is no edge joining the two endpoints, replace the path by joining the two endpoints by a edge 2.

Now there are only edge 1 s, and lone edge 2 s. Now dividing the graph 1 into paths of edge 1 , and dealing with several small cases, we are done.

Problem 2.4.36 (Iran TST 2009 P6)**.** We have a closed path that goes from one vertex to another neighboring vertex, on the vertices of a $n \times n$ square which pass throgugh each vertex exactly once. Prove that we have two adjacent vertices such that if we cut the path at these two points then the length of each open paths is at least $n^2/4$.

Solution [grobber]**.** After drawing the closed path we notice that it divides the grid into a tree like structure, which we definitely need to explore!

After drawing the tree with the unit squares as vertices, we try to reformulate our problem in our new tree. We want to find an edge of this tree that will divide the tree into two parts, and those two part should have perimeter more than $\frac{n^2}{4}$.

We know of Pick's Theorem which lets us relate area with perimiter of such polygons. Which we use to do some bounding. We also need to show that if every vertex in a tree has degree at most 4, then there is an edge that divides the tree into two "big" portions, which we will use.

Problem 2.4.37 (Iran TST 2006)**.** Suppose we have a simple polygon (that is it does not intersect itself, but not necessarily convex). Show that this polygon has a diagonal which is completely inside the polygon and the two arcs it creates on the polygon perimeter (the two arcs have 2 vertices in common) both have at least one third of the vertices of the polygon.

Solution [grobber]**.** First we triangulate the polygon. Now we take the diagonal that divides the polygon into two arcs whose difference in length is the least. We show that it satisfies the requirement.

Solution [induction]**.** After the triangulation, we can also do induction on the number of triangles, by removing a "outer" triangle.

Solution [graph transformation]**.** We can convert the problem into a graph by considering the triangles as vertices and joining two vertices if they share a common side. Then this graph is a tree of which all vertices have degree at most 3. And we know that in any graph with maximum degree 3, there is way to remove an edge to divide the graph into two subgraphs with at least $\frac{n}{3}$ vertices.

Problem 2.4.38 (OC Chap2 P2). Arutyun and Amayak perform a magic trick as follows. A spectator writes down on a board a sequence of N (decimal) digits. Amayak covers two adjacent digits by a black disc. Then Arutyun comes and says both closed digits (and their order). For which minimal N can this trick always work? NOTE: Arutyun and Amayak have a strategy determined beforehand.

> **Solution.** We have to actually find a bijection between all of the combinations the spectator can create, and all of the combinations that Arutyun might see when he comes back. Which tells us to use "Perfect Matching" tricks.

> **Solution** [Existence]**.** For this trick to always work, they have to make a bijection from a set of N digits with two covered, to an unique set of N digits. Consider a bijection from the set of $0-9$ strings with length N to the set of $0-9$ strings with length N with 2 adjacent digits unknown. There exist a bijection iff the two sets satisfy Hall's Marriage Theorem. By double counting we get the value of N from here.

Problem 2.4.39 (Simurgh 2019 P3). Call a graph *symmetric*, if one can put its vertices on the plane such that it becomes symmetric wrt a line (which doesn't pass through any vertex). Find the minimum value of k such that (the edges of) every graph on 100 vertices, can be decomposed into k symmetric subgraph.

Problem 2.4.40 (RMM 2020 P3). Let $n \geq 3$ be an integer. In a country there are n airports and n airlines operating two-way flights. For each airline, there is an odd integer $m \geq 3$, and m distinct airports $c_1, \ldots, c_m$, where the flights offered by the airline are exactly those between the following pairs of airports: c_1 and c_2; c_2 and c_3; $\ldots$; c_{m-1} and c_m; c_m and c_1.

Prove that there is a closed route consisting of an odd number of flights where no two flights are operated by the same airline.

> **Solution** [Weird Induction]**.** Fix one vertice, merge all neighbors with it that has a unique airline between them.

> **Solution** [Element of Time]**.** Add one edge from each cycle one at a time, without creating a cycle. Our objective is to show that when we reach the maximum stage where one edge creates a cycle, that cycle is of odd length.

2.5 Game Theory

- Zawad's Game Theory Pset

2.5.1 Games

Definition (Nimbers)— Nimbers are simply 'Nim values' which are assigned to a game configuration - these values are written as $0, *1, *2, *3 \ldots$ We shall first describe how to obtain the Nim values for the game Squaring the Number. First, the Nim value of $n = 0$ is assigned 0, since it is a state in which neither player has a valid move. We then recursively adopt the following rule for each n : `find all the possible moves from` n `and pick the smallest Nim value which does not occur among all these possible moves.`

> **Theorem 2.5.1** (Sprague-Grund Theorem) — The *SpragueGrundy theorem* states that every impartial game under the normal play convention is equivalent to a nimber.

Game (Chip Firing Game)— Let $G = (V, E)$ be a graph without any loops or multiedges. Let a number of s_i chips be stacked on vertex i. The game follows with the player choosing a vertex i, taking d_i chips from it $(s_i - d_i > 0)$, and sending one chip to each of the neighbors of the vertex where d_i is the degree of i. The Problem of this game is to determine when the game will be infinte.

- If N is the total number of edges in G, and S is the total number of chips, then

Game (Cutting a stack in half)— Given a number of stacks, at his/her move, a player can choose a stack with even number stones, and divide it in two stacks with the same number of stones.

Game (Cutting a stack in several)— Given a number of stacks, at his/her move, a player can choose a stack, and divide it in several stacks with the same number of stones.

2.5.2 Problems

Problem 2.5.1 (USAMO 2008 P5). Three non-negative real numbers r_1 , r_2 , r_3 are written on a blackboard. These numbers have the property that there exist integers a_1 , a_2 , a_3 , not all zero, satisfying $a_1r_1 + a_2r_2 + a_3r_3 = 0$. We are permitted to perform the following operation: find two numbers x , y on the blackboard with $x \leq y$, then erase y and write $y - x$ in its place. Prove that after a finite number of such operations, we can end up with at least one 0 on the blackboard.

> **Solution.** When can't get info out of the reals, try the integers. Observe the integers, and check if they have any invariant. Rule of thumb of finding an invariant.

Problem 2.5.2 (USAMO 2014 P1). Let k be a positive integer. Two players A and B play a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with A moving first. In his move, A may choose two adjacent hexagons in the grid which are empty and place a counter in both of them. In his move, B may choose any counter on the board and remove it. If at any time there are k consecutive grid cells in a line all of which contain a counter, A wins. Find the minimum value of k for which A cannot win in a finite number of moves, or prove that no such minimum value exists.

> **Solution.** Trying to block A. We see that if we could alternately color the points black and white, we could've found some strategy for B. But the triangle grid doesnt seem very friendly. How can we color the triangles? And don't forget the details idiot.

Problem 2.5.3 (Indian TST 2004). The game of pebbles is played as followed: Initially there is one pebble at $(0,0)$. In a move one can remove the pebble at (i,j) and put one pebble each at $(i+1,j)$ and $(i,j+1)$, given that both $(i+1,j)$ and $(i,j+1)$ were empty. Prove that at any point in the game, there will be a pebble at some lattice point (a,b) with $a+b \leq 3$.

> **Solution.** Two from one, means if the weight is reduced by half in the second level, then the sum would be the same.

Problem 2.5.4 (ISL 1998 C7). A solitaire game is played on an $m \times n$ rectangular board, using mn markers which are white on one side and black on the other. Initially, each square of the board contains a marker with its white side up, except for one corner square, which contains a marker with its black side up. In each move, one may take away one marker with its black side up, but must then turn over all markers which are in squares having an edge in common with the square of the removed marker. Determine all pairs (m,n) of positive integers such that all markers can be removed from the board.

Solution. If we remove one marker, then this cell becomes useless. So the neighbors to this cell will act like they are not connected to this cell. Now if a cell is connected to w white cells, and b black cells, then the resulting board state will have $b-w$ more cells. Now only this info doesn't build up an invariant. Notice that as we are doing moves, we are reducing neighborhood relations as well, in other words, neighborhood relations decrease by $b+w$. So if we consider the sum $W+E$ where W is the number of all white cells, and E is the number of all neighborhood relations, we get an invariant on this value.

Problem 2.5.5 (ARO 1999 P10.1)**.** There are three empty jugs on a table. Winnie the pooh, Rabbit, and Piglet put walnuts in the jugs one by one. They play successively, with the initial determined by a draw. Thereby Winnie the pooh plays either in the first or second jug, Rabbit in the second or third, and Piglet in the first or third. The player after whose move there are exactly 1999 walnuts loses the games. Show that Winnie the pooh and Piglet can cooperate so as to make Rabbit lose.

Problem 2.5.6 (USAMO 2004, P4)**.** Alice and Bob play a game on a 6 by 6 grid. On his or her turn, a player chooses a rational number not yet appearing in the grid and writes it in an empty square of the grid. Alice goes first and then the players alternate. When all squares have numbers written in them, in each row, the square with the greatest number in that row is colored black. Alice wins if she can then draw a line from the top of the grid to the bottom of the grid that stays in black squares, and Bob wins if she can't. (If two squares share a vertex, Alice can draw a line from one to the other that stays in those two squares.) Find, with proof, a winning strategy for one of the players.

Problem 2.5.7 (RMM 2019 P1)**.** Amy and Bob play the game. At the beginning, Amy writes down a positive integer on the board. Then the players take moves in turn, Bob moves first. On any move of his, Bob replaces the number n on the blackboard with a number of the form $n-a^2$, where a is a positive integer. On any move of hers, Amy replaces the number n on the blackboard with a number of the form n^k, where k is a positive integer. Bob wins if the number on the board becomes zero. Can Amy prevent Bobs win?

Solution. Decent.

Problem 2.5.8 (ISL 2015 C4)**.** Let n be a positive integer. Two players A and B play a game in which they take turns choosing positive integers $k \leq n$. The rules of the game are:

1. A player cannot choose a number that has been chosen by either player on any previous turn.
2. A player cannot choose a number consecutive to any of those the player has already chosen on any previous turn.

3. The game is a draw if all numbers have been chosen; otherwise the player who cannot choose a number anymore loses the game.

The player A takes the first turn. Determine the outcome of the game, assuming that both players use optimal strategies.

> **Solution.** Look at the simplest things, first produce data like a good boy, and then see what A has to do to win, or at least draw that he can't because B is an asshole.

Problem 2.5.9 (ISL 2012 C4). Players A and B play a game with $N \geq 2012$ coins and 2012 boxes arranged around a circle. Initially A distributes the coins among the boxes so that there is at least 1 coin in each box. Then the two of them make moves in the order $B, A, B, A, \ldots$ by the following rules:

1. On every move of his B passes 1 coin from every box to an adjacent box.
2. On every move of hers A chooses several coins that were not involved in B's previous move and are in different boxes. She passes every coin to and adjacent box.

Player A's goal is to ensure at least 1 coin in each box after every move of hers, regardless of how B plays and how many moves are made. Find the least N that enables her to succeed.

> **Solution.** Investigate B's move, see how and where he can make 0's

Problem 2.5.10 (ISL 2009 C1). Consider 2009 cards, each having one gold side and one black side, lying on parallel on a long table. Initially all cards show their gold sides. Two player, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those which showed gold now show black and vice versa. The last player who can make a legal move wins.

1. Does the game necessarily end?
2. Does there exist a winning strategy for the starting player?

Remark. Trying out small cases doesn't help in this problem. Rather exploring what inevitably has to happen helps to notice patterns.

> **Solution.** The first part of the problem is trivial induction usage.
> For the second half, notice that card 1 has to chosen only once. And cards $2, \ldots 50$ have to chosen an even number of times each. Again, card 51 has to be taken an odd number of times. Inductively, cards $1, 51 \ldots 1951$ were chosen odd number of times each, and all other cards were chosen even number of times each. Since that means the parity of total number of moves in this game is even.

2.6 Combinatorial Geometry

- Combinatorial Geometry - Maria Monk (MOP 2010)

Stuck? Try These

- Consider the convex hull made up of the points.
- Consider the extreme points: smallest or highest x or y coordinate.
- Find the triangle (quadrilateral, pentagon, etc.) with the vertices being the points from your set S, so that the area of the triangle is minimal/maximal.

Theorem 2.6.1 (Helly's Theorem) — Let $X_1, ..., X_n$ be a finite collection of convex subsets of $\mathbb{R}^d$, with $n > d$. If the intersection of every $d+1$ of these sets is nonempty, then the whole collection has a nonempty intersection; that is,

$$\bigcap_{j=1}^{n} X_j \neq \varnothing$$

- Sperner's Lemma - Moor Xu

Theorem 2.6.2 (Sperner's Lemma) — Given a triangle ABC, and a triangulation $\mathcal{T}$ of the triangle, the set S of vertices of $\mathcal{T}$ is colored with three colors in such a way that

1. A, B, and C are colored $1, 2$, and 3 respectively.
2. Each vertex on an edge of ABC is to be colored only with one of the two colors of the ends of its edge. For example, each vertex on AC must have a color either 1 or 3.

Then there exists a triangle from $\mathcal{T}$, whose vertices are colored with the three different colors. More precisely, there must be an odd number of such triangles.

Proof. Consider a graph G built from the triangulation $\mathcal{T}$ as follows:

The vertices of G are the members of $\mathcal{T}$ plus the area outside the triangle. Two vertices are connected with an edge if their corresponding areas share a common border with an edge 1–2.

Note that on the interval AB there is an odd number of borders colored 1–2. Therefore, the vertex of G corresponding to the outer area has an odd degree.

But since in a finite graph there is an even number of vertices with odd degree, in the remaining graph, excluding the outer area, there is an odd number of vertices with odd degree corresponding to members of $\mathcal{T}$.

It can be easily seen that the only possible degree of a triangle from $\mathcal{T}$ is $0, 1$, or 2, and that the degree 1 corresponds to a triangle colored with the three colors $1, 2$, and 3.

Thus we have obtained a slightly stronger conclusion, which says that in a triangulation $\mathcal{T}$ there is an odd number (and at least one) of full-colored triangles.

Theorem 2.6.3 (Brouwer fixed-point theorem) — Let B^n be the nth dimensional ball. Then any continuous map $f : B^n \to B^n$ has a fixed point.

Theorem 2.6.4 (Monsky's theorem) — If we triangulate a square with triangles of equal area, then there must be an even number of triangles used.

Theorem 2.6.5 (Pick's Theorem) — A simple polygon P has all of its vertices on the lattice points of xy grid. If it's area is A, the number of lattice points inside the polygon is i and the number of lattice points on the boundary is b, then **Pick's theorem** states that

$$A = i + \frac{b}{2} - 1$$

2.6.1 Problems

Problem 2.6.1 (ARO 2013 P9.4). N lines lie on a plane, no two of which are parallel and no three of which are concurrent. Prove that there exists a non-self-intersecting broken line $A_1A_2A_3 \ldots A_N$ with N parts, such that on each of the N lines lies exactly one of the N segments of the line.

Problem 2.6.2 (EGMO 2017 P3). There are 2017 lines in the plane such that no three of them go through the same point. Turbo the snail sits on a point on exactly one of the lines and starts sliding along the lines in the following fashion: she moves on a given line until she reaches an intersection of two lines. At the intersection, she follows her journey on the other line turning left or right, alternating her choice at each intersection point she reaches. She can only change direction at an intersection point. Can there exist a line segment through which she passes in both directions during her journey?

> **Solution.** The condition that tells us to go either right or left, seems very non-rigorous. So to rigorize this condition, instead of using right or left condition in the direction, we consider whats on our right and left. (INTUITION) After some experiment we see (not all of us) that if we color the plane with two colors in a way where every neighboring regions have different colors, we find some interesting stuff. (CREATIVITY) With this we are done. Color the Plane

Problem 2.6.3 (ISL 2006 C2). Let P be a regular 2006 -gon. A diagonal is called good if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P. The sides of P are also called good.

Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

> **Solution.** The straight way, induction.

> **Solution.** The intuitive way, bijection. There are at most n good triangles, there are $2n$ edges, so a mapping that takes a two edges to a single good triangle must exist. Finding it is not that hard.

Problem 2.6.4 (ARO 2014 P9.3). In a convex n -gon, several diagonals are drawn. Among these diagonals, a diagonal is called good if it intersects exactly one other diagonal drawn (in the interior of the n -gon). Find the maximum number of good diagonals.

Solution. There can be two cases, two good diagonals intersecting each other, and no two good diagonals intersecting each other. In the first case, we just use induction, and in the later, all of the good diagonals create a "triangulation" of the polygon, which gives us the numbers.

Problem 2.6.5 (ISL 2013 C2, IMO 2013 P2). A configuration of 4027 points in the plane is called Colombian if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is good for a Colombian configuration if the following two conditions are satisfied:

1. No line passes through any point of the configuration.
2. No region contains points of both colors.

Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

Solution. Obviously a n00b would think about induction. The only problem occurs when the convex hull completely consists of red points. In this case, after some investigation, we should get the sandwiching two points idea.

Solution. Another way of inductive approach is like this, as the problem condition says that no region contains points of both colors, which means if we connect any two red and blue points, some line must bisect this segment. Now it is known that there is non intersecting partition of the points in to red-blue segments. So suppose in such a partition, we draw bisectors of each segments. Now there will be some holes in this proof. We see that to fill these holes, we have to focus on two red points with their respective blue partners, and draw the two bisectors in a way that separates the two red points form the blue points. So to remove further holes, we get the sandwiching idea.

Problem 2.6.6 (ILL 1985). Let A and B be two finite disjoint sets of points in the plane such that no three distinct points in $A \cup B$ are collinear. Assume that at least one of the sets A, B contains at least five points. Show that there exists a triangle all of whose vertices's are contained in A or in B that does not contain in its interior any point from the other set.

Solution. Concentrating on one of the sets five points such that there is no other points of the same set inside the hull of those five points.

Problem 2.6.7 (APMO 1999 P5)**.** Let S be a set of $2n+1$ points in the plane such that no three are collinear and no four concyclic. A circle will be called "Good" if it has 3 points of S on its circumference, $n-1$ points in its interior and $n-1$ points in its exterior. Prove that the number of good circles has the same parity as n.

Solution. When thinking about induction, got a feeling that double counting with the number of good circles going through pairs of points might be useful, because a good circle will be counted three times, if we can show that every pair has odd number of good circles, we are done. So, take a pair. Now we need to 'sort' the points somehow. See that, we can't sort the points in a trivial way with numbers, so moving to angles. Now setting conditions for a point inside of a circle in terms of angles, we see amazing patter, and an easy way to calculate the number of good circle of that pair of points.

Problem 2.6.8 (ISL 2014 C1)**.** Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. The rectangle R is to be dissected into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect R into at least $n+1$ smaller rectangles.

Solution. Work with the largest continuous segments, and their endpoints.

Problem 2.6.9 (ISL 2007 C2)**.** A rectangle D is partitioned in several (≥ 2) rectangles with sides parallel to those of D. Given that any line parallel to one of the sides of D, and having common points with the interior of D, also has common interior points with the interior of at least one rectangle of the partition; prove that there is at least one rectangle of the partition having no common points with D's boundary.

Solution. There existing such a rectangle means that there is a rectangular region inside of the original rectangle. So what if we walked along the segments, and cut a smaller rectangle from the inside of the rectangle? Like the way in the game.

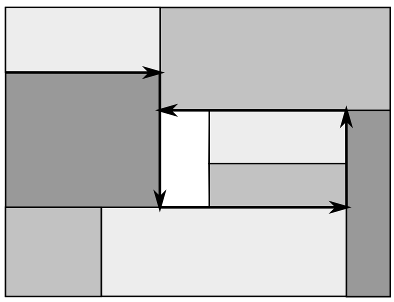


Figure 2.6.1

Solution. Starting from one corner, and taking the opposite corner of the rectangle containing that corner, we use infinite decent to reach a contradiction.

Solution. Using ISL 2014 C1 as a lemma.

> **Solution.** Take one side of the square. Take a "sandwiched" rectangle touching that side. If no such rectangle exists, then it's just a special case that can be dealt with ease.

Problem 2.6.10 (ISL 2003 C2). Let $D_1, D_2 \ldots, D_n$ be closed discs in the plane. (A closed disc is the region limited by a circle, taken jointly with this circle.) Suppose that every point in the plane is contained in at most 2003 discs D_i. Prove that there exists a disc D_k which intersects at most $7 \cdot 2003 - 1 = 14020$ other discs D_i.

> **Solution.** Just go with the natural idea.

Problem 2.6.11 (ISL 2003 C3). Let $n \geq 5$ be a given integer. Determine the greatest integer k for which there exists a polygon with n vertices (convex or not, with non-selfintersecting boundary) having k internal right angles.

> **Solution.** double count

Problem 2.6.12 (Tournament of Towns 2015S S4).

> **Solution.** Just use what's the most natural, POP, on one vertex point.

Problem 2.6.13 (USAMO 2007 P2). A square grid on the Euclidean plane consists of all points (m, n), where m and n are integers. Is it possible to cover all grid points by an infinite family of discs with non-overlapping interiors if each disc in the family has radius at least 5?

Problem 2.6.14 (MEMO 2015 T4). Let N be a positive integer. In each of the N^2 unit squares of an $N \times N$ board, one of the two diagonals is drawn. The drawn diagonals divide the $N \times N$ board into K regions. For each N, determine the smallest and the largest possible values of K.

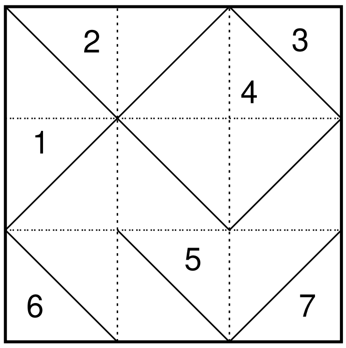


Figure 2.6.2

> **Solution.** An Algorithmic Approach: Consider each diagonal as 0 or 1, prove that the maximum configuration is the one with alternating $0, 1$s and the minimum one is the one with all 0s.

> **Solution.** A Counting Approach: Just count and bound with the minimum areas of the regions.

Problem 2.6.15. In every cells of a $m \times n$ grid, one of the two diagonals are drawn. Prove that there exist a path on these diagonals from left to right or from up to bottom of the grid.

> **Solution.** First remove the cycles, then take the largest path from left to right, and use induction.

Problem 2.6.16 (`Math Price for Girls 2017 P4`)**.** A lattice point is a point in the plane whose two coordinates are both integers. A lattice line is a line in the plane that contains at least two lattice points. Is it possible to color every lattice point red or blue such that every lattice line contains exactly 2017 red lattice points? Prove that your answer is correct.

> **Solution.** Transfinite induction.

Problem 2.6.17 (`China TST 2016 T3P2`)**.** In the coordinate plane the points with both coordinates being rational numbers are called rational points. For any positive integer n, is there a way to use n colours to colour all rational points, every point is coloured one colour, such that any line segment with both endpoints being rational points contains the rational points of every colour?

> **Solution.** Transfinite induction

Problem 2.6.18 (`IGO 2018 A3`)**.** Find all possible values of integer $n > 3$ such that there is a convex n-gon in which, each diagonal is the perpendicular bisector of at least one other diagonal.

> **Solution.** Taking maximum terminal triangle.

Problem 2.6.19 (`Lithuania ??`)**.** Prove that in every polygon there is a diagonal that cuts off a triangle and lies completely within the polygon.

Problem 2.6.20 (`Romanian TST 2008 T1P4`)**.** Prove that there exists a set S of $n - 2$ points inside a convex polygon P with n sides, such that any triangle determined by 3 vertices of P contains exactly one point from S inside or on the boundaries.

> **Solution.** Checking small cases inductively quickly shows a construction.

Problem 2.6.21 (Iran TST ??). In an isosceles right-angled triangle shaped billiards table, a ball starts moving from one of the vertices adjacent to hypotenuse. When it reaches to one side then it will reflect its path. Prove that if we reach to a vertex then it is not the vertex at initial position

Problem 2.6.22 (APMO 2018 P4). Let ABC be an equilateral triangle. From the vertex A we draw a ray towards the interior of the triangle such that the ray reaches one of the sides of the triangle. When the ray reaches a side, it then bounces off following the law of reflection, that is, if it arrives with a directed angle α, it leaves with a directed angle $180° - \alpha$. After n bounces, the ray returns to A without ever landing on any of the other two vertices. Find all possible values of n.

> **Solution.** Reflect the whole board when just reflecting the ball doesn't seem to be helping. GLOBAL

Problem 2.6.23 (ISL 2007 C5). In the Cartesian coordinate plane define the strips $S_n = \{(x, y)|n \leq x < n + 1\}$, $n \in \mathbb{Z}$ and color each strip black or white. Prove that any rectangle which is not a square can be placed in the plane so that its vertices have the same color.

> **Solution.** Proceed step by step. See what happens if the parity of a, b are different. Then the case with two coprimes. In this case, we want to tilt the rectangle to some extent where the desired result is achieved. We just need to show that this is possible. A bit of wishful thinking and a bit of algebra does the rest.

Problem 2.6.24 (APMO 2018 P3). A collection of n squares on the plane is called tri-connected if the following criteria are satisfied:

1. All the squares are congruent.
2. If two squares have a point P in common, then P is a vertex of each of the squares.
3. Each square touches exactly three other squares.

How many positive integers n are there with $2018 \leq n \leq 3018$, such that there exists a collection of n squares that is tri-connected?

> **Solution.** Play around to find that $6k$ for $k > 4$ is good. Then play around a little bit more for a different construction. Another construction for $6k$ gives rise to a construction for $10k$. Which integers can be written as a sum of $6k$ and $10k$?

Problem 2.6.25 (Iran 2005). A simple polygon is one where the perimeter of the polygon does not intersect itself (but is not necessarily convex). Prove that a simple polygon P contains a diagonal which is completely inside P such that the diagonal divides the perimeter into two parts both containing at least $\frac{n}{3} - 1$ vertices. (Do not count the vertices which are endpoints of the diagonal.)

Solution. Triangulate.

Problem 2.6.26 (ISL 2008 C3). In the coordinate plane consider the set S of all points with integer coordinates. For a positive integer k, two distinct points a, $B \in S$ will be called k-friends if there is a point $C \in S$ such that the area of the triangle ABC is equal to k. A set $T \subset S$ will be called k-clique if every two points in T are k-friends. Find the least positive integer k for which there exits a k-clique with more than 200 elements.

Solution. When does $ax + by = c$ have integer solution? Fix one point as origin and check other points friendliness with other points.

Problem 2.6.27 (ISL 2015 C2). We say that a finite set $\mathcal{S}$ of points in the plane is balanced if, for any two different points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is centre-free if for any three different points A, B and C in $\mathcal{S}$, there is no points P in $\mathcal{S}$ such that $PA = PB = PC$.

1. Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.
2. Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of n points.

Solution. Simple, think about circles, then think about "center-free" in a graph theoritic manner.

2.6.2 Chessboard Pieces

Lemma 2.6.6 — What is the maximum number of knights that can be placed on a chessboard such that no two knights attack each other?

Solution. A knight's move always changes the color of the cell.

Problem 2.6.28 (IMO 2018 P4). A site is any point (x, y) in the plane such that x and y are both positive integers less than or equal to 20.

Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones with Amy going first. On her turn, Amy places a new red stone on an unoccupied site such that the distance between any two sites occupied by red stones is not equal to $\sqrt{5}$. On his turn, Ben places a new blue stone on any unoccupied site. (A site occupied by a blue stone is allowed to be at any distance from any other occupied site.) They stop as soon as a player cannot place a stone.

Find the greatest K such that Amy can ensure that she places at least K red stones, no matter how Ben places his blue stones.

Solution. Using the maximum knight problem as a lemma.

Problem 2.6.29. How many rooks can be placed on an $n \times n$ board such that each rook attacks at most one other rook?

Solution. Use graphs with one set of degrees being rows, and the other set of degrees being columns.

Problem 2.6.30 (Eight queens puzzle). How many queens can be placed on an $n \times n$ board such that no queen attacks another queen?

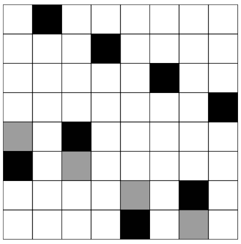


Figure 2.6.3

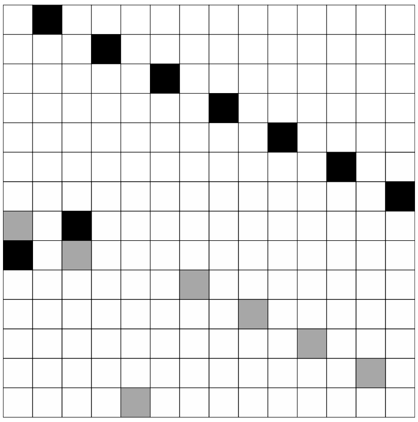


Figure 2.6.4

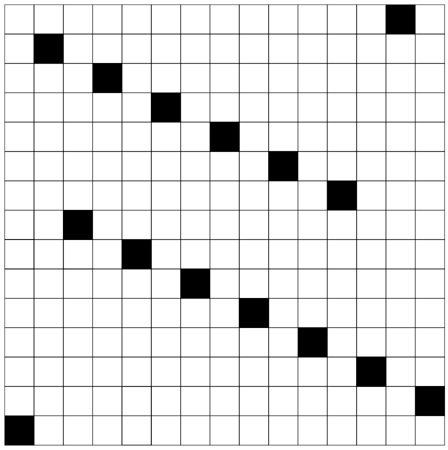


Figure 2.6.5

Problem 2.6.31 (Serbia National D2P2). How many queens can be placed on an $n \times n$ board such that each queen attacks at most one other queen?

Problem 2.6.32 (BdMO 2019 P10). Define a new chess piece named warrior. it can either go three steps forward and one step to the side, or t2wo steps forward and two steps to the side in some orientation. In a 2020×2020 chessboard, prove that the mazimum number of warriors so that none of them attack each other is leass htan or equal to $\frac{2}{5}$ of the number of cells.

> **Solution.** Color and partition

Problem 2.6.33 (RMM 2019 P4). Prove that for every positive integer n there exists a (not necessarily convex) polygon with no three collinear vertices, which admits exactly n diff-ferent triangulations.
(A triangulation is a dissection of the polygon into triangles by interior diagonals which have no common interior points with each other nor with the sides of the polygon)

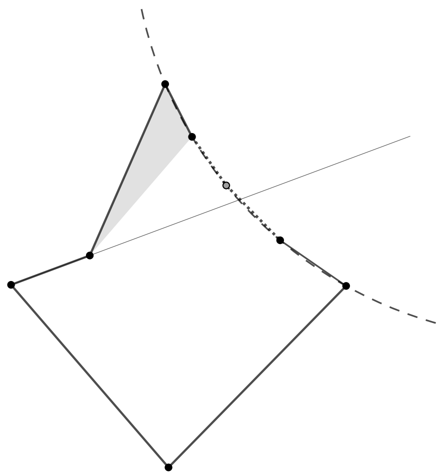


Figure 2.6.6: Fixes

Problem 2.6.34 (China TST 2015 T1D2P1). Prove that : For each integer $n \geq 3$, there exists the positive integers $a_1 < a_2 < \cdots < a_n$, such that for $i = 1, 2, \cdots, n-2$, With a_i, a_{i+1}, a_{i+2} may be formed as a triangle side length , and the area of the triangle is a positive integer.

> **Solution.** First of all we dont need to limit us to integers, we can work with rationals. We want to build a_4 from a_1, a_2, a_3. with $a_4 > a_3$ while keeping the area rational i.e. keeping the height and base rational.

Problem 2.6.35 (Codeforces 1158D). You are given n points on the plane, and a sequence S of length $n-2$ consisting of L and R. You need to generate a sequence of the points $a_1, a_2 \ldots a_n$ such that

- the polyline $a_1 a_2 \ldots a_n$ is not self intersecting.
- the directed segment $a_{i+1}a_{i+2}$ is on the left side of the the directed segment $a_i a_{i+1}$ if $S_i = L$, and on the right side if $S_i = R$.

2.7 Sequences

2.7.1 Lemmas

Theorem 2.7.1 (Van der Waerden's Theorem) — For any given positive integers r and k, there is some number N such that if the integers $\{1, 2, ..., N\}$ are colored, each with one of r different colors, then there are at least k integers in arithmetic progression all of the same color.

Theorem 2.7.2 (Catalan Recursion) — The infinite series defined as following:

$$a_0 = a_1 = 1, \;\; a_n = \prod_{i=0}^{n} a_i a_{n-i+1} = a_0 a_{n-1} + a_1 a_{n-2} \cdots + a_{n-1} a_0$$

has the general term

$$a_n = C_n = \boxed{\frac{1}{n+1}\binom{2n}{n}}$$

2.7.2 Problems

- Sequences - Alexander Remorov

Problem 2.7.1 (ISL 1990). Assume that the set of all positive integers is decomposed into r (disjoint) subsets $A_1 \cup A_2 \cup \cdots \cup A_r = \mathbb{N}$. Prove that one of them, say A_i, has the following property: There exists a positive m such that for any k one can find numbers $a_1, a_2, \ldots, a_k$ in A_i with $0 < a_{j+1} - a_j \leq m$, $(1 \leq j \leq k-1)$.

Problem 2.7.2 (Dividing the integers into arithmetic progressions, Erdos). Let $d_1, d_2, \ldots, d_k$ be differences of k arithmetic progressions that partition $\mathbb{N}$. Show that $d_i = d_j$ for some i, j.

Problem 2.7.3 (APMO 1999 P1). Find the smallest positive integer n with the following property: there does not exist an arithmetic progression of 1999 real numbers containing exactly n integers.

Solution. If the difference is $\frac{p}{q}$, then we can arrange the sequence in a way that we can get both exactly $\left\lfloor \frac{1999}{q} \right\rfloor$ and $\left\lceil \frac{1999}{q} \right\rceil$ integers.

So what we want is to find smallest integer n for which there is a k such that

$$n+1 \leq \frac{1999}{k} \quad \text{and} \quad \frac{1999}{k+1} \leq n-1$$

So that $\frac{1999}{k}$ skips over n. After a bit of calculation, we get $n = 70, k = 28$ are the solutions we want.

Problem 2.7.4 (APMO 1999 P2). Let $a_1, a_2, \ldots$ be a sequence of real numbers satisfying $a_{i+j} \leq a_i + a_j$ for all $i, j = 1, 2, \ldots$. Prove that

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \geq a_n$$

for each positive integer n.

Solution [jgnr]. We will prove this by induction. Note that the inequality holds for $n = 1$. Assume that the inequality holds for $n = 1, 2, \ldots, k$, that is,

$$a_1 \geq a_1, \quad a_1 + \frac{a_2}{2} \geq a_2, \quad a_1 + \frac{a_2}{2} + \frac{a_3}{3} \geq a_3, \quad \ldots \quad a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_k}{k} \geq a_k.$$

Sum them up:

$$ka_1 + (k-1)\frac{a_2}{2}a_2 + \cdots + \frac{a_k}{k} \geq a_1 + a_2 + \cdots + a_k.$$

Add $a_1 + \ldots + a_k$ to both sides:

$$(k+1)\left(a_1 + \frac{a_2}{2} + \cdots + \frac{a_k}{k}\right) \geq (a_1 + a_k) + (a_2 + a_{k-1}) + \cdots + (a_k + a_1) \geq ka_{k+1}.$$

Divide both sides by $k+1$:

$$a_1 + \frac{a_2}{2} + \cdots + \frac{a_k}{k} \geq \frac{ka_{k+1}}{k+1},$$

i.e.

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \geq a_n.$$

Problem 2.7.5 (ISL 1994 A1). Let for each nonnegative integer n,

$$a_0 = 1994 \quad a_{n+1} = \frac{a_n^2}{a_n + 1}$$

Prove that $1994 - n$ is the greatest integer less than or equal to a_n, $0 \leq n \leq 998$

Solution [t0rajir0u]. Rewrite the condition as

$$a_{n+1} = a_n - 1 + \frac{1}{a_n + 1}$$

Which gives us

$$a_k = 1994 - k + \frac{1}{a_{k-1}+1} + \frac{1}{a_{k-2}+1} + \cdots + \frac{1}{1994+1}$$

So we need to bound the fraction part below 1 for a_{998}. By induction, it is atmost

$$\frac{1}{997} + \frac{1}{998} \cdots + \frac{1}{1995}$$

Which is trivial to prove.

Problem 2.7.6 (ISL 2007 C4). Let $A_0 = (a_1, \ldots, a_n)$ be a finite sequence of real numbers. For each $k \geq 0$, from the sequence $A_k = (x_1, \ldots, x_k)$ we construct a new sequence A_{k+1} in the following way.

1. We choose a partition $\{1, \ldots, n\} = I \cup J$, where I and J are two disjoint sets, possibly empty, such that the expression

$$\left| \sum_{i \in I} x_i - \sum_{j \in J} x_j \right|$$

attains the smallest value. If there are several such partitions, one is chosen arbitrarily.

2. We set $A_{k+1} = (y_1, \ldots, y_n)$ where $y_i = x_i + 1$ if $i \in I$, and $y_i = x_i - 1$ if $i \in J$.

Prove that for some k, the sequence A_k contains an element x such that $|x| \geq \dfrac{n}{2}$.

Solution. Suppose the contrary. Now, since A_i can only attain finite values, So $A_i = A_j$ for some i, j. Now, we are taking about changes here, so we need to think of some invariants. Firstly the sum, it's not much of an help, because it doesn't give us much control. So kinda sum-ish invariant with a bit more control is the sum of squares. We combine these two ideas.

Problem 2.7.7 (ISL 2009 A6). Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the sub-sequences

$$s_{s_1},\ s_{s_2},\ s_{s_3},\ \ldots \quad \text{and} \quad s_{s_1+1},\ s_{s_2+1},\ s_{s_3+1},\ \ldots$$

are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

Solution. First notice that the two arithmetic sequences has the same common difference. Then notice that the diffrences of the original sequence is bounded.

Another advice, give everything names. After naming the smallest difference and the largest differnce, we get two different inequalities, from where we deduce that the difference is constant.

Problem 2.7.8 (ISL 2013 N2). Assume that k and n are two positive integers. Prove that there exist positive integers $m_1, \ldots, m_k$ such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right) \cdots \left(1 + \frac{1}{m_k}\right).$$

Solution. Just induct, and think wishfully.

Problem 2.7.9 (USAMO 2007 P1). n be a positive integer. Define a sequence by setting $a_1 = n$ and, for each $k > 1$, letting a_k be the unique integer in the range $0 \leq a_k \leq k - 1$ for which $a_0 + a_1 \cdots + a_k$ is divisible by k. Prove that for any n the sequence a_i eventually becomes constant.

Solution. Investigate and done.

Problem 2.7.10 (APMO 2015 P3). A sequence of real numbers $a_0, a_1, \ldots$ is said to be good if the following three conditions hold.

1. The value of a_0 is a positive integer.
2. For each non-negative integer i we have $a_{i+1} = 2a_i + 1$ or $a_{i+1} = \dfrac{a_i}{a_i + 2}$
3. There exists a positive integer k such that $a_k = 2014$.

Find the smallest positive integer n such that there exists a good sequence $a_0, a_1, \ldots$ of real numbers with the property that $a_n = 2014$.

Solution. We will rename the sequence and call it $\{s_i\}$, with $s_0 = x \in \mathbb{Z}$. Now let,

$$s_i = \frac{a_i x + b_i}{c_i x + d_i}$$

At the beginning we have $a_0 = d_0 = 1, b_0 = c_0 = 0$. It is easy to prove that

$$a_{i+1} + c_{i+1} = 2(a_i + c_i)$$

$$b_{i+1} + d_{i+1} = 2(b_i + d_i)$$

So it follows that

$$a_i + c_i = 2^i = b_i + d_i$$

Also, by induction (which is easy to prove), we have

$$a_i - b_i = d_i - c_i = 1$$

Suppose for some k, $s_k = 2014$. So,

$$\frac{a_k x + b_k}{c_k x + d_k} = 2014 \tag{2.3}$$

$$\implies x = \frac{2014 d_k - b_k}{a_k - 2014 c_k} = \frac{2015 d_k - 2^k}{2^k - 2015 c_k} \tag{2.4}$$

$$= \frac{2015(d_k - c_k)}{2^k - 2015 c_k} - 1 \tag{2.5}$$

But $\gcd(2015, 2^k - 2015c_k) = 1$, which implies $2^k - 2015 - c_k = 1$. Solving for k with CRT gives us $60|k$.
Now we have to prove that there is a sequence with $s_{60} = 2014$. Solving (1), $s_0 = 2014$, and,

$$a_{60} = \frac{2014 \cdot 2^{60} + 1}{2015} \qquad b_{60} = \frac{2014 \cdot 2^{60} - 2014}{2015}$$

$$c_{60} = \frac{2^{60} - 1}{2015} \qquad d_{60} = \frac{2^{60} + 2014}{2015}$$

We show that we can make (a_{60}, c_{60}) from $(a_0, c_0) = (1, 0)$. We prove it by induction,

that (a_k, c_k) can take any form $(2^k - i, i)$ with $i \in \{0, 1, \ldots 2^k - 1\}$.

Problem 2.7.11 (ISL 2008 A4). For an integer m, denote by $t(m)$ the unique number in $\{1, 2, 3\}$ such that $m + t(m)$ is a multiple of 3. A function $f : \mathbb{Z} \to \mathbb{Z}$ satisfies $f(-1) = 0$, $f(0) = 1$, $f(1) = -1$ and $f(2^n + m) = f(2^n - t(m)) - f(m)$ for all integers m, $n \geq 0$ with $2^n > m$. Prove that $f(3p) \geq 0$ holds for all integers $p \geq 0$.

Solution. We begin by listing values of $f(n)$ for $n \leq 16$, and immediately it strikes us that:

1. if $-1 \leq x \leq 2^{2m} - 1$ then the maximal value is $f(2^{2m} - 1)$ the minimal value is $f(2^{2m} - 2)$
2. if $-1 \leq x \leq 2^{2m+1}$ then the maximal value is $f(2^{2m+1} - 2)$ the minimal value is $f(2^{2m+1} - 1)$

After which we are done by induction.

2.7.3 Recurrent Sequences

- WOOT 2010-11 Recursion

Theorem 2.7.3 — Sum of Geometric Sequences Every recurrent sequence can be written as a sum of some geometric sequences. Given a recurrent sequence,

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_k x_{n-k}$$

Then x_n can be written as

$$x_n = c_1 r_1^n + c_2 r_2^n + \cdots + c_l r_l^n$$

For all c_i if r_i are the roots of the *characteristic polynomial* of the recursion. Which is:

$$\boxed{x^k - a_1 x^{k-1} - a_2 x^{k-2} \cdots - a_k = 0}$$

If there are double roots, say $r_1 = r_2 = r_3$, then we instead have,

$$x_n = \boxed{c_1 r_1^n + c_2 n\, r_2^n + c_3 n^2\, r^n} \cdots + c_l r_l^n$$

Reversely, we can say that a sequence defined by a sum of geometric recurrent series is a recursion.

Lemma 2.7.4 — Let F_n be the nth Fibonacci number. Then the following holds:

$$F_n^2 + F_{n+1}^2 = F_{2n+1}$$

Proof. Expanding the general form of the terms, and showing that $a_n = F_n^2 + F_{n+1}^2 - F_{2n+1}$ is a recursion by Theorem 2.7.3.

Theorem 2.7.5 (title=Repertoire Method) — Given a recurrent function defined by

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$$

We plug in different values for $f(n)$, for example, $f(n) = 1, n, 2n$ etc. for which the values are known from the recursion, and then solve for A, B, C.

2.8 Exploring Configurations

Problems where there is some kind of a configuration is given, the question usually asks to proof or find some specific properties of the configuration.

2.8.1 Problems

Problem 2.8.1 (ARO 2018 P11.5). On the table, there're 1000 cards arranged on a circle. On each card, a positive integer was written so that all 1000 numbers are distinct.

First, Vasya selects one of the card, remove it from the circle, and do the following operation: If on the last card taken out was written positive integer k, count the k^{th} clockwise card not removed, from that position, then remove it and repeat the operation. This continues until only one card left on the table.

Is it possible that, initially, there's a card A such that, no matter what other card Vasya selects as first card, the one that left is always card A?

> **Solution.** Consider the numbering,
>
> $$1, 1001! + 1, 1002! + 1, \ldots 1998! + 1, 1999! + 2$$
>
> It's easy to check that it works.

Remark. We want to find a configuration, where one card, let's call it a, gets skipped over all the time. Now, controlling the skipping for every move is kinda hard. Instead of doing that, we want to control only one move that skips a, and all other moves will go to the next card in the clockwise rotation. And the card before a will skip over a, and land of the next card. That being said, the construction is rather trivial.

Problem 2.8.2 (APMO 2017 P1). We call a 5-tuple of integers arrangeable if its elements can be labeled a, b, c, d, e in some order so that $a - b + c - d + e = 29$. Determine all 2017 -tuples of integers $n_1, n_2, n_3 \ldots n_{2017}$ such that if we place them in a circle in clockwise order, then any 5-tuple of numbers in consecutive positions on the circle is arrangeable.

> **Solution.** The annoying part is the $a - b + c - d + e = 29$ condition, as 29 is too random. Can we do something to make this sum equal to a nicer interger, possibly 0?

Problem 2.8.3 (ISL 2004 C1). There are 10001 students at an university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Find all possible values of k so that the following conditions are satisfied:

1. Each pair of students are in exactly one club.
2. For each student and each society, the student is in exactly one club of the society.
3. Each club has an odd number of students. In addition, a club with $2m+1$ students (m is a positive integer) is in exactly m societies.

Solution. Just Double-Counting.

Problem 2.8.4 (ISL 2002 C1). Let n be a positive integer. Each point (x, y) in the plane, where x and y are non-negative integers with $x + y < n$, is coloured red or blue, subject to the following condition: if a point (x, y) is red, then so are all points (x', y') with $x' \leq x$ and $y' \leq y$. Let A be the number of ways to choose n blue points with distinct x -coordinates, and let B be the number of ways to choose n blue points with distinct y -coordinates. Prove that $A = B$.

Problem 2.8.5 (USAMO 2012 P2). A circle is divided into 432 congruent arcs by 432 points. The points are colored in four colors such that some 108 points are colored Red, some 108 points are colored Green, some 108 points are colored Blue, and the remaining 108 points are colored Yellow. Prove that one can choose three points of each color in such a way that the four triangles formed by the chosen points of the same color are congruent.

Solution. Double counting saves the day :) The trick is to rotate ;)

Problem 2.8.6 (European Mathematics Cup 2018 P1). Call a partition of n a set $a_1, \dots a_k$ with $a_1 \leq a_2 \cdots \leq a_k$ and $a_1 + a_2 \cdots + a_k =$. A partition of a positive integer is 'even' if all of its elements are even numbers. Similarly, a partition is 'odd' if all of its elements are odd. Determine all positive integers n such that the number of even partitions of n is equal to the number of odd partitions of n.

Solution. Bijection.

Problem 2.8.7 (ISL 2015 C1). In Lineland there are $n \geq 1$ towns, arranged along a road running from left to right. Each town has a left bulldozer (put to the left of the town and facing left) and a right bulldozer (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a left and right bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, bulldozers are quite unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

Let A and B be two towns, with B to the right of A. We say that town A can sweep town B away if the right bulldozer of A can move over to B pushing off all bulldozers it meets. Similarly town B can sweep town A away if the left bulldozer of B can move over to A pushing off all bulldozers of all towns on its way.

Prove that there is exactly one town that cannot be swept away by any other one.

> **Solution.** Focus on the heaviest bulldozer.

Problem 2.8.8 (APMO 2008 P2). Students in a class form groups each of which contains exactly three members such that any two distinct groups have at most one member in common. Prove that, when the class size is 46 , there is a set of 10 students in which no group is properly contained.

> **Solution.** Taking the maximum set that follows the "in which no group is properly contained" rule. Now the elements that are *not* in this set, we can connect this element to only one of the pairs from the set. Now defining a bijection, and counting the elements, we are done.

Problem 2.8.9 (IMO SL 1985). A set of 1985 points is distributed around the circumference of a circle and each of the points is marked with 1 or -1. A point is called "good" if the partial sums that can be formed by starting at that point and proceeding around the circle for any distance in either direction are all strictly positive. Show that if the number of points marked with -1 is less than 662 , there must be at least one good point.

> **Solution.** First thing to notice, the number $3 * 661 + 2 = 1985$. And these numbers are completely random. So what if we try to replace 1985 by n ? Will the condition still hold?

Problem 2.8.10 (IMO 2011 P4). Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \cdots, 2^{n-1}$. We are to place each of the n weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed. Determine the number of ways in which this can be done.

> **Solution.** Writing the whole process as a sum, we see that only 2^0 is the odd term here, if we remove that we can divide by 2 to get a recursive formula.

Solution. Calculating wrt to the last placed weight.

Solution. Getting recursive formula considering the position of 2^{n-1}.

Problem 2.8.11 (USAMO 2008 P3). Let n be a positive integer. Denote by S_n the set of points (x, y) with integer coordinates such that

$$|x| + \left|y + \frac{1}{2}\right| < n.$$

A path is a sequence of distinct points $(x_1, y_1), (x_2, y_2), \ldots, (x_\ell, y_\ell)$ in S_n such that, for $i = 2, \ldots, \ell$, the distance between (x_i, y_i) and (x_{i-1}, y_{i-1}) is 1 (in other words, the points (x_i, y_i) and (x_{i-1}, y_{i-1}) are neighbors in the lattice of points with integer coordinates). Prove that the points in S_n cannot be partitioned into fewer than n paths (a partition of S_n into m paths is a set $\mathcal{P}$ of m nonempty paths such that each point in S_n appears in exactly one of the m paths in $\mathcal{P}$).

Solution. Graph + Partition, coloring is just natural. Again, the edges join two neighbor lattice points, so checkerboard coloring. But checkerboard doesn't do much good. So the next thing we try is to apply some derivations of it, pseudo!!! Well, overkill.

Solution. For all n, induction is very natural. The optimal partition (the most beautiful one) and the longest path in it, say P , gives us a way to perform induction. As always, we suppose a partition with $n-1$ paths. As there are a lot of partitions, we need to choose a certain partition, say $\mathbb{M}$. Again as our goal is to include P in $\mathbb{M}$. So suppose that the set with all the points in P is A. And further more, suppose that in $\mathbb{M}$ there is a path Q with $|Q \cap A|$ being maximal among all other partitions of the points. Some some easy case work shows that we must have $P \in \mathbb{M}$.

Problem 2.8.12 (USAMO 2013 P2). For a positive integer $n \geq 3$ plot n equally spaced points around a circle. Label one of them A , and place a marker at A. One may move the marker forward in a clockwise direction to either the next point or the point after that. Hence there are a total of $2n$ distinct moves available; two from each point. Let a_n count the number of ways to advance around the circle exactly twice, beginning and ending at A , without repeating a move. Prove that $a_{n-1} + a_n = 2^n$ for all $n \geq 4$.

Solution. Problems where there are multiple possible value of a function regardless of the current position, one of dealing with these is to assigning labels of these possible values to each points of the function, and this will give a combinatorial model and a way to deal it with bijection.

Solution. First investigate the problem condition, $a_n + a_{n-1} = 2^n$, now, 2^n means the number of differently coloring every point black or white, and the left side is the number of such paths for n and $n-1$. Which means we should try to color the points and see what happens.

Proof. EChen's solution: In this problem, the main obstacle seems to be the circle condition. And on top of that, on can land on the starting point. So things are pretty messed up here. What we want to do is to make things a little bit more easy to deal with. So our best option is to change the problem so that we get the similar problem with a different explanation. So we change the condition circle with matrix, 2 round with 2 rows. n points with n entries in each rows. What we get now is the same problem, just a bit easier to deal with. We call this Tweak The Problem strategy.

Problem 2.8.13. 10 persons went to a bookstore. It is known that: Every person has bought 3 kinds on books and for every 2 persons, there is at least one kind of books which they both have bought. Let m_i be the number of the persons who bought the i^{th} kind of books and $M = \max\{m_i\}$ Find the smallest possible value of M.

Problem 2.8.14 (ARO 2006 11.3)**.** On a 49×69 rectangle formed by a grid of lattice squares, all $50 \cdot 70$ lattice points are colored blue. Two persons play the following game: In each step, a player colors two blue points red, and draws a segment between these two points. (Different segments can intersect in their interior.) Segments are drawn this way until all formerly blue points are colored red. At this moment, the first player directs all segments drawn - i. e., he takes every segment AB, and replaces it either by the vector $\overrightarrow{AB}$, or by the vector $\overrightarrow{BA}$. If the first player succeeds to direct all the segments drawn in such a way that the sum of the resulting vectors is $\overrightarrow{0}$, then he wins; else, the second player wins.

Which player has a winning strategy?

Proof. The basic idea comes from wishing that the first player might be able to copy the second player to "nullify" his moves. But since this isn't always possible, because no nice symmetry exists on the board, the idea of coloring the board with dominoes and copying moves wrt the dominoes comes.

Problem 2.8.15 (ISL 2002 C3)**.** Let n be a positive integer. A sequence of n positive integers (not necessarily distinct) is called full if it satisfies the following condition: for each positive integer $k \geq 2$, if the number k appears in the sequence then so does the number $k-1$, and moreover the first occurrence of $k-1$ comes before the last occurrence of k. For each n, how many full sequences are there?

Proof. After guessing the ans, the first thing that I did was to draw a level based graph. Suppose that a full sequence has k different entries. Then the top level contains the positions of k in the sequence sorted from left to right. The next level contains the positions of $k-1$ in the sequence sorted so, and so on till the last level. What I noticed is that if we draw arrows pointing from a larger integer to a smaller integer, the only arrows (or more like relations between entries of the sequence) we need to worry about are the arrows pointing left to right in each levels, and the arrows from the last entry of level i to the first entry of level $i+1$. After this, if we try with a smaller case, we see that this leads to a bijection from the set of sequences of length n with n different integers to the set of full-sequences of length n.

Solution. Another bijection approach is as followed, in a full-sequence, on first run, go from right to left, placing integers starting with 1 onwards on the 1's in the sequence. on the second run continue counting and placing integers on the 2's and so on.

Solution. Another idea is to prove $a_n = na_{n-1}$. To do this, remove the rightmost 1 and do some casework.

Problem 2.8.16 (ISL 1994 C2). In a certain city, age is reckoned in terms of real numbers rather than integers. Every two citizens x and x' either know each other or do not know each other. Moreover, if they do not, then there exists a chain of citizens $x = x_0, x_1, \ldots, x_n = x'$ for some integer $n \geq 2$ such that x_{i-1} and x_i know each other. In a census, all male citizens declare their ages, and there is at least one male citizen. Each female citizen provides only the information that her age is the average of the ages of all the citizens she knows. Prove that this is enough to determine uniquely the ages of all the female citizens.

Solution. Describing the problem using matrix and vector spaces, the problem reduces to well known theorems of linear algebra.

Problem 2.8.17 (ISL 2003 C1). Let A be a 101-element subset of the set $S = \{1, 2, \ldots, 1000000\}$. Prove that there exist numbers $t_1, t_2, \ldots, t_{100}$ in S such that the sets

$$A_j = \{x + t_j \mid x \in A\}, \qquad j = 1, 2, \ldots, 100$$

are pairwise disjoint.

Solution. just count...

Problem 2.8.18 (EGMO 2017 P5). Let $n \geq 2$ be an integer. An n-tuple $(a_1, a_2, \ldots, a_n)$ of not necessarily different positive integers is expensive if there exists a positive integer k such that

$$(a_1 + a_2)(a_2 + a_3) \dots (a_{n-1} + a_n)(a_n + a_1) = 2^{2k-1}$$

a) Find all integers $n \geq 2$ for which there exists an expensive n-tuple.

b) Prove that for every odd positive integer m there exists an integer $n \geq 2$ such that m belongs to an expensive n-tuple.

Remark. gutaguti solution

Solution. All odd n works, you can prove for even n by $+1, -1$ addition. For the second part, start with the odd number, move on both side, you will eventually reach 1.

Problem 2.8.19 (USAMO 2006 P2). For a given positive integer k find, in terms of k, the minimum value of N for which there is a set of $2k+1$ distinct positive integers that has sum greater than N but every subset of size k has sum at most $\frac{N}{2}$.

Solution. Compactness is the optimal decision.

Problem 2.8.20 (MOP Problem). Prove that for any positive integer c, there exists an integer n such that n has more 1's in its binary expansion than $n^2 + c$ does.

Solution. For $x = 2^a - 1$, x and x^2 have the same number of $1's$. So does $x = 2^a - 2^b$. But what if increase the number of $1's$ in this x by substracting 1? Let $x = 2^a - 2^b - 1$. This might work if we can choose nice a, b's

Problem 2.8.21 (EGMO 2015 P2). A domino is a 2×1 or 1×2 tile. Determine in how many ways exactly n^2 dominoes can be placed without overlapping on a $2n \times 2n$ chessboard so that every 2×2 square contains at least two uncovered unit squares which lie in the same row or column.

Solution. Notice how each of the four kind of dominoes needs to be in a group. So if we separated them into blocks, inverstigation shows that there can only be 4 blocks and each strictly attached to the sides. The reason why this is happening is pretty obvious. Now those blocks create two paths between the two opposite vertices of the square. This gives our desired bijection.

Problem 2.8.22 (USA TST 2006 P5). Let n be a given integer with n greater than 7 , and let $\mathcal{P}$ be a convex polygon with n sides. Any set of $n-3$ diagonals of $\mathcal{P}$ that do not intersect in the interior of the polygon determine a triangulation of $\mathcal{P}$ into $n-2$ triangles. A triangle in the triangulation of $\mathcal{P}$ is an interior triangle if all of its sides are diagonals of $\mathcal{P}$.

Express, in terms of n, the number of triangulations of $\mathcal{P}$ with exactly two interior triangles, in closed form.

> **Solution.** Just mindless calculation...

Problem 2.8.23 (ISL 2010 C3). 2500 chess kings have to be placed on a 100×100 chessboard so that

1. no king can capture any other one (i.e. no two kings are placed in two squares sharing a common vertex);
2. each row and each column contains exactly 25 kings.

Find the number of such arrangements. (Two arrangements differing by rotation or symmetry are supposed to be different.)

> **Solution.** In a 2×2 box, one can place only one king. So we divide the board in that way, and explore...

Problem 2.8.24 (USA Winter TST 2018 P3). At a university dinner, there are 2017 mathematicians who each order two distinct entrées, with no two mathematicians ordering the same pair of entrées. The cost of each entrée is equal to the number of mathematicians who ordered it, and the university pays for each mathematician's less expensive entrée (ties broken arbitrarily). Over all possible sets of orders, what is the maximum total amount the university could have paid?

> **Solution** [Grid Representation]**.** We put the information in a grid in the usual manner. We take the configuration for which the score (which we define to be the total amount the university has to pay) is maximum.
>
> > ***Definition***— Let C_i be the price of the dish i. Let P_j be the pair of dishes j ordered.
>
> We now "sort" the grid in the following way:
>
> - Sort the columns first, from the least expensive to the most.
> - Now we have 2017 binary strings as rows. We sort them decreasingly from top to bottom.
>
> We end up with something like this:

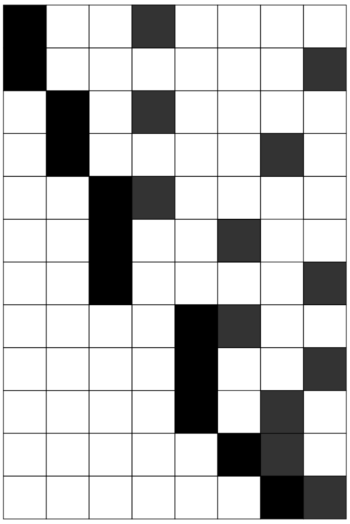


Figure 2.8.1

Definition— We make some sets:

$$A_i = \{x | x \text{ ordered dish } i \text{ but didn't order any of the dishes } j < i\}$$

Let $a_i = |A_i|$. We also call A_i block, the xth rows with $x \in A_i$. It is easy to see that the committee has to pay for dish i a_i times.

Take a j such that $a_j = 0$. If $j < k$, let

$$S_j = \{x \mid \text{In the } j^{th} \text{ column, there is a } 1 \text{ in the set of the rows from } A_x\}$$

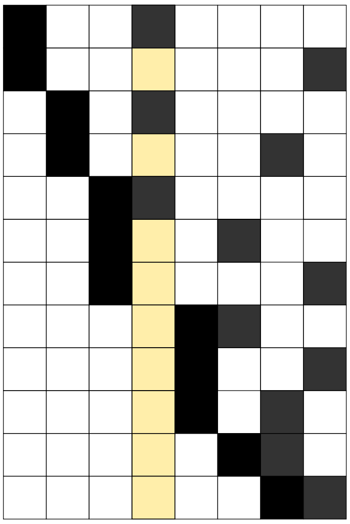


Figure 2.8.2

In the example of the previous diagram, $S_4 = \{1, 2, 3\}$

Take the largest element of S_j, namely t. We want to move the 1 in the A_t block of column j to another column to the right. Notice that this won't decrease the score, because for each of the mathematician after the block A_t, the score would be non decreasing. If we could do this, we could do this inductively, moving the jth column to the left each move, eventually making it disappear.

So suppose we can't do this. So there is no column on the right of t, that does not have a 1 in the block A_t. It is straightforward to deduce that $N = t + a_t$.

From there, we can say:

$$a_{t+m} \leq a_t - m \text{ and } a_{t-m} \leq a_t + m$$

So,

$$\begin{aligned} 2017 = \sum a_i &\leq (a_t + t - 1) + \dots 1 \\ \implies 2017 &\leq \frac{N(N-1)}{2} \\ \implies N &\geq 65 \end{aligned}$$

We know that the score, $S = \sum C_k a_k$. where a_i is strictly decreasing and C_i is non decreasing. So the maximum sum would have $C_i = K$ for some constant (by rearrangement inequality). And to have K maximum, we need N minimum or $N = 64$.

And if $j > k$, the same reason holds. So $N = 65, a_1 = 63, a_2 = 62 \dots a_{63} = 1$. But we need another $a_i = 1$, that has to be on its own.

Remark. Easy to think in grids, but it was quite difficult to formulate the solution rigorously, I still am not 100% convinced myself. Next time when I feel like it, I will reconstruct the solution. Roughly it is: take the final 0 value columns, prove that there is only one such, and prove the bounds on the solution before using that instead. Then use general bounding to find the maximum. It's not hard, it's just I don't have time now. GAH!! HSC!!

Remark. Didn't read other solutions too, gonna give a todo

2.8.1.1 Conway's Soldiers

Problem 2.8.25 (`ISL 1993 C5`). On an infinite chessboard, a solitaire game is played as follows: at the start, we have n^2 pieces occupying a square of side n. The only allowed move is to jump over an occupied square to an unoccupied one, and the piece which has been jumped over is removed. For which n can the game end with only one piece remaining on the board?

Solution. We want to find an invariant. So we need to find a weight for each of the cells such that any two consecutive cells' values equals to the values of the two cells on the two sides. Some mind bashing gives the idea of mod 3. And a construction for the other n's can be easily generated after some casework.

Problem 2.8.26 (ARO 1999 P4). A frog is placed on each cell of a $n \times n$ square inside an infinite chessboard (so initially there are a total of $n \times n$ frogs). Each move consists of a frog A jumping over a frog B adjacent to it with A landing in the next cell and B disappearing (adjacent means two cells sharing a side). Prove that at least $\left[\frac{n^2}{3}\right]$ moves are needed to reach a configuration where no more moves are possible.

Solution. In the final stage, no two neighboring cells are occupied. Could we double count the number of frogs with this information? What about the number of frogs in the original $n \times n$ board? Another small information needed for this is that we need 2 moves to "empty" a 2×2 board.

2.8.1.2 Triominos

Problem 2.8.27 (ARO 2011 P10.8). A 2010×2010 board is divided into corner-shaped figures of three cells. Prove that it is possible to mark one cell in each figure such that each row and each column will have the same number of marked cells.

Solution. First we will mark the corner pieces of the triominos. Then shift the mark to either of the legs. Our objective is to show that we can always do this. First if we only focus on the rows, we can easily show that this can be done using some counting. To show that we can do the same for columns as well, we create a graph from columns and rows to triominos which should be operated on, and using Hall's Marriage we pove the result.

Problem 2.8.28 (St. Petersburg 2000). On an infinite checkerboard are placed 111 non-overlapping corners, L-shaped figures made of 3 unit squares. Suppose that for any corner, the 2×2 square containing it is entirely covered by the corners. Prove that one can remove some number between 1 and 110 of the corners so that the property will be preserved.

2.8.1.3 Dominos

Problem 2.8.29. An $m \times n$ rectangular grid is covered by dominoes. Prove that the vertices of the grid can be coloured using three colours so that any two vertices a distance 1 apart are colored with different colours if and only if their segment lies on the boundary of a domino.

Solution. Create a graph with the midpoints of the dominos.

2.8.2 Clearly Bijection

Problem 2.8.30 (APMO 2017 P3). Let $A(n)$ denote the number of sequences $a_1 \geq a_2 \geq \cdots \geq a_k$ of positive integers for which $\sum_{i=1}^{k} a_k = n$ and each $a_i + 1$ is a power of two.

Let $B(n)$ denote the number of sequences $b_1 \geq b_2 \geq \cdots \geq b_k$ of positive integers for which $\sum_{i=1}^{k} b_k = n$ and each inequality $b_j \geq 2b_{j+1}$ holds $(j = 1, 2 \ldots m - 1)$.

Prove that $|A(n)| = |B(n)|$ for every positive integer.

Solution. A sequence of the first type can be rewritten as:

$$n = x_1 + 3x_2 + 7x_3 \cdots + (2^i - 1)x_i + \ldots (2^k - 1)x_k$$

Where x_i are non-negative integers. This motivates us to find a way to represent b_i as sums of $(2^i - 1)x_i$. Then since $b_j \geq 2b_{j+1}$, we write: $b_i = 2b_{i-1} + x_i$ with x_i being non-negative integers.

Problem 2.8.31 (ISL 2008 C4). Let n and k be positive integers with $k \geq n$ and $k - n$ an even number. Let $2n$ lamps labeled $1, 2 \ldots 2n$ be given, each of which can be either on or off. Initially, all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on).

Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n + 1$ through $2n$ are all off.

Let M be number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps $n + 1$ through $2n$ are all off, but where none of the lamps $n + 1$ through $2n$ is ever switched on.

Determine $\dfrac{N}{M}$.

Solution. These type of problems most of the time have bijection or algo solutions. Think of a way to perform bijection from the set $S\{M\} \to S\{N\}$. Find an algorithm to get a sequence of the first type from a sequence of the second type.

Problem 2.8.32 (USAMO 1996 P4). An n -term sequence $(x_1, x_2, \ldots, x_n)$ in which each term is either 0 or 1 is called a binary sequence of length n. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n.

Solution. These type of problems cries for a nice bijection. That is a way to get from $a \to b$ and vice versa. What if there is no $0, 0, 1, 1$? Or what if there is no $0, 1, 0$? What is an one way bijection?

2.8.3 Coloring Problems

Problem 2.8.33 (EGMO 2017 P2). Find the smallest positive integer k for which there exists a colouring of the positive integers $\mathbb{Z}_{>0}$ with k colours and a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ with the following two properties:

1. For all positive integers m, n of the same colour, $f(m+n) = f(m) + f(n)$.
2. There are positive integers m, n such that $f(m+n) \neq f(m) + f(n)$.

In a colouring of $\mathbb{Z}_{>0}$ with k colours, every integer is coloured in exactly one of the k colours. In both (i) and (ii) the positive integers m, n are not necessarily distinct.

> **Solution.** Firstly a modular coloring shows that $1 < k \leq 2$. For $k = 2$ we do some trivial case works.

Problem 2.8.34 (ISL 2002 C2). For n an odd positive integer, the unit squares of an $n \times n$ chessboard are coloured alternately black and white, with the four corners coloured black. A it tromino is an L-shape formed by three connected unit squares. For which values of n is it possible to cover all the black squares with non-overlapping trominos? When it is possible, what is the minimum number of trominos needed?

> **Solution.** First find the first ans and a configuration that works. Then guess the second ans, and see from where that might come from, usually these anses come from some special set of problems, where bijection is applicable.

Problem 2.8.35 (Codeforces 101954/G). Two Knights are given on a chessboard, one black one white. Which player has a winning possibility?

> **Solution.** A knight's move always changes the color of the cell.

Problem 2.8.36 (ARO 1993 P10.4). Thirty people sit at a round table. Each of them is either smart or dumb. Each of them is asked: "Is your neighbor to the right smart or dumb?" A smart person always answers correctly, while a dumb person can answer both correctly and incorrectly. It is known that the number of dumb people does not exceed F. What is the largest possible value of F such that knowing what the answers of the people are, you can point at at least one person, knowing he is smart?

> **Solution.** We see that the strings of truth only exist either when all people are dumb or the last one is the truthful one. Now we take the longest such string, and this sting has to be of the second kind. To prove this, we use bounding with the given constraint.

Problem 2.8.37 (Tournament of Towns 2015S S6). An Emperor invited 2015 wizards to a festival. Each of the wizards knows who of them is good and who is evil, however the Emperor doesnt know this. A good wizard always tells the truth, while an evil wizard can tell the truth or lie at any moment. The Emperor gives each wizard a card with a single question, maybe different for different wizards, and after that listens to the answers of all wizards which are either yes or no. Having listened to all the answers, the Emperor expels a single wizard through a magic door which shows if this wizard is good or evil. Then the Emperor makes new cards with questions and repeats the procedure with the remaining wizards, and so on. The Emperor may stop at any moment, and after this the Emperor may expel or not expel a wizard. Prove that the Emperor can expel all the evil wizards having expelled at most one good wizard.

Solution. There is only one problem with the cyclic arrangement, that is what if all the answers are 'yes'? We get rid of this problem by trying small case with $n = 3$ and trying the most simple way to connect this strategy to any n. Simplicity is the key.

Problem 2.8.38 (ISL 2007 C1). Let $n > 1$ be an integer. Find all sequences $a_1, a_2, \dots a_{n^2+n}$ satisfying the following conditions:

1. $a_i \in \{0, 1\}$ for all $1 \leq i \leq n^2 + n$
2. for all $0 \leq i \leq n^2 - n$

$$a_{i+1} + a_{i+2} + \dots + a_{i+n} < a_{i+n+1} + a_{i+n+2} + \dots + a_{i+2n}$$

Solution. $n + 1$ blocks of n, each strictly greater than the previous one. means the sums of the blocks have to be $0, 1, \dots n$. construction's easy from examples of $2, 3$.

Problem 2.8.39 (ISL 2016 C2). Find all positive integers n for which all positive divisors of n can be put into the cells of a rectangular table under the following constraints: each cell contains a distinct divisor;
the sums of all rows are equal; and
the sums of all columns are equal.

Solution. Check the sizes.

Problem 2.8.40 (ISL 2007 C3). Find all positive integers n for which the numbers in the set $S = \{1, 2, \dots, n\}$ can be colored red and blue, with the following condition being satisfied: The set $S \times S \times S$ contains exactly 2007 ordered triples (x, y, z) such that:

1. the numbers x, y, z are of the same color
2. the number $x + y + z$ is divisible by n.

Solution. Trying out small cases, noticing patter. It doesn't matter 'which' numbers are red, but 'how' many numbers are red.

Problem 2.8.41 (ISL 2014 C4). Construct a tetromino by attaching two 2×1 dominoes along their longer sides such that the midpoint of the longer side of one domino is a corner of the other domino. This construction yields two kinds of tetrominoes with opposite orientations. Let us call them S- and Z-tetrominoes, respectively.

Assume that a lattice polygon P can be tiled with S-tetrominoes. Prove that no matter how we tile P using only S- and Z-tetrominoes, we always use an even number of Z-tetrominoes.

Solution. So after we are determined to do coloring, it is not very hard to come up with a coloring. Start from stracth type coloring. Color one square at a time, this might take several tries.

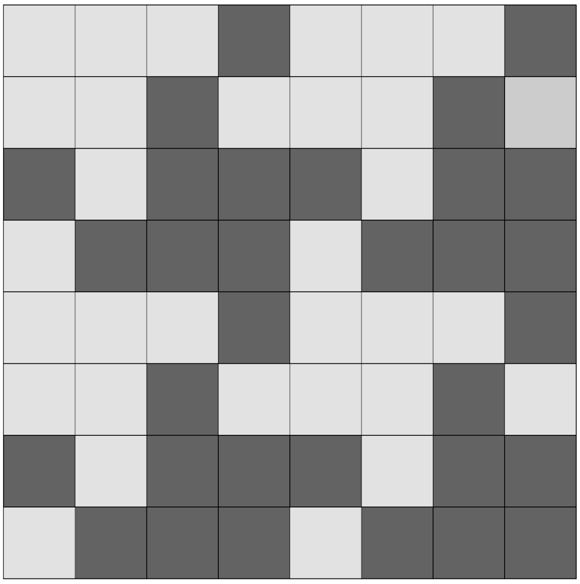


Figure 2.8.3

2.9 Linear Algebra

- Algebraic Techniques - Yufei Zhao

Lemma 2.9.1 (Multiplication Order) —

$$(AB)^T = B^T A^T$$

$$ABCD = A(BC)D$$

Theorem 2.9.2 (Linear map that reverses the order of multiplication) — Let F be a field, and m a positive integer. The only F-linear maps $s : \mathrm{M}_m(F) \to \mathrm{M}_m(F)$ which satisfy

$$s(X_k X_{k-1} \ldots X_1) = s(X_1)\, s(X_2) \ldots s(X_k) \quad \forall k \in \mathbb{N},\ X_i \in \mathrm{M}_m(F)$$

are maps of the form $s(P) = UP^T U^{-1} \quad \forall P \in \mathrm{M}_m(F)$ for U an invertible $m \times m$ matrix over F.

Proof. Let s be a map that satisfies the problem condition. Let t be another linear map defined by $t(P) = s(P)^T \quad \forall P \in \mathrm{M}(F)$.
This map t also satisfies

$$t(X_k X_{k-1} \ldots X_1) = t(X_1)\, t(X_2) \ldots t(X_k) \quad \forall k \in \mathbb{N},\ X_i \in \mathrm{M}_m(F)$$

The rest uses advanced algebra :cold_sweat: something called `Noether-Skolem theorem`... maybe later...

Problem 2.9.1 (`ISL 2009 C3`). Let n be a positive integer. Given a sequence $\varepsilon_1, \ldots, \varepsilon_{n-1}$ with $\varepsilon_i = 0$ or $\varepsilon_i = 1$ for each $i = 1, \ldots, n-1$, the sequences $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$ are constructed by the following rules:

$$a_0 = b_0 = 1, \quad a_1 = b_1 = 7,$$

$$a_{i+1} = \begin{cases} 2a_{i-1} + 3a_i, & \text{if } \varepsilon_i = 0, \\ 3a_{i-1} + a_i, & \text{if } \varepsilon_i = 1, \end{cases} \quad \text{for each } i = 1, \ldots, n-1,$$

$$b_{i+1} = \begin{cases} 2b_{i-1} + 3b_i, & \text{if } \varepsilon_{n-i} = 0, \\ 3b_{i-1} + b_i, & \text{if } \varepsilon_{n-i} = 1, \end{cases} \quad \text{for each } i = 1, \ldots, n-1.$$

Prove that $a_n = b_n$.

Solution [Algebraic, darij grinberg]. We'll first translate the problem in matrix language, then find a linear transformation to prove it.

Definition— Let $M_2(\mathbb{Q})$ be the ring of 2×2 matrices over the rational numbers. Definte two matrices $A, B \in M_2(\mathbb{Q})$ by $A = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}$.
For every $i \in \{1, 2, ..., n-1\}$, define a matrix $K_i \in \mathrm{M}_2(\mathbb{Q})$ by

$$K_i = \varepsilon_i B + (1 - \varepsilon_i) A$$

This clearly yields that $K_i = A$ if $\varepsilon_i = 0$, and that $K_i = B$ if $\varepsilon_i = 1$.
For every $i \in \{1, 2, ..., n-1\}$, we have $\begin{pmatrix} a_{i+1} \\ a_i \end{pmatrix} = K_i \begin{pmatrix} a_i \\ a_{i-1} \end{pmatrix}$. By induction we have

$$\begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = K_{n-1} K_{n-2} ... K_1 \begin{pmatrix} 7 \\ 1 \end{pmatrix}$$

$$a_n = \begin{pmatrix} 1 & 0 \end{pmatrix} K_{n-1} K_{n-2} ... K_1 \begin{pmatrix} 7 \\ 1 \end{pmatrix}$$

We need to prove that

$$\begin{pmatrix} 1 & 0 \end{pmatrix} K_{n-1} K_{n-2} ... K_1 \begin{pmatrix} 7 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} K_1 K_2 ... K_{n-1} \begin{pmatrix} 7 \\ 1 \end{pmatrix}$$

In order to do this, it is clearly enough to define some map $s : \mathrm{M}_2(\mathbb{Q}) \to \mathrm{M}_2(\mathbb{Q})$ which satisfies

1. $s(K_{n-1} K_{n-2} ... K_1) = K_1 K_2 ... K_{n-1}$
2. Every matrix $P \in \mathrm{M}_2(\mathbb{Q})$ satisfies $\begin{pmatrix} 1 & 0 \end{pmatrix} P \begin{pmatrix} 7 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} s(P) \begin{pmatrix} 7 \\ 1 \end{pmatrix}$

Let U be the invertible matrix $\begin{pmatrix} 7 & 1 \\ 1 & 2 \end{pmatrix} \in \mathrm{M}_2(\mathbb{Q})$. Let $s : \mathrm{M}_2(\mathbb{Q}) \to \mathrm{M}_2(\mathbb{Q})$ be the map defined by

$$s(P) = U P^T U^{-1}, \quad \forall P \in M_2(\mathbb{Q})$$

It's easy to check with computation that this mapping satisfies (2). We need to show it satisfies (1) too.
First let's list some properties of s. We have,

1. $s(I_2) = I_2$ where I_2 is the identity matrix.
2. $s(XY) = s(Y)s(X)$, trivial by Lemma 2.9.1
3. $s(A) = A$ and $s(B) = B$, easy to show with some computation.
4. It follows that $s(K_i) = K_i$.

(1) trivially follows from these properties. So s satisfies the problem.

Remark. The above solution followed a rather standard procedure (translating linear recurrences into matrix multiplication - this is the same trick that solves many problems about Fibonacci numbers) until the point where we "guessed" the matrix U and the map s. How did we do that?

The motivation is the following: We need a map s which satisfies (1) and (2). We forget about (2) for a moment, and try to satisfy (1) only.

The easiest way to ensure that (1) holds for every choice of n and $\varepsilon_1, \varepsilon_2, ..., \varepsilon_{n-1}$ is to choose s as a linear map satisfying $s(A) = A$ and $s(B) = B$ (this immediately guarantees that $s(K_i) = K_i$ for every i, because $K_i = \varepsilon_i B + (1 - \varepsilon_i) A$ is a linear combination of A and B) and satisfying $s(X_k X_{k-1} ... X_1) = s(X_1) s(X_2) ... s(X_k)$ for any $k \in \mathbb{N}$ and any 2×2 matrices X_1, X_2, ..., X_k.

This condition $s(X_k X_{k-1} ... X_1) = s(X_1) s(X_2) ... s(X_k)$ is fulfilled, for example, when the map s has the form $s(P) = UP^T U^{-1}$ for every $P \in \mathrm{M}_2(\mathbb{Q})$ for U an invertible 2×2 matrix. Actually it is fulfilled only in this case, as I explain further below, but as for now let us at least agree that $s(P) = UP^T U^{-1}$ for every $P \in \mathrm{M}_2(\mathbb{Q})$ is a good point to start.

So now we are searching for a 2×2 matrix U such that the map s defined by $s(P) = UP^T U^{-1}$ for every $P \in \mathrm{M}_2(\mathbb{Q})$ satisfies $s(A) = A$, $s(B) = B$ and (2). These conditions give linear equations on the entries of this matrix U, and the only matrix U which solves all of them is (up to scaling) $\begin{pmatrix} 7 & 1 \\ 1 & 2 \end{pmatrix} \in \mathrm{M}_2(\mathbb{Q})$. It is now clear how to proceed from here.

Solution [Bijection, evan chen]. Let $A_i = 2^i a_i$. So the recursive relations now are:

$$A_{i+1} = 8A_{i-1} + 6A_i \text{ or } 12A_{i-1} + 2A_i$$

Now we build a bijective relation. Imagine n rooms in a row with $n-1$ doors, numbered ϵ_i, between them. We have 14 colors, $\{\star, 1, 2, \ldots 13\}$ to paint the rooms. So the two sides of the doors will get different coloring, let these two colors be i, j. We need to follow some certain rules:

1. If i is $\star$, then j can be any color.
2. If the door is labeled with 0, and $i \in \{\star, j-2, j-1, j, j+1, j+2\} \pmod{13}$
3. If the door is labeled with 1, $i \in \{\star, j\}$.

We prove that A_i is the number of ways the first i rooms can be painted (and B_i the number of ways the last i rooms).
First, the door between the rooms $i, i+1$ was labeled 0. If $A_i \neq \star$, then we have 6 ways to paint $i+1$. From there comes $6A_i$ ways of painting the first $i+1$ rooms. If $A_i = \star$, then there are 14 ways of painting $i+1$. But 6 of those ways have already been counted. And the rest of the 8 new colors will come from A_{i-1}. So $8A_{i-1}$ ways from there. In total:

$$A_{i+1} = 8A_{i-1} + 6A_i$$

And if the door was labeled 1, then $6, 8$ becomes $2, 12$. And our result follows.

Remark. We first have to settle to search for a bijective solution. So we ask the question, "a_i's are the number of ways to do something. How can we define that?"

Now the coefficients of the recurresions doesn't add up. It would be better if we had them being equal. So we do that, nice and easy.

So the coefficients now add up to 14. So for the $i+1$-th object, we have 14 choices, and those choices get divided depending on the choice for the i-th object. From here, it is natural to think about those rules.

2.10 Double Counting and Other Algebraic Methods

Problem 2.10.1 (ISL 2012 C3). In a 999×999 square table some cells are white and the remaining ones are red. Let T be the number of triples (C_1, C_2, C_3) of cells, the first two in the same row and the last two in the same column, with C_1, C_3 white and C_2 red. Find the maximum value T can attain.

Solution. Expicitly count the value, and bound it using ineq...

2.10.1 Probabilistic Methods

Problem 2.10.2 (ISL 2006 C3). Let S be a finite set of points in the plane such that no three of them are on a line. For each convex polygon P whose vertices are in S, let $a(P)$ be the number of vertices of P, and let $b(P)$ be the number of points of S which are outside P. A line segment, a point, and the empty set are considered as convex polygons of 2, 1, and 0 vertices respectively. Prove that for every real number x

$$\sum_P x^{a(P)}(1-x)^{b(P)} = 1,$$

where the sum is taken over all convex polygons with vertices in S.

Solution. This can be done by strong induction and double counting. Counting for every possibile subsets with every points inside of the convex hull and at least one point on the convex hull outside of the set.

Solution. The beautiful solution on the other hand uses probability. Color each point black or white, then translate the condition in terms of probability. :D

2.11 Bounding

Problem 2.11.1 (RMM 2019 P3). Given any positive real number ε, prove that, for all but finitely many positive integers v, any graph on v vertices with at least $(1+\varepsilon)v$ edges has two distinct simple cycles of equal lengths. (Recall that the notion of a simple cycle does not allow repetition of vertices in a cycle.)

Solution. If suppose there are x cycles at some point, each distinct in size. **Double-counting** the number of edges in those cycles: there are at least $\frac{x^2}{2}$ edges. Since there are at most $2n$ edges in the graph, there is one edge that is contained in at least $\frac{x^2}{4n}$ cycles.
Now, if we delete that edge, and keep doing that until there is no cycle left in the graph, how many steps might we need to make?
Notice that $x < 4n$. If we let $4n = c$, we can rephrase our question:

Lemma 2.11.1 — Given $m < c$, let $a_{i\geq 0}$ be a sequence defined by:

$$a_0 = m, \;\; a_{i+1} = a_i - \left\lceil \frac{x^2}{c} \right\rceil$$

It is clear that eventually for some t, a_t will become 0. What is the upper bound for t?

Suppose we are at x now. The decrement we need to make now is $k := \frac{x^2}{c}$. The decrement we need to make when we are at $\frac{x}{2}$ is $\frac{k}{\sqrt{2}}$. AND at each step in going from $x \to \frac{x}{2}$ changes the decrement very little. So we can take the average decrement of all the steps and calculate the number of steps with that fixed.
So, essentially **Greedy Algorithm** From this intuition, and some calculation, we get the idea of thinking about steps needed to cross an interval of $\sqrt{\frac{c}{4}}$

Solution. It is easier to control cycles in a graph where there is no cycle! And the best kind of spanning tree is a BFS tree.
Now, every edge that is not in this tree creates a cycle. We know that the lengths of these cycles are all distinct. So we can lower bound the sum of the lengths. Whenever we want to bound something, it is always a good idea to look for a way to represent the variables differently. This is where the "jumping over vertices" idea comes from.

Solution. Think of the cycles as binary strings. Addition of these cycles are XOR operation. Now, let M be the set of all cycle lenghts. If we take some cycles and add them, they correspond to some subset of M. This will give us an upper bound for the sum of some lengths from M.
Now, we want these sums to be distinct. How to do that? We are adding cycles together right? So we need to bring order in there. We will only take cycles from a certain set, so that it holds. These cycles need to be linearly independent of themeselvs. That gives us the construction for the set.
After this, the rest is just bounding.

Problem 2.11.2 (ISL 2018 C5). Let k be a positive integer. The organising commitee of a tennis tournament is to schedule the matches for $2k$ players so that every two players play once, each day exactly one match is played, and each player arrives to the tournament site the day of his first match, and departs the day of his last match. For every day a player is present on the tournament, the committee has to pay 1 coin to the hotel. The organisers want to design the schedule so as to minimise the total cost of all players' stays. Determine this minimum cost.

2.12 CP Algorithms

- CP Algorithms

2.12.1 Cycle Finding Algorithms

Algorithm (Floyd's cycle-finding algorithm)— This algorithm finds a cycle by using two pointer. These pointers move over the sequence at different speeds. **In each iteration the first pointer advances to the next element, but the second pointer advances two elements.** It's not hard to see, that if there exists a cycle, the second pointer will make at least one full cycle and then meet the first pointer during the next few cycle loops. *If the cycle length is λ and the μ is the first index at which the cycle starts, then the algorithm will run in $O(\lambda + \mu)$ time.*

This algorithm is also known as **tortoise and the hare algorithm**, based on the tale in which a tortoise (here a slow pointer) and a hare (here a faster pointer) make a race.

It is actually possible to determine the parameter λ and μ using this algorithm (also in $O(\lambda+\mu)$ time and $O(1)$ space), but here is just the simplified version for finding the cycle at all. The algorithm and returns true as soon as it detects a cycle.

```
def floyd(f, x0, n):
    tortoise = x0
    hare = x0
    while(tortoise != hare):
        tortoise = f(tortoise, n)
        hare = f(hare, n)
        hare = f(hare, n)
    return True
```

Algorithm (Brent's algorithm)— Brent uses a similar algorithm as Floyd. It also uses two pointer. But instead of advancing the pointers by one and two respectably, we advance them in powers of two. As soon as $2^i$is greater than λ and μ, we will find the cycle.

```
def f(x, n):
    return (x*x + 3)%n

def brent(f, x0, n):
    tortoise = x0
    hare = f(x0, n)
    l = 1
    while(tortoise != hare):
        tortoise = f(tortoise, n)
        for i in range(l):
            hare = f(hare, n)
            if(hare == tortoise):
```

```
                    return True
            l *= 2
    return True
```

2.13 Permutations

Problem 2.13.1 (USAMO 2017 P5). Let $m_1, m_2, \ldots, m_n$ be a collection of n positive integers, not necessarily distinct. For any sequence of integers $A = (a_1, \ldots, a_n)$ and any permutation $w = w_1, \ldots, w_n$ of $m_1, \ldots, m_n$, define an A-inversion of w to be a pair of entries w_i, w_j with $i < j$ for which one of the following conditions holds:

$$a_i \geq w_i > w_j, \quad w_j > a_i \geq w_i, \quad w_i > w_j > a_i$$

Show that, for any two sequences of integers $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$, and for any positive integer k, the number of permutations of $m_1, \ldots, m_n$ having exactly k A-inversions is equal to the number of permutations of $m_1, \ldots, m_n$ having exactly k B-inversions.

Solution. Notice that if we take B as a sequence with all elements greater than all w_i, then we have the B-inversions to be normal inversions wrt M. So we need to show that there exists a bijection between A-inversion and normal inversion.

So we can either show that there for a permutation w with k normal inversions, there is a permutation p with k A-inversions. But we soon figure out it is pretty hard.

If we try the other way, show that for every w with k A-inversions, there is a p with the same k normal inversions, and if we can show injectivity, we will be done. It turns out that this is much easier.

Problem 2.13.2 (ISL 2008 C2). Let $n \in \mathbb{N}$ and A_n set of all permutations $(a_1, \ldots, a_n)$ of the set $\{1, 2, \ldots, n\}$ for which

$$k \mid 2(a_1 + \cdots + a_k), \text{ for all } 1 \leq k \leq n.$$

Find the number of elements of the set A_n.

Solution. First we try some smaller cases: $|A_1| = 1$, $|A_2| = 2$, $|A_3| = 6$, $|A_4| = 12$, which has a clear pattern. So we proceed with induction.

With induction, we focus on a_n only, it can have values eiher $n, 1$ or $\frac{n+1}{2}$. But the later case is impossible, and so we only have two options for a_n, which gives us our desired inductive relation.

2.14 Unsorted Problems

Problem 2.14.1 (ARO 2018 10.3). A positive integer k is given. Initially, N cells are marked on an infinite checkered plane. We say that the cross of a cell A is the set of all cells lying in the same row or in the same column as A. By a turn, it is allowed to mark an unmarked cell A if the cross of A contains at least k marked cells. It appears that every cell can be marked in a sequence of such turns. Determine the smallest possible value of N.

> **Solution.** First find the construction.

Problem 2.14.2 (ARO 2018 P9.5). On the circle, 99 points are marked, dividing this circle into 99 equal arcs. Petya and Vasya play the game, taking turns. Petya goes first; on his first move, he paints in red or blue any marked point. Then each player can paint on his own turn, in red or blue, any uncolored marked point adjacent to the already painted one. Vasya wins, if after painting all points there is an equilateral triangle, all three vertices's of which are colored in the same color. Could Petya prevent him?

> **Solution.** Think of what Petya must do to prevent immediate losing.

Problem 2.14.3 (ISL 2004 C2). Let n and k be positive integers. There are given n circles in the plane. Every two of them intersect at two distinct points, and all points of intersection they determine are pairwise distinct (i. e. no three circles have a common point). No three circles have a point in common. Each intersection point must be colored with one of n distinct colors so that each color is used at least once and exactly k distinct colors occur on each circle. Find all values of $n \geq 2$ and k for which such a coloring is possible.

Problem 2.14.4 (ISL 2004 C3). The following operation is allowed on a finite graph: Choose an arbitrary cycle of length 4 (if there is any), choose an arbitrary edge in that cycle, and delete it from the graph. For a fixed integer $n \geq 4$, find the least number of edges of a graph that can be obtained by repeated applications of this operation from the complete graph on n vertices's (where each pair of vertices's are joined by an edge).

> **Solution.** Walk backwards. or the same thing with Bipartite Graphs.

Problem 2.14.5 (Iran TST 2012 P4). Consider $m+1$ horizontal and $n+1$ vertical lines ($m, n \geq 4$) in the plane forming an $m \times n$ table. Cosider a closed path on the segments of this table such that it does not intersect itself and also it passes through all $(m-1)(n-1)$ interior vertices's (each vertex is an intersection point of two lines) and it doesn't pass through any of outer vertices. Suppose A is the number of vertices's such that the path passes through them straight forward, B number of the table squares that only their two opposite sides are used in the path, and C number of the table squares that none of their sides is used in the path. Prove that $A = B - C + m + n - 1$.

Problem 2.14.6 (AoPS). Given $2n+1$ irrational numbers, prove that one can pick n from them s.t. no two of the choosen n sum up to a rational number.

Solution. Use a graph theory representation.

Problem 2.14.7 (Bulgarian IMO TST 2004, Day 3, Problem 3). Prove that among any $2n+1$ irrational numbers there are $n+1$ numbers such that the sum of any k of them is irrational, for all $k \in \{1, 2, 3, \ldots, n+1\}$.

Solution. We first create a set B such that any linear combination of the elements in it are irrational. Then for convenience, we add 1 to it, so that now the sum equals to 0 of any linear combinations. An algorithm for building it comes into our mind, which leaves some other original elements, which we then later add to the final solution set A along with the elements in the set B except 1.

Problem 2.14.8 (ISL 1997 P4). An $n \times n$ matrix whose entries come from the set $S = \{1, 2, ..., 2n-1\}$ is called a "silver matrix" if, for each $i = 1, 2, ..., n$, the i -th row and the i -th column together contain all elements of S. Show that:

1. there is no silver matrix for $n = 1997$;
2. silver matrices exist for infinitely many values of n.

Solution. Proving that for odd n 's isn't hard. Then A small try-around with $n = 2, 4$, we see a pattern that leads to a construction for 2^n

Problem 2.14.9. A rectangle is completely partitioned into smaller rectangles such that each smaller rectangles has at least one integral side. Prove that the original rectangle also has at least one integral side.

Solution. Try a special grid system with $.5 \times .5$ boxes.

Solution. Consider the number of corners in the rectangle.

Problem 2.14.10 (ISL 2004 C5). A and B play a game, given an integer N, A writes down 1 first, then every player sees the last number written and if it is n then in his turn he writes $n+1$ or $2n$, but his number cannot be bigger than N. The player who writes N wins. For which values of N does B win?

> **Solution.** Trying with smaller cases, it's easy. Using most important game theory trick.

Problem 2.14.11 (ISL 2006 C1). We have $n \geq 2$ lamps $L_1, L_2 \dots L_n$ in a row, each of them being either on or off. Every second we simultaneously modify the state of each lamp as follows: if the lamp L_i and its neighbors (only one neighbor for $i = 1$ or $i = n$, two neighbors for other i) are in the same state, then L_i is switched off; otherwise, L_i is switched on. Initially all the lamps are off except the leftmost one which is on.

1 Prove that there are infinitely many integers n for which all the lamps will eventually be off.
2 Prove that there are infinitely many integers n for which the lamps will never be all off

Problem 2.14.12 (ISL 2006 C4). A cake has the form of an $n \times n$ square composed of n^2 unit squares. Strawberries lie on some of the unit squares so that each row or column contains exactly one strawberry; call this arrangement $\mathbb{A}$.

Let $\mathbb{B}$ be another such arrangement. Suppose that every grid rectangle with one vertex at the top left corner of the cake contains no fewer strawberries of arrangement $\mathbb{B}$ than of arrangement $\mathbb{A}$. Prove that arrangement $\mathbb{B}$ can be obtained from $\mathbb{A}$ by performing a number of switches, defined as follows:

A switch consists in selecting a grid rectangle with only two strawberries, situated at its top right corner and bottom left corner, and moving these two strawberries to the other two corners of that rectangle.

> **Solution.** When the first approach fails, don't throw that idea yet. Stick to it, as it is most probably the closest to a correct solution. Taking the smallest rectangle with 0 's equal to 1 's, we see that we can 'shrink' the rectangle. Which leads to a solution instantly.

Problem 2.14.13 (ISL 2014 C2). We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b , then we erase these numbers and write the number $a+b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m.

Solution. When you know that the problem can be solved using invariants, go through all of the possible invariants (from the rules of thumb). Don't give up on one so quickly. And product and sum are actually more close than you think. Because if you are told to prove some bound on the sum, then product can come very handy. After all there is AM-GM to connect sum and product.

Problem 2.14.14 (ISL 2016 C3). Let n be a positive integer relatively prime to 6. We paint the vertices's of a regular n -gon with three colours so that there is an odd number of vertices's of each colour. Show that there exists an isosceles triangle whose three vertices's are of different colours.

Solution. Double Count with the number of points of each colors.

Problem 2.14.15 (Iran TST 2002 P3). A "2-line" is the area between two parallel lines. Length of "2-line" is distance of two parallel lines. We have covered unit circle with some "2-lines". Prove sum of lengths of "2-lines" is at least 2.

Solution. Consider the "2-line" of the largest length.

Problem 2.14.16 (ARO 2008 P9.5). The distance between two cells of an infinite chessboard is defined as the minimum number to moves needed for a king to move from one to the other. On the board are chosen three cells on pairwise distances equal to 100. How many cells are there that are at the distance 50 from each of the three cells?

Problem 2.14.17 (USAMO 1986 P2). During a certain lecture, each of five mathematicians fell asleep exactly twice. For each pair of mathematicians, there was some moment when both were asleep simultaneously. Prove that, at some moment, three of them were sleeping simultaneously.

Problem 2.14.18 (Mexican Regional 2014 P6). Let $A = n \times n$ be a $\{0, 1\}$ matrix, where each row is different. Prove that you can remove a column such that the resulting $n \times (n-1)$ matrix has n different rows.

Solution. Try to represent the sets in a nicer way, with graph. or. Induction on the number of columns deleted and the number or different rows being there.

Problem 2.14.19 (IMO 2017 P5). An integer $N \geq 2$ is given. A collection of $N(N+1)$

soccer players, no two of whom are of the same height, stand in a row. Show that Sir Alex can always remove $N(N-1)$ players from this row leaving a new row of $2N$ players in which the following N conditions hold:

(1) no one stands between the two tallest players,

(2) no one stands between the third and fourth tallest players,

$\vdots$

(N) no one stands between the two shortest players.

> **Solution.** $N(N+1)$, rows, removing $\ldots$ these things just begs for to be arranged in a systematic order. As arranging thing in a matrix is the simplest way, we arrange the bad-bois in a $N \cdot (N+1)$ matrix. Now finding the algorithm is not very hard.

Problem 2.14.20 (ISL 1990 P3). Let $n \geq 3$ and consider a set E of $2n-1$ distinct points on a circle. Suppose that exactly k of these points are to be colored black. Such a coloring is good if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly n points from E. Find the smallest value of k so that every such coloring of k points of E is good.

> **Solution.** Creating a graph and using Alternating Chains Technique

Problem 2.14.21 (USAMO 1999 P1). Some checkers placed on an $n \times n$ checkerboard satisfy the following conditions:

1 every square that does not contain a checker shares a side with one that does;
2 given any pair of squares that contain checkers, there is a sequence of squares containing checkers, starting and ending with the given squares, such that every two consecutive squares of the sequence share a side.

Prove that at least $(n^2-2)/3$ checkers have been placed on the board.

> **Solution.** As the problem simply seems to exist, we can't count how much contribution a checker cntaining square contributes to the whole board. So we place one at a time and see the changes.

Generalization 2.14.21.1 (USAMO 1999 P1 generalization). Find the smallest positive integer m such that if m squares of an $n \times n$ board are colored, then there will exist 3 colored squares whose centers form a right triangle with sides parallel to the edges of the board.

Problem 2.14.22 (ISL 2013 C1). Let n be an positive integer. Find the smallest integer k with the following property; Given any real numbers $a_1, \cdots, a_d$ such that $a_1+a_2+\cdots+a_d = n$ and $0 \leq a_i \leq 1$ for $i = 1, 2, \cdots, d$, it is possible to partition these numbers into k groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.

> **Solution.** Think about the worst case where d is the minimum and the ans is d , it would only be possible if each $a_i > \frac{1}{2}$ but this can't be true, so, the ans is $2n-1$. Now the ques should become obvious.

Problem 2.14.23 (Brazilian Olympic Revenge 2014). Let n a positive integer. In a $2n \times 2n$ board, $1 \times n$ and $n \times 1$ pieces are arranged without overlap. Call an arrangement maximal if it is impossible to put a new piece in the board without overlapping the previous ones. Find the least k such that there is a maximal arrangement that uses k pieces.

> **Solution.** Intuition gives that there is at least one n -mino in each row. But we can easily guess that there is no maximal arrangement with $2n$ minos. Suppose in a maximal arrangement, there are no vertical n -mino, that means there are more than $2n+1$ n-minos. So suppose that there is at least one vertical suppose that it lies in a column i between 1 and n. Then we have that there is at least one n -mino in each column in between 1 and i. If there is one in between 1 and $2n$, say j , then there is one in each of the columns on the right side of it. Then we count horizontal n -minos, we show that $2n+1$ is the answer.

Problem 2.14.24 (ISL 2008 C1). In the plane we consider rectangles whose sides are parallel to the coordinate axes and have positive length. Such a rectangle will be called a box. Two boxes intersect if they have a common point in their interior or on their boundary. Find the largest n for which there exist n boxes $B_1, B_2 \ldots B_n$ such that B_i and B_j intersect if and only if $i \not\equiv j \pm 1 \pmod n$.

> **Solution.** Instead of focusing on building the boxes from only one side (i.e. starting with $1, 2 \ldots$, we should include n in our investigation, and follow from both direction, (i.e. $1, 2 \ldots$ and $\ldots, n-1, n$).

Problem 2.14.25 (USAMO 2008 P4). Let $\mathcal{P}$ be a convex polygon with n sides, $n \geq 3$. Any set of $n-3$ diagonals of $\mathcal{P}$ that do not intersect in the interior of the polygon determine a triangulation of $\mathcal{P}$ into $n-2$ triangles. If $\mathcal{P}$ is regular and there is a triangulation of $\mathcal{P}$ consisting of only isosceles triangles, find all the possible values of n.

> **Solution.** Its not hard after getting the ans.

Problem 2.14.26 (ARO 2016 P3). We have a sheet of paper, divided into 100×100 unit squares. In some squares, we put right-angled isosceles triangles with $leg = 1$ (Every triangle lies in one unit square and is half of this square). Every unit grid segment (boundary too) is under one leg of a triangle. Find maximal number of unit squares, that don't contains any triangles.

> **Solution.** What is the minimum number of triangles you can use in a row? Create a good row one at a time

Problem 2.14.27 (India TST 2013 Test 3, P1). For a positive integer n , a *Sum-Friendly Odd Partition* of n is a sequence $(a_1, a_2 \dots a_k)$ of odd positive integers with $a_1 \leq a_2 \leq \cdots \leq a_k$ and $a_1 + a_2 + \cdots + a_k = n$ such that for all positive integers $m \leq n$, m can be uniquely written as a subsum $m = a_{i_1} + a_{i_2} + \cdots + a_{i_r}$. (Two subsums $a_{i_1} + a_{i_2} + \cdots + a_{i_r}$ and $a_{j_1} + a_{j_2} + \cdots + a_{j_s}$ with $i_1 < i_2 < \cdots < i_r$ and $j_1 < j_2 < \cdots < j_s$ are considered the same if $r = s$ and $a_{i_l} = a_{j_l}$ for $1 \leq l \leq r$.) For example, $(1, 1, 3, 3)$ is a *sum-friendly odd partition* of 8. Find the number of sum-friendly odd partitions of 9999.

> **Solution.** Firstly we explore one SFOP at a time. Which gives us a way to tell what a_{i+1} is going to be by looking at $a_1 \dots a_i$.

Problem 2.14.28 (IMO 2011 P2). Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A windmill is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q , takes over as the new pivot, and the line now rotates clockwise about Q , until it next meets a point of $\mathcal{S}$. This process continues indefinitely.

Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

> **Solution.** Some workaround gives us the idea that the starting line has to be kinda "in between" the points. Formal words could be: the line should divide the set of points in two sets so that the two sets have equal number of points. Once we take a such line, we see that after every move we get a new line which has similar properties of the first line.

> **Solution.** So moral of the story is that if you get some vague idea that something has to satisfy something-ish, remove the -ish part, and try with a formal assumption.

Problem 2.14.29 (IOI 2016 P5). A computer bug has a permutation P of length $2^k = N$ that changes any string added to a DS according to the permutation, i.e. it makes $S[i] =$

$S[P[i]]$. Your task it to find the permutation in the following ways:

1 You can add at most $n \log_2 n$ N bit binary strings to the DS.
2 You can ask at most $n \log_2 n$, in the form of N bit binary strings. The answer will be "true" if the string exists in the DS after the Bug had changed the strings and "no" otherwise.

> **Solution.** Typical Divide and Conquer approach. You want to do the same thing for $N = \frac{N}{2}$, and to do so you need to tell exactly what the first $\frac{N}{2}$ terms of the permutation are. To do this, you can use at most N questions. This is easy, you first add strings with only one bit present in the first $\frac{N}{2}$ positions, and then ask N questions with only one bit in every N positions. This maps the first $\frac{N}{2}$ numbers of the permutation to a set of $\frac{N}{2}$ integers. And we can proceed by induction now.

Problem 2.14.30 (`ISL 2001 C6`). For a positive integer n define a sequence of zeros and ones to be balanced if it contains n zeros and n ones. Two balanced sequences a and b are neighbors if you can move one of the $2n$ symbols of a to another position to form b. For instance, when $n = 4$, the balanced sequences 01101001 and 00110101 are neighbors because the third (or fourth) zero in the first sequence can be moved to the first or second position to form the second sequence. Prove that there is a set S of at most $\frac{1}{n+1}\binom{2n}{n}$ balanced sequences such that every balanced sequence is equal to or is a neighbor of at least one sequence in S.

Problem 2.14.31 (`ISL 1998 C4`). Let $U = \{1, 2, \ldots, n\}$, where $n \geq 3$. A subset S of U is said to be split by an arrangement of the elements of U if an element not in S occurs in the arrangement somewhere between two elements of S. For example, 13542 splits $\{1, 2, 3\}$ but not $\{3, 4, 5\}$. Prove that for any $n - 2$ subsets of U, each containing at least 2 and at most $n - 1$ elements, there is an arrangement of the elements of U which splits all of them.

> **Solution.** If we try to apply induction, we see that the sets with 2 and $n - 1$ elements create problems, so we handle them first.

Problem 2.14.32 (`USA TST 2009 P1`). Let m and n be positive integers. Mr. Fat has a set S containing every rectangular tile with integer side lengths and area of a power of 2. Mr. Fat also has a rectangle R with dimensions $2^m \times 2^n$ and a 1×1 square removed from one of the corners. Mr. Fat wants to choose $m + n$ rectangles from S, with respective areas $2^0, 2^1, \ldots, 2^{m+n-1}$, and then tile R with the chosen rectangles. Prove that this can be done in at most $(m + n)!$ ways.

> **Solution.** The fact that this can be done in $(m + n)!$ asks for a bijective proof. Now an intuition gives us that we have to sort the tiles wrt the missing square in some way. Now since the numbers

Problem 2.14.33 (ARO 2016 P1). There are 30 teams in **NBA** and every team play 82 games in the year. Bosses of **NBA** want to divide all teams on Western and Eastern Conferences (not necessarily equally), such that the number of games between teams from different conferences is half of the number of all games. Can they do it?

> **Solution.** You want to divide something. Check the parity.

Problem 2.14.34 (AoPS). Each edge of a polyhedron is oriented with an arrow such that at each vertex, there is at least one arrow leaving the vertex and at least one arrow entering the vertex. Prove that there exists a face on the polyhedron such that the edges on its boundary form a directed cycle.

> **Solution.** The trick which is used to prove Euler's Polyhedron theorem.

Problem 2.14.35 (ISL 2014 C3). Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is peaceful if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 unit squares.

> **Solution.** Guessing the "Correct" ans is the challenge, think of the worst case you can produce.

Problem 2.14.36 (APMO 2012 P2). Into each box of a $n \times n$ square grid, a real number greater than or equal to 0 and less than or equal to 1 is inserted. Consider splitting the grid into 2 non-empty rectangles consisting of boxes of the grid by drawing a line parallel either to the horizontal or the vertical side of the grid. Suppose that for at least one of the resulting rectangles the sum of the numbers in the boxes within the rectangle is less than or equal to 1, no matter how the grid is split into 2 such rectangles. Determine the maximum possible value for the sum of all the $n \times n$ numbers inserted into the boxes. Find the ans for k-dimension grids too.

> **Solution.** As the maximal rectangle defines other smaller rectangles in it, we take that.

Problem 2.14.37 (Indian Postal Coaching 2011). Consider 2011^2 points arranged in the form of a 2011×2011 grid. What is the maximum number of points that can be chosen among them so that no four of them form the vertices's of either an isosceles trapezium or a rectangle whose parallel sides are parallel to the grid lines?

Solution. Since we need to maintain the relation of perpendicular bisectors, we focus on perp bisectors and the points on one line only and then count.

Problem 2.14.38 (ISL 2010 C2). On some planet, there are 2^N countries ($N \geq 4$). Each country has a flag N units wide and one unit high composed of N fields of size 1×1, each field being either yellow or blue. No two countries have the same flag. We say that a set of N flags is diverse if these flags can be arranged into an $N \times N$ square so that all N fields on its main diagonal will have the same color. Determine the smallest positive integer M such that among any M distinct flags, there exist N flags forming a diverse set.

Solution. Using induction we see that if we have found the value of M for $N-1$, then possibly the value for M_N is twice as large than M_{N-1}. With some further calculation, we see that if we have $2 * M_{N-1} - 1 = M_N$, then we can pick half of them and apply induction and still be left with a 'lot' of flags to choose the Nth element of the diverse set.

After that the only work left is to proof for $N = 4$. Which is easy casework.

Solution. Another way to prove the ans, is to prove the bound for any non-diverse set. In this case, we use hall's marriage to prove the contradiction.

Problem 2.14.39 (Iran TST 2007 P2). Let A be the largest subset of $\{1, \ldots, n\}$ such that for each $x \in A$, x divides at most one other element in A. Prove that

$$\frac{2n}{3} \leq |A| \leq \left\lceil \frac{3n}{4} \right\rceil.$$

Solution. Partition the set optimally.

Problem 2.14.40 (India IMO Camp 2017). Find all positive integers n s.t. the set $\{1, 2, \ldots, 3n\}$ can be partitioned into n triplets (a_i, b_i, c_i) such that $a_i + b_i = c_i$ for all $1 \leq i \leq n$.

Problem 2.14.41 (ISL 2012 C2). Let $n \geq 1$ be an integer. What is the maximum number of disjoint pairs of elements of the set $\{1, 2, \ldots, n\}$ such that the sums of the different pairs are different integers not exceeding n ?

Solution. As Usual, first find the ans. Using double counting is quite natural. Working with small cases easily gives a construction.

Problem 2.14.42 (CodeForces 989C).

Problem 2.14.43 (CodeForces 989B).

Problem 2.14.44 (ISL 2011 A5). Prove that for every positive integer n , the set $\{2, 3, \dots, 3n+1\}$ can be partitioned into n triples in such a way that the numbers from each triple are the lengths of the sides of some obtuse triangle.

> **Solution.** What is the best way to choose the side lengths of an obtuse triangle? Obviously by maintaining some strict rules to get the third side from the first two sides and making the rules invariant. One way of doing this is to take $(a, b, a+b-1)$.
> After that, some (literally this is the hardest part of the problem) experiment to find a construction. First, we try to partition the set into tuples of our desired form, but we soon realize that that cant be done so easily. So we try a little bit of different approach and make one tuple different from the others. Luckily this approach gives us a nice construction.

Problem 2.14.45 (Iran TST 2017 D1P1). In the country of Sugarland, there are 13 students in the IMO team selection camp. 6 team selection tests were taken and the results have came out. Assume that no students have the same score on the same test. To select the IMO team, the national committee of math Olympiad have decided to choose a permutation of these 6 tests and starting from the first test, the person with the highest score between the remaining students will become a member of the team. The committee is having a session to choose the permutation.

Is it possible that all 13 students have a chance of being a team member?

> **Solution.** If a student is in x^{th} place in a test t_y , and he has a chance to get into the team iff the $1^th, 2^th \dots x-1^th$ persons in test t_y are already in the team. So $x \le 5$. Make a $6 \cdot 6$ grid with place $\cdot$ test. WHY?? Because it makes the best sense among other possible choices of the grid. A little bit of work produces a configuration where every student has a chance to get into the team.

Problem 2.14.46 (ISL 2009 C2). For any integer $n \ge 2$, let $N(n)$ be the maximum number of triples (a_i, b_i, c_i) , $i = 1, 2 \dots, N(n)$, consisting of nonnegative integers a_i , b_i and c_i such that the following two conditions are satisfied:

1 $a_i + b_i + c_i = n$ for all $i = 1, \dots, N(n)$,
2 If $i \ne j$ then $a_i \ne a_j$, $b_i \ne b_j$ and $c_i \ne c_j$

Determine $N(n)$ for all $n \geq 2$.

> **Solution.** Find an upper bound. Its easy. Then with some experiment, we see that this upper bound is achievable. So our next task is to find a construction. As it is related to 3 , we first try with $n = 3k$. Some experiment and experience gives us a construction.

Problem 2.14.47. Let n be an integer. What is the maximum number of disjoint pairs of elements of the set $\{1, 2, \ldots, n\}$ such that the sums of the different pairs are different integers not exceeding n ?

Problem 2.14.48 (ISL 2002 C6). Let n be an even positive integer. Show that there is a permutation $(x_1, x_2 \ldots x_n)$ of $(1, 2 \ldots n)$ such that for every $1 \leq i \leq n$, the number x_{i+1} is one of the numbers $2x_i, 2x_i - 1, 2x_i - n, 2x_i - n - 1$. Hereby, we use the cyclic subscript convention, so that x_{n+1} means x_1.

Some experiments show that our graph has more than 2 incoming and outgoing degree in all vertexes expect the first and last vertexes. So our lemma wont work yet. To make use of our lemma we take a graph with half of the vertexes of our original graph and make each vertex v_{2k} represent two integers: $(2k - 1, 2k)$. Simple argument shows that this graph has an Euler Circuit, and surprisingly this itself is sufficient, as we can follow this circuit to get every integers in the interval $[1, n]$.

Problem 2.14.49 (USA TST 2017 P1). In a sports league, each team uses a set of at most t signature colors. A set S of teams is color-identifiable if one can assign each team in S one of their signature colors, such that no team in S is assigned any signature color of a different team in S.

For all positive integers n and t, determine the maximum integer $g(n, t)$ such that: In any sports league with exactly n distinct colors present over all teams, one can always find a color-identifiable set of size at least $g(n, t)$.

> **Solution.** First, guess the answer, then try taking the minimal set.

Problem 2.14.50 (Putnam 2017 A4). $2N$ students take a quiz in which the possible scores are $0, 1 \ldots 10$. It is given that each of these scores appeared at least once, and the average of their scores is 7.4. Prove that the students can be divided into two sets of N student with both sets having an average score of 7.4.

> **Solution.** We take a set $S_1 = \{0, 1 \dots 10\}$. Basically we have to partition the set of $2N$ into two equal sets with equal sum. So we pair S , and other leftovers and see what happens.

Problem 2.14.51 (ISL 2005 C3). Consider a $m \times n$ rectangular board consisting of mn unit squares. Two of its unit squares are called adjacent if they have a common edge, and a path is a sequence of unit squares in which any two consecutive squares are adjacent. Two paths are called non-intersecting if they don't share any common squares.

Each unit square of the rectangular board can be colored black or white. We speak of a coloring of the board if all its mn unit squares are colored.

Let N be the number of colorings of the board such that there exists at least one black path from the left edge of the board to its right edge. Let M be the number of colorings of the board for which there exist at least two non-intersecting black paths from the left edge of the board to its right edge.

Prove that $N^2 \geq M \times 2^{mn}$.

> **Solution.** Bijective relation problem, the condition has $\times$, means we find a combinatorial model for the R.H.S. which is a pair of boards satisfying conditions. We want to show a surjection from this model to the model on the L.H.S.

Problem 2.14.52 (Result by Erdos). Given two *different* sequence of integers $(a_1, a_2 \dots a_n), (b_1, b_2, \dots$ such that two $\frac{n(n-1)}{2}$-tuples

$$a_1 + a_2, a_1 + a_3 \dots a_{n-1}a_n \quad \text{and} \quad b_1 + b_2, b_1 + b_3 \dots b_{n-1}b_n$$

are equal upto permutation. Prove that $n = 2^k$ for some k.

Problem 2.14.53 (A reformulation of Catalan's Numbers). Let $n \geq 3$ students all have different heights. In how many ways can they be arranged such that the heights of any three of them are not from left to right in the order: medium, tall, short?

> **Solution.** The proof uses derivatives to construct a polynomial similar to a **Maclaurin Series**.

Problem 2.14.54. There are n cubic polynomials with three distinct real roots each. Call them $P_1(x), P_2(x), \dots, P_n(x)$. Furthermore for any two polynomials P_i, P_j, $P_i(x)P_j(x) = 0$

has exactly 5 distinct real roots. Let S be the set of roots of the equation

$$P_1(x)P_2(x)\dots P_n(x) = 0$$

. Prove that

1 If for each a, b there is exactly one $i \in \{1, \dots n\}$ such that $P_i(a) = P_i(b) = 0$, then $n = 7$.
2 If $n > 7$, $|S| = 2n + 1$.

Problem 2.14.55 (Serbia TST 2017 P2). Initially a pair (x, y) is written on the board, such that exactly one of it's coordinates is odd. On such a pair we perform an operation to get pair $(\frac{x}{2}, y + \frac{x}{2})$ if $2|x$ and $(x + \frac{y}{2}, \frac{y}{2})$ if $2|y$. Prove that for every odd $n > 1$ there is a even positive integer $b < n$ such that starting from the pair (n, b) we will get the pair (b, n) after finitely many operations.

> **Solution.** Finding a construction through investigation and realizing that the infos and operations on x only defines the changes are enough for this problem.

Problem 2.14.56 (Serbia TST 2017 P4). We have an $n \times n$ square divided into unit squares. Each side of unit square is called unit segment. Some isosceles right triangles of hypotenuse 2 are put on the square so all their vertices's are also vertices's of unit squares. For which n it is possible that every unit segment belongs to exactly one triangle (unit segment belongs to a triangle even if it's on the border of the triangle)?

> **Solution.** Finding n is even, seeing 4 fails...

Problem 2.14.57 (China MO 2018 P2). Let n and k be positive integers and let

$$T = \{(x, y, z) \in \mathbb{N}^3 \mid 1 \le x, y, z \le n\}$$

be the length n lattice cube. Suppose that $3n^2 - 3n + 1 + k$ points of T are colored red such that if P and Q are red points and PQ is parallel to one of the coordinate axes, then the whole line segment PQ consists of only red points.

Prove that there exists at least k unit cubes of length 1, all of whose vertices's are colored red.

> **Solution.** The inductive solution is tedious, and since we have to count the number of "good" boxes, we can try double counting. Explicitly counting all the "good" boxes.

Problem 2.14.58 (China MO 2018 P5). Let $n \geq 3$ be an odd number and suppose that each square in a $n \times n$ chessboard is colored either black or white. Two squares are considered adjacent if they are of the same color and share a common vertex and two squares a, b are considered connected if there exists a sequence of squares $c_1, \ldots, c_k$ with $c_1 = a, c_k = b$ such that c_i, c_{i+1} are adjacent for $i = 1, 2, \ldots, k-1$.

Find the maximal number M such that there exists a coloring admitting M pairwise disconnected squares.

> **Solution.** It's not hard to get the ans, now that the answer is guesses, and we have tried to prove with induction and couldn't find anything good, we try double counting. We notice that all the connected components in the $n \times n$ are planar graphs. Now we use Euler's theorem on Planar Graphs to find a value of M wrt to other values, and we double count the other values.

Problem 2.14.59 (USAMO 2006 P2). For a given positive integer k find, in terms of k, the minimum value of N for which there is a set of $2k+1$ distinct positive integers that has sum greater than N but every subset of size k has sum at most $\frac{N}{2}$.

> **Solution.** The best or simple looking set is the set of consecutive integers. So if there are some 'holes', we can fill them up to some extent, this opens two sub-cases.

Problem 2.14.60 (USAMO 2005 P1). Determine all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.

Problem 2.14.61 (USAMO 2005 P5). A mathematical frog jumps along the number line. The frog starts at 1, and jumps according to the following rule: if the frog is at integer n, then it can jump either to $n+1$ or to $n+2^{m_n+1}$ where 2^{m_n} is the largest power of 2 that is a factor of n. Show that if $k \geq 2$ is a positive integer and i is a nonnegative integer, then the minimum number of jumps needed to reach $2^i k$ is greater than the minimum number of jumps needed to reach 2^i.

> **Solution.** The main idea is to notice that the operation only uses powers of 2. And it depends on only the power of 2 in the integers, and in the sequence of 2-powers, the operation is very nice.

Problem 2.14.62 (ISL 1991 P10). Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2, \ldots, k$ in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is equal to 1.

Problem 2.14.63. A robot has n modes, and programmed as such: in mode i the robot will go at a speed of $i$ms^{-1} for i seconds. At the beginning of its journey, you have to give it a permutation of $\{1, 2, \ldots n\}$. What is the maximum distance you can make the robot go?

Problem 2.14.64. A slight variation of the previous problem, in this case, the problem goes at a speed of $(n-1)$ms^{-1} for i seconds in mode i.

Problem 2.14.65. m people each ordered n books but because Ittihad was the mailman, he messed up. Everyone got n books but not necessarily the one they wanted you need to fix this. To go to a house from another house it takes one hour. You can carry one book with you during any trip (at most one). You know who has which books and all books are different (i,e, $n * m$ different books). Prove that you can always finish the job in $m * (n + \frac{1}{2})$ hours

> **Solution.** Thinking about the penultimate step, when we have to go to a house empty handed. Thinking in this way gives us a way to pair the houses up, and since pairing...

> **Solution.** Another way to do this is to convert it to a multi-graph. Now go to a house and return with a book means removing two edges from that vertex. We play around with it for sometime

Problem 2.14.66. There are n campers in a camp and they will try to solve a IMO P6 but everyone has a confidence threshold (they will solve the problem by group solving). For example Laxem has threshold 5. I.e. if he's in the group, the group needs to contain at least 5 people (him included). A group is 'confident' when everyone of the team is confident. Now MM wants to make a list of possible "perfect confident" groups. I.e. groups that are confident but adding anyone else will destroy the confidence. How long can his list be?

Problem 2.14.67 (`timus 1862`).

Problem 2.14.68 (`ARO 2014 P9.7`). In a country, mathematicians chose an $\alpha > 2$ and issued coins in denominations of 1 ruble, as well as α^k rubles for each positive integer k. α was chosen so that the value of each coins, except the smallest, was irrational. Is it possible that any natural number of rubles can be formed with at most 6 of each denomination of coins?

Problem 2.14.69 (Saint Petersburg 2001). The number n is written on a board. A and B take turns, each turn consisting of replacing the number n on the board with $n-1$ or $\left\lfloor \frac{n+1}{2} \right\rfloor$. The player who writes the number 1 wins. Who has the winning strategy?

> **Solution.** Recursively building the losing positions.

Problem 2.14.70 (ARO 2011 P11.6). There are more than n^2 stones on the table. Peter and Vasya play a game, Peter starts. Each turn, a player can take any prime number less than n stones, or any multiple of n stones, or 1 stone. Prove that Peter always can take the last stone (regardless of Vasya's strategy).

Problem 2.14.71 (ARO 2007 P9.7). Two players by turns draw diagonals in a regular $(2n+1)$-gon $(n > 1)$. It is forbidden to draw a diagonal, which was already drawn, or intersects an odd number of already drawn diagonals. The player, who has no legal move, loses. Who has a winning strategy?

> **Solution.** Turning the diagonals as vertices, and connection being intersections, we get a graph to play the game on. We then count the degrees.

Problem 2.14.72. After tiling a 6×6 box with dominoes, prove that a line parallel to the sides of the box can be drawn that this line doesn't cut any dominoes.

> **Solution.** Double count how many lines "cut" a domino, and domino number.

Problem 2.14.73. There are 100 points on the plane. You have to cover them with discs, so that any two disks are at a distance of 1. Prove that you can do this in such a way that the total diameter of the disks is < 100.

> **Solution.** As the number 100 is very random, we suspect that is true for all values. So we can use induction

Problem 2.14.74 (ARO 2014 P10.8). Given are n pairwise intersecting convex k-gons on the plane. Any of them can be transferred to any other by a homothety with a positive coefficient. Prove that there is a point in a plane belonging to at least $1 + \frac{n-1}{2k}$ of these

k-gons.

> **Solution.** The most natural such point should be a vertex of a polygon. And these kinda problems use PHP more often, so we will have to divide by k somewhere. Again to find the polygon to use the PHP we will have to divide by n also. So we want to have nk in the denominator. We change the term to achieve this and Ta-Da! we get a fine term to work with.

Problem 2.14.75 (IOI 2018 P1).

Problem 2.14.76 (USAMO 2005 P1). Determine all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.

Problem 2.14.77 (USAMO 2005 P4). Legs L_1, L_2, L_3, L_4 of a square table each have length n, where n is a positive integer. For how many ordered 4-tuples (k_1, k_2, k_3, k_4) of nonnegative integers can we cut a piece of length k_i from the end of leg L_i $(i = 1, 2, 3, 4)$ and still have a stable table?

(The table is stable if it can be placed so that all four of the leg ends touch the floor. Note that a cut leg of length 0 is permitted.)

Problem 2.14.78 (USAMO 2006 P2). For a given positive integer k find, in terms of k, the minimum value of N for which there is a set of $2k + 1$ distinct positive integers that has sum greater than N but every subset of size k has sum at most $\frac{N}{2}$.

Problem 2.14.79 (USAMO 2006 P5). A mathematical frog jumps along the number line. The frog starts at 1, and jumps according to the following rule: if the frog is at integer n, then it can jump either to $n + 1$ or to $n + 2^{m_n+1}$ where 2^{m_n} is the largest power of 2 that is a factor of n. Show that if $k \geq 2$ is a positive integer and i is a nonnegative integer, then the minimum number of jumps needed to reach $2^i k$ is greater than the minimum number of jumps needed to reach 2^i.

Problem 2.14.80 (USAMO 2009 P2). Let n be a positive integer. Determine the size of the largest subset of $\{-n, -n + 1, \ldots, n - 1, n\}$ which does not contain three elements a, b, c (not necessarily distinct) satisfying $a + b + c = 0$.

Problem 2.14.81 (IMO 1979 P3). Two circles in a plane intersect. A is one of the points of intersection. Starting simultaneously from A two points move with constant speed, each travelling along its own circle in the same sense. The two points return to A simultaneously after one revolution. Prove that there is a fixed point P in the plane such that the two points are always equidistant from P.

Chapter 3

Algebra

3.1 Functional Equations

- Equating terms:

  1. **Pseudo-symmetry**: If an equation is almost but not completely symmetrical, what happens if you change the order of the variables and compare with what you started with?
  2. **Fudging**: Can you change one variable so as to alter the equation only slightly? If so, compare with what you started with.
  3. **Self-cancelation**: Can you make two terms in the same functional equation cancel each other out?
  4. These are the most mechanical ways of getting the same value to show up multiple times, but each problem has its own tricks. If you see an interesting expression pop up, always ask yourself whether you can get it to pop up in a slightly different way too.

- Induction, Cauchy

  1. If you want to solve a functional equation over the integers or over the rationals, it often helps to inductively calculate something like $f(nx)$ in terms of $f(x)$
  2. A slight variation: if the domain or range of f is $\mathbb{N}$, definitely look at induction! In addition to asking what is $f(1)$, you can also ask when is $f(n) = 1$
  3. If you want to show $f(x) \geq y$, it suffices to show $f(x) \geq y - \epsilon$ for all $\epsilon > 0$. Can you find progressively tighter ways of bounding $f(x)$ and then apply this argument? Many of the hardest functional equations use this kind of idea.

- Injective, Surjective, Bijective

  1. There are many variations on injectivity and you should not be too fixated on the form used here. The main idea is this: if you can show a relationship between $f(x)$ and $f(y)$, what can conclude about x and y ?
     a) If f is injective and $f(x) = f(y)$, then $x = y$
     b) If f is increasing and $f(x) > f(y)$, then $x > y$
     c) Often it helps to start with a weaker version of injectivity: if $f(x) = f(y) = 0$, then $x = y$
     d) Even if f is not injective, we can often still end up with something useful. For example, if $f(x) = x^2$ is a valid solution, we will not be able to show $f(x)$ is injective, but perhaps we can show that if $f(x) = f(y)$, then $x = \pm y$. That is almost as good.
     e) If you can show any kind of injectivity results, it is often useful to set $x = f(z)$ for some arbitrary z.

2. Surjectivity is a little less common but it still comes in a couple flavours. The main idea is this: is there some nasty expression in your equation that you wish could be replaced by x? If so, prove that expression is surjective, and you are good to go.

a) Usually the nasty expression will be f itself.

b) It does not have to be though. For example, if you could show f (blah) has some nice property, then a good follow-up would be to show that blah is surjective.

Can't Start? Try These

1 GUESS THE POSSIBLE SOLUTIONS.

2 SUBSTITUTION

a) Try EVERY possible substitutions, and write them in a list, dont think during this time.

b) Now think what these results give you.

c) Find values of $f(0), f(1), f(2), f(-x)$ etc.

d) Tweak the function a little bit, do substitution again.

e) Assume some other functions according to the solutions, substitute them to make the fe easier to get info out of.

3 PROPERTIES OF THE FUNCTION

a) Try proving INJECTIVITY, SURJECTIVITY etc.

b) Look for Injectivity or Surjectivity of $f(x) - f(y)$.

4 Assume for the sake of contradiction that the value of the function is greater or smaller than the estimated value at some point.

5 Sometimes consider the difference of two values of f.

Stuck? Try These

1 Proving that $f(x) - x$ is injective might come handy in some cases.

2 If you're *NOT* able to make one side of the equation equal to 0, try to make it equal to any real or some particular real. (pco 169 P11)

3 Sometimes in integer functions, divisibility of the type $f(1)^{k-1} \mid f(x)^k$ helps.

4 Durr... I want things to cancel.

3.1.1 Problems

Problem 3.1.1 (EGMO 2012 P3). Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(yf(x+y)+f(x)\right) = 4x + 2yf\left(x+y\right)$$

for all $x, y \in \mathbb{R}$.

Problem 3.1.2 (pco 169 P11). Find all $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x, y the following holds:

$$f(x)^2 + 2yf(x) + f(y) = f(y + f(x))$$

Problem 3.1.3 (IMO 1994 P5). Let S be the set of all real numbers strictly greater than -1. Find all functions $f : S \to S$ satisfying the two conditions:

1. $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all x, y in S;
2. $\frac{f(x)}{x}$ is strictly increasing on each of the two intervals $-1 < x < 0$ and $0 < x$.

Problem 3.1.4 (ISL 1994 A4). Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{R}^+$ the subset of all positive ones. Let α and β be given elements in $\mathbb{R}$, not necessarily distinct. Find all functions $f : \mathbb{R}^+ \mapsto \mathbb{R}$ such that:

$$f(x)f(y) = y^\alpha f\left(\frac{x}{2}\right) + x^\beta f\left(\frac{y}{2}\right) \forall x, y \in \mathbb{R}^+.$$

Problem 3.1.5 (IMO 2017 P2). Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for any real numbers x and y,

$$f(f(x)f(y)) + f(x + y) = f(xy).$$

Problem 3.1.6 (ISL 2008 A1). Find all functions $f : (0, \infty) \mapsto (0, \infty)$ (so f is a function from the positive real numbers) such that

$$\frac{\left(f(w)\right)^2 + \left(f(x)\right)^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers w, x, y, z, satisfying $wx = yz$.

Problem 3.1.7 (pco 169 P15). Find all $a \in \mathbb{R}$ for which there exists a non-constant function $f : (0,1] \to \mathbb{R}$ such that

$$a + f(x + y - xy) + f(x)f(y) \le f(x) + f(y)$$

for all $x, y \in (0,1]$

Problem 3.1.8 (pco 168 P18). Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

for all $x, y \in \mathbb{R}$

Problem 3.1.9 (ISL 2011 A3). Determine all pairs (f, g) of functions from the set of real numbers to itself that satisfy

$$g(f(x + y)) = f(x) + (2x + y)g(y)$$

for all real numbers x and y.

Problem 3.1.10 (ISL 2005 A2). We denote by $\mathbb{R}^+$ the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ which have the property:

$$f(x)f(y) = 2f(x + yf(x))$$

for all positive real numbers x and y.

> **Solution.** Let's first substitute. If there existed some x such that $f(x) < 1$, we could find a nice substitution. But that leads to a contradiction. So what if we could do something like this for the other cases, $f(x) < 2$ and $f(x) > 2$?

Problem 3.1.11 (ISL 2005 A4). Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) + f(x)f(y) = f(xy) + 2xy + 1$ for all real numbers x and y.

> **Solution.** Substitution.

Problem 3.1.12 (Iran TST T2P1). Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the following conditions:

1. $x + f(y + f(x)) = y + f(x + f(y)) \quad \forall x, y \in \mathbb{R}$
2. The set $I = \left\{ \frac{f(x) - f(y)}{x - y} \mid x, y \in \mathbb{R}, x \neq y \right\}$ is an interval.

Problem 3.1.13 (169 P20)**.** Let a be a real number and let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying: $f(0) = \frac{1}{2}$ and

$$f(x+y) = f(x)f(a-y) + f(y)f(a-x)$$

$\forall x, y \in \mathbb{R}$. Prove that f is constant

Problem 3.1.14 (Vietnam 1991)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which

$$\frac{1}{2}f(xy) + \frac{1}{2}f(xz) - f(x)f(yz) \geq \frac{1}{4}$$

> **Solution.** Just substitute.

Problem 3.1.15. Suppose that f and g are two functions defined on the set of positive integers and taking positive integer values. Suppose also that the equations $f(g(n)) = f(n) + 1$ and $g(f(n)) = g(n) + 1$ hold for all positive integer n. Prove that $f(n) = g(n)$ for all positive integer n.

> **Solution.** Durrr... I want things to cancel... Hint: You want to show $f(n) - g(n) = 0$.

Problem 3.1.16 (ISL 2002 A1)**.** Find all functions f from the reals to the reals such that

$$f(f(x) + y) = 2x + f(f(y) - x)$$

for all real x, y.

> **Solution.** On one of our substitution, we see that there is surjectivity in the equation. So trying to show injectivity is the most intuitive move after that. Again, we have x on the outside, so we need to make x, a once and b once. but we have $f(y) - x$ which we need to eleminate, keeping y constant. We can make it either a or b since we already have $f(a) = f(b)$. And again we can take whatever value we want for $f(y)$.

Problem 3.1.17 (ISL 2001 A1)**.** Let T denote the set of all ordered triples (p, q, r) of nonnegative integers. Find all functions $f : T \to \mathbb{R}$ satisfying

$$f(p,q,r) = \begin{cases} 0 & \text{if } pqr = 0, \\ 1 + \frac{1}{6}(f(p+1,q-1,r) + f(p-1,q+1,r) & \\ +f(p-1,q,r+1) + f(p+1,q,r-1) & \\ +f(p,q+1,r-1) + f(p,q-1,r+1)) & \text{otherwise} \end{cases}$$

for all nonnegative integers p, q, r.

Solution. First let us guess the ans. For all points on the 3 sides, our function gives 0. We get $f(1,1,1) = 1$. We get $f(1,1,2) = f(1,2,1) = f(2,1,1) = \frac{3}{2}$. We get $f(1,1,3) = \frac{9}{5}$. We get $f(1,2,2) = \frac{12}{5}$. Now, since for $pqr = 0$, we have $f = 0$, we need the expression pqr on the numerator. And we kinda guess that the denominator is $p + q + r$. From here the guess is obvious.
Now proving that this solution is the only solution. Let the solution be g. Define, $h := f - g$. Our aim is to prove that $h = 0$ for all inputs.

Problem 3.1.18 (RMM 2019 P5). Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x + yf(x)) + f(xy) = f(x) + f(2019y),$$

for all real numbers x and y.

Solution. After getting $f(yf(0)) = f(y2019)$, one should think of proving that either f is constant, all zero except 0, or linear. How to do this?

Problem 3.1.19 (APMO 2015 P2). Let $S = \{2, 3, 4, \ldots\}$ denote the set of integers that are greater than or equal to 2. Does there exist a function $f : S \to S$ such that

$$f(a)f(b) = f(a^2b^2) \text{ for all } a, b \in S \text{ with } a \neq b?$$

Solution. Try to break the symmetry, add another variable.

Problem 3.1.20 (ISL 2015 A2). Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$.

Solution. It just flows.

Problem 3.1.21 (ISL 2015 A4). Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation

$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$

for all real numbers x and y.

Solution. When you don't know any heavy techniques, just plug in simple values into the function, and write down all of the equations in a list.

Problem 3.1.22 (ISL 2012 A5). Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the conditions

$$f(1 + xy) - f(x + y) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R},$$

and $f(-1) \neq 0$.

Solution. In FE, always look back to what you have, and what things can you make from those.

Problem 3.1.23 (ISL 2012 A1). Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a + b + c = 0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

(Here $\mathbb{Z}$ denotes the set of integers.)

Solution. Go with the flow.

Problem 3.1.24 (ISL 2012 A1). Find all function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following equality holds

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

where $\lfloor a \rfloor$ is greatest integer not greater than a.

Solution. Go with the flow.

Problem 3.1.25 (ISL 2008 A3). Let $S \subseteq \mathbb{R}$ be a set of real numbers. We say that a pair (f, g) of functions from S into S is a Spanish Couple on S, if they satisfy the following conditions:

1. Both functions are strictly increasing, i.e. $f(x) < f(y)$ and $g(x) < g(y)$ for all x, $y \in S$ with $x < y$;
2. The inequality $f(g(g(x))) < g(f(x))$ holds for all $x \in S$.

Decide whether there exists a Spanish Couple

- on the set $S = \mathbb{N}$ of positive integers;
- on the set $S = \{a - \frac{1}{b} : a, b \in \mathbb{N}\}$

Solution. Inspecting the fe immediately gives us $g(g(x)) < f(x)$. Any attempt at constructing a pair for $\mathbb{N}$, fails because eventually $g(x)$ becomes larger than $f(x)$. So there might be something with how g grows that restricts the construction.

This idea motivates us to further inspect the fe. We eventually notice that $g^n(x) < f(x)$ for all $n \in \mathbb{N}$, and that means $S = \mathbb{N}$ just won't work.

Now we begin to suspect that this is definitely why the second set has be constructed that way. Thinking about the basic construction where $f(x) = x + 1$, we see that if we think of $g(x)$ as moving x by some step to the right on the number line, we get a better picture of how $f(g(g(x)))$ behaves.

And that motivates our solution:

$$\boxed{f(x) = x + 1, \quad g\left(a - \frac{1}{b}\right) = a - \frac{1}{b + 3^a}}$$

Problem 3.1.26 (ISL 2007 A4). Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$f\left(x + f\left(y\right)\right) = f\left(x + y\right) + f\left(y\right)$$

for all pairs of positive reals x and y. Here, $\mathbb{R}^+$ denotes the set of all positive reals.

Solution [substitution]. Set $x = f(y)$, we have

$$f\left(2f(y)\right) = f(2y) + 2f(y)$$
$$\text{and, } f(x + kf(y)) = f(x + ky) + kf(y)$$

Now setting $y = 2f(y)$, we get

$$\begin{aligned} f\left(x + f(2f(y))\right) &= f\left(x + 2f(y)\right) + f(2f(y)) \\ &= f(x + 2y) + 4f(y) + f(2y) \end{aligned}$$

$$\begin{aligned} \text{Also, } f(x + f(2f(y))) &= f(x + f(2y) + 2f(y)) \\ &= f(x + 4y) + 2f(y) + f(2y) \end{aligned}$$

$$\implies f(x + 4y) = f(x + 2y) + 2f(y) = f(x + 2f(y))$$

$$\boxed{\therefore f(x + 4y) = f(x + 2f(y))} \ \forall y \in \mathbb{R}^+ \tag{3.1}$$

If f is injective, then we have $\boxed{f(x) = 2x}$. If not, then suppose $f(a) = f(b)$. Substituting $y = a, b$ we get

$$f(x + a) = f(x + b) \ \forall x \in \mathbb{R}^+$$

Combining it with (1), we get that $f(x)$ is a constant function, and so $\boxed{f = 0}$.

Solution [cauchy, Raja Oktovin]. For any positive real numbers z, we have that

$$f(x+f(y))+z=f(x+y)+f(y)+z$$

$$f(f(x+f(y))+z)=f(f(x+y)+f(y)+z)$$
$$f(x+f(y)+z)+f(x+f(y))=f(x+y+f(y)+z)+f(x+y)$$
$$f(x+y+z)+f(y)+f(x+y)+f(y)=f(x+2y+z)+f(y)+f(x+y)$$
$$f(x+y+z)+f(y)=f(x+2y+z)$$

$$\boxed{f(a)+f(b)=f(a+b)}$$

and by Cauchy in positive reals, then $f(x)=\alpha x$ for all $x\in(0,\infty)$. Now it's easy to see that $\alpha=2$, then $f(x)=2x$ for all positive real numbers x.

Problem 3.1.27 (ISL 2007 A2). Consider those functions $f:\mathbb{N}\mapsto\mathbb{N}$ which satisfy the condition

$$f(m+n)\geq f(m)+f(f(n))-1$$

for all $m,n\in\mathbb{N}$. Find all possible values of $f(2007)$.

Solution. Substituting $n=1$, we have

$$f(m+1)\geq f(m),\quad f(m+1)\geq f(f(m))$$

So f is non decreasing. We prove that $f(n)\leq n+1$. Suppose not, so there exists a n for which

$$\begin{aligned}&f(n)\geq n+2\\ \implies\ &f(2^kn)\geq 2^kn+2^k+1\qquad\forall k\end{aligned}$$

Since f grows without any bounds, there is a t such that $f(2^t+1)>1$. Then we have,

$$\begin{aligned}f(2^tn+2^t+1)&\geq f(2^t+1)+f(f(2^tn))-1\\&\geq f(2^t+1)-1+f(2^tn+2^t+1)\\&>f(2^tn+2^t+1)\end{aligned}$$

A contradiction.

So we get a nice bounding for $f(2007)$, that is $\{1,2,\dots 2008\}$. It's time to construct functions for that. For $1\leq k\leq 2007$, consider the functions

$$f(n)=\begin{cases}1&\text{if }n<k\\ n-k+1&\text{if }n\geq k\end{cases}$$

And for 2008, consider the function

$$f(n)=\begin{cases}n+1&\text{if }2007|n\\ n&\text{otherwise}\end{cases}$$

> The constructions comes from the idea that we want $f(f(n)) = f(n)$, and we can use 1's to push our values to the right.

3.1.2 Weird Ones

Problem 3.1.28 (ISL 2009 A3). Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths

$$a, f(b) \text{ and } f(b + f(a) - 1).$$

(A triangle is non-degenerate if its vertices are not collinear.)

Solution. $f(1) > 1 \implies f$ is periodic $\implies$ repeatation $\implies$ contradiction. $f(2) > 2 \implies$ strictly increasing $\implies$ repeatation.

Problem 3.1.29 (USA TST 2018 P2). Find all functions $f : \mathbb{Z}^2 \to [0, 1]$ such that for any integers x and y,

$$f(x, y) = \frac{f(x-1, y) + f(x, y-1)}{2}.$$

Solution. We know that the function has to be a constant function. So it is a intuitive idea considering the difference of two values of the function. Again as we wish to show that this difference is 0, we have to use either equality of limit. As equality is quite ambiguous in this problem, we approach with limits. We see that $f(x, y)$ can be written as a term depending on the values of the 3rd quarter of the plane with (x, y) as its origin. With infinite values in our hand, we try bounding.

Problem 3.1.30 (IMEO 2020 P3). Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all positive real x, y holds

$$xf(x) + yf(y) = (x + y)f\left(\frac{x^2 + y^2}{x + y}\right)$$

Solution. Note that

$$\frac{xf(x) + yf(y)}{x + y} = f\left(\frac{x^2 + y^2}{x + y}\right)$$

Which looks just like a linear form. Also since we know that the solution is probably only $f(x) = ax + b$, we pursue this idea.

It can be proved that all the rational points lie on a line. Now what if a irrational lies outside of the line?

Solution [MarkBcc168, geometric]. Our first step is to do Vieta Jumping on the functional equation. The main result from this is the following.

Claim— $af(a) - bf(b) = (a-b)f(a+b)$ for any $a, b \in \mathbb{R}^+$.

Proof. Fix $t, k \in \mathbb{R}^+$ where $t < k$. Let $g(x) = \frac{x^2+t^2}{x+t}$. Since $\lim_{x\to\infty} g(x) = \infty$ and $g(t) = t$, there exists a such that $g(a) = \frac{t^2+a^2}{t+a} = k$. Moreover, by Vieta's Jumping, $g(k-a) = k$. Thus, by considering

$$\begin{array}{rcrcl} P(t,a) & \implies & tf(t) + af(a) & = & (a+t)f(k) \qquad \text{and} \\ P(t,k-a) & \implies & tf(t) + (k-a)f(k-a) & = & (k-a+t)f(k), \end{array}$$

we get that

$$af(a) - (k-a)f(k-a) = (2a-k)f(k)$$

for any a, k such that $k > a$. Thus, by setting $b = k - a$, we get the desired claim.

Now let's do some geometry. Let $P_x = (x, f(x))$. Then the claim above implies that P_a, P_b, P_{a+b} are colinear whenever $a, b \in \mathbb{R}^+$.

Fix any pairwise distinct $a, b, c \in \mathbb{R}^+$. Set $A = P_a$, $B = P_b$, $C = P_c$, $D = P_{b+c}$, $E = P_{c+a}$, and $F = P_{a+b}$. By the claim, $D \in BC$, $E \in CA$, $F \in AB$, and AD, BE, CF are concurrent at P_{a+b+c}. However, we claim that

Claim— D, E, F are colinear.

Proof. If A, B, C are colinear, then the result is trivial. Otherwise, we use Menelaus theorem. Observe that

$$\frac{\overline{BD}}{\overline{DC}} = \frac{b-(b+c)}{(b+c)-c} = -\frac{b}{c}$$

hence multiplying the result cyclically gives the result.

By Ceva's theorem, both D, E, F being colinear and AD, BE, CF being concurrent can only happen when A, B, C are colinear. This means that $(a, f(a))$, $(b, f(b))$, $(c, f(c))$ are colinear for any $a, b, c \in \mathbb{R}^+$. This concludes, say by varying c, that f must be linear.

3.2 FE cantonmathguy Seclected Problems

1. Determine all functions $f : \mathbb{Q} \to \mathbb{Q}$ satisfying $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{Q}$.

2. Let $a_1, a_2, \ldots$ be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer n the numbers $a_1, a_2, \ldots, a_n$ leave n different remainders upon division by n. Prove that every integer occurs exactly once in the sequence $a_1, a_2, \ldots$.

3. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying
$$f(x)f(y) = f(x+y) + xy$$
for all real x and y.

4. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a+b+c = 0$, the following equality holds:
$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

5. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying
$$f(x) + f(y) = f(x+y) \quad \text{and} \quad f(xy) = f(x)f(y)$$
for all $x, y \in \mathbb{R}$.

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$, the following equality holds
$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$
where $\lfloor a \rfloor$ is the greatest integer not greater than a.

7. ★ Let k be a real number. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying
$$|f(x) - f(y)| \leq k(x-y)^2$$
for all real x and y.

8. Let $f : \mathbb{N} \to \mathbb{N}$ be a function, and suppose that positive integers k and c satisfy
$$f^k(n) = n + c$$
for all $n \in \mathbb{N}$, where f^k denotes f applied k times. Show that $k \mid c$.

9. Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying
$$f(f(f(n))) + f(f(n)) + f(n) = 3n$$
for every positive integer n.

10. Let S be the set of integers greater than 1. Find all functions $f : S \to S$ such that (i) $f(n) \mid n$ for all $n \in S$, (ii) $f(a) \geq f(b)$ for all $a, b \in S$ with $a \mid b$.

11. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all pairs of real numbers x and y.

12. ★ Let T denote the set of all ordered triples (p, q, r) of nonnegative integers. Find all functions $f : T \to \mathbb{R}$ satisfying

$$f(p,q,r) = \begin{cases} 0 & \text{if } pqr = 0, \\ 1 + \frac{1}{6}(f(p+1, q-1, r) + f(p-1, q+1, r) & \\ +f(p-1, q, r+1) + f(p+1, q, r-1) & \\ +f(p, q+1, r-1) + f(p, q-1, r+1)) & \text{otherwise} \end{cases}$$

for all nonnegative integers p, q, r.

13. Determine all strictly increasing functions $f : \mathbb{N} \to \mathbb{N}$ satisfying $nf(f(n)) = f(n)^2$ for all positive integers n.

14. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$.

15. Find all real-valued functions f defined on pairs of real numbers, having the following property: for all real numbers a, b, c, the median of $f(a, b), f(b, c), f(c, a)$ equals the median of a, b, c.

16. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that, for all positive integer n, we have $f(f(n)) < f(n + 1)$.

17. Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that, for any $w, x, y, z \in N$,

$$f(f(f(z)))f(wxf(yf(z))) = z^2 f(xf(y))f(w).$$

Show that $f(n!) \geq n!$ for every positive integer n.

18. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(n!) = f(n)!$ for all positive integers n and such that $m - n$ divides $f(m) - f(n)$ for all distinct positive integers m, n.

19. Find all functions f from the reals to the reals such that

$$(f(a) + f(b))(f(c) + f(d)) = f(ac + bd) + f(ad - bc)$$

for all real a, b, c, d.

20. Determine all functions f defined on the natural numbers that take values among the natural numbers for which
$$(f(n))^p \equiv n \pmod{f(p)}$$
for all $n \in \mathbb{N}$ and all prime numbers p.

21. Let $n \geq 4$ be an integer, and define $[n] = \{1, 2, \dots, n\}$. Find all functions $W : [n]^2 \to \mathbb{R}$ such that for every partition $[n] = A \cup B \cup C$ into disjoint sets,
$$\sum_{a \in A} \sum_{b \in B} \sum_{c \in C} W(a,b)W(b,c) = |A||B||C|.$$

22. ★ Find all infinite sequences $a_1, a_2, \dots$ of positive integers satisfying the following properties: (a) $a_1 < a_2 < a_3 < \cdots$, (b) there are no positive integers i, j, k, not necessarily distinct, such that $a_i + a_j = a_k$, (c) there are infinitely many k such that $a_k = 2k - 1$.

23. Show that there exists a bijective function $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that for all $m, n \in \mathbb{N}_0$,
$$f(3mn + m + n) = 4f(m)f(n) + f(m) + f(n)$$

24. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying
$$f(f(m) + n) + f(m) = f(n) + f(3m) + 2014$$
for all integers m and n.

25. Let $n \geq 3$ be a given positive integer. We wish to label each side and each diagonal of a regular n-gon $P_1 \dots P_n$ with a positive integer less than or equal to r so that:

    a) every integer between 1 and r occurs as a label;
    b) in each triangle $P_iP_jP_k$ two of the labels are equal and greater than the third.

    Given these conditions:

    a) Determine the largest positive integer r for which this can be done.
    b) For that value of r, how many such labellings are there?

26. ★ Suppose that f and g are two functions defined on the set of positive integers and taking positive integer values. Suppose also that the equations $f(g(n)) = f(n) + 1$ and $g(f(n)) = g(n) + 1$ hold for all positive integer n. Prove that $f(n) = g(n)$ for all positive integer n.

27. Find all the functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ satisfying the relation

28. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation
$$f(x + f(x + y)) + f(xy) = x + f(x + y) + yf(x)$$
for all real numbers x and y.

29. Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the sub-sequences

$$s_{s_1}, s_{s_2}, s_{s_3}, \ldots \qquad \text{and} \qquad s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \ldots$$

are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

30. Find all functions f from $\mathbb{N}_0$ to itself such that

$$f(m+f(n)) = f(f(m)) + f(n)$$

for all $m, n \in \mathbb{N}_0$.

31. ★ Consider a function $f : \mathbb{N} \to \mathbb{N}$. For any $m, n \in \mathbb{N}$ we write $f^n(m) = \underbrace{f(f(\ldots f}_{n}(m)\ldots))$.

Suppose that f has the following two properties:

a) if $m, n \in \mathbb{N}$, then $\frac{f^n(m)-m}{n} \in \mathbb{N}$;
b) The set $\mathbb{N} \setminus \{f(n) \mid n \in \mathbb{N}\}$ is finite.

Prove that the sequence $f(1)-1, f(2)-2, f(3)-3, \ldots$ is periodic.

32. Let $\mathbb{N}$ be the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ that satisfy the equation

$$f^{abc-a}(abc) + f^{abc-b}(abc) + f^{abc-c}(abc) = a+b+c$$

for all $a, b, c \geq 2$.

33. Let $2\mathbb{Z}+1$ denote the set of odd integers. Find all functions $f : \mathbb{Z} \to 2\mathbb{Z}+1$ satisfying

$$f(x+f(x)+y) + f(x-f(x)-y) = f(x+y) + f(x-y)$$

for every $x, y \in \mathbb{Z}$.

3.3 Polynomials

3.3.1 Techniques to remember

Stuck? Try These

1 A polynomial with odd degree always has at least one real root.

2 If a polynomial with even degree has a negative value on its graph, then it has at least one real root.

3 Roots of unity divide a polynomial in parts like congruence classes.

4 MODULUS SIGN: use Triangle Inequality.

5 They say, In Poly, chase ROOTS.

6 $x^n f(\frac{1}{x})$ has the same coefficients as $f(x)$, but in opposite order.

Theorem 3.3.1 (Lagrange Interpolation Theorem) — Given n real numbers, there exist a polynomial with at most $n-1$ degree such that the graph of the polynomial goes through all of the points.

Theorem 3.3.2 (Finite Differences) — This is the descrete form of derivatives. The first finite difference of a function f is defined as $g(x) := f(x+1) - f(x)$.
$n+1$th finite difference of a n degree polynomial: For any polynomial $P(x)$ of degree at most n the following equation holds:

$$\sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} P(i) = 0$$

Remark. This can be used

1. to reduce the degree of a polynomial, manipulate the coefficients etc.
2. to solve recurrences, where the recurrence equation is a bit messy and contains a lot of previous values. Like this recurrence is quite messy to solve as it is, but if we take the first finite difference here, it becomes easy:

$$f(x) = \frac{1}{3} f(x+1) + \frac{2}{3} f(x-1) + 1$$

It's like solving for the first derivative and then finding the original function.

3.3.2 General Problems

Problem 3.3.1 (USA TST 2014 P4). Let n be a positive even integer, and let $c_1, c_2, \ldots, c_{n-1}$ be real numbers satisfying

$$\sum_{i=1}^{n-1} |c_i - 1| < 1.$$

Prove that

$$2x^n - c_{n-1}x^{n-1} + c_{n-2}x^{n-2} - \cdots - c_1x^1 + 2$$

has no real roots.

> **Solution.** A polynomial has no real root means that the polynomial completely lies in either of the two sides of the x-axis. So in this case, we have to prove that $P(x) > 0$. So we gotta try to bound. Again, to make $|c_i - 1|$ a little bit more approachable, we assign $b_i = c_i - 1$ and write $P(x)$ in terms of b_i. Now, how to bring the modulus sign in our polynomial? Oh, we have triangle ineq for those kinda work :0

Problem 3.3.2 (USAMO 2002 P3). Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

> **Solution.** (i) If we have $n + 1$ points, we have an unique polynomial through them.
> (ii) If we have one positive value of a polynomial and one negative value, then there exists a real root between that two values.

Problem 3.3.3 (China TST 1995 P5). A and B play the following game with a polynomial of degree at least 4:

$$x^{2n} + \_\ x^{2n-1} + \_\ x^{2n-2} + \ldots + \_\ x + 1 = 0$$

A and B take turns to fill in one of the blanks with a real number until all the blanks are filled up. If the resulting polynomial has no real roots, A wins. Otherwise, B wins. If A begins, which player has a winning strategy?

> **Solution.** Not always (actually in very few cases) the first move decides the winning strategy. In this case, if B could make the last move, he would definitely win. But as he can't, consider the final two moves. Again "Waves".

Problem 3.3.4 (Zhao Polynomials). A set of n numbers are considered to be k-cool if $a_1 + a_{k+1} \cdots = a_2 + a_{k+2} \cdots = \cdots = a_k + a_{2k} \ldots$. Suppose a set of 50 numbers are $3, 5, 7, 11, 13, 17$-cool. Prove that every element of that set is 0.

Solution. Equivalence class :0 roots of unity :0 :0 :0

Problem 3.3.5 (All Russian Olympiad 2016, Day2, Grade 11, P5). Let n be a positive integer and let $k_0, k_1, \ldots, k_{2n}$ be nonzero integers such that $k_0 + k_1 + \cdots + k_{2n} \neq 0$. Is it always possible to find a permutation $(a_0, a_1, \ldots, a_{2n})$ of $(k_0, k_1, \ldots, k_{2n})$ so that the equation

$$a_{2n}x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_0 = 0$$

has no integer roots?

Solution. The degree is $2n$, and we have to find a zero, so proving/disproving the existence of negative value of $P(x)$ is enough. If all the values of $P(x)$ are to be positive, the leading coefficient must be very big...

Problem 3.3.6 (Zhao Poly). Let $f(x)$ be a monic polynomial with degree n with distinct zeroes $x_1, x_2, ..., x_n$. Let $g(x)$ be any monic polynomial of degree $n-1$. Show that

$$\sum_{j=1}^{n} \frac{g(x_j)}{f'(x_j)} = 1$$

where $f'(x_i) = \prod_{j \neq i}(x_i - x_j)$

Solution. Lagrange's Interpolation

Problem 3.3.7 (ARO 2018 P11.1). The polynomial $P(x)$ is such that the polynomials $P(P(x))$ and $P(P(P(x)))$ are strictly monotone on the whole real axis. Prove that $P(x)$ is also strictly monotone on the whole real axis.

Problem 3.3.8 (Serbia 2018 P4). Prove that there exists a uniqe $P(x)$ polynomial with real coefficients such that

$$xy - x - y \mid (x+y)^{1000} - P(x) - P(y)$$

for all real x, y.

Solution. Substitution.

3.3.3 Root Hunting

Problem 3.3.9 (Putnam 2017 A2). Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

3.3.4 NT Polynomials

Problem 3.3.10 (Iran TST 2009 P4). Find all polynomials f with integer coefficient such that, for every prime p and natural numbers u and v with the condition:

$$p \mid uv - 1$$

we always have

$$p \mid f(u)f(v) - 1$$

Solution. Notice that we can disregard v by considering it $\frac{1}{u}$, and the condition won't be affected, because primes allow multiplicative inverses. After this observation the problem is almost solved.

Problem 3.3.11 (Iran TST 2004 P6). p is a polynomial with integer coefficients and for every natural n we have $p(n) > n$. x_k is a sequence that: $x_1 = 1, x_{i+1} = p(x_i)$ for every N one of x_i is divisible by N. Prove that $p(x) = x + 1$

Solution. Notice that $\{x_i\}$ becomes periodic mod any prime. Now, we start by showing that $P(1) = 2$. We have, $P(x)$ has to be even. If it is > 2 then what happens? what if we take $N = P(1) - 1$?

Problem 3.3.12 (ISL 2006 N4). Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

Solution. Suppose that there are more than n fixed points. So at least one of them cant be a fixed point of P. Use that. Follow.

Problem 3.3.13 (ISL 2012 A4). Let f and g be two nonzero polynomials with integer coefficients and $\deg f > \deg g$. Suppose that for infinitely many primes p the polynomial $pf + g$ has a rational root. Prove that f has a rational root.

Solution. dunno

3.3.5 Fourier Transformation

Lemma 3.3.3 (Dealing with binomial terms with a common factor) — Let n, k be two integers, and let z be a kth root of unity other than 1. Then,

$$\binom{n}{0} + \binom{n}{k} + \binom{n}{2k} + \cdots = \frac{(1+z^1)^n + (1+z^2)^n + \cdots + (1+z^k)^n}{k}$$

Proof. For j not divisible by k,

$$z^j \left(\sum_{i=1}^{k} z^i \right) = 0$$

3.3.6 Irreducibility

- Summer Camp 2015 Handout
- Yufei Zhao's Handout

Stuck? Try These: What can be showed to prove Irreducibility

- Writing $f = g \cdot h$ and equating coefficients
- If the polynomial involves some prime, it's often useful to try factoring modulo that prime
- If the last coefficient is a prime, then there are some obvious bounds on the roots
- If there are bounds on the coefficients, then try root bounding

Lemma 3.3.4 (Bounds On Roots) — P is a monic polynomial. Suppose $P(0) \neq 0$ and at most one complex root of P has absolute value at least 1. Then P is irreducible.

Lemma 3.3.5 — P is a monic polynomial. Suppose that $|P(0)|$ is prime, and all complex roots of P have absolute value greater than 1. Then P is irreducible.

Lemma 3.3.6 (Leading Coefficient is LARGE) — Let $P(x) = b_nx^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0$ such that

$$|b_n| > |b_{n-1}| + |b_{n-2}| + \cdots + |b_0|$$

Then every root α of P is **strictly inside of the unit circle**, i.e. $|\alpha| < 1$.

i.e. If the first coefficient of the polynomial is very large, then all of the roots lie inside the unit circle.

Lemma 3.3.7 (Coefficients form a Decreasing Sequence) — Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a real polynomial. Such that,

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0$$

Then any complex z of $P(x)$ satisfies $|z| \leq 1$

i.e. If the coefficients form a decreasing sequence then all of the roots lie on or inside the unit circle.

Lemma 3.3.8 (Constant is LARGE) — Let $P(x) = a_nx^n + a_{n-1}x^{n-1} \cdots + a_1x + a_0$ be a polynomial over integers. Where, a_0 is a prime, and

$$|a_0| > |a_n| + |a_{n-1}| + \cdots + |a_1|$$

Prove that $P(x)$ is irreducible.

Theorem 3.3.9 (Rouché's Theorem) — Let f, g be analytic functions on and inside a simple closed curve $\mathcal{C}$. Suppose that

$$|f(z)| > |g(z)|$$

for all points z on $\mathcal{C}$. Then f and $f + g$ have the same number of zeroes (counting multiplicities) interior to $\mathcal{C}$

Theorem 3.3.10 (Perron's Criterion) — Let $P(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \dots a_1 x + a_0$ be a polynomial over integers such that

$$|a_{n-1}| > 1 + |a_{n-2}| + |a_{n-3}| + \cdots + |a_1| + |a_0|$$

Then $P(x)$ is irreducible.

Remark. The crucial idea behind the proof is that $|a_0| \geq 1$, and if the polynomial is reducible, then there are at least two roots $|z| \geq 1$.

Proof [Bounding Roots]. Let $P(z) = 0$ for some $|z| = 1, z \in \mathbb{C}$. That means we have

$$\begin{aligned} -a_{n-1}z^{n-1} &= z^n + a_{n-2}z^{n-2} \dots a_1 z + a_0 \\ \implies |a_{n-1}| &= |z^n + a_{n-2}z^{n-2} \dots a_1 z + a_0| \\ &\leq |1| + |a_{n-2}| \cdots + |a_0| \end{aligned}$$

Which is a contradiction. So $|z| \neq 1$.

We know that there exist a root z that has an absolute value greater than 1. We prove that there is only one such root of $P(x)$.

First, let $P(x) = (x-z)Q(x)$, where $|z| > 1$, and $Q(x) = x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0$. So we have,

$$P(x) = (x-z)Q(x)$$
$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \dots a_1 x + a_0 = x^n + (b_{n-2} - z)x^{n-1} + \cdots + (b_0 - zb_1)x + zb_0$$

$$\begin{aligned} \implies |b_{n-2} - z| &> 1 + |b_{n-3} - zb_{n-2}| + \cdots + |b_0 - zb_1| + |zb_0| \\ |b_{n-2}| + |z| &> 1 - |b_{n-3}| + |z||b_{n-2}| + \dots |b_0| - |z||b_1| + |z||b_0| \\ |b_{n-2}| + |z| &= (|z| - 1)(|b_{n-2}| + |b_{n-3} \dots |b_0|) + |b_{n-2}| + 1 \\ 1 &> |b_{n-2}| + |b_{n-3} \dots |b_0| \end{aligned}$$

And by Lemma 3.3.6, $Q(x)$ does not have any root $|z| > 1$.

Theorem 3.3.11 (Perron's Criterion's Generalization (Dominating Term)) — Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a complex polynomial, such that its a_k term is dominant, that is,

$$|a_k| > |a_0| + |a_1| + \dots |a_{k-1}| + |a_{k+1}| + \cdots + |a_n|$$

for some $0 \leq k \leq n$. Then exactly k roots of P lies strictly inside of the unit circle, and the other $n - k$ roots of P lies strictly outside of the unit circle.

Proof. A direct application of Theorem 3.3.9.

Lemma 3.3.12 (Bound on roots) — Let $f(x) = a_n x^n + a_{n-1}x^{n-1} \cdots + a_1 x + a_0$ be an integer polynomial. Suppose that $a_n \geq 1, a_{n-1} \geq 0$ and $a_i \leq H$ for some positive constant H and $i = 0, 1, \ldots n-2$. Then any complex zero α of $f(x)$ has either *nonpositive real part*, or satisfies

$$|\alpha| < \frac{1 + \sqrt{1+4H}}{2}$$

Proof. Suppose z is a root such that $|z| > 1$ and $\operatorname{Re} z > 0$. Then we have

$$\begin{aligned}\left|\frac{f(z)}{z^n}\right| &\geq \left(a_n - \frac{a_{n-1}}{z}\right) - H\left(\frac{1}{|z|^2} + \frac{1}{|z|^3} \cdots \frac{1}{|z|^n}\right) \\ &> \operatorname{Re}\left(a_n + \frac{a_{n-1}}{z}\right) - \frac{H}{|z|^2 - |z|} \\ &\geq 1 - \frac{H}{|z|^2 - |z|} \\ &= \frac{|z|^2 - |z| - H}{|z|^2 - |z|} \\ &\geq 0\end{aligned}$$

Whenever

$$|z| \geq \frac{1 + \sqrt{1+4H}}{2}$$

Theorem 3.3.13 (Cohn's Criterion) — Suppose p is a prime number, expressed as $\overline{p_n p_{n-1} \cdots p_1 p_0}$ in base $b \geq 2$. Then the polynomial

$$f(x) = p_n x^n + p_{n-1}x^{n-1} \cdots + p_1 x + p_0$$

is irreducible.

Problem 3.3.14 (ISL 2005 A1). Find all pairs of integers a, b for which there exists a polynomial $P(x) \in \mathbb{Z}[X]$ such that product $(x^2 + ax + b) \cdot P(x)$ is a polynomial of a form

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$$

where each of $c_0, c_1, \ldots, c_{n-1}$ is equal to 1 or -1.

Solution. The idea of bounding the roots using the coefficients.

Problem 3.3.15. Let $P(x)$ be a polynomial with real coefficients, and $P(x) \geq 0$ for all $x \in \mathbb{R}$. Prove that there exists two polynomials $R, S \in \mathbb{Q}$ such that

$$P(x) = R(x)^2 + Q(x)^2$$

Problem 3.3.16 (Romanian TST 2006 P2). Let p a prime number, $p \geq 5$. Find the number of polynomials of the form

$$x^p + px^k + px^l + 1, \quad k > l, \quad k, l \in \{1, 2, \cdots, p-1\},$$

which are irreducible in $\mathbb{Z}[X]$.

> **Solution.** Taking mod p, we have that $x^p + 1 \equiv (x+1)^p (\bmod\ p)$. Now we can try equating terms or plug in some values to check for equality.

Problem 3.3.17 (Romanian TST 2003 P5). Let $f \in \mathbb{Z}[X]$ be an irreducible polynomial over the ring of integer polynomials, such that $|f(0)|$ is not a perfect square. Prove that if the leading coefficient of f is 1 (the coefficient of the term having the highest degree in f) then $f(X^2)$ is also irreducible in the ring of integer polynomials.

> **Solution.** If $f(x^2) = g(x)h(x)$, plugging $-x$ gives us $g(x)h(x) = g(-x)h(-x)$. So we should look at the common roots of $h(x)$ and $h(-x)$. And it is straightforward from here.

> **Solution.**

3.4 Inequalities

- Olympiad Inequalities - Thomas J. Mildorf
- $A < B$ - Keran Kedlaya
- Convexity - Po Shen Loh
- Brief Intro to Ineqs - Evan Chen

Definition (Majorizes)— Given two sequences of real numbers $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$, we say (x_n) *majorizes* (y_n), written $(x_n) \succ (y_n)$ if

$$x_1 + x_2 \cdots + x_n = y_1 + y_2 \cdots + y_n \text{ and,}$$
$$x_1 + x_2 \cdots + x_k \geq y_1 + y_2 \cdots + y_k \quad \forall 1 \leq k \leq n-1$$

Definition (Mean Values)— Given n *positive reals* $a_2 \dots a_n$, we have

$$\text{Arithmetic Mean : } \quad \frac{\sum a_i}{n} = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

$$\text{Geometric Mean : } \quad \sqrt[n]{\prod a_i} = \sqrt[n]{a_1 a_2 \dots a_n}$$

$$\text{Quadratic Mean : } \quad \sqrt{\frac{\sum a_i^2}{n}} = \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}}$$

$$\text{Harmonic Mean : } \quad \frac{n}{\sum \frac{1}{a_i}} = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

Definition (Convexity)— The function f is **convex** in interval I iff for all $a, b \in I$ and for all $t < 1$,

$$tf(a) + (1-t)f(b) \geq f\left(ta + (1-t)b\right)$$

Which if put in words, means that the line segment joining $(a, f(a))$ and $(b, f(b))$ lies completely above the graph of the function.

The function is convex in interval I if f' is **increasing** in I or f'' is **positive** in I.

3.4.1 Basic Inequalities

Theorem 3.4.1 (Triangle Inequality) — For any complex numbers $a_1, a_2 \dots a_n$ the following holds:

$$|a_1 + a_2 \cdots + a_k| \leq |a_1| + |a_2| + \cdots + |a_k|$$

Theorem 3.4.2 (QM-AM-GM-HM) — Given n positive real numbers $x_1, x_2, \dots x_n$, the following relation holds:

$$\sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}} \geq \frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n} \geq \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Theorem 3.4.3 (Weighted AM-GM) — If $a_1, a_2, \dots, a_n$ are nonnegative real numbers, and $\lambda_1, \lambda_2, \dots, \lambda_n$ are nonnegative real numbers (the "weights") which sum to 1, then

$$\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n \geq a_1^{\lambda_1} a_2^{\lambda_2} \dots a_n^{\lambda_n}$$

Equality holds if and only if $a_i = a_j$ for all integers i, j such that $\lambda_i \neq 0$ and $\lambda_j \neq 0$. We obtain the unweighted form of AM-GM by setting

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = \frac{1}{n}$$

Theorem 3.4.4 (Muirhead's Inequality) — If $a_1, a_2, \dots a_n$ are real positive reals, and $(x_n) \succ (y_n)$, then we have,

$$\sum_{\text{sym}} a_1^{x_1} a_2^{x_2} \cdots a_n^{x_n} \geq \sum_{\text{sym}} a_1^{y_1} a_2^{y_2} \cdots a_n^{y_n}$$

Theorem 3.4.5 (Jensen's Inequality) — Let $x_1, \dots, x_n \in \mathbb{R}$ and let $\alpha_1, \dots, \alpha_n \geq 0$ satisfy $\alpha_1 + \cdots + \alpha_n = 1$. If f is a Convex Function, we have:

$$\alpha_1 f(x_1) + \alpha_2 f(x_2) \cdots + a_n f(x_n) \geq f(\alpha_1 x_1 + \alpha_2 x_2 \cdots + \alpha_n x_n)$$

If f is a Concave Function, the sign of the fuction is reversed.

Theorem 3.4.6 (Karamata's Inequality) — If f is convex and $(x_n) \succ (y_n)$, then

$$f(x_1) + f(x_2) \cdots + f(x_n) \geq f(y_1) + f(y_2) \cdots + f(x_n)$$

The reverse inequality holds if f is concave.

Theorem 3.4.7 (Tangent Line Trick: Best-Nice bound for function) — Even if a fucntion is not convex or concave for us to use Jensen's Inequality, we can still find a number a such that for our required interval I, f stays above (or below) the tangent line to f at $(a, f(a))$, that is

$$\boxed{f(x) \geq f(a) + f'(a)(x-a)}$$

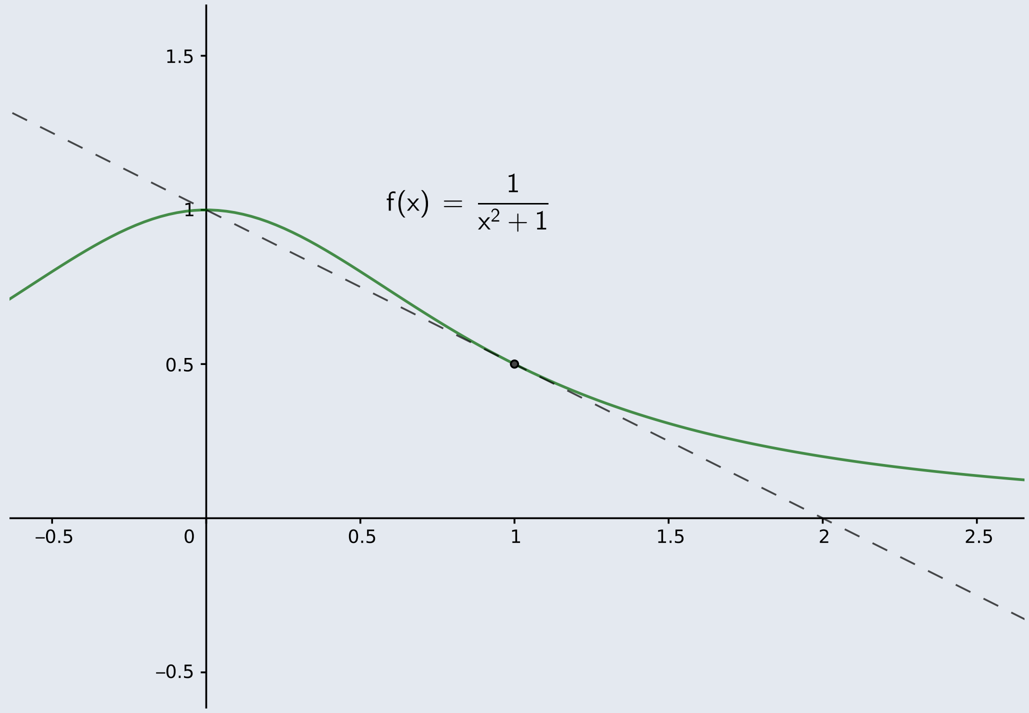


Figure 3.4.1: The tangent line at $(1, f(1))$ is the best fit line in interval $[0, 2]$

Theorem 3.4.8 ($n-1$ Equal Values) — Let $a_1, a_2, \ldots a_n$ be real numbers with $a_1 + a_2 \ldots a_n$ fixed. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with exactly one *inflection point*. If

$$f(a_1) + f(a_2) + \cdots + f(a_n)$$

achieves a maximal or minimal value, then $n-1$ of the a_i are equal to each other.

Lemma 3.4.9 (Power function convexity) — If $(x_i), (y_i), (m_i)$ be three sequences of real numbers, $x, y \in \mathbb{R}$ and $p > 1$. Then for $\alpha \in (0,1)$,

$$(x+y)^p \;\le\; \alpha^{1-p} x^p \;+\; (1-\alpha)^{1-p} y^p$$

$$\sum (x_i+y_i)^p m_i \;\le\; \alpha^{1-p} \sum x_i^p m_i \;+\; (1-\alpha)^{1-p} \sum y_i^p m_i$$

Equality holds iff

$$\frac{x}{y} = \frac{x_i}{y_i} = \frac{\alpha}{1-\alpha}$$

Proof. Because $f(x) = x^p$ is a convex function,

$$(\alpha a + (1-\alpha)b)^p < \alpha a^p + (1-\alpha) b^p$$

So setting $\alpha a = x$ and $(1-\alpha)b = y$, we get the first inequality. The second one is just an extention of the first one.

Theorem 3.4.10 (Minkowski) — If $x_1, x_2 \ldots x_n$, $y_1, y_2, \ldots y_n$ and $m_1, m_2, \ldots m_n$ be three sequence of real numbers and $p > 1$, then

$$\left(\sum (x_i+y_i)^p\, m_i\right)^{\frac{1}{p}} \le \left(\sum x_i^p m_i\right)^{\frac{1}{p}} + \left(\sum y_i^p m_i\right)^{\frac{1}{p}}$$

Equality holds iff (x_i) and (y_i) are proportional.

Proof. Let us write,

$$A = \left(\sum x_i^p m_i\right)^{\frac{1}{p}}, \; B = \left(\sum y_i^p m_i\right)^{\frac{1}{p}}$$

From the second equation of Lemma 3.4.9, for $\alpha \in (0,1)$ we have

$$\sum (x_i+y_i)^p m_i \le \alpha^{1-p} A^p + (1-\alpha)^{1-p} B^p$$

Setting α be such that $\dfrac{A}{B} = \dfrac{\alpha}{1-\alpha}$, by the equality case of the first equation of the lemma we get,

$$\sum (x_i+y_i)^p m_i \;\le\; \alpha^{1-p} A^p + (1-\alpha)^{1-p} B^p = (A+B)^p$$

Theorem 3.4.11 (Young's Inequality) — If $a, b > 0$ and $p, q > 0$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Equality occurs when $a^p = b^q$.

Proof. Consider the function $f(x) = e^x$, it is convex. So we have

$$e^{\frac{1}{p}x+\frac{1}{q}y} \leq e^{\frac{1}{p}x} + e^{\frac{1}{q}y}$$

If we let $a = e^{\frac{x}{p}}, b = e^{\frac{y}{q}}$, we are done. Equality occurs when $x = y$.

Theorem 3.4.12 (Weighted Power Mean) — Let $a_1, a_2, \dots a_n$ and $w_1, w_2, \dots + w_n$ be positive real numbers with $w_1 + w_2 \dots w_n = 1$. For real number r, we define,

$$P(r) = \begin{cases} (w_1 a_1^r + w_2 a_2^r \cdots + w_n a_n^r)^{1/r} & r \neq 0 \\ a_1^{w_1} a_2^{w_2} \dots a_n^{w_n} & r = 0 \end{cases}$$

If $r > s$, then $P(r) \geq P(s)$. Equality occurs iff $a_1 = a_2 = \cdots = a_n$

Proof. First we show that, $\lim_{r\to 0} P(r) = P(0)$. Using L'Hopital's law,

$$\begin{aligned}
\lim_{r\to 0} \ln P(r) &= \lim_{r\to 0} \frac{\ln \sum w_i a_i^r}{r} = \lim_{r\to 0} \frac{\frac{\sum w_i a_i^r \ln a_i}{\sum w_i a_i^r}}{1} \\
&= \lim_{r\to 0} \frac{\sum w_i a_i^r \ln a_i}{\sum w_i a_i^r} \\
&= \sum \ln a_i^{w_i} \\
&= \ln P(0)
\end{aligned}$$

First we prove that for positive $r < s$, we have $P(r) \leq P(s)$. Define the function

$$f : \mathbb{R}^+ \to \mathbb{R}^+ \quad f(x) = x^{\frac{s}{r}}$$

This is a convex function, and so by Jensen's Inequality we have

$$\begin{aligned}
f\left(\sum_{i=1}^n w_i a_i^r\right) &\leq \sum_{i=1}^n w_i f(a_i^r) \\
\implies \sqrt[\frac{r}{s}]{\sum_{i=1}^n w_i a_i^r} &\leq \sum_{i=1}^n w_i a_i^s
\end{aligned}$$

Theorem 3.4.13 (Cauchy-Schward Inequality) — For any real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$,

$$\left(a_1^2 + a_2^2 \ldots a_n^2\right)\left(b_1^2 + b_2^2 \ldots b_n^2\right) \geq \left(a_1 b_1 + a_2 b_2 \ldots a_n b_n\right)^2$$

with equality when there exist constants μ, λ not both zero such that for all $1 \leq i \leq n$, $\mu a_i = \lambda b_i$.

The inequality sometimes appears in the following form.

Theorem 3.4.14 (Cauchy-Schwarz Inequality Complex form) — Let $a_1, a_2 \ldots, a_n$ and $b_1, b_2 \ldots, b_n$ be complex numbers. Then

$$\left(\left|a_1^2\right| + \left|a_2^2\right| \ldots \left|a_n^2\right|\right)\left(\left|b_1^2\right| + \left|b_2^2\right| \ldots \left|b_n^2\right|\right) \geq \left|a_1 b_1 + a_2 b_2 \ldots a_n b_n\right|^2$$

Theorem 3.4.15 (Titu's Lemma) — For positive reals $a_1, a_2 \ldots a_n$ and $b_1, b_2 \ldots b_n$ the following holds:

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{\left(a_1 + a_2 + \cdots + a_n\right)^2}{\left(b_1 + b_2 + \cdots + b_n\right)}$$

Theorem 3.4.16 (Holder's Inequality) — If $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, \ldots, z_1, z_2, \ldots, z_n$ are nonnegative real numbers and $\lambda_a, \lambda_b, \ldots, \lambda_z$ are nonnegative reals with sum of 1, then

$$a_1^{\lambda_a} b_1^{\lambda_b} \ldots z_1^{\lambda_z} + \cdots + a_n^{\lambda_a} b_n^{\lambda_b} \ldots z_n^{\lambda_z} \leq \left(a_1 + \cdots + a_n\right)^{\lambda_a}\left(b_1 + \cdots + b_n\right)^{\lambda_b} \ldots \left(z_1 + \cdots + z_n\right)^{\lambda_z}$$

Theorem 3.4.17 (Popoviciu's inequality) — Let f be a convex function on and interval $I \in \mathbb{R}$. Then for any numbers $x, y, z \in I$,

$$f(x) + f(y) + f(z) \; + 3f\left(\frac{x+y+z}{3}\right) \; \geq 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{y+z}{2}\right) + 2f\left(\frac{z+x}{2}\right)$$

3.4.2 Tricks

Some tricks to try

1. Replace trigonometric functions by reals, and translate the problem

2. **Smoothing**, replace two variable while keeping something invariant, to make the inequality sharper.

3. **Convexity**, differentiate to check convexity, if the second derivative is positive on some interval, then the function is convex on that interval except probably at the endpoints, and concave otherwise.

   An example is $\ln \dfrac{1-x}{x}$. It is convex in $\left(0, \frac{1}{2}\right]$ and concave in $\left[\frac{1}{2}, 1\right)$

4. If there is product, and if the problem is 'ad-hoc'y, then apply $AM - GM$ and $\ln$ to see if there is something to play with.

Definition (Homogeneous Expression)— Expression $F(a_1, a_2 \dots a_n)$ is said to be homogeneous of degree k if and only if there exists real k such that for every $t > 0$ we have

$$t^k F(a_1, a_2 \dots a_n) = F(ta_1, ta_2 \dots ta_n)$$

If an expression is homogeneous, then the following can be assumed (only one at a time):

$\boxed{\sum_{i=1}^{n} a_i = 1}$ $\boxed{\prod_{i=1}^{n} a_i = 1}$ $\boxed{\sum_{i=1}^{n} a_i^2 = 1}$ $\boxed{\sum_{Cyc} a_i a_{i+1} = 1}$ $\boxed{a_1 = 1 \text{ or for some } i,\ a_i = 1}$

Theorem 3.4.18 (Substitutions) — Some common substitution techniques:

1. For the condition $abc = 1$, set $a = \dfrac{x}{y},\ b = \dfrac{y}{z},\ c = \dfrac{z}{x}$

2. If given $xyx = x + y + z + 2 \implies \dfrac{1}{x+1} + \dfrac{1}{y+1} + \dfrac{1}{z+1} = 1$ then there exists a, b, c such that

$$x = \frac{b+c}{a},\ y = \frac{a+c}{b},\ z = \frac{b+a}{c}$$

3. $2xyz + xy + yz + zx = 1$ is just the inverse of the previous condition.

4. $x^2 + y^2 + z^2 = xyz + 4$ and $|x|,\ |y|,\ |z| \geq 2$ implies the existence of a, b, c such that

$$abc = 1 \text{ and } x = a + \frac{1}{a},\ y = b + \frac{1}{b},\ z = c + \frac{1}{c}$$

   In fact even if only $\max(|x|, |y|, |z|) > 2$ is given, the result still holds.

Lemma 3.4.19 — The following inequality holds for every positive integer n

$$2\sqrt{n+1} - 2\sqrt{n} < \sqrt{\frac{1}{n}} < 2\sqrt{n} - 2\sqrt{n-1}$$

Lemma 3.4.20 — Given 4 posititve real numbers $a < b < c < d$. Call the score of a permutation a_1, a_2, a_3, a_4 of the four given reals be equal to the real

$$\left|\frac{a_1}{a_2} - \frac{a_3}{a_4}\right|$$

Then the minimum score over all permutaions is $\left|\frac{a}{c} - \frac{b}{d}\right|$

3.4.3 Problems

Problem 3.4.1 (APMO 1991 P3). Let $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$. Show that

$$\frac{a_1^2}{a_1+b_1} + \frac{a_2^2}{a_2+b_2} + \cdots + \frac{a_n^2}{a_n+b_n} \geq \frac{a_1+a_2+\cdots+a_n}{2}$$

Problem 3.4.2 (ISL 2007 A3). Let n be a positive integer, and let x and y be a positive real number such that $x^n + y^n = 1$. Prove that

$$\left(\sum_{k=1}^{n} \frac{1+x^{2k}}{1+x^{4k}}\right) \cdot \left(\sum_{k=1}^{n} \frac{1+y^{2k}}{1+y^{4k}}\right) < \frac{1}{(1-x)\cdot(1-y)}.$$

Solution. We need to turn the ugly fraction $\dfrac{1+x^{2k}}{1+x^{4k}}$ into something more usable. Also the right hand size looks almost like the sum of geometric series $\sum_{i=1}^{n} \frac{1}{x}$. Which motivates us to look for a t such that

$$\frac{1+x^{2k}}{1+x^{4k}} \leq \frac{1}{x^t}$$

But by Karamata's inequality, we know that $t = k$, as $x + x^3 \leq 1 + x^4$ for all positive x. Where equality holds iff $x = 1$

$$\begin{aligned}
\left(\sum_{k=1}^{n} \frac{1+x^{2k}}{1+x^{4k}}\right) \cdot \left(\sum_{k=1}^{n} \frac{1+y^{2k}}{1+y^{4k}}\right) &< \left(\sum_{k=1}^{n} \frac{1}{x^k}\right) \cdot \left(\sum_{k=1}^{n} \frac{1}{y^k}\right) \\
&= \frac{1-x^n}{(1-x)x^n} \cdot \frac{1-y^n}{(1-y)y^n} \\
&= \frac{y^n}{(1-x)x^n} \cdot \frac{x^n}{(1-y)y^n} = \frac{1}{(1-x)\cdot(1-y)}
\end{aligned}$$

Problem 3.4.3 (ISL 2009 A2). Let a, b, c be positive real numbers such that $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = a + b + c$. Prove that:

$$\frac{1}{(2a+b+c)^2} + \frac{1}{(a+2b+c)^2} + \frac{1}{(a+b+2c)^2} \leq \frac{3}{16}.$$

Problem 3.4.4 (ARO 2018 P11.2). Let $n \geq 2$ and $x_1, x_2, \ldots, x_n$ positive real numbers. Prove that

$$\frac{1+x_1^2}{1+x_1x_2} + \frac{1+x_2^2}{1+x_2x_3} + \ldots + \frac{1+x_n^2}{1+x_nx_1} \geq n$$

Solution. The inequality says sum is greater, so if the product is greater, then we are done by AM-GM.

Problem 3.4.5 (Turkey TST 2017 P5). For all positive real numbers a, b, c with $a+b+c=3$, show that

$$a^3b + b^3c + c^3a + 9 \geq 4(ab + bc + ca)$$

Solution. Always try the most simple ineq possible, AM-GM

Problem 3.4.6 (IMO 2012 P2). Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2a_3 \cdots a_n = 1$. Prove that

$$(1+a_2)^2(1+a_3)^3 \cdots (1+a_n)^n > n^n.$$

Solution. The main idea is to look for the ans of the ques, $(1+a_k)^k >=?$. We have k^{th} power. So if we can get a k term sum inside of the brackets, we can get a clean term for $?$ from AM-GM. And 1 seems like it's crying to be partitioned. So we write the term as $\left(a_k + \frac{1}{k-1} + \cdots + \frac{1}{k-1}\right)$

Solution. *Looks at the $a_2a_3 \ldots a_n = 1$ condition*
Hey, we have a substitution for this one, why not try it out...
darn it, i still have to do the partition thing to cancel out the powers $>($

Problem 3.4.7 (ISL 1998 A1). Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n < 1$. Prove that

$$\frac{a_1a_2 \cdots a_n \left[1 - (a_1 + a_2 + \cdots + a_n)\right]}{(a_1 + a_2 + \cdots + a_n)(1-a_1)(1-a_2)\cdots(1-a_n)} \leq \frac{1}{n^{n+1}}.$$

Solution. Simplifying and making it symmetric, we get to the inequality

$$\prod_{i=1} n\frac{1-a_i}{a_i} \geq n^{n+1}$$

Now approaching similarly as this problem, we get to the solution.

Problem 3.4.8 (ISL 2001 A1). Let a, b, c be positive real numbers so that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Solution. Substitute.

Problem 3.4.9 (ISL 1999 A1). Let $n \geq 2$ be a fixed integer. Find the least constant C such the inequality

$$\sum_{i<j} x_i x_j \left(x_i^2 + x_j^2\right) \leq C \left(\sum_i x_i\right)^4$$

holds for any $x_1, \ldots, x_n \geq 0$ (the sum on the left consists of $\binom{n}{2}$ summands). For this constant C, characterize the instances of equality.

Solution. Follow the ineq sign and remember AM-GM.

Problem 3.4.10 (ISL 2017 A1). Let $a_1, a_2, \ldots a_n, k$, and M be positive integers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = k \quad \text{and} \quad a_1 a_2 \cdots a_n = M$$

If $M > 1$, prove that the polynomial

$$P(x) = M(x+1)^k - (x+a_1)(x+a_2)\cdots(x+a_n)$$

has no positive roots.

Solution. The same idea used in this and this, spreading an expression to perform AM-GM on it.

Problem 3.4.11 (ISL 2016 A1). Let a, b, c be positive real numbers such that $\min(ab, bc, ca) \geq 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1.$$

Solution. Try the simpler version with two variables first. Now you can use this discovery with a little bit of cleverness to solve the problem. The clever part is to notice that 4 variable ineq is more solvable than a 3 variable one.

Problem 3.4.12 (ISL 2016 A2). Find the smallest constant $C > 0$ for which the following statement holds: among any five positive real numbers a_1, a_2, a_3, a_4, a_5 (not necessarily distinct), one can always choose distinct subscripts i, j, k, l such that

$$\left|\frac{a_i}{a_j} - \frac{a_k}{a_l}\right| \leq C.$$

> **Solution.** Simplify the problem to get the ans first. Think about what is the smallest such value for any given 4 positive reals.

Problem 3.4.13 (ISL 2004 A1). Let $n \geq 3$ be an integer. Let t_1, t_2, ..., t_n be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n) \left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n} \right).$$

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

> **Solution.** Easy solution by induction. For a more elegant solution, write the right side as sum of paired factors. Finding when the inequality breaks and relating it to the end statement.

Problem 3.4.14 (ISL 1996 A2). Let $a_1 \geq a_2 \geq \ldots \geq a_n$ be real numbers such that for all integers $k > 0$,

$$a_1^k + a_2^k + \ldots + a_n^k \geq 0.$$

Let $p = \max\{|a_1|, \ldots, |a_n|\}$. Prove that $p = a_1$ and that

$$(x - a_1) \cdot (x - a_2) \cdots (x - a_n) \leq x^n - a_1^n$$

for all $x > a_1$.

> **Solution.** After the first part, apply AM-GM on the whole left side, this not gonna work, since we can't bound $\sum a_i$ wrt a_1. So what if we divide both side by $(x - a_1)$ and then apply AM-GM?

3.4.3.1 Smoothing And Convexity

Some usual tricks

1. Bring x, y closer, keeping $x+y$ constant.
2. If we need to smoothen up the value xy, then take $\ln$ on both side.
3. Work with different variables.

Problem 3.4.15 (USAMO 1998 P3). Let $a_0, a_1, \cdots, a_n$ be numbers from the interval $(0, \pi/2)$ such that

$$\tan\left(a_0 - \frac{\pi}{4}\right) + \tan\left(a_1 - \frac{\pi}{4}\right) + \cdots + \tan\left(a_n - \frac{\pi}{4}\right) \geq n-1$$

Prove that

$$\tan a_0 \tan a_1 \cdots \tan a_n \geq n^{n+1}$$

Solution. Setting $b_i = \tan(a_i)$, we get

$$\sum_{i=0}^{n} \frac{b_i - 1}{b_i + 1} \geq n-1 \implies \sum_{i=0}^{n} \frac{1}{b_i+1} \leq 1$$

We need to show that geometric mean of b_i is greater than n. But we know that the harmonic mean is smaller than geometric mean. So we want to prove

$$n \leq \frac{n+1}{\sum \frac{1}{b_i}} \implies \sum_{i=0}^{n} \frac{1}{b_i} \leq \frac{n+1}{n}$$

If we let $\dfrac{1}{b_i+1} = c_i$, then $\dfrac{1}{b_i} = \dfrac{c_i}{1-c_i}$, which is a convex function, since

$$f(x) = \frac{x}{1-x}, \quad f'(x) = \frac{1}{(x-1)^2} \quad \text{which is increasing for } x \in [0,1)$$

So $\displaystyle\sum_{i=0}^{n} \frac{1}{b_i}$ is maximized when all the c_i's are equal to $\dfrac{1}{n}$.

Problem 3.4.16 (USAMO 1974 P2). Prove that if a, b, and c are positive real numbers, then

$$a^a b^b c^c \geq (abc)^{(a+b+c)/3}$$

Solution. Do $\ln$, and obvious by Jensen.

Problem 3.4.17 (India 1995). Let $x_1, x_2, ..., x_n > 0$ be real numbers such that $x_1 + x_2 + x_3 + ... + x_n = 1$. Prove the inequality

$$\frac{x_1}{\sqrt{1-x_1}} + \frac{x_2}{\sqrt{1-x_2}} + ... + \frac{x_n}{\sqrt{1-x_n}} \geq \sqrt{\frac{n}{n-1}}$$

Solution. easy smoothing

Problem 3.4.18 (Vietnam 1998). $x_1, x_2 ... x_n$ are real numbers such that

$$\frac{1}{x_1 + 1998} + \cdots + \frac{1}{x_n + 1998} = \frac{1}{1998}$$

Prove that

$$\frac{\sqrt[n]{x_1 \dots x_n}}{n-1} \geq 1998$$

Solution. Translate the given expression in a nicer way with new variables...

Problem 3.4.19 (IMO 1974 P2). The variables a, b, c, d, traverse, independently from each other, the set of positive real values. What are the values which the expression

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

takes?

Solution. $\frac{x}{y+c} \leq \frac{x}{y} \leq \frac{x}{y-c}$

Problem 3.4.20 (Bulgaria 1995). Given n real number $x_1, x_2, ..., x_n \in [0, 1]$. Prove the following inequality

$$(x_1 + x_2 + \cdots + x_n) - (x_1x_2 + x_2x_3 + \cdots + x_nx_1) \leq \left\lfloor \frac{n}{2} \right\rfloor$$

3.5 Ad-Hocs

Problem 3.5.1 (ISL 2014 A2). Define the function $f : (0, 1) \to (0, 1)$ by

$$f(x) = x^2, \text{ for } x \geq \frac{1}{2} \text{ and } x + \frac{1}{2}, \text{ for } x < \frac{1}{2}$$

Let a and b be two real numbers such that $0 < a < b < 1$. We define the sequences an and bn by $a_0 = a, b_0 = b$, and $a_n = f(a_{n-1}), b_n = f(b_{n-1})$ for $n > 0$. Show that there exists a positive integer n such that

$$(a_n - a_{n-1})(b_n - b_{n-1}) < 0$$

.

Problem 3.5.2 (ARO 2018 P10.1). Determine the number of real roots of the equation

$$|x| + |x + 1| + \cdots + |x + 2018| = x^2 + 2018x - 2019$$

Problem 3.5.3 (European Mathematics Cup 2018 P3). Find all $k > 1$ such that there exists a set S such that,

1. There exists $N > 0$ such that, if $x \in S, then x < N$.
2. If $a, b \in S$, and $a > b$, then $k(a - b) \in S$

> **Solution.** Find some constraints such as, $k(a - b) \not> a$, S has a smallest element. These two combined with a sequence of decreasing elements of S is enough to solve this problem.

Problem 3.5.4 (APMO 2018 P2). Let $f(x)$ and $g(x)$ be given by

$$f(x) = \frac{1}{x} + \frac{1}{x-2} + \frac{1}{x-4} + \cdots + \frac{1}{x-2018}$$

$$g(x) = \frac{1}{x-1} + \frac{1}{x-3} + \frac{1}{x-5} + \cdots + \frac{1}{x-2017}$$

Prove that $|f(x) - g(x)| > 2$ for any non-integer real number x satisfying $0 < x < 2018$.

> **Solution** [Subtract, manipulate]. See that for $\epsilon < 1$, for $x = 2k + \epsilon$ it's true, if it's true for $x = 2k + 1 + \epsilon$. Then for $x = 2k + 1 + \epsilon$, sustitute the value to find a common term appearing in all of those equations. So if that term were to be greater than 2, we would be done. How do we test that? Take the first derivative to find the minima.

Problem 3.5.5 (`ISL 2015 A1`). Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \geq \frac{ka_k}{a_k^2 + (k-1)}$$

for every positive integer k. Prove that $a_1 + a_2 + \ldots + a_n \geq n$ for every $n \geq 2$.

> **Solution.** Simplify the inequality. And then sum it up.

Problem 3.5.6 (`ISL 2015 A3`). Let n be a fixed positive integer. Find the maximum possible value of

$$\sum_{1 \leq r < s \leq 2n} (s - r - n)x_r x_s,$$

where $-1 \leq x_i \leq 1$ for all $i = 1, \cdots, 2n$.

> **Solution.** The expression is weird, and beautiful. Now if we write the expression as a single variable function, we see that $x_i \in 1, -1$. Now, there is $x_i x_j$ in the expression. So we need to multiply two expressions. Again, see that $s - r - n$ can be rewritten as $-(n - s) - (r)$. Now, how do we get an expression like $x_i x_j$ which can be found in squares, with a coefficient $(n - s)$ and r? By summing it up r times, simple.

Problem 3.5.7 (`ISL 2010 A3`). Let $x_1, \ldots, x_{100}$ be nonnegative real numbers such that $x_i + x_{i+1} + x_{i+2} \leq 1$ for all $i = 1, \ldots, 100$ (we put $x_{101} = x_1, x_{102} = x_2$). Find the maximal possible value of the sum $S = \sum_{i=1}^{100} x_i x_{i+2}$.

> **Solution.** Bound a small portion of the large sum.

Problem 3.5.8 (`ISL 2005 A3`). Four real numbers p, q, r, s satisfy $p+q+r+s = 9$ and $p^2 + q^2 + r^2 + s^2 = 21$. Prove that there exists a permutation (a, b, c, d) of (p, q, r, s) such that $ab - cd \geq 2$.

> **Solution.** Put p, q, r, s in order, find which permutation must satisfy the condition. Since we know, $\sum_{sym} pq = 30$, what can we say about the largest sum? How do we get $pq - rs$ with the equations given to us? What can we do to make the conditions met?

3.5.1 Factorization

Problem 3.5.9 (USAMO 2013 P4). Find all real numbers $x, y, z \geq 1$ satisfying

$$\min(\sqrt{x+xyz}, \sqrt{y+xyz}, \sqrt{z+xyz}) = \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

Solution. Replacement is never a bad idea to try out. But the main part is not replacement, but it's factorization. I don't yet know how to find such factorization, but let's find out.

3.5.2 Bounding

Problem 3.5.10 (ISL 2004 A2). Let a_0, a_1, a_2, ... be an infinite sequence of real numbers satisfying the equation

$$a_n = |a_{n+1} - a_{n+2}|$$

for all $n \geq 0$, where a_0 and a_1 are two different positive reals. Can this sequence a_0, a_1, $a_2 \ldots$ be bounded?

> **Solution.** In bounding problems, name the bounds, then focus on them. **Another thing: In reals, a variable does not necessarily need to be equal to the bound.**

3.5.3 Manipulation

Problem 3.5.11 (ISL 2011 A2). Determine all sequences $(x_1, x_2, \dots, x_{2011})$ of positive integers, such that for every positive integer n there exists an integer a with

$$\sum_{j=1}^{2011} j x_j^n = a^{n+1} + 1$$

> **Solution.** Manipulate the data.
> Since for all n the statement holds, we can guess there is bounding involved. Can we bound x_i or a? Tweak the terms and see if there is something nice to work with.

Problem 3.5.12 (ISL 2014 A1). Let $a_0 < a_1 < a_2 \dots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + a_2 + \cdots + a_n}{n} \leq a_{n+1}.$$

> **Solution.** Manipulate the data.
> Either directly, or using the "Δ method"

Problem 3.5.13 (Putnam 2011 A2). Let $a_1, a_2, \dots$ and $b_1, b_2, \dots$ be sequences of positive real numbers such that $a_1 = b_1 = 1$ and $b_n = b_{n-1}a_n - 2$ for $n = 2, 3, \dots$. Assume that the sequence (b_j) is bounded. Prove that

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 \cdots a_n}$$

converges, and evaluate S

> **Solution.** Look for partial sum. And in limit problems on contests, it is always a good idea to think about $\epsilon_n = l - S_n$

Problem 3.5.14 (Putnam 2013 A3). Suppose that the real numbers $a_0, a_1, \dots, a_n$ and x, with $0 < x < 1$, satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} = 0.$$

Prove that there exists a real number y with $0 < y < 1$ such that

$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

Solution. How do you show $\exists$ root $\in I$ if you don't want to construct it? Also those geometric sums are begging to be expanded...

Problem 3.5.15 (GQMO 2020 P3). We call a set of integers *special* if it has 4 elements and can be partitioned into 2 disjoint subsets $\{a, b\}$ and $\{c, d\}$ such that $ab - cd = 1$. For every positive integer n, prove that the set $\{1, 2, \ldots, 4n\}$ cannot be partitioned into n disjoint special sets.

Solution [Multiply 'em All]**.** Each special set must have exactly two evens and two odds. Now, consider the products of all even numbers and all odd numbers. Clearly the product of the odd parts of each set will be much smaller than the product of the even parts.

Problem 3.5.16 (Korean Summer Program TST 2016 1). Find all real numbers $x_1, \ldots, x_{2016}$ that satisfy the following equation for each $1 \le i \le 2016$. (Here $x_{2017} = x_1$.)

$$x_i^2 + x_i - 1 = x_{i+1}$$

Chapter 4

Geometry

4.1 First Portion

Lemma 4.1.1 — Let the incircle and excircle (opposite to A) of $\triangle ABC$ meet BC at D and E resp. Suppose F is the antipode of D wrt the incircle.

1. Prove that A, F, E are collinear.

2. M be the midpoint of DE. Prove that MI meets AD at it's midpoint.

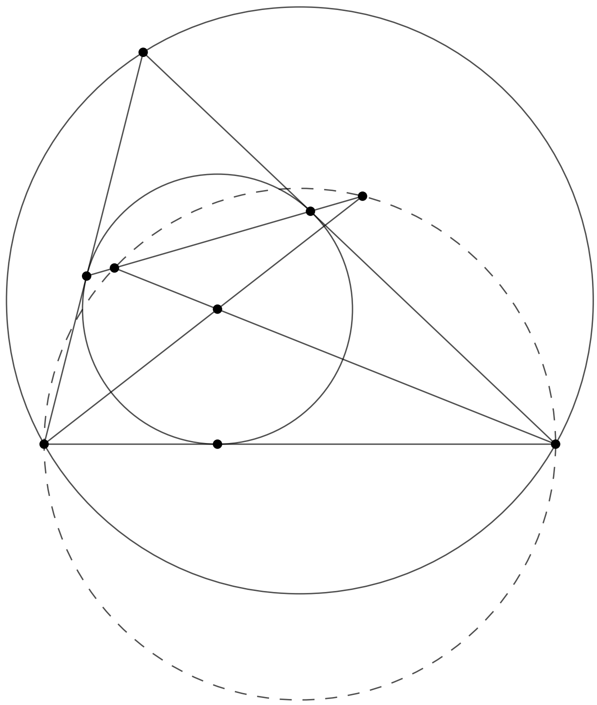


Figure 4.1.1

Lemma 4.1.2 — Let the incircle of $\triangle ABC$ meets AB and AC at X and Y resp. BI and CI meet XY at P and Q respectively. Prove that $BPQC$ is cyclic. (In fact $BP \perp CP$ and $BQ \perp CQ$)

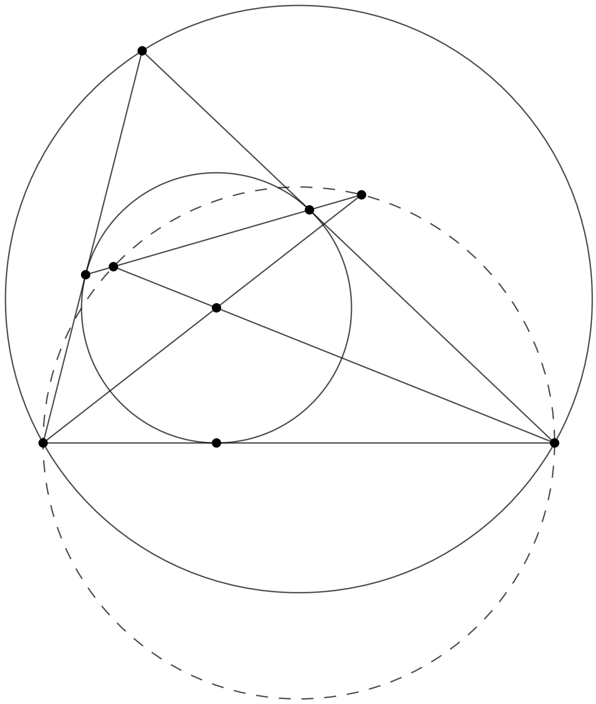


Figure 4.1.2

Lemma 4.1.3 — AD is an altitude of $\triangle ABC$. E, F are on AC, AB so that AD, BE, CF are concurrent. Prove $\angle EDA = \angle FDA$.

Lemma 4.1.4 — Let AD be an altitude of $\triangle ABC$ and $E \in \odot ABC$ so that $AE \parallel BC$. Prove that D, G, E are collinear where G is the centroid of $\triangle ABC$.

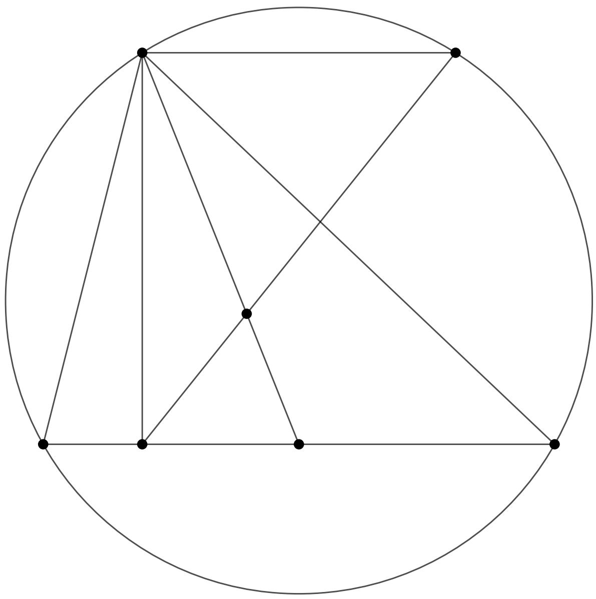


Figure 4.1.3

Problem 4.1.1. Let O be the circumcenter of $\triangle ABC$ and A', B', C' are reflections of O on BC, CA, AB resp. Prove that AA', BB', CC' are concurrent.

Problem 4.1.2. Let D, E are on sides AC, AB of $\triangle ABC$ resp. such that $BE = CD$. Let $\odot ABC \cap \odot ADE = P$. Prove that $PB = PC$.

Problem 4.1.3. Let a line PQ touch circle S_1 and S_2 at P and Q resp. Prove that the radical axis of S_1 and S_2 passes through the midpoint of PQ.

Problem 4.1.4. Let $\omega_1, \omega_2, omega_3$ are 3 circles. Prove that the 3 radical axis of ω_1 and ω_2,ω_2 and ω_3,ω_3 and ω_1 are either concurrent or parallel.

Problem 4.1.5. Two equal-radius circles ω_1 and ω_2 are centered at points O_1 and O_2 . A point X is reflected through O_1 and O_2 to get points A_1 and A_2. The tangents from A_1 to ω_1 touch ω_1 at points P_1 and Q_1, and the tangents from A_2 to ω_2 touch ω_2 at points P_2 and Q_2. If P_1Q_1 and P_2Q_2 intersect at Y , prove that Y is equidistant from A_1 and A_2.

Problem 4.1.6. Let BD, CE be the altitudes of $\triangle ABC$ and M be the midpoint of BC. If the ray MH meet $\odot ABC$ at point K, prove that AK, BC, DE are concurrent.

Problem 4.1.7. Two circle ω and Γ touches one another internally at P with ω inside of Γ. Let AB be a chord of Γ which touches ω at D. Let $PD \cap \Gamma = Q$. Prove that $QA = QB$.

Problem 4.1.8. Let AD be a symmedian of $\triangle ABC$ with D on $\odot ABC$. Let M be the midpoint of AD. Prove that $\angle BMD = \angle CMD$ and A, M, O, D are cyclic where O is the circumcenter of $\triangle ABC$.

Problem 4.1.9. Let A, B be two fixed points and let P be varying point such that $\frac{PA}{PB}$ is constant. Prove that the locus of P is a circle.

Problem 4.1.10. Prove that $r_1 + r_2 + r_3 = 4R + r$ $(R, r, r_1, r_2, r_3$ are the circumradius, inradius and three exradiuses respectively of a triangle)

Problem 4.1.11. Let M be the midpoint of the altitude BE in $\triangle ABC$ and suppose that the excircle opposite to B touches AC at Y. Then MY goes through the incenter I.

Problem 4.1.12. Let ABC be a triangle, and draw isosceles triangles $\triangle DBC, \triangle AEC, \triangle ABF$ external to $\triangle ABC$ (with $BC; CA; AB$ as their respective bases). Prove that the lines through $A; B; C$ perpendicular to $EF; FD; DE$, respectively, are concurrent.

Problem 4.1.13. In a triangle ABC we have $AB = AC$. A circle which is internally tangent with the circumscribed circle of the triangle is also tangent to the sides $AB; AC$ in the points P, respectively Q. Prove that the midpoint of PQ is the center of the inscribed circle of the triangle ABC

Problem 4.1.14. **Nagel Point** N: If the Excircles of ABC touch $BC; CA; AB$ at $D; E; F$, then the intersection point of $AD; BE; CF$ is called the **Nagel Point** N. Prove that

1. $I; G; N$ are collinear. (G centroid, I incenter.)
2. $GN = 2 \cdot IG$.
3. **Speiker center** S: The incircle of the medial triangle is called the Speiker circle, and its center is **Speiker center** S. Prove that S is the midpoint of IN.

4.2 Second Portion

Problem 4.2.1. Let PB and PC are tangent to $\odot ABC$. Let D, E, F are projection of A on BC, PB, PC resp. Prove that $AD^2 = AE \times AF$.

Problem 4.2.2. Let D and E are on AB and AC s.t $DE \parallel BC$. P is an arbitrary point inside $\triangle ADE$. $PB, PC \cap DE = F, G$. Let $\odot PDG \cap \odot PFE = Q$. Prove that A, P, Q are collinear.

Problem 4.2.3. Let AB and CD be chords in a circle of center O with A, B, C, D distinct , and with the lines AB and CD meeting at a right angle at point E. Let also M and N be the midpoints of AC and BD respectively . If $MN \perp OE$, prove that $AD \parallel BC$

Problem 4.2.4. Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at A and B. Let $M \in AB$. A line through M (different from AB) cuts circles $\mathcal{C}_1$ and $\mathcal{C}_2$ at Z, D, E, C respectively such that $D, E \in ZC$. Perpendiculars at B to the lines EB, ZB and AD respectively cut circle $\mathcal{C}_2$ in F, K and N. Prove that $KF = NC$.

Problem 4.2.5. Let D be a point on side AC of triangle ABC. Let E and F be points on the segments BD and BC respectively, such that $\angle BAE = \angle CAF$. Let P and Q be points on BC and BD respectively, such that EP and FQ are both parallel to CD. Prove that $\angle BAP = \angle CAQ$.

Problem 4.2.6. In the non-isosceles triangle ABC an altitude from A meets side BC in D . Let M be the midpoint of BC and let N be the reflection of M in D. The circumcirle of triangle AMN intersects the side AB in $P \neq A$ and the side AC in $Q \neq A$. Prove that $AN,\ BQ$ and CP are concurrent.

Problem 4.2.7. In triangle ABC, the interior and exterior angle bisectors of $\angle BAC$ intersect the line BC in D and E, respectively. Let F be the second point of intersection of the line AD with the circumcircle of the triangle ABC. Let O be the circumcenter of the triangle $ABC$and let D' be the reflection of D in O. Prove that $\angle D'FE = 90$.

Problem 4.2.8. Let $ABCD$ be a convex quadrilateral such that the line BD bisects the angle ABC. The circumcircle of triangle ABC intersects the sides AD and CD in the points P and Q, respectively. The line through D and parallel to AC intersects the lines BC and BA at the points R and S, respectively. Prove that the points P, Q, R and S lie on a common circle.

Problem 4.2.9. The incircle of triangle ABC touches BC, CA, AB at points A_1, B_1, C_1, respectively. The perpendicular from the incenter I to the median from vertex C meets the line A_1B_1 in point K. Prove that CK is parallel to AB.

Problem 4.2.10. Let X be an arbitrary point inside the circumcircle of a triangle ABC. The lines BX and CX meet the circumcircle in points K and L respectively. The line LK intersects BA and AC at points E and F respectively. Find the locus of points X such that the circumcircles of triangles AFK and AEL touch.

Problem 4.2.11. Let BD be a bisector of triangle ABC. Points I_a, I_c are the incenters of triangles ABD, CBD respectively. The line I_aI_c meets AC in point Q. Prove that $\angle DBQ = 90°$.

Problem 4.2.12. Given right-angled triangle ABC with hypotenuse AB. Let M be the midpoint of AB and O be the center of circumcircle ω of triangle CMB. Line AC meets ω for the second time in point K. Segment KO meets the circumcircle of triangle ABC in point L. Prove that segments AL and KM meet on the circumcircle of triangle ACM.

Problem 4.2.13. Let BN be median of triangle ABC. M is a point on BC. S lies on BN such that $MS \parallel AB$. P is a point such that $SP \perp AC$ and $BP \parallel AC$. MP cuts AB at Q. Prove that $QB = QP$.

Problem 4.2.14. Let $ABCD$ be a convex quadrilateral with AB parallel to CD. Let P and Q be the midpoints of AC and BD, respectively. Prove that if $\angle ABP = \angle CBD$, then $\angle BCQ = \angle ACD$.

Problem 4.2.15. Point P lies inside a triangle ABC. Let D, E and F be reflections of the point P in the lines BC, CA and AB, respectively. Prove that if the triangle DEF is equilateral, then the lines AD, BE and CF intersect in a common point.

Problem 4.2.16. Let $\triangle ABC$ be an acute angled triangle. The circle with diameter AB intersects the sides AC and BC at points E and F respectively. The tangents drawn to the circle through E and F intersect at P. Show that P lies on the altitude through the vertex C.

Problem 4.2.17. Let γ be circle and let P be a point outside γ. Let PA and PB be the tangents from P to γ (where $A, B \in \gamma$). A line passing through P intersects γ at points Q and R. Let S be a point on γ such that $BS \parallel QR$. Prove that SA bisects QR

Problem 4.2.18. Given is a convex quadrilateral $ABCD$ with $AB = CD$. Draw the triangles ABE and CDF outside $ABCD$ so that $\angle ABE = \angle DCF$ and $\angle BAE = \angle FDC$. Prove that the midpoints of $\overline{AD}$, $\overline{BC}$ and $\overline{EF}$ are collinear

Problem 4.2.19. Let P be a point out of circle C. Let PA and PB be the tangents to the circle drawn from C. Choose a point K on AB . Suppose that the circumcircle of triangle PBK intersects C again at T. Let P' be the reflection of P with respect to A. Prove that

$$\angle PBT = \angle P'KA$$

Problem 4.2.20. Consider a circle C_1 and a point O on it. Circle C_2 with center O, intersects C_1 in two points P and Q. C_3 is a circle which is externally tangent to C_2 at R and internally tangent to C_1 at S and suppose that RS passes through Q. Suppose X and Y are second intersection points of PR and OR with C_1. Prove that QX is parallel with SY.

Problem 4.2.21. In triangle ABC we have $\angle A = \frac{\pi}{3}$. Construct E and F on continue of AB and AC respectively such that $BE = CF = BC$. Suppose that EF meets circumcircle of $\triangle ACE$ in K. ($K \not\equiv E$). Prove that K is on the bisector of $\angle A$

Problem 4.2.22. In triangle ABC, $\angle A = 90°$ and M is the midpoint of BC. Point D is chosen on segment AC such that $AM = AD$ and P is the second meet point of the circumcircles of triangles $\Delta AMC, \Delta BDC$. Prove that the line CP bisects $\angle ACB$

Problem 4.2.23. Let C_1, C_2 be two circles such that the center of C_1 is on the circumference of C_2. Let C_1, C_2 intersect each other at points M, N. Let A, B be two points on the circumference of C_1 such that AB is the diameter of it. Let lines AM, BN meet C_2 for the second time at A', B', respectively. Prove that $A'B' = r_1$ where r_1 is the radius of C_1.

Problem 4.2.24. Given a triangle ABC, let P lie on the circumcircle of the triangle and be the midpoint of the arc BC which does not contain A. Draw a straight line l through P so that l is parallel to AB. Denote by k the circle which passes through B, and is tangent to l at the point P. Let Q be the second point of intersection of k and the line AB (if there is no second point of intersection, choose $Q = B$). Prove that $AQ = AC$.

Problem 4.2.25. Let $ABCD$ be a cyclic quadrilateral in which internal angle bisectors $\angle ABC$ and $\angle ADC$ intersect on diagonal AC. Let M be the midpoint of AC. Line parallel to BC which passes through D cuts BM at E and circle $ABCD$ in F ($F \neq D$). Prove that $BCEF$ is parallelogram

Problem 4.2.26. The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that, if $AD = BE$, then the triangle ABC is right-angled

Problem 4.2.27. $ABCD$ is a cyclic quadrilateral inscribed in the circle Γ with AB as diameter. Let E be the intersection of the diagonals AC and BD. The tangents to Γ at the points C, D meet at P. Prove that $PC = PE$

Problem 4.2.28. The quadrilateral $ABCD$ is inscribed in a circle. The point P lies in the interior of $ABCD$, and $\angle PAB = \angle PBC = \angle PCD = \angle PDA$. The lines AD and BC meet at Q, and the lines AB and CD meet at R. Prove that the lines PQ and PR form the same angle as the diagonals of $ABCD$

Problem 4.2.29. Let $ABCD$ be a cyclic quadrilateral with opposite sides not parallel. Let X and Y be the intersections of AB, CD and AD, BC respectively. Let the angle bisector of $\angle AXD$ intersect AD, BC at E, F respectively, and let the angle bisectors of $\angle AYB$ intersect AB, CD at G, H respectively. Prove that $EFGH$ is a parallelogram.

Problem 4.2.30. Triangle ABC is given with its centroid G and cicumcentre O is such that GO is perpendicular to AG. Let A' be the second intersection of AG with circumcircle of triangle ABC. Let D be the intersection of lines CA' and AB and E the intersection of lines BA' and AC. Prove that the circumcentre of triangle ADE is on the circumcircle of triangle ABC

Problem 4.2.31. Let M be the midpoint of the side AC of $\triangle ABC$. Let $P \in AM$ and $Q \in CM$ be such that $PQ = \frac{AC}{2}$. Let (ABQ) intersect with BC at $X \neq B$ and (BCP) intersect with BA at $Y \neq B$. Prove that the quadrilateral $BXMY$ is cyclic.

Problem 4.2.32. Let be given a triangle ABC and its internal angle bisector BD $(D \in BC)$. The line BD intersects the circumcircle Ω of triangle ABC at B and E. Circle ω with diameter DE cuts Ω again at F. Prove that BF is the symmedian line of triangle ABC.

Problem 4.2.33. ΔABC is a triangle such that $AB \neq AC$. The incircle of ΔABC touches BC, CA, AB at D, E, F respectively. H is a point on the segment EF such that $DH \perp EF$. Suppose $AH \perp BC$, prove that H is the orthocenter of ΔABC.

Problem 4.2.34. Let ABC be a triangle and let P be a point on the angle bisector AD, with D on BC. Let E, F and G be the intersections of AP, BP and CP with the circumcircle of the triangle, respectively. Let H be the intersection of EF and AC, and let I be the ntersection of EG and AB. Determine the geometric place of the intersection of BH and CI when P varies

Problem 4.2.35. Let $D; E; F$ be the points on the sides $BC; CA; AB$ respectively, of $\triangle ABC$. Let $P; Q; R$ be the second intersection of $AD; BE; CF$ respectively, with the cricumcircle of $\triangle ABC$.

Show that

$$\frac{AD}{PD} + \frac{BE}{QE} + \frac{CF}{RF} \geq 9$$

Problem 4.2.36. Points D and E lie on sides AB and AC of triangle ABC such that $DE \parallel BC$. Let P be an arbitrary point inside ABC. The lines PB and PC intersect DE at F and G, respectively. If O_1 is the circumcenter of PDG and O_2 is the circumcenter of PFE, show that $AP \parallel O_1O_2$.

Problem 4.2.37. Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$

Problem 4.2.38. Let O and I be the circumcenter and incenter of triangle ABC, respectively. Let ωA be the excircle of triangle ABC opposite to A; let it be tangent to AB, AC, BC at K, M, N, respectively. Assume that the midpoint of segment KM lies on the circumcircle of triangle ABC. Prove that $O; N; I$ are collinear.

Problem 4.2.39. Let $ABCD$ be a cyclic quadrilateral. Let $AB \cap CD = P$ and $AD \cap BC = Q$. Let the tangents from Q meet the circumcircle of $ABCD$ at E and F. Prove that $P; E; F$ are collinear.

4.3 Orthocenter–Circumcircle–NinePoint Circle

Notes everyone need to memorize by heart top to bottom:

1. Circles - Yufei Zhao
2. Big Picture - Yufei Zhao
3. POP - Yufei Zhao
4. 3 Lemmas - Yufei Zhao

Definition (The usual notations)— Unless stated otherwise, we assume that $\triangle ABC$ is an arbitrary triangle, with circumcenter O, orthocenter H, ω is the circumcircle. Usually, DEF is the orthic triangle in this chapter. MNP is the median triangle.

Lemma 4.3.1 (Collinearity with antipode and center) — Let A' be the antipode of A in $\odot ABC$. Let $BDEC$ be a cyclic quadrilateral with $D \in AB$ and $E \in AC$. Let P be the center of $BDEC$. Also, let $X = BE \cap CD$. Then A', P, X are collinear.

Solution. Using "The Big Picture" property to show that if $Q = \odot ADE \cap \odot ABC$, then P, X, Q collinear and $PQ \perp AQ$. Which implies that P, A', Q are collinear.

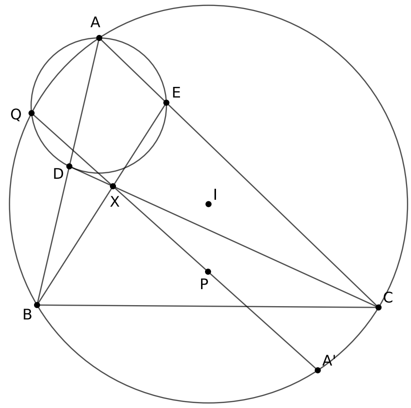


Figure 4.3.1

Lemma 4.3.2 (Weird point Y) — Let Y be a point on AC such that $\triangle CBY \sim CAB$. Let E bet the foot of B on AC, let N be the midpoint of AB. Then NE, CO, BE are concurrent.

Solution. Draw te circles, $BNKO$ and $BKYC$.

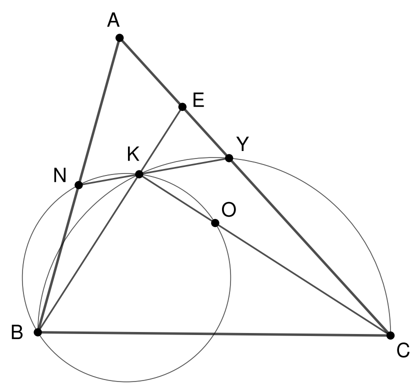


Figure 4.3.2

4.3.1 Problems

Problem 4.3.1 (Balkan MO 2017 P3). Let t_B and t_C be the tangents ω at B and C, they meet at L. The straight lines passing through B, C and parallel to AC, AB intersects t_C, t_B at points D, E respectively. $T = AC \cap \odot BDC, S = AB \cap \odot CBE$. Prove that ST, AL, and BC are concurrent.

> **Solution.** We have $\triangle ABT \sim \triangle ACB \sim ASC$, which leads to $BT \parallel CS$, and AL becomes the median of $\triangle ABT$.

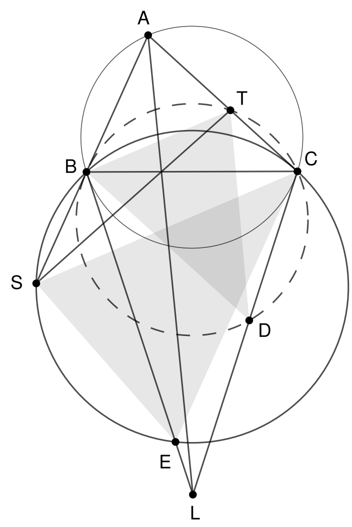


Figure 4.3.3

Problem 4.3.2 (USAMO 2014 P5). Let P be the second intersection of $\odot AHC$ with the internal bisector of $\angle BAC$. Let X be the circumcenter of triangle APB and Y the orthocenter of triangle APC. Prove that the length of segment XY is equal to the circumradius of triangle ABC.

> **Solution.** No length conditions given, yet we need to prove that two lengths are equal. *Parallelogram* !!! Just need to prove that $Y \in \odot ABC$ and $YD \perp AB$

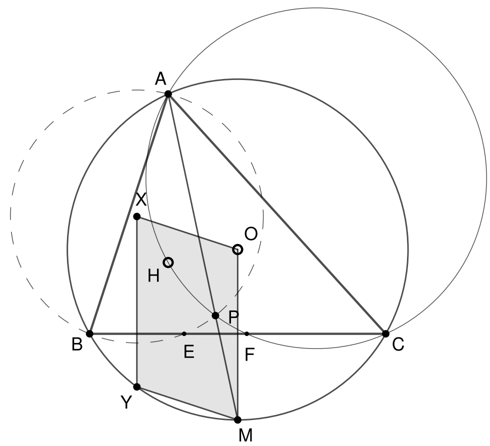


Figure 4.3.4

Problem 4.3.3 (Saudi Arab 2015). P is a point. (K) is the circle with diameter AP. (K) cuts CA, AB again at E, F. PH cuts (K) again at G. Tangent line at E, F of (K) intersect at T. M is midpoint of BC. L is the point on MG such that $AL \parallel MT$. Prove that $LA \perp LH$.

> **Solution** [Phantom Point]. Take $L' = MG \cap AZYH$, then use spiral similarity to show that $AL' \parallel MT$.

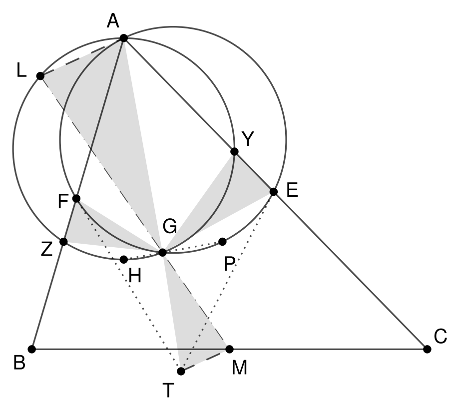


Figure 4.3.5

Problem 4.3.4 (Bewarish 1). Let DEF be the orthic triangle, and let $EF \cap BC = P$. Let the tangent at A to $\odot ABC$ meet BC at Q. Let T be the reflection of Q over P. Let K be the orthogonal projection of H on AM. Prove that $\angle OKT = 90$.

Solution. Spiral similarity from O to get rid of Q and T. Then spiral similarity again from P to get a trivial circle.

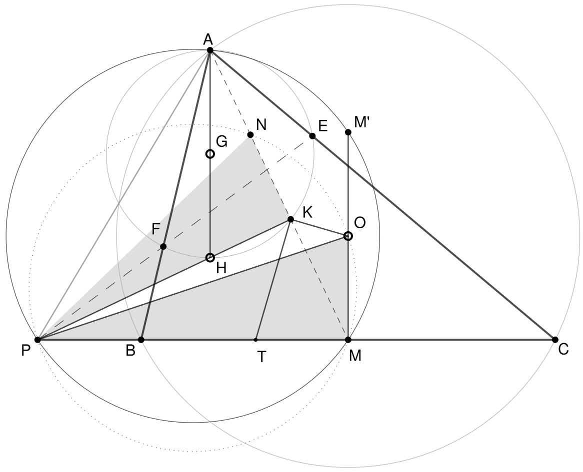


Figure 4.3.6

Problem 4.3.5 (Saudi Arab 2015). P lies on (O). The line passes through P and parallel to BC cuts CA at E. K is circumcenter of triangle PCE and L is nine point center of triangle PBC. Prove that the line passes through L and parallel to PK, always passes through a fixed point when P moves.

Solution [Construction]**.** Notice that if we reflect P over L to get P', then $OP = AH$ and $OP \perp BC$ where O is the circumcenter of $\odot ABC$. Which trivially implies that the line throught L passes throught the midpoint of $P'D$ where D is the reflection of H over BC.

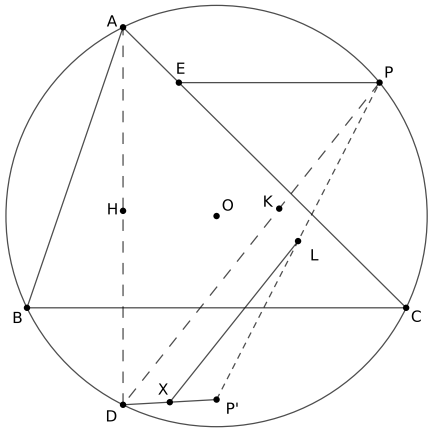


Figure 4.3.7

Problem 4.3.6 (Saudi Arab 2015). Altitude AH, H lies on BC. P is a point that lies on bisector $\angle BAC$ and P is inside triangle ABC. Circle diameter AP cuts (O) again at G. L is projection of P on AH. Assume that GL bisects HP. Prove that P is incenter of ABC.

Solution [Angle Chase]**.** Since $\angle APL = \angle ABD = \angle AGD$, G, L, M are collinear. Let $E \in BC$ and $PE \perp BC$. Then E also lies on DG.
Again we have, $\triangle DPE \sim \triangle DGP$. Which implies $DP = DB = DC$.

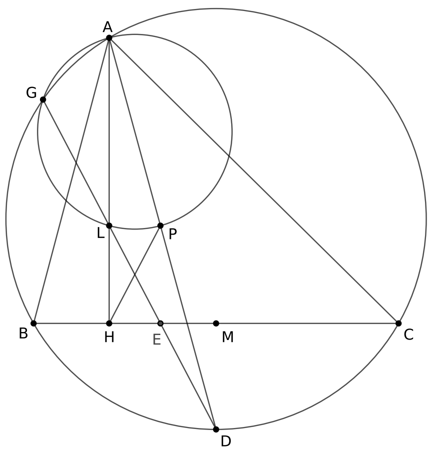


Figure 4.3.8

Problem 4.3.7 (Saudi Arab 2015). M lies on small arc $\overline{BC}$. P lies on AM. Circle diameter MP cuts (O) again at N. MO cuts circle diameter MP again at Q. AN cuts circle diameter MP again at R. Prove that $\angle PRA = \angle PQA$.

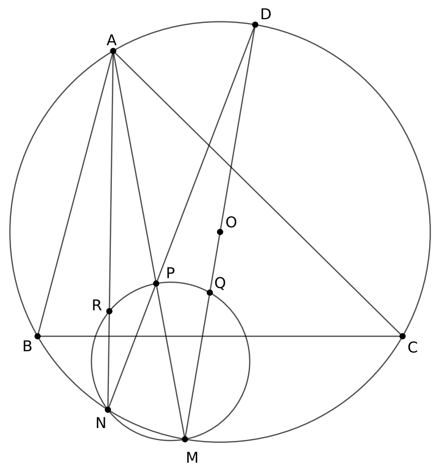


Figure 4.3.9

Solution [Angle Chase]**.** Let $MO \cap \odot ABC = D$. Becase $NP \perp MN$, we have N, P, D collinear, and $APQD$ cyclic. So, $\triangle APQ \sim \triangle ANM \sim \triangle APR$.

Problem 4.3.8 (Saudi Arab 2015). Let ABC be right triangle with hypotenuse BC, bisector BE, E lies on CA. Assume that circumcircle of triangle BCE cuts segment AB again at F. K is projection of A on BC. L lies on segment AB such that $BL = BK$. Prove that $\dfrac{AL}{AF} = \sqrt{\dfrac{BK}{BC}}$.

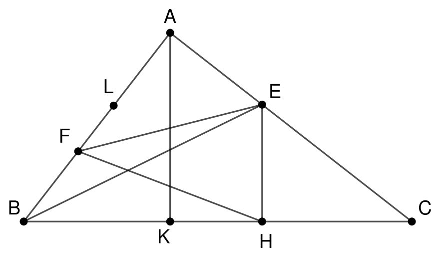


Figure 4.3.10

Problem 4.3.9 (Saudi Arab 2015). AD is diameter of (O). M, N lie on BC such that $OM \parallel AB$, $ON \parallel AC$. DM, DN cut (O) again at P, Q. Prove that $BC = DP = DQ$.

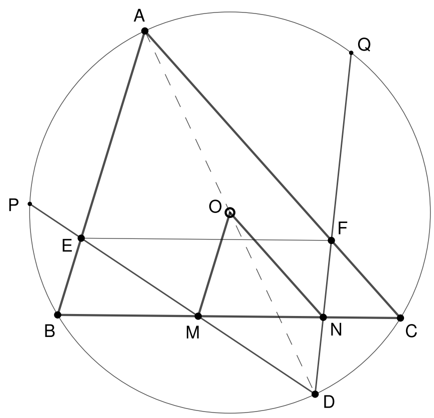


Figure 4.3.11

Solution. We will prove that $MB = MD$. Since $OD = OA$, we have $EM = MD$. And since $\triangle EBD$ is a right triangle, $MB = ME = MD$. And so the arcs PB and DC are equal.

Problem 4.3.10 (IGO 2017 A3)**.** Let O be the circumcenter of $\triangle ABC$. Line CO intersects the altitude through A at point K. Let P, M be the midpoints of AK, AC respectively. If PO intersects BC at Y , and the circumcircle of $\triangle BCM$ meets AB at X, prove that $BXOY$ is cyclic

Solution. We want to prove that $\angle POX = \angle AMX$. But we also notice by Reim's theorem that $\angle BOY = \angle BMC$, which leads to $\angle XPO = \angle XAM$.

Now, we want to show that

$$\frac{XA}{AP} = \frac{XM}{OM}$$

Which is simple length chase.

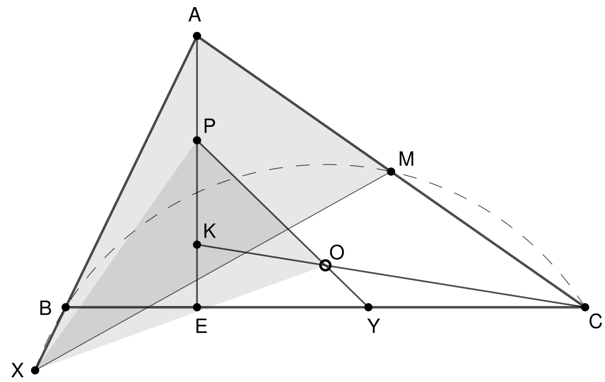


Figure 4.3.12

Problem 4.3.11. Let $EF \parallel BC$ be two points on the circumcircle. Let D be the center of HE, and let K be the point on AB for which $OK \parallel AF$. Prove that $DK \perp DC$.

Solution. The main part of the solution is to get rid of OK in some way. We take T to be the point such that $T \in OK$ and $\angle OTC = 90$. And let $S = CT \cap \odot ABC$. Then we have, $ES \perp AB$, and if we can show that D lies on $\odot PTC$, we are done. But then that is straightforward as $DT \parallel FS \parallel CH$ and $DP \parallel EQ$.

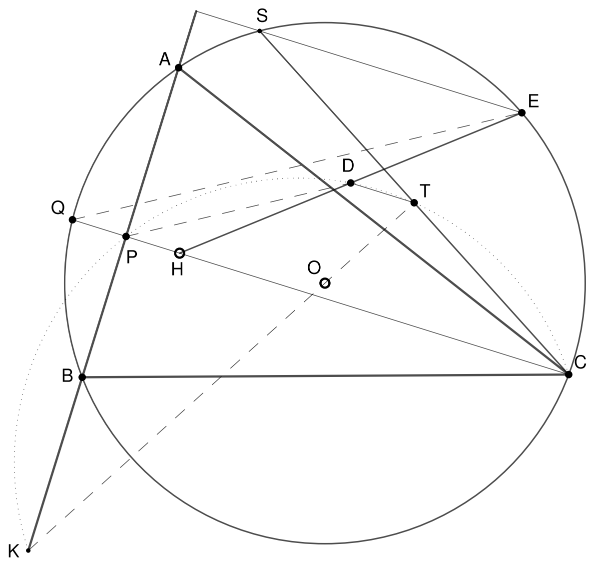


Figure 4.3.13

Problem 4.3.12 (Turkey TST 2018 P4). In a non-isosceles acute triangle ABC, D is the midpoint of BC. The points E and F lie on AC and AB, respectively, and the circumcircles of CDE and AEF intersect in P on AD. The angle bisector from P in triangle EFP intersects EF in Q. Prove that the tangent line to the circumcircle of AQP at A is perpendicular to BC.

Solution [angle chase]. Note that AQ is the angle bisector of $\angle BAC$. Using this fact, we can easily prove that $\angle HAQ = \angle APQ$.

Solution [inversion]. Inverting around A with radius $\sqrt{AP \cdot AD}$ sends EF to the circumcircle of ABC and P to D. Since AQ bisects $\angle BAC$, we have $DQ' \perp BC$.

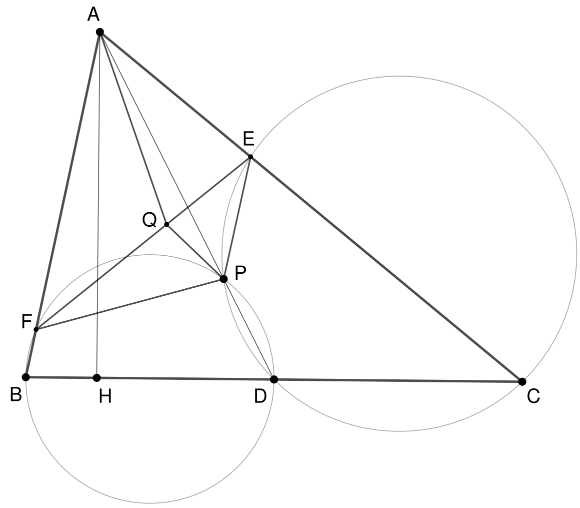


Figure 4.3.14

Problem 4.3.13 (USA Winter TST 2020 P2). Two circles Γ_1 and Γ_2 have common external tangents ℓ_1 and ℓ_2 meeting at T. Suppose ℓ_1 touches Γ_1 at A and ℓ_2 touches Γ_2 at B. A circle Ω through A and B intersects Γ_1 again at C and Γ_2 again at D, such that quadrilateral $ABCD$ is convex.

Suppose lines AC and BD meet at point X, while lines AD and BC meet at point Y. Show that T, X, Y are collinear.

Solution [Radical Axis]. It is easy to see that X lies on the radical axis of Γ_1 and Γ_2. Let $B' = l_1 \cap \Gamma_2$ and $A' = l_2 \cap \Gamma_1$. Let $C' = A'X \cap \Gamma_1$ and $D' = B'X \cap \Gamma_2$. Let $A'C \cap AC' = Z$.

We have $AD'CB'$ and $A'DC'B$ cyclic. Also T, D, C' and T, D', C are collinear. Which implies $A'D'CB$ and $ADC'B'$ are cyclic too.

Applying pascal on $AAC'CA'A'$, we have T, X, Z are collinear.

Now, it is easy to see that Z, Y, T lie on the radical axis of $A'D'CB$ and $ADC'B'$. So we have T, X, Y, Z collinear.

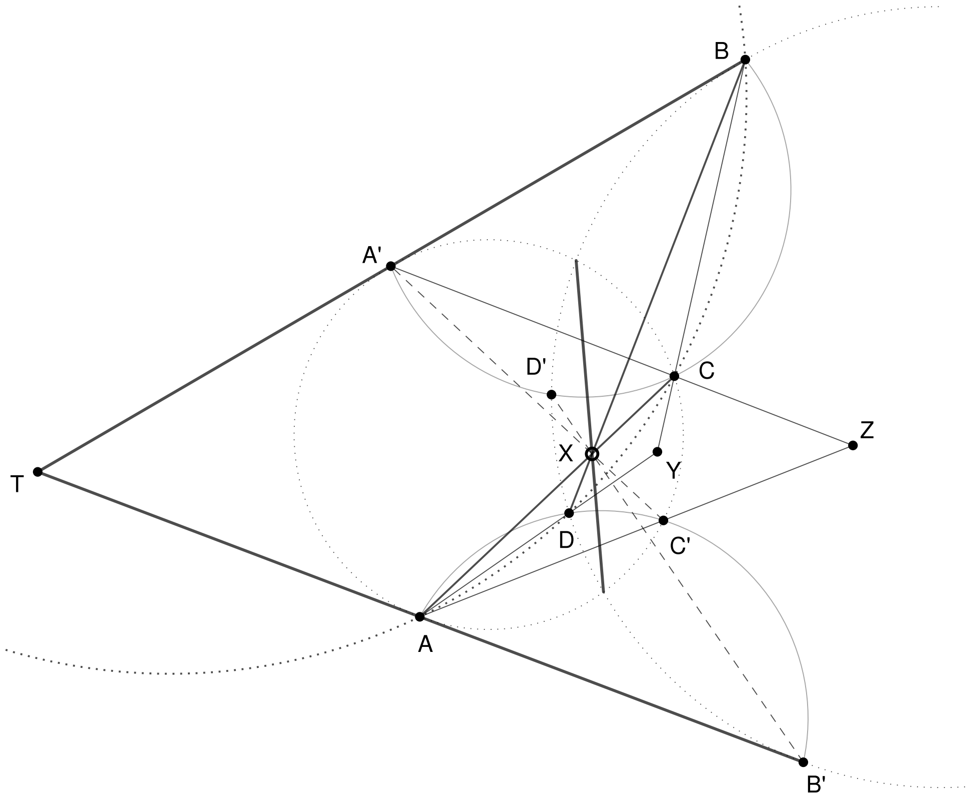


Figure 4.3.15

Solution [mOvInG pOiNtS, by shawnee03]. Fix Γ_1 and Γ_2 (and hence ℓ, T, A, B) and animate X linearly on ℓ. Then

- C moves projectively on Γ_1 (it is the image of the perspectivity through A from ℓ to Γ_1) and thus has degree 2, and similarly for D.
- $\overline{AD}$ has degree at most $0+2=2$, and similarly for $\overline{BC}$.
- $Y = \overline{AD} \cap \overline{BC}$ has degree at most $2+2=4$.
- The collinearity of T, X, Y has degree at most $0+1+4=5$.

Thus it suffices to verify the problem for six different choices of X. We choose:

- $\ell \cap \ell_1$: here Y approaches A as X approaches $\ell \cap \ell_1$.
- $\ell \cap \ell_2$: here Y approaches B as X approaches $\ell \cap \ell_2$.
- $\ell \cap \overline{AB}$: here Y approaches $\ell \cap \overline{AB}$ as X approaches $\ell \cap \overline{AB}$.
- the point at infinity along ℓ: here $Y = T$.
- the two intersections of Γ_1 and Γ_2: here $Y = X$.

(The final two cases may be chosen because we know that there exists a choice of A, B, C, D for which $ABCD$ is convex; this forces Γ_1 and Γ_2 to intersect.)

Generalization 4.3.13.1 (USA Winter TST 2020 P2). Let $ABCD$ be a cyclic quadrilateral, $X = AC \cap BD$, and $Y = AB \cap CD$. Let T be a point on line XY, Γ_1 be the circle through A and C tangent to TA, and Γ_2 be the circle through B and D tangent to TD. Then Γ_1 and Γ_2 are viewed at equal angles from T.

Solution [Length Chase, by a1267ab]. If the radiuses of Γ_1 and Γ_2 are r_1, r_2, then we have to show,

$$\frac{TA}{r_1} = \frac{TD}{r_2}$$

We have,

$$r_1 = \frac{AB}{2\sin\angle TAB},\ r_2 = \frac{CD}{2\sin\angle TDC}$$

To get the sine ratios, we compare the areas of $\triangle TAB$ and $\triangle TDC$. We have,

$$\frac{TA \cdot AB}{TD \cdot CD} \frac{\sin\angle TAB}{\sin\angle TDC} = \frac{[TAB]}{[TCD]} = \frac{[XAB]}{[XCD]} = \frac{AB^2}{CD^2}$$

$$\implies \frac{r_1}{TA} = \frac{r_2}{TD}$$

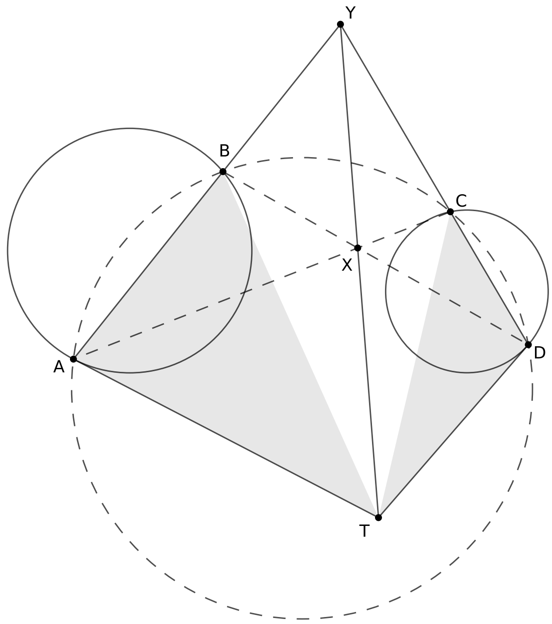


Figure 4.3.16

Problem 4.3.14 (USAJMO 2018 P3). Let $ABCD$ be a quadrilateral inscribed in circle ω with $\overline{AC} \perp \overline{BD}$. Let E and F be the reflections of D over lines BA and BC, respectively, and let P be the intersection of lines BD and EF. Suppose that the circumcircle of $\triangle EPD$ meets ω at D and Q, and the circumcircle of $\triangle FPD$ meets ω at D and R. Show that $EQ = FR$.

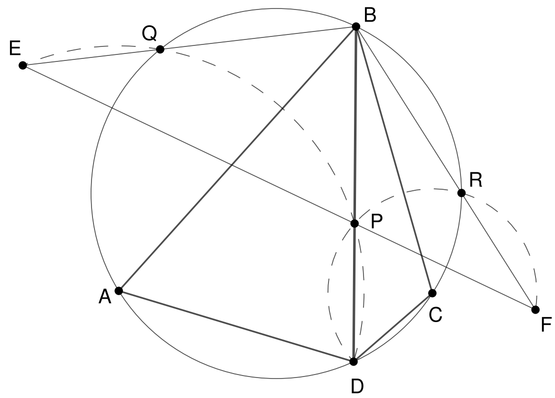


Figure 4.3.17

Problem 4.3.15 (IRAN 3rd Round 2016 P1). Let ABC be an arbitrary triangle, P is the intersection point of the altitude from C and the tangent line from A to the circumcircle. The bisector of angle A intersects BC at D. PD intersects AB at K, if H is the orthocenter then prove : $HK \perp AD$

Solution. Finding a set of Collinear points.

Problem 4.3.16. Let $\triangle ABC$ be a triangle. F, G be arbitrary points on AB, AC. Take D, E midpoint of BF, CG. Show that the nine-point centers of $\triangle ABC$, $\triangle ADE$, $\triangle AFG$ are collinear.

Problem 4.3.17 (IGO 2017 A4). Three circles W_1, W_2 and W_3 touches a line l at A, B, C respectively (B lies between A and C). W_2 touches W_1 and W_3. Let l_2 be the other common external tangent of W_1 and W_3. l_2 cuts W_2 at X, Y. Perpendicular to l at B intersects W_2 again at K. Prove that KX and KY are tangent to the circle with diameter AC.

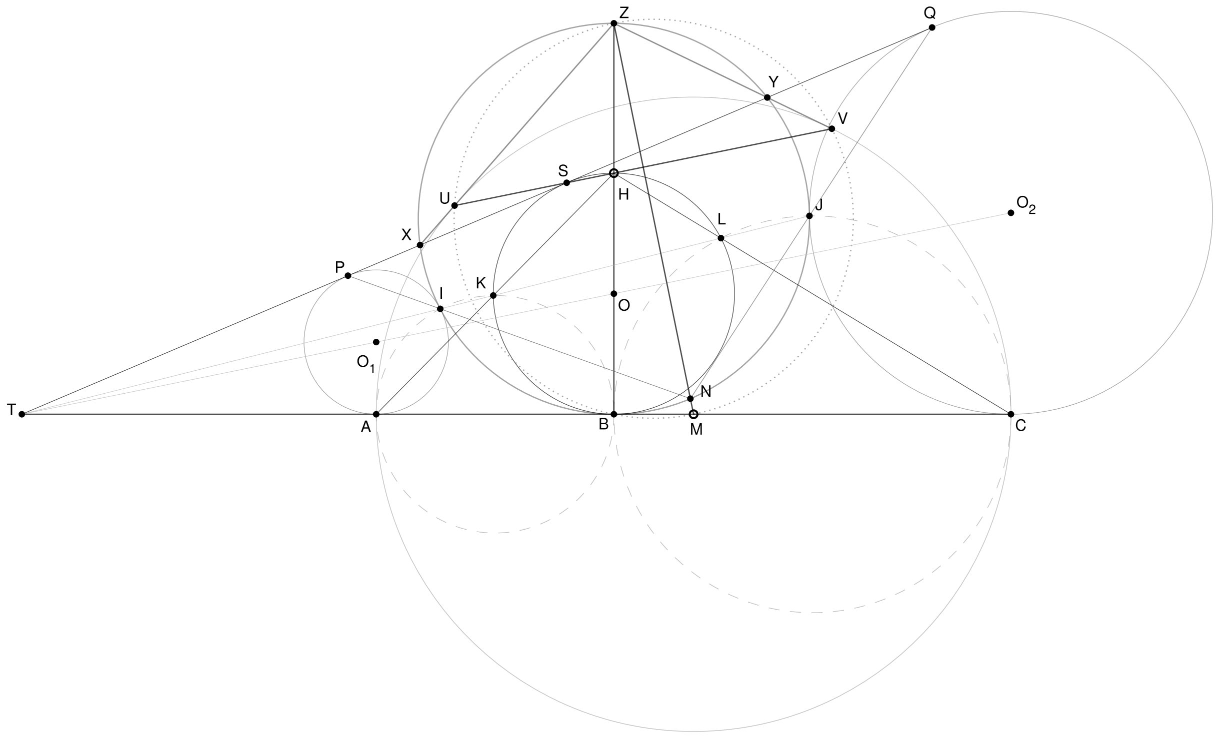


Figure 4.3.18

Solution. Using the names of the vertices in the diagram, we let UV be a segment parallel to O_1, O_2. Step by step we prove that

1. O is the center of SHB, then O, O_1, O_2 are collinear.
2. ZM bisects $\angle XZY$.
3. $BUZVM$ is cyclic.
4. $ZU^2 = ZH.ZB$.

Problem 4.3.18 (IGO 2017 A2). We have six pairwise non-intersecting circles that the radius of each is at least one (no circle lies in the interior of any other circle). Prove that the radius of any circle intersecting all the six circles, is at least one.

> **Solution.** We first expand the circles so that they touch each other in a ring like shape. Then we take the largest diameter of the convex hexagon with the centers. We show that any circle that intersects those two cirlces must have radius at least 1.

Problem 4.3.19 (IGO 2017 A4). Quadrilateral $ABCD$ is circumscribed around a circle. Diagonals AC, BD are not perpendicular to each other. The angle bisectors of angles between these diagonals, intersect the segments AB, BC, CD and DA at points K, L, M and N. Given that $KLMN$ is cyclic, prove that so is $ABCD$.

> **Solution.** If we let K', L', M', N' be the points where the incenter touches the sides, then we wish to prove that $K = K'$ and so on. To prove this, we first prove that KL, MN, AC are concurrent.
>
> Then we prove that $K'L'$ and $M'N'$ also passes through the same point. This lets us use the lemmas of complete cyclic quadrilaterals.

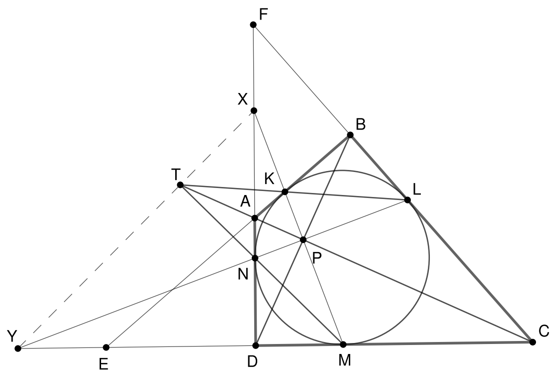


Figure 4.3.19

Problem 4.3.20 (ARO 2018 P10.2). Let $\triangle ABC$ be an acute-angled triangle with $AB < AC$. Let M and N be the midpoints of AB and AC, respectively; let AD be an altitude in this triangle. A point K is chosen on the segment MN so that $BK = CK$. The ray KD meets the circumcircle Ω of ABC at Q. Prove that C, N, K, Q are concyclic.

Problem 4.3.21 (ARO 2014 P9.4). Let M be the midpoint of the side AC of acute-angled triangle ABC with $AB > BC$. Let Ω be the circumcircle of ABC. The tangents to Ω at the points A and C meet at P, and BP and AC intersect at S. Let AD be the altitude of the triangle ABP and ω the circumcircle of the triangle CSD. Suppose ω and Ω intersect at $K \neq C$. Prove that $\angle CKM = 90°$.

Problem 4.3.22 (APMO 1999 P3). Let Γ_1 and Γ_2 be two circles intersecting at P and Q. The common tangent, closer to P, of Γ_1 and Γ_2 touches Γ_1 at A and Γ_2 at B. The tangent of Γ_1 at P meets Γ_2 at C, which is different from P, and the extension of AP meets BC at R. Prove that the circumcircle of triangle PQR is tangent to BP and BR.

Problem 4.3.23 (Simurgh 2019 P2). Let ABC be an isosceles triangle, $AB = AC$. Suppoe that Q is a point such that $AQ = AB,\ AQ||BC$. Let P be the foot of perpendicular line from Q to BC. Prove that the circle with diameter PQ is tangent to the circumcircle of ABC.

Problem 4.3.24 (European Mathematics Cup 2018 P2). Let ABC be a triangle with $|AB| < |AC|$. Let k be the circumcircle of $\triangle ABC$ and let O be the center of k. Point M is the midpoint of the arc $\widehat{BC}$ of k not containing A. Let D be the second intersection of the perpendicular line from M to AB with k and E be the second intersection of the perpendicular line from M to AC with k.

Points X and Y are the intersections of CD and BE with OM respectively. Denote by k_b and k_c circumcircles of triangles BDX and CEY respectively. Let G and H be the second intersections of k_b and k_c with AB and AC respectively. Denote by k_a the circumcircle of triangle AGH.

Prove that O is the circumcenter of $\triangle O_aO_bO_c$, where O_a, O_b, O_c are the centers of k_a, k_b, k_c respectively.

Problem 4.3.25 (RMM 2019 P2). Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. Let E be the midpoint of AC. Denote by ω and Ω the circumcircles of the triangles ABE and CDE, respectively. Let P be the crossing point of the tangent to ω at A with the tangent to Ω at D. Prove that PE is tangent to Ω.

Problem 4.3.26 (IGO 2018 A5). $ABCD$ is a cyclic quadrilateral. A circle passing through A, B is tangent to segment CD at point E. Another circle passing through C, D is tangent to AB at point F. Point G is the intersection point of AE, DF, and point H is the intersection point of BE, CF. Prove that the incenters of triangles AGF, BHF, CHE, DGE lie on a circle.

Solution [juckter]. The cases where two opposite sides of $ABCD$ are parallel are easily dealt with. Let $X = AB \cap CD$. Then $XE^2 = XA \cdot XB = XC \cdot XD = XF^2$, so $XE = XF$. Reflect E through X onto E', and notice that $XE^2 = XC \cdot XD$ implies $(C, D; E, E') = -1$. Because $\angle EFE' = 90^\circ$ (which follows from $XE = XF = XE'$) it follows that FE bisects $\angle CFD$ and analogously EF bisects $\angle AEB$. It then follows easily that G and H are symmetric about EF.

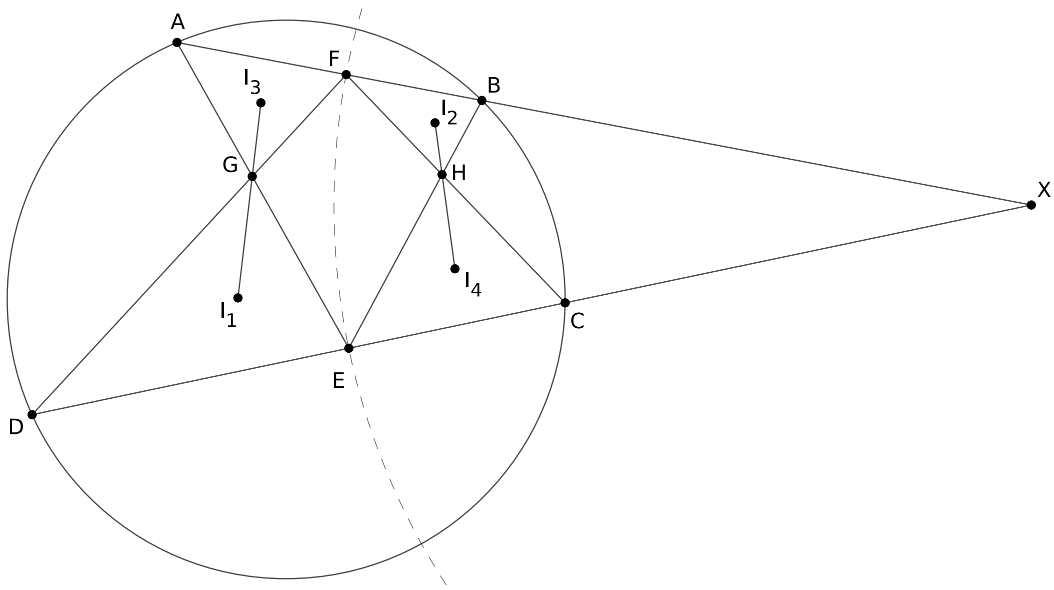


Figure 4.3.20: Problem 4.3.26 IGO 2018 A5

Now let I_1, I_2, I_3 and I_4 be the incenters of AGF, DGE, CHE, BHF respectively. Then I_1I_2 and I_3I_4 are the external bisectors of angles EGF and EHF respectively, and by symmetry about EF these lines intersect at a (possibly ideal) point $X \in EF$.

Finally, we may angle chase to find that E, I_1, I_2, F and E, I_3, I_4, F are quadruples of concyclic points. If I_1I_2 is parallel to I_3I_4 then we may easily conclude by symmetry about the perpendicular bisector of EF. Otherwise by Power of a Point from X we have $XI_1 \cdot XI_2 = XE \cdot XF = XI_3 \cdot XI_4$, so I_1, I_2, I_3, I_4 are concyclic, as desired.

Problem 4.3.27 (`ISL 2011 G8`). Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.

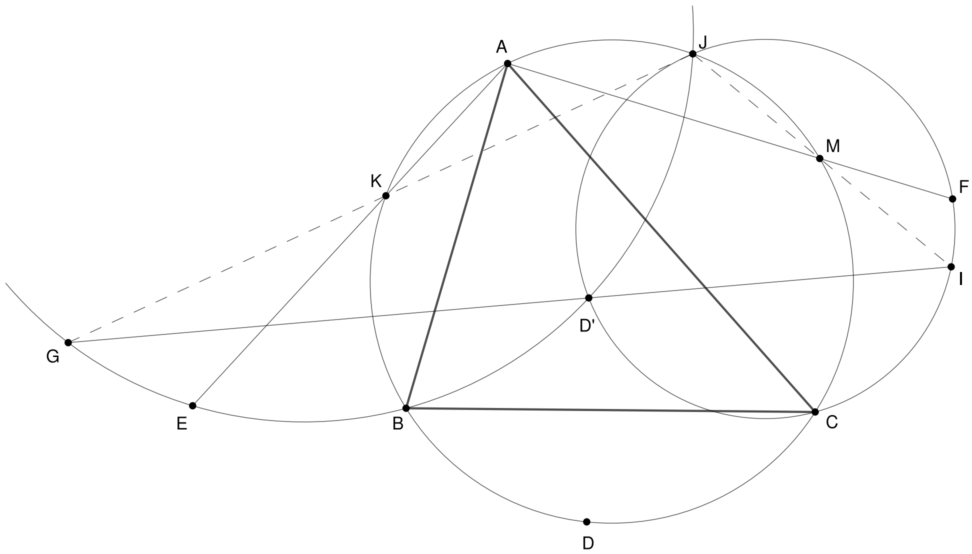


Figure 4.3.21

Solution. The main problem here are the reflected lines. We need to somehow know more about them. So we come up with some ways to construct the three lines without drawing the tangent l, which leads us to the reflection of D over the three sides idea. And after doing some angle chasing to find out the angles of the triangle, we begin to see relationships between the reflection points and the vertices of the triangle.

Problem 4.3.28 (ELMO 2019 P3)**.** Let ABC be a triangle such that $\angle CAB > \angle ABC$, and let I be its incentre. Let D be the point on segment BC such that $\angle CAD = \angle ABC$. Let ω be the circle tangent to AC at A and passing through I. Let X be the second point of intersection of ω and the circumcircle of ABC. Prove that the angle bisectors of $\angle DAB$ and $\angle CXB$ intersect at a point on line BC.

Solution [Angle Chase]**.** Suppose the bisector of $\angle BAD$ meet BC at G'. Then we have,

$$\angle BG'A = \frac{\angle A - \angle B}{2}$$
$$\therefore \angle CG'A = \angle B + \angle BG'A$$
$$= \frac{A+B}{2}$$

$$\implies CG' = CA$$
$$\therefore \angle G'ID = \angle B$$

Now, let M be the midpoint of the minor arc BC. Let $G = XM \cap BC$. So we have

$$\triangle MGI \sim \triangle MIX \implies \angle MIG = \angle MXI$$

Let $XI \cap \odot ABC = N \neq X$. Since AC is tangent to $\odot AXI$, $NC \parallel AM$. Which means

$$\angle MXI = \angle B = \angle MIG$$

Which completes our proof by implying that $G' \equiv G$.

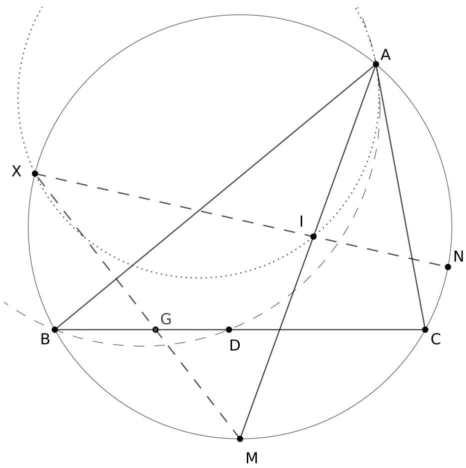


Figure 4.3.22

Problem 4.3.29 (ISL 2014 G5). Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90°$. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and

$$\angle CHS - \angle CSB = 90°, \quad \angle THC - \angle DTC = 90°.$$

Prove that line BD is tangent to the circumcircle of triangle TSH.

Solution. First construct using nice circles, then prove the center is on AH using angle bisector theorem.

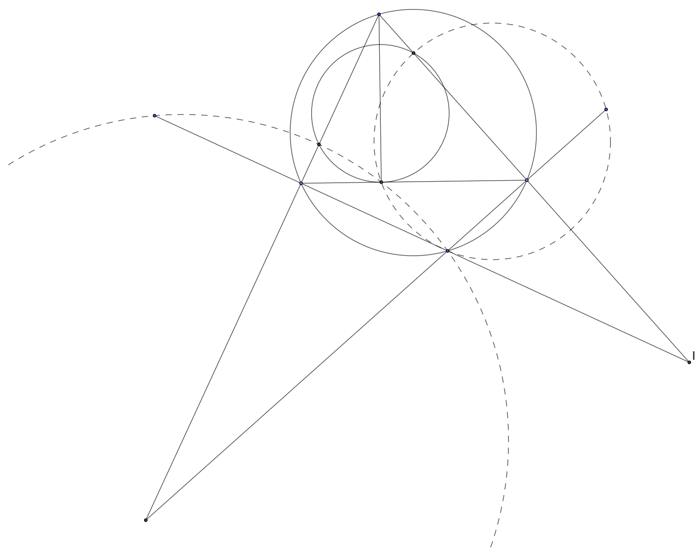


Figure 4.3.23: Construction

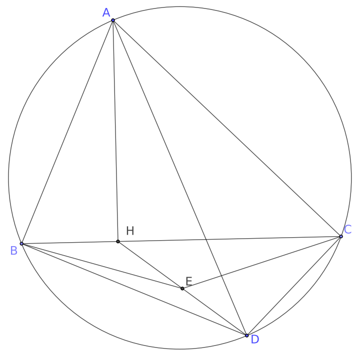


Figure 4.3.24: Lemma

Problem 4.3.30 (ISL 2014 G7). Let ABC be a triangle with circumcircle Ω and incentre I. Let the line passing through I and perpendicular to CI intersect the segment BC and the arc BC (not containing A) of Ω at points U and V , respectively. Let the line passing through U and parallel to AI intersect AV at X, and let the line passing through V and parallel to AI intersect AB at Y . Let W and Z be the midpoints of AX and BC, respectively. Prove that if the points I, X, and Y are collinear, then the points I, W, and Z are also collinear.

Solution. Draw a nice diagram, and use the parallel property to find circles.

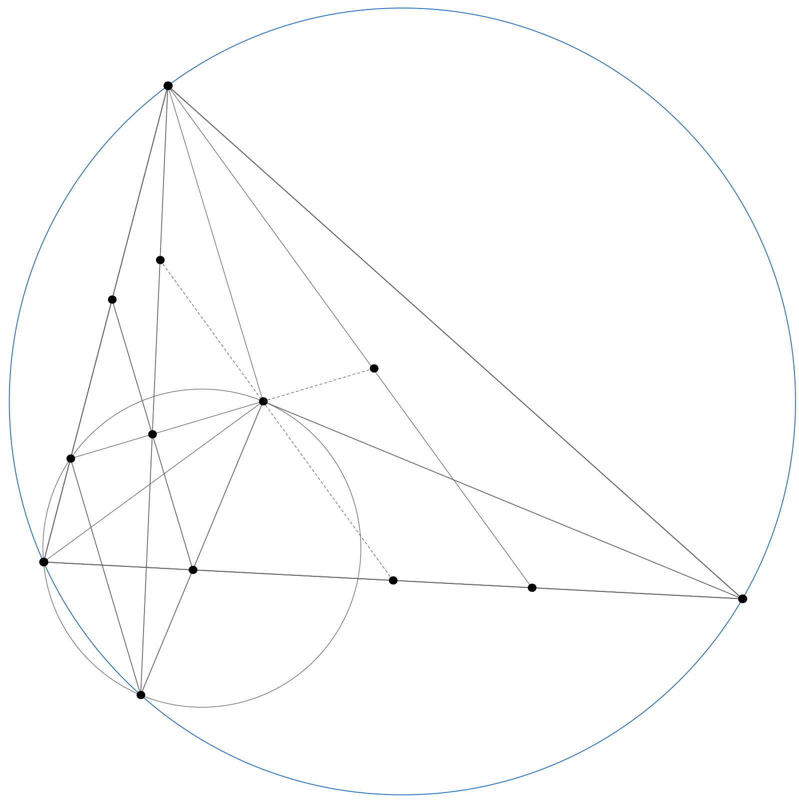


Figure 4.3.25: ISL 2014 G7

Problem 4.3.31 (ISL 2015 G6). Let ABC be an acute triangle with $AB > AC$. Let Γ be its cirumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = 90°$ and let K be the point on Γ such that $\angle HKQ = 90°$. Assume that the points A, B, C, K and Q are all different and lie on Γ in this order.

Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

Solution. Draw the tangent line, and find angles.

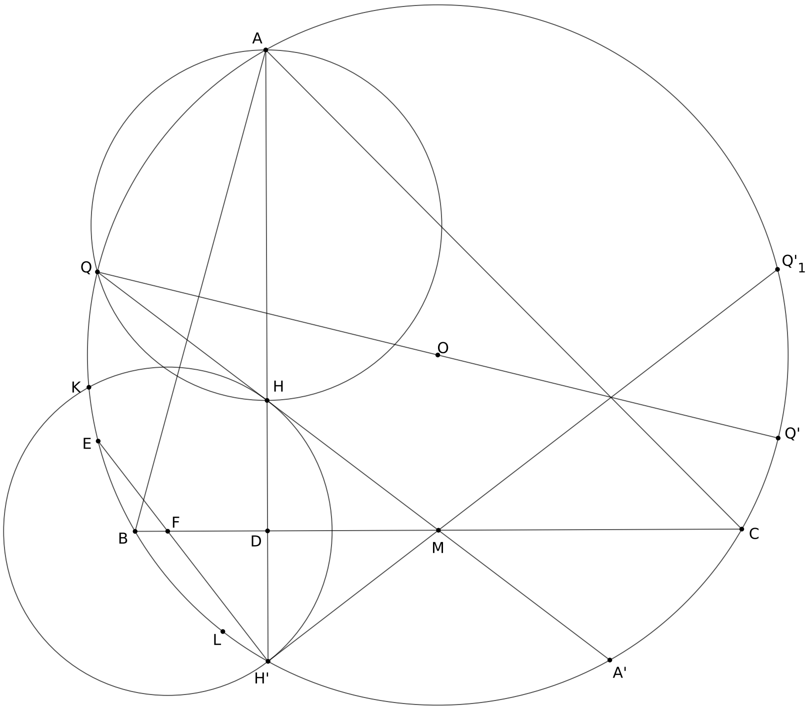


Figure 4.3.26: ISL 2015 G6

Problem 4.3.32 (ISL 2015 G5). Let ABC be a triangle with $CA \neq CB$. Let D, F, and G be the midpoints of the sides AB, AC, and BC respectively. A circle Γ passing through C and tangent to AB at D meets the segments AF and BG at H and I, respectively. The points H' and I' are symmetric to H and I about F and G, respectively. The line $H'I'$ meets CD and FG at Q and M, respectively. The line CM meets Γ again at P. Prove that $CQ = QP$.

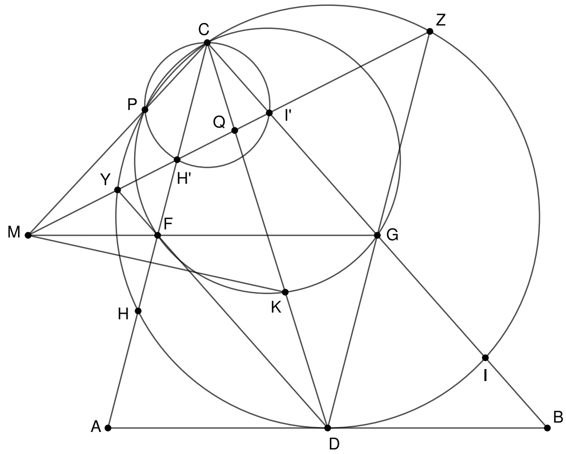


Figure 4.3.27: ISL 2015 G5

Problem 4.3.33 (ISL 2010 G5). Let $ABCDE$ be a convex pentagon such that $BC \parallel AE$, $AB = BC + AE$, and $\angle ABC = \angle CDE$. Let M be the midpoint of CE, and let O be the circumcenter of triangle BCD. Given that $\angle DMO = 90°$, prove that $2\angle BDA = \angle CDE$.

Solution. First try to construct the point. Do this the long way, then find a easier way that includes B, C, not B, A to do that. Then try to translate what 90 degree condition into angles, and take midpoints, since we have midpoints involved.

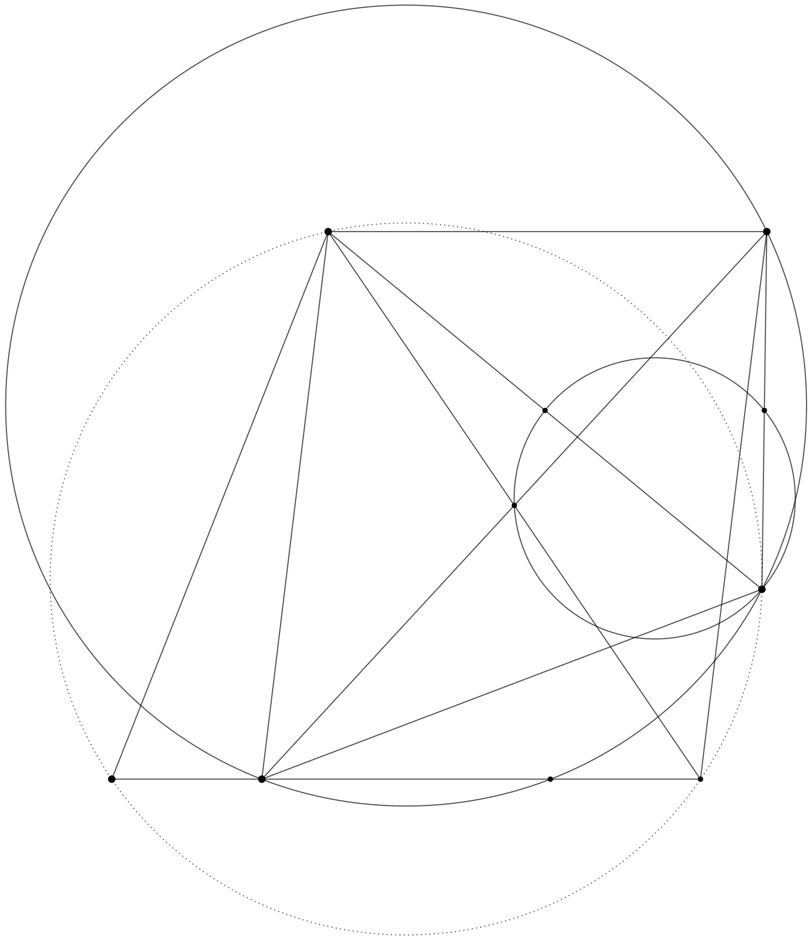


Figure 4.3.28: ISL 2010 G5

Problem 4.3.34 (IGO 2019 A5). Let points A, B and C lie on the parabola Δ such that the point H, orthocenter of triangle ABC, coincides with the focus of parabola Δ. Prove that by changing the position of points A, B and C on Δ so that the orthocenter remain at H, inradius of triangle ABC remains unchanged.

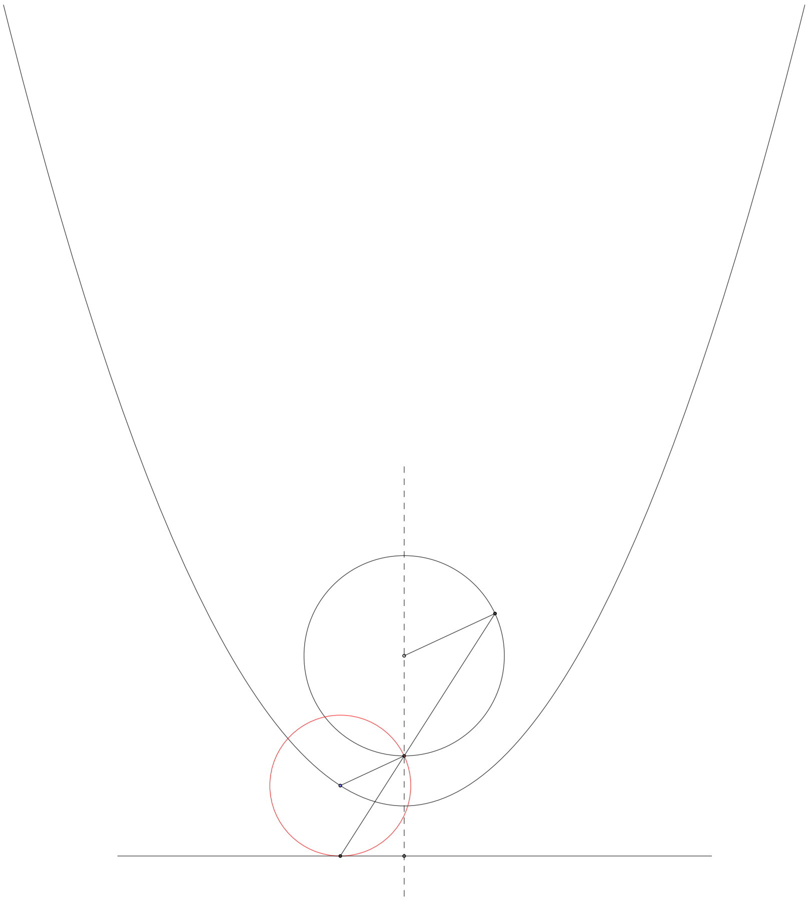


Figure 4.3.29: IGO 2019 A5

Solution. I think the idea for inversion should have been pretty natural after finding that the incircle is fixed.

Problem 4.3.35 (Iran 3rd Round 2015 P5). Let ABC be a triangle with orthocenter H and circumcenter O. Let R be the radius of circumcircle of $\triangle ABC$. Let A', B', C' be the points on $\overrightarrow{AH}, \overrightarrow{BH}, \overrightarrow{CH}$ respectively such that $AH.AA' = R^2, BH.BB' = R^2, CH.CC' = R^2$. Prove that O is incenter of $\triangle A'B'C'$.

Solution. The condition easily leads to a nice construction of the points. It should be trivial to figure that the construction is really important. Also, noticing a similarity among the triangles is really important.

4.3.2 The line parallel to BC

Definition (Important Points)— Let the line parallel to BC through O meet AB, AC at D, E. Let K be the midpoint of AH, M be the midpoint of BC. F be the feet of A-altitude on BC and let H' be the reflection of H on F. Let O' be the circumcenter of KBC.

Lemma 4.3.3 — $\angle DKC = \angle EKB = 90°$

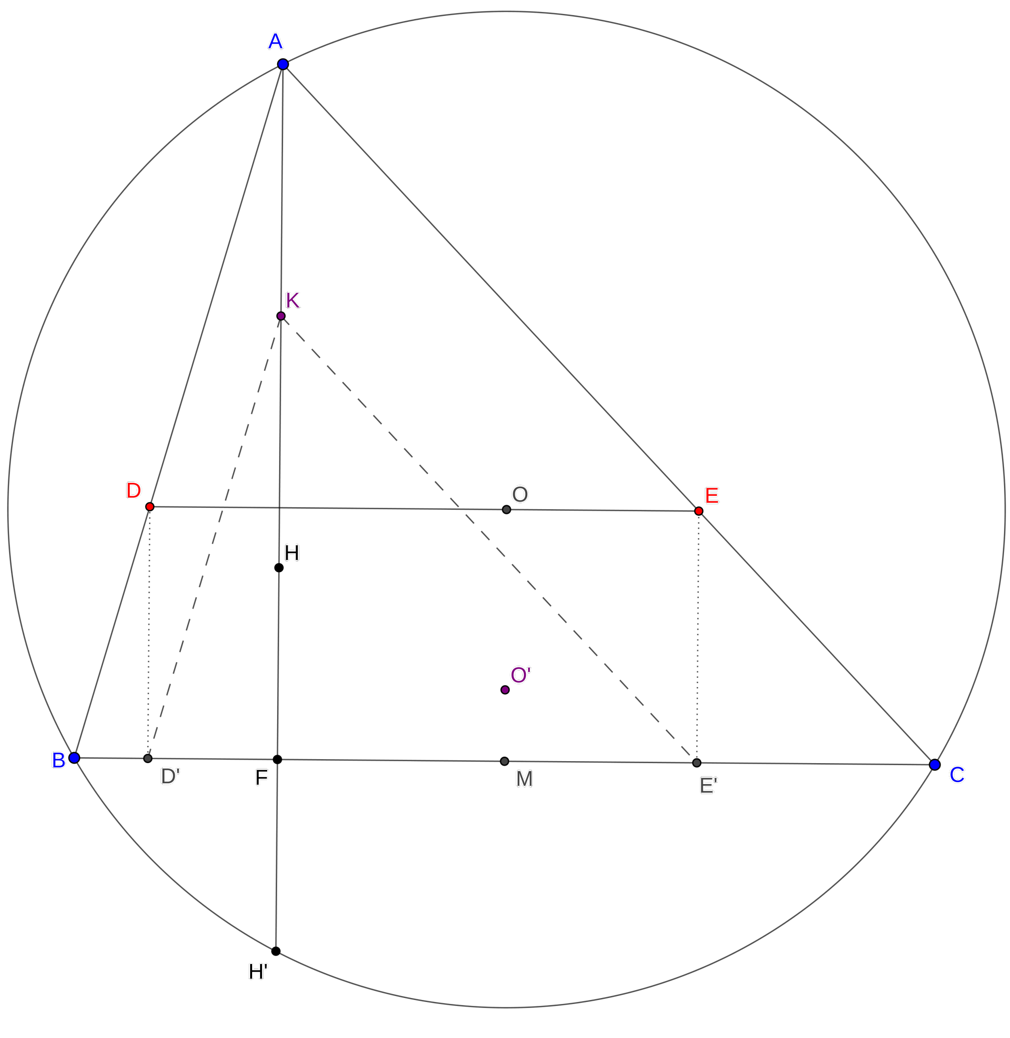


Figure 4.3.30

Lemma 4.3.4 — CD, BE, OH', AM, KO' are concurrent. (by **??**)

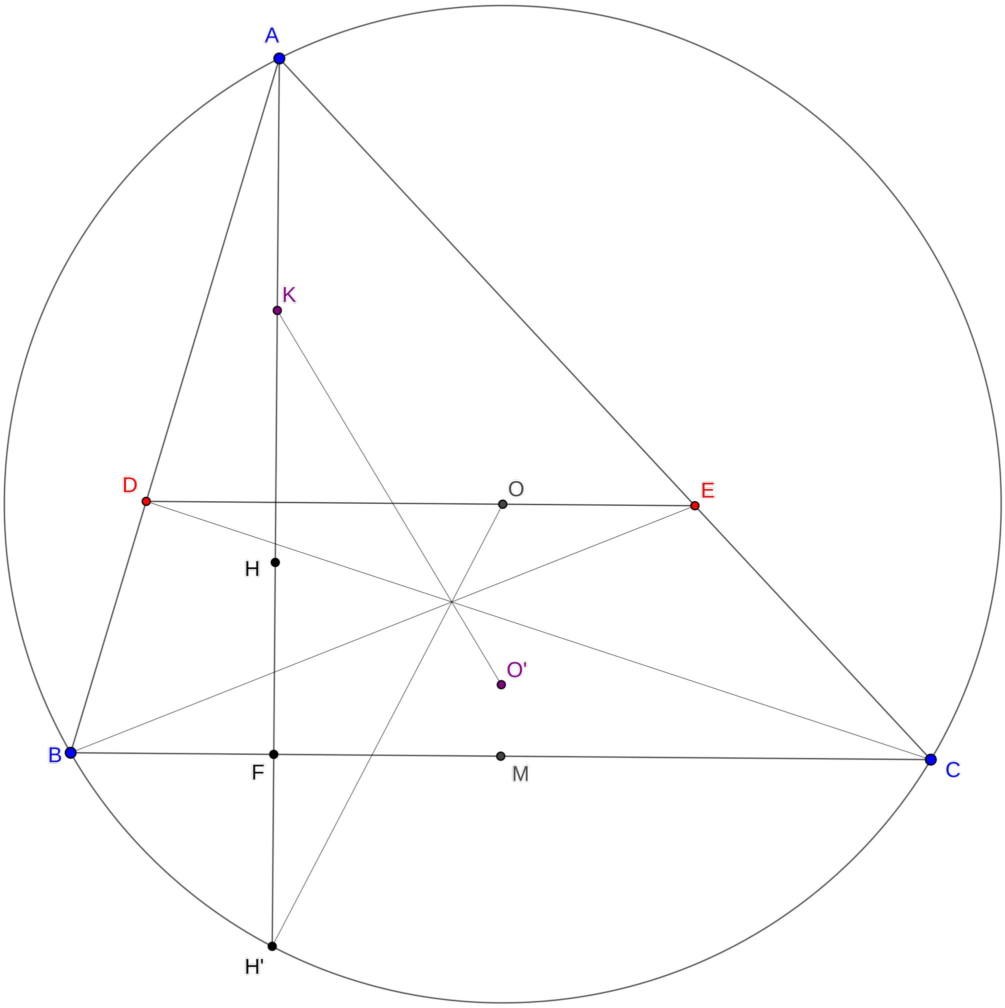


Figure 4.3.31

Problem 4.3.36 (InfinityDots MO Problem 3). Let $\triangle ABC$ be an acute triangle with circumcenter O and orthocenter H. The line through O parallel to BC intersect AB at D and AC at E. X is the midpoint of AH. Prove that the circumcircles of $\triangle BDX$ and $\triangle CEX$ intersect again at a point on line AO.

Solution. Just using Lemma 4.3.2 to get another pair of circle where we can apply radical axis arguments.

Solution. Noticing that the resulting point is the isogonal conjugate of a well defined point,

Lemma 4.3.5 — Let P, Q be on AB, AC resp. such that $PQ \parallel BC$. And let A' be such that $A' \in \odot ABC, AA' \parallel BC$. Let $CP \cap BQ = X$, and let the perpendicular bisector of BC meet PQ at Y. Prove that A', X, Y are collinear.

Solution. No angles... Do Lengths...

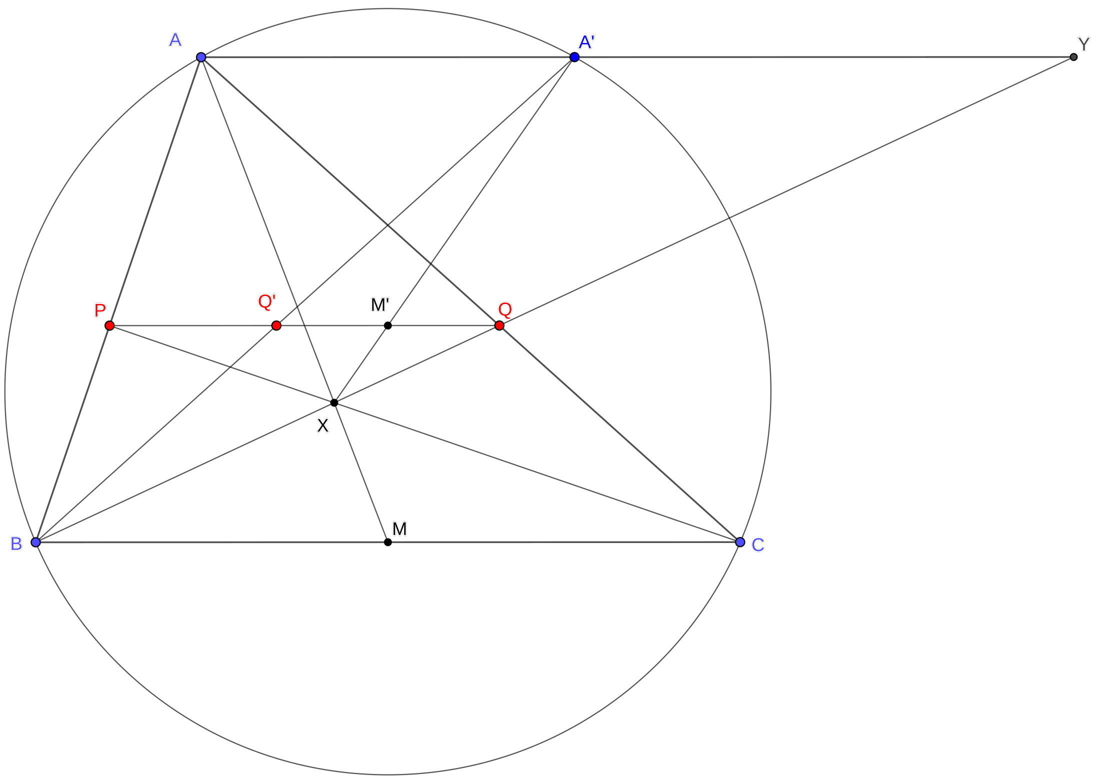


Figure 4.3.32

Problem 4.3.37 (ARO 2018 P11.4). $P \in AB$, $Q \in AC$, $PQ \parallel BC$, $BQ \cap CP = X$. A' is the reflection of A wrt BC. $A'X \cap \odot APQ = Y$. Prove that $\odot BYC$ is tangent to $\odot APQ$.

Solution. Of co it can be solved using angle chase, Lemma 4.3.2 makes it almost trivial.

Problem 4.3.38 (buratinogigle). Let (O) be a circle and E, F are two points inside (O). $(K), (L)$ are two circles passing though E, F and tangent internally to (O) at A, D, respectively. AE, AF cut (O) again at B, C, respectively. BF cuts CE at G. Prove that reflection of A though EF lies on line DG.

Rephrasing the problem as such: In the setup of Lemma 4.3.2, let $A'X \cap \odot ABC = Z$, then $\odot PQZ$ is tangent to $\odot ABC$.

Solution. Simple angle chase.

Solution. Another solution to this is by taking D as a phantom point.

Solution. Another solution is with cross ratios

4.3.3 Simson Line and Stuffs

Lemma 4.3.6 (Simson Line Parallel) — Let P be a point on the circumcircle, let P' be the reflection of P on BC and let $PP' \cap \Omega = Q$, and let l_p be the Simson line of P. Prove that $l_p \parallel AD \parallel HP'$.

Lemma 4.3.7 (Simson Line Angle) — Given triangle ABC and its circumcircle (O). Let E, F be two arbitrary points on (O). Then the angle between the Simson lines of two points E and F is half the measure of the arc EF.

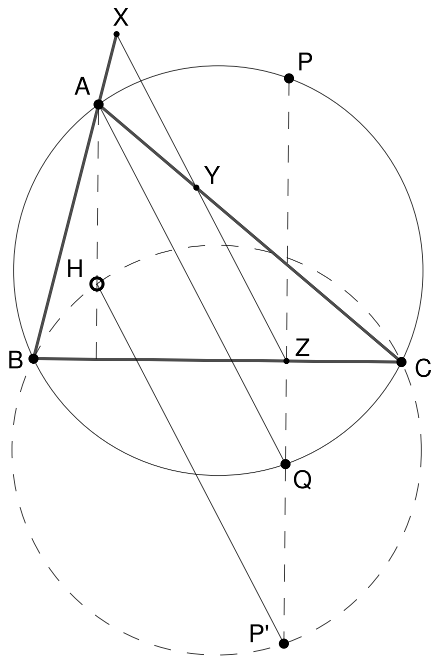


Figure 4.3.33: Lemma 4.3.6

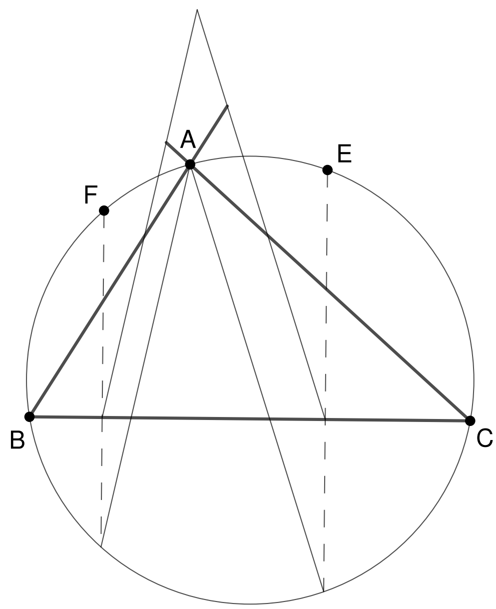


Figure 4.3.34: Lemma 4.3.7

4.3.4 Euler Line

Theorem 4.3.8 (Perspectivity Line with Orthic triangle is perpendicular to Euler line) — Let DEF be the orthic triangle. Then $BC \cap EF, CA \cap FD, AB \cap ED$ are collinear, and the line is perpendicular to the Euler line. In fact this line is the radical axis of the Circumcircle and the NinePoint circle

Lemma 4.3.9 — DEF is orthic triangle of ABC, XYZ is the orthic triangle of DEF. Prove that the perspective point of ABC and XYZ lies on the Euler line of ABC

Proof. Thinking the stuff wrt to the incircle and using cross ratio.

Lemma 4.3.10 (Perpendicular on Euler Line) — Let E, F be the feet of altitudes from B, C, and let M, N be the midpoints of AC, AB. $D = MN \cap AB$. Prove that AF is perpendicular to the euler line.

Proof. We know that $MNEF$ lie on a circle. So D is the radical center of $\odot MNEF, \odot AMN, \odot AEF$.

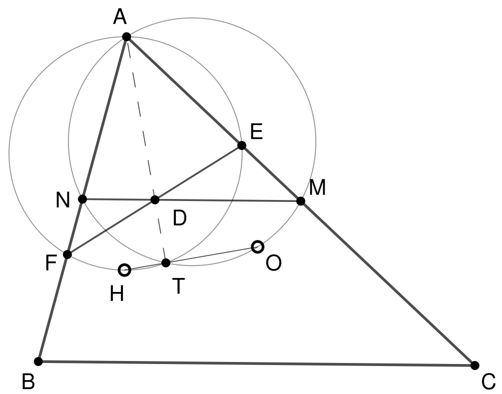


Figure 4.3.35

Problem 4.3.39 (CHKMO 2014 P4). Let $\triangle ABC$ be a scalene triangle, and let D and E be points on sides AB and AC respectively such that the circumcircles of triangles $\triangle ACD$ and $\triangle ABE$ are tangent to BC. Let F be the intersection point of BC and DE. Prove that AF is perpendicular to the Euler line of $\triangle ABC$.

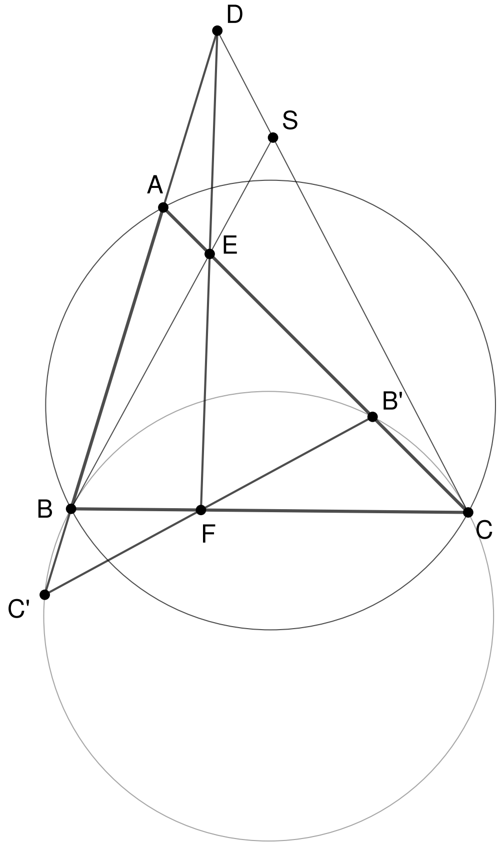


Figure 4.3.36

Solution. Let B', C' be the points on AC, AB such that $BB' = BA$, $CA = CC'$. By simple angle chasing, we can show that BE, CD are tangent to $\odot BB'CC'$.

Now by Pascal's theorem on $BBB'CCC'$, we can show that D, E, F are collinear where $F = BC \cap B'C'$. Then by Lemma 4.3.10, we have that $AF \perp HO$.

Lemma 4.3.11 (Extension of CHKMO) — AS, BB', CC' are concurrent.

4.3.5 Assorted Diagrams

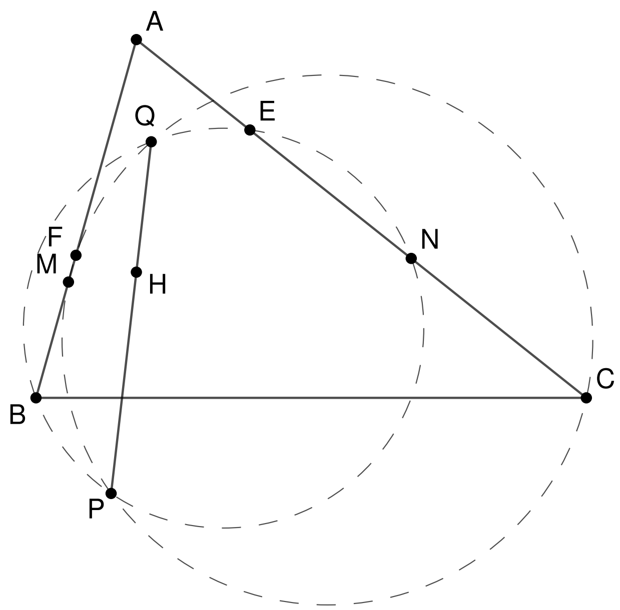


Figure 4.3.37: H lies on the line, circles vary

4.4 Cevian and Circumcevian Triangles

4.4.1 Circumcevian Triangle

Theorem 4.4.1 (Hagge's circles) — Let P be a point on the plane of $\triangle ABC$, let Ω be the circumcircle. Let A_1, B_1, C_1 be the intersections of AP, BP, CP with Ω for the second time. Let A_2, B_2, C_2 be the reflections of A_1, B_1, C_1 wrt BC, CA, AB. Prove that H, A_2, B_2, C_2 lie on a circle. This circle is called the ***P*-Hagge's Circle**.

Solution. Either using the dual of Hagge's Circle, or using the reflection points of A, B, C wrt the isogonal conjugate of P. And using Lemma 1.1 to finish.

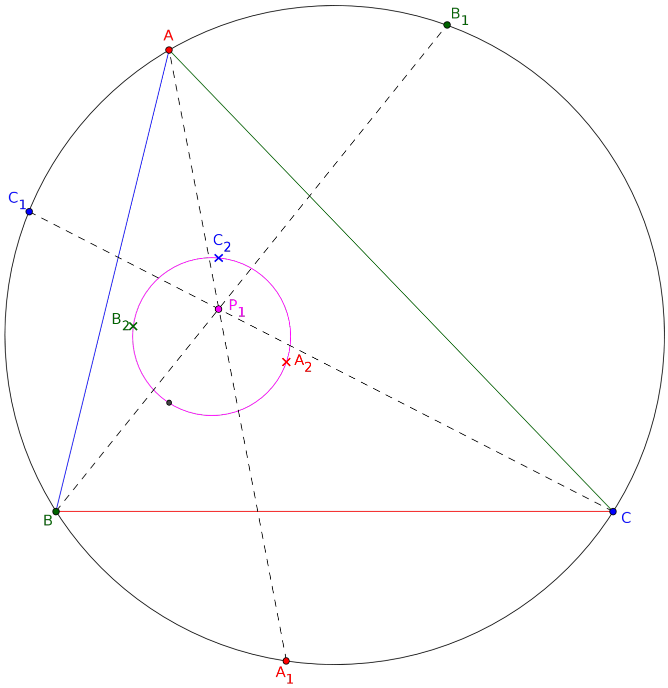


Figure 4.4.1: P-Hagge Circle

Corollary 4.4.2 — $\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2$.

Solution. Straightforward use of Lemma 1.2.

Corollary 4.4.3 — If AH, BH, CH meet $\odot A_2B_2C_2H$ at A_3, B_3, C_3, then A_2A_3, B_2B_3, C_2B_3 meet at P.

Solution. Simple angle chase and similarity transformation.

Corollary 4.4.4 — If I is the incenter of $\odot A_2B_2C_2$, K is the reflection of H over I, AK, BK, CK meet $\odot A_2B_2C_2$ at A_4, B_4, C_4, then A_4A_3, B_4B_3, C_4B_3 are concurrent.

Solution. Simple angle chasing and trig-ceva.

Problem 4.4.1 (China TST D2P2, Dual of the Hagge's Circle theorem). Let ω be the circumcircle of $\triangle ABC$. P is an interior point of $\triangle ABC$. A_1, B_1, C_1 are the intersections of AP, BP, CP respectively and A_2, B_2, C_2 are the symmetrical points of A_1, B_1, C_1 with respect to the midpoints of side BC, CA, AB. Show that the circumcircle of $\triangle A_2B_2C_2$ passes through the orthocenter of $\triangle ABC$. Further proof that if this circle's center is O_1, then $HOPO_1$ is a parallelogram.

Solution. Construct Parallelograms. You have to prove two angles are equal. Reflection the smaller trig wrt one of the midpoints.

Problem 4.4.2 (China TST 2011, Quiz 2, D2, P1). Let AA', BB', CC' be three diameters of the circumcircle of an acute triangle ABC. Let P be an arbitrary point in the interior of $\triangle ABC$, and let D, E, F be the orthogonal projection of P on BC, CA, AB, respectively. Let X be the point such that D is the midpoint of $A'X$, let Y be the point such that E is the midpoint of $B'Y$, and similarly let Z be the point such that F is the midpoint of $C'Z$. Prove that triangle XYZ is similar to triangle ABC.

Solution. A straightforward application of Lemma 6.1 using the O-Hagge's Circle.

4.4.2 Cevian Triangle

Lemma 4.4.5 (Isogonal Conjugate Lemma) — Let a circle ω meet the sides of triangle ABC at A_1, A_2; B_1, B_2; C_1, C_2. Let P_1, P_2 be the miquel points of ABC wrt $A_1B_1C_1, A_2B_2C_2$ resp. Then P_1, P_2 are isogonal conjugates.

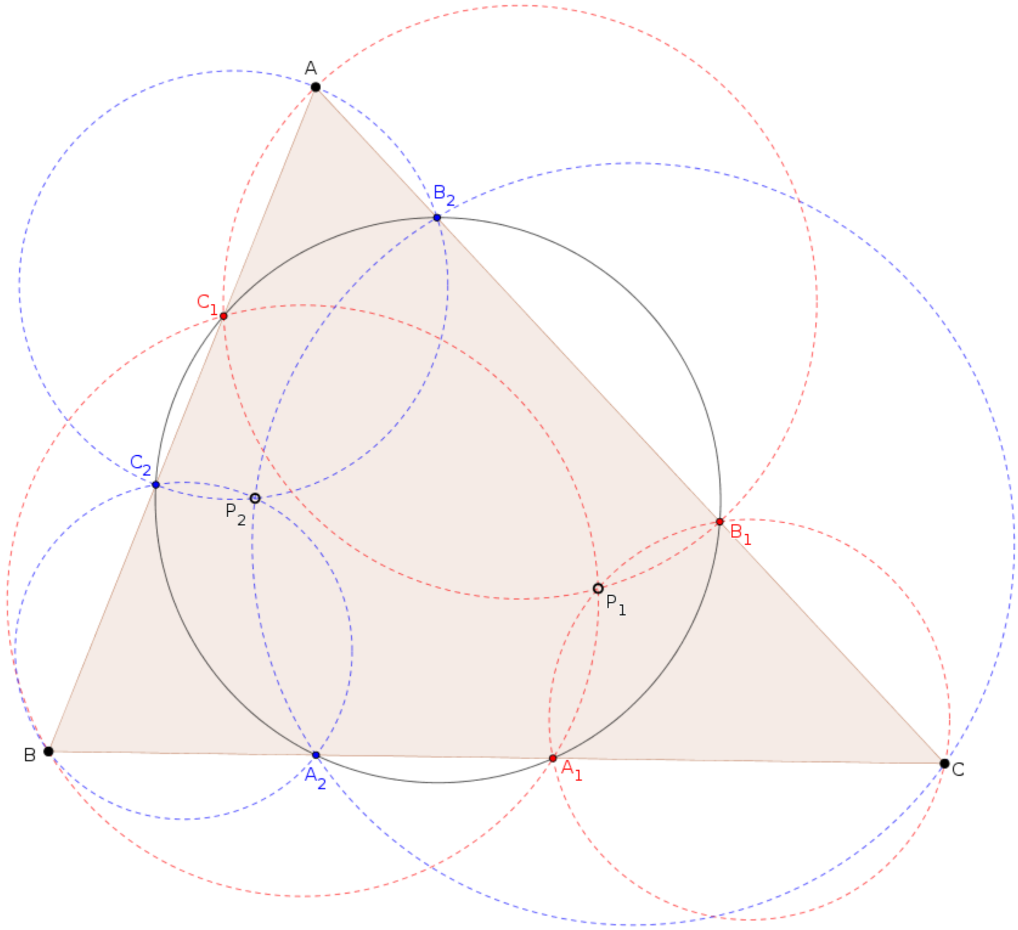


Figure 4.4.2: The two round points are isogonal conjugates.

Theorem 4.4.6 (Terquem's Cevian Theorem) — Let a circle ω meet the sides of triangle ABC at A_1, A_2; B_1, B_2; C_1, C_2. If AA_1, BB_1, CC_1 are concurrent, then so are AA_2, BB_2, CC_2

.

Theorem 4.4.7 (Mannheim's Theorem) — Let ABC be a triangle, and let L, M, N be points on BC, CA, AB respectively. Let A', B', C' be points on $(AMN), (BNL), (CLM)$, and denote $K \equiv AA' \cap BB'$. Then if $K \in CC'$, A', B', C', K are concyclic.

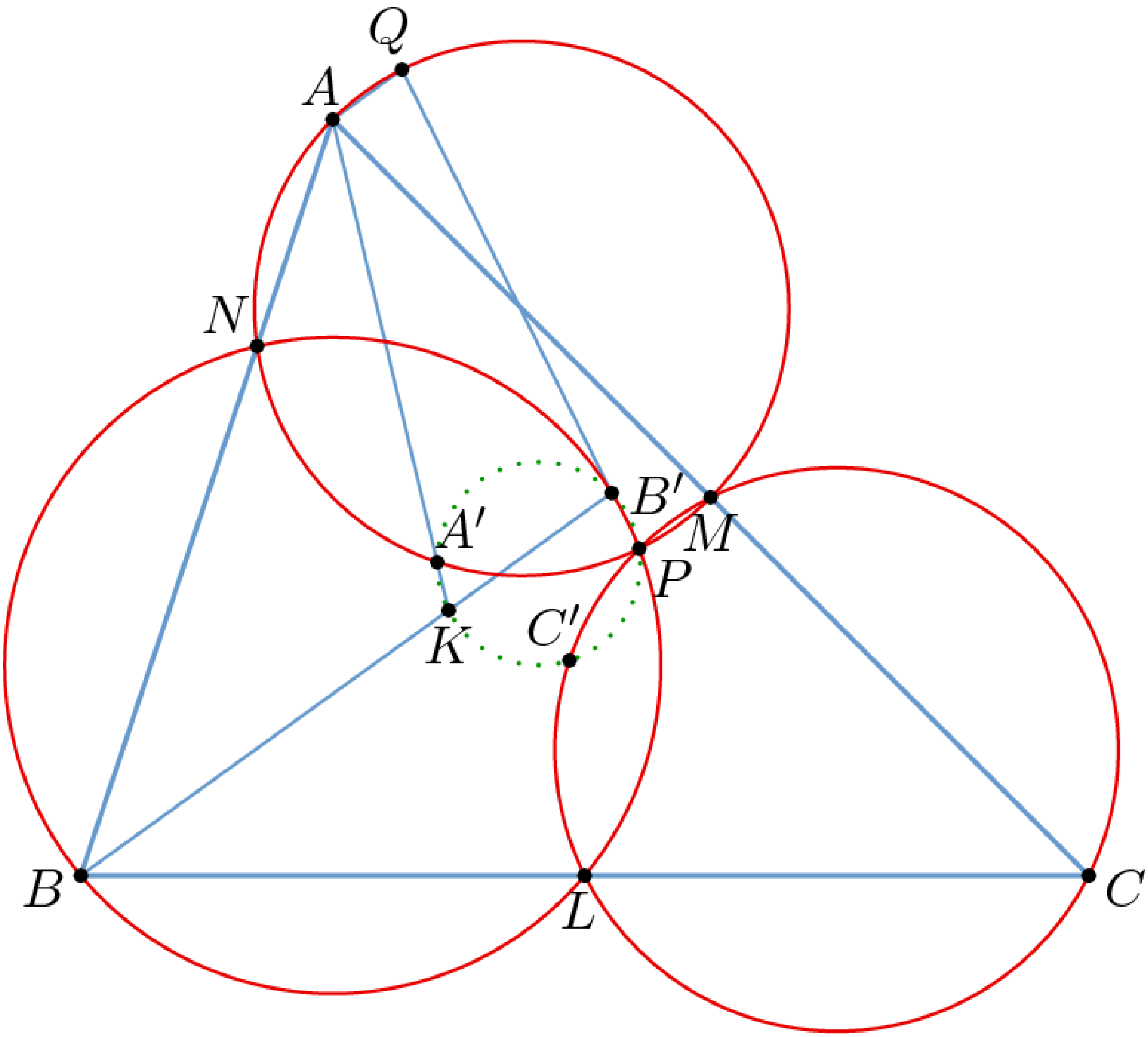


Figure 4.4.3: Mannheim's Theorem

Theorem 4.4.8 (Mannheim's Theorem's Converse) — Let ABC be a triangle, and let L, M, N be points on BC, CA, AB respectively. Let A', B', C' be points on $(AMN), (BNL), (CLM)$, and denote $K \equiv AA' \cap BB'$. Then if A', B', C', K are concyclic, $C' \in CK$.

Theorem 4.4.9 (Brocard Points) — Brocard Points are points inside a triangle such that

$$\angle PAB = \angle PBC = \angle PCA = \omega$$

and

$$\angle QCB = \angle QBA = \angle QAC = \omega.$$

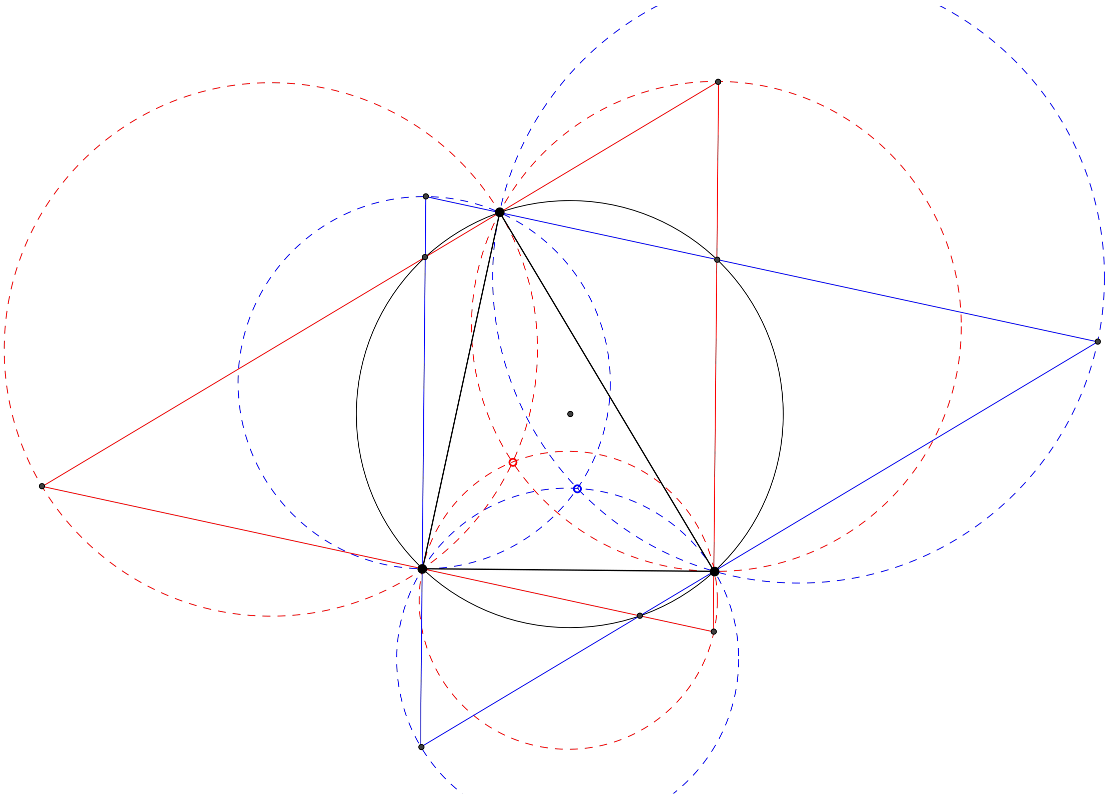


Figure 4.4.4: Brocard Points

Problem 4.4.3 (Rioplatense Olympiad 2013 Problem 6). Let ABC be an acute-angled scalene triangle, with centroid G and orthocenter H. The circle with diameter AH cuts the circumcircle of BHC at A', distinct from H. Analogously define B', C'. Prove that A', B', C', G are concyclic.

Problem 4.4.4 (Iran 3rd Round Training 2016). ABC is an acute triangle and H, O are its orthocenter and circumcenter respectively. If AO, BO, CO intersect BH, CH, AH at X, Y, Z respectively,then prove that H, X, Y, Z lie on a circle

Solution. Using Brocard Point

Solution. Using Mannheim's Theorem

Theorem 4.4.10 (Jacobi's Theorem) — Suppose that D, E, F are points such that AE, AF are isogonal wrt $\angle BAC$. Similarly with D, E, F. Then AD, BE, CF are concurrent.

4.5 Centers of inside and outside

Definition (Incenter and Co.)— Let $\triangle ABC$ be an ordinary triangle, I is its incenter, D, E, F are the touch points of the incenter with BC, CA, AB and D', E', F' are the reflections of D, E, F wrt I. Let the I_a, I_b, I_c excircles touch BC, CA, AB at D_1, E_1, F_1.

Let M_a, M_b, M_c be the midpoints of the smaller arcs BC, CA, AB, and M_A, M_B, M_C be the midpoints of the major arcs BC, CA, AB. M are the midpoint of BC. Let A' be the antipode of A wrt $\odot ABC$.

Let (I_a) touch BC, CA, AB at D_A, E_A, F_A. So, $D_A \equiv D_1$.

Call EF, 'A-tangent line', and DE, DF similarly. And call E_AF_A 'A_A-tangent line.

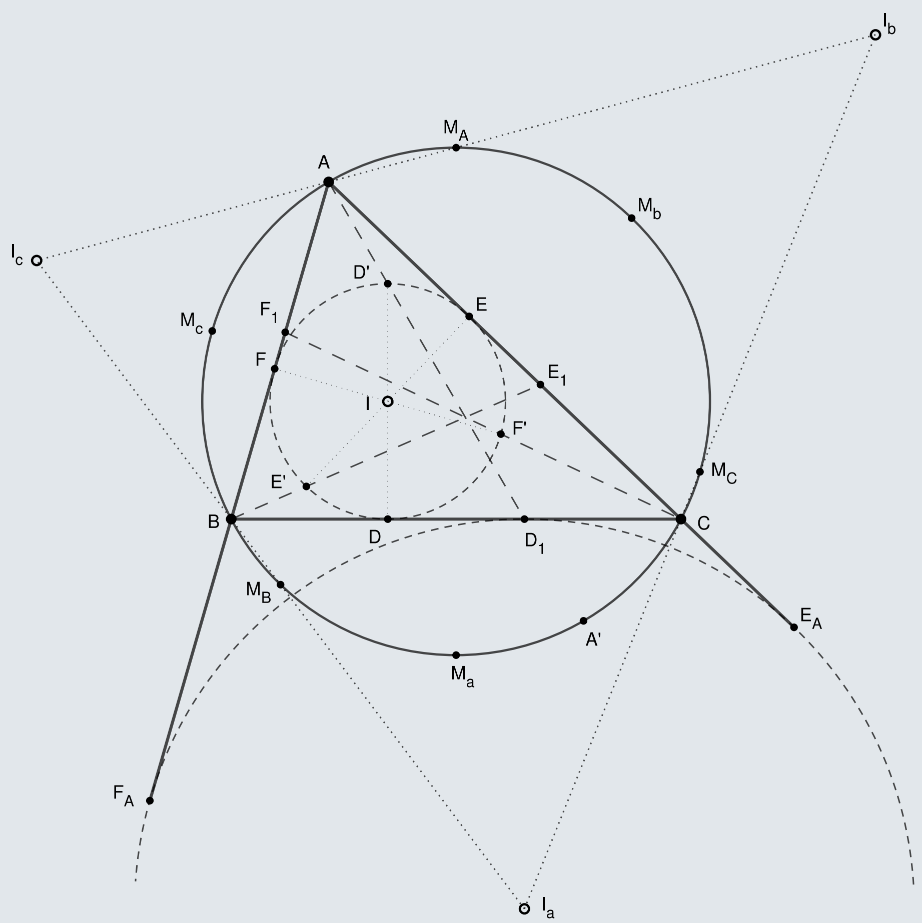


Figure 4.5.1: All the primary points related to the incenter and the excenters

Lemma 4.5.1 (Antipode and Incenter) — $A'I, \odot ABC, \odot AEIF$ are concurrent at Y_A. And Y_A, D, M_a are collinear.

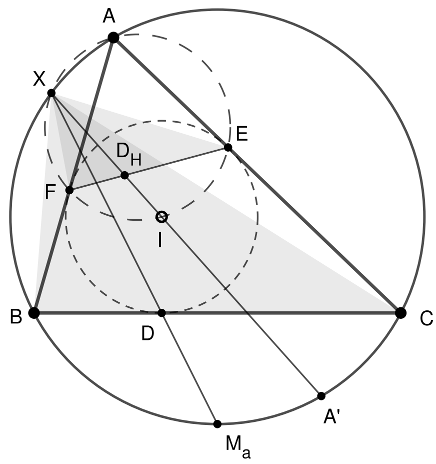


Figure 4.5.2: Lemma 4.5.1

Lemma 4.5.2 — $DD_H \perp EF$, then D_H, I, A' are collinear.

Lemma 4.5.3 —

$$\frac{FD_H}{D_HE} = \frac{BD}{DC}$$

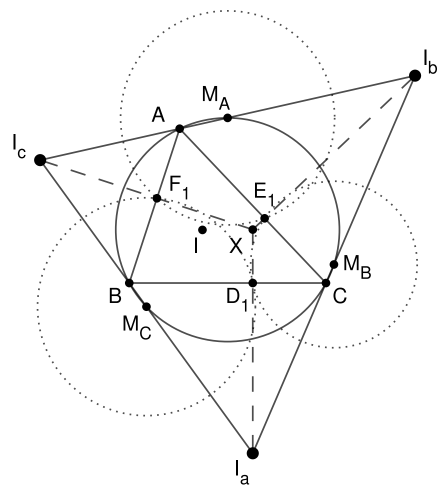


Figure 4.5.3: Lemma 4.5.4

Lemma 4.5.4 (Arc Midpoint as Centers) —

$$M_AE_1 = M_AF_1 \quad M_BF_1 = M_BD_1 \quad M_CD_1 = M_CE_1$$

Moreover, I, O are the orthocenter and the circumcenter of $\triangle I_aI_bI_c$

Lemma 4.5.5 (Incircle Touchpoint and Cevian) — Let a cevian be AX and let I_1, I_2 be the incirlces of $\triangle ABX, \triangle ACX$. Then D, I_1, I_2, X are concyclic. And the other common tangent of $\odot I_1$ and $\odot I_2$ goes through D.

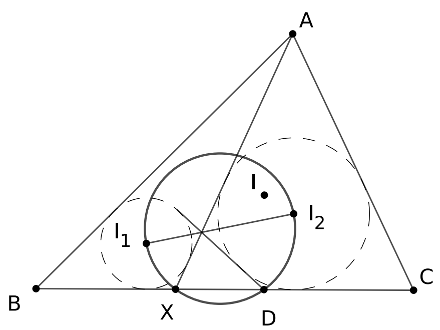


Figure 4.5.4: Lemma 4.5.5

Lemma 4.5.6 (Apollonius Circle and Incenter, ISL 2002 G7) — Let ω_a be the circle that goes through B, C and is tangent to (I) at X. Then XD', EF, BC are concurrent and X, D, I_a are collinear. The same properties is held if the roles of incenter and excenter are swapped.

- The circle BXC is tangent to (I)
- X lies on the Apollonius Circle of $(B, C; D, G)$.
- XD bisects $\angle BXC$.

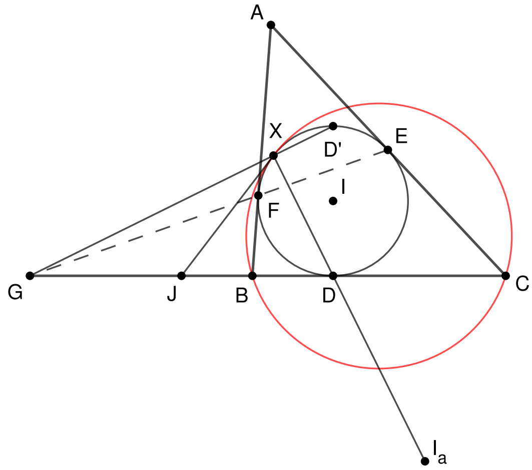


Figure 4.5.5: Lemma 4.5.6

Lemma 4.5.7 (Line parallel to BC through I) — Let E, F be the intersection of the B, C angle bisectors with AC, AB. Then the tangent to $\odot ABC$ at A, EF and the line through I parallel to BC are concurrent.

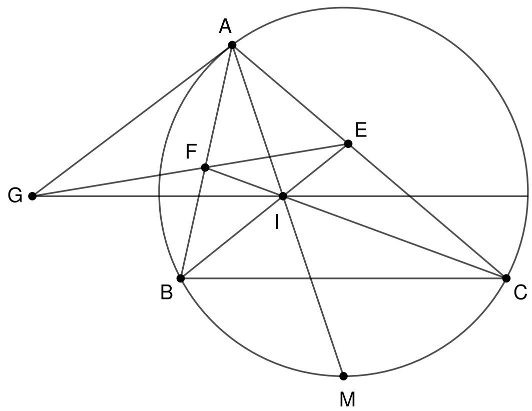


Figure 4.5.6: Lemma 4.5.7

Lemma 4.5.8 (Midline Concurrency with Incircle Touchpoints) — AI, B, B_A-tangent lines and C-mid-line are concurrent. And, if the concurrency point is X, then $CS \perp AI$

Theorem 4.5.9 (Paul Yui Theorem) — B-tangent line, C_A-tangent line, and AH are concurrent.

Lemma 4.5.10 (Concurrent Lines in Incenter) — Let $AD \cap (I) = G, AD' \cap (I) = H$. Let the line through D' parallel to BC meet AB, AC at B', C'. Then AM, EF, GH, DD', BC', CB' are concurrent.

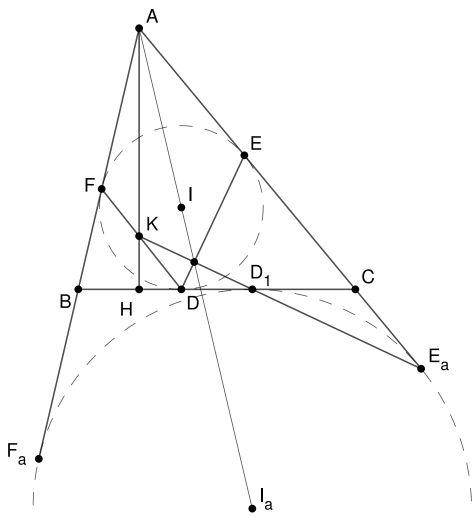


Figure 4.5.7: Lemma 4.5.8 & Theorem 4.5.9

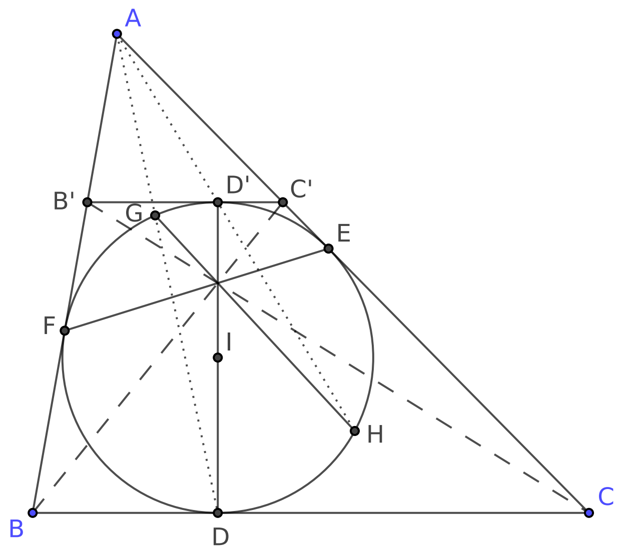


Figure 4.5.8: Lemma 4.5.10 The lines are concurrent.

Lemma 4.5.11 (Insimilicenter) — The *insimilicenter* is the positive homothety center of the circumcircle and the incircle. It is also the *isogonal conjugate of the Nagel Point* wrt $\triangle ABC$.

Solution. Let T be the A-mixtilinear touchpoint. If $AT \cap (I) = A'$, then if we can show that $A'B'$ arc has angle $\angle C$, where $A'B' \parallel AB$, we are done.

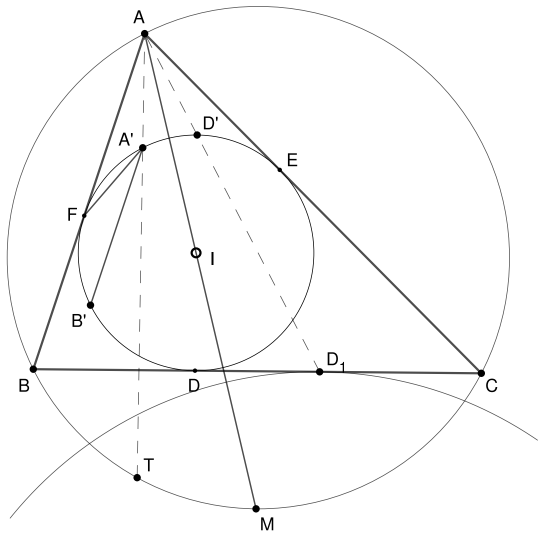


Figure 4.5.9

Problem 4.5.1 (Application of Aollonius Circle and Incenter Lemma). Let triangle ABC, incircle (I), the A-excircle (I_a) touches BC at M. IM intersects (I_a) at the second point X. Similarly, we get Y, Z. Prove that AX, BY, CZ are concurrent.

Extension, by buratinogigle: Triangle ABC and XYZ are homothetic with center I is incenter of ABC. Excircles touches BC, CA, AB at D, E, F. XD, YE, FZ meets excircles again at U, V, W. Prove that AU, BV, CW are concurrent.

Definition (Isodynamic Points)— Let ABC be a triangle, and let the angle bisectors of $\angle A$ meet BC at X, Y. Call ω_a the circumcircle of $\triangle AXY$. Define ω_b, ω_c similarly. The first and second isodynamic points are the points where the three circles $\omega_a, \omega_b, \omega_c$ meet. I.e. these two points are the intersections of the three Apollonius circles. These two points satisfy the following relations:

1. $PA \sin A = PB \sin B = PC \sin C$
2. They are the isogonal congugates of the Fermat Points, and they lie on the 'Brocard Axis'

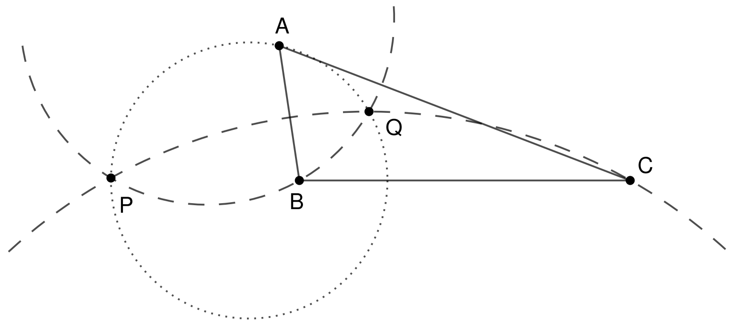


Figure 4.5.10

Theorem 4.5.12 (Pedal Triangles of Isodynamic Points) — Prove that the pedal triangles of the isodynamic points are equilateral triangles. Also, Inverting around the Isodynamic Points trasnform $\triangle ABC$ into an equilateral triangle.

Problem 4.5.2 (China TST 2018 T1P3). Circle ω is tangent to sides AB,AC of triangle ABC at D,E respectively, such that $D \neq B$, $E \neq C$ and $BD + CE < BC$. F,G lies on BC such that $BF = BD$, $CG = CE$. Let DG and EF meet at K. L lies on minor arc DE of ω, such that the tangent of L to ω is parallel to BC. Prove that the incenter of $\triangle ABC$ lies on KL.

Solution. Using Lemma 4.3.1, in the touch triangle of ω.

Problem 4.5.3. Given a triangle ABC with circumcircle Γ. Points E and F are the foot of angle bisectors of B and C, I is incenter and K is the intersection of AI and EF. Suppose that N be the midpoint of arc BAC. Circle Γ intersects the A -median and circumcircle of AEF for the second time at X and S. Let S' be the reflection of S across AI and J' be the second intersection of circumcircle of $AS'K$ and AX. Prove that quadrilateral $TJ'IX$ is cyclic.

Solution [Reim and lemmas]. Since $NI \cap \odot ABC = T$, the mixtilinear touchpoint, if we can show that $AT || IJ'$, we will be done. Instead of working with S' and J', we reflect them back and work with S, J. Then we need to prove that $IJ||AD'$ where D' is the reflection of D over I.

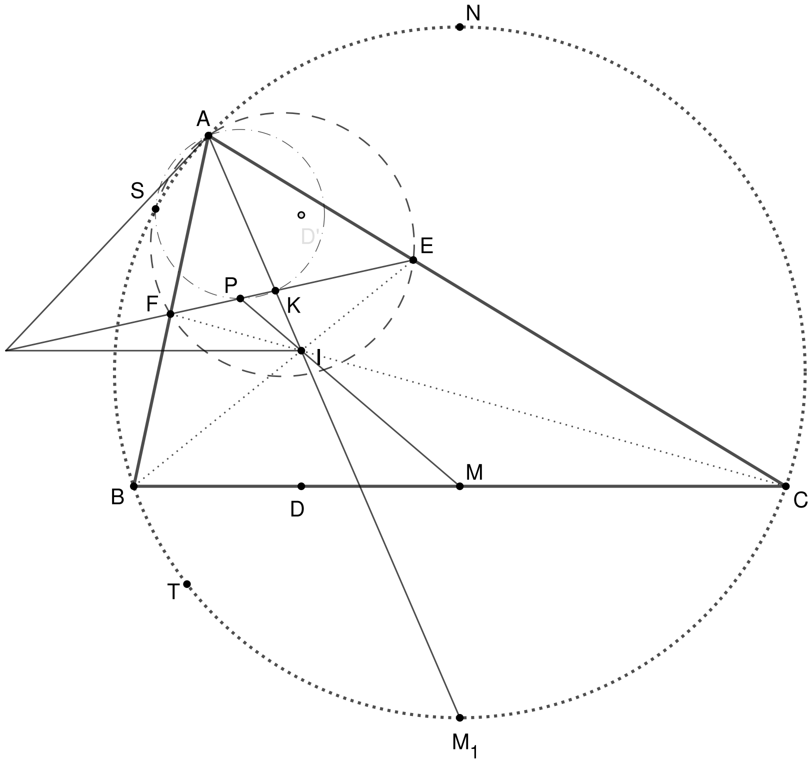


Figure 4.5.11

From Lemma 4.5.7 we know that the A symmedian, EF, IM are concurrent at a point P. We prove that $P \equiv J$. For that we need to show that P lies on $\odot AKS$.

If $\odot AKP \cap AB, AC = U, V$, it is sufficient to prove that

$$\frac{UF}{FB} = \frac{VE}{EC}$$

We have:

$$\frac{UF}{KU} = \frac{\sin FAP}{\sin KFA} \quad \frac{VE}{KV} = \frac{\sin EAP}{\sin KEA}$$

$$\begin{aligned} \therefore \frac{UF}{VE} &= \frac{\sin FAP}{\sin EAP} \frac{\sin KEA}{\sin KFA} = \frac{\sin CAM}{\sin BAM} \frac{AF}{AE} \\ &= \frac{BA}{CA} \frac{AF}{AE} = \frac{BA}{AE} \frac{AF}{CA} = \frac{BC}{EC} \frac{CF}{BC} \\ &= \frac{BF}{EC} \end{aligned}$$

Problem 4.5.4 (Vietnamese TST 2018 P6.a). Triangle ABC circumscribed (O) has A-excircle (I_a) that touches AB, BC, AC at F, D, E, resp. M is the midpoint of BC. Circle with diameter MI_a cuts DE, DF at K, H. Prove that (BDK), (CDH) have an intersecting point on (I_a).

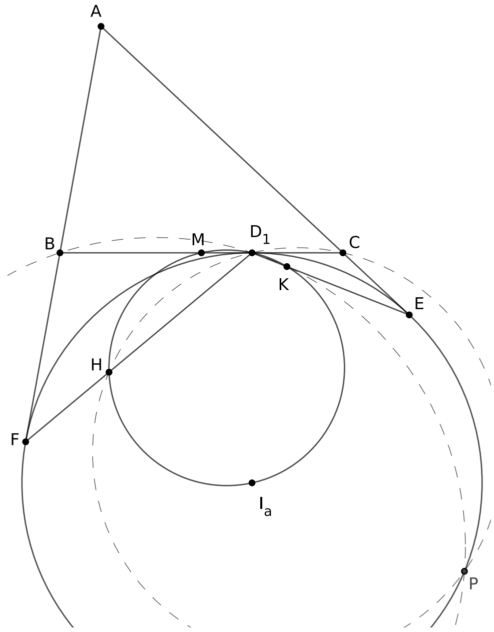


Figure 4.5.12

Problem 4.5.5 (After Inverting Around D). MD is a line, I_a is an arbitrary point such that $DI_a \perp MD$. l is the perpendicular bisector of DI_a. F, E are arbitrary points on l. $B = I_aF \cap MD, C = I_aE \cap MD$ $H = FD \cap MI_a, K = DE \cap MI_a$. Then BK, CH, l are concurrent.

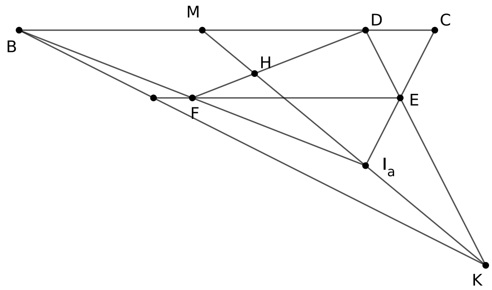


Figure 4.5.13: After inverting around D

Solution. It is straightforward using Puppus's Theorem on lines BDC and HI_aK.

Solution [Synthetic: Length Chase].

Lemma— Let G, H, B', C' be defined the same way in Lemma 3.2. Prove that F lies on the radical axis of $\odot D'GI, D'C'H$. By extension prove that B lies on the radical axis of $\odot D'B'I, D'C'H$

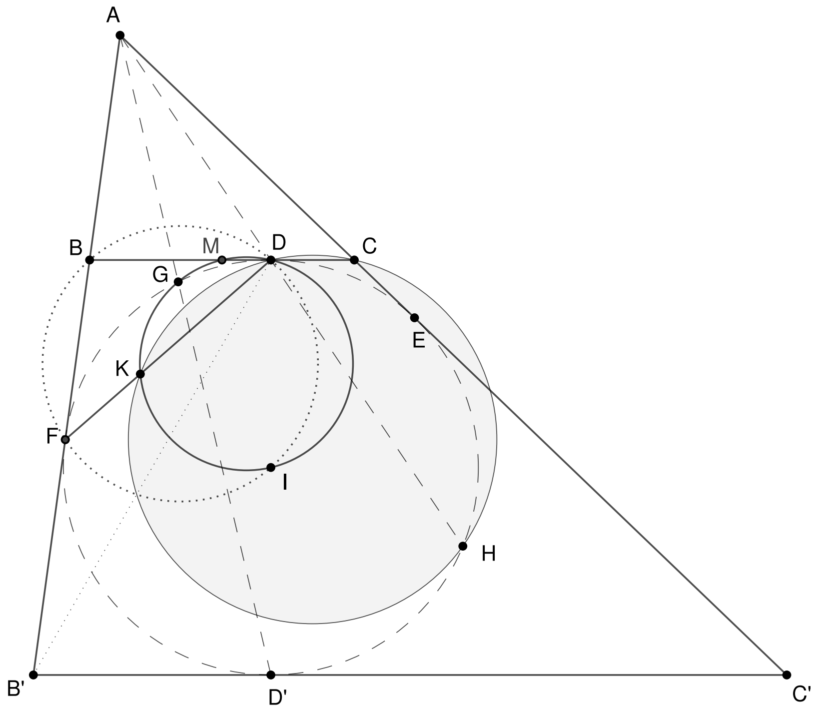


Figure 4.5.14: Vietnamese TST 2018 P6.a

We prove the first part, and the second part follows using spiral similarity.
Suppose $K \in FD \cap \odot KDI$. Due to spiral similarity on $\odot KDI, \odot(I)$, we have $\triangle GFK \sim \triangle GD'I$. Which implies:

$$\frac{FK}{GF} = \frac{ID}{GD'} \implies FK = ID\frac{GF}{GD'}$$

Now, if $KDCE$ is to be cyclic, we need to have $\triangle HFK \sim \triangle HDC$. So we need,

$$\frac{FK}{HF} = \frac{DC}{HD} \implies FK = DC\frac{HF}{HD}$$

Combining two equations:

$$\frac{GF}{GD'} \cdot \frac{ID}{DC} = \frac{HF}{HD}$$

Now, using Ptolemy's theorem in $\square FDEH$, we have,

$$FD \cdot EH + DE \cdot FH = DH \cdot EF$$
$$EH \cdot \frac{FD}{FH} + DE = EF \cdot \frac{DH}{FH}$$
$$2\,\frac{DE}{EF} = \frac{DH}{FH}$$

Similarly from $\square FGED'$ we get,

$$2\,\frac{D'E}{EF} = \frac{GD'}{FG}$$

Combining these two equations gives us the desired result.

Generalization 4.5.5.1 (Vietnamese TST 2018 P6.a Generalization).
Let ABC be a triangle. The points D, E, F are on the lines BC, CA, AB respectively. The circles (AEF), (CFD), (CDE) have a common point P. A circle (K) passes through P, D meet DE, DF again at Q, R respectively. Prove that the circles (DBQ), (DCR) and (DEF) are coaxial.

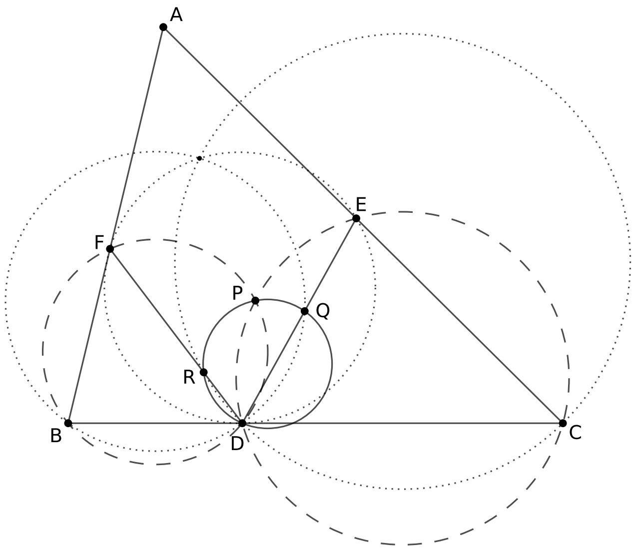


Figure 4.5.15: Vietnamese TST 2018 P6.a Generalization

Solution [Inversion]. Invert around D, and use Pappu's Theorem as in Problem 4.5.4.

Remark. The synthetic solution of Problem 4.5.4 can't be reproduced here maybe because here we don't have A, P, D collinear, and we can't have harmonic quadrilaterals either.

Theorem 4.5.13 (Poncelet's Porism) — Poncelet's porism (sometimes referred to as Poncelet's closure theorem) states that whenever a polygon is inscribed in one conic section and circumscribes another one, the polygon must be part of an infinite family of polygons that are all inscribed in and circumscribe the same two conics.

Problem 4.5.6 (IMO 2013 P3). Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.

Solution. Straightforward use of **??**

Problem 4.5.7 (buratinogigle's proposed probs for Arab Saudi team 2015). Let ABC be acute triangle with $AB < AC$ inscribed circle (O). Bisector of $\angle BAC$ cuts (O) again at D. E is reflection of B through AD. DE cuts BC at F. Let (K) be circumcircle of triangle BEF. BD, EA cut (K) again at M, N, reps. Prove that $\angle BMN = \angle KFM$.

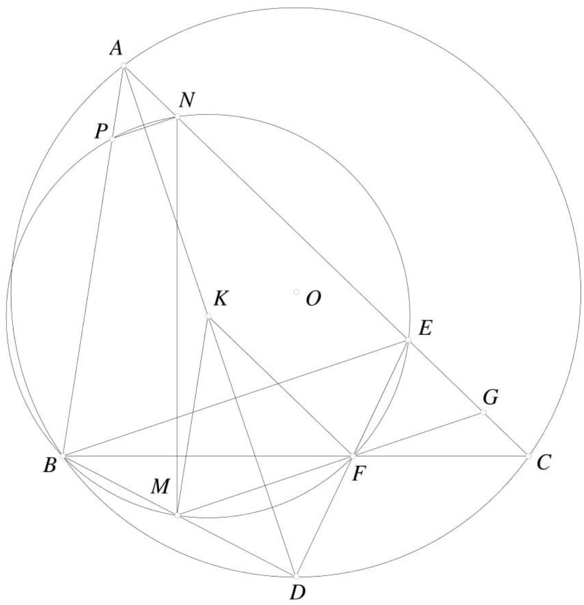


Figure 4.5.16

Problem 4.5.8 (USAMO 1999 P6). Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles.

Problem 4.5.9 (Serbia 2018 P1). Let $\triangle ABC$ be a triangle with incenter I. Points P and Q are chosen on segments BI and CI such that $2\angle PAQ = \angle BAC$. If D is the touch point of incircle and side BC prove that $\angle PDQ = 90$.

> **Solution.** Straightforward Trig application.

Problem 4.5.10 (Iran TST T2P5). Let ω be the circumcircle of isosceles triangle ABC ($AB = AC$). Points P and Q lie on ω and BC respectively such that $AP = AQ$ .AP and BC intersect at R. Prove that the tangents from B and C to the incircle of $\triangle AQR$ (different from BC) are concurrent on ω.

Problem 4.5.11. Let a point P inside of $\triangle ABC$ be such that the following condition is satisfied

$$\frac{AP+BP}{AB} = \frac{BP+CP}{BC} = \frac{CP+AP}{CA}$$

Lines AP, BP, CP intesect the circumcirle again at A', B', C'. Prove that ABC and A', B', C' have the same incircle.

Solution. After finiding the point P, we get a lot of ideas.

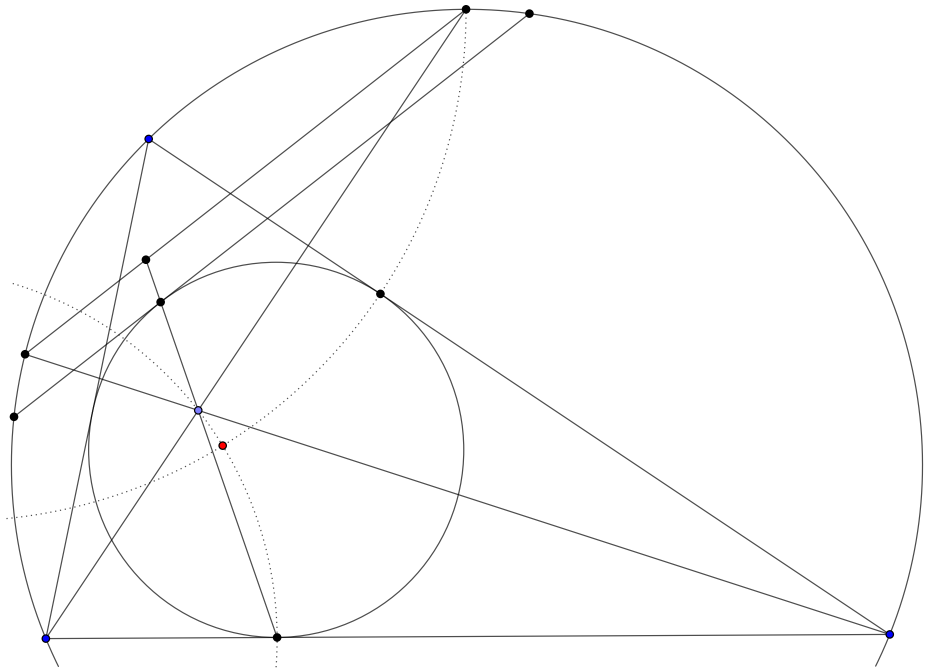


Figure 4.5.17: two lines are parallel

Problem 4.5.12 (Iran TST 2018 P3). In triangle ABC let M be the midpoint of BC. Let ω be a circle inside of ABC and is tangent to AB, AC at E, F, respectively. The tangents from M to ω meet ω at P, Q such that P and B lie on the same side of AM. Let $X \equiv PM \cap BF$ and $Y \equiv QM \cap CE$. If $2PM = BC$ prove that XY is tangent to ω.

Solution. Work backwards

Problem 4.5.13 (Iran TST 2018 P4). Let ABC be a triangle ($\angle A \neq 90°$). BE, CF are the altitudes of the triangle. The bisector of $\angle A$ intersects EF, BC at M, N. Let P be a point such that $MP \perp EF$ and $NP \perp BC$. Prove that AP passes through the midpoint of BC.

Problem 4.5.14 (APMO 2018 P1). Let H be the orthocenter of the triangle ABC. Let M and N be the midpoints of the sides AB and AC, respectively. Assume that H lies inside the quadrilateral $BMNC$ and that the circumcircles of triangles BMH and CNH are tangent to each other. The line through H parallel to BC intersects the circumcircles of the triangles BMH and CNH in the points K and L, respectively. Let F be the intersection point of MK and NL and let J be the incenter of triangle MHN. Prove that $FJ = FA$.

Problem 4.5.15 (ISL 2006 G6). Circles w_1 and w_2 with centres O_1 and O_2 are externally tangent at point D and internally tangent to a circle w at points E and F respectively. Line t is the common tangent of w_1 and w_2 at D. Let AB be the diameter of w perpendicular to t, so that A, E, O_1 are on the same side of t. Prove that lines AO_1, BO_2, EF and t are concurrent.

Solution. This

Lemma 4.5.14 (Tangential Quadrilateral Incenters) — Let $ABCD$ be a tangential quadrilateral. Let I_1, I_2 be the incenters of $\triangle ABD, \triangle BCD$. Then $(I_1), (I_2)$ is tangent to BD at the same point.

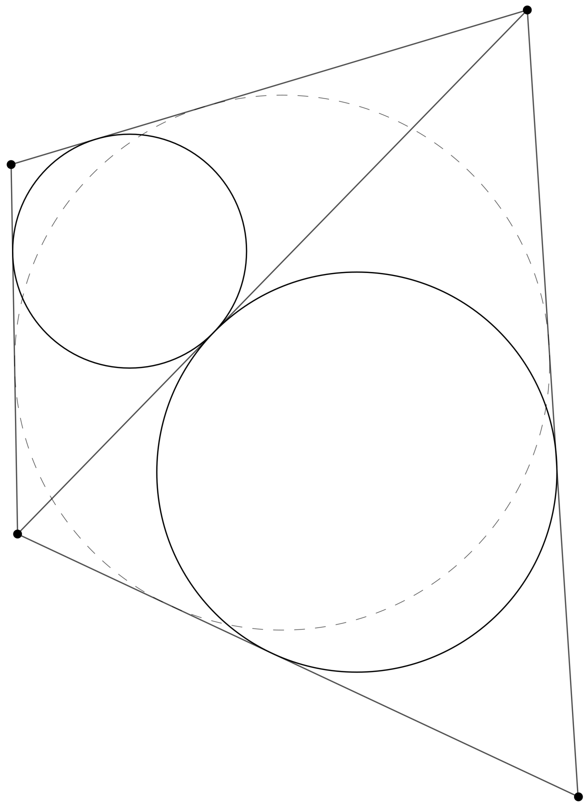


Figure 4.5.18

Problem 4.5.16 (Four Incenters in a Tangential Quadrilateral). Let $ABCD$ be a quadrilateral. Denote by X the point of intersection of the lines AC and BD. Let I_1, I_2, I_3, I_4 be the centers of the incircles of the triangles XAB, XBC, XCD, XDA, respectively. Prove that the quadrilateral $I_1I_2I_3I_4$ has a circumscribed circle if and only if the quadrilateral $ABCD$ has an inscribed circle.

Solution. There is a lot going on in this figure, firstly, the J_1, J_2 and M, then K, then $\angle I_4ME = \angle I_3ME$. Connecting them with the lemma.

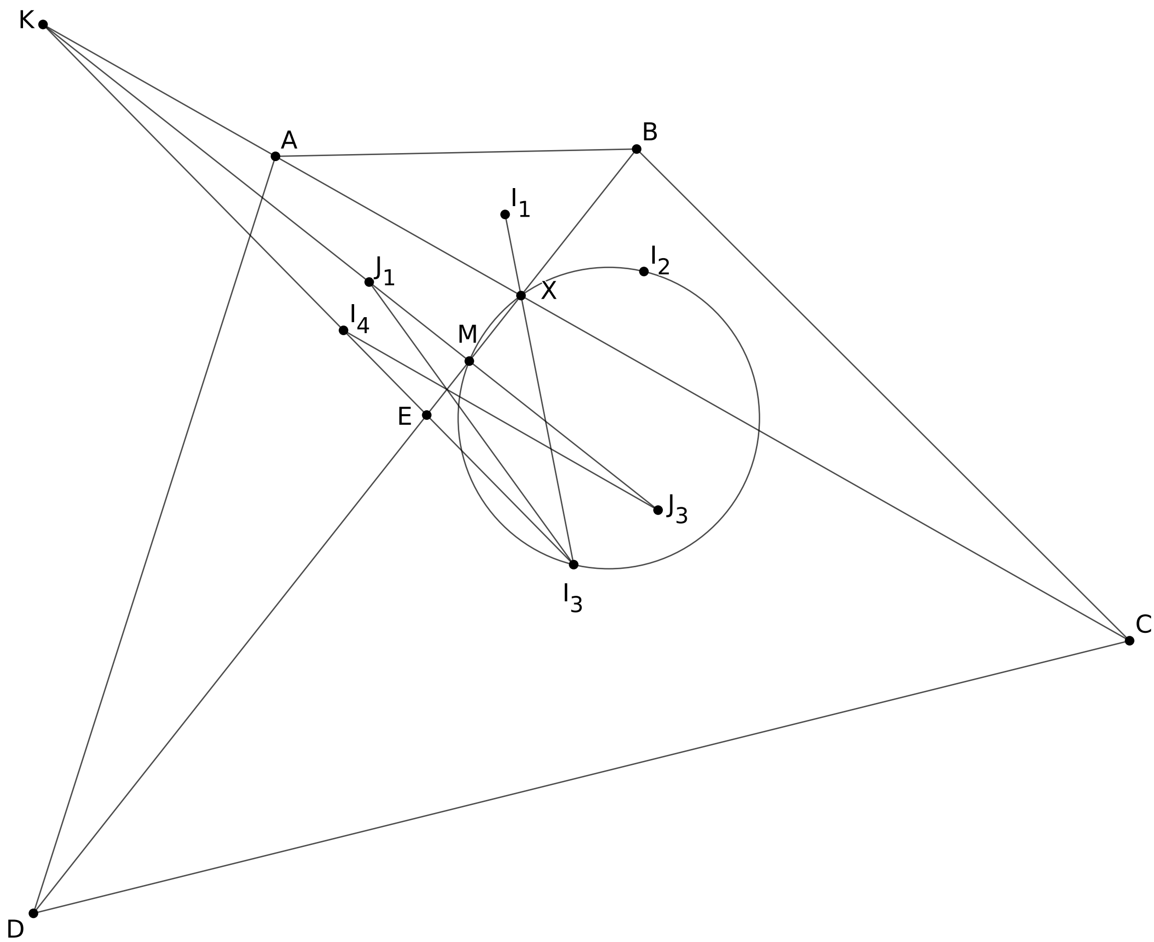


Figure 4.5.19

Problem 4.5.17 (Geodip). Let G be the centeroid. Dilate $\odot I$ from G with constant -2 to get I'. Then I' is tangent to the circumcircle.

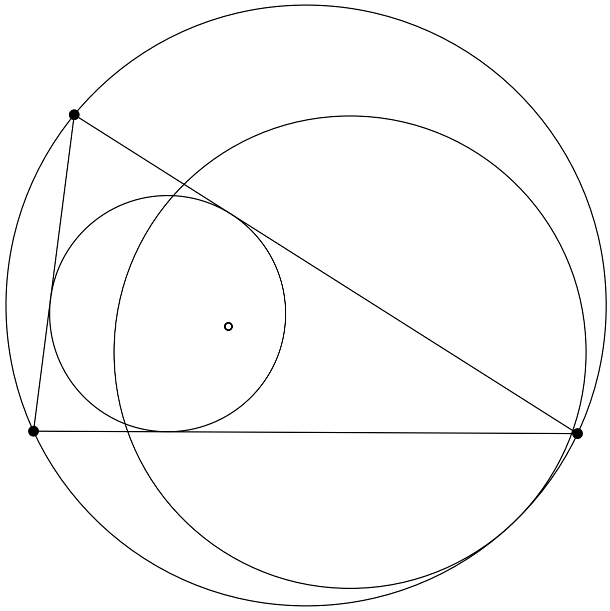


Figure 4.5.20

Theorem 4.5.15 (Fuhrmann Circle) — Let X', Y', Z' be the midpoints of the arcs not containing A, B, C of $\odot ABC$. Let X, Y, Z be the reflections of these points on the sides. Then $\odot XYZ$ is called the **Fuhrmann Circle**. The orthocenter H and the nagel point N lies on this circle, and HN is a diameter of this circle.
Furthermore, AH, BH, CH cut the circle for the second time at a distance $2r$ from the vertices.

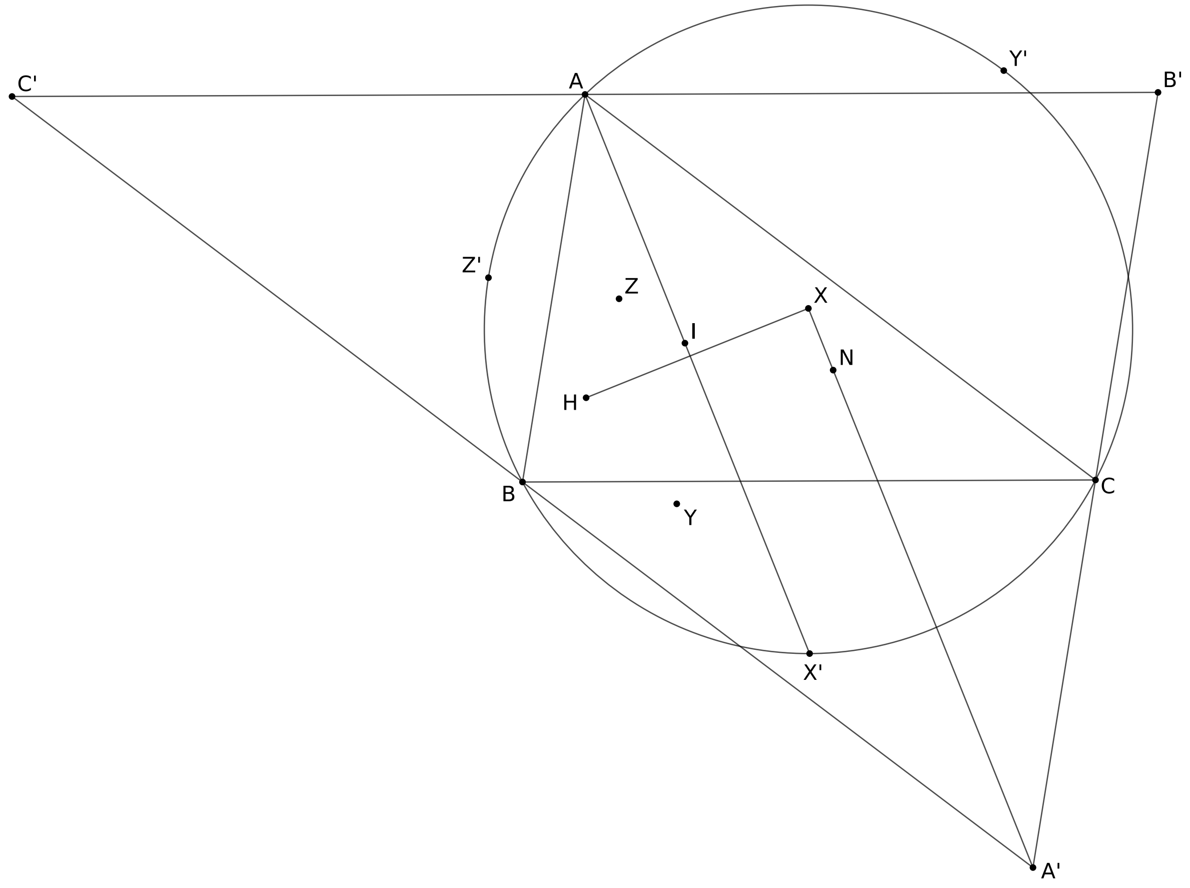


Figure 4.5.21: Fuhrmann Circle

Problem 4.5.18 (Iran TST 2008 P12). In the acute-angled triangle ABC, D is the intersection of the altitude passing through A with BC and I_a is the excenter of the triangle with respect to A. K is a point on the extension of AB from B, for which $\angle AKI_a = 90^\circ + \frac{3}{4}\angle C$. I_aK intersects the extension of AD at L. Prove that DI_a bisects the angle $\angle AI_aB$ iff $AL = 2R$. (R is the circumradius of ABC)

Solution.

Lemma 4.5.16 (Polars in Incircle) — In the acute angled triangle ABC, I is the incenter and DEF is the touch triangle. Let EF meet $\odot ABC$ at P, Q such that E lies inside F, Q. If QD meets $\odot ABC$ for the second time at U, prove that AU is the polar line of P wrt (I).

Proof [Projective]. We have:

$$\begin{aligned}(B, C; D, EF \cap BC) &= Q\,(B, C; P, U)\\ &= A(E, F; P, U)\\ &= -1\end{aligned}$$

Which means $(P, AU \cap EF; E, F)$ is harmonic.

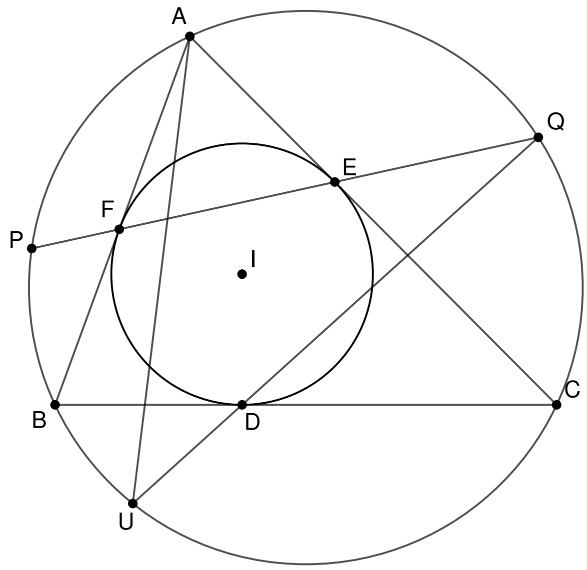


Figure 4.5.22

Problem 4.5.19 (ISL 2019 G6). In the acute angled triangle ABC, I is the incenter and DEF is the touch triangle. Let EF meet $\odot ABC$ at P, Q such that E lies inside F, Q. Prove that

$$\angle APD + \angle AQD = \angle PIQ$$

Solution. Since $PI \cap AU$ at X from Lemma 4.5.16, we have $AFPX$ is cyclic. And so

$$\angle FAX = \angle FIX$$

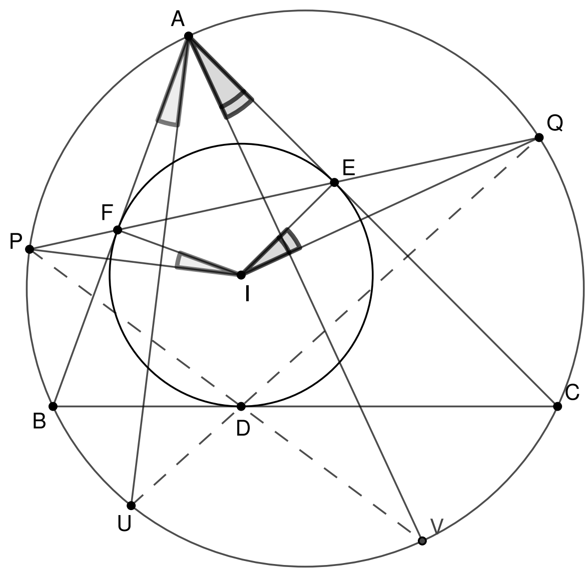


Figure 4.5.23

After some more angle chasing, we reach our goal.

4.5.1 Feurbach Point

Definition (Feurbach Point)— The point where the nine point circle touches the incircle is called the *Feurbach Point*.

Theorem 4.5.17 (It Exists!) — The nine point circle touches the incircle and the excircles.

Proof [Inversion]. Let D, D' be the incircle and the A-excircle touchpoints with BC. Let M, N, P be the midpoints of BC, CA, AB resp. Also let $B'C'$ be the reflection of BC on AI. Now let N', P' be the intersection points of MN, MP with $B'C'$.

We invert around M with radius $MD = \frac{b-c}{2}$. We prove that the image of $\odot MNP$ after the inversion is $B'C'$. And since (I) and (I_a) are orthogonal to (M), we will be done.

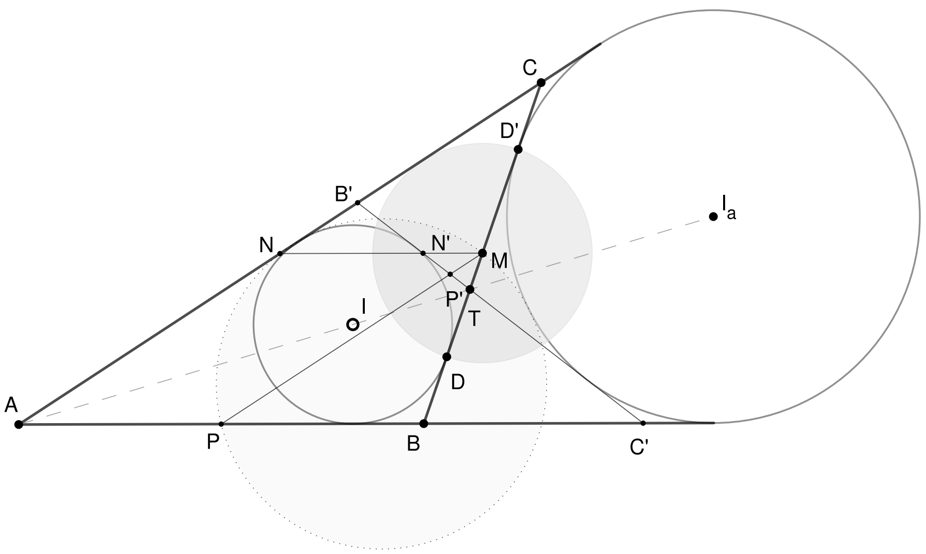


Figure 4.5.24

Wlog, assume that $b \geq c$.

$$B'N = AB' - AN = c - \frac{b}{2} \qquad NN' = B'N \cdot \frac{AC'}{AB'}$$

$$MN' = MN - NN' \qquad = \frac{2c-b}{2} \cdot \frac{b}{2}$$

$$MN' \cdot MN = \frac{c}{2}\left(\frac{c}{2} - \frac{b}{c} \cdot \frac{2c-b}{2}\right) = \frac{b-c^2}{2}$$

Which concludes the proof.

Theorem 4.5.18 (Construction of Feurbach Point) — Let D be the incenter touch point with BC. Let M, L be the midpoints of BC and AI. Let D_1, D' be the reflections of D over I and M. Let K, P be the refecltions of D_1, D over L and AI. Let Q be the intersection of AD_1 with the incircle.

Then D_1K and MP meet at F on the incircle, which is the Feurbach Point of $\triangle ABC$.

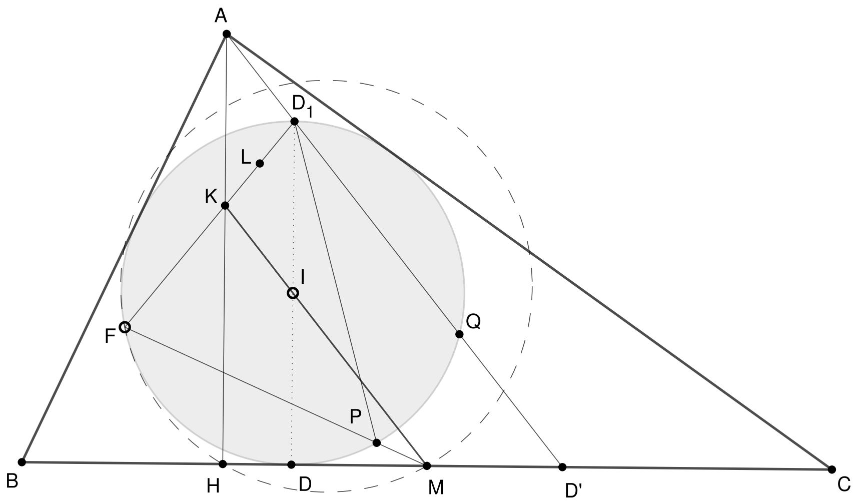


Figure 4.5.25

Proof. It is easy to see that the tangents at P and M to the incircle and the nine point circle are parallel. So if we let $F = MP \cap (I)$, then we have F is the Feurbach point.

And since MQ is tangent to (I), we also have $(F, P; D, Q) = -1$. But notice that

$$\begin{aligned} D_1(A, I; L, P) &= D_1(K, P; Q, I) \\ &= -1 \end{aligned}$$

So D_1K passes through F.

4.5.2 Assorted Diagrams

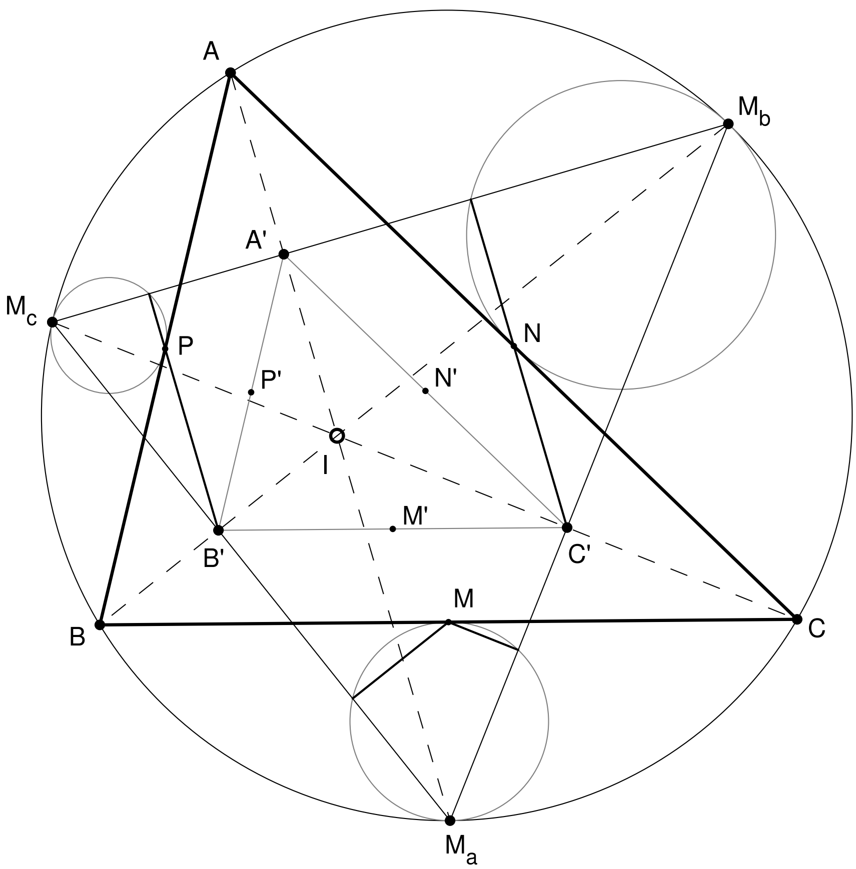


Figure 4.5.26: The smaller circles touches the side and the circumcircle

4.6 Conjugates

4.6.1 Isogonal Conjugate

Theorem 4.6.1 (Isogonal Line Lemma) — Let AP, AQ are isogonal lines with respect to $\angle BAC$. Let $BP \cap CQ = F$ and $BQ \cap CP = E$. Then AE, AF are isogonal lines with respect to $\angle BAC$.

Proof.

$$\begin{aligned} A(B, F; P, X) = (B, F; P, X) &= C(B, Q; E, X) \\ &= (B, Q; E, X) = (X, E; Q, B) \end{aligned}$$

So if we define a projective transformation that swaps isogonal lines wrt $\angle BAC$, we see AE, AF are conjugates of each other.

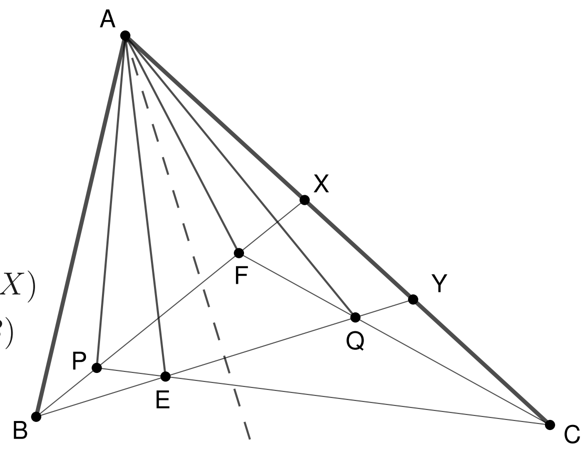


Figure 4.6.1

Problem 4.6.1 (India Postals 2015 Set 2). Let $ABCD$ be a convex quadrilateral. In the triangle ABC let I and J be the incenter and the excenter opposite the vertex A, respectively. In the triangle ACD let K and L be the incenter and the excenter opposite the vertex A, respectively. Show that the lines IL and JK, and the bisector of the angle BCD are concurrent.

Solution. Using Theorem 4.6.1

Lemma 4.6.2 — Let ω_1, ω_2 be two circles such that ω_1 passes through A, B and is tangent to AC at A. ω_2 is defined similarly by swapping B with C. $\omega_1 \cap \omega_2 = X$.
Let γ_1, γ_2 be two circles such that γ_1 passes through A, B and is tangent to BC at B. γ_2 is defined similarly by swapping B with C. $\gamma_1 \cap \gamma_2 = Y$.
Then X, Y are isogonal conjugates wrt $\triangle ABC$.

Lemma 4.6.3 (Isogonality in quadrilateral) — For a point X, its isogonal conjugate wrt a quadrilateral $ABCD$ exists iff

$$\measuredangle BXA + \measuredangle DXC = 180^\circ$$

Solution. Draw the cirles, look for similarity.

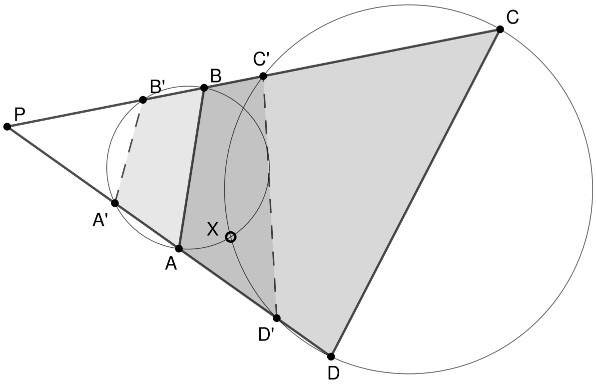


Figure 4.6.2: Isogonality in quadrilateral

Lemma 4.6.4 (Ratio) — Given a $\triangle ABC$ with isogonal conjugate P, Q. Let AP, AQ cut the circumcircle of $\triangle ABC$ again at U, V, respectively and let $D \equiv AP \cap BC$. Then

$$\frac{AQ}{QV} = \frac{PD}{DU}$$

Proof. By using cross ratio:

$$\begin{aligned}(A, F; Q, V) &= C(A, F; Q, V)\\ &= C(D, A; P, V*) = (D, A; P, V*)\\ &= (A, D; V*, P)\end{aligned}$$

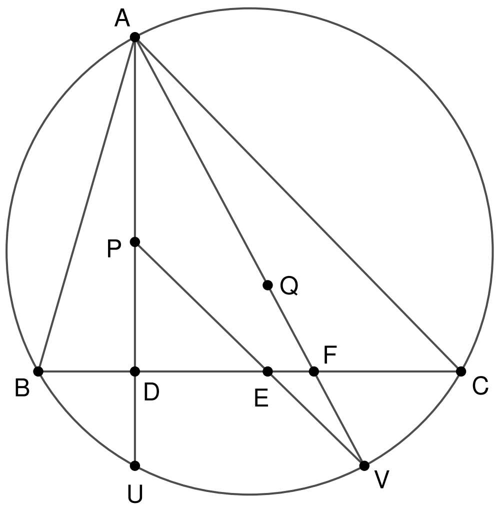


Figure 4.6.3

4.6.1.1 Symmedians

Definition (Symmedians)— In $\triangle ABC$, let T_a, T_b, T_c be the meet points of the tangents at A, B, C. Let $\triangle N_aN_bN_c$ be the cevian triangle of AT_a, BT_b, CT_c. Let S be the symmedian point of $\triangle ABC$. Let M_a, M_b, M_c be the midpoints of BC, CA, AB.

> **Lemma 4.6.5** (Most Important Symmedian Property) — Let the circles tangent to AC, AB at A and passes through B, C respectively meet at T' for the second time. Let $AT_a \cap \odot ABC = A'$. Let the tangents to $\odot ABC$ at A, A' meet BC at T. Prove that, $A, T', T_a,$ and T, T', O are collinear.

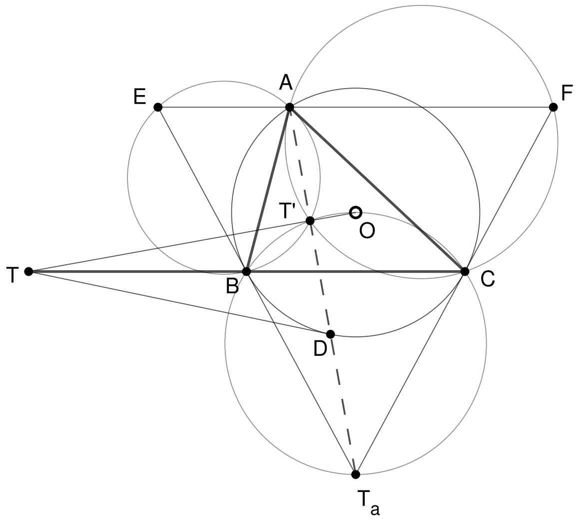


Figure 4.6.4: T' is quite special!

Problem 4.6.2 (USAMO 2008 P2). Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of BC, CA, and AB, respectively. Let the perpendicular bisectors of AB and AC intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.

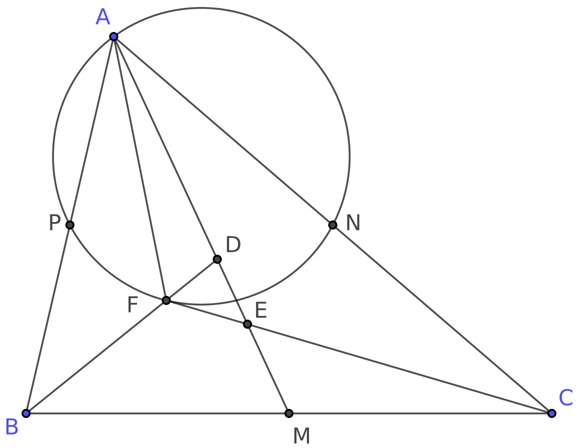


Figure 4.6.5: USAMO 2008 P2

> **Solution** [Phantom Point]. First assume $F \in BD$, and $F = T'$ (Where T' comes from Lemma 4.6.5, and prove that $F \in CE$.)

> **Solution** [Isogonal Conjugate]. Construct the isogonal conjugate of F, which is the intersection of the circles touching BC and passing through A, B and A, C.

> **Solution.** Using Theorem 4.6.1 by taking the reflections of B, C over D, F

Problem 4.6.3 (IRAN TST 2015 Day 3, P3). AH is the altitude of triangle ABC and H' is the reflection of H trough the midpoint of BC. If the tangent lines to the circumcircle of ABC at B and C, intersect each other at X and the perpendicular line to XH' at H', intersects AB and AC at Y and Z respectively, prove that $\angle ZXC = \angle YXB$.

Problem 4.6.4 (Two Symmedian Points). Let E, F be the feet of B, C-altitudes. Let K, K_A be the symmedian points of $\triangle ABC, \triangle AEF$. Prove that $KK_A \perp BC, KK_A \cap BC = P$ and $KK_A = KP$

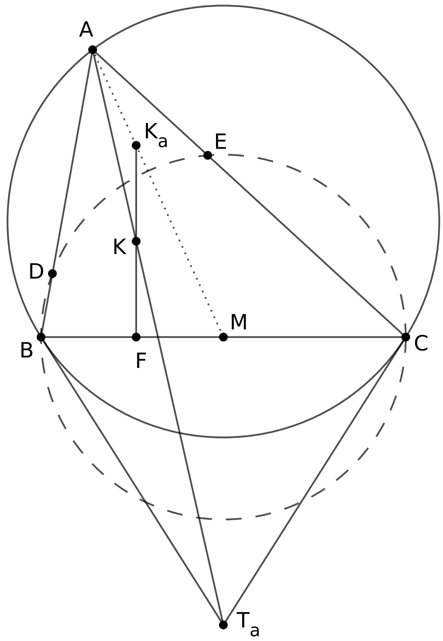


Figure 4.6.6: $KK_A \perp BC$

4.6.2 Isotonic Conjugate

Theorem 4.6.6 (Isotonic Lemma) — Let M be the midpoint of BC, and PQ such that Q is the reflection of P on M. Two points Q, R on AP, AQ, $BQ \cap CR = X$, $BR \cap CQ = Y$. Then AX, AY are isotonic wrt BC.

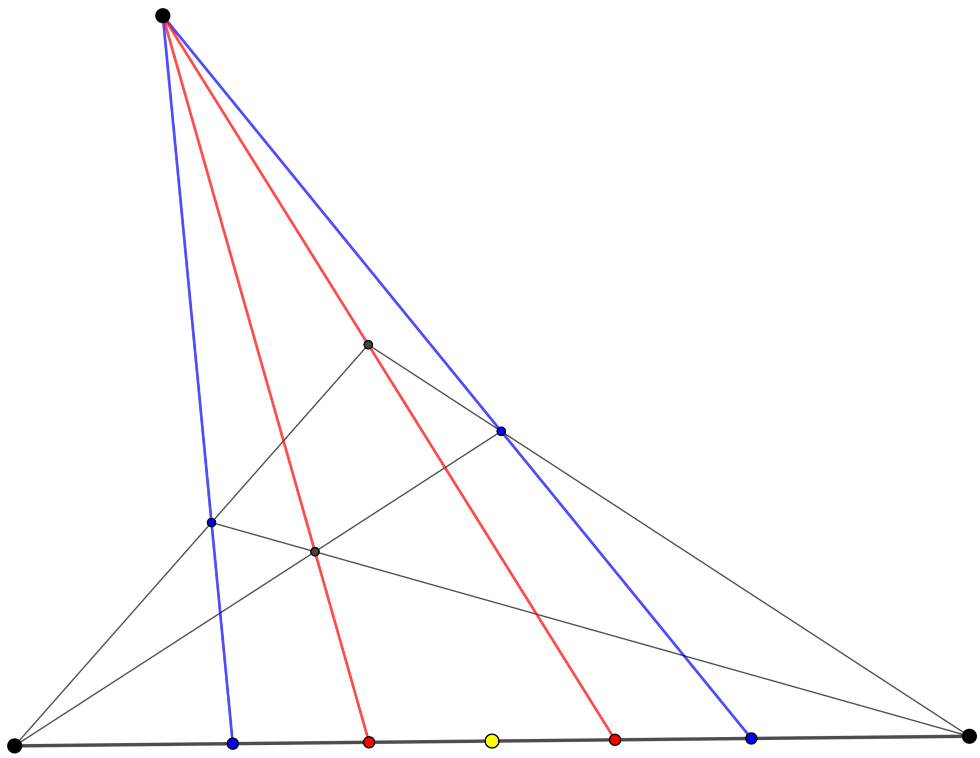


Figure 4.6.7

Problem 4.6.5 (IGO 2014 S5). Two points P and Q lying on side BC of triangle ABC and their distance from the midpoint of BC are equal.The perpendiculars from P and Q to BC intersect AC and AB at E and F,respectively.M is point of intersection PF and

EQ.If H_1 and H_2 be the orthocenters of triangles BFP and CEQ, respectively, prove that $AM \perp H_1H_2$.

Solution. We first show that the slope of H_1H_2 is fixed, and then show that AM is fixed where we use isotonic lemma, and finally show that these two lines are perpendicular.

4.6.3 Reflection

Lemma 4.6.7 (Homothety and Reflection) — Let two oppositely oriented congruent triangles be $\triangle ABC, \triangle DEF$. Prove that the midpoints of AD, BE, CF are collinear.

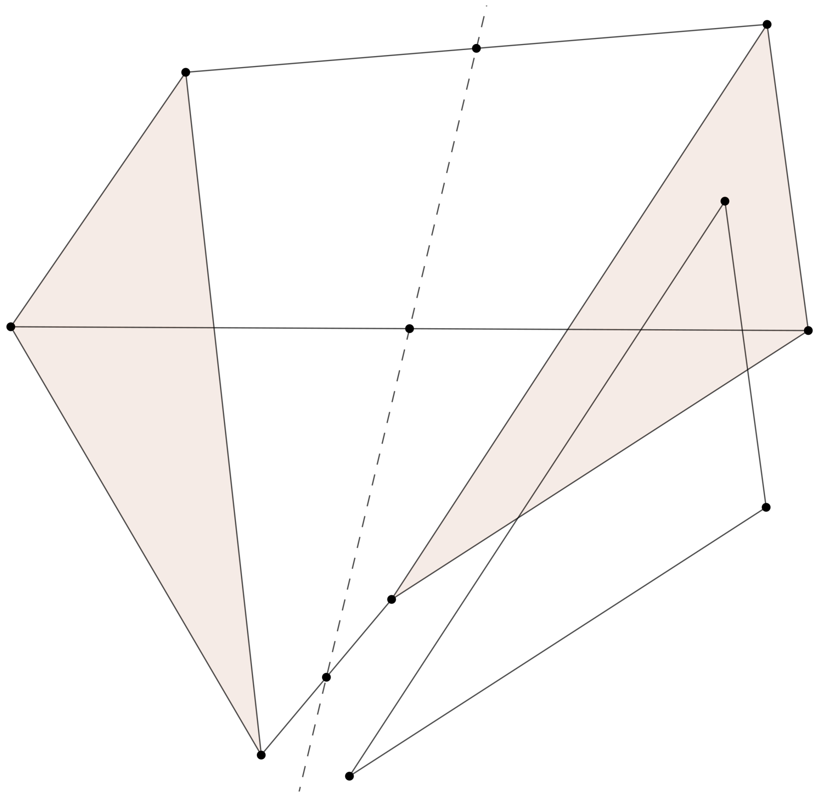


Figure 4.6.8: Oppositely oriented congruent triangles

Problem 4.6.6 (`Autumn Tournament, 2012`). Let two oppositely oriented equilateral triangles be $\triangle ABC, \triangle DEF$. What is the least possible value of $\max(AD, BE, CF)$?

4.7 Circles that are really touching

Definition (Mixtilinear Circle)— Let $\triangle ABC$ be an ordinary triangle, I is its incenter, D is the touch points of the incenter with BC. Let ω be the mixtilinear incircle. Let it touch CA, AB at E, F. Furthermore, let $\omega \cap \odot ABC \equiv T$. Let M_a, M_b, M_c be the midpoints of the smaller arcs BC, CA, AB, and M_A, M_B, M_C be the midpoints of the major arcs BC, CA, AB.

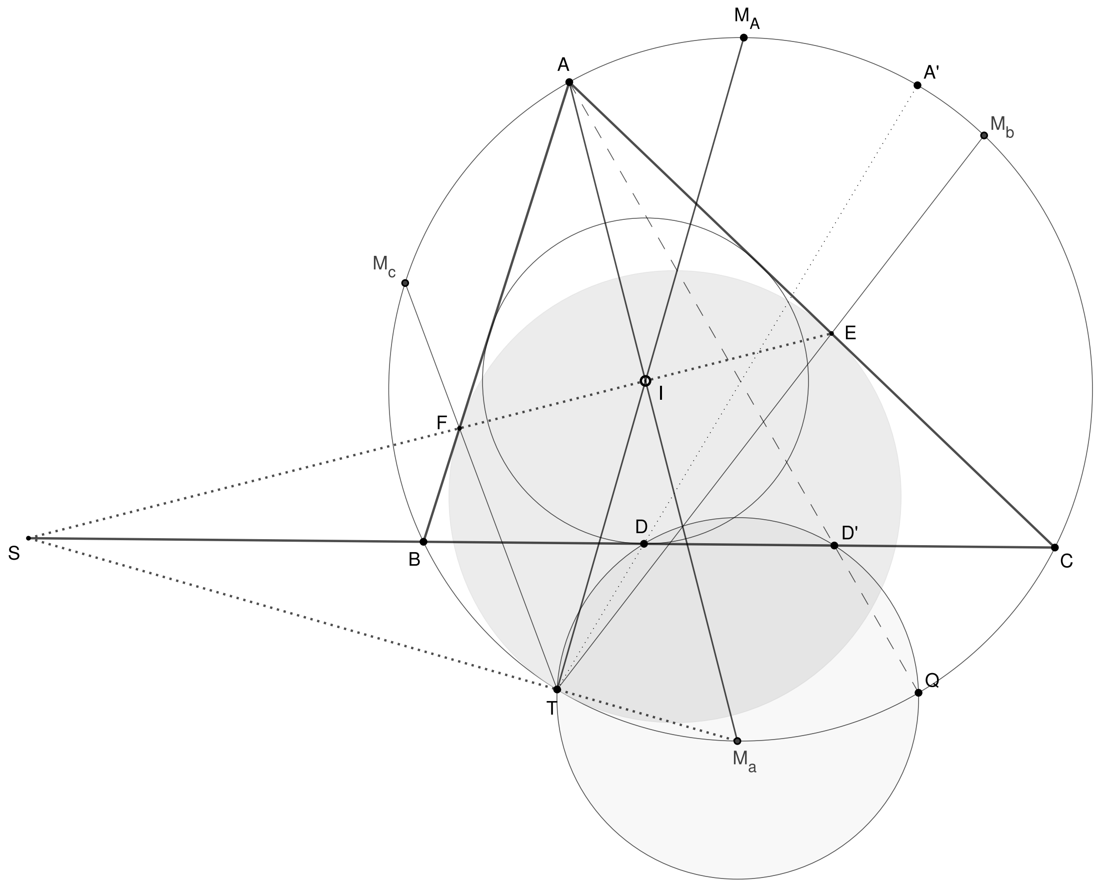


Figure 4.7.1: Mixtilinear Incircle: more than meets the eye

Proof. E, I, F are collinear. Consider the circle $TI'EC$ and do so T, I, M_A are collinear. Consider the circle $TIEC$ and apply reim's theorem and angle chasing. Also, proving that AC touches $\odot TCD$ proves that T, D, A' collinear.

Lemma 4.7.1 — The bundle $(A, T; M_b, M_c)$ is harmonic. And TA is a symmedian of $\triangle TM_cM_b$.

$$\frac{TM_c}{M_cA} = \frac{TM_b}{M_bA}$$

Lemma 4.7.2 (ISL 1999 G8) — Let X be a variable point on the arc AB, and let O_1 and O_2 be the incenters of the triangles CAX and CBX. Then X, O_1, O_2 and T lie on a circle.

Solution. Using similarity and Lemma 4.7.

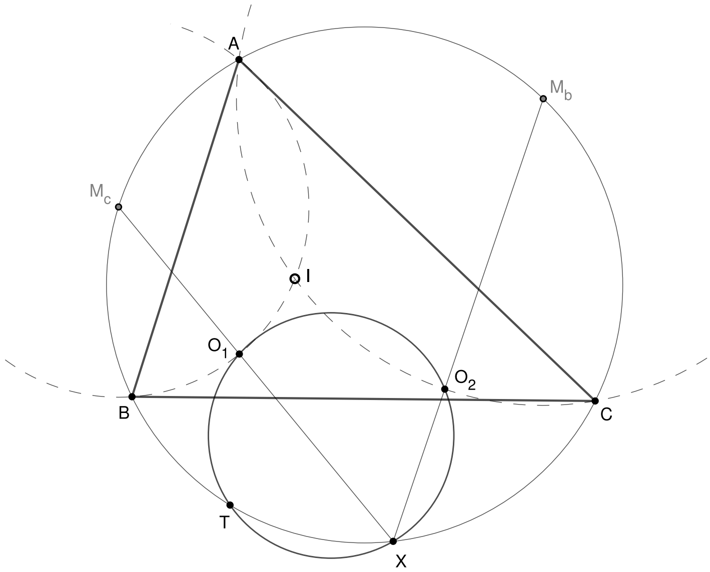


Figure 4.7.2: The two incenters are cyclic with T, X

Problem 4.7.1 (AoPS1). Let $ABCD$ be a quadrilateral inscribed in a circle, such that the inradius of $\triangle ABC$ and ACD are the same. Let T be the touchpoint of A-mixtilinear incircle of the triangle ABD with $\odot ABCD$. Let I_1, I_2 be the incenters of the triangles ABC, ACD respectively. Show that I_1I_2 and the tangents of A, T wrt $\odot ABCD$ are concurrent.

Solution. We know from Lemma 4.7, that TI_1I_2C is cylic. Moreover from the given condition, we know that AC bijects I_1, I_2. Which gives us $I_1I_2 \parallel TC$. Using that we can show that if $S = I_1I_2 \cap M_cM_b$, then TS is tangent to $\odot ABC$, which also gives us SA is tangent to $\odot ABC$.

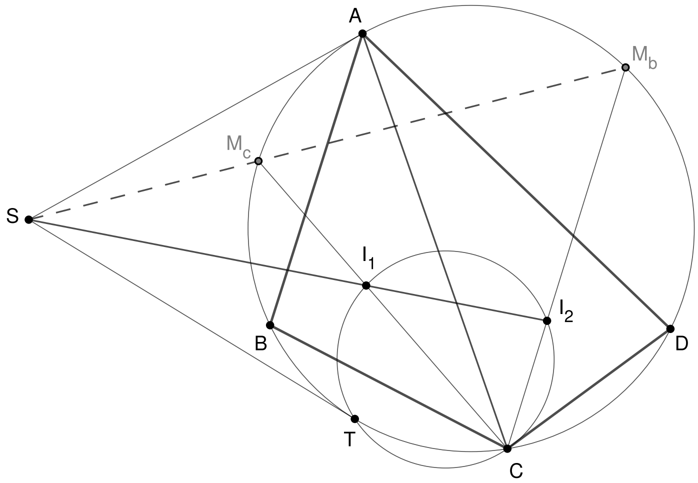


Figure 4.7.3

Definition (Generalization of Mixtilinear Incirlce)— Consider $\triangle ABC$ and let M_c, M_b are midpoints of arcs AB, AC. Let E, F on AB, AC such that $EF \parallel M_bM_c$. Let EM_b, FM_c meet (ABC) second time at P, Q. Consider two intersection points E', F' of $(EFPQ)$ with AB, AC different from E, F. Then $EF' \cap E'F$ is the incenter of ABC.

In other words, if P lies on the arc BC, and $M_bP \cap AC = E$, $M_cP \cap AB = F'$, then E, F', I are collinear.

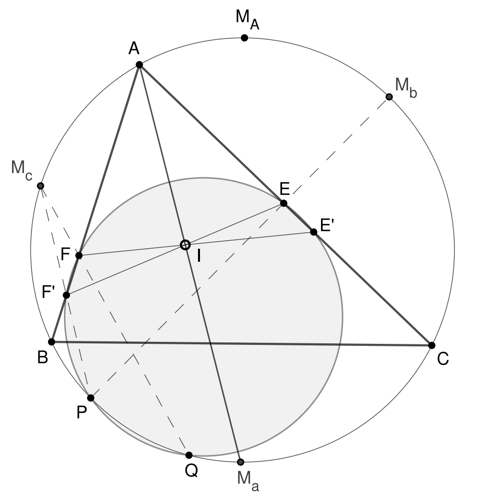


Figure 4.7.4

Problem 4.7.2 (Archer - EChen M1P3). Let the incenter touch BC at D. Let $AI \cap BC = E$, $AI \cap \odot ABC = F$. Prove that $\odot DEF \cap \odot ABC = T$, the mixtilinear touch-point. Now let, $\odot DEF \cap \odot(I_a) = S_1, S_2$. Prove that AT goes through either S_1 or S_2.

Solution. We already know that T, D, E, F are cyclic. Then, let $S = AT \cap \odot DEF$, then we have,

$$\angle FSE = \angle FDE = \angle FD'E$$

Which means S' is the reflection of D' over AI, which definitely lies on (I_a).

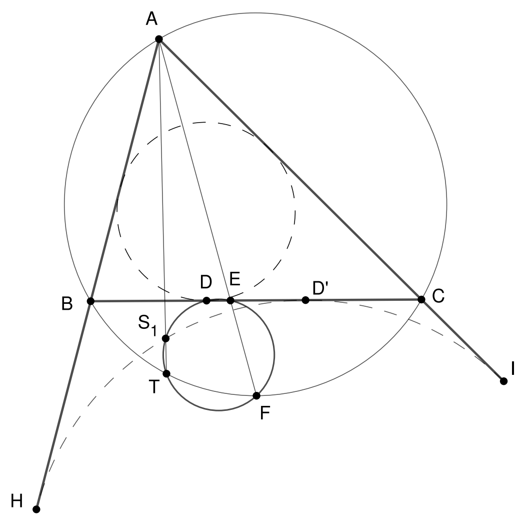


Figure 4.7.5: Problem 4.7.2

Problem 4.7.3. Let the B-mixtilinear and C-mixtilinear circles touch BC at X_B, X_C respectively. Then X_B, X_C, T, M_a lie on a circle.

Solution. We use mixtilinear related lemma to show that, $ST \cdot SM_a = SX_b \cdot SX_c$. We can do that by angle chase to show that $SI^2 = SX_b \cdot SX_c$.

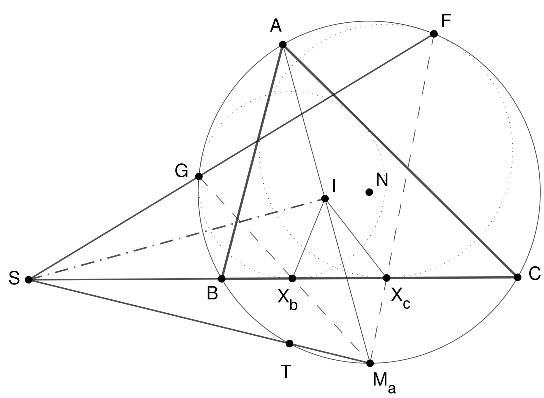


Figure 4.7.6

Theorem 4.7.3 (Root BC Inversion in Mixtilinear Circles) — If we invert around A with the radius $\sqrt{bc}$, the mixtilinear incircle goes to the excicle.

Proof. We use Problem 4.7.2 to show the results.

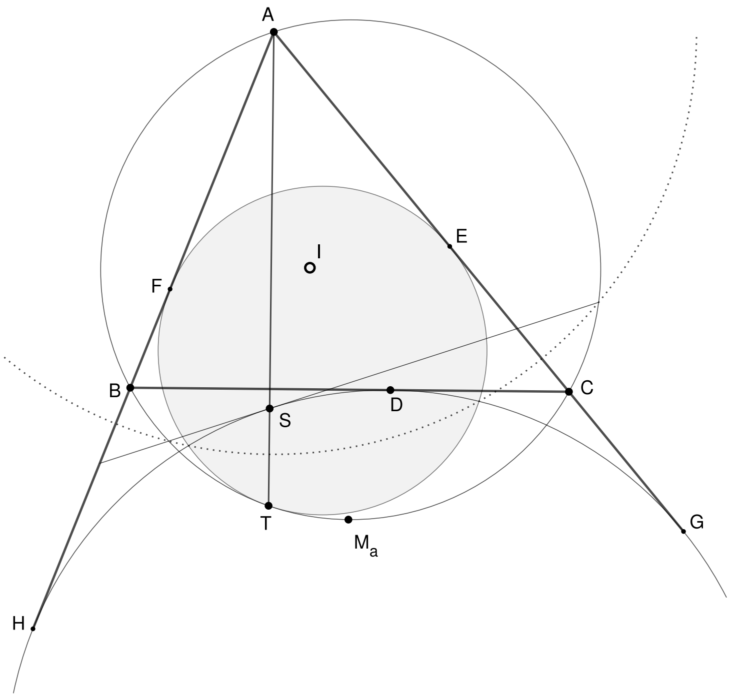


Figure 4.7.7

Problem 4.7.4 (Taiwan TST 2014 T3P3). Let M be any point on the circumcircle of $\triangle ABC$. Suppose the tangents from M to the incircle meet BC at two points X_1 and X_2. Prove that T, M, X_1, X_2 lie on a circle.

Definition (Curvilinear Incircle)— Let $ABCD$ be a cyclic quadrilateral. AC meets BD at X. We call the circle that touches AX, BX and the circumcircle from the inside a curvilinear incircle. The curvilinear incircle touch AX, BX at P, Q resp. And let the incircle of $\triangle ABD$ be I. Then P, Q, I are collinear.

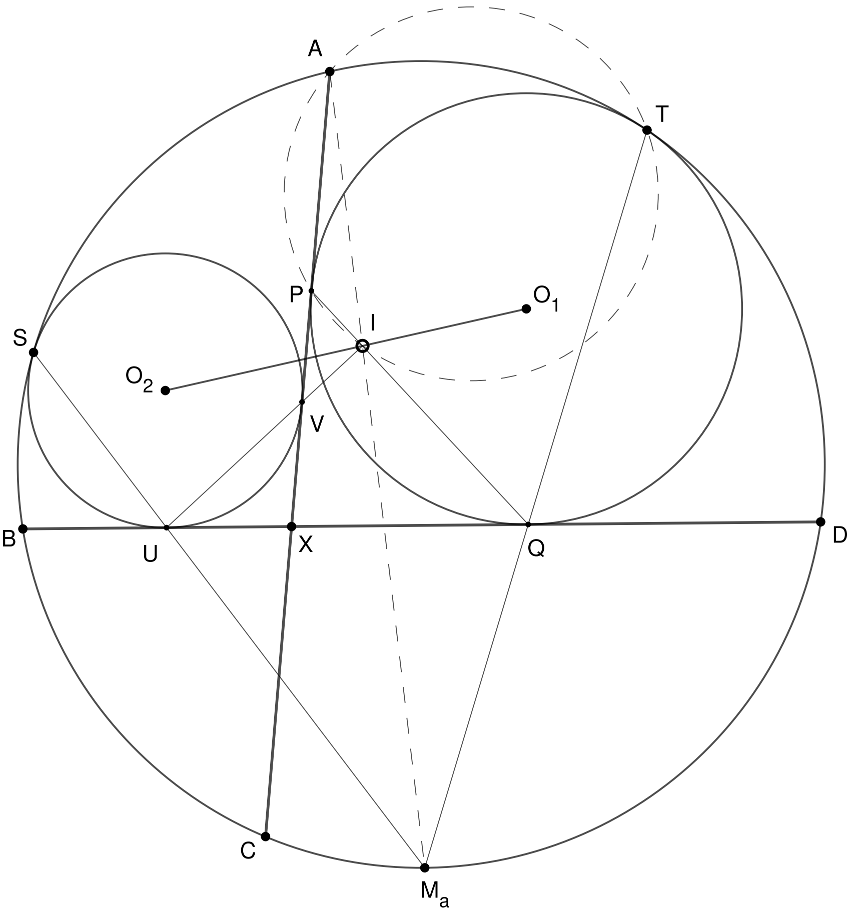


Figure 4.7.8

Theorem 4.7.4 (Sawayama and Thebault's theorem) — Let O_1, O_2 be the centers of the two curvilinear incircles on BD and AC. Then O_1, I, O_2 are collinear.

Solution. Notice that this is similar to the circles $TIEC$ and $TIFB$ in the mixtilinear circle figures.

4.8 Circles and Radical Axises

Problem 4.8.1. In $\triangle ABC$, H is the orthocenter, and AD, BE are arbitrary cevians. Let ω_1, ω_2 denote the circles with diameters AD, BE resp. HD, HE meet ω_1, ω_2 again at F, G. DE meet ω_1, ω_2 again at P_1, P_2. FG meet ω_1, ω_2 again at Q_1, Q_2. P_1H, P_2H meet ω_1, ω_2 at R_1, R_2 and Q_1H, Q_2H meet ω_1, ω_2 at S_1, S_2. $P_1Q_1 \cap P_2Q_2 \equiv X$ and $R_1S_1 \cap R_2S_2 \equiv Y$. Prove that X, Y, H are collinear.

Solution. Too much info...

Lemma 4.8.1 (Pseudo Miquel's Theorem) — In a $\triangle ABC$ let E, F be points on AC, AB and D be a point on $\odot(ABC)$. Let $X = \odot(BFD) \cap \odot(CED)$ then E, F, X are collinear.

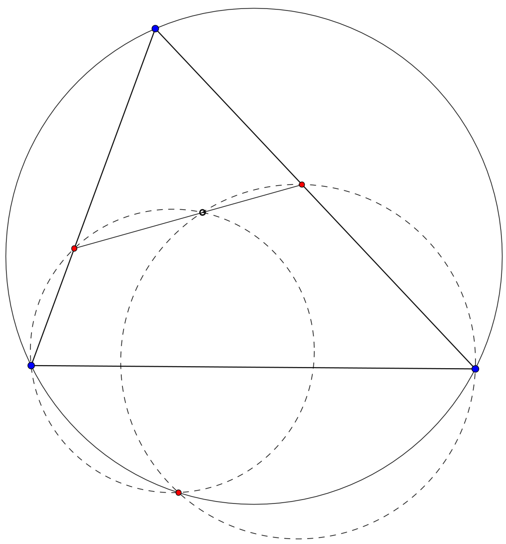


Figure 4.8.1: Notice the collinearity

Problem 4.8.2 (buratinogigle's proposed problems for Arab Saudi team 2015). Let ABC be a triangle and (K) is a circle that touches segments CA, AB at E, F, reps. M, N lie on (K) such that BM, CN are tangent to (K). G, H are symmetric of A through E, F. The circle passes through G and touches to (K) at N that cuts CA again at S. The circle passes through H and touches (K) at M that cuts AB again at T. Prove that the line passes through K and perpendicular to ST always passes through a fixed point when (K) changes.

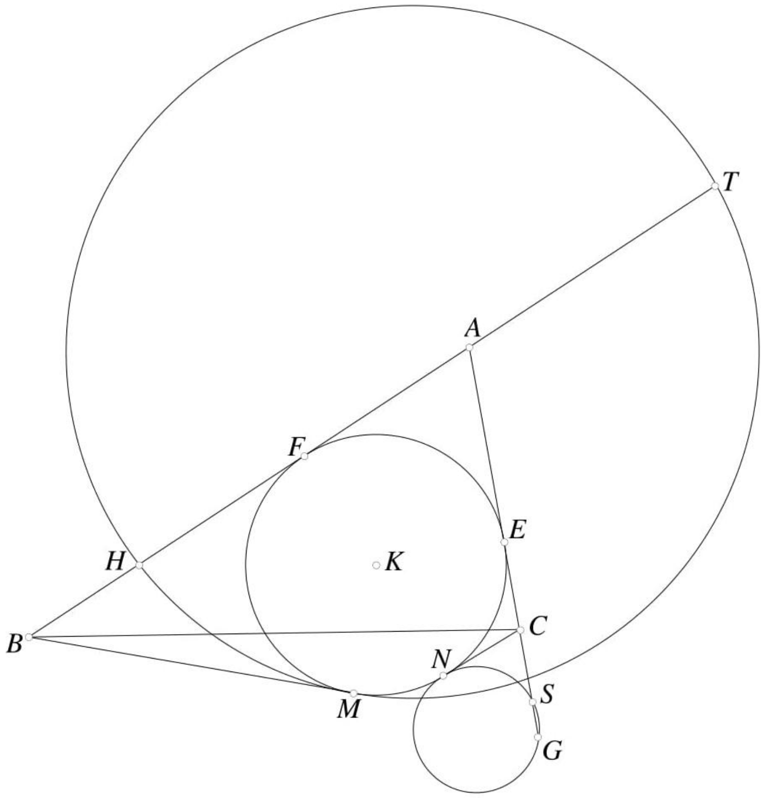


Figure 4.8.2

Problem 4.8.3 (ISL 2002 G8). Let two circles S_1 and S_2 meet at the points A and B. A line through A meets S_1 again at C and S_2 again at D. Let M, N, K be three points on the line segments CD, BC, BD respectively, with MN parallel to BD and MK parallel to BC. Let E and F be points on those arcs BC of S_1 and BD of S_2 respectively that do not contain A. Given that EN is perpendicular to BC and FK is perpendicular to BD prove that $\angle EMF = 90°$.

Solution. When one single property can produce a lot others, and we need to prove this property, assume the property to be true and work backwards.

Problem 4.8.4 (APMO 1999 P3). Let Γ_1 and Γ_2 be two circles intersecting at P and Q. The common tangent, closer to P, of Γ_1 and Γ_2 touches Γ_1 at A and Γ_2 at B. The tangent of Γ_1 at P meets Γ_2 at C, which is different from P, and the extension of AP meets BC at R. Prove that the circumcircle of triangle PQR is tangent to BP and BR.

Problem 4.8.5 (USA TST 2019 P1). Let ABC be a triangle and let M and N denote the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Let X be a point such that $\overline{AX}$ is tangent to the circumcircle of triangle ABC. Denote by ω_B the circle through M and B tangent to $\overline{MX}$, and by ω_C the circle through N and C tangent to $\overline{NX}$. Show that ω_B and ω_C intersect on line BC.

Solution [Spiral Similarity]. Let $\omega_C \cap BC = P$. If we extend NP to meet AB at R, we get $XANR$ cyclic. Similarly, if $\odot XAM \cap AC = Q$, then we have to prove $QM \cap NR = P$.

Suppose $QM \cap NR = P'$. Then by spiral similarity, X takes $Q \to M$ and $N \to R$. It also takes $Q \to N$ and $M \to R$. So $XMP'R$ is cyclic. We now show that $XMPR$ is also cyclic, which will prove $P = P'$.

Let $T = \odot ABC \cap \odot XAN$. By spiral similarity, T takes $R \to B$ and $N \to C$. It also takes $R \to N$ and $B \to C$, which means $RBPT$ is cyclic.

By spiral similarity, we have, $\triangle TXA \sim \triangle TNC$, $\triangle TXN \sim \triangle TAC$, $\triangle TBA \sim \triangle TPN$ Which implies,

$$\frac{XN}{TN} = \frac{AC}{TC}, \frac{XA}{TA} = \frac{NC}{TC}$$

$$\implies \frac{XN}{XA} = 2\frac{TN}{TA}$$

And so,

$$\frac{AB}{TA} = \frac{NP}{TN} \quad \implies \frac{2AM}{NP} = \frac{TA}{TN} = \frac{XA}{XN}2$$

$$\implies \frac{AM}{NP} = \frac{XA}{XN}$$

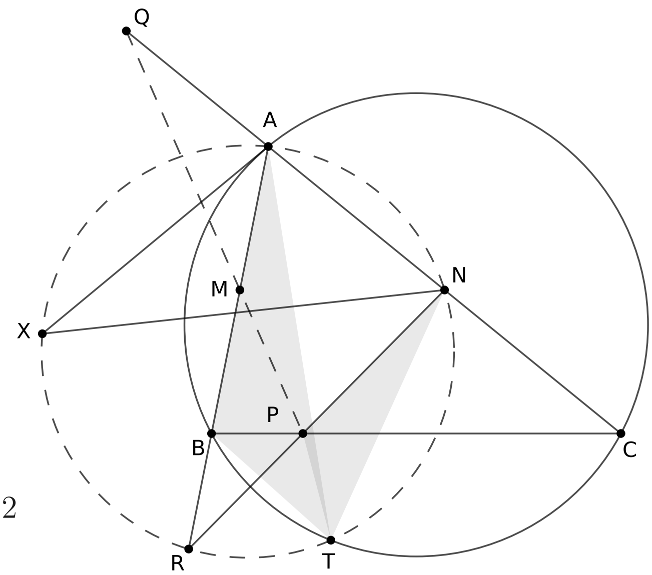


Figure 4.8.3

Which means $\triangle XAM \sim \triangle XNP$ since $\angle XAM = \angle XNP$, which concludes the proof.

Solution [Clever Observation]. Reflect A over X to A'. Draw the circle with center X with radius XA. Call it ω. Let $P = \omega \cap \odot ABC$. Let $Q = A'B \cap \omega$.

We will show that M, P, B, Q are cyclic, and XM is tangent to the circle.

First, we have $AQ \perp A'B$. So $MB = MQ$. Now,

$$\begin{aligned}\measuredangle MPQ &= \measuredangle APQ - \measuredangle APM \\ &= \measuredangle AA'Q - \measuredangle ANM \\ &= \measuredangle AXM - \measuredangle XAM \\ &= \measuredangle AMX \\ &= \measuredangle MBQ\end{aligned}$$

So M, Q, P, B is cyclic. Also since $MQ = MB$, and $XM \parallel BQ$, XM is tangent to $\odot MPBQ$, and $\odot MPBQ = \omega_B$.

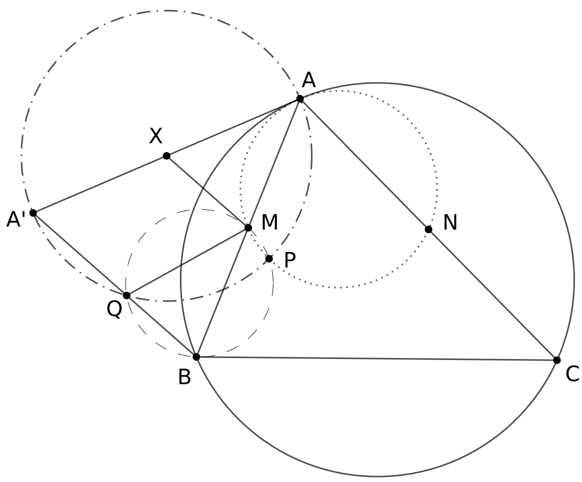


Figure 4.8.4

Similarly ω_C passes through P, and by Miquel's theorem, their intersection lies on BC.

Problem 4.8.6 (USA TST 2019 P1 parallel problem). Pick a point X such that AX is parallel to BC. Let M, N be the midpoints of AB, AC. Let w_b be the circle passing through M and B tangent to (AXB) and define w_c similarly. Show that w_b, w_c intersect on (AMN).

Solution. Doing a $\sqrt{\frac{bc}{2}}$ inversion in Problem 4.8 one ends up with this parallel problem.

Problem 4.8.7 (Sharygin 2010 P3). Points A', B', C' lie on sides BC, CA, AB of triangle ABC. for a point X one has $\angle AXB = \angle A'C'B' + \angle ACB$ and $\angle BXC = \angle B'A'C' + \angle BAC$. Prove that the quadrilateral $XA'BC'$ is cyclic.

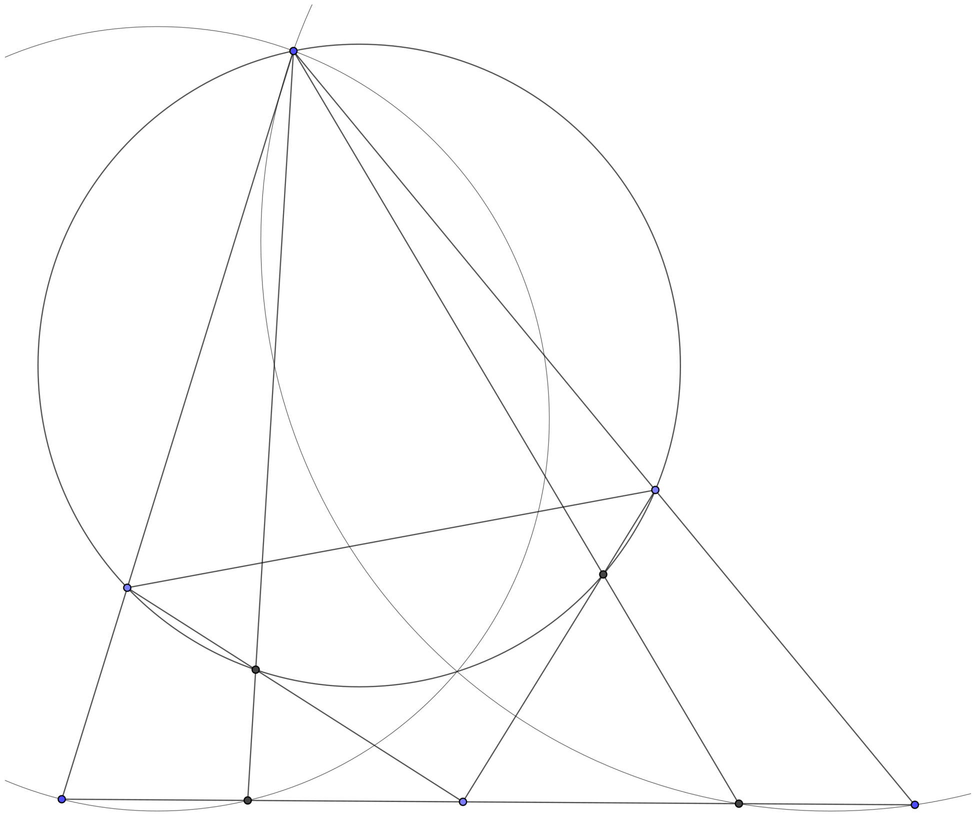


Figure 4.8.5

Problem 4.8.8 (IMO 2018 P6). A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point X lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \quad \text{and} \quad \angle XBC = \angle XDA.$$

Prove that $\angle BXA + \angle DXC = 180^\circ$.

Proof. Let $P = AB \cap CD,\ Q = AD \cap BC$

From the first condition, we get that $\dfrac{AB}{BC} = \dfrac{AD}{DC}$, implying that the angle bisectors of $\angle DAB, \angle DCB$ meet on BD.

And from the second condition, we have $X = \odot QBD \cap \odot PAC$

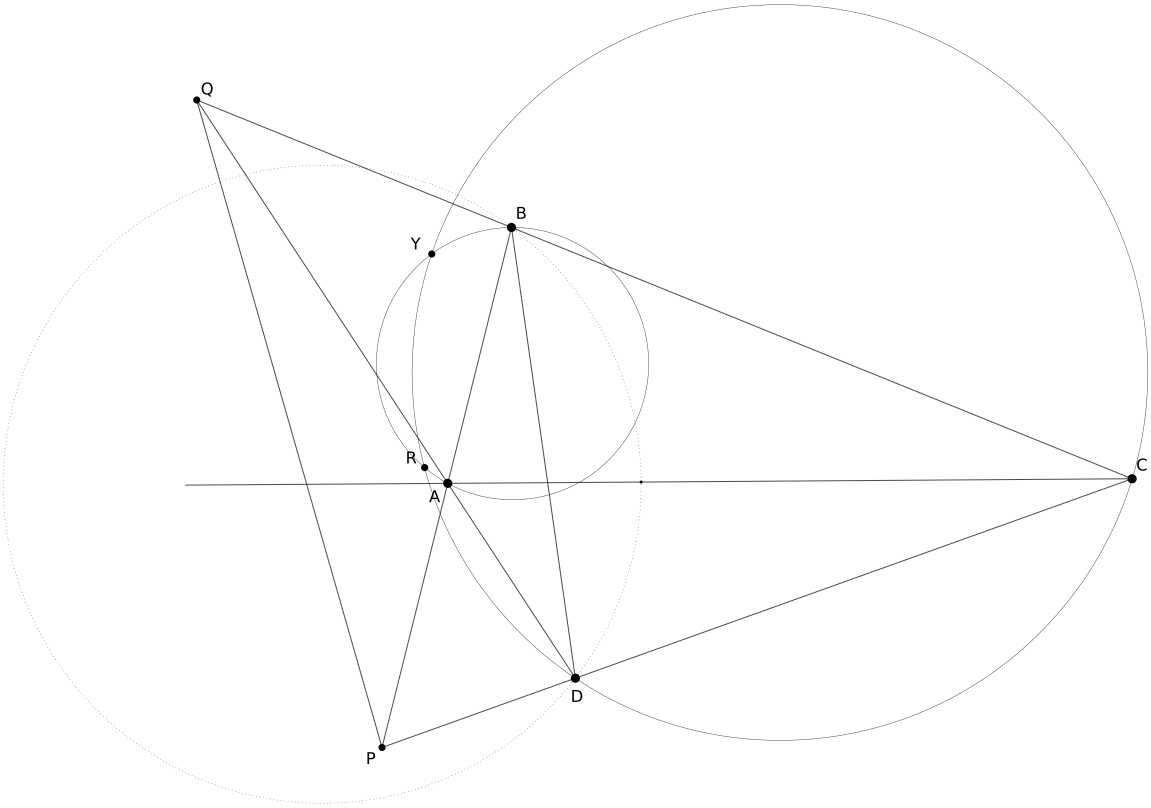


Figure 4.8.6: IMO 2018 P6, Simple Angle-Chase proof.

Let us define the point R such that AR, CR are isogonal to AC wrt to $\angle DAB, \angle DCB$ respectively. In $\triangle RAC$, we have, the bisectors of $\angle RAC, \angle RCA$ meet on the line BRD, meaning that RB bisects $\angle ARC$.

Let $\odot ARM \cap \odot DRC = Y$. We have,

$$\begin{aligned} \measuredangle AYC &= \measuredangle AYR + \measuredangle RYC \\ &= \measuredangle ABR + \measuredangle RDP \\ &= \measuredangle BPD \\ \implies & \square PAYC \text{ is cyclic.} \end{aligned}$$

And,

$$\begin{aligned} \measuredangle BYD &= \measuredangle BYR + \measuredangle RYD \\ &= \measuredangle BAR + \measuredangle RCD \\ &= \measuredangle CAD + \measuredangle BCA \\ &= \measuredangle CQD \\ \implies & \square QBYD \text{ is cyclic.} \end{aligned}$$

So, $Y \equiv X$. So,

$$\measuredangle BYA + \measuredangle DYC = \measuredangle BRA + \measuredangle DRC = \measuredangle BRA + \measuredangle ARD = 180^\circ$$

Problem 4.8.9 (Sharygin 2010). In $\triangle ABC$, let AL_a, AM_a be the external and internal bisectors of $\angle A$ with L_a, M_a lying on BC. Let ω_a be the reflection of the circumcircle of $\triangle AL_aM_a$wrt the midpoint of BC. Let ω_a be defined similarly. Prove that ω_a, ω_b are tangent to each other iff $\triangle ABC$ is a right-angled triangle.

4.9 Complete Quadrilateral + Spiral Similarity

Lemma 4.9.1 — Three lines, l_a, l_b, l_c, origin at point P. Two circles ω_1, ω_2 passing through P meet the lines at A_1, B_1, C_1; A_2, B_2, C_2 resp. Let A_3 be the reflection of A_2 on A_1. Define B_3, C_3 similarly. Then $PA_3B_3C_3$ are concyclic.

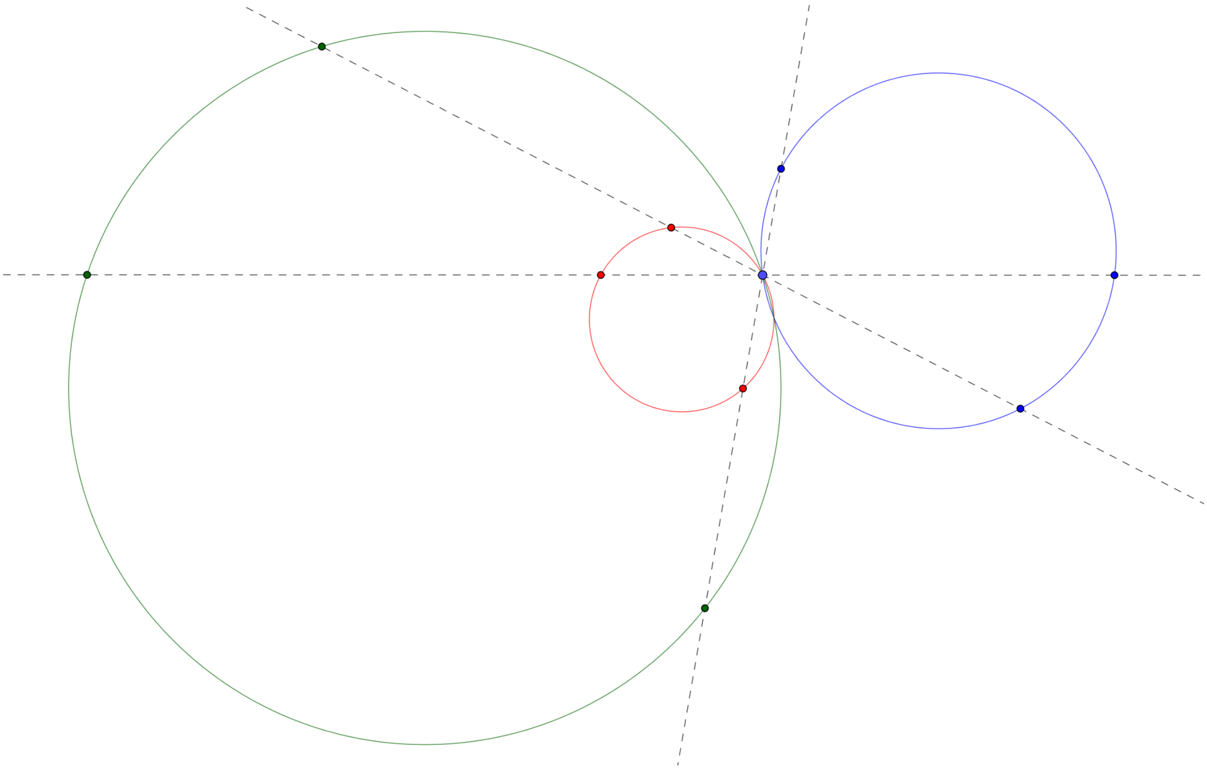


Figure 4.9.1: Spiral Similarity Lemma 1: the Blue points have been reflected wrt to the Red points to get the Green points

Problem 4.9.1 (ISL 2009 G4). Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G and H.

Solution. This problem generalizes to this

Problem 4.9.2 (All Russian 2014 Grade 10 Day 1 P4). Given a triangle ABC with $AB > BC$, let Ω be the circumcircle. Let M, N lie on the sides AB, BC respectively, such that $AM = CN$. Let K be the intersection of MN and AC. Let P be the incenter of the triangle AMK and Q be the K-excenter of the triangle CNK. If R is midpoint of the arc ABC of Ω then prove that $RP = RQ$.

Lemma 4.9.2 — Let E and F be the intersections of opposite sides of a convex quadrilateral $ABCD$. The two diagonals meet at P. Let M be the foot of the perpendicular from P to EF. Show that $\angle BMC = \angle AMD$. And PM is the bisector of angles $\angle AMC, \angle BMD$.

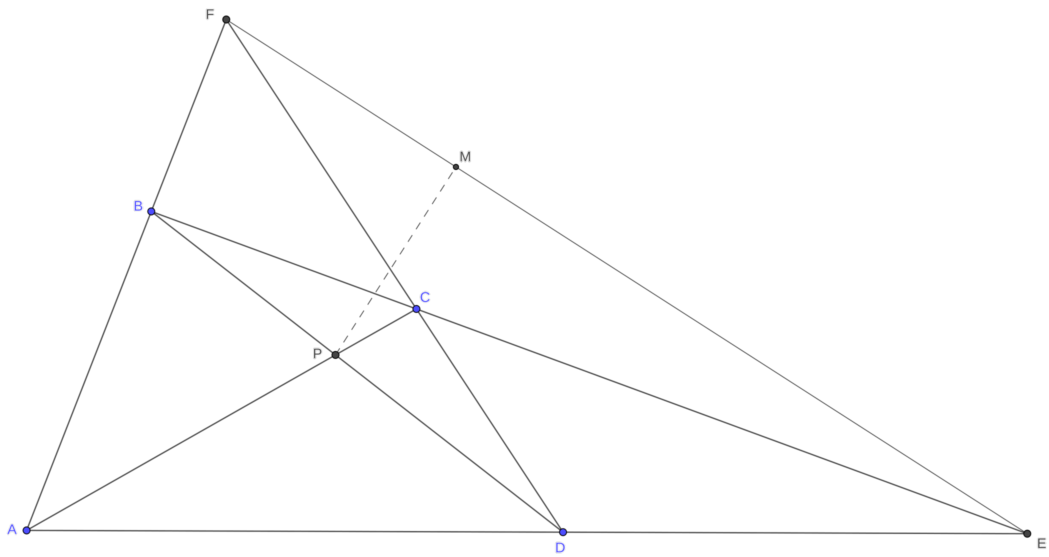


Figure 4.9.2

Theorem 4.9.3 (Newton-Gauss Line) — Among the points A, B, C, D no three are collinear. The lines AB and CD intersect at E, and BC and DA intersect at F. Prove that either the circles with diameters AC, BD, EF pass through two common points, or no two of them have any common point.
The previous can be stated differently: The midpoints of AC, BD, EF are collinear and this line is called "Newton-Gauss Line".

Solution. Either by E.R.I.Q. Lemma, Length Chase, or configurations like Varignon Parallelogram

Problem 4.9.3. Let $ABCD$ incribed (O) and a point so-called M .Call X, Y, Z, T, U, V are the projection of M onto AB, BC, CD, DA, CA, BD respectively. Call I, J, H the midpoints of XZ, UV, YT respectively. Prove that N, P, Q are collinear.

Solution. Divide the problem in cases, and prove the easiest case first.

Lemma 4.9.4 — In a cyclic quadrilateral $ABCD$, $AC \cap BD = P$, $AD \cap BC = Q$, $AB \cap CD = R$. S, T are the midpoints of PQ, PR. And a point X is on ST. Prove that the power of X wrt $ABCD$ is XP^2.

Solution. Using polar argument wrt P

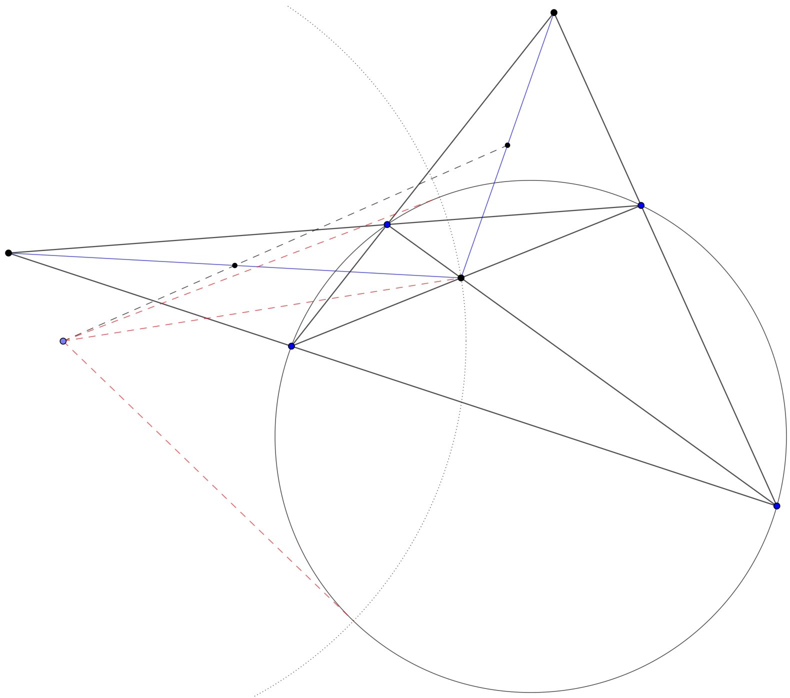


Figure 4.9.3

Problem 4.9.4 (USA TST 2000 P2). Let $ABCD$ be a cyclic quadrilateral and let E and F be the feet of perpendiculars from the intersection of diagonals AC and BD to AB and CD, respectively. Prove that EF is perpendicular to the line through the midpoints of AD and BC.

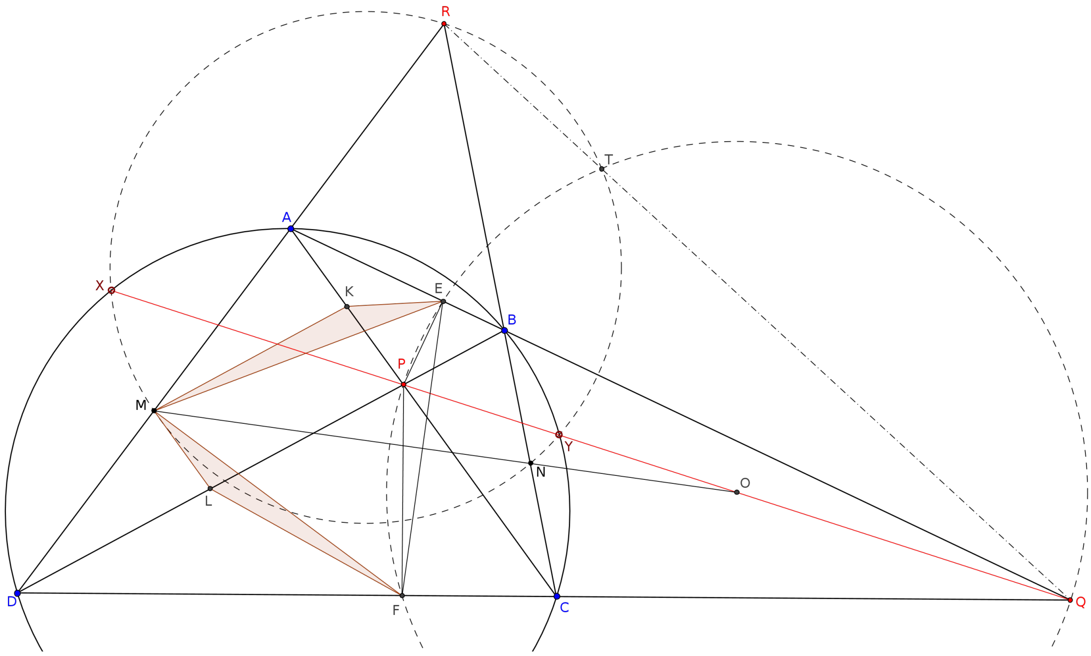


Figure 4.9.4: USA TST 2000 P2

> **Solution.** First solution is using some properties of the complete quad and angle bash the angle $\measuredangle(MN, EF)$

> **Solution.** Second solution is to notice the two brow triangles and proving them congruent.

Problem 4.9.5. Let 2 equal circle $(O_1), (O_2)$ meet each other at P, Q. O be the midpoint of PQ. 2 line through P meet the circles at $A,\ B,\ C,\ D,\ (A, C \in (O_1);\ B, D \in (O_2))$. M, N be midpoint of AD, BC. Prove that M, N, O are collinear.

Problem 4.9.6 (AoPS)**.** In $\triangle ADE$ a circle with center O, passes through A, D meets AE, ED respectively at B, C, $BD \cap AC = G$, line OG meets $\odot ADE$ at P. Prove that $\triangle PBD, \triangle PAC$ has the same incenter (preferably without using inversion).

Problem 4.9.7 (Archer - EChen M1P2)**.** Let a circle ω centered at A meet BC at D, E, such that B, D, E, C all lie on BC in that order. Let ω meet $\odot ABC$ at F, G such that A, F, B, C, G lie on the circle in that order. Let $\odot BFD \cap AB = K$, $\odot CGE \cap AC = L$. Prove that FK, GL, AO are concurrent.

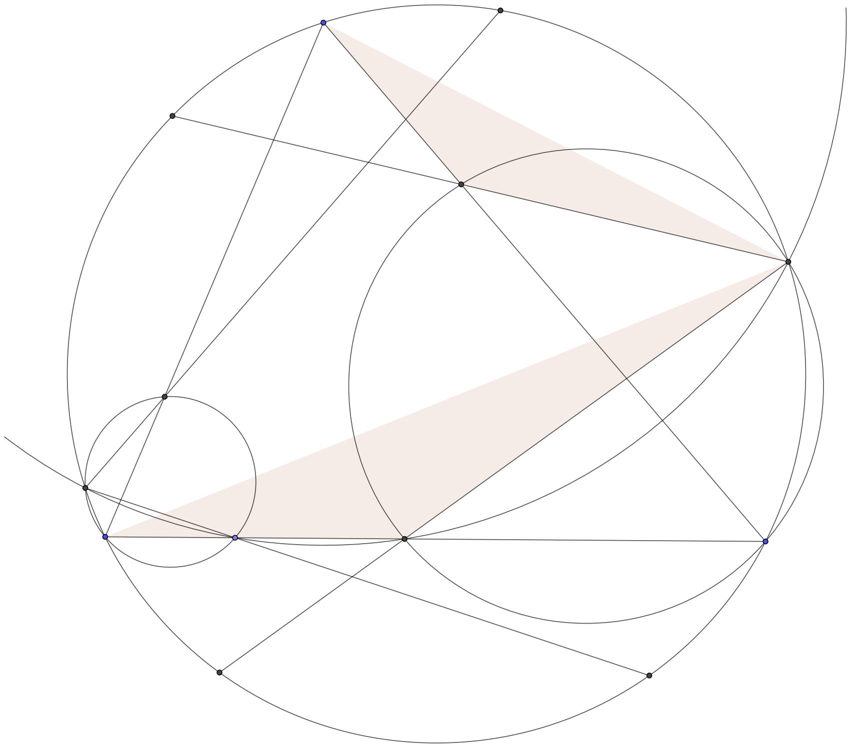


Figure 4.9.5

Problem 4.9.8 (Sharygin 2012 P22). A circle ω with center I is inscribed into a segment of the disk, formed by an arc and a chord AB. Point M is the midpoint of this arc AB, and point N is the midpoint of the complementary arc. The tangents from N touch ω in points C and D. The opposite sidelines AC and BD of quadrilateral $ABCD$ meet in point X, and the diagonals of $ABCD$ meet in point Y. Prove that points X, Y, I and M are collinear.

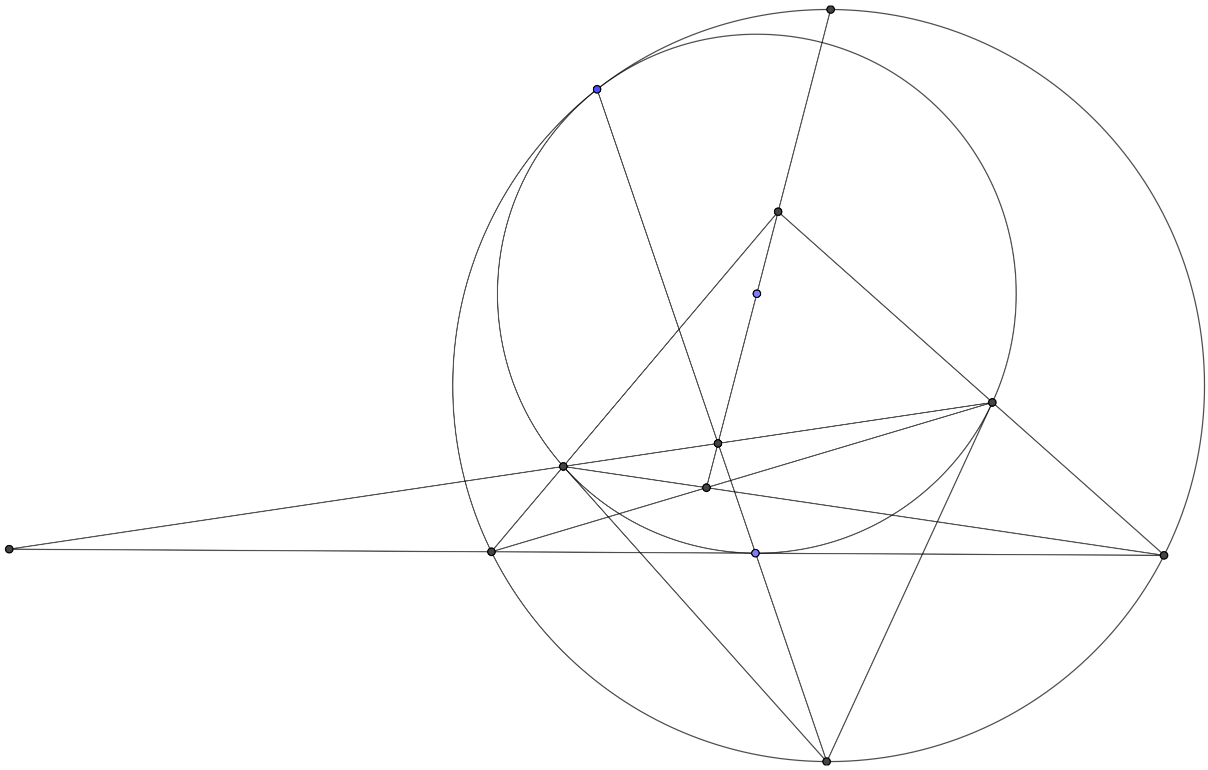


Figure 4.9.6

Solution. La Hire

Problem 4.9.9 (Sharygin 2012 P21). Two perpendicular lines pass through the orthocenter of an acute-angled triangle. The sidelines of the triangle cut on each of these lines two segments: one lying inside the triangle and another one lying outside it. Prove that the product of two internal segments is equal to the product of two external segments.

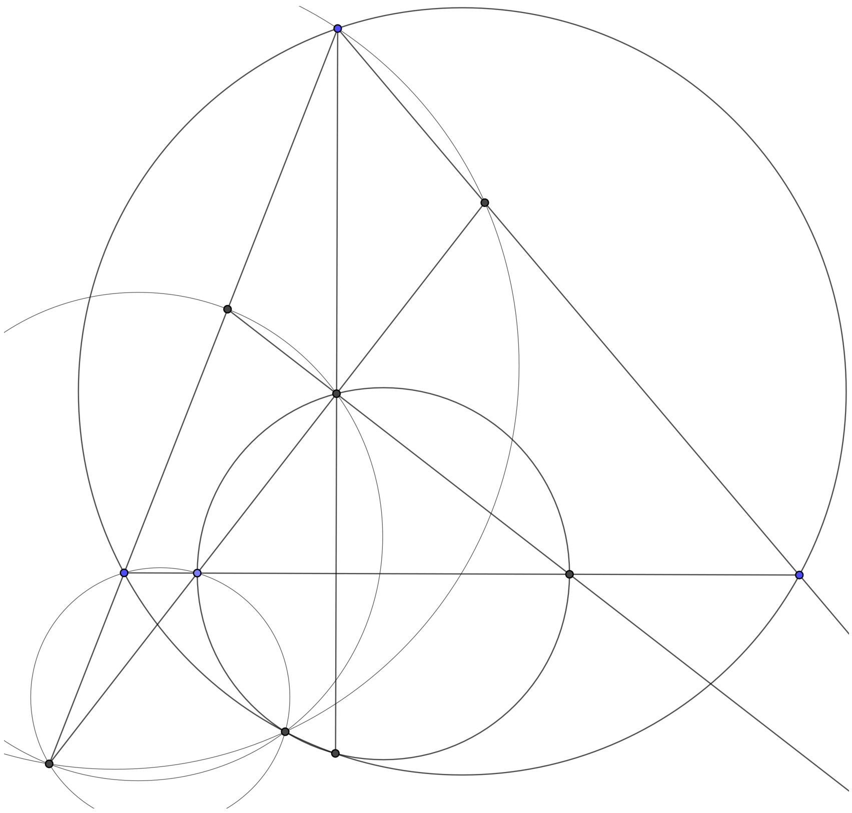


Figure 4.9.7

Solution. Spiral Similarity

Problem 4.9.10 (Iran TST 2004 P4). Let M, M' be two conjugates point in triangle ABC (in the sense that $\angle MAB = \angle M'AC, \dots$). Let P, Q, R, P', Q', R' be foots of perpendiculars from M and M' to BC, CA, AB. Let $E = QR \cap Q'R'$, $F = RP \cap R'P'$ and $G = PQ \cap P'Q'$. Prove that the lines AG, BF, CE are parallel.

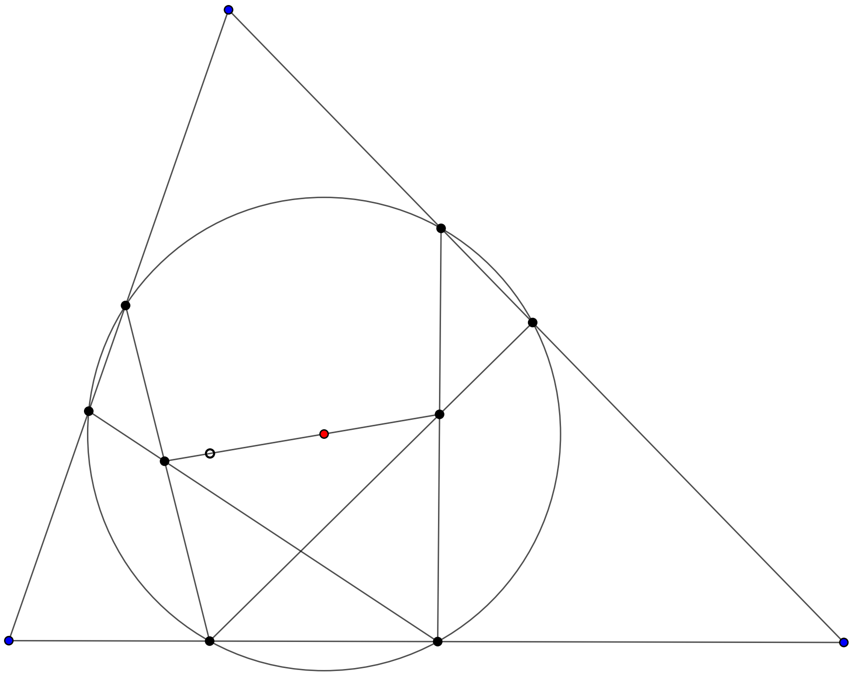


Figure 4.9.8: The points are collinear, by Zhao Lemmas

Problem 4.9.11 (Iran TST 2018 D2P6). Consider quadrilateral $ABCD$ inscribed in circle ω. $P \equiv AC \cap BD$. E, F lie on sides AB, CD respectively such that $A\hat{P}E = D\hat{P}F$. Circles ω_1, ω_2 are tangent to ω at X, Y respectively and also both tangent to the circumcircle of $\triangle PEF$ at P. Prove that:

$$\frac{EX}{EY} = \frac{FX}{FY}$$

4.10 Projective Geometry

- Cross Ratio - Zarathustra Brady
- Desargues' Involution Theorem - MarkBcc168

4.10.1 Definitions

Definition (Projective Plane)— The *projective plane* $\mathbb{P}^2$ is a *set of lines* passing through an observation point O in three dimensional space. A *porjective line* is a plane passing through O, and a *projective point* is a line passing throught O.

Definition (Coordinates in Projective Planes)— A point in a projective plane $\mathbb{P}^2$ has coordinates $(p : q : r)$. If $r = 0$, we say that the point is an *infinite point*. Every line in $\mathbb{P}^2$ can also be described with $(p : q : r)$ in a sense that this line (which is a plane passing through O) has the equation

$$pa + qb + rc = 0$$

Definition (Projection)— We can define *projection* of $\mathbb{P}^2$ on some plane A^2 not passing through O (for simplicity we will take the plane $z = 1$) by associating

$$P = (p : q : r) \in \mathbb{P}^2 \to P' = \left(\frac{p}{r} : \frac{q}{r} : 1\right) \in A^2$$

If $r = 0$, then we say P' is an infinite point with slope $\dfrac{q}{p}$.

A projective line l with coordinates $p : q : r$ gets associated to a line $l \in A^2$ likewise. The line at infinity is the line associated with the projective line passing through O and parallel to A^2.

Definition (Projective Line and Inversive Plane)— Every point in a *projective line* $\mathbb{P}^1$ has a coordinate $(s : t)$ which corresponds to the ordinary point $x = \dfrac{s}{t}$. The point at infinity will have $t = 0$. If we let s, t be complex numbers then the projective line is called the *inversive plane*.

Definition (Cross Ratio)— If four points A, B, C, D lie on a line, their *cross ratio* is defined as

$$(A, B; C, D) = \frac{AC}{BC} \div \frac{AD}{BD}$$

If four lines l_1, l_2, l_3, l_4 pass through a point, then their cross ratio is

$$(l_1, l_2; l_3, l_4) = \frac{\sin \angle l_1 l_3}{\sin \angle l_2 l_3} \div \frac{\sin \angle l_1 l_4}{\sin \angle l_2 l_4}$$

If four points on the inversive plane has the complex coordinate a, b, c, d, then the cross ratio is defined by

$$(A, B; C, D) = \frac{a-c}{b-c} \div \frac{a-d}{c-d}$$

Definition (Möbius Transformation)— A *Möbius Transformation* is defined by a transformation f_M of the inversive plane by a two by two matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with non zero determiant as followed:

$$(s : t) \in \mathbb{P}^1 \to (sa + tb \ : \ sc + td)$$

Which is in ordinary coordinates:

$$f_M(z) = \frac{az + b}{cz + d}$$

A Möbius transformation can be thought as a *matrix transformation* of the projective line (which can be thought as a plane) and then projection on an ordinary line.

Definition (Harmonic Conjugate Map)— For any points A, B on $\mathbb{P}^1$, we define

$$h_{A,B}(C) = D \text{ if } (A, B; C, D) = -1$$

A harmonic cojugate is a Möbius transformation. And a Möbius transformation that is also an *involution*, i.e. that has $f(f(x)) = x$, is a harmonic conjugate.

Definition (Circle Points)— The circle points are the points in $\mathbb{P}^2$ with coordinates $\alpha = (1 : i : 0)$ and $\alpha' = (-i : 1 : 0)$. These two points are both infinite and imaginary. And *every circle* passes through these two points.

Definition (Coharmonic Points)— Three pairs of points $\{A, A'\}, \{C, C'\}, \{E, E'\}$ on the same line are called *coharmonic points* iff there exists a pair of points $\{M, N\}$ on the line such that

$$(M, N; A, B) = (M, N; C, D) = (M, N; E, F) = -1$$

Definition (Involution)— If there exists a point X on l such that for the Möbius Transformation $f : l \to l$, such that $f(f(X)) = X$, then f is an *involution*.

Theorem 4.10.1 (Properties of Coharmonic Points) — If A, B, C, A', B', C' lie on a line, no three the same and $A \neq X$, then the following are equivalent:

1. $\{A, A'\}, \{B, B'\}, \{C, C'\}$ are coharmonic.

2. There is a Möbius Transformation with $f(A) = A', f(B) = B', f(C) = C'$ which is an involution.
3. $(A, A'; B, C) = (A', A; B', C')$
4. $\frac{AC'}{C'B}\frac{BA'}{A'C}\frac{CB'}{B'A} = -1$
5. $(A, A'; C, C') = (A, A'; C, B)\ (A, A'; C, B')$

Theorem 4.10.2 (Invertible Function on a line) — If f is an *invertible function* from a line to itself that is defined by some geometric procedure that has no *configuration mess*, then f preserves cross ratio, and is a Möbius Transformation. Similarly, an invertible Möbius Transformation is an involution on the line.

4.10.2 Cross Ratio

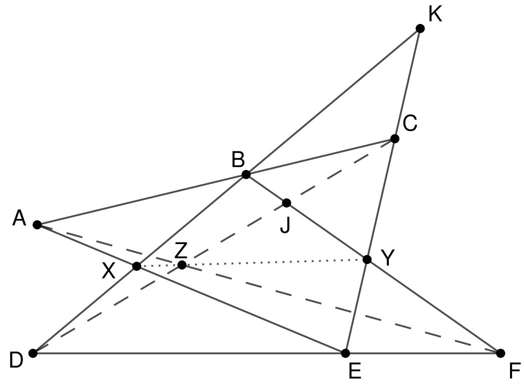


Figure 4.10.1

> **Theorem 4.10.3** (Pappus's Hexagon Theorem) — Let A, B, C be on a line, and let D, E, F be on another line. Let $X = AE \cap BD, Y = BF \cap CE, Z = CD \cap AF$. Then X, Y, Z are on a line.

> **Proof.** Let $CD \cap BF = J, DE \cap BD = K$. We have,
>
> $$(D, Z;\ J, C) \stackrel{F}{=} (AB \cap FD, A;\ B, C)$$
> $$\stackrel{C}{=} (D, X;\ B, K) \stackrel{Y}{=} (D, XY \cap DC;\ J, C)$$

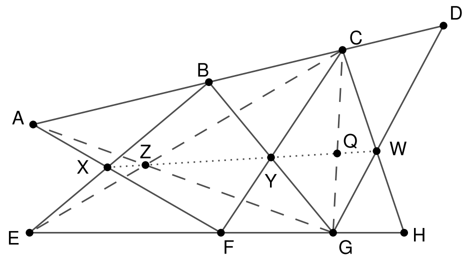


Figure 4.10.2: $(A, B; C, D) = (E, F; G, H)$

> **Theorem 4.10.4** (Cross Ratio Equality) — Let A, B, C, D be on a line, and let E, F, G, H be on another line. Let $X = AF \cap BE, Y = BG \cap CF, Z = CH \cap DG$. Then X, Y, Z are on a line if and only if $(A, B; C, D) = (E, F; G, H)$.

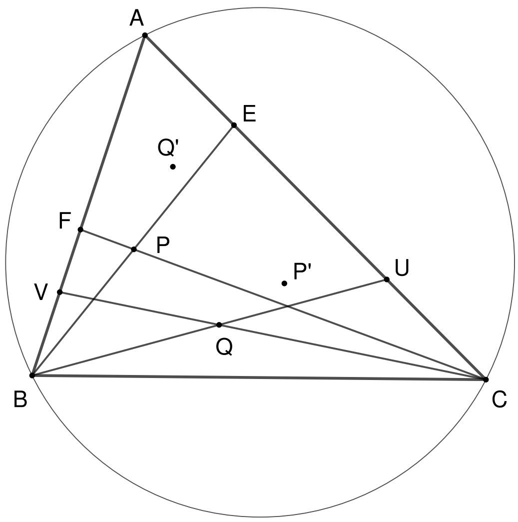


Figure 4.10.3

Problem 4.10.1 (Isogonal Conjugate and cevians). Let P, Q be two points inside $\triangle ABC$. Let P', Q' be the isogonal conjugates of P, Q. Let DEF, WUV be the cevian triangles of P, Q. Prove that iff $Q' \in EF$, then $P' \in UV$.

4.10.3 Involution

Theorem 4.10.5 (Involution on a line) — An involution on a line l is an inversion around some point on l.

Theorem 4.10.6 (Involution on a conic) — Let $f : \mathcal{C} \to \mathcal{C}$ be an involution. Let $f(X) = X'$ for all $X \in \mathcal{C}$. Then all XX' pass through a fixed point P.

Solution [Polar line, Pascal]. Let l be the polar line of P wrt $\mathcal{C}$. Let l' be the line parallel to l passing through P. Let $l' \cap \mathcal{C} = \{X, X'\}$. We show that, if $A, B \in \mathcal{C}$, and A', B' are the second intersection of AP, BP with $\mathcal{C}$, then, $\{A, A'\}, \{B, B'\}$ and $\{X, X'\}$ are coharmonic points wrt $\mathcal{C}$, that is, for a point $Q \in \mathcal{C}$,

$$(QX, QX'; QA, QB) = (QX', QX; QA', QB')$$

Let $XA, XB \cap l = A_1, B_1$, and $XA', XB' \cap l = A_1', B_1'$. Since the line $(XA \cap X'A', XA' \cap X'A)$ is the polar of $XX' \cap AA'$, X', A', A_1' are collinear. Similarly for X', B', B_1'. Let T be the polar point of XX'.

We have, $T, XB \cap X'A$, $XA \cap X'B$ collinear by Pascal's theorem on hexagon $XXABX'X'$. Let, $T' = (T, XB \cap X'A, XA \cap X'B) \cap l'$. We have,

$$\frac{XT'}{T'X} = \frac{B_1T}{TA_1'} = \frac{A_1T}{TB_1'} \implies \frac{TB_1}{TA_1} = \frac{TA_1'}{TB_1'}$$

Now, we have

$$\begin{aligned} X(X, X'; A, B) &= (T, \infty; A_1, B_1) \\ &= \frac{TA_1}{TB_1} = \frac{TB_1'}{TA_1'} \qquad = (\infty, T; A_1', B_1') \\ &= X(X', X; A', B') \end{aligned}$$

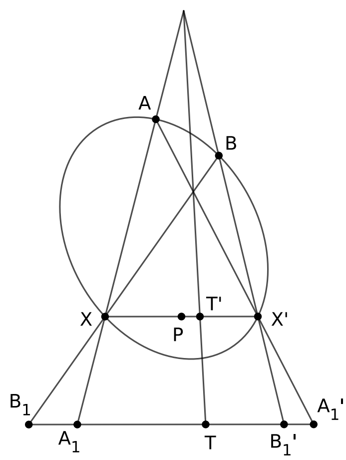


Figure 4.10.4

Which concludes the proof.

Solution [Inversion]. First project the conic to a circle, then invert $\mathcal{C} \to l$ across a point P on $\mathcal{C}$. The goal is to show that for every conjugate pair $X, X' \in \mathcal{C}$ and their image after inversion $X_1, X_1' \in l$, $\odot PX_1X_1'$ passes through a fixed point. By Theorem 4.10.5, we know that there is a point K on l that inverts X_1 to X_1'. So the circles PX_1X_1' have radical axis PK. Which concludes the proof.

Theorem 4.10.7 (Three Conic Law) — Let A, B, C, D be any four points, no three on a line. Let l be a line passing through at most one of them. For a point P on l, define $f(P) = P'$, where P' is the second intersection of l with the conic passing through A, B, C, D, P'. Then f is an involution.

Then for any three points $X, Y, Z \in l$, $\{X, f(X)\}$, $\{Y, f(Y)\}$, $\{Z, f(Z)\}$ are coharmonic, i.e. $\{X, f(X)\}$ are conjugate pairs of an involution on l,

Theorem 4.10.8 (Desargues' Involution Theorem) — Let $ABCD$ be a quadrilateral, let a conic $\mathcal{C}$ pass through A, B, C, D. And let a line l intersect $(AB, CD), (AD, BC), (AC, BD), \mathcal{C}$ at $(X_1, X_2), (Y_1, Y_2), (Z_1, Z_2), (W_1, W_2)$. Then

$$\{X_1, X_2\}, \{Y_1, Y_2\}, \{Z_1, Z_2\}, \{W_1, W_2\}$$

are *coharmonic points* i.e. they are reciprocal pairs of some involution on l.

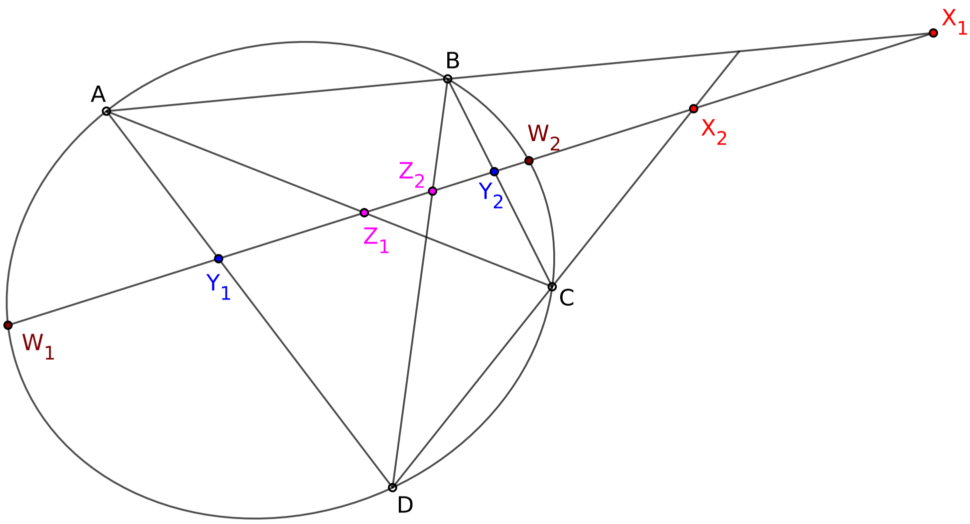


Figure 4.10.5: Desargues' Involution Theorem

Proof. Apply the *Three Conic Law* on l with points A, B, C, D.

Theorem 4.10.9 (Degenerate Desargues' Involution: 2 Points) — Let A, B, be two points on a conic $\mathcal{C}$, let a line l meet $AB, \mathcal{C}$ and the tangents at A, B to $\mathcal{C}$ at $X, (W_1, W_2), (Y_1, Y_2)$. Then $(X, X), (W_1, W_2), (Y_1, Y_2)$ are reciprocal pairs of an involution on l.

Definition (Involution on Pencil)— Let P be a point on the plane. Let $\mathcal{L}$ be the set of all line containing P. Then $f : \mathcal{L} \to \mathcal{L}$ is an *involution on a pencil* of lines if and only if.

1. For every $\overline{PA}, \overline{PB}, \overline{PC}, \overline{PD} \in \mathcal{L}$, we have
$$(\overline{PA}, \overline{PB}; \overline{PC}, \overline{PD}) = (f(\overline{PA}), f(\overline{PB}); f(\overline{PC}), f(\overline{PD}))$$

2. $f(f(\ell)) = \ell$ for every $\ell \in \mathcal{L}$. Furthermore, we call a pair $(\ell, f(\ell))$ reciprocal pair.

Theorem 4.10.10 (Dual of Desargues' Involution Theorem) — Let P, A, B, C, D be points on a plane with $\overline{AB} \cap \overline{CD} = E, \overline{AD} \cap \overline{BC} = F$. Let a conic $\mathcal{C}$ tangent to lines AB, CD, AD, BC. Let $\overline{PX}, \overline{PY}$ are the tangent line from P to $\mathcal{C}$. Then

$$(\overline{PX}, \overline{PY}), (\overline{PA}, \overline{PC}), (\overline{PB}, \overline{PD}), (\overline{PE}, \overline{PF})$$

are reciprocal pairs of some involution on pencil of lines through P.

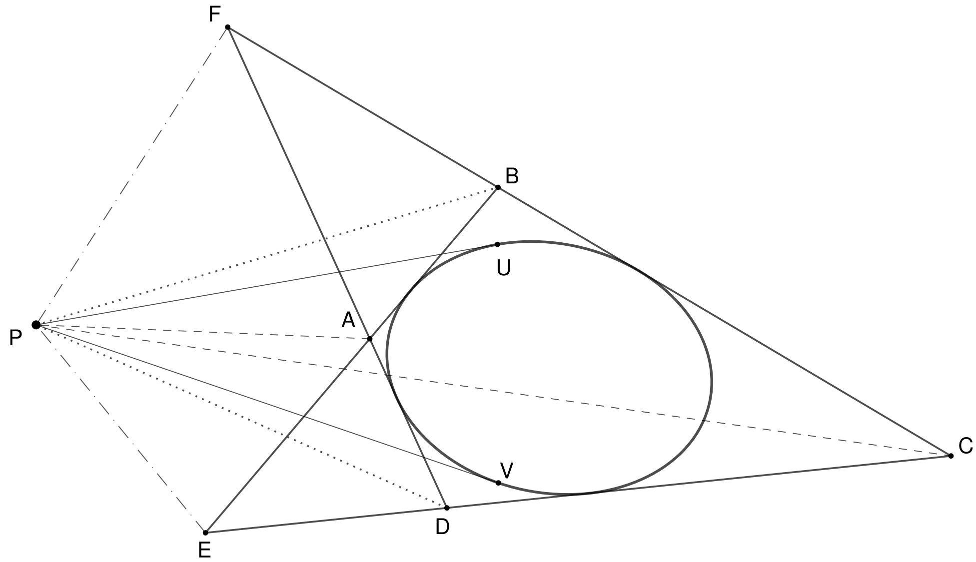


Figure 4.10.6

4.10.4 Inversion

TelvCohls $\sqrt{bc}$ inversion problem collection

Lemma 4.10.11 — WRT a circle ω with center O the polar of a point A can be constructed as the radical axis of ω and the circle with diameter OA.

Lemma 4.10.12 — $\measuredangle(a, b) = \measuredangle AOB$

Theorem 4.10.13 (Pascal's Theorem for Octagons: A special case) — Let $ABCDA'B'C'D'$ be a octagon inscribed in a conic section. If the points:

$$AD \cap BC,\ AC' \cap BB',\ AD' \cap CA',\ BD' \cap DA',\ DB' \cap CC'$$

are collinear, then so are the points

$$A'D' \cap B'C',\ A'C \cap B'B,\ A'D \cap C'A,\ B'D \cap D'A,\ D'B \cap C'C$$

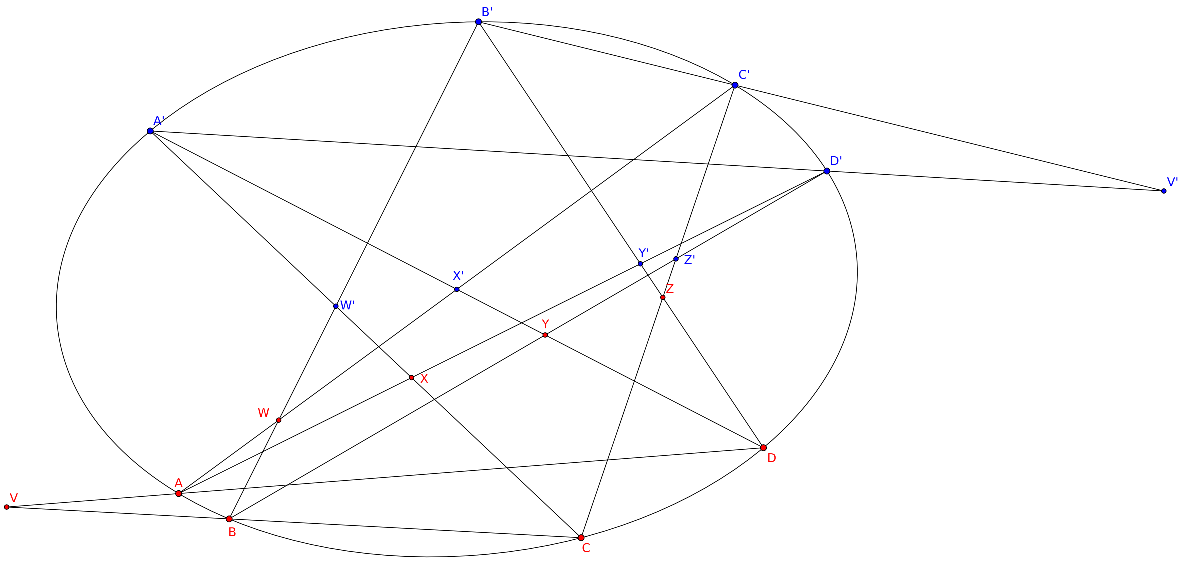


Figure 4.10.7: If the small Red points are collinear, then the Blue ones are too.

Theorem 4.10.14 (Inscribed Conic in Pascal's theorem) — $A_1A_2A_3A_4A_5A_6$ be a hexagon inscribed in a conic section. Then the hexagon formed by

$$A_1A_3 \cap A_2A_6,\ A_2A_4 \cap A_1A_3,\ A_2A_4 \cap A_3A_5,\ A_3A_5 \cap A_4A_6,\ A_5A_1 \cap A_4A_6,\ A_1A_5 \cap A_2A_6$$

has an inscribed conic section.

Problem 4.10.2. Let $ABCD$ have an incircle (I). Let (I) meet AB, BC, CD, DA at M, N, P, Q. Let K, L be the circumcenters of AMN, APQ. $KL \cap BD = R$, $AI \cap MQ = J$. Prove that $RA = RJ$.

Problem 4.10.3. Let the A mixtilinear incircle (O) of $\triangle ABC$ meet $\odot ABC, AC, AB$ at P, E, F. Let M be the BC arc midpoint. Let $\mathcal{H}$ be the conic that goes through E, F, O, P, M meet $\odot ABC$ at X, Y. Prove that AA, XY, EF are concurrent.

Problem 4.10.4 (Iran 3rd Round G4)**.** Let ABC be a triangle with incenter I. Let K be the midpoint of AI and $BI \cap \odot(\triangle ABC) = M, CI \cap \odot(\triangle ABC) = N$. points P, Q lie on AM, AN respectively such that $\angle ABK = \angle PBC, \angle ACK = \angle QCB$. Prove that P, Q, I are collinear.

Solution. Since we are dealing with collinearity, and usually we use harmonic bundle in these cases to show collinearity. But in this problem, there is no harmonic bundle. So we use cross ratio...

Generalization 4.10.4.1 (Iran 3rd Round G4 Generalized version)**.** Let ABC be a triangle inscribed in circle (O) and P, Q are two isogonal conjugate points. PB, PC cut (O) again at M, N. QA cuts MN at K. L is isogonal conjugate of K. LB, LC cut AM, AN at S, T, resp. Prove that S, Q, T are collinear.

Lemma 4.10.15 — Too long, can't explain, loot at the figure. The dotted lines go through that concurrency point.

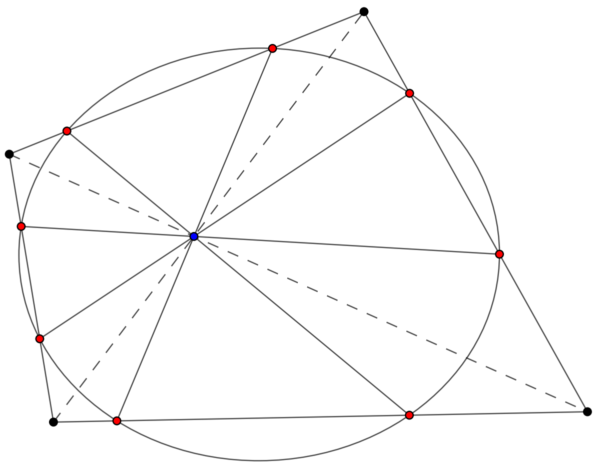


Figure 4.10.8: Everything concurrs

Lemma 4.10.16 (Construction of Involution Center on Line) — Given a line l, four points A, B, A', B' such that A, A' and B, B' are two conjugate pairs of some involution i.e. some inversion on l, find the center O of inversion.

4.10.5 Problems

Problem 4.10.5 (Dunno). Let E, F be on the lines AC, AB of $\triangle ABC$. Let P be a point on EF. Let Q be the intersection of the lines through E, F and parallel to BP and CP respectively. Prove that, as P moves along EF, Q moves along a line.

Solution [Cross Ratio]. Let $X \in BF, Y \in CE$ such that $\frac{BX}{XF} = \frac{CE}{EA}, \frac{CY}{YA} = \frac{BF}{FA}$. With trivial calculation, we have $XY \parallel BC$ We show that Q, X, Y are collinear. For that we will show $FU \parallel EV$ where $U = BP \cap XY, V = CP \cap XY$. And by reverse Pappu's theorem on FPE, UQV, we will have U, Q, V collinear.

Let $K, L = BP, CP \cap XY$. Also, $S = BC \cap EF$. Then we have,

$$(S, P; F, E) \stackrel{B}{=} (\infty, U; X, K) \stackrel{C}{=} (\infty, V; L, Y)$$

$$\implies \frac{XU}{UK} = \frac{LV}{VY}$$

But we have,

$$\frac{FX}{XB} = \frac{FL}{LC} = \frac{AE}{EC}$$

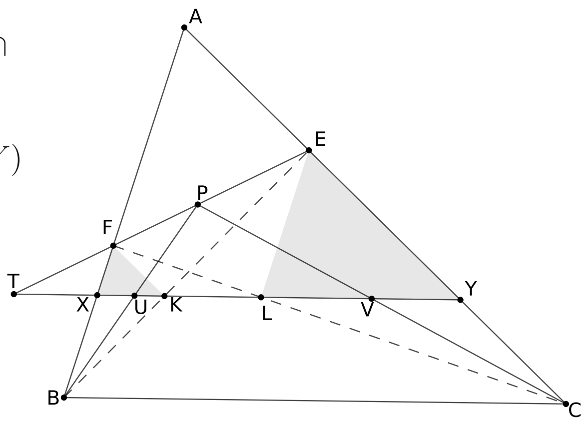


Figure 4.10.9

So, $FX \parallel EL$, and similarly, $FK \parallel EY$. So by similarity, we have $FU \parallel EV$.

Lemma 4.10.17 (Conic through orthocenter and vertices) — Let $ABCD$ be a quadrilateral. Let G, H be the orthocenters of $\triangle ABC$ and $\triangle DBC$. Then A, B, C, D, G, H all lie on a conic.

Proof. Since $line(AC \cap BD, BG \cap CH)$ is perpendicular to BC, we have $AG \cap DH, AC \cap BD, BG \cap CH$ collinear. So by reverse Pascal's theorem, A, D, G, H, B, C lie on a conic.

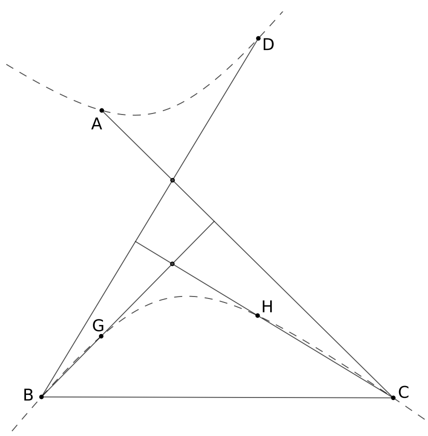


Figure 4.10.10

Lemma 4.10.18 (Orthogonal Hyperbola) — Let H be the orthocenter of ABC. Let $\mathcal{C}$ be a conic through A, B, C, H. If XYZ is a triangle with vertices in $\mathcal{C}$, then the orthocenter W of XYZ lies also on $\mathcal{C}$. Also, the asymptotes of $\mathcal{C}$ are orthogonal.

Solution. As in Lemma 4.10.17, the orthocenter ot XBC lies on $\mathcal{C}$ too. So the orthocenter of XYC lies on $\mathcal{C}$ and so does the orthocenter of XYZ.

Now we show that asymptotes of $\mathcal{C}$ are orthogonal.

Let the two infinity points on $\mathcal{C}$ be s, t. Consider the triangle Ast. Let its orthocenter be B. Then we have, $tB \perp sA$. But tB is parallel to the asymptote through t and sA is parallel to the asymptote through s. So the asymptotes themselves are orthogonal.

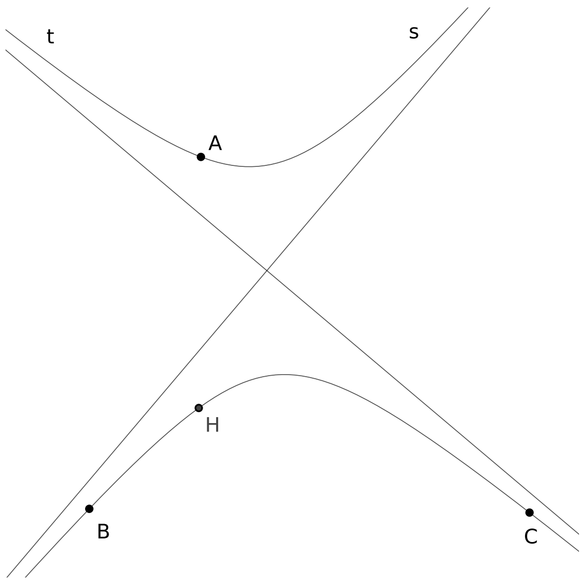


Figure 4.10.11: Hyperbola through A, B, C, H

4.10.6 Projective Constructions

Construction 1 (Second Intersection of Line with Conic)— Given four points A, B, C, D, no three collinear, and a point P on a line l passing through at most one of the four points, construct the point $P' \in l$ such that A, B, C, D, P, P' line on the same conic.

> **Solution.** Let $AP \cap BC = X$, $l \cap CD = Y$, $XY \cap AD = Z$. Then by Pascal's Hexagrummum Mysticum Theorem, we have, $P' = BZ \cap l$

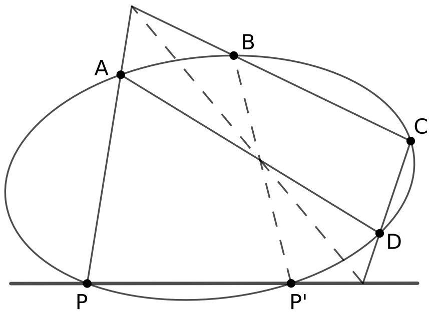


Figure 4.10.12

Construction 2 (Conic touching conic)— Given a conic $\mathcal{C}$, and two points A, B on it, and C inside of it. Construct the conic $\mathcal{H}$ that is tangent to $\mathcal{C}$ at A, B and passes through C.

> **Solution.** Draw the two tangest at A, B which meet at X. Take an arbitrary line passing through X that intersects AC, BC at Y, Z. Take $D = BY \cap AZ$. Then D lies on $\mathcal{H}$ by Pascal. Construct another point E similarly and draw the conic.

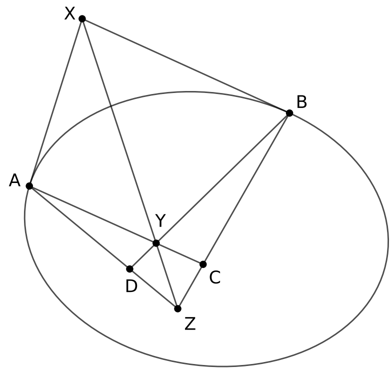


Figure 4.10.13

Construction 3 (Inconic of a quadrilateral)— Given a convex quadrilateral $ABCD$. $P = AC \cap BD$, $S \in AD, T \in BC$ such that S, P, T are collinear. Construct the conic that touches AB, CD, and also touches AD, BC at S, T respectively.

> **Solution** [the_Construction]. Draw the polar line l of P wrt to the quadrilateral. Let $Z = BC \cap l$. Let $ZS \cap AB = U$, $ZT \cap CD = V$. Then $SSUUTTVV$ is our desired conic.

Proof. If $U, V \in CD, AB$ such that UV passes through P, and if the conic passing through U, V and tangent to AD, BC at S, T intersects CD at U' again, then $SV, U'T, DB$ are concurrent. So to show our construction works, we just need to prove that U, V, P are collinear.

Since Pascal's theorem works on $SVBTUD$, we know S, V, B, T, U, D lie on a conic $\mathcal{H}$ and l is the pole of P wrt $\mathcal{H}$. Now, applying Pascal's theorem on $TDVUBS$, and quadrilateral theorem on $BTUD$ and $BVSD$, we have, $ST \cap UV \in AC$, which is P. So we are done.

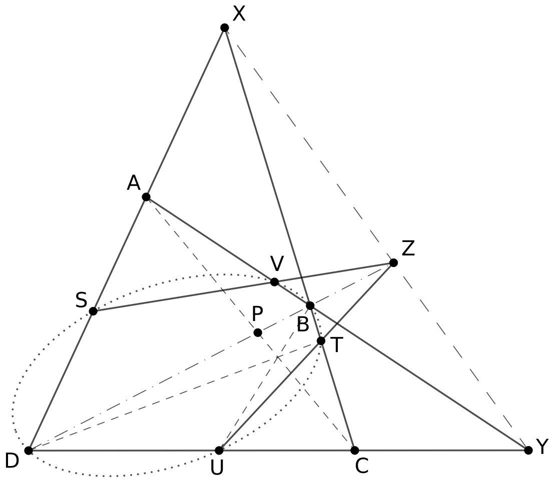


Figure 4.10.14

Construction 4 (Sharygin Olympiad 2010)— A conic $\mathcal{C}$ passing through the vertices of $\triangle ABC$ is drawn, and three points A', B', C' on its sides BC, CA, AB are chosen. Then the original triangle is erased. Prove that the original triangle can be constructed iff AA', BB', CC' are concurrent.

Solution [the_Construction]**.** Draw $B'C'$. It intersects the circle at X_1, X_2. Draw the conic $\mathcal{H}$ that is tangent to $\mathcal{C}$ at X_1, X_2 and passes through A'. Then BC is tangent to $\mathcal{H}$ at A'.

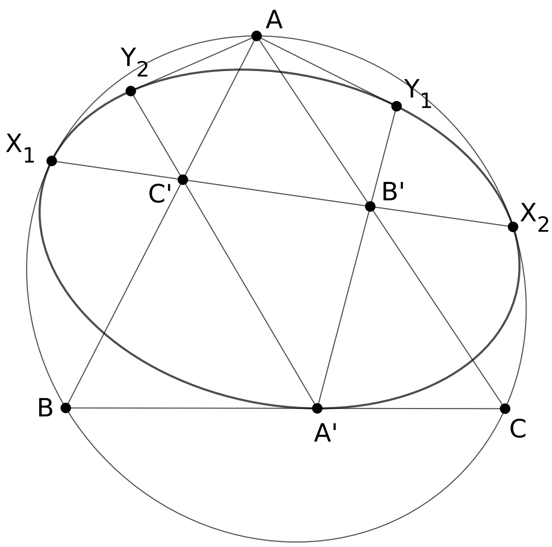


Figure 4.10.15

Proof. The only if part is easy to prove. Becase if AA', BB', CC' aren't concurrent, then we can get multiple triangles ABC. So suppose that they are concurrent.

Now we define some intersetion points.

$$\begin{array}{ccccc} W_1 & = & BB' & \cap & \mathcal{C} \\ S & = & X_1X_1 & \cap & AW_1 \\ T & = & X_1B & \cap & AX_2 \\ U & = & X_1X_1 & \cap & BC \\ V & = & X_2X_2 & \cap & BC \\ R & = & X_2X_2 & \cap & AW_1 \\ Y_1 & = & A'B' & \cap & SR \end{array}$$

T, S, B' are collinear by Pascal's theorem on $BX_1X_1X_2AW_1$. T, B', V are similarly collinear for $AX_2X_2X_1BC$. And similarly R, B', U are collinear.

We will prove that $\mathcal{H}$ is an inconic of $SRVU$ that goes through A', X_1, X_2.

For a point X on UV, define $f : UV \to UV$ such that $f(X)$ is the second intersection of the conic $X_1X_1X_2X_2X$ $(X_1X_1 = SU, X_2X_2 = RV)$ with UV. f is an involution by **??**.

Suppose A_1 is the intersection with the inconic of $SRUV$ through X_1, X_2 and UV. Let $A_2 = X_1X_2 \cap UV$. Then $f(A_1) = A_1, f(A_2) = A_2, f(B) = C$.

Which means, $A(B, C; A_1, A_2) = -1$. Which means $A_1 = A'$. So, $X_1X_2A'X_2X_2$ is an inconic of $SRVU$, just as we wanted.

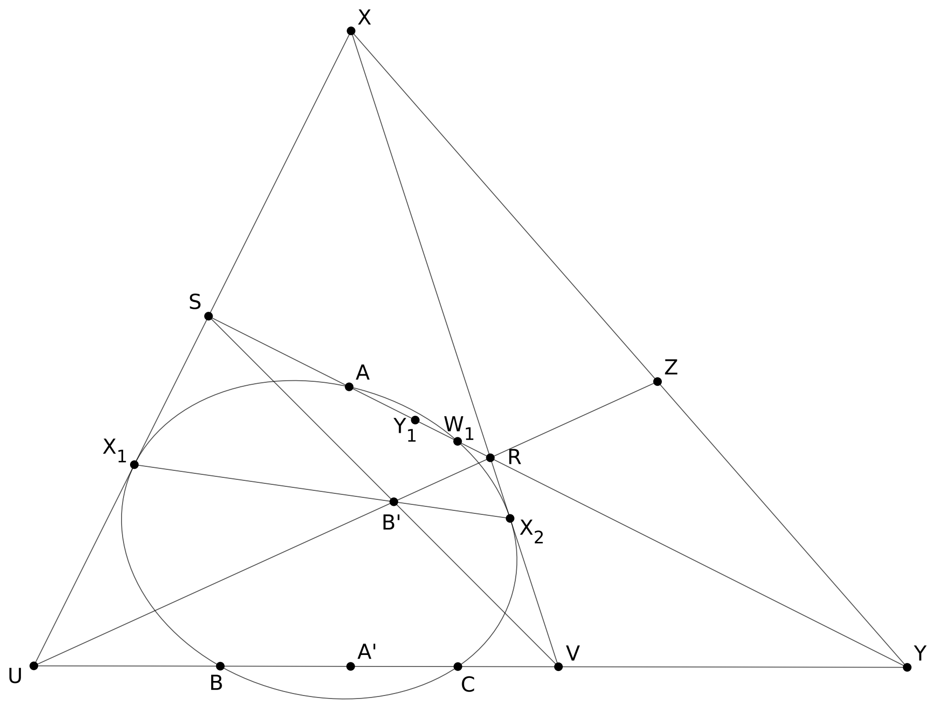


Figure 4.10.16

Construction 5 (Focus and Directrix of a Parabola)— First draw two parallel segments on the parabola, join their midpoints to get the line parallel to the axis. Then draw the main axis and find out the tip of the parabola. Then draw $f(x) = \frac{x}{2}$ line through P. And find the foot of the intersection of it with the parabola. It is the focus.

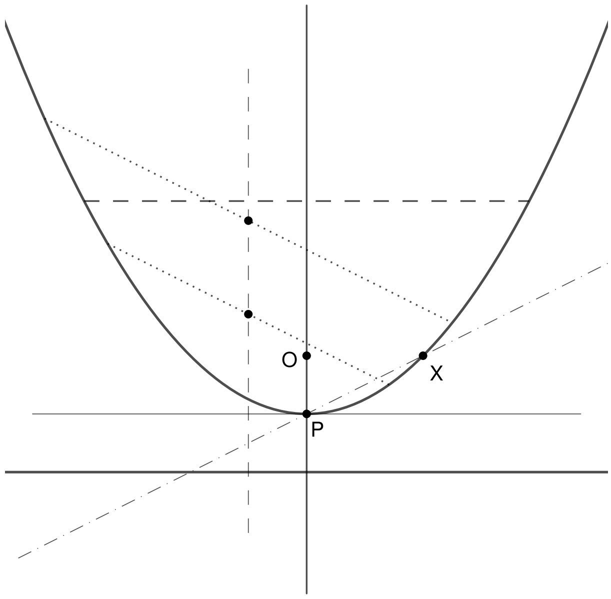


Figure 4.10.17

4.11 Parallelogram Stuff

Theorem 4.11.1 (Maximality of the Area of a Cyclic Quadrilateral) — Among all quadrilaterals with given side lengths, the cyclic one has maximal area.

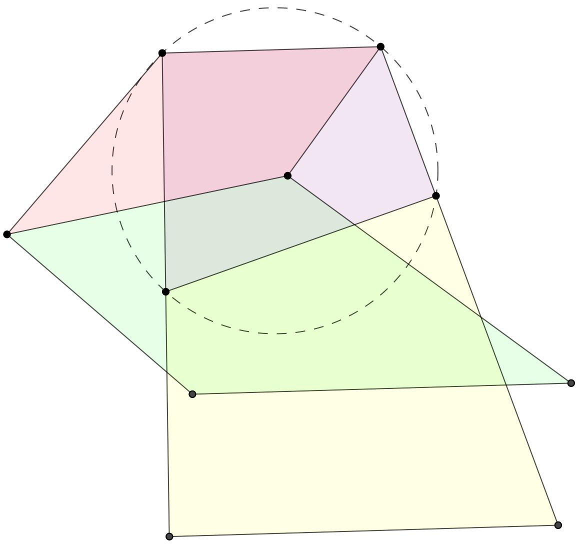


Figure 4.11.1: The cyclic quad has the maximal area

Problem 4.11.1 (IOM 2017 P1). Let $ABCD$ be a parallelogram in which angle at B is obtuse and $AD > AB$. Points K and L on AC such that $\angle ADL = \angle KBA$(the points A, K, C, L are all different, with K between A and L). The line BK intersects the circumcircle ω of ABC at points B and E, and the line EL intersects ω at points E and F. Prove that $BF \parallel AC$.

Simplify: Make the diagram easier to draw.

Problem 4.11.2 (USA TST 2006 P6). Let ABC be a triangle. Triangles PAB and QAC are constructed outside of triangle ABC such that $AP = AB$ and $AQ = AC$ and $\angle BAP = \angle CAQ$. Segments BQ and CP meet at R. Let O be the circumcenter of triangle BCR. Prove that $AO \perp PQ$.

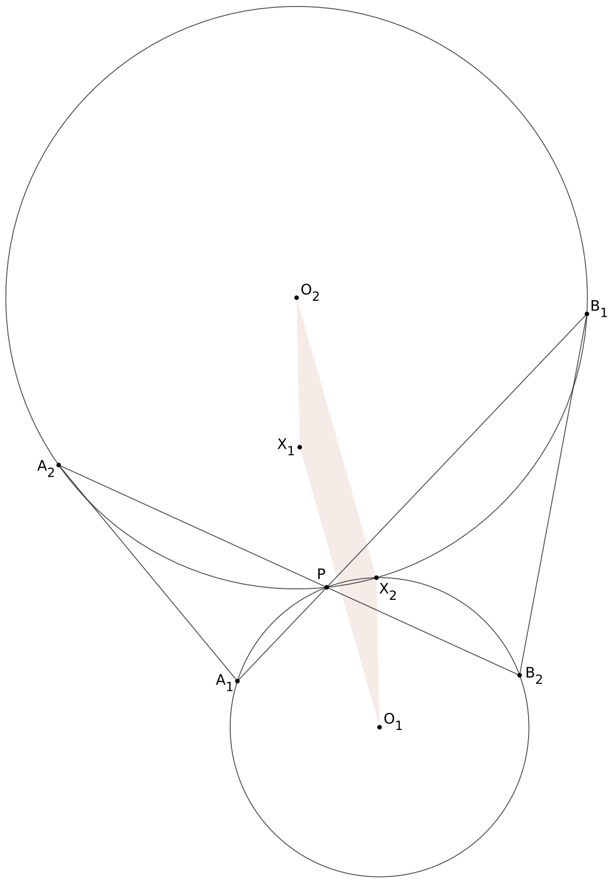


Figure 4.11.2: USA TST 2006 P6, That is a prallelogram

4.12 Length Relations

Lemma 4.12.1 (E.R.I.Q. (Equal Ration in Quadrilateral) Lemma) — Let $A_1, B_1, C_1;\ A_2, B_2, C_2$ be two sets of collinear points such that

$$\frac{A_1B_1}{B_1C_1} = \frac{A_2B_2}{B_2C_2} = k$$

. Let points A, B, C be on A_1A_2, B_1B_2, C_1, C_2 such that:

$$\frac{A_1A}{A_2A} = \frac{B_1B}{B_2B} = \frac{C_1C}{C_2C}$$

Then we have,

$$A, B, C \text{ are collinear and, } \frac{AB}{BC} = k$$

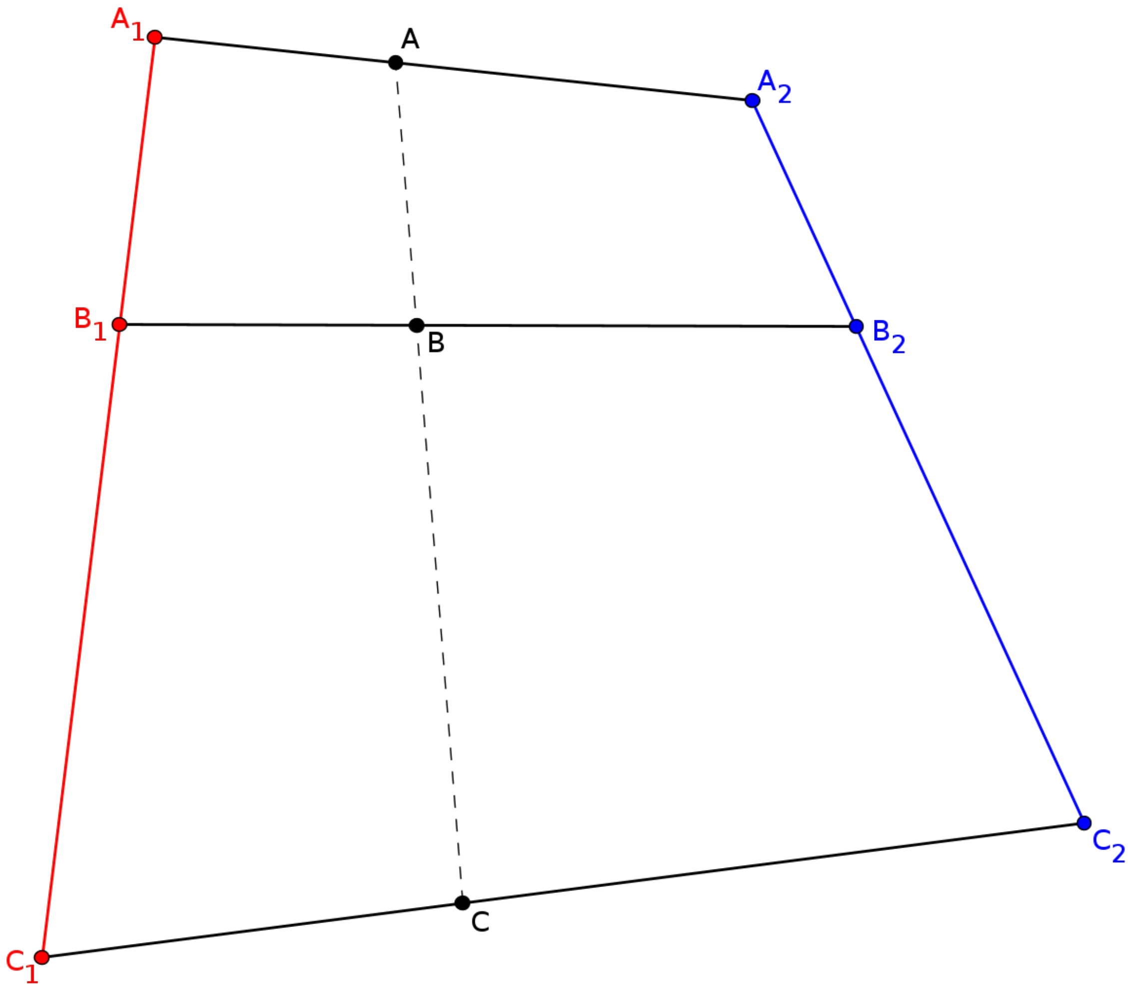


Figure 4.12.1: E.R.I.Q. Lemma

Solution. A great use of this problem is in proving some midpoints collinear. Line in Newton-Gauss Line and some other such problems (1, 2, 3) where it is asked to proved that some midpoints are collinear.

Lemma 4.12.2 (Steiner's Isogonal Cevian Lemma) — In $\triangle ABC$, AA_1, AA_2 are two isogonal cevians, with $A_1, A_2 \in BC$. Then we have

$$\frac{BA_1}{A_1C} \times \frac{BA_2}{A_2C} = \frac{BA^2}{AC^2}$$

Theorem 4.12.3 — Let P_1, P_2 be two isogonal conjugates wrt $\triangle ABC$. Then if the Pedal triangle of P_1 is homological wrt to $\triangle ABC$ then so is the Pedal triangle of P_2.

Theorem 4.12.4 (Erdos-Mordell Theorem (Forum Geometricorum Volume 1 (2001) 7-8)) — If from a point O inside a given $\triangle ABC$ perpendiculars OD, OE, OF are drawn to its sides, then $OA + OB + OC \geq 2(OD + OE + OF)$. Equality holds if and only if $\triangle ABC$ is equilateral.

Apparently nothing is needed except "Ptolemy's Theorem". Think of a way to connect OA with OE, OF and the sides of the triangle. As it is the most natural to use AB, AC, we have to deal with BE, CF too. And dealing with lengths is the easiest when we have similar triangles. So we do some construction.

Problem 4.12.1 (ISL 2011 G7). Let $ABCDEF$ be a convex hexagon all of whose sides are tangent to a circle ω with centre O. Suppose that the circumcircle of triangle ACE is concentric with ω. Let J be the foot of the perpendicular from B to CD. Suppose that the perpendicular from B to DF intersects the line EO at a point K. Let L be the foot of the perpendicular from K to DE. Prove that $DJ = DL$.

Solution. There are a LOT of equal lenths, equal angles, and we have a perpendicularity lemma working as well. Why don't we try cosine :0

4.13 Pedal Triangles

Definition (Pedal Triangles)— Let P be an arbitrary point, let $\triangle A_1B_1C_1$ be its pedal triangle wrt $\triangle ABC$. Let A', B', C' and A_0, B_0, C_0 be the feet of the altitudes and the midpoints of $\triangle ABC$.

$$B_1C_1 \cap B_0C_0 = A_2,\ C_1A_1 \cap C_0A_0 = B_2,\ A_1B_1 \cap A_0B_0 = C_2$$
$$B'C' \cap B_0C_0 = A_3,\ C'A' \cap C_0A_0 = B_3,\ A'B' \cap A_0B_0 = C_3$$

Theorem 4.13.1 (Fontene's First Theorem) — A_1A_2, B_1B_2, C_1C_2 are concurrent at the intersection of $\odot A_1B_1C_1$ and $\odot A_0B_0C_0$

Lemma 4.13.2 — $A'A_3, B'B_3, C'C_3$ and A_0A_3, B_0B_3, C_0C_3 concur at the nine point circle of $\triangle ABC$.

Theorem 4.13.3 (Fontene's Second Theorem) — Let the concurrency point in the first theorem be Q. Then, if the line OP is fixed and P moves along that line, Q will stay fixed.

The previous result leads to another beautiful result:

Lemma 4.13.4 — Suppose a varying point P is chosen on the Euler Line of $\triangle ABC$. Then the pedal circle of P wrt $\triangle ABC$ intersects the 9p circle at a fixed point which is the Euler Reflection Point of the median triangle.

4.14 Unsorted Problems

Problem 4.14.1. In $\triangle ABC$, I is the incenter, D is the touch point of the incenter with BC. $AD \cap \odot ABC \equiv X$. The tangents line from X to $\odot I$ meet $\odot ABC$ at Y, Z. Prove that YZ, BC and the tangent at A to $\odot ABC$ concur.

Problem 4.14.2 (IRAN TST 2017 Day 1, P3). In triangle ABC let I_a be the A-excenter. Let ω be an arbitrary circle that passes through A, I_a and intersects the extensions of sides AB, AC (extended from B, C) at X, Y respectively. Let S, T be points on segments I_aB, I_aC respectively such that $\angle AXI_a = \angle BTI_a$ and $\angle AYI_a = \angle CSI_a$. Lines BT, CS intersect at K. Lines KI_a, TS intersect at Z. Prove that X, Y, Z are collinear.

Problem 4.14.3 (IRAN TST 2015 Day 3, P2). In triangle ABC(with incenter I) let the line parallel to BC from A intersect circumcircle of $\triangle ABC$ at A_1 let $AI \cap BC = D$ and E is tangency point of incircle with BC let $EA_1 \cap \odot(\triangle ADE) = T$ prove that $AI = TI$.

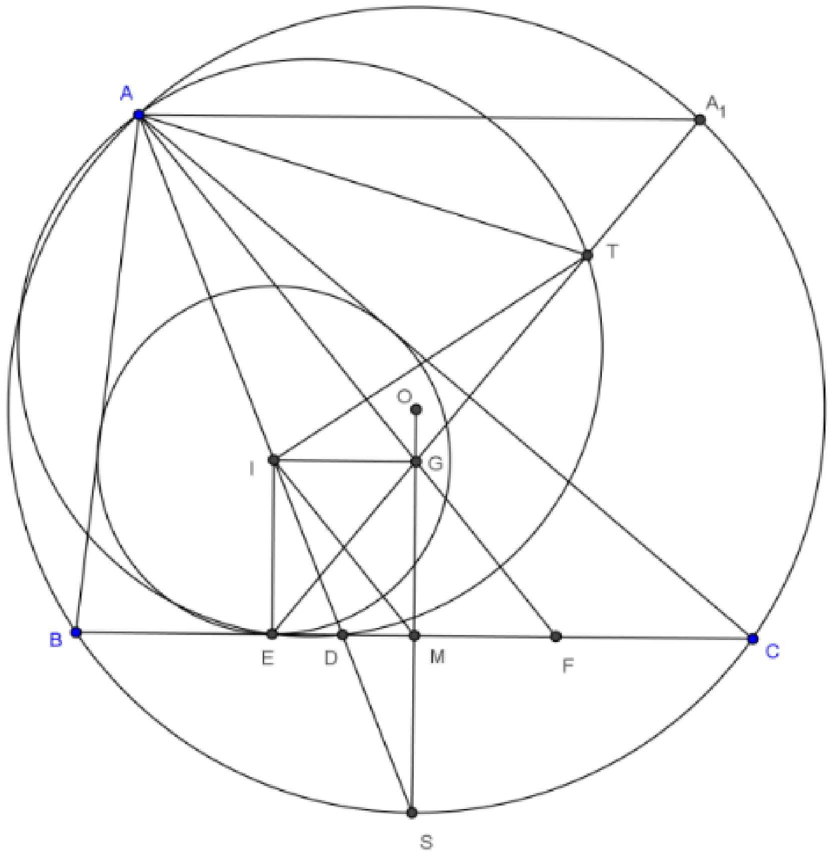


Figure 4.14.1: IRAN TST 2015 Day 3, P2

Problem 4.14.4 (Generalization of Iran TST 2017 P5). Let ABC be triangle and the points P, Q lies on the side BC s.t B, C, P, Q are all different. The circumcircles of triangles ABP and ACQ intersect again at G. AG intersects BC at M. The circumcircle of triangle APQ intersects AB, AC again at E, F , respectively. EP and FQ intersect at T. The lines through M and parallel to AB, AC, intersect EP, FQ at X, Y, respectively. Prove that the circumcircles of triangle TXY and APQ are tangent to each other.

Problem 4.14.5 (ARMO 2013 Grade 11 Day 2 P4). Let ω be the incircle of the triangle ABC and with center I. Let Γ be the circumcircle of the triangle AIB. Circles ω and Γ intersect at the point X and Y. Let Z be the intersection of the common tangents of the circles ω and Γ. Show that the circumcircle of the triangle XYZ is tangent to the circumcircle of the triangle ABC.

Problem 4.14.6 (AoPS). Let ABC be a triangle with incircle (I) and A-excircle (I_a). $(I), (I_a)$ are tangent to BC at D, P, respectively. Let $(I_1), (I_2)$ be the incircle of triangles

APC, APB, respectively, $(J_1), (J_2)$ be the reflections of $(I_1), (I_2)$ wrt midpoints of AC, AB. Prove that AD is the radical axis of (J_1) and (J_2).

Problem 4.14.7 (AoPS). Let ABC be a $A-$right-angled triangle and $MNPQ$ a square inscribed into it, with M, N onto BC in order $B - M - N - C$, and P, Q onto CA, AB respectively. Let $R = BP \cap QM, S = CQ \cap PN$. Prove that $AR = AS$ and RS is perpendicular to the $A-$inner angle bisector of $\triangle ABC$.

Problem 4.14.8 (AoPS). P is an arbitrary point on the plane of $\triangle ABC$ and let $\triangle A'B'C'$ be the cevian triangle of P WRT $\triangle ABC$. The circles $\odot(ABB')$ and $\odot(ACC')$ meet at A, X. Similarly, define the points Y and Z WRT B and C. Prove that the lines AX, BY, CZ concur at the isogonal conjugate of the complement of P WRT $\triangle ABC$.

Problem 4.14.9 (AoPS). Given are $\triangle ABC, L$ is Lemoine point,L_a, L_b, L_c are three Lemoine point of triangles LBC, LCA, LAB prove that AL_a, BL_b, CL_a are concurrent!

A question: What is the locus of point P such that AL_a, BL_b, CL_a are concurrent with L_a, L_b, L_c are three 'Lemoine points' of triangles PBC, PCA, PAB?

Problem 4.14.10 (AoPS). Let ABC be a triangle inscribed circle (O). Let (O') be the circle wich is tangent to the circle (O) and the sides CA, AB at D and E, F, respectively. The line BC intersects the tangent line at A of (O), EF and AO' at T, S and L, respectively. The circle (O) intersects AS again at K. Prove that the circumcenter of triangle AKL lies on the circumcircle of triangle ADT.

Problem 4.14.11. Let P and Q be isogonal conjugates of each other. Let $\triangle XYZ, \triangle KLM$ be the pedal triangles of P and Q wrt $\triangle ABC$. (X, K lie on BC; Y, L lie on CA; Z, M lie on AB) Prove that YM, ZL, PQ are concurrent.

Problem 4.14.12 (2nd Olympiad of Metropolises). Let $ABCDEF$ be a convex hexagon which has an inscribed circle and a circumscribed circle. Denote by ω_A, ω_B, ω_C, ω_D, ω_E, and ω_F the inscribed circles of the triangles FAB, ABC, BCD, CDE, DEF,and EFA,respectively. Let l_{AB} be the external common tangent of ω_A and ω_B other than the line AB; lines $l_{BC}, l_{CD}, l_{DE}, l_{EF}$, and l_{FA} are analogously defined. Let A_1 be the intersection point of the lines l_{FA} and l_{AB}; B_1 be the intersection point of the lines l_{AB} and l_{BC}; points C_1, D_1, E_1, and $F1$ are analogously defined. Suppose that $A_1B_1C_1D_1E_1F_1$ is a convex hexagon. Show that its diagonals A_1D_1, B_1E_1, and C_1F_1 meet at a single point.

Problem 4.14.13 (ISL 2016 G6). Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC < 90^\circ$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at E and F respectively, and meet each other at point P. Let M be the midpoint of AC and let ω be the circumcircle of triangle BPD. Segments BM and DM intersect ω again at X and Y respectively. Denote by Q the intersection point of lines XE and YF. Prove that $PQ \perp AC$.

4.15 Problems

Problem 4.15.1 (IRAN 3rd Round 2016 P2). Let ABC be an arbitrary triangle. Let E, F be two points on AB, AC respectively such that their distance to the midpoint of BC is equal. Let P be the second intersection of the triangles ABC, AEF circumcircles . The tangents from E, F to the circumcircle of AEF intersect each other at K. Prove that : $\angle KPA = 90$

Problem 4.15.2 (IRAN 2nd Round 2016 P6). Let ABC be a triangle and X be a point on its circumcircle. Q, P lie on a line BC such that $XQ \perp AC, XP \perp AB$. Let Y be the circumcenter of $\triangle XQP$. Prove that ABC is equilateral triangle if and if only Y moves on a circle when X varies on the circumcircle of ABC

Problem 4.15.3 (AoPS). Consider ABC with orthic triangle $A'B'C'$, let $AA' \cap B'C' = E$ and E' be reflection of E wrt BC. Let M be midpoint of BC and O be circumcenter of $E'B'C'$. Let M' be projection of O on BC and N be the intersection of a perpendicular to $B'C'$ through E with BC. Prove that $MM' = 1/4MN$.

Problem 4.15.4 (IRAN 3rd Round 2010 D3, P5). In a triangle ABC, I is the incenter. D is the reflection of A to I. the incircle is tangent to BC at point E. DE cuts IG at P (G is centroid). M is the midpoint of BC. Prove that $AP||DM$ and $AP = 2DM$.

Problem 4.15.5 (IRAN 3rd Round 2011 G5). Given triangle ABC, D is the foot of the external angle bisector of A, I its incenter and I_a its A-excenter. Perpendicular from I to DI_a intersects the circumcircle of triangle in A'. Define B' and C' similarly. Prove that AA', BB' and CC' are concurrent.

Problem 4.15.6 (AoPS3). I is the incenter of ABC, $PI, QI \perp BC$, PA, QA intersect BC at DE. Prove: $IADE$ is on a circle.

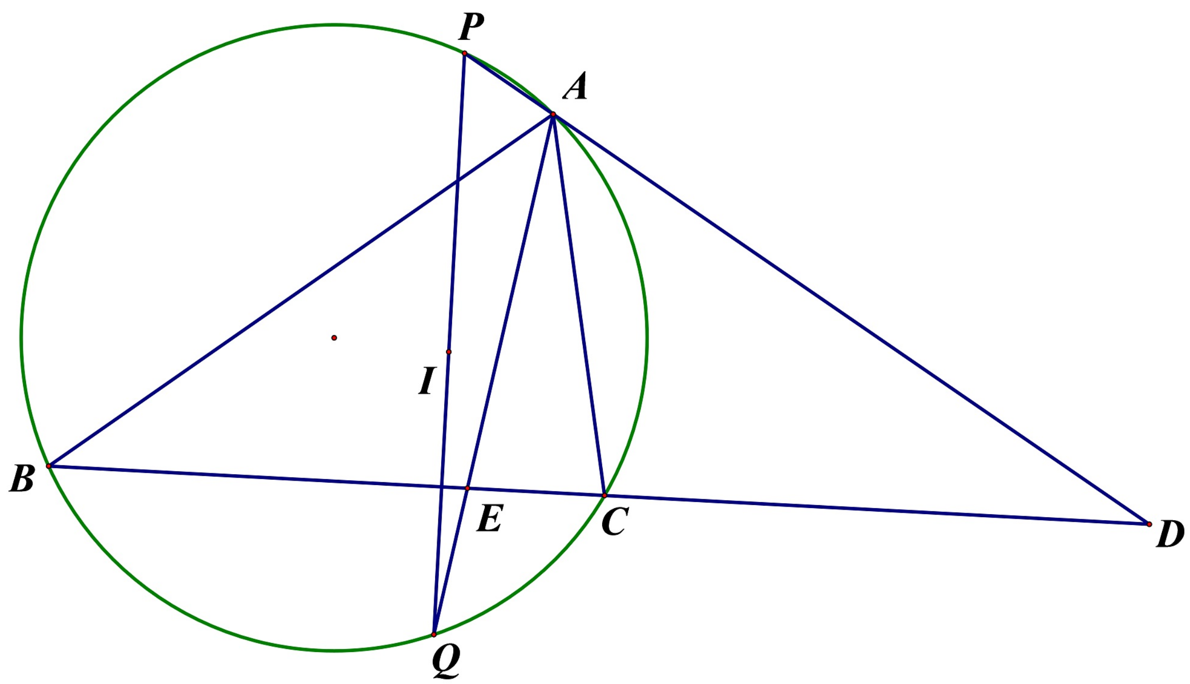


Figure 4.15.1: AoPS3

Problem 4.15.7 (AoPS4). Given a triangle ABC, the incircle (I) touch BC, CA, AB at D, E, F respectively. Let AA_1, BB_1, CC_1 be $A, B, C - altitude$ respectively. Let N be the orthocenter of the triangle AEF. Prove that N is the incenter of AB_1C_1

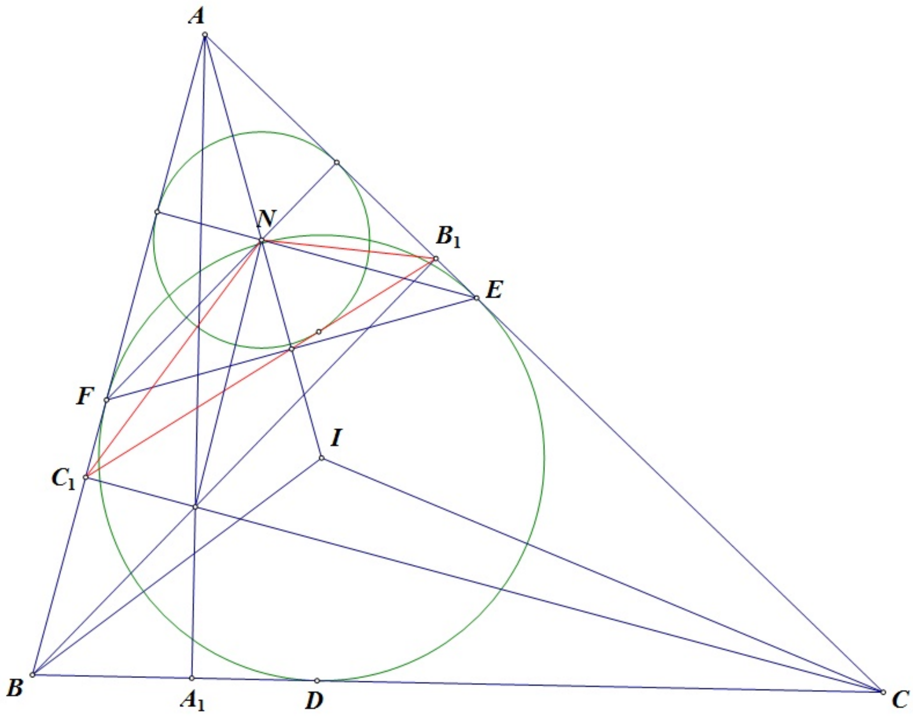


Figure 4.15.2: AoPS4

Problem 4.15.8 (IRAN TST 2015 Day 2, P3). $ABCD$ is a circumscribed and inscribed quadrilateral. O is the circumcenter of the quadrilateral. E, F and S are the intersections of AB, CD; AD, BC and AC, BD respectively. E' and F' are points on AD and AB such that $\angle AEE' = \angle E'ED$ and $\angle AFF' = \angle F'FB$. X and Y are points on OE' and OF' such that $\frac{XA}{XD} = \frac{EA}{ED}$ and $\frac{YA}{YB} = \frac{FA}{FB}$. M is the midpoint of arc BD of (O) which contains A. Prove that the circumcircles of triangles OXY and OAM are coaxial with the circle with diameter OS.

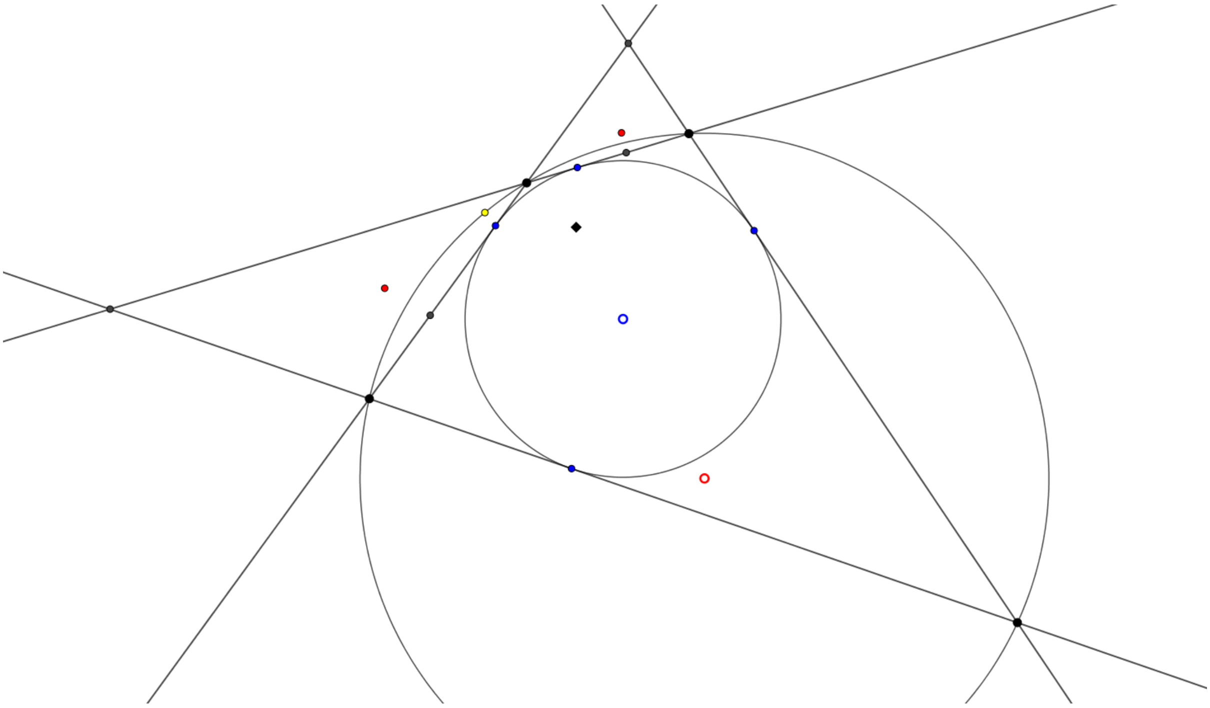


Figure 4.15.3: Actual Prob

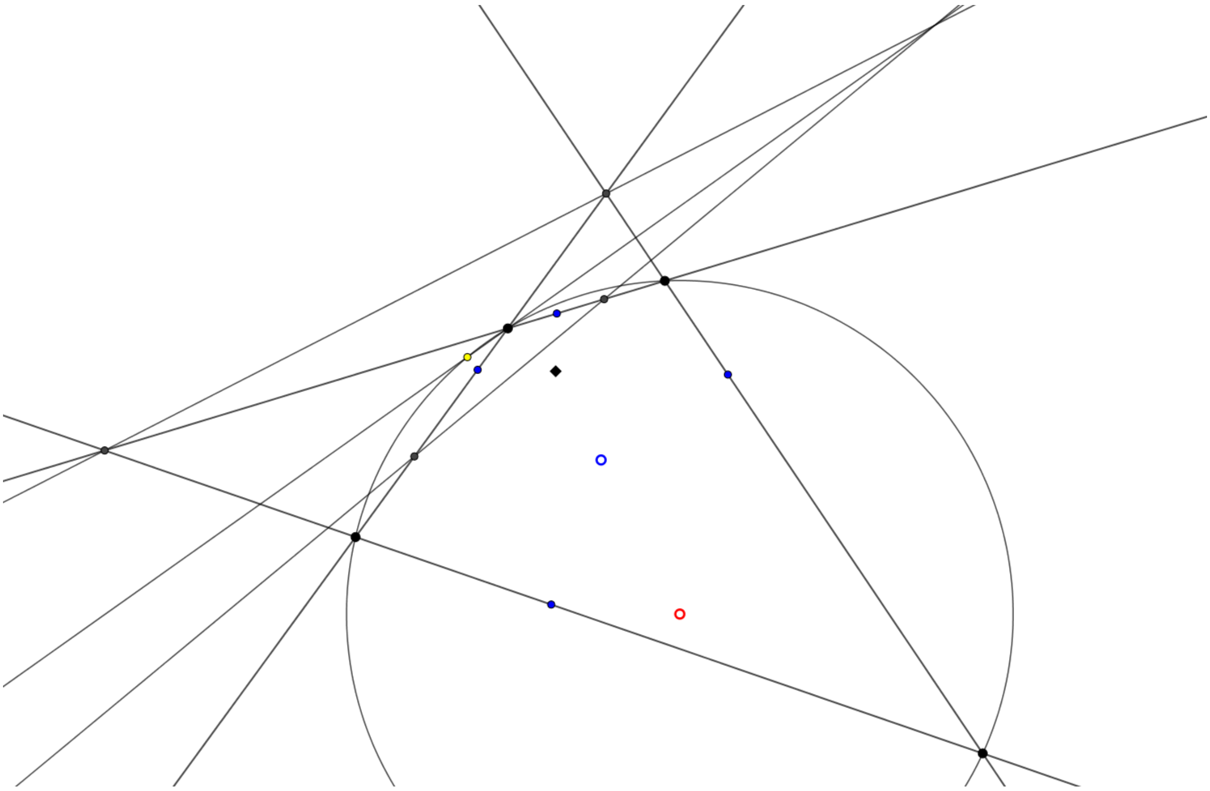


Figure 4.15.4: Inverted

Problem 4.15.9 (USA TST 2017 P2). Let ABC be an acute scalene triangle with circumcenter O, and let T be on line BC such that $\angle TAO = 90^\circ$. The circle with diameter $\overline{AT}$ intersects the circumcircle of $\triangle BOC$ at two points A_1 and A_2, where $OA_1 < OA_2$. Points B_1, B_2, C_1, C_2 are defined analogously.

1. Prove that $\overline{AA_1}$, $\overline{BB_1}$, $\overline{CC_1}$ are concurrent.
2. Prove that $\overline{AA_2}$, $\overline{BB_2}$, $\overline{CC_2}$ are concurrent on the Euler line of triangle ABC.

Problem 4.15.10 (AoPS2). Let ABC be a triangle with circumcenter O and altitude AH. AO meets BC at M and meets the circle (BOC) again at N. P is the midpoint of MN. K is the projection of P on line AH. Prove that the circle (K, KH) is tangent to the circle (BOC).

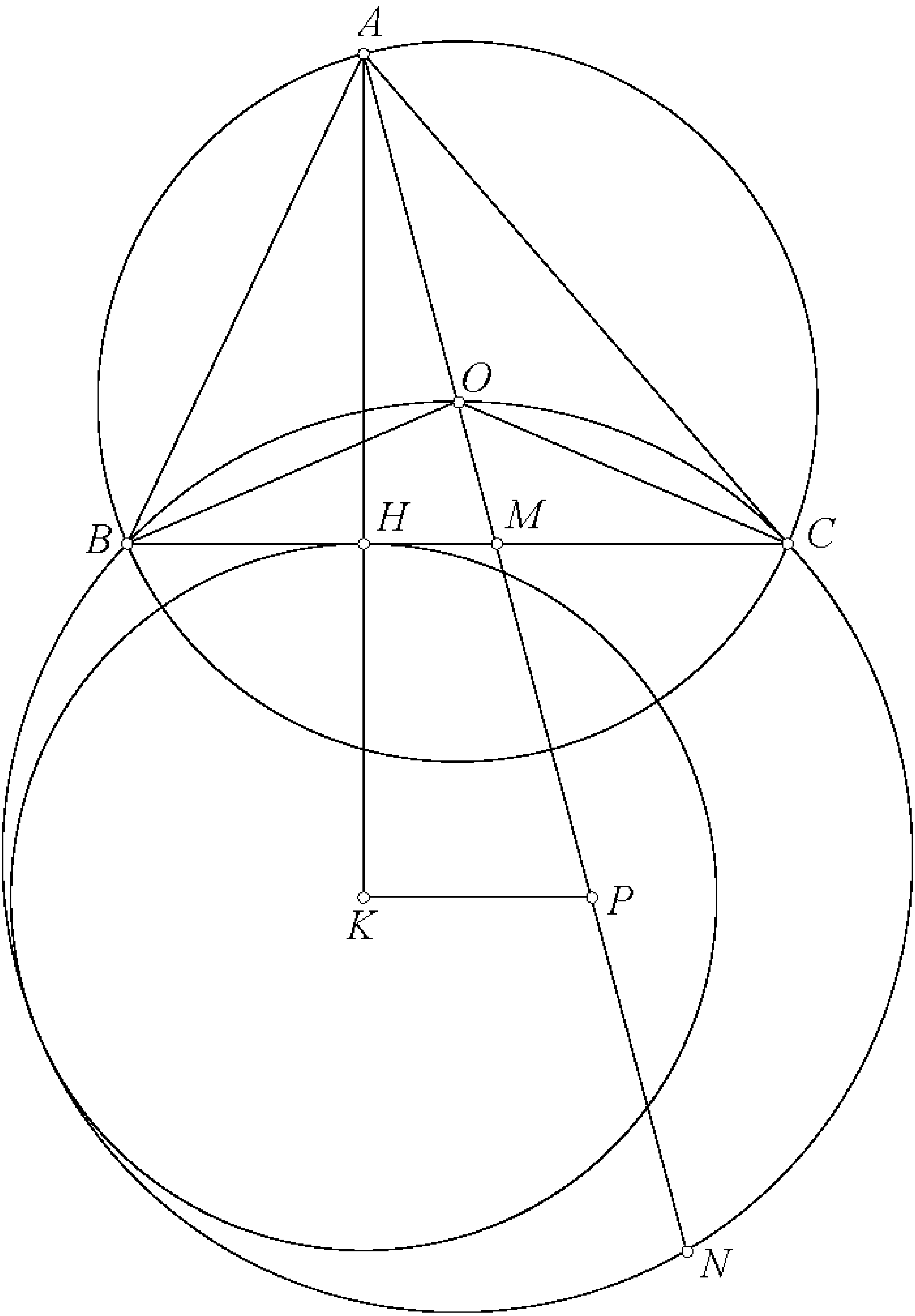


Figure 4.15.5: AoPS2

Solution. Inversion all the way...

Problem 4.15.11 (AoPS5). Let ABC be a triangle inscribed in (O) and P be a point. Call P' be the isogonal conjugate point of P. Let A' be the second intersection of AP' and (O). Denote by M the intersection of BC and $A'P$. Prove that $P'M \parallel AP$.

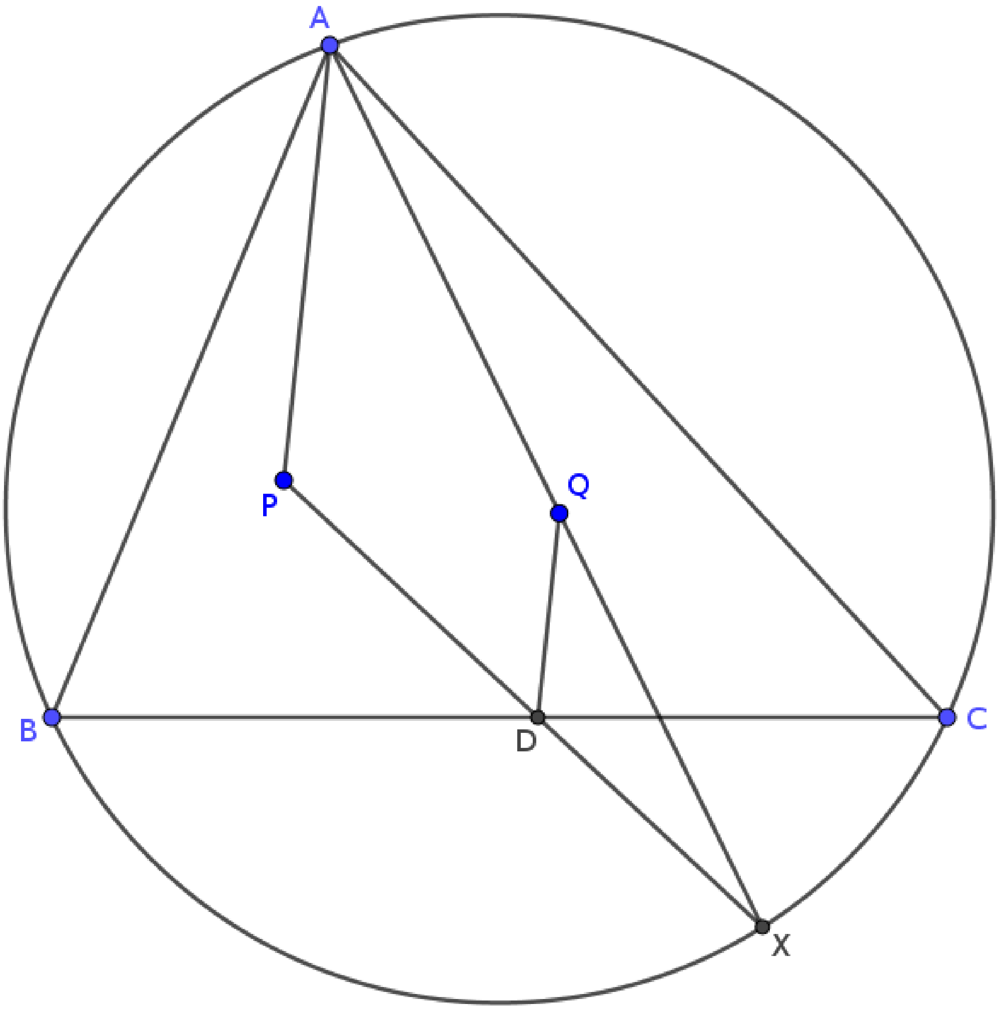


Figure 4.15.6: AoPS5

Problem 4.15.12 (AoPS). I is the incenter of a non-isosceles triangle $\triangle ABC$. If the incircle touches BC, CA, AB at A_1, B_1, C_1 respectively, prove that the circumcentres of the triangles $\triangle AIA_1$, $\triangle BIB_1$, $\triangle CIC_1$ are collinear.

Problem 4.15.13 (AoPS). Given $\triangle ABC$ and a point P inside. AP cuts BC at M. Let M', A' be the reflection of M, A in the perpendicular bisector of BC. $A'P$ cuts the perpendicular bisector of BC at N. Let Q be the isogonal conjugate of P in triangle ABC. Prove that $QM' \parallel AN$.

Problem 4.15.14 (IRAN 3rd Round 2016 G6). Given triangle $\triangle ABC$ and let D, E, F be the foot of angle bisectors of A, B, C ,respectively. M, N lie on EF such that $AM = AN$. Let H be the foot of A-altitude on BC.
Points K, L lie on EF such that triangles $\triangle AKL, \triangle HMN$ are correspondingly similar (with the given order of vertices's) such that $AK \not\parallel HM$ and $AK \not\parallel HN$. Show that: $DK = DL$.

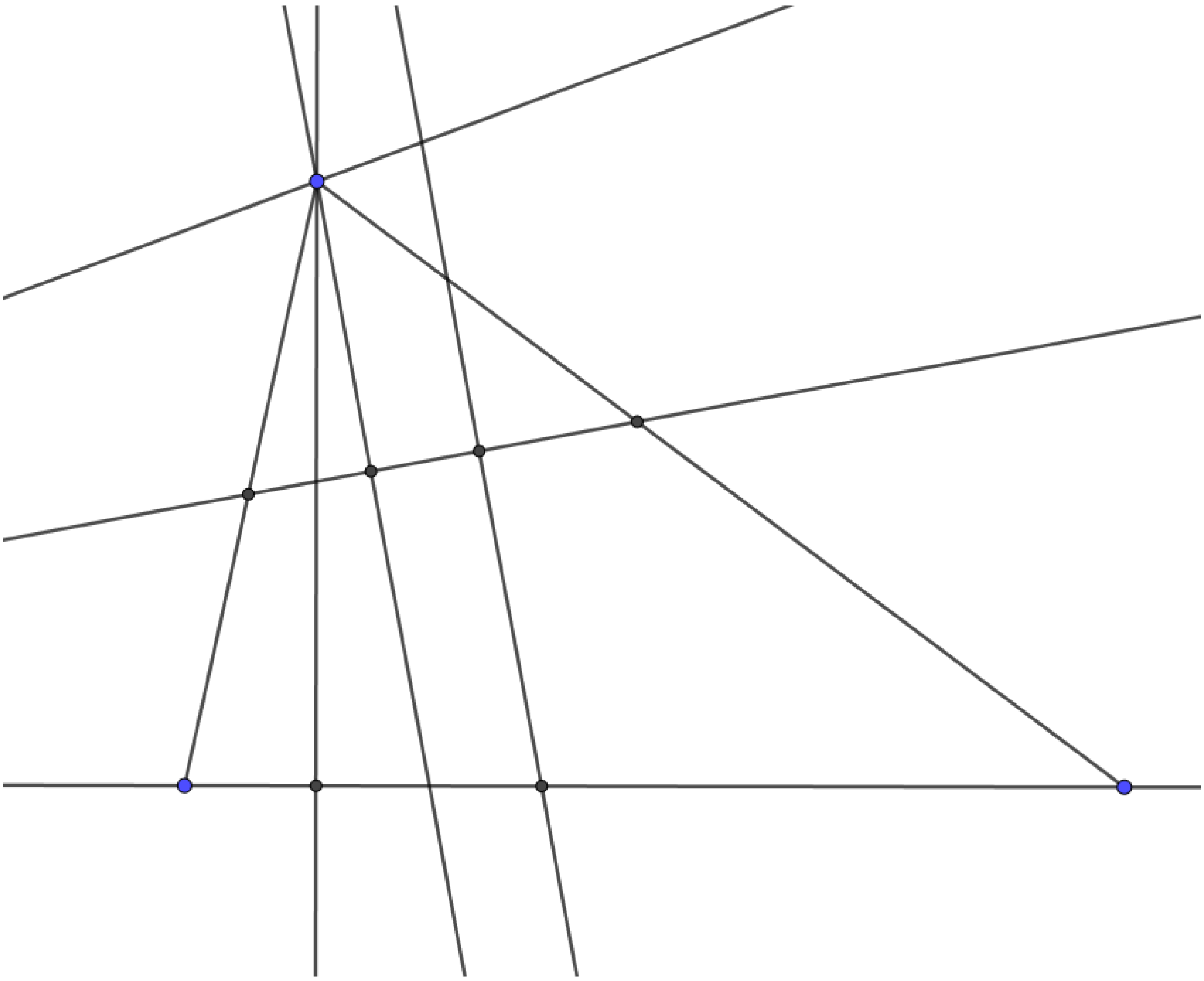


Figure 4.15.7: IRAN 3rd Round 2016 G6

Problem 4.15.15 (Iran TST 2017 T3 P6). In triangle ABC let O and H be the circumcenter and the orthocenter. The point P is the reflection of A with respect to OH. Assume that P is not on the same side of BC as A. Points E, F lie on AB, AC respectively such that $BE = PC$, $CF = PB$. Let K be the intersection point of AP, OH. Prove that

$\angle EKF = 90°$.

Spiral Similarity (points on AB, AC with some properties)

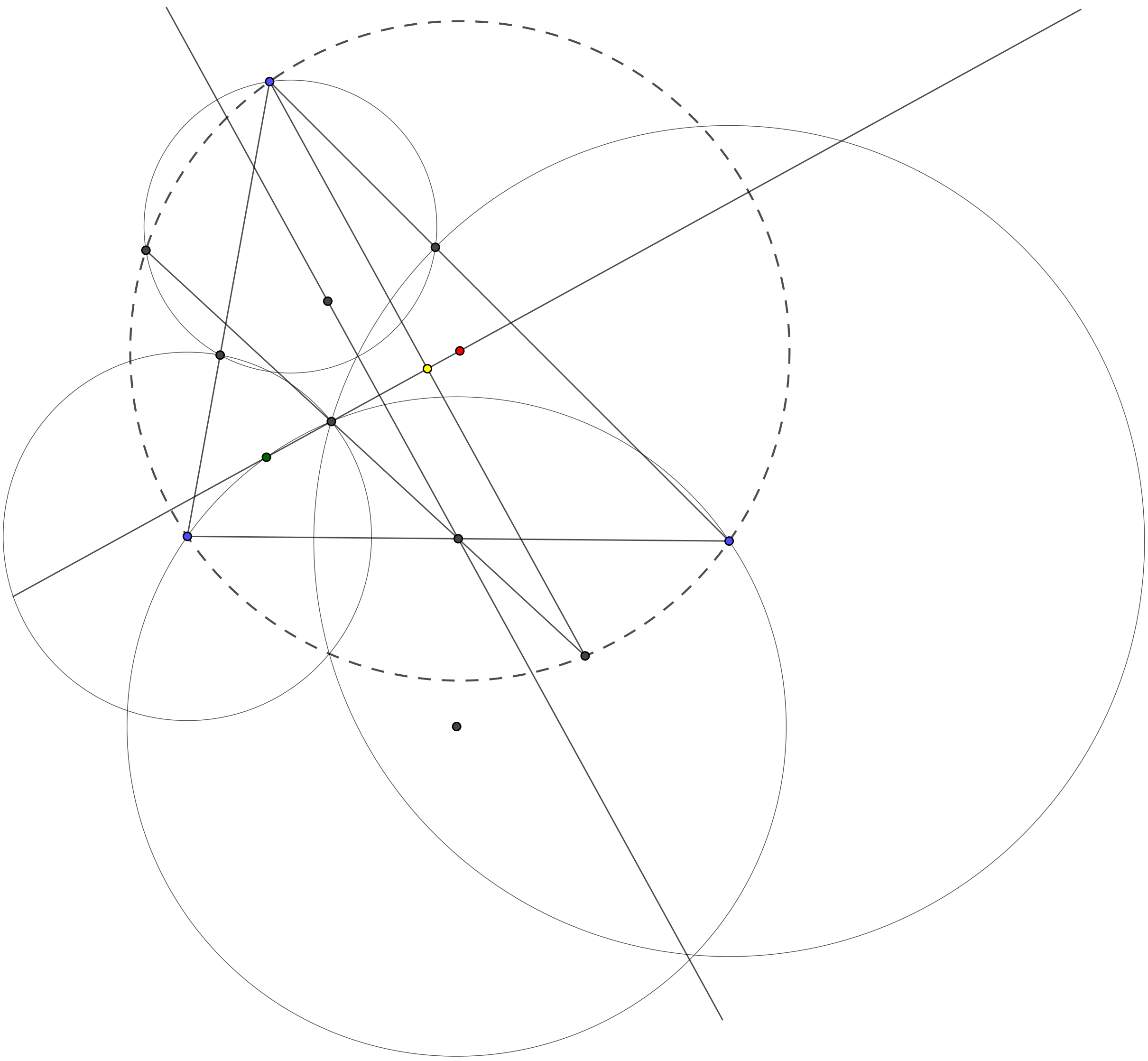


Figure 4.15.8: Iran TST 2017 T3 P6

Problem 4.15.16 (`IRAN 3rd Round 2010 D3, P6`). In a triangle ABC, $\angle C = 45°$. AD is the altitude of the triangle. X is on AD such that $\angle XBC = 90 - \angle B$ (X is inside of the triangle). AD and CX cut the circumcircle of ABC in M and N respectively. Ff the tangent to $\odot ABC$ at M cuts AN at P, prove that P, B and O are collinear.

Cross-Ratio

Problem 4.15.17 (Iran TST 2014 T1P6). I is the incenter of triangle ABC. perpendicular from I to AI meet AB and AC at B' and C' respectively. Suppose that B'' and C'' are points on half-line BC and CB such that $BB'' = BA$ and $CC'' = CA$. Suppose that the second intersection of circumcircles of $AB'B''$ and $AC'C''$ is T. Prove that the circumcenter of AIT is on the BC.

projective, inversion

Solution. Too many collinearity, need to prove concurrency, what else can come into mind except projective approach.

Solution. Too many incenter related things, $\sqrt{bc}$-inversion :o

Problem 4.15.18 (APMO 2014 P5). Circles ω and Ω meet at points A and B. Let M be the midpoint of the arc AB of circle ω (M lies inside Ω). A chord MP of circle ω intersects Ω at Q (Q lies inside ω). Let ℓ_P be the tangent line to ω at P, and let ℓ_Q be the tangent line to Ω at Q. Prove that the circumcircle of the triangle formed by the lines ℓ_P, ℓ_Q and AB is tangent to Ω.

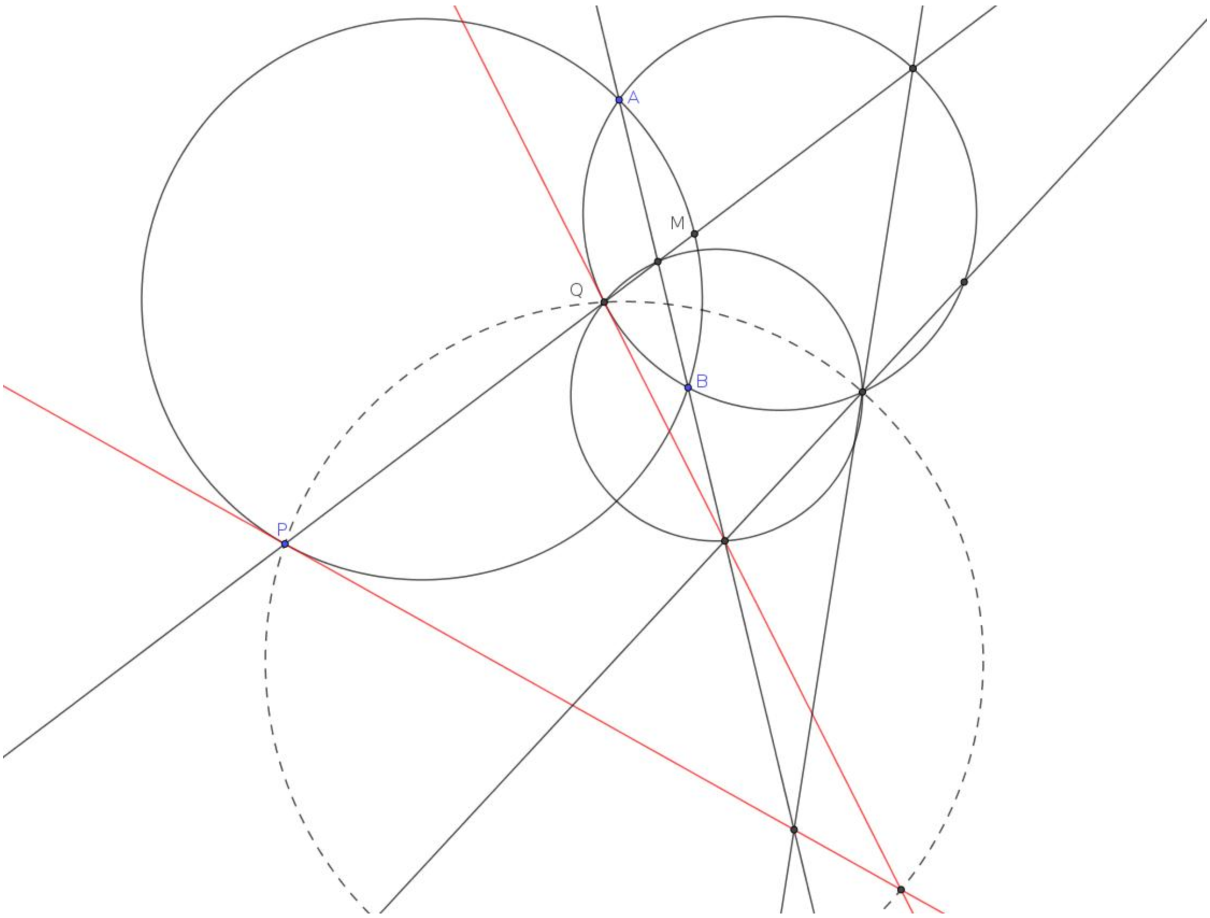


Figure 4.15.9: APMO 2014 P5

Problem 4.15.19. Let ABC be a triangle, D, E, F are the feet of the altitudes, $DF \cap BE \equiv P, DE \cap CF \equiv Q$. Prove that the perpendicular from A to PQ goes through the reflection of O on BC.

projective

Solution. Projective approach.

Problem 4.15.20 (RMM 2018 P6). Fix a circle Γ, a line ℓ to tangent Γ, and another circle Ω disjoint from ℓ such that Γ and Ω lie on opposite sides of ℓ. The tangents to Γ from a variable point X on Ω meet ℓ at Y and Z. Prove that, as X varies over Ω, the circumcircle of XYZ is tangent to two fixed circles.

inversion

Solution. Too many circles, plus tangency, what else other than inversion? After the inversion the problem turns into a pretty obvious work-around problem.

Problem 4.15.21 (AoPS6). Let O and I be the circumcenter and incenter of ΔABC. Draw circle ω so that $B, C \in \omega$ and ω touches (I) internally at P. AI intersects BC at X. Tangent at X to (I) which is different from BC, intersects tangent at P to (I) at S. $SA \cap (O) = T \neq A$. Prove that $\angle ATI = 90°$

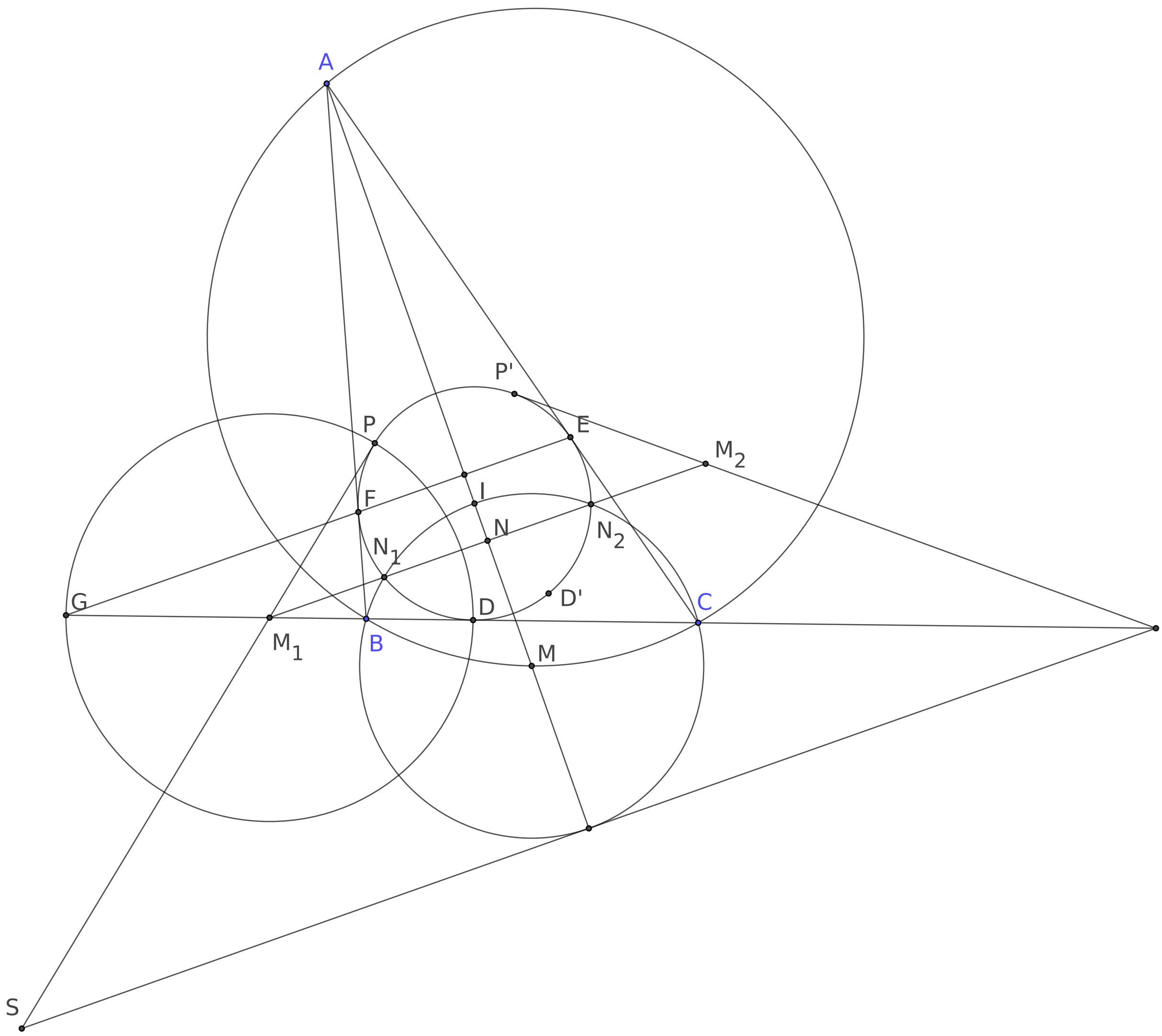


Figure 4.15.10: Solution 1

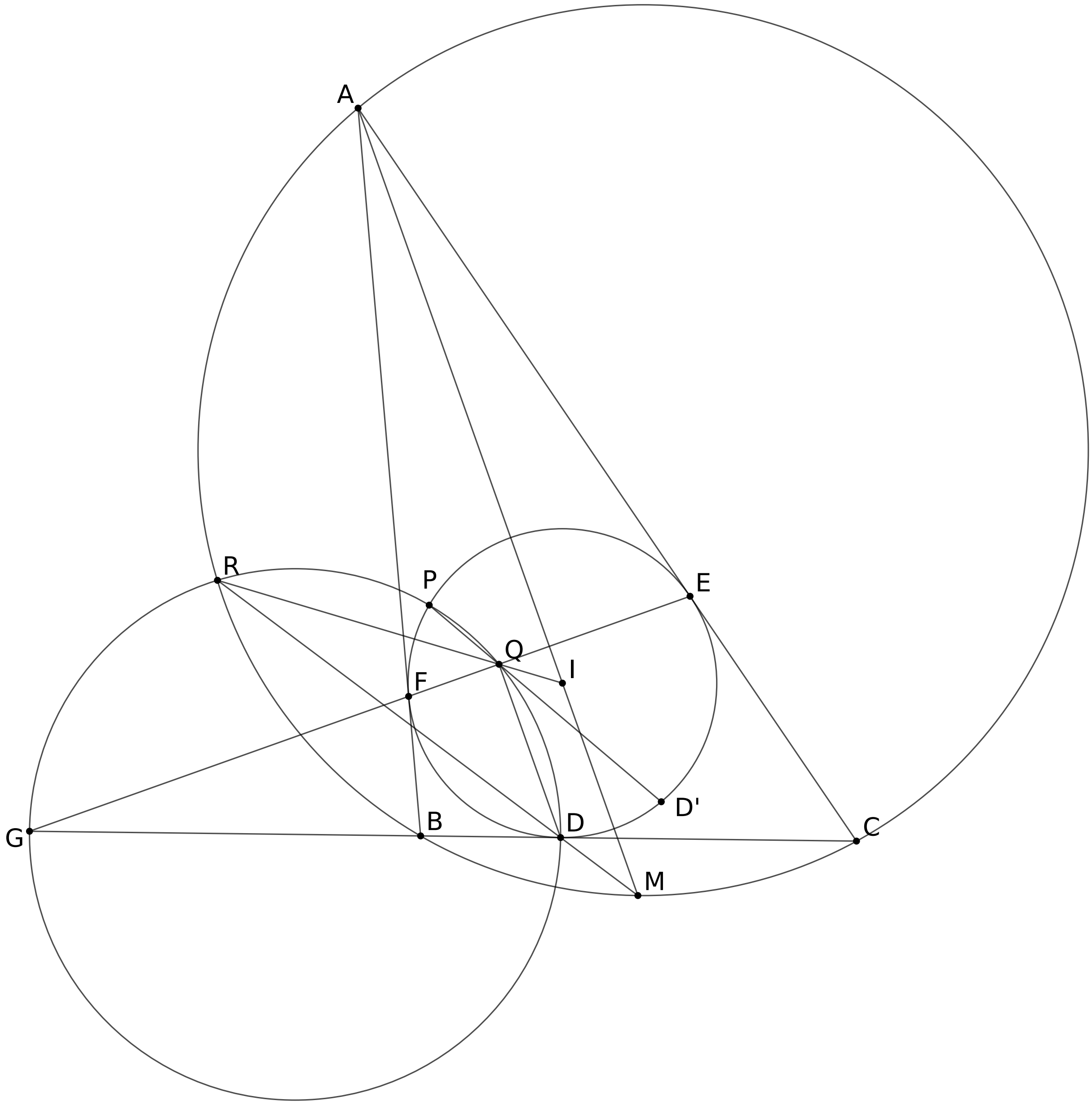


Figure 4.15.11: Solution 2

Problem 4.15.22 (AoPS7). Let ABC be a triangle with incenter I and circumcircle Γ. Let the line through I perpendicular to AI meet AB at E and AC at F. Let the circumcircles of triangles AIB and AIC intersect the circumcircle of triangle AEF ω again at points M and N, and let ω intersect Γ again at Q. Prove that AQ, MN, and BC are concurrent.

Problem 4.15.23 (AoPS). Given a circle (O) with center O and A, B are 2 fixed points on (O). E lies on AB. C, D are on (O) and CD pass through E. P lies on the ray DA, Q lies on the ray DB such that E is the midpoint of PQ. Prove that the circle passing through C and touch PQ at E also pass through the midpoint of AB

Problem 4.15.24 (WenWuGuangHua Mathematics Workshop). O_B, O_C are the B and C mixtilinear centers respectively. (O_B) touches BC, AB at X_B, Y_B respectively, and $X_BY_B \cap O_BO_C$ at Z_B. Define X_C, Y_C, Z_C similarly. Prove that if $BZ_C \cap CZ_B = T$, then AT is the A-angle bisector.

Problem 4.15.25 (All Russia 1999 P9.3). A triangle ABC is inscribed in a circle S. Let A_0 and C_0 be the midpoints of the arcs BC and AB on S, not containing the opposite vertex, respectively. The circle S_1 centered at A_0 is tangent to BC, and the circle S_2 centered at C_0 is tangent to AB. Prove that the incenter I of $\triangle ABC$ lies on a common tangent to S_1 and S_2.

Problem 4.15.26 (All Russia 2000 P11.7). A quadrilateral $ABCD$ is circumscribed about a circle ω. The lines AB and CD meet at O. A circle ω_1 is tangent to side BC at K and to the extensions of sides AB and CD, and a circle ω_2 is tangent to side AD at L and to the extensions of sides AB and CD. Suppose that points O, K, L lie on a line. Prove that the midpoints of BC and AD and the center of ω also lie on a line.

Problem 4.15.27 (All Russia 2000 P9.3). Let O be the center of the circumcircle ω of an acute-angle triangle ABC. A circle ω_1 with center K passes through A, O, C and intersects AB at M and BC at N. Point L is symmetric to K with respect to line NM. Prove that $BL \perp AC$.

Problem 4.15.28 (WenWuGuangHua Mathematics Workshop).

1. AD, BE, CF are concurrent cevians. Angle bisectors of $\angle ADB$ and $\angle AEB$ meet at C_0. Again the angle bisectors of $\angle ADC$ and $\angle AFC$ meet at B_0. And bisectors of $\angle BEC$ and $\angle BFC$ meet at A_0. Prove that AA_0, BB_0, CC_0 are concurrent.
2. Angle bisectors of $\angle AEB$ and $\angle AFC$ meet at D_0, of $\angle BFC$ and BDA meet at E_0, and of $\angle CEB$ and $\angle CDA$ meet at F_0. Prove that DD_0, EE_0, FF_0 are concurrent.

Solution. As this problem is purely made up with lines, we can do a projective transformation to simplify the problem. And as there are perpendicularity at D, E, F, we make D, E, F the feet of the altitudes of $\triangle ABC$. Then the angle bisector properties get replaced by simpler properties wrt DEF.

Problem 4.15.29 (WenWuGuangHua Mathematics Workshop). Generalization: Let AD, BE, CF be any cevians concurrent at T. $AD \cap EF = A'$, $BE \cap DF = B'$, $CF \cap DE = C'$, $B'A' \cap AC = X$, $B'A' \cap BC = Y$, $C'X \cap EF = Z$. Prove that T, Y, Z are collinear.

Problem 4.15.30 (AoPS). On circumcircle of triangle ABC, T and K are midpoints of arcs BC and BAC respectively . And E is foot of altitude from C on AB . Point P is on extension of AK such that PE is perpendicular to ET . Prove that $PC = CK$.

Problem 4.15.31 (USJMO 2018 P3). Let $ABCD$ be a quadrilateral inscribed in circle ω with $\overline{AC} \perp \overline{BD}$. Let E and F be the reflections of D over lines BA and BC, respectively, and let P be the intersection of lines BD and EF. Suppose that the circumcircle of $\triangle EPD$ meets ω at D and Q, and the circumcircle of $\triangle FPD$ meets ω at D and R. Show that $EQ = FR$.

Problem 4.15.32 (All Russia 2002 P11.6). The diagonals AC and BD of a cyclic quadrilateral $ABCD$ meet at O. The circumcircles of triangles AOB and COD intersect again at K. Point L is such that the triangles BLC and AKD are similar and equally oriented. Prove that if the quadrilateral $BLCK$ is convex, then it has an incircle.

Problem 4.15.33 (WenWuGuangHua Mathematics Workshop). Let O_B, O_C be the B, C mixtilinear excircles. O meet CA, CB at X_C, Y_C and O_B meet BA, BC at X_B, Y_B. Let I_C be the C-excircle. $I_C Y_B$ meet $O_B O_C$ at T. Prove that $BT \perp O_B O_C$

> **Solution.** From what we have to prove, we find two circles, from where we get another circle. This circle suggests that we try power of point.

Problem 4.15.34 (Iran TST 2018 T1P3). In triangle ABC let M be the midpoint of BC. Let ω be a circle inside of ABC and is tangent to AB, AC at E, F, respectively. The tangents from M to ω meet ω at P, Q such that P and B lie on the same side of AM. Let $X \equiv PM \cap BF$ and $Y \equiv QM \cap CE$. If $2PM = BC$ prove that XY is tangent to ω.

Problem 4.15.35 (Iran TST 2018 T1P4). Let ABC be a triangle ($\angle A \neq 90^\circ$). BE, CF are the altitudes of the triangle. The bisector of $\angle A$ intersects EF, BC at M, N. Let P be a point such that $MP \perp EF$ and $NP \perp BC$. Prove that AP passes through the midpoint of BC.

> **Solution.** :'3 kala para na T_T

Problem 4.15.36 (Iran TST 2018 T3P6). Consider quadrilateral $ABCD$ inscribed in circle ω. $AC \cap BD = P$. E, F lie on sides AB, CD, respectively such that $\angle APE = \angle DPF$. Circles ω_1, ω_2 are tangent to ω at X, Y respectively and also both tangent to the circumcircle of PEF at P. Prove that:

$$\frac{EX}{EY} = \frac{FX}{FY}$$

Solution. fucking beautiful.

Problem 4.15.37 (ISL 2006 G6). Circles ω_1 and ω_2 with centres O_1 and O_2 are externally tangent at point D and internally tangent to a circle ω at points E and F respectively. Line t is the common tangent of ω_1 and ω_2 at D. Let AB be the diameter of ω perpendicular to t, so that A, E, O_1 are on the same side of t. Prove that lines AO_1, BO_2, EF and t are concurrent.

Problem 4.15.38 (ISL 2006 G7). In a triangle ABC, let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB, respectively, and T_a, T_b, T_c be the midpoints of the arcs BC, CA, AB of the circumcircle of ABC, not containing the vertices's A, B, C, respectively. For $i \in a, b, c$, let w_i be the circle with M_iT_i as diameter. Let p_i be the common external common tangent to the circles w_j and w_k (for all $i, j, k = a, b, c$) such that w_i lies on the opposite side of p_i than w_j and w_k do.

Prove that the lines p_a, p_b, p_c form a triangle similar to ABC and find the ratio of similitude

Problem 4.15.39 (ISL 2006 G9). Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC, respectively. The circumcircles of triangles $AB_1C_1,\ BC_1A_1,\ CA_1B_1$ intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2, respectively ($A_2 \neq A, B_2 \neq B, C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

Solution. In this type of "Miquel's Point and the intersections of the circumcircles" related problems, it is useful to think about the second intersections of the lines joining the first intersections and the Miquel's Point with the main circle.

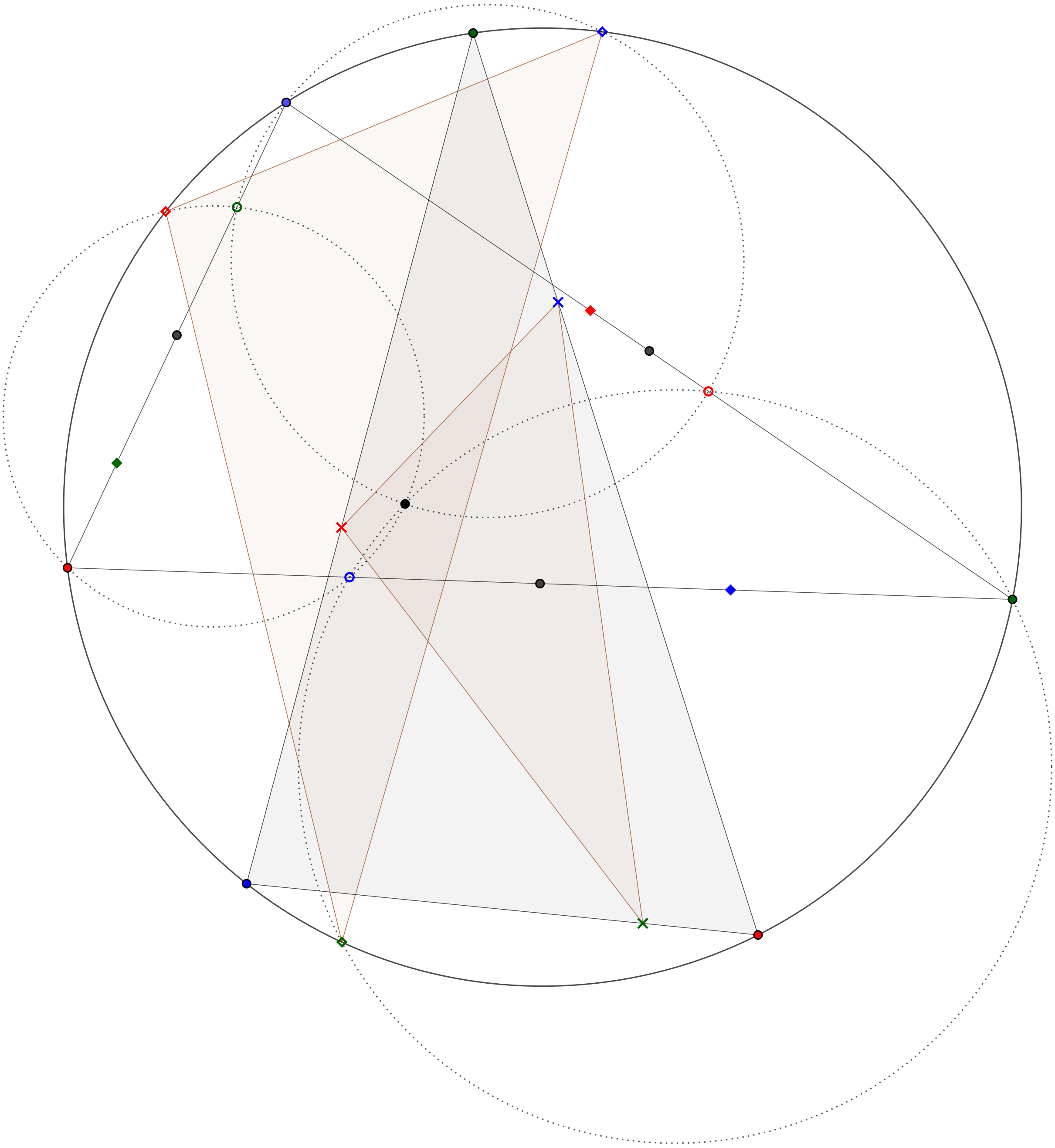


Figure 4.15.12: IMO Shortlist G9

Problem 4.15.40 (Iran TST 2017 P5). In triangle ABC, arbitrary points P, Q lie on side BC such that $BP = CQ$ and P lies between B, Q. The circumcircle of triangle APQ intersects sides AB, AC at E, F respectively. The point T is the intersection of EP, FQ. Two lines passing through the midpoint of BC and parallel to AB and AC, intersect EP and FQ at points X, Y respectively. Prove that the circumcircle of triangle TXY and triangle APQ are tangent to each other.

Problem 4.15.41. Let X be the touchpoint of the incircle with BC and let AX meet $\cdot ABC$ at D. The tangents from D to the incircle meet $\cdot ABC$ at E, F. Prove that the tangent to the circumcircle at A, EF and BC are concurrent.

Problem 4.15.42 (ISL 2012 G8). Let ABC be a triangle with circumcircle ω and ℓ a line without common points with ω. Denote by P the foot of the perpendicular from the center of ω to ℓ. The side-lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove that the circumcircles of the triangles AXP, BYP and CZP have a common point different from P or are mutually tangent at P.

Solution. Using Cross ratio and Desergaus's Involution Theorem.

Problem 4.15.43. Suppose an involution on a line l sending X, Y, Z to X', Y', Z'. Let l_x, l_y, l_z be three lines passing through X, Y, Z respectively. And let $X_0 = l_y \cap l_z,\ Y_0 = l_x \cap l_z,\ Z_0 = l_x \cap l_y$. Then X_0X', Y_0Y', Z_0Z' are concurrent.

Problem 4.15.44 (USAMO 2018 P5). In convex cyclic quadrilateral $ABCD$, we know that lines AC and BD intersect at E, lines AB and CD intersect at F, and lines BC and DA intersect at G. Suppose that the circumcircle of $\triangle ABE$ intersects line CB at B and P, and the circumcircle of $\triangle ADE$ intersects line CD at D and Q, where C, B, P, G and C, Q, D, F are collinear in that order. Prove that if lines FP and GQ intersect at M, then $\angle MAC = 90°$.

Problem 4.15.45 (Japan MO 2017 P3). Let ABC be an acute-angled triangle with the circumcenter O. Let D, E and F be the feet of the altitudes from A, B and C, respectively, and let M be the midpoint of BC. AD and EF meet at X, AO and BC meet at Y, and let Z be the midpoint of XY. Prove that A, Z, M are collinear.

Problem 4.15.46 (ISL 2002 G1). Let B be a point on a circle S_1, and let A be a point distinct from B on the tangent at B to S_1. Let C be a point not on S_1 such that the line segment AC meets S_1 at two distinct points. Let S_2 be the circle touching AC at C and touching S_1 at a point D on the opposite side of AC from B. Prove that the circumcenter of triangle BCD lies on the circumcircle of triangle ABC.

Problem 4.15.47 (ISL 2002 G2). Let ABC be a triangle for which there exists an interior point F such that $\angle AFB = \angle BFC = \angle CFA$. Let the lines BF and CF meet the sides AC and AB at D and E respectively. Prove that

$$AB + AC \geq 4DE.$$

Solution. Pari nai.

Problem 4.15.48 (ISL 2002 G3). The circle S has center O, and BC is a diameter of S. Let A be a point of S such that $\angle AOB < 120°$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and at F. Prove that I is the incenter of the triangle CEF.

Problem 4.15.49 (ISL 2002 G4). Circles S_1 and S_2 intersect at points P and Q. Distinct points A_1 and B_1 (not at P or Q) are selected on S_1. The lines A_1P and B_1P meet S_2 again at A_2 and B_2 respectively, and the lines A_1B_1 and A_2B_2 meet at C. Prove that, as A_1 and B_1 vary, the circumcentres of triangles A_1A_2C all lie on one fixed circle.

Problem 4.15.50 (ISL 2002 G7). The incircle Ω of the acute-angled triangle ABC is tangent to its side BC at a point K. Let AD be an altitude of triangle ABC, and let M be the midpoint of the segment AD. If N is the common point of the circle Ω and the line KM (distinct from K), then prove that the incircle Ω and the circumcircle of triangle BCN are tangent to each other at the point N.

Problem 4.15.51 (Japan MO 2017 P3). Let ABC be an acute-angled triangle with the circumcenter O. Let D, E and F be the feet of the altitudes from A, B and C, respectively, and let M be the midpoint of BC. AD and EF meet at X, AO and BC meet at Y, and let Z be the midpoint of XY. Prove that A, Z, M are collinear.

Problem 4.15.52 (India TST). ABC triangle, D, E, F touchpoints, M midpoint of BC, K orthocenter of $\triangle AIC$, prove that $MI \perp KD$

Problem 4.15.53 (ISL 2009 G3). Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelogram. Prove that $GR = GS$.

> **Solution.** Point Circle, distance same means Power same wrt point circles.

Problem 4.15.54 (ARO 2018 P11.6). Three diagonals of a regular n-gon prism intersect at an interior point O. Show that O is the center of the prism.

(The diagonal of the prism is a segment joining two vertices's not lying on the same face of the prism.)

Problem 4.15.55 (ISL 2011 G4). Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.

Problem 4.15.56. Given 3 circle, construct another circle that is tangent to these three circles.

> **Solution.** A trick to remember: decreasing the radius's of some circles doesn't effect much.

Problem 4.15.57. Let $ABCD$ be a convex quadrilateral, let $AD \cap BC = P$. Let O, O'; H, H' be the circumcentres and orthocenter of $\triangle PCD, \triangle PAB$. $\odot DOC$ is tangent to $\odot AD'B$, if and only if $\odot DHC$ is tangent to $\odot AH'B$

Problem 4.15.58 (Iran MO 3rd round 2017 mid-terms Geometry P3). Let ABC be an acute-angle triangle. Suppose that M be the midpoint of BC and H be the orthocenter of ABC. Let $F \equiv BH \cap AC$ and $E \equiv CH \cap AB$. Suppose that X be a point on EF such that $\angle XMH = \angle HAM$ and A, X are in the distinct side of MH. Prove that AH bisects MX.

4.16 Research Stuffs for later

Problem 4.16.1 (AoPS). Let ABC be a triangle with incenter I. L_a, L_b, L_c are symmedian points of triangles IBC, ICA, IAB. Let X, Y, Z be the reflections of I through L_a, L_b, L_c.

- Prove that AX, BY, CZ and OI are concurrent.
- Let I_a, I_b, I_c be the excenters of ABC. Prove that I_aX, I_bY, I_cZ are concurrent at a point P and isogonal conjugate of P with respect to triangle $I_aI_bI_c$ lies on Euler line of ABC.

Problem 4.16.2 (buratinogigle Tough P1). Let ABC be a triangle inscribed in circle (O) with A-excircle (J). Circle passing through A, B touches (J) at M. Circle passing through A, C touches (J) at N. BM cuts CN at P. Prove that AP passes through tangent point of A-mixtilinear incircle with (O).

4.17 Big Pictures

4.17.1 Parallel lines to the A-median

Definition (Skeleton Diagram)— In a triangle ABC, M is the center of BC. B', C' are on $\odot ABC$ such that $BB' \parallel CC' \parallel AM$. Let R be the point on BC such that $\odot B'C'R$ is tangent to BC. Draw the two circles through BB' and CC' that are tangent to BC. Also let $S = BC \cap B'C'$, and let G be such that SG is tangent to $\odot ABC$ at G. Also let M_a, M_A be the minor and major arc midpoints of $\widehat{BC}$.

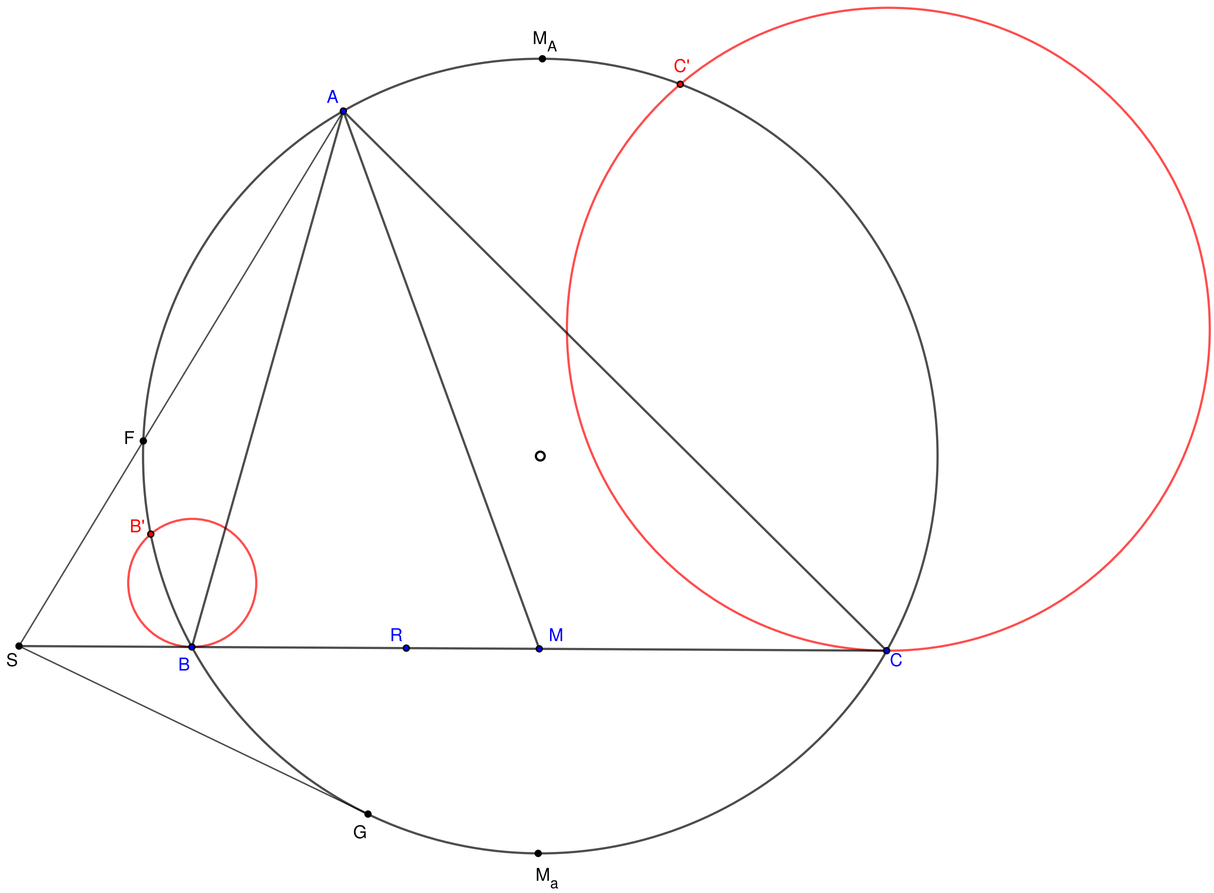


Figure 4.17.1: Skeleton

Lemma 4.17.1 — M_A, R, G are collinear

Proof. Let $R' = GM_A \cap BC$. Then do some angle chansing to show that $SR' = SG$.

Definition (Further Construction)— Now let $M_AR \cap AM = H, M_AR \cap \odot ABC = G$. Let ω_b, ω_c be the circles through BB' and CC' and tangent to BC. Let I be the intersection of the line through M_a parallel to AM and ω_x. Let $J = AM \cap \omega_x$. Let $J \cap \omega_b = X, J \cap \omega_c = Y$. Let $AS \cap \odot ABC = F$.

Lemma 4.17.2 — SG is tangent to $\odot ABC$

Lemma 4.17.3 — R, M, M_a, G is cyclic, call it ω_x

Lemma 4.17.4 — I, G, O are collinear, and $IG = IJ$

Lemma 4.17.5 — BX, CY, AM, RM_A are concurrent.

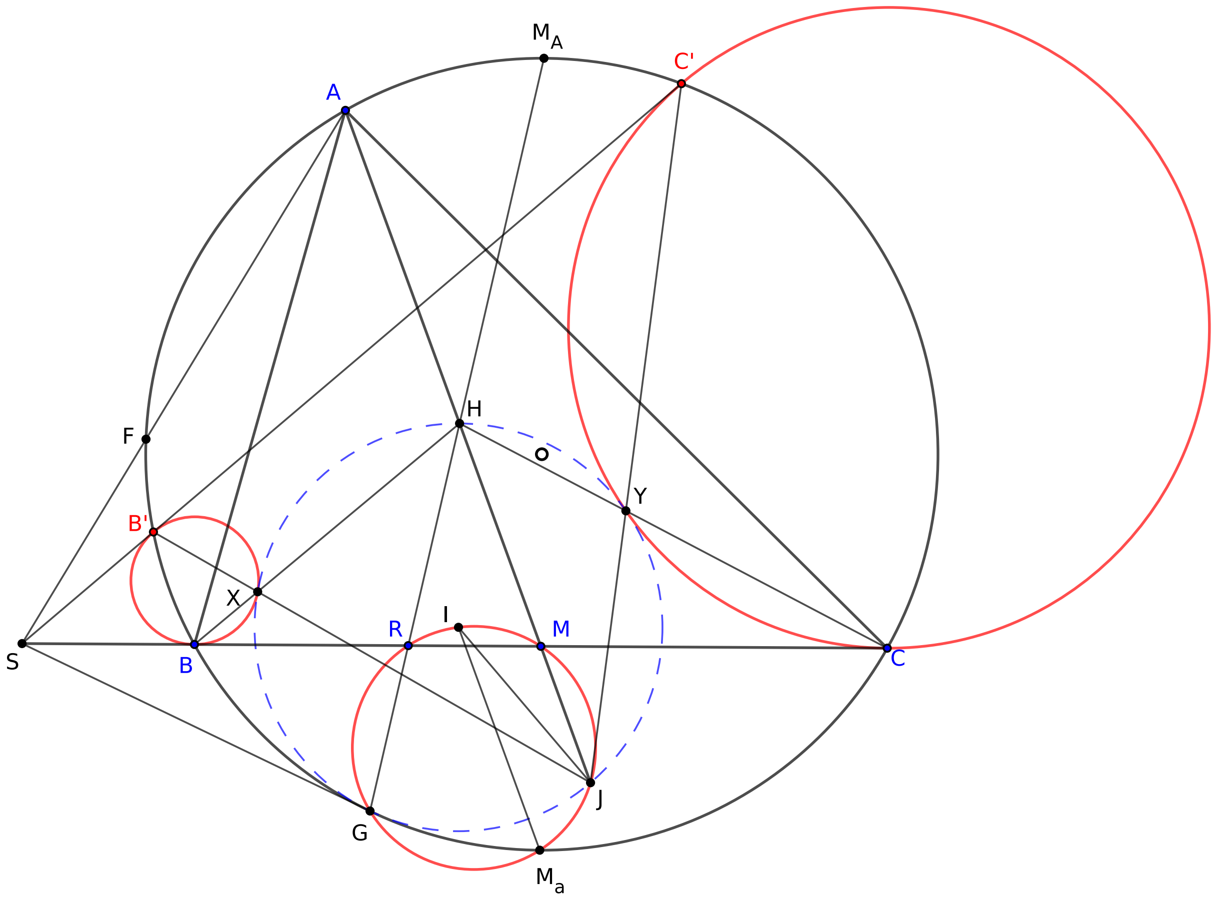


Figure 4.17.2

Problem 4.17.1 (IGO 2019 Advanced P4). $XHYG$ is cyclic, and tangent to ω_b, ω_c and $\odot ABC$. And I is the center of this circle.

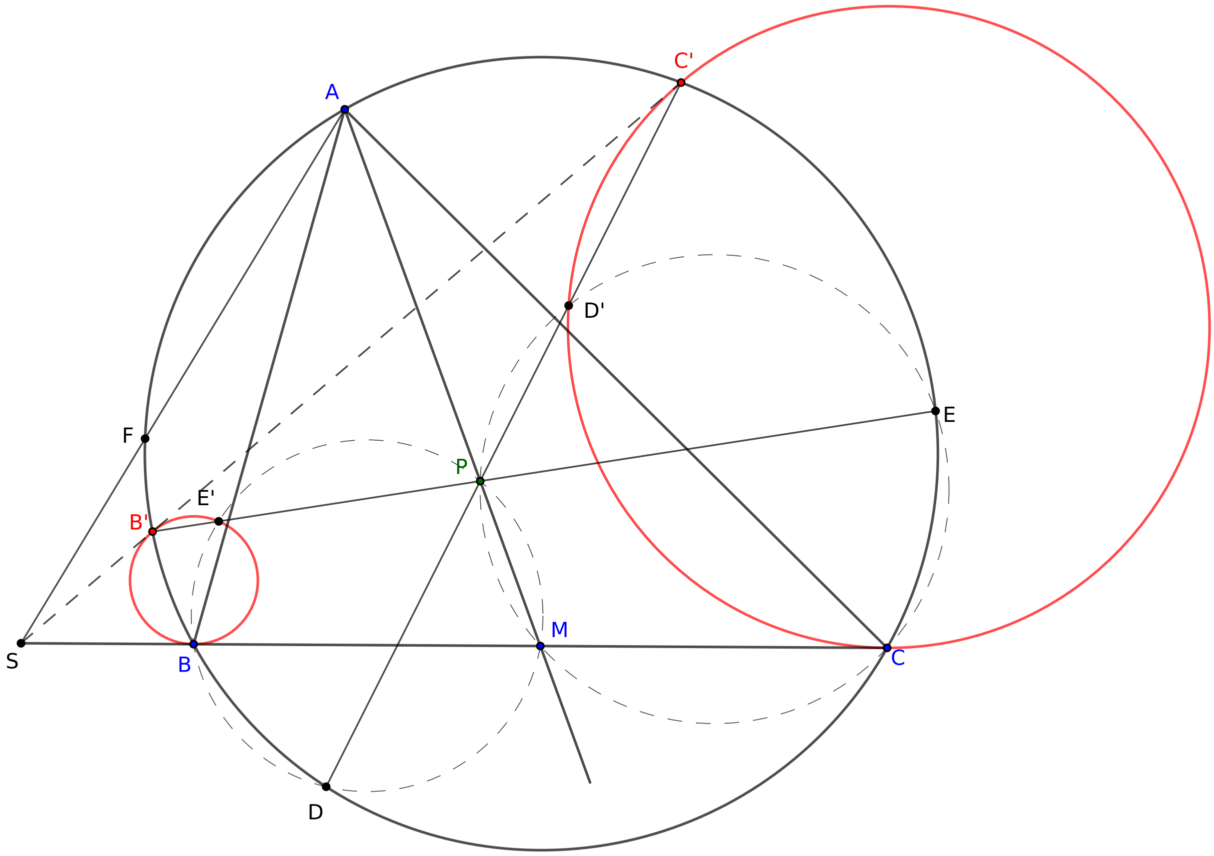


Figure 4.17.3

Problem 4.17.2 (APMO 2019 P3). A variable point P is selected in the line segment AM. The circumcircles of triangles BPM and CPM intersect Γ again at points D and E, respectively. The lines DP and EP intersect (a second time) the circumcircles to triangles CPM and BPM at X and Y, respectively. Prove that $AFXY$ is cyclic.

Chapter 5

Number Theory

5.1 Tricks

"Every object in the problem's universe is important, and should be considered when approaching the problem."

- permutation type problem
- do there exist...
- proving identities
- Dunno
- divisibility by primes and prime divisors stuff

1. Add. Everything. Up.
2. Infinitude of primes:
   a) Eulerian infinitude trick
   b) For large enough numbers, there is a larger prime divisor
   c) Assuming contradiction, if there are any number co-prime to the product of the primes, then that must be 1.

5.1.1 Digit Sum or Product

When dealing with the sum of the digits or the product of them, to find the construction it is very important to consider 0 and 1's in the number.

5.1.2 Diophantine Equations

1. finding some solutions
2. trying modular cases

3. making some variables depended on other variables
4. putting constrains on variables which would make the problems easier
5. if there are infinitely many solutions, can you find a construction?
6. factorize (this is BIG)
7. In these problems, investigation, induction, recursion, constructions etc. are essentials

5.1.3 Sequences

5.1.4 NT Functions

5.1.5 Construction Problems

5.1.6 Sets satisfying certain properties

5.1.7 Other Small Techniques to Remember

1. $a-b$ stays invariant upon addition, just as $\frac{a}{b}$ stays invariant upon multiplication.

5.2 Lemmas

Lemma 5.2.1 (Maybe not useful at all) — Let $p \geq 5$ be a prime number. Prove that if $p \mid a^2 + ab + b^2$, then

$$p^3 \mid a + b^p - a^p - b^p$$

Lemma 5.2.2 (Non-zero digits in base b) — If $b \geq 2$ and $b^n - 1 \mid a$ then there exist at least n non-zero digits in the representation of a in base b

Theorem 5.2.3 (Frobenius Coin Problem, extended Chicken McNugget) — Given a bunch of coprime integers, one can write all integers after a certain limit as a linear combination of these integers.

Formally, for any integers $a_1, a_2 \ldots a_n$ such that $\gcd(a_1, a_2 \ldots a_n) = 1$, there exists positive integers m such that for any integer $M \geq m$, there are non-negative integers $b_1, b_2 \ldots b_n$ such that

$$\sum_{i=1}^{n} b_i a_i = M$$

Remark. If $n = 2$ then $m = a_1 a_2 - a_1 - a_2$.

If $n \geq 2$, then there doesn't exist an explicit formula, but if $\{a_i\}$ are in arithmetic progression, $(a_i = a_1 + (i-1)d)$ then

$$m = \lfloor * \rfloor \frac{a-2}{n-1} a + (d-1)(a-1) - 1$$

Theorem 5.2.4 (Beatty's Theorem or Rayleigh Theorem) — If a, b are two irrational numbers such that $\frac{1}{a} + \frac{1}{b} = 1$, then the two sets $\{\lfloor * \rfloor\, ia\}$ and $\{\lfloor * \rfloor\, ib\}$, where i are the positive integers, form a partition of the set of natural numbers.

Remark. Express the two conditions in numeric terms assuming the contrary.

Proof. **No integer belongs to both sets:**

Suppose there exists some j, k, m such that $j = \lfloor ka \rfloor = \lfloor mb \rfloor$. Then the inequalities will hold:

$$j \leq ka < j + 1 \text{ and } j \leq mb < j + 1$$

Which leads to

$$j < ka < j + 1 \text{ and } j < mb < j + 1$$

From there, it follows that

$$\frac{j}{a} < k < \frac{j+1}{a} \text{ and } \frac{j}{b} < m < \frac{j+1}{b}$$

Adding them gives

$$j < k + m < j + 1$$

Which is false.

Every integer belongs to one set:

Suppose there is are integers j, k, m such that $\lfloor ka \rfloor < j < \lfloor ka + a \rfloor$ and $\lfloor mb \rfloor < j < \lfloor mb + b \rfloor$ Which can be written as

$$ka < j < ka + a - 1 \text{ and } mb < j < mb + b - 1$$

It follows from here that,

$$k < \frac{j}{a} < k + 1 - \frac{1}{a} \text{ and } m < \frac{j}{b} < m + 1 - \frac{1}{b}$$

Adding them gives us

$$k + m < j < k + m + 1$$

Which, again, is absurd. So the theorem is proved.

Lemma 5.2.5 — Let x, y be co-prime. Then

$$\gcd(z, xy) = \gcd(z, x)\gcd(z, y) = \gcd(z \bmod x, x)\gcd(z \bmod y, y)$$

$$\implies \gcd(r(a, b), xy)) = \gcd(a, x)\gcd(b, y)$$

(here $r(a, b)$ denotes the smallest integer that satisfies $r(a, b) \equiv a \bmod x,\ r(a, b) \equiv b \bmod y$)

Lemma 5.2.6 (NT version of Erdos Theorem) — Let $0 < a_1 < a_2 < \cdot \cdot \cdot < a_{(mn+1)}$ be $mn + 1$ integers. Prove that you can select either $m + 1$ of them no one of which divides any other, or $n + 1$ of them each dividing the following one.

Theorem 5.2.7 (Prime divisors of an integer polynomial) — If $P(x) \in \mathbb{Z}[x]$, then the set of primes, $P = \{p : p \mid P(x)\}$ is infinite.

5.2.1 Modular Arithmatic Theorems and Useful Results

Theorem 5.2.8 (Wolstenholme's Theorem) — For all prime p the following relation is true:

$$p^2 \,\Big|\, 1 + \frac{1}{2} + \frac{1}{3} \cdots \frac{1}{p-1} = \sum_{i=1}^{p-1} \frac{1}{i}$$

Corollary 5.2.9 —

$$p \,\Big|\, 1 + \frac{1}{2^2} + \frac{1}{3^2} \cdots \frac{1}{(p-1)^2} = \sum_{i=1}^{p-1} \frac{1}{i^2}$$

Corollary 5.2.10 — If $p > 3$ is a prime, then

$$\binom{2p}{p} \equiv 2 \ (mod\ p^3)$$

$$\binom{2p-1}{p-1} \equiv 1 \ (mod\ p^3)$$

Lemma 5.2.11 —

$$\frac{1}{(p-i)!} \equiv (-1)^i (i-1)! \ (mod\ p)$$

$$\binom{p-1}{k} \equiv (-1)^k \ (mod\ p)$$

Problem 5.2.1.

$$\binom{p^{n+1}}{p} \equiv p^n \ (mod\ p^{2n+3})$$

5.3 Orders Modulo a Prime and Related Stuffs

- Order Modulo a Prime - Evan Chen
- Another source of the proof of Zsigmondy's Theorem

5.3.1 Cyclotomic Polynomials

- Elementary Properties of Cyclotomic Polynomials - Yimin Ge

Definition (Cyclotomic Formulas)— For any integer n, we have

$$X^n - 1 = \prod_{d|n} \Phi_d(X)$$

In particular, if p is a prime then

$$\Phi_p(X) = \frac{X^p - 1}{X - 1} = X^{p-1} + X^{p-2} + \cdots + 1$$

Definition (Mobius Function)— The Mobius function μ maps the natural numbers to the set $\{-1, 0, 1\}$. μ can be defined in multiple ways:

- Let ζ_i be the primitive roots of n^{th} cyclotomic polynomial, then $\mu(n) = \sum \zeta_i$
- $$\mu(n) = \begin{cases} 1 & \text{if } n \text{ is square free and has an even number of prime divisors} \\ -1 & \text{if } n \text{ is square free and has an odd number of prime divisors} \\ 0 & \text{if } n \text{ has a prime square divisor} \end{cases}$$

Lemma 5.3.1 (Sum of Mobius functions of divisors) — Let n be a positive integer. Then

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases}$$

Proof. The divisors of n with a prime square divisor will contribute 0. If $n = 1$ it's trivial. So let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$. Then,

$$\begin{aligned}\sum_{d|n} \mu(d) &= \sum_{\{b_1 \dots b_t\} \subseteq \{1,2 \dots k\}} p_{b_1} \dots p_{b_t} \\ &= \binom{k}{0} - \binom{k}{1} + \dots + (-1)^k \binom{k}{k} \\ &= 0\end{aligned}$$

Theorem 5.3.2 (Mobius Inverison Formula) — Suppose $F, f : \mathbb{Z}^+ \to \mathbb{Z}^+$ are functions such that

$$F(n) = \sum_{d|n} f(d) \implies f(n) = \sum_{d|n} \mu(d)\, F\left(\tfrac{n}{d}\right)$$

$$F(n) = \prod_{d|n} f(d) \implies f(n) = \prod_{d|n} F\left(\tfrac{n}{d}\right)^{\mu(d)}$$

Proof.

$$\begin{aligned}\sum_{d|n} \mu(d) \sum_{t|\frac{n}{d}} f(t) &= \sum_{t|n} f(t) \sum_{d|n,\ t|\frac{n}{d}} \mu(d) \\ &= \sum_{t|n} f(t) \sum_{d|\frac{n}{t}} \mu(d) \\ &= f(n) \quad [\because \textit{Lemma } 5.3.1]\end{aligned}$$

Lemma 5.3.3 (Prime Divisors of Cyclotomic Polynomials) — If $a \in \mathbb{Z}$ such that $\Phi_n(a) \neq 0$ and for some prime p,

$$\Phi_n(a) \equiv 0 \pmod{p}$$

Then either

- $p \equiv 1 \pmod{n}$, or
- $p|n$

Lemma 5.3.4 — If p does not divide m, then

$$\Phi_{pm}(x)\Phi_m(x) = \Phi_m(x^p)$$

5.3.2 Quadratic Residue

Theorem 5.3.5 (Quadratic Reciprocity) — Let p and q be two prime numbers. Then using Legendre symbol, we have

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$$

Lemma 5.3.6 (2 is a quadratic residue) —

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$$

Proof. Take $a \not\equiv 0 \pmod p$. Let $t_i \equiv ia \pmod p$ the least positive residues for $1 \leq i \leq \dfrac{p-1}{2}$. Let $r_1, r_2 \dots r_m$ be the t_i's that are smaller than $\dfrac{p-1}{2}$, and let $s_1, s_2 \dots s_n$ be the t_i's that are larger than $\dfrac{p-1}{2}$. Then we have,

$$\{r_1, r_2, \dots r_m, p - s_1, p - s_2, \dots p - s_n\} = \{1, 2, \dots \frac{p-1}{2}\}$$

So, multiplying gives us,

$$a^{\frac{p-1}{2}} \equiv (-1)^n = \left(\frac{a}{p}\right) \pmod p$$

Plugging in $a = 2$, we get Lemma 5.3.6.

Lemma 5.3.7 (Quadratic non-residue mod infitely many primes) — There are infinitely many primes p for every non-square integers a such that a is a non-quadratic residue $\pmod p$

Proof. We can assume that a is square free. So $a = 2^s p_1 p_2 \dots p_r$ for some $s \in \{0, 1\}$ and $r \geq 0$. Assume the contrary, that there are only finitely many primes, namely $q_1, q_2 \dots q_m$ for which a is a quadratic non residue. We have three cases:

Case 1: $s = 1, r = 0$. In that case, we have infinitely many prime number, q such that $q \equiv 3 \pmod 8$, and $x^2 \not\equiv a \pmod q$ using Lemma 5.3.6.

Case 2: $s = 1, r > 0$. Take t, a quadratic non-residue $\pmod p_r$. Consider the following congruences:

$$\begin{aligned}
x &\equiv 1 \pmod{q_i} \text{ for all } q_i \\
x &\equiv 1 \pmod{8} \\
x &\equiv 1 \pmod{p_i} \text{ for } i = 1, 2 \dots r-1 \\
x &\equiv t \pmod{p_r}
\end{aligned}$$

By Chinese Remainder Theorem and Dirichlet theorem, it follows that there exists infinitely many prime numbers N that satisfies all of the congruences. Now using Jacobi symbol and Law of Reciprocity, it follows that,

$$\begin{aligned}
\left(\frac{2}{N}\right) &= 1 \\
\left(\frac{p_i}{N}\right) &= 1 \text{ for } i = 1, 2 \dots r-1 \\
\left(\frac{q_i}{N}\right) &= 1 \text{ for all } q_i \\
\left(\frac{p_r}{N}\right) &= -1
\end{aligned}$$

Therefore, $\left(\frac{a}{N}\right) = -1$ Which is a contradiction, since $q_i \not| N$.

Case 3: $s = 0, r > 0$, this is similar to the above case.

5.3.2.1 Quadratic Residue

Theorem 5.3.8 — Let p be an odd prime. Then,

1. The product of two quadratic residue is a quadratic residue.
2. The product of two quadratic non-residue is a quadratic residue.
3. The product of a qudratic residue and a quadratic non-residue is a quadratic non-residue

Theorem 5.3.9 — For an odd prime p and any two integers a, b, we have $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$

Theorem 5.3.10 (Euler's Criterion) — Let p be an odd prime. Then,

$$a^{\frac{p-1}{2}} \equiv \left(\frac{a}{p}\right) (mod\ p)$$

Theorem 5.3.11 — Let $(a, b) = 1$. Then every prime divisors of $a^2 + b^2$ is either 2 or a prime of the form $4k + 1$.

Theorem 5.3.12 (Gauss's Criterion) — Let p be a prime number and a be an integer coprime to p. Let $\mu(a)$ be the number of integers $x \in \{a, a * 2, \ldots a * \frac{p-1}{2}\}$ such that $x(mod\ p) > \frac{p}{2}$. Then

$$\left(\frac{a}{p}\right) = -1^{\mu(a)}$$

Theorem 5.3.13 — The smallest quadratic non-residue of an odd prime p is a prime which is less than $\sqrt{p} + 1$

Theorem 5.3.14 (Quadratic Residue Law) — Using the usual Legendre Symbol, for two prime numbers p, q we have:

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$$

Definition (Jacobi Symbol)— Let $a, n = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$. We define Jacobi symbol as

$$\left(\frac{a}{n}\right) = \prod_{i=1}^{k} \left(\frac{a}{p_i}\right)^{\alpha_i}$$

Note. Jacobi symbol is not as accurate as Legendre symbol. $\left(\frac{a}{n}\right) = -1$ means that a is a quadratic non-residue of n, but $= 1$ doesnt necessarily mean that a is a quadratic residue of

n.

Theorem 5.3.15 — Let $a, n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$, then a is a quadratic residue of n iff it is a quaadratic residue of every $p_i^{\alpha_i}$.

Theorem 5.3.16 — If an integer is a quadratic residue of every prime, then it is a square.

5.3.3 Zsigmondy's Theorem

- Zsigmondy's Theorem's Proof, has some useful lemmas

Theorem 5.3.17 (Zsigmondy's Theorem) — Let $a, b \in \mathbb{N}$ such that $\gcd(a, b) = 1$ and $n \in \mathbb{N}, n > 1$. Then there exists a prime division of $a^n - b^n$ that does not divide $a^k - b^k$ for all $1 \leq k < n$ *except* in the following cases:

- $2^n - 1^n$
- $n = 2$ and $a + b$ is a power of 2.

5.3.4 Problems

Problem 5.3.1 (`CGMO 2016 P3`). Let m and n are relatively prime integers and $m > 1, n > 1$. Show that:There are positive integers a, b, c such that $m^a = 1 + n^b c$, and n and c are relatively prime.

Problem 5.3.2 (`ISL 2004 N4`). Let k be a fixed integer greater than 1, and let $m = 4k^2 - 5$. Show that there exist positive integers a and b such that the sequence (x_n) defined by

$$x_0 = a, \quad x_1 = b, \quad x_{n+2} = x_{n+1} + x_n \quad \text{for} \quad n = 0, 1, 2, \dots,$$

has all of its terms relatively prime to m.

Solution. Let's play with some integer n and Fibonacci sequences $\bmod n$. What should we take the value of n? As $11 = 16 - 5$, let's take it first. We see that the period of Fibonacci sequences $\bmod 11$ is at most 10. From here it is natural to make a conjecture that for prime n's, the period probably is $n - 1$. We also design a proof that there is a sequence which doesn't contain any of n's products.

So let's see if it works for all primes. No it doesn't, breaks at 7. How is 7 so different than 11? The most straightforward guess is that probably $7 \nmid 4k^2 - 5$ for any k. And it is true. So what's so special about the primes dividing $4k^2 - 5$? Writing it in modular arithmetic manner, $4k^2 \equiv 5 \pmod{p}$. Wait, 5 is a quadratic residue $\bmod p$? But isn't $\sqrt{5}$ related to Fibonacci sequences? What's the general formula for a Fibonacci sequence starting with a, b? Wait, now that explains why the period of the Fibonacci sequences $\bmod$ these primes is $p - 1$.

5.4 Primes

Problem 5.4.1 (ISL 2013 N5). Fix an integer $k > 2$. Two players, called Ana and Banana, play the following game of numbers. Initially, some integer $n \geq k$ gets written on the blackboard. Then they take moves in turn, with Ana beginning. A player making a move erases the number m just written on the blackboard and replaces it by some number m' with $k \leq m' < m$ that is coprime to m. The first player who cannot move anymore loses.

An integer $n \geq k$ is called good if Banana has a winning strategy when the initial number is n, and bad otherwise.

Consider two integers $n, n' \geq k$ with the property that each prime number $p \leq k$ divides n if and only if it divides n'. Prove that either both n and n' are good or both are bad.

> **Solution.** Every idea that naturally follows lead to a solution, so after getting the idea of working on a single equivalence class is enough, we face the problem that "big primes" cause the trouble. So can we get rid of them by making some minimal number that arent "contaminated" by big primes?

Problem 5.4.2 (ISL 2014 N4). Let $n > 1$ be a given integer. Prove that infinitely many terms of the sequence $(a_k)_{k\geq 1}$, defined by

$$a_k = \left\lfloor \frac{n^k}{k} \right\rfloor,$$

are odd. (For a real number x, $\lfloor x \rfloor$ denotes the largest integer not exceeding x.)

> **Solution.** First we take a prime, doesnt work, then we take two primes, one being 2 (Since we need it in the bottom), but that doesn't work either. Then take n instead of 2, because we want the 2's in the numerator vanish. Surprisingly this works.

Problem 5.4.3 (ISL 2012 N3). Determine all integers $m \geq 2$ such that every n with $\frac{m}{3} \leq n \leq \frac{m}{2}$ divides the binomial coefficient $\binom{n}{m-2n}$.

> **Solution.** Investigate, and find out when $n | \binom{n}{m-3n}$.

Problem 5.4.4 (ISL 2004 N1). Let $\tau(n)$ denote the number of positive divisors of the positive integer n. Prove that there exist infinitely many positive integers a such that the equation $\tau(an) = n$ does not have a positive integer solution n.

Solution. if $p \not| n$, $\tau(pn) = \tau(p)\tau(n) = 2\tau(n) = n$ which is true for n
let's try p^k, and $p^t||n$,

$$\tau(p^{k+t}m) = (k+t+1)\tau(m) = p^t m$$

for $k+1 < p$, if $t = 0$, we don't really have an obvious contradiction.
So what if $k+1 = p$, then, $t = 0, 1$
if $t = 0$, $p\tau(m) = m$ but $(p, m) = 1$
and if $t = 1$,

$$(p+1)\tau(m) = pm$$

but trivial bounding shows that $m <= 4$, for which there is no solution by case check

Problem 5.4.5 (USA TST 2014 P2)**.** Let $a_1, a_2, a_3, \dots$ be a sequence of integers, with the property that every consecutive group of a_i's averages to a perfect square. More precisely, for every positive integers n and k, the quantity

$$\frac{a_n + a_{n+1} + \cdots + a_{n+k-1}}{k}$$

is always the square of an integer. Prove that the sequence must be constant (all a_i are equal to the same perfect square).

Solution. The most obvious fact of this sequence is that every element has to be the same residue mod every prime. Using $p = 3$ we see that if one element is divisible by p, every other elements are divisible by p as well. As this is true, we try to prove it. Again we use the most obvious facts we can get from the sequence.

Problem 5.4.6 (ISL 2012 N6)**.** Let x and y be positive integers. If $x^{2^n} - 1$ is divisible by $2^n y + 1$ for every positive integer n, prove that $x = 1$.

Solution. First notice the important things about the primes dividing $b_n = x^{2^n} - 1$. Another important thing to notice is that if $p|b_i$, then $ord_p(b_i) = 2^j$. So we get another bound for the primes.
The most natural thing is to ask now if the primes that have the property $2||p-1$ are infinite, because then we will have an obvious contradiction. Which leads to the following lemma.
Another approach is to taking a N such that $a_N = 2^N y + 1$ becomes congruent to a_1, and show a_1 is actually congruent to some constant, but since N is arbitrary, we are done.

Lemma 5.4.1 (Primes in a recursive sequence) — Let $a_n = 2^n y + 1$ be a sequence of positive intergers. Prove that there are infinitely many prime numbers such that $p \equiv -1 \pmod{p}$ and $p|a_i$ for some i.

Solution. Somehow we want to deploy Euclid. Let's first group the desired primes in a set T.
So we ask ourselves what is the most natural thing for $a_n \equiv 1 \pmod 4$ for some n to have a prime divisor congruent to -1 mod 4? If we can factor a_n into congruent to -1 mod 4 parts. We notice that $a_1 \equiv -1 \pmod 4$. So that should be a good starting point.
Now we want to find some n such that $a_1 | a_n$ and $p \in T$, $p \nmid \frac{a_n}{a_1}$, or wishfully, $p | \frac{a_n}{a_1} - 1$

Problem 5.4.7 (China TST 2018 T2P4)**.** Let k, M be positive integers such that $k-1$ is not squarefree. Prove that there exist a positive real α, such that $\lfloor \alpha \cdot k^n \rfloor$ and M are coprime for any positive integer n.

Solution. Think about what α actually represent in the numberline. It's the ratio that represents how a coprime number of M is from k^n. We think wishfully and hope that α is something like $A + B\frac{1}{k-1}$. Why $\frac{1}{k-1}$? Because this number is "nice" when multiplied by k^n, because it lets us strech the "same ratio" motivation for all k^n. With a bit of workaround to find suitable values for A, B is needed.

Problem 5.4.8 (China TST 2013 T2P2)**.** Prove that: there exists a positive constant K, and an integer series $\{a_n\}$, satisfying:

1. $0 < a_1 < a_2 < \cdots < a_n < \cdots$
2. For any positive integer n, $a_n < 1.01^n K$
3. For any finite number of distinct terms in $\{a_n\}$, their sum is not a perfect square

Remark. The basic idea is to make any sum of elements have a prime divisor with odd power. We want an extension of the set $\{p, p^3, p^5 \ldots\}$ for some prime p.

Solution. Take a prime number p which satisfies

$$p^2 < 1.01^p$$

Such prime numbers exist because the function $f(x) = x^{\frac{2}{p}}$ is strictly decreasing.
Now let $i = pk + j$, and let

$$a_i = a_{pk+j} = jp^{2k+2} + p^{2k+1}$$

And let $K = p^3$. Since $p^2 < 1.01^p$, we have

$$a_i < p^{2k+3} = (p^2)^k \cdot p^3 < 1.01^{pk} p^3 < 1.01^i K$$

Also, it's easy to check that sum of every finite subset of this set has an odd power of p, so no sum is a perfect square.

5.5 NT Functions and Polynomials

Remark. The main idea for most NT FEs is to take aid from LARGE integers on the left side of the divisibility.

Problem 5.5.1 (ISL 2013 N1). Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m and n.

> **Solution.** Go with the flow.

Problem 5.5.2 (ISL 2010 N5). Find all functions $g : \mathbb{N} \to \mathbb{N}$ such that

$$(g(m) + n)(g(n) + m)$$

is a perfect square for all $m, n \in \mathbb{N}$.

> **Solution.** Playig around with some primes give us that for every "big" primes, we need to have every residue class present in the range of g. Now with this fact, we can prove the injectivity as well. Now we want to show that $g(n+1) = g(n) + 1$. How to show that? We can show that by saying that no prime p exits such that $p|g(n+1) - g(n)$, in other words, again the residue classes.

Problem 5.5.3 (ISL 2004 N3). Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$\left(f(m)^2 + f(n)\right) \middle| \left(m^2 + n\right)^2$$

for any two positive integers m and n.

> **Solution.** First get $f(1) = 1$, then using casework, get $f(p-1) = p - 1$. Then let $m = (p-1)$, and use the division algorithm to reduce the dividend.

Remark. In nt-fe s with "divisible" condition, one occurring idea is to get an infinite set of integers with the desired output value, and somehow keep those numbers only in the divisor, and make them vanish from the dividend. Most of the time, using division algorithm to reduce the dividend does the trick.

Problem 5.5.4 (ISL 2009 N5). Let $P(x)$ be a non-constant polynomial with integer coefficients. Prove that there is no function T from the set of integers into the set of integers such that the number of integers x with $T^n(x) = x$ is equal to $P(n)$ for every $n \geq 1$, where T^n denotes the n-fold application of T.

Solution. Write down what the problem is actually saying. Integer functions are like cycles, use that.

Problem 5.5.5 (ISL 2019 N4). Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $a, b \in \mathbb{N}, a + b < C$ for some constant C, we have

$$a + f(b) | a^2 + bf(a)$$

Solution. The main idea is to get primes on the left hand side somehow. We want to show for primes, $p | f(p)$.

5.6 Diophantine Equations

Problem 5.6.1 (APMO 1999 P4). Determine all pairs (a, b) of integers with the property that the numbers $a^2 + 4b$ and $b^2 + 4a$ are both perfect squares.

Solution. Easy case work assuming positive or negative values for a, b.

Problem 5.6.2 (All Squares). Prove that there are infinitely many pairs of positive integers (x, y) satisfying that $x + y, x - y, xy + 1$ are all perfect squares.

Solution. Just work it out.

Problem 5.6.3 (ISL 2010 N3). Find the smallest number n such that there exist polynomials $f_1, f_2, \ldots, f_n$ with rational coefficients satisfying

$$x^2 + 7 = f_1(x)^2 + f_2(x)^2 + \ldots + f_n(x)^2.$$

Solution. Find the obvious answer, which is very small, so we can probably case work it out. We find the case for 3 a bit challenging. But we plan to show that $7a^2$ can't be written as a sum of 3 squares. Which numbers can be written as the sum of 3 squares? Investigate...

Problem 5.6.4 (ISL 2014 N5). Find all triples (p, x, y) consisting of a prime number p and two positive integers x and y such that $x^{p-1} + y$ and $x + y^{p-1}$ are both powers of p.

Solution. The old school trick, replacement. Assume $x < y$ and replace y, and then compare the power of p.

5.7 Divisibility

Problem 5.7.1 (ISL 2002 N6). Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that
$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$
is itself an integer.

Solution [Manipulation].

Lemma— For $a, m, n \in \mathbb{N}$, we have
$$a^n + a^2 - 1 \mid a^{m-i} + (-1)^i \left(a^{n-1} - a^{n-2} + \cdots + (-1)^{i+1} a^{n-i}\right) + (-1)^i (a-1)$$

Proof. We proceed by induction on i. For $i = 0$, it is true. So assume for $i < k$, it is true.

$$\begin{aligned}
a^n + a^2 - 1 \mid \quad & a^{m-i} + (-1)^i \left(a^{n-1} - a^{n-2} + \cdots + (-1)^{i+1} a^{n-i}\right) + (-1)^i (a-1) \\
\implies a^n + a^2 - 1 \mid \quad & a^{m-i} + (-1)^i \left(a^{n-1} - a^{n-2} + \cdots + (-1)^{i+1} a^{n-i}\right) \\
& + (-1)^i (a-1) - (-1)^i (a^n + a^2 - 1) \\
& = a^{m-i} + (-1)^i \left(-a^n + a^{n-1} - a^{n-2} + \cdots + (-1)^{i+1} a^{n-i}\right) \\
& + (-1)^{i+1} (a^2 - a) \\
\therefore a^n + a^2 - 1 \mid \quad & a^{m-i-1} + (-1)^{i+1} \left(a^{n-1} - a^{n-2} + \cdots + (-1)^{i+1} a^{n-i}\right) \\
& + (-1)^{i+1} (a-1)
\end{aligned}$$

For $i = n$, we have,
$$a^n + a^2 - 1 \mid a^{m-n} + (-1)^n \left(\frac{a^n + 1}{a+1}\right) + (-1)^n (a-1)$$
Clearly, $m \geq n$. Let $m = nk + q$. If n is even,
$$\begin{aligned}
a^n + a^2 - 1 &\mid a^{m-n} + a - 1 + \left(\frac{a^n + 1}{a+1}\right) \\
\implies a^n + a^2 + 1 &\mid a^q + a - 1 + k \left(\frac{a^n + 1}{a+1}\right)
\end{aligned}$$
Which is not possible since the polynomial on the right side has degree at most $n - 1$, and can't be 0 for all $a \in \mathbb{Z}$.

So, n is odd. Then we have,

$$a^n + a^2 - 1 \mid a^{m-n} + a - 1 - \left(\frac{a^n+1}{a+1}\right) - 2(a-1)$$
$$\implies a^n + a^2 - 1 \mid a^q + a - 1 - k\left(\frac{a^n+1}{a+1}\right) - 2k(a-1) = P(a)$$

Since $P(a)$ has degree at most $n-1$, $P(a) = 0$ for all $a \in \mathbb{Z}$.

$$P(a) = a^q + a - 1 - k\left(\frac{a^n+1}{a+1}\right) - 2k(a-1)$$
$$\therefore a^q + a - 1 = k\left(\frac{a^n+1}{a+1}\right) + 2k(a-1)$$

Comparing the coefficients and degrees on both sides, we have, $q = n - 1 = 2,\ k = 1$. Which gives us the only solution $(m, n) = (5, 3)$.

Problem 5.7.2 (Iran 3rd Round 2016 N3). A sequence $P = \{a_n\}$ is called a Permutation of natural numbers (positive integers) if for any natural number m, there exists a unique natural number n such that $a_n = m$.

We also define $S_k(P)$ as: $S_k(P) = a_1 + a_2 + \cdots + a_k$ (the sum of the first k elements of the sequence).

Prove that there exists infinitely many distinct Permutations of natural numbers like $P_1, P_2, \cdots$ such that:

$$\forall k, \forall i < j : S_k(P_i) | S_k(P_j)$$

Solution. Instead of giving a construction for the sequence we prove that for a given permutation P we can find another permutation Q such that the partial sums of P divide the corresponding partial sums of Q.

As we try to build Q from P, we have a constraint of divisibility. And we need to make sure every integer i gets to be in Q. For that to be always possible, we need a special property of P to be true. Finding out that property is the main task of this problem.

After the property is determined, we need to add in some more details, that is, for the induction to work, we need to maintain that property in Q as well.

5.8 Modular Arithmatic

Theorem 5.8.1 (Thue's Lemma) — Let $n > 1$ be an integer and a be an integer co-prime to n. Then there are integers x, y with $0 < |x|, |y| < \sqrt{n}$ so that

$$x \equiv ay \pmod{n}$$

Such a solution (x, y) is called a "small solution" sometimes.

Proof. Let $r = \lfloor \sqrt{n} \rfloor$ i.e. r is the unique integer for which $r^2 \leq n < (r+1)^2$ The number of pairs (x, y) so that $0 \leq x, y \leq r$ is $(r+1)^2$ which is greater than n. Then there must be two different pairs (x_1, y_1) and (x_2, y_2) so that

$$\begin{aligned} x_1 - ay_1 &\equiv x_2 - ay_2 \pmod{n} \\ x_1 - x_2 &\equiv a(y_1 - y_2) \pmod{n} \end{aligned}$$

Let $x = x_1 - x_2$ and $y = y_1 - y_2$, and we get $x \equiv ay \pmod{n}$. Now, we need to show that $0 < |x|, |y| < r$ and $x, y \neq 0$. Certainly, if one of x, y is zero, the other is zero as well. If both x and y are zero, that would mean that two pairs (x_1, y_1) and (x_2, y_2) are actually same. That is not the case, and so both x, y can not be 0. Therefore, none of x or y is 0, and we are done.

Theorem 5.8.2 (Generalization of Thue's Lemma) — Let α and β are two real numbers so that $\alpha\beta \geq p$. Then for an integer x, there are integers a, b with $0 < |a| < \alpha$ and $0 < |b| < \beta$ so that

$$a \equiv xb \pmod{p}$$

And we can even make this lemma a two dimensional one.

Theorem 5.8.3 (Fermat's 4n+1 Theorem) — Every prime of the form $4n + 1$ can be written as the sum of squares of two coprime integers.

Proof. We know that there is an x such that

$$x^2 \equiv -1 \pmod{p}$$

And by Theorem 5.8.1, there are a, b with $0 < |a|, |b| < \sqrt{n}$ for which

$$\begin{aligned} a &\equiv xb \pmod{p} \\ a^2 &\equiv x^2b^2 \pmod{p} \\ a^2 + b^2 &\equiv 0 \pmod{p} \end{aligned}$$

Since $a^2 + b^2 < 2p$, we are done.

Theorem 5.8.4 (General Fermat's 4n+1 Theorem) — Let $n \in \{1, 2, 3\}$. If $-n$ is a quadratic residue modulo p, then there exists a, b such that $a^2 + nb^2 = p$

Theorem 5.8.5 (Factors are of the same form) — If $D \in \{1, 2, 3\}$ and $n = x^2 + Dy^2$ for some $x \perp y$, then all of the factors of n are of the form $a^2 + Db^2$.

Proof. This is because the product of two numbers of such form is the same form as them:

$$\begin{aligned}\left(a^2 + Db^2\right)\left(c^2 + Dd^2\right) &= (ac - Dbd)^2 + D\,(ad + bc)^2 \\ &= (ac + Dbd)^2 + D\,(ad - bc)^2\end{aligned}$$

And by Theorem 5.8.4 the prime factors of n are of the same form. And so all factors of n are of the same form.

Theorem 5.8.6 (Quadratic Residue -3) — -3 is a quadratic residue of modulo p iff p is of the form $3k + 1$.

Proof. The only if part is easy with Thue's Lemma. For the if part, we have

$$\left(\frac{p}{3}\right)\left(\frac{3}{p}\right) = (-1)^{\left(\frac{p-1}{2}\right)\left(\frac{3-1}{2}\right)} = (-1)^{\left(\frac{p-1}{2}\right)}$$

Then we casework on $p \equiv 1, -1 \pmod 4$ to show that in either case,

$$\left(\frac{-3}{p}\right) = 1$$

Problem 5.8.1 (Thue's Lemma Note). Let p be prime number, prove that there exists x, y such that $p = 2x^2 + 3y^3$ iff $p \equiv 5, 11 \pmod{24}$.

Solution. We need to show that $\frac{-3}{2}$ is a quadratic residue mod p, and the rest will follow from Thue's Lemma. We do that using the quadratic residue rules for 2 and -3.

5.9 Unsorted Problems

Problem 5.9.1. Let n be an odd integer, and let $S = \{x \mid 1 \le x \le n,\ (x,n) = (x+1,n) = 1\}$. Prove that

$$\prod_{x \in S} x = 1 \ (mod\ n)$$

Problem 5.9.2 (Iran TST 2015, P4). Let n is a fixed natural number. Find the least k such that for every set A of k natural numbers, there exists a subset of A with an even number of elements which the sum of its members is divisible by n.

> **Solution.** Odd-Even, so lets first try for odd n's. It is quite easy.
>
> So now, for evens, lets first try the simplest kind of evens. As we need a set with an even number of elements, this tells us to pair things up. We can try to partition A into pairs of *e*-*e*'s and *o*-*o*'s. This gives us our desired result.

Problem 5.9.3 (Iran TST 2015 P11). We call a permutation $(a_1, a_2 \ldots a_n)$ of the set $\{1, 2 \ldots n\}$ "good" if for any three natural numbers $i < j < k$,

$$n \nmid a_i + a_k - 2a_j$$

find all natural numbers $n \ge 3$ such that there exist a "good" permutation of the set $\{1, 2 \ldots n\}$.

> **Solution.** Looking for "possibilities" for the first element, we get some more restrictions for the values of other terms.

Problem 5.9.4 (ISL 2004 N2). The function $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(n) = \sum_{k=1}^{n} \gcd\ (k, n)$, $n \in \mathbb{N}$.

1. Prove that $f(mn) = f(m)f(n)$ for every two relatively prime $m, n \in \mathbb{N}$.
2. Prove that for each $a \in \mathbb{N}$ the equation $f(x) = ax$ has a solution.
3. Find all $a \in \mathbb{N}$ such that the equation $f(x) = ax$ has a unique solution.

> **Solution.** Why not casually try to multiply $f(m)$ and $f(n)$?? And also find a formula for $n =$ prime power.

Problem 5.9.5 (Balkan MO 2017 P1). Find all ordered pairs of positive integers (x, y) such that: $x^3 + y^3 = x^2 + 42xy + y^2$.

Problem 5.9.6 (`Balkan MO 2017 P3`). Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$n + f(m) \mid f(n) + nf(m)$$

for all $n, m \in \mathbb{N}$.

> **Solution.** Check sizes and bound for large n.

Problem 5.9.7 (`Iran MO 3rd Round N3`). Let $p > 5$ be a prime number and $A = \{b_1, b_2 \dots b_{\frac{p-1}{2}}\}$ be the set of all quadratic residues modulo p, excluding zero. Prove that there doesn't exist any natural a, c satisfying $\gcd\ (ac, p) = 1$ such that set $B = \{x \mid x = ay + c, y \in A\}$ and set A are disjoint modulo p.

> **Solution.** Sum it up.

> **Solution.** For every integer a, b and prime p such that, $\gcd(a, p) = \gcd(b, p) = 1$, there exist (x, y) such that $x^2 \equiv ay^2 + c \pmod{p}$.

> **Solution.** For a prime p, there exists an integer x such that x and $x+1$ both are quadratic residues $\pmod{p}$.

Problem 5.9.8 (`All Russia 2014 P9.5`). Define $m(n)$ to be the greatest proper natural divisor of n. Find all $n \in \mathbb{N}$ such that $n + m(n)$ is a power of 10.

Problem 5.9.9 (`ISL 2000 N1`). Determine all positive integers $n \geq 2$ that satisfy the following condition: for all a and b relatively prime to n we have $a \equiv b \pmod{n}$ iff $ab \equiv 1 \pmod{n}$.

> **Solution.** Don't forget the details.

Problem 5.9.10 (`ISL 2000 N3`). Does there exist a positive integer n such that n has exactly 2000 prime divisors (not necessarily distinct) and $n \mid 2^n + 1$?

> **Solution.** Goriber Bondhu Induction. As the number 2000 seems so out of the place, we replace 2000 by k. Now suppose that for some k, the condition works. For simplicity let $k = p^i$ for some i, as it is quite clear that there is another prime q that divides $2^k + 1$, let $k' = kq$. So k' also satisfies the condition. So it is quite intuitive to think that for every x there exist some p and i for which $2^{p^i} + 1$ has x prime factors. So we search for such p.

Problem 5.9.11 (USAMO 2001 P5). Let S be a set of integers (not necessarily positive) such that

1. There exist $a, b \in S$ with $\gcd(a, b) = \gcd(a-2, b-2) = 1$;
2. If x and y are elements of S (possibly equal), then $x^2 - y \in S$

Prove that S is the set of all integers.

Solution. One possible intuition could be trying to make the problem statement a little bit more stable, like the term $x^2 - y$ is not so symmetric. So trying to make it a little bit more symmetric can come handy.

Solution. If $c, x, y \in S$ then we can easily see that $A(x^2-y^2)-c \in S$ for all $A \in \mathbb{Z}$. We take this a little too far and show that if $c, x, y, u, v \in S$, then $A(x^2-y^2)+B(u^2-v^2)-c \in S$ for all $A, B \in \mathbb{Z}$. So if we can find such x, y, u, v such that $\gcd(x^2-y^2, u^2-v^2) = 1$, we are almost done by Frobenius Coin Problem. So we start looking for integers that can be obtained from a, b. After some playing around we get the feeling (or maybe not) that we need one more pair. Again playing around for some time we find three pairs. FCP gives us an upper bound for all integers that are not in S. Easily we include them in S.

Problem 5.9.12 (Vietnam TST 2017 P2). For each positive integer n, set $x_n = \binom{2n}{n}$

1. Prove that if $\frac{2017^k}{2} < n < 2017^k$ for some positive integer k then $2017 \mid x_n$.
2. Find all positive integer $h > 1$ such that there exist positive integers N, T such that the sequence (x_n) for $n > N$, is periodic $(bmodh)$ with period T.

Problem 5.9.13 (Vietnam 2017 TST P6). For each integer $n > 0$, a permutation $(a_1, a_2 \ldots a_{2n})$ of $1, 2 \ldots 2n$ is called *beautiful* if for every $1 \le i < j \le 2n$, $a_i + a_{n+i} = 2n+1$ and $a_i - a_{i+1} \not\equiv a_j - a_{j+1} \pmod{2n+1}$ (suppose that $a_i = a_{2n+i}$).

1. For $n = 6$, point out a *beautiful* permutation.
2. Prove that there exists a *beautiful* permutation for every n.

Solution. Trial and Error.

Problem 5.9.14 (BrMO 2008). Find all sequences $a_{i}{}_{i=0}^{\infty}$ of rational numbers which follow the following conditions:

1. $a_n = 2a_{n-1}^2 - 1$ for all $n > 0$

2. $a_i = a_j$ for some $i, j > 0$, $i \neq j$

Solution. Trial and Error. Don't forget that you only need the numbers to be rational.

Problem 5.9.15 (USA TST 2000 P4). Let n be a positive integer. Prove that

$$\sum_{i=0}^{n} \binom{n}{i}^{-1} = \frac{n+1}{2^{n+1}} \left(\sum_{i=0}^{n+1} \frac{2^i}{i} \right)$$

Solution. *Positive Integer*, nuff said.

Problem 5.9.16 (USA TST 2000 P3). Let p be a prime number. For integers r, s such that $rs(r^2 - s^2)$ is not divisible by p, let $f(r, s)$ denote the number of integers $1 \leq n \leq p$ such that $\{\frac{rn}{p}\}$ and $\{\frac{sn}{p}\}$ are either both less than $\frac{1}{2}$ or both greater than $\frac{1}{2}$. Prove that there exists $N > 0$ such that for $p \geq N$ and all r, s,

$$\lceil * \rceil \frac{(p-1)}{3} \leq f(r, s) \leq \lfloor * \rfloor \frac{2(p-1)}{3}$$

Problem 5.9.17 (China TST 2005). Let n be a positive integer and $f_n = 2^{2^n} + 1$. Prove that for all $n \geq 3$, there exists a prime factor of f_n which is larger than $2^{n+2}(n+1)$ [Stronger Version: $2^{n+4}(n+1)$].

Problem 5.9.18 (IRAN TST 2009 P2). Let a be a fixed natural number. Prove that the set of prime divisors of $2^{2^n} + a$ for $n = 1, 2, 3 \ldots$ is infinite.

Problem 5.9.19 (USAMO 2004 P2). Suppose $a_1, a_2 \ldots a_n$ are integers whose greatest common divisor is 1. Let S be a set of integers with the following properties:

1. $a_i \in S$

2. For $i, j = 1, 2 \ldots n$ (not necessarily distinct), $a_i - a_j \in S$.

3. For any integers $x, y \in S$, if $x + y \in S$, then $x - y \in S$.

Prove that S must be equal to the set of all integers.

Solution. First we see that if $d = \gcd\ (x, y)$ and $x, y \in S$ then $d \in S$. So all we have to do is to find two x, y with $d = 1$.

Problem 5.9.20 (USAMO 2008 P1). Prove that for each positive integer n, there are pairwise relatively prime integers $k_0, k_1 \dots k_n$, all strictly greater than 1, such that $k_0 k_1 \dots k_{n-1}$ is the product of two consecutive integers.

Solution. *Positive Integer n* nuff said.

Problem 5.9.21 (USAMO 2007 P5). Prove that for every nonnegative integer n, the number $7^{7^n} + 1$ is the product of at least $2n + 3$ (not necessarily distinct) primes.

Solution. When you try to apply induction, always name the hypothesis. In this case name $7^{7^n} + 1 = a_n$. And try to relate a_n with a_{n+1}.

Problem 5.9.22 (IMO 1998 P3). For any positive integer n, let $\tau(n)$ denote the number of its positive divisors (including 1 and itself). Determine all positive integers m for which there exists a positive integer n such that $\frac{\tau(n^2)}{\tau(n)} = m$.

Solution. Easy go with the flow.

Problem 5.9.23 (USA TST 2002 P2). Let $p > 5$ be a prime number. For any integer x, define

$$f_p(x) = \sum_{k=1}^{p-1} \frac{1}{(px + k)^2}$$

Prove that for any two integers x, y,

$$p^3 \mid f(x) - f(y)$$

Solution. If there is $f(x) - f(y)$ for all x, y, then always make one of those equal to 0 or in other words make one side constant

Solution. All fractional sum problems should be solved by some expression manipulation.

Solution. Sometimes try adding things up.

Problem 5.9.24 (USAMO 2012 P4). Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ (where $\mathbb{Z}^+$ is the set of positive integers) such that $f(n!) = f(n)!$ for all positive integers n and such that $(m-n)$ divides $f(m) - f(n)$ for all distinct positive integers m, n.

Positive Integer n nuff said.

Problem 5.9.25 (USAMO 2013 P5). Given positive integers m and n, prove that there is a positive integer c such that the numbers cm and cn have the same number of occurrences of each non-zero digit when written in base ten.

What if we make cm and cn have the same digits, occuring same number of time, when the digits are sorted? This will make the things a whole lot easier. Again for more simplicity, what if the arrangement of the digits in both of these numbers are "almost" the same? Like, if the digits are in blocks and if decomposed into such blocks, we get the same set for both of those problems? This idea of simplicity is more that enough to "Simplify" a problem. Call this strategry Simplify.

Problem 5.9.26 (ISL 2015 N4). Suppose that $a_0, a_1, \cdots$ and $b_0, b_1, \cdots$ are two sequences of positive integers such that $a_0, b_0 \geq 2$ and

$$a_{n+1} = \gcd(a_n, b_n) + 1, \qquad b_{n+1} = \operatorname{lcm}(a_n, b_n) - 1.$$

Show that the sequence a_n is eventually periodic; in other words, there exist integers $N \geq 0$ and $t > 0$ such that $a_{n+t} = a_n$ for all $n \geq N$.

Like most NT probs, pure investigation. We see that the function a_n is mostly decreasing, but it is increasing as well. But the increase rate is not greater than the decrease rate. After some time playing around, we see that the value of a_n rises gradually and then suddenly drops. But the peak value of a_n doesn't seem to increase. Well, this is true, we prove that. After that, we are mostly done, we just show that eventually the least value of a_n becomes stable as well. We use the intuitions we get from working around.

Problem 5.9.27 (ISL 2015 N3). Let m and n be positive integers such that $m > n$. Define $x_k = \frac{m+k}{n+k}$ for $k = 1, 2, \ldots, n+1$. Prove that if all the numbers $x_1, x_2, \ldots, x_{n+1}$ are integers, then $x_1 x_2 \ldots x_{n+1} - 1$ is divisible by an odd prime.

As there are powers of 2, we use the powers of 2.

Problem 5.9.28 (USA TST 2018 P1). Let $n \geq 2$ be a positive integer, and let $\sigma(n)$ denote the sum of the positive divisors of n. Prove that the n^{th} smallest positive integer relatively prime to n is at least $\sigma(n)$, and determine for which n equality holds.

even-odd

Pretty starightforward, as the ques suggests, there are fewer than n coprimes in the interval $[1, \sigma(n)]$, we directly show this. [as consttructions don't seem to be trivial/ easy to get] Inclusion/Exclusion all the way. But remember, not checking floors can get you doomed.

Problem 5.9.29 (APMO 2014 P3). Find all positive integers n such that for any integer k there exists an integer a for which $a^3 + a - k$ is divisible by n.

factorize, quadratic residue, complete residue class

Solution. You have to show that the set $\{x \mid x \equiv a^3 + a \ (mod\ p)$ is equal to the set $\{1 \dots p-1\}$. So some properties shared by a set as a whole must be satisfied by both of the sets. The quickest such properties that come into mind are the summation of the set and the product of the set. While the former doesnt help out much, the later seems promising. Where we get a nice relation that we have to satisfy:

$$\prod_{i=0}^{p-1} a^2 + 1 \equiv 1 \ (mod\ p)$$

One way of concluding from here is to use the quadratic residue ideas, or using the fact that $x^2 + 1 = (x+i)(x-i)$. The later requires some higher tricks tho.

Solution. The most natural way must be to show that for any prime p we will find two integers $p \nmid (a-b)$, and $p \mid a^2 + b^2 + ab + 1$, factorizing the later and getting two squares and a constant gives us our desired result.

Problem 5.9.30 (APMO 2014 P1). For a positive integer m denote by $S(m)$ and $P(m)$ the sum and product, respectively, of the digits of m. Show that for each positive integer n, there exist positive integers $a_1, a_2, \dots, a_n$ satisfying the following conditions:

$$S(a_1) < S(a_2) < \cdots < S(a_n) \text{ and } S(a_i) = P(a_{i+1}) \quad (i = 1, 2, \dots, n).$$

(We let $a_{n+1} = a_1$.)

Solution. 1 is the only integer that increases the sum, but doesnt change the product. This may seem trivial, but on problems like this where both the sum and the product of the digits of a number are concerned, this tiny little fact can change everything.

Problem 5.9.31 (RMM 2018 P4). Let a, b, c, d be positive integers such that $ad \neq bc$ and $gcd(a, b, c, d) = 1$. Let S be the set of values attained by $\gcd(an + b, cn + d)$ as n runs through the positive integers. Show that S is the set of all positive divisors of some positive integer.

Problem 5.9.32 (ISL 2011 N1). For any integer $d > 0$, let $f(d)$ be the smallest possible integer that has exactly d positive divisors (so for example we have $f(1) = 1, f(5) = 16$, and $f(6) = 12$). Prove that for every integer $k \geq 0$ the number $f\left(2^k\right)$ divides $f\left(2^{k+1}\right)$.

Solution. Construct the function for 2^n.

Problem 5.9.33 (ISL 2004 N1). Let $\tau(n)$ denote the number of positive divisors of the positive integer n. Prove that there exist infinitely many positive integers a such that the equation $\tau(an) = n$ does not have a positive integer solution n.

Solution. Infinitely many, divisor, what else should come to mind except prime powers...

Problem 5.9.34 (USAMO 2018 P4). Let p be a prime number and let $a_1, a_2 \dots a_p$ be integers. Prove that there exists an integer k s.t. the $S = \{a_i + ik\}$ has at least $\frac{p}{2}$ elements modulo p

Solution. As the only thing that is holding ourselves down is the equivalence of any two elements of S, we investigate it furthur. It is a good idea to represent by graphs.

Problem 5.9.35 (ISL 2009 N1). Let n be a positive integer and let $a_1, a_2, a_3, \dots, a_k$ ($k \geq 2$) be distinct integers in the set $1, 2, \dots, n$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, 2, \dots, k-1$. Prove that n does not divide $a_k(a_1 - 1)$.

Problem 5.9.36 (ISL 2009 N2). A positive integer N is called balanced, if $N = 1$ or if N can be written as a product of an even number of not necessarily distinct primes. Given positive integers a and b, consider the polynomial P defined by $P(x) = (x + a)(x + b)$.

1. Prove that there exist distinct positive integers a and b such that all the number $P(1)$, $P(2)$,..., $P(50)$ are balanced.
2. Prove that if $P(n)$ is balanced for all positive integers n, then $a = b$

Problem 5.9.37 (USA TSTST 2015 P5). Let $\varphi(n)$ denote the number of positive integers less than n that are relatively prime to n. Prove that there exists a positive integer m for which the equation $\varphi(n) = m$ has at least 2015 solutions in n.

Solution. When does the equation has multiple solutions? Suppose $m = \prod_{i=1}^{t} p_i^{\alpha_i}(p_i - 1)$ then $\Phi(n) = m$ has multiple solutions if for some $p's$ in m, their product is one less from another prime. Which gives us necessary intuition to construct a m for which there are A LOT of solutions for the equation.

Problem 5.9.38 (Iran 2018 T1P1). Let $A_1, A_2, ..., A_k$ be the subsets of $\{1, 2, 3, ..., n\}$ such that for all $1 \leq i, j \leq k$:$A_i \cap A_j \neq \varnothing$. Prove that there are n distinct positive integers $x_1, x_2, ..., x_n$ such that for each $1 \leq j \leq k$:

$$lcm_{i \in A_j} \{x_i\} > lcm_{i \notin A_j} \{x_i\}$$

Solution. Main part of the problem is to notice that the first $|A_i|$ columns of the matrix has 1 from all of the rows. Which triggers the idea of giving one prime to every row, and count x_i's with them.

Problem 5.9.39 (Iran TST 2018 T2P4). Call a positive integer "useful but not optimized " (!), if it can be written as a sum of distinct powers of 3 and powers of 5. Prove that there exist infinitely many positive integers which they are not "useful but not optimized".

e.g. 37 is a "useful but not optimized" number since $37 = (3^0 + 3^1 + 3^3) + (5^0 + 5^1)$

Problem 5.9.40 (ISL 2014 N2). Determine all pairs (x, y) of positive integers such that

$$\sqrt[3]{7x^2 - 13xy + 7y^2} = |x - y| + 1.$$

Solution. Always factor, before everything else.

Problem 5.9.41 (ISL 2014 N1). Let $n \geq 2$ be an integer, and let A_n be the set

$$A_n = \{2^n - 2^k \mid k \in \mathbb{Z},\ 0 \leq k < n\}.$$

Determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of A_n .

Solution. In every problem, conjecture from smaller case, and check if the conjecture is true in bigger cases.

Problem 5.9.42 (ISL 2002 N1). What is the smallest positive integer t such that there exist integers $x_1, x_2, \ldots, x_t$ with

$$x_1^3 + x_2^3 + \ldots + x_t^3 = 2002^{2002}\,?$$

Solution. $1000 + 1000 + 1 + 1$.

Problem 5.9.43 (ISL 2002 N2). Let $n \geq 2$ be a positive integer, with divisors $1 = d_1 < d_2 < \ldots < d_k = n$. Prove that $d_1d_2 + d_2d_3 + \ldots + d_{k-1}d_k$ is always less than n^2, and determine when it is a divisor of n^2.

Solution. In problems with all of the divisors of n involved, it is a good choice to substitute $d_i = \frac{n}{d_{k-i+1}}$. That way, you get the exact same set, represented differently, with n involved. And $\sum_{i=1}^{n} \frac{1}{i*(i+1)} = \frac{n}{n+1}$

Problem 5.9.44 (Japan MO 2017 P2, TST Mock 2018). Let N be a positive integer. There are positive integers $a_1, a_2, \cdots, a_N$ and all of them are not multiples of 2^{N+1}. For each integer $n \geq N+1$, set a_n as below:

If the remainder of a_k divided by 2^n is the smallest of the remainder of $a_1, \cdots, a_{n-1}$ divided by 2^n, set $a_n = 2a_k$. If there are several integers k which satisfy the above condition, put the biggest one.

Prove the existence of a positive integer M which satisfies $a_n = a_M$ for $n \geq M$.

Solution. Things must go far...

Problem 5.9.45 (ISL 2002 N3). Let $p_1, p_2, \ldots, p_n$ be distinct primes greater than 3. Show that $2^{p_1p_2\cdots p_n} + 1$ has at least 4^n divisors.

Problem 5.9.46 (Japan MO 2017 P1). Let a, b, c be positive integers. Prove that $lcm(a, b) \neq lcm(a + c, b + c)$.

Problem 5.9.47 (ISL 2009 N3). Let f be a non-constant function from the set of positive integers into the set of positive integer, such that $a - b$ divides $f(a) - f(b)$ for all distinct positive integers a, b. Prove that there exist infinitely many primes p such that p divides $f(c)$ for some positive integer c.

Solution. Notice if $f(1) = 1$, we can easily prove the result, so assume that $f(1) = c$. Now see that, if we can somehow, create another function g from the domain and range of f with the same properties as f, and with $g(1) = 1$, we will be done. So to do this, we need to perform some kind of division by c.

Problem 5.9.48 (ARO 2018 P9.1). Suppose $a_1, a_2, \ldots$ is an infinite strictly increasing sequence of positive integers and $p_1, p_2, \ldots$ is a sequence of distinct primes such that $p_n \mid a_n$ for all $n \geq 1$. It turned out that $a_n - a_k = p_n - p_k$ for all $n, k \geq 1$. Prove that the sequence $(a_n)_n$ consists only of prime numbers.

Problem 5.9.49 (ARO 2018 P10.4). Initially, a positive integer is written on the blackboard. Every second, one adds to the number on the board the product of all its nonzero digits, writes down the results on the board, and erases the previous number. Prove that there exists a positive integer which will be added infinitely many times.

> **Proof.** Using Bounding and the

Problem 5.9.50 (APMO 2008 P4). Consider the function $f : \mathbb{N}_0 \to \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of all non-negative integers, defined by the following conditions :

$$f(0) = 0, \quad f(2n) = 2f(n) \ \text{ and } \ f(2n+1) = n + 2f(n) \quad \text{for all } n \geq 0$$

1. Determine the three sets $L = \{n|f(n) < f(n+1)\}$, $E = \{n|f(n) = f(n+1)\}$, and $G = \{n|f(n) > f(n+1)\}$.
2. For each $k \geq 0$, find a formula for $a_k = \max\{f(n) : 0 \leq n \leq 2^k\}$ in terms of k.

Problem 5.9.51 (ISL 2009 N4). Find all positive integers n such that there exists a sequence of positive integers a_1, a_2,. . ., a_n satisfying:

$$a_{k+1} = \frac{a_k^2 + 1}{a_{k-1} + 1} - 1$$

for every k with $2 \leq k \leq n - 1$.

> **Solution.** Rewriting the condition, and doing some parity check. Then assuming the contrary and taking extreme case.

Problem 5.9.52 (ISL 2015 N2). Let a and b be positive integers such that $a! + b!$ divides $a!b!$. Prove that $3a \geq 2b + 2$.

> **Solution.** Size Chase

Problem 5.9.53 (USA TST 2019 P2). Let $\mathbb{Z}/n\mathbb{Z}$ denote the set of integers considered modulo n (hence $\mathbb{Z}/n\mathbb{Z}$ has n elements). Find all positive integers n for which there exists a bijective function $g : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, such that the 101 functions

$$g(x), \quad g(x) + x, \quad g(x) + 2x, \quad \ldots, \quad g(x) + 100x$$

are all bijections on $\mathbb{Z}/n\mathbb{Z}$.

Solution. A very nice problem. We get the motivation by trying the cases for 2, 3 replacing 101. In the case of 2, we just consider the sum $\sum g(x)$. We get that $2 \nmid n$. So in the case of 3, we conjecture that $3 \nmid n$. But we can't prove this similarly as before. Whats the most common 'sum-type' invariant after the normal sum? Sum of the Squares.

Now that we have proved that $(6, n) = 1$, most probably our conjecture is correct. So lets try for any k, we need to show that $(k!, n) = 1$. In the case of 3, we used the 2nd power sum. So probably to prove that $k \nmid n$ we need to take the $(k-1)$th power sum.

Now the real thing begins. In the case of 3, doesn't the modular sum equation looks something like the first finite difference? This rings a bell that whenever there is powers involved, we should consider using the derivatives. For $(k-1)$th power, the $(k-1)$th derivative that is.

Another thing here, in the case of 3, we did something like

$$\sum (g(x) + 2x)^2 - \sum (g(x) + x)^2 \equiv \sum (g(x) + x)^2 - \sum (g(x))^2 \equiv 0 \pmod{n}$$

We try something similar again.

Problem 5.9.54 (Tuymaada 2016, P5). The ratio of prime numbers p and q does not exceed 2 $(p \neq q)$. Prove that there are two consecutive positive integers such that the largest prime divisor of one of them is p and that of the other is q.

Problem 5.9.55 (ISL 2016 N5). Let a be a positive integer which is not a perfect square, and consider the equation

$$k = \frac{x^2 - a}{x^2 - y^2}.$$

Let A be the set of positive integers k for which the equation admits a solution in $\mathbb{Z}^2$ with $x > \sqrt{a}$, and let B be the set of positive integers for which the equation admits a solution in $\mathbb{Z}^2$ with $0 \leq x < \sqrt{a}$. Show that $A = B$.

Solution. Building x_2, y_2 from x_1, y_1 in the most simple and dumb way.

Problem 5.9.56 (Simurgh 2019 P1). Prove that there exists a 10×10 table of 'different' positive integers such that, if we define r_i, s_i be te product of the elements of the ith row and ith column respectively, then $r_1, r_2 \ldots r_{10}$ and $s_1, s_2 \ldots s_{10}$ form a non-constant arithmetic progression.

Solution. We want to keep things simple. The simpliest arithmetic progression is the $a, 2a, 3a\dots$ one. Again, we have $r_1 r_2 \dots r_{10} = s_1 s_2 \dots s_{10}$, we can wish that $r_i = s_i$. With these two assumptions, we can hope that we will find a table with the two sequences being a constant arithmetic progression.

Problem 5.9.57 (APMO 2017 P4). Call a rational number r powerful if r can be expressed in the form $\frac{p^k}{q}$ for some relatively prime positive integers p, q and some integer $k > 1$. Let a, b, c be positive rational numbers such that $abc = 1$. Suppose there exist positive integers x, y, z such that $a^x + b^y + c^z$ is an integer. Prove that a, b, c are all powerful.

Problem 5.9.58 (USA TST 2010 P9). Determine whether or not there exists a positive integer k such that $p = 6k + 1$ is a prime and

$$\binom{3k}{k} \equiv 1 \pmod{p}$$

Solution. $\binom{3k}{k} \equiv 1 \implies \binom{3k}{2k} \equiv 1$ which implies,

$$\binom{3k}{0} + \binom{3k}{k} + \binom{3k}{2k} + \binom{3k}{3k} \equiv 4 \ (mod\ p)$$

Which gives the idea to find how the following term works in mod p

$$\sum_{i=0}^{\infty} \binom{n}{ki} \text{ for any arbitrary } k$$

From **??** we know a nice way of representing it with the k^{th} roots of unity. Roots of unity are primitive roots mod prime.

Problems

Theorems

Definitions

Strategies