BCS Question Bank

AHSAN

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Contents

Chapter 1

Geometry

1.1 First Portion

Lemma 1.1.1 — *Let the incircle and excircle (opposite to A) of $\triangle ABC$ meet BC at D and E resp. Suppose F is the antipode of D wrt the incircle.*

1. *Prove that A, F, E are collinear.*
2. *M be the midpoint of DE. Prove that MI meets AD at it's midpoint.*

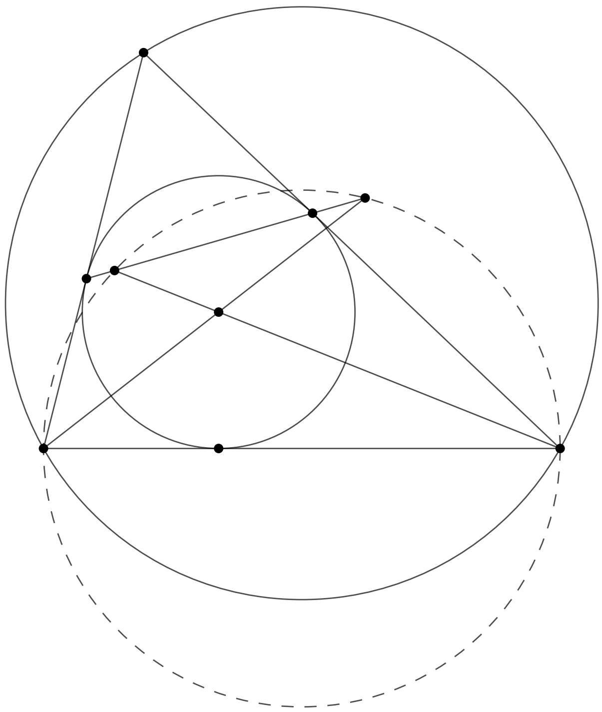

Figure 1.1

Lemma 1.1.2 — *Let the incircle of $\triangle ABC$ meets AB and AC at X and Y resp. BI and CI meet XY at P and Q respectively. Prove that $BPQC$ is cyclic. (In fact $BP \perp CP$ and $BQ \perp CQ$)*

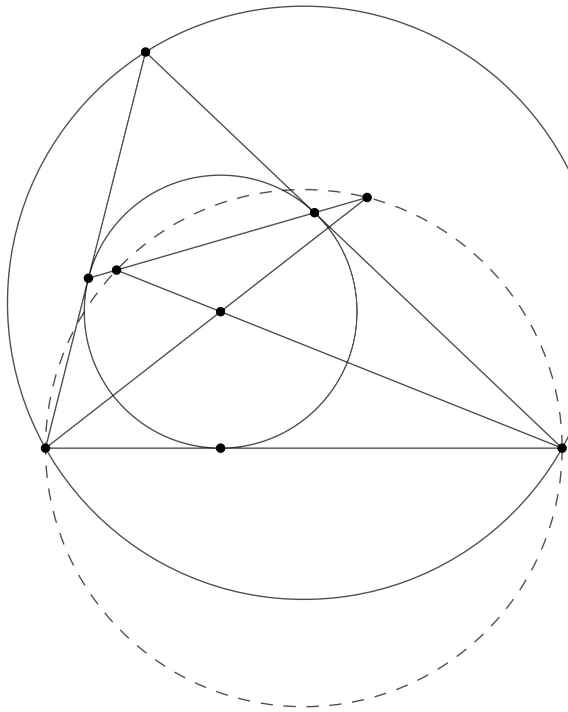

Figure 1.2

Lemma 1.1.3 — *AD is an altitude of $\triangle ABC$. E, F are on AC, AB so that AD, BE, CF are concurrent. Prove $\angle EDA = \angle FDA$.*

Lemma 1.1.4 — *Let AD be an altitude of $\triangle ABC$ and $E \in \odot ABC$ so that $AE \parallel BC$. Prove that D, G, E are collinear where G is the centroid of $\triangle ABC$.*

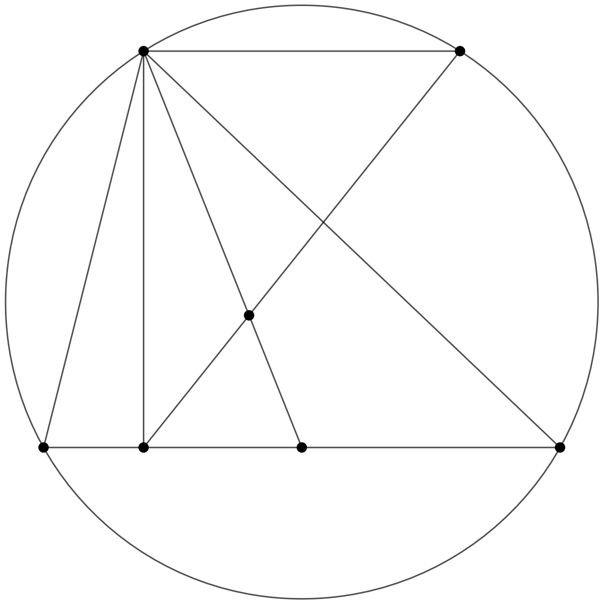

Figure 1.3

Problem 1.1.1 () **:** Let O be the circumcenter of $\triangle ABC$ and A', B', C' are reflections of O on BC, CA, AB resp. Prove that AA', BB', CC' are concurrent.

Problem 1.1.2 () **:** Let D, E are on sides AC, AB of $\triangle ABC$ resp. such that $BE = CD$. Let $\odot ABC \cap \odot ADE = P$. Prove that $PB = PC$.

Problem 1.1.3 () **:** Let a line PQ touch circle S_1 and S_2 at P and Q resp. Prove that the radical axis of S_1 and S_2 passes through the midpoint of PQ.

Problem 1.1.4 () **:** Let $\omega_1, \omega_2, omega_3$ are 3 circles. Prove that the 3 radical axis of ω_1 and ω_2,ω_2 and ω_3,ω_3 and ω_1 are either concurrent or parallel.

Problem 1.1.5 () **:** Two equal-radius circles ω_1 and ω_2 are centered at points O_1 and O_2 . A point X is reflected through O_1 and O_2 to get points A_1 and A_2. The tangents from A_1 to ω_1 touch ω_1 at points P_1 and Q_1, and the tangents from A_2 to ω_2 touch ω_2 at points P_2 and Q_2. If P_1Q_1 and P_2Q_2 intersect at Y , prove that Y is equidistant from A_1 and A_2.

Problem 1.1.6 () **:** Let BD, CE be the altitudes of $\triangle ABC$ and M be the midpoint of BC. If the ray MH meet $\odot ABC$ at point K, prove that AK, BC, DE are concurrent.

Problem 1.1.7 () **:** Two circle ω and Γ touches one another internally at P with ω inside of Γ. Let AB be a chord of Γ which touches ω at D. Let $PD \cap \Gamma = Q$. Prove that $QA = QB$.

Problem 1.1.8 () **:** Let AD be a symmedian of $\triangle ABC$ with D on $\odot ABC$. Let M be the midpoint of AD. Prove that $\angle BMD = \angle CMD$ and A, M, O, D are cyclic where O is the circumcenter of $\triangle ABC$.

Problem 1.1.9 () **:** Let A, B be two fixed points and let P be varying point such that $\frac{PA}{PB}$ is constant. Prove that the locus of P is a circle.

Problem 1.1.10 () **:** Prove that $r_1 + r_2 + r_3 = 4R + r$ (R, r, r_1, r_2, r_3 are the circumradius, inradius and three exradiuses respectively of a triangle)

Problem 1.1.11 () **:** Let M be the midpoint of the altitude BE in $\triangle ABC$ and suppose that the excircle opposite to B touches AC at Y. Then MY goes through the incenter I.

Problem 1.1.12 () **:** Let ABC be a triangle, and draw isosceles triangles $\triangle DBC, \triangle AEC, \triangle ABF$ external to $\triangle ABC$ (with $BC; CA; AB$ as their respective bases). Prove that the lines through $A; B; C$ perpendicular to $EF; FD; DE$, respectively, are concurrent.

Problem 1.1.13 () **:** In a triangle ABC we have $AB = AC$. A circle which is internally tangent with the circumscribed circle of the triangle is also tangent to the sides $AB; AC$ in the points P, respectively Q. Prove that the midpoint of PQ is the center of the inscribed circle of the triangle ABC

Problem 1.1.14 () **:** **Nagel Point** N: If the Excircles of ABC touch $BC; CA; AB$ at $D; E; F$, then the intersection point of $AD; BE; CF$ is called the **Nagel Point** N. Prove that

1. $I; G; N$ are collinear. (G centroid, I incenter.)
2. $GN = 2 \cdot IG$.
3. **Speiker center** S: The incircle of the medial triangle is called the Speiker circle, and it's center is **Speiker center** S. Prove that S is the midpoint of IN.

1.2 Second Portion

Problem 1.2.1 () : Let PB and PC are tangent to $\odot ABC$. Let D, E, F are projection of A on BC, PB, PC resp. Prove that $AD^2 = AE \times AF$.

Problem 1.2.2 () : Let D and E are on AB and AC s.t $DE \parallel BC$. P is an arbitrary point inside $\triangle ADE$. $PB, PC \cap DE = F, G$. Let $\odot PDG \cap \odot PFE = Q$. Prove that A, P, Q are collinear.

Problem 1.2.3 () : Let AB and CD be chords in a circle of center O with A, B, C, D distinct , and with the lines AB and CD meeting at a right angle at point E. Let also M and N be the midpoints of AC and BD respectively . If $MN \perp OE$, prove that $AD \parallel BC$

Problem 1.2.4 () : Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at A and B. Let $M \in AB$. A line through M (different from AB) cuts circles $\mathcal{C}_1$ and $\mathcal{C}_2$ at Z, D, E, C respectively such that $D, E \in ZC$. Perpendiculars at B to the lines EB, ZB and AD respectively cut circle $\mathcal{C}_2$ in F, K and N. Prove that $KF = NC$.

Problem 1.2.5 () : Let D be a point on side AC of triangle ABC. Let E and F be points on the segments BD and BC respectively, such that $\angle BAE = \angle CAF$. Let P and Q be points on BC and BD respectively, such that EP and FQ are both parallel to CD. Prove that $\angle BAP = \angle CAQ$.

Problem 1.2.6 () : In the non-isosceles triangle ABC an altitude from A meets side BC in D . Let M be the midpoint of BC and let N be the reflection of M in D. The circumcirle of triangle AMN intersects the side AB in $P \neq A$ and the side AC in $Q \neq A$. Prove that AN, BQ and CP are concurrent.

Problem 1.2.7 () : In triangle ABC, the interior and exterior angle bisectors of $\angle BAC$ intersect the line BC in D and E, respectively. Let F be the second point of intersection of the line AD with the circumcircle of the triangle ABC. Let O be the circumcenter of the triangle $ABC$and let D' be the reflection of D in O. Prove that $\angle D'FE = 90$.

Problem 1.2.8 () : Let $ABCD$ be a convex quadrilateral such that the line BD bisects the angle ABC. The circumcircle of triangle ABC intersects the sides AD and CD in the points P and Q, respectively. The line through D and parallel to AC intersects the lines BC and BA at the points R and S, respectively. Prove that the points P, Q, R and S lie on a common circle.

Problem 1.2.9 () **:** The incircle of triangle ABC touches BC, CA, AB at points A_1, B_1, C_1, respectively. The perpendicular from the incenter I to the median from vertex C meets the line A_1B_1 in point K. Prove that CK is parallel to AB.

Problem 1.2.10 () **:** Let X be an arbitrary point inside the circumcircle of a triangle ABC. The lines BX and CX meet the circumcircle in points K and L respectively. The line LK intersects BA and AC at points E and F respectively. Find the locus of points X such that the circumcircles of triangles AFK and AEL touch.

Problem 1.2.11 () **:** Let BD be a bisector of triangle ABC. Points I_a, I_c are the incenters of triangles ABD, CBD respectively. The line I_aI_c meets AC in point Q. Prove that $\angle DBQ = 90°$.

Problem 1.2.12 () **:** Given right-angled triangle ABC with hypotenuse AB. Let M be the midpoint of AB and O be the center of circumcircle ω of triangle CMB. Line AC meets ω for the second time in point K. Segment KO meets the circumcircle of triangle ABC in point L. Prove that segments AL and KM meet on the circumcircle of triangle ACM.

Problem 1.2.13 () **:** Let BN be median of triangle ABC. M is a point on BC. S lies on BN such that $MS \parallel AB$. P is a point such that $SP \perp AC$ and $BP \parallel AC$. MP cuts AB at Q. Prove that $QB = QP$.

Problem 1.2.14 () **:** Let $ABCD$ be a convex quadrilateral with AB parallel to CD. Let P and Q be the midpoints of AC and BD, respectively. Prove that if $\angle ABP = \angle CBD$, then $\angle BCQ = \angle ACD$.

Problem 1.2.15 () **:** Point P lies inside a triangle ABC. Let D, E and F be reflections of the point P in the lines BC, CA and AB, respectively. Prove that if the triangle DEF is equilateral, then the lines AD, BE and CF intersect in a common point.

Problem 1.2.16 () **:** Let $\triangle ABC$ be an acute angled triangle. The circle with diameter AB intersects the sides AC and BC at points E and F respectively. The tangents drawn to the circle through E and F intersect at P. Show that P lies on the altitude through the vertex C.

Problem 1.2.17 () **:** Let γ be circle and let P be a point outside γ. Let PA and PB be the tangents from P to γ (where $A, B \in \gamma$). A line passing through P intersects γ at points Q and R. Let S be a point on γ such that $BS \parallel QR$. Prove that SA bisects QR

Problem 1.2.18 () **:** Given is a convex quadrilateral $ABCD$ with $AB = CD$. Draw the triangles ABE and CDF outside $ABCD$ so that $\angle ABE = \angle DCF$ and $\angle BAE = \angle FDC$. Prove that the midpoints of $\overline{AD}$, $\overline{BC}$ and $\overline{EF}$ are collinear

Problem 1.2.19 () **:** Let P be a point out of circle C. Let PA and PB be the tangents to the circle drawn from C. Choose a point K on AB . Suppose that the circumcircle of triangle PBK intersects C again at T. Let P' be the reflection of P with respect to A. Prove that

$$\angle PBT = \angle P'KA$$

Problem 1.2.20 () **:** Consider a circle C_1 and a point O on it. Circle C_2 with center O, intersects C_1 in two points P and Q. C_3 is a circle which is externally tangent to C_2 at R and internally tangent to C_1 at S and suppose that RS passes through Q. Suppose X and Y are second intersection points of PR and OR with C_1. Prove that QX is parallel with SY.

Problem 1.2.21 () **:** In triangle ABC we have $\angle A = \frac{\pi}{3}$. Construct E and F on continue of AB and AC respectively such that $BE = CF = BC$. Suppose that EF meets circumcircle of $\triangle ACE$ in K. ($K \not\equiv E$). Prove that K is on the bisector of $\angle A$

Problem 1.2.22 () **:** In triangle ABC, $\angle A = 90°$ and M is the midpoint of BC. Point D is chosen on segment AC such that $AM = AD$ and P is the second meet point of the circumcircles of triangles $\Delta AMC, \Delta BDC$. Prove that the line CP bisects $\angle ACB$

Problem 1.2.23 () **:** Let C_1, C_2 be two circles such that the center of C_1 is on the circumference of C_2. Let C_1, C_2 intersect each other at points M, N. Let A, B be two points on the circumference of C_1 such that AB is the diameter of it. Let lines AM, BN meet C_2 for the second time at A', B', respectively. Prove that $A'B' = r_1$ where r_1 is the radius of C_1.

Problem 1.2.24 () **:** Given a triangle ABC, let P lie on the circumcircle of the triangle and be the midpoint of the arc BC which does not contain A. Draw a straight line l through P so that l is parallel to AB. Denote by k the circle which passes through B, and is tangent to l at the point P. Let Q be the second point of intersection of k and the line AB (if there is no second point of intersection, choose $Q = B$). Prove that $AQ = AC$.

Problem 1.2.25 () **:** Let $ABCD$ be a cyclic quadrilateral in which internal angle bisectors $\angle ABC$ and $\angle ADC$ intersect on diagonal AC. Let M be the midpoint of AC. Line parallel to BC which passes through D cuts BM at E and circle $ABCD$ in F ($F \neq D$). Prove that $BCEF$ is parallelogram

Problem 1.2.26 () **:** The side BC of the triangle ABC is extended beyond C to D so that $CD = BC$. The side CA is extended beyond A to E so that $AE = 2CA$. Prove that, if $AD = BE$, then the triangle ABC is right-angled

Problem 1.2.27 () **:** $ABCD$ is a cyclic quadrilateral inscribed in the circle Γ with AB as diameter. Let E be the intersection of the diagonals AC and BD. The tangents to Γ at the points C, D meet at P. Prove that $PC = PE$

Problem 1.2.28 () **:** The quadrilateral $ABCD$ is inscribed in a circle. The point P lies in the interior of $ABCD$, and $\angle PAB = \angle PBC = \angle PCD = \angle PDA$. The lines AD and BC meet at Q, and the lines AB and CD meet at R. Prove that the lines PQ and PR form the same angle as the diagonals of $ABCD$

Problem 1.2.29 () **:** Let $ABCD$ be a cyclic quadrilateral with opposite sides not parallel. Let X and Y be the intersections of AB, CD and AD, BC respectively. Let the angle bisector of $\angle AXD$ intersect AD, BC at E, F respectively, and let the angle bisectors of $\angle AYB$ intersect AB, CD at G, H respectively. Prove that $EFGH$ is a parallelogram.

Problem 1.2.30 () **:** Triangle ABC is given with its centroid G and cicumcentre O is such that GO is perpendicular to AG. Let A' be the second intersection of AG with circumcircle of triangle ABC. Let D be the intersection of lines CA' and AB and E the intersection of lines BA' and AC. Prove that the circumcentre of triangle ADE is on the circumcircle of triangle ABC

Problem 1.2.31 () **:** Let M be the midpoint of the side AC of $\triangle ABC$. Let $P \in AM$ and $Q \in CM$ be such that $PQ = \frac{AC}{2}$. Let (ABQ) intersect with BC at $X \neq B$ and (BCP) intersect with BA at $Y \neq B$. Prove that the quadrilateral $BXMY$ is cyclic.

Problem 1.2.32 () **:** Let be given a triangle ABC and its internal angle bisector BD ($D \in BC$). The line BD intersects the circumcircle Ω of triangle ABC at B and E. Circle ω with diameter DE cuts Ω again at F. Prove that BF is the symmedian line of triangle ABC.

Problem 1.2.33 () **:** ΔABC is a triangle such that $AB \neq AC$. The incircle of ΔABC touches BC, CA, AB at D, E, F respectively. H is a point on the segment EF such that $DH \perp EF$. Suppose $AH \perp BC$, prove that H is the orthocenter of ΔABC.

Problem 1.2.34 () **:** Let ABC be a triangle and let P be a point on the angle bisector AD, with D on BC. Let E, F and G be the intersections of AP, BP and CP with the circumcircle of the triangle, respectively. Let H be the intersection of EF and AC, and let I be the ntersection of EG and AB. Determine the geometric place of the intersection of BH and CI when P varies

Problem 1.2.35 () **:** Let $D; E; F$ be the points on the sides $BC; CA; AB$ respectively, of $\triangle ABC$. Let $P; Q; R$ be the second intersection of $AD; BE; CF$ respectively, with the cricumcircle of $\triangle ABC$.

Show that
$$\frac{AD}{PD} + \frac{BE}{QE} + \frac{CF}{RF} \geq 9$$

Problem 1.2.36 () **:** Points D and E lie on sides AB and AC of triangle ABC such that $DE \parallel BC$. Let P be an arbitrary point inside ABC. The lines PB and PC intersect DE at F and G, respectively. If O_1 is the circumcenter of PDG and O_2 is the circumcenter of PFE, show that $AP \parallel O_1O_2$.

Problem 1.2.37 () **:** Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$

Problem 1.2.38 () **:** Let O and I be the circumcenter and incenter of triangle ABC, respectively. Let ωA be the excircle of triangle ABC opposite to A; let it be tangent to AB, AC, BC at K, M, N, respectively. Assume that the midpoint of segment KM lies on the circumcircle of triangle ABC. Prove that $O; N; I$ are collinear.

Problem 1.2.39 () **:** Let $ABCD$ be a cyclic quadrilateral. Let $AB \cap CD = P$ and $AD \cap BC = Q$. Let the tangents from Q meet the circumcircle of $ABCD$ at E and F. Prove that $P; E; F$ are collinear.

1.3 Orthocenter–Circumcircle–NinePoint Circle

1. Circles - Yufei Zhao
2. Big Picture - Yufei Zhao
3. POP - Yufei Zhao
4. 3 Lemmas - Yufei Zhao

1.3.1 some figures, might be good for something, i dunno

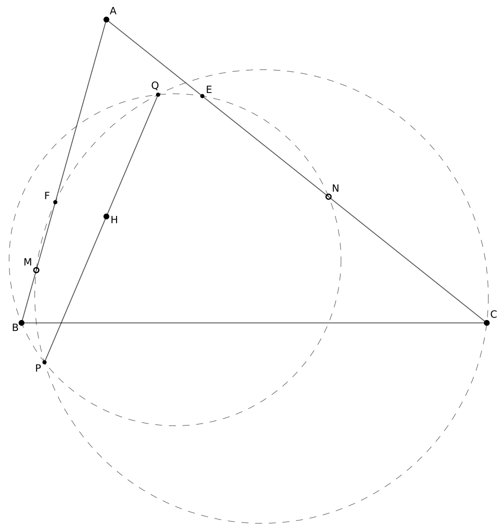

Figure 1.4: *H* lies on the line, Circles vary

1.3.2 Problems

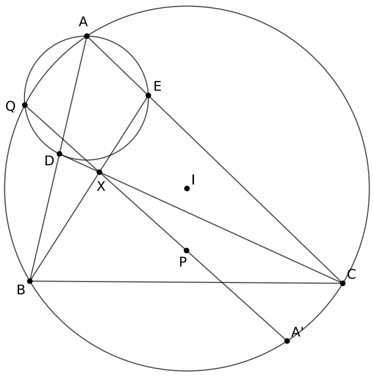

Figure 1.5

Lemma 1.3.1 (Collinearity with antipode and center) — *Let A' be the antipod of A in $\odot ABC$. Let $BDEC$ be a cyclic quadrilateral with $D \in AB$ and $E \in AC$. Let P be the center of $BDEC$. Also, let $X = BE \cap CD$. Then A', P, X are collinear.*

Solution. Using "The Big Picture" property to show that if $Q = \odot ADE \cap \odot ABC$, then P, X, Q collinear and $PQ \perp AQ$. Which implies that P, A', Q are collinear. $\square$

Problem 1.3.1 (Balkan MO 2017 P3) **:** Consider an acute-angled triangle ABC with $AB < AC$ and let ω be its circumscribed circle. Let t_B and t_C be the tangents to the circle ω at points B and C, respectively, and let L be their intersection. The straight line passing through the point B and parallel to AC intersects t_C in point D. The straight line passing through the point C and parallel to AB intersects t_B in point E. The circumcircle of the triangle BDC intersects AC in T, where T is located between A and C. The circumcircle of the triangle BEC intersects the line AB (or its extension) in S, where B is located between S and A. Prove that ST, AL, and BC are concurrent.

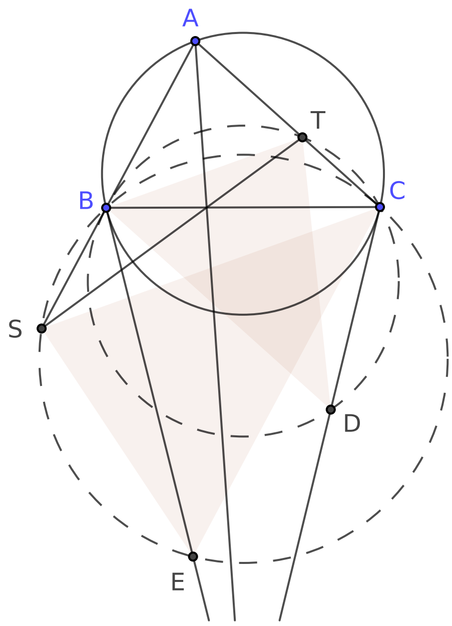

Figure 1.6

Solution. You could've thought like: symmedian is there & and $\triangle ABC \rightarrow \triangle ACS$ & concurrent $\implies$ parallel lines and median concurrency?
Just need to prove $BT \parallel CS$. $\square$

Problem 1.3.2 (USAMO 2014 P5) **:** Let ABC be a triangle with orthocenter H and let P be the second intersection of $\odot AHC$ with the internal bisector of $\angle BAC$. Let X be the circumcenter of triangle APB and Y the orthocenter of triangle APC. Prove that the length of segment XY is equal to the circumradius of triangle ABC.

> ***Solution.*** No length conditions given, yet we need to prove that two lengths are equal. Parallelogram !!!
> Just need to prove that $Y \in \odot ABC$ & $YD \perp AB$ □

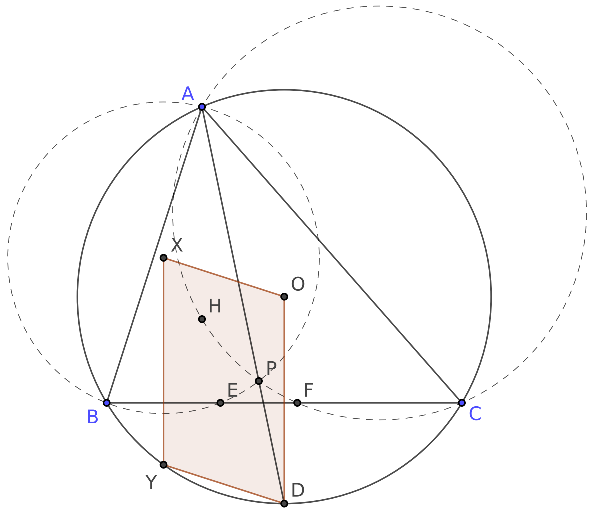

Figure 1.7

Problem 1.3.3 (Bewarish 1) **:** Let DEF be the orthic triangle, and let $EF \cap BC = P$. Let the tangent at A to $\odot ABC$ meet BC at Q. Let T be the reflection of Q over P. Let K be the orthogonal projection of H on AM. Prove that $\angle OKT = 90$.

> ***Solution.*** Spiral similarity to get rid of Q and T. Then spiral similarity again to find a trivial circle. □

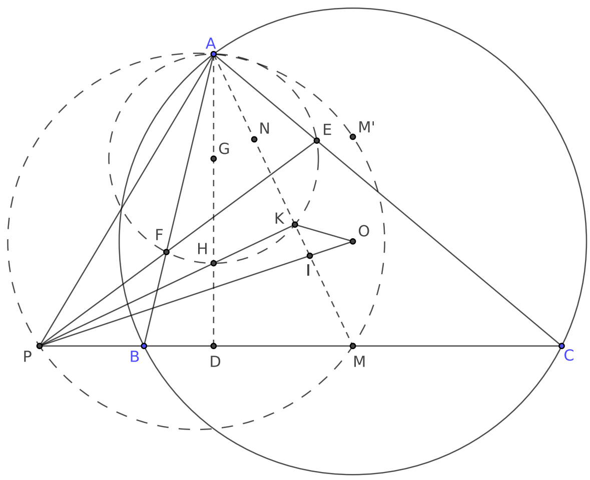

Figure 1.8

Problem 1.3.4 (buratinogigle's proposed problems for Arab Saudi team 2015) **:** Let ABC be a triangle with orthocenter H. P is a point. (K) is the circle with diameter AP. (K) cuts CA, AB again at E, F. PH cuts (K) again at G. Tangent line at E, F of (K) intersect at T. M is midpoint of BC. L is the point on MG such that $AL \parallel MT$. Prove that $LA \perp LH$.

Solution [Phantom Point]**.** Take $L' = MG \cap AZYH$, then use spiral similarity to show that $AL' \parallel MT$. □

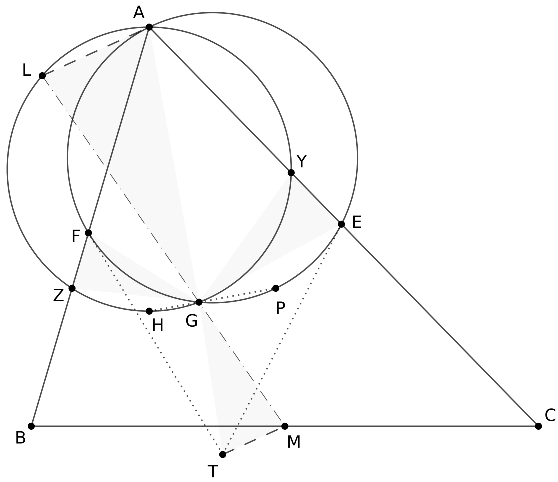

Figure 1.9

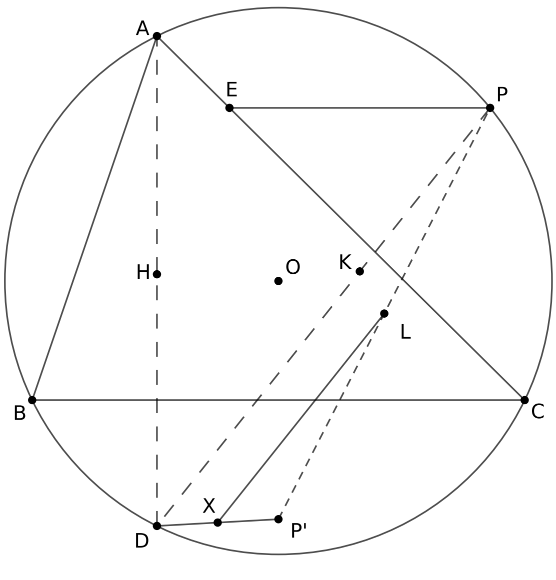

Figure 1.10

Problem 1.3.5 (buratinogigle's proposed problems for Arab Saudi team 2015) **:** Let ABC be a triangle inscribed circle (O). P lies on (O). The line passes through P and parallel to BC cuts CA at E. K is circumcenter of triangle PCE and L is nine point center of triangle PBC. Prove that the line passes through L and parallel to PK, always passes through a fixed point when P moves.

Solution [Construction]**.** Notice that if we reflect P over L to get P', then $OP = AH$ and $OP \perp BC$ where O is the circumcenter of $\odot ABC$. Which trivially implies that the line throught L passes throught the midpoint of $P'D$ where D is the reflection of H over BC. □

Problem 1.3.6 (buratinogigle's proposed problems for Arab Saudi team 2015) **:** Let ABC be acute triangle inscribed circle (O), altitude AH, H lies on BC. P is a point that lies on bisector $\angle BAC$ and P is inside triangle ABC. Circle diameter AP cuts (O) again at G. L is projection of P on AH. Assume that GL bisects HP. Prove that P is incenter of ABC.

Solution [Angle Chase]**.** Since $\angle APL = \angle ABD = \angle AGD$, G, L, M are collinear. Let $E \in BC$ and $PE \perp BC$. Then E also lies on DG.
Again we have, $\triangle DPE \sim \triangle DGP$. Which implies $DP = DB = DC$. □

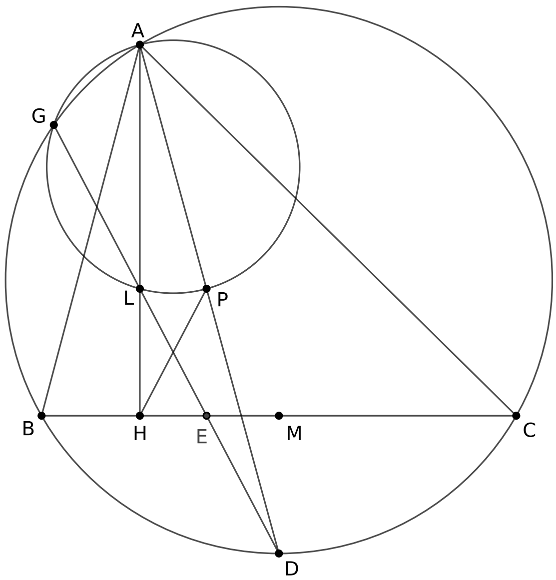

Figure 1.11

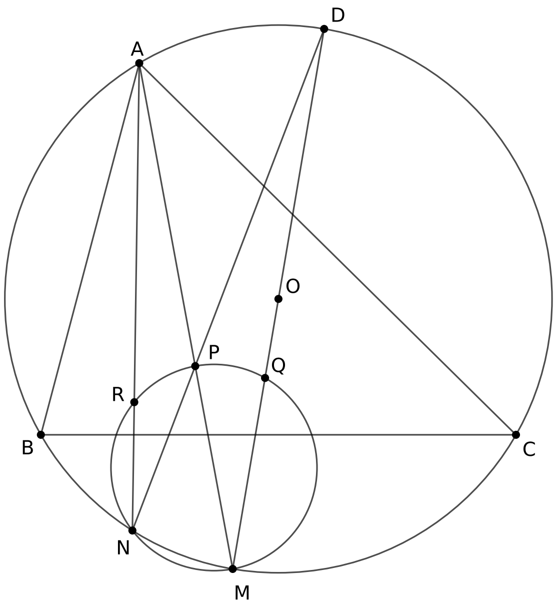

Figure 1.12

Problem 1.3.7 (buratinogigle's proposed problems for Arab Saudi team 2015) **:** Let ABC be an acute triangle inscribed circle (O). M lies on small arc $\overline{BC}$. P lies on AM. Circle diameter MP cuts (O) again at N. MO cuts circle diameter MP again at Q. AN cuts circle diameter MP again at R. Prove that $\angle PRA = \angle PQA$.

Solution [Angle Chase]**.** Let $MO \cap \odot ABC = D$. Becase $NP \perp MN$, we have N, P, D collinear, and $APQD$ cyclic.
So, $\triangle APQ \sim \triangle ANM \sim \triangle APR$. □

Problem 1.3.8 (buratinogigle's proposed problems for Arab Saudi team 2015) **:** Let ABC be right triangle with hypotenuse BC, bisector BE, E lies on CA. Assume that circumcircle of triangle BCE cuts segment AB again at F. K is projection of A on BC. L lies on segment AB such that $BL = BK$. Prove that $\frac{AL}{AF} = \sqrt{\frac{BK}{BC}}$.

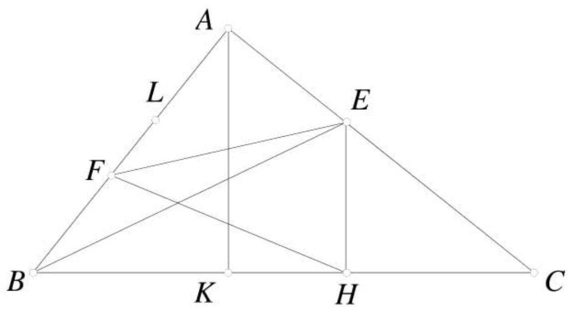

Figure 1.13

Problem 1.3.9 (buratinogigle's proposed problems for Arab Saudi team 2015) **:** Let ABC be acute triangle inscribed circle (O). AD is diameter of (O). M, N lie on BC such that $OM \parallel AB$, $ON \parallel AC$. DM, DN cut (O) again at P, Q. Prove that $BC = DP = DQ$.

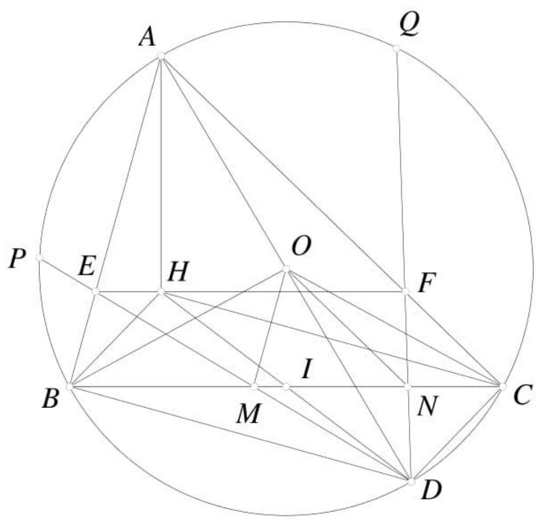

Figure 1.14

Problem 1.3.10 () **:** Let $\triangle ABC$ be a triangle. F, G be arbitrary points on AB, AC. Take D, E midpoint of BF, CG. Show that the center of nine-point circle of $\triangle ABC$, $\triangle ADE$, $\triangle AFG$ are collinear.

Problem 1.3.11 (IGO 2017 Advance P3) **:** L et O be the circumcenter of $\triangle ABC$. Line CO intersects the altitude through A at point K. Let P, M be the midpoints of AK, AC respectively.

If PO intersects BC at Y , and the circumcircle of $\triangle BCM$ meets AB at X, prove that $BXOY$ is cyclic

Solution. There is no easily measurable angles, in this case use projective geometry. And since we still don't have any easy angles, we look for the second way of concyclicity, POP □

Problem 1.3.12 (Turkey TST 2018 P4) **:** In a non-isosceles acute triangle ABC, D is the midpoint of BC. The points E and F lie on AC and AB, respectively, and the circumcircles of CDE and AEF intersect in P on AD. The angle bisector from P in triangle EFP intersects EF in Q. Prove that the tangent line to the circumcircle of AQP at A is perpendicular to BC.

Solution. Inverting around A. □

Problem 1.3.13 (USA Winter TST 2020 P2) **:** Two circles Γ_1 and Γ_2 have common external tangents ℓ_1 and ℓ_2 meeting at T. Suppose ℓ_1 touches Γ_1 at A and ℓ_2 touches Γ_2 at B. A circle Ω through A and B intersects Γ_1 again at C and Γ_2 again at D, such that quadrilateral $ABCD$ is convex.

Suppose lines AC and BD meet at point X, while lines AD and BC meet at point Y. Show that T, X, Y are collinear.

Solution [Radical Axis]**.** It is easy to see that X lies on the radical axis of Γ_1 and Γ_2. Let $B' = l_1 \cap \Gamma_2$ and $A' = l_2 \cap \Gamma_1$. Let $C' = A'X \cap \Gamma_1$ and $D' = B'X \cap \Gamma_2$. Let $A'C \cap AC' = Z$.

We have $AD'CB'$ and $A'DC'B$ cyclic. Also T, D, C' and T, D', C are collinear. Which implies $A'D'CB$ and $ADC'B'$ are cyclic too.

Applying pascal on $AAC'CA'A'$, we have T, X, Z are collinear.

Now, it is easy to see that Z, Y, T lie on the radical axis of $A'D'CB$ and $ADC'B'$. So we have T, X, Y, Z collinear.

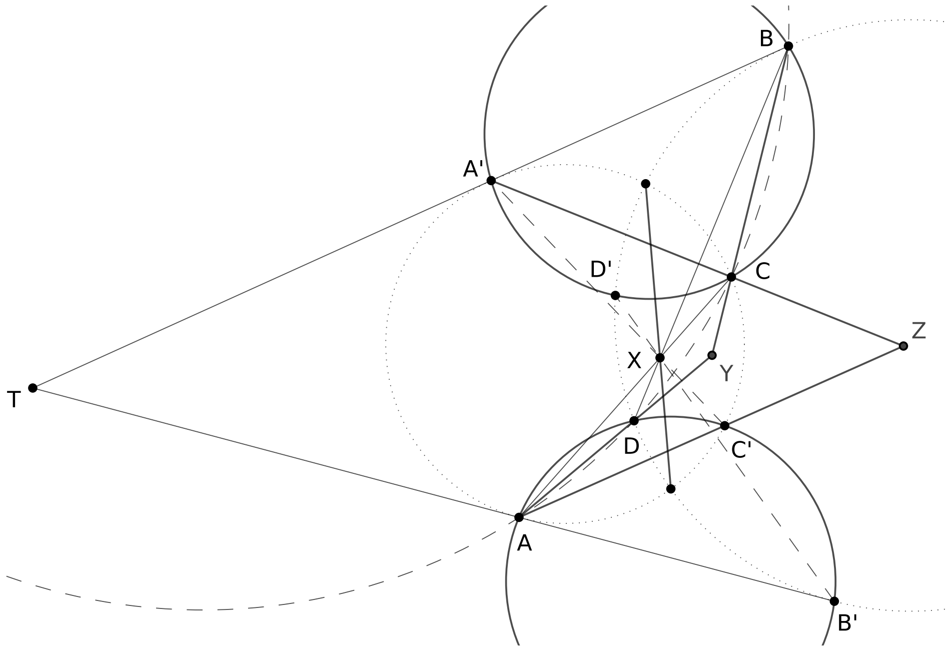

Figure 1.15

□

Solution [mOvInG pOiNtS, by shawnee03]**.** Fix Γ_1 and Γ_2 (and hence ℓ, T, A, B) and animate X linearly on ℓ. Then

- C moves projectively on Γ_1 (it is the image of the perspectivity through A from ℓ to Γ_1) and thus has degree 2, and similarly for D.
- $\overline{AD}$ has degree at most $0 + 2 = 2$, and similarly for $\overline{BC}$.
- $Y = \overline{AD} \cap \overline{BC}$ has degree at most $2 + 2 = 4$.
- The collinearity of T, X, Y has degree at most $0 + 1 + 4 = 5$.

Thus it suffices to verify the problem for six different choices of X. We choose:

- $\ell \cap \ell_1$: here Y approaches A as X approaches $\ell \cap \ell_1$.
- $\ell \cap \ell_2$: here Y approaches B as X approaches $\ell \cap \ell_2$.
- $\ell \cap \overline{AB}$: here Y approaches $\ell \cap \overline{AB}$ as X approaches $\ell \cap \overline{AB}$.
- the point at infinity along ℓ: here $Y = T$.
- the two intersections of Γ_1 and Γ_2: here $Y = X$.

(The final two cases may be chosen because we know that there exists a choice of A, B, C, D for which $ABCD$ is convex; this forces Γ_1 and Γ_2 to intersect.) □

Generalization 1.3.13.1 (USA Winter TST 2020 P2) **:** Let $ABCD$ be a cyclic quadrilateral, $X = AC \cap BD$, and $Y = AB \cap CD$. Let T be a point on line XY, Γ_1 be the circle through A and C tangent to TA, and Γ_2 be the circle through B and D tangent to TD. Then Γ_1 and Γ_2 are viewed at equal angles from T.

Solution [Length Chase, by a1267ab]**.** If the radiuses of Γ_1 and Γ_2 are r_1, r_2, then we have to show,

$$\frac{TA}{r_1} = \frac{TD}{r_2}$$

We have,

$$r_1 = \frac{AB}{2\sin\angle TAB},\ r_2 = \frac{CD}{2\sin\angle TDC}$$

To get the sine ratios, we compare the areas of $\triangle TAB$ and $\triangle TDC$. We have,

$$\frac{TA \cdot AB\ \sin\angle TAB}{TD \cdot CD\ \sin\angle TDC} = \frac{[TAB]}{[TCD]} = \frac{[XAB]}{[XCD]} = \frac{AB^2}{CD^2}$$

$$\implies \frac{r_1}{TA} = \frac{r_2}{TD}$$

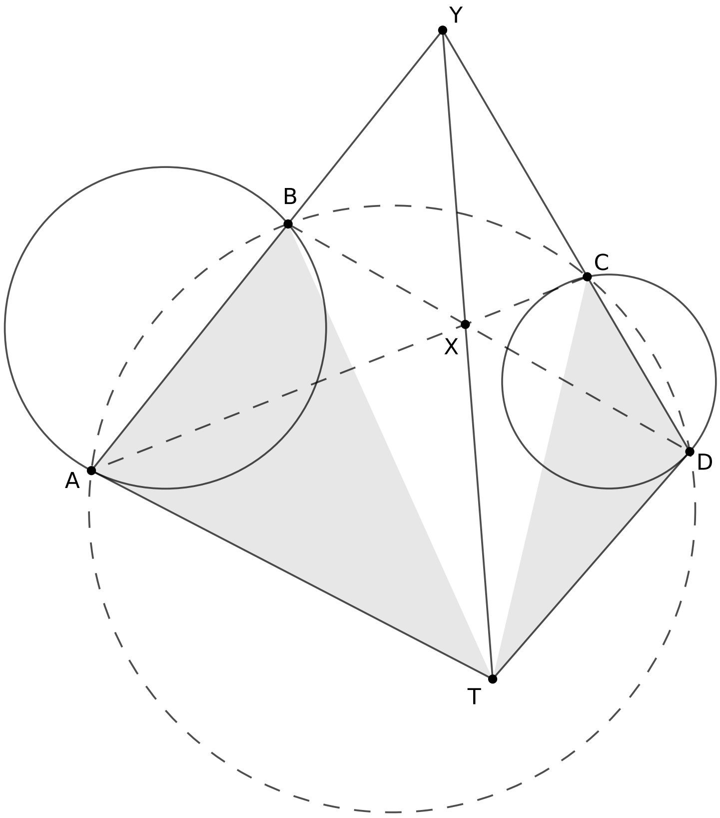

Figure 1.16

□

Problem 1.3.14 (IRAN 3rd Round 2016 P1) **:** Let ABC be an arbitrary triangle, P is the intersection point of the altitude from C and the tangent line from A to the circumcircle. The bisector of angle A intersects BC at D. PD intersects AB at K, if H is the orthocenter then prove : $HK \perp AD$

Solution. Finding a set of Collinear points. □

Problem 1.3.15 (IGO 2017 Advance P4) **:** T hree circles W_1, W_2 and W_3 touches a line l at A, B, C respectively (B lies between A and C). W_2 touches W_1 and W_3. Let l_2 be the other common external tangent of W_1 and W_3. l_2 cuts W_2 at X, Y. Perpendicular to l at B intersects W_2 again at K. Prove that KX and KY are tangent to the circle with diameter AC.

Solution. Finding a Orthocenter Figure in these circle simplifies the problem a lot. □

Problem 1.3.16 (2017 IGO Advanced P2) **:** We have six pairwise non-intersecting circles that the radius of each is at least one (no circle lies in the interior of any other circle). Prove that the radius of any circle intersecting all the six circles, is at least one.

Problem 1.3.17 (ARO 2018 P10.2) **:** Let $\triangle ABC$ be an acute-angled triangle with $AB < AC$. Let M and N be the midpoints of AB and AC, respectively; let AD be an altitude in this triangle. A point K is chosen on the segment MN so that $BK = CK$. The ray KD meets the circumcircle Ω of ABC at Q. Prove that C, N, K, Q are concyclic.

Problem 1.3.18 (ARO 2014 P9.4) **:** Let M be the midpoint of the side AC of acute-angled triangle ABC with $AB > BC$. Let Ω be the circumcircle of ABC. The tangents to Ω at the points A and C meet at P, and BP and AC intersect at S. Let AD be the altitude of the triangle ABP and ω the circumcircle of the triangle CSD. Suppose ω and Ω intersect at $K \neq C$. Prove that $\angle CKM = 90°$.

Problem 1.3.19 (APMO 1999 P3) **:** Let Γ_1 and Γ_2 be two circles intersecting at P and Q. The common tangent, closer to P, of Γ_1 and Γ_2 touches Γ_1 at A and Γ_2 at B. The tangent of Γ_1 at P meets Γ_2 at C, which is different from P, and the extension of AP meets BC at R. Prove that the circumcircle of triangle PQR is tangent to BP and BR.

Problem 1.3.20 (Simurgh 2019 P2) **:** Let ABC be an isosceles triangle, $AB = AC$. Suppoe that Q is a point such that $AQ = AB$, $AQ||BC$. Let P be the foot of perpendicular line from Q to BC. Prove that the circle with diameter PQ is tangent to the circumcircle of ABC.

Problem 1.3.21 (European Mathematics Cup 2018 P2) **:** Later

Problem 1.3.22 (RMM 2019 P2) **:** Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. Let E be the midpoint of AC. Denote by ω and Ω the circumcircles of the triangles ABE and CDE, respectively. Let P be the crossing point of the tangent to ω at A with the tangent to Ω at D. Prove that PE is tangent to Ω.

Problem 1.3.23 (IGO 2018 A5) **:** $ABCD$ is a cyclic quadrilateral. A circle passing through A, B is tangent to segment CD at point E. Another circle passing through C, D is tangent to AB at point F. Point G is the intersection point of AE, DF, and point H is the intersection point of BE, CF. Prove that the incenters of triangles AGF, BHF, CHE, DGE lie on a circle.

Solution [juckter]**.** The cases where two opposite sides of $ABCD$ are parallel are easily dealt with. Let $X = AB \cap CD$. Then $XE^2 = XA \cdot XB = XC \cdot XD = XF^2$, so $XE = XF$. Reflect E through X onto E', and notice that $XE^2 = XC \cdot XD$ implies $(C, D; E, E') = -1$. Because $\angle EFE' = 90°$ (which follows from $XE = XF = XE'$) it follows that FE bisects $\angle CFD$ and analogously EF bisects $\angle AEB$. It then follows easily that G and H are symmetric about EF.

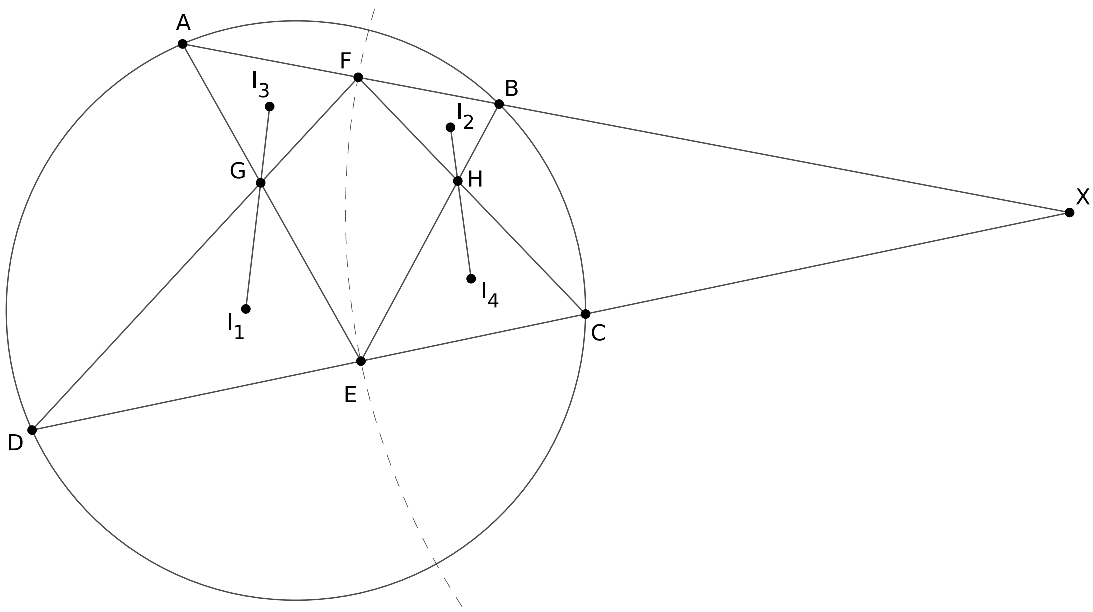

Figure 1.17: Problem 1.3.23 IGO 2018 A5

Now let I_1, I_2, I_3 and I_4 be the incenters of AGF, DGE, CHE, BHF respectively. Then I_1I_2 and I_3I_4 are the external bisectors of angles EGF and EHF respectively, and by symmetry about EF these lines intersect at a (possibly ideal) point $X \in EF$. Finally, we may angle chase to find that E, I_1, I_2, F and E, I_3, I_4, F are quadruples of concyclic points. If I_1I_2 is parallel to I_3I_4 then we may easily conclude by symmetry about the perpendicular bisector of EF. Otherwise by Power of a Point from X we have $XI_1 \cdot XI_2 = XE \cdot XF = XI_3 \cdot XI_4$, so I_1, I_2, I_3, I_4 are concyclic, as desired. □

Problem 1.3.24 (ISL 2011 G8) **:** Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.

Solution. Find the translated triangle circumscribed in $\odot ABC$. Once you find the properties of this triangle and the relations between this and the common touch point, the problem becomes obvious. □

Problem 1.3.25 (ELMO 2019 P3) **:** Let ABC be a triangle such that $\angle CAB > \angle ABC$, and let I be its incentre. Let D be the point on segment BC such that $\angle CAD = \angle ABC$. Let ω be the circle tangent to AC at A and passing through I. Let X be the second point of intersection of ω and the circumcircle of ABC. Prove that the angle bisectors of $\angle DAB$ and $\angle CXB$ intersect at a point on line BC.

Solution [Angle Chase]**.** Suppose the bisector of $\angle BAD$ meet BC at G'. Then we have,

$$\begin{aligned}\angle BG'A &= \frac{\angle A - \angle B}{2}\\ \therefore \angle CG'A &= \angle B + \angle BG'A\\ &= \frac{A+B}{2}\\ \\ \implies CG' &= CA\\ \therefore \angle G'ID &= \angle B\end{aligned}$$

Now, let M be the midpoint of the minor arc BC. Let $G = XM \cap BC$. So we have

$$\triangle MGI \sim \triangle MIX \implies \angle MIG = \angle MXI$$

Let $XI \cap \odot ABC = N \neq X$. Since AC is tangent to $\odot AXI$, $NC \parallel AM$. Which means

$$\angle MXI = \angle B = \angle MIG$$

Which completes our proof by implying that $G' \equiv G$.

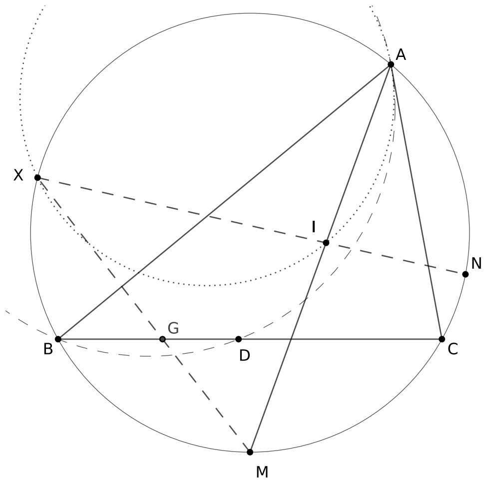

Figure 1.18

□

Problem 1.3.26 (ISL 2014 G5) **:** Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90°$. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and

$$\angle CHS - \angle CSB = 90°, \quad \angle THC - \angle DTC = 90°.$$

Prove that line BD is tangent to the circumcircle of triangle TSH.

Solution. First construct using nice circles, then prove the center is on AH using angle bisector theorem.

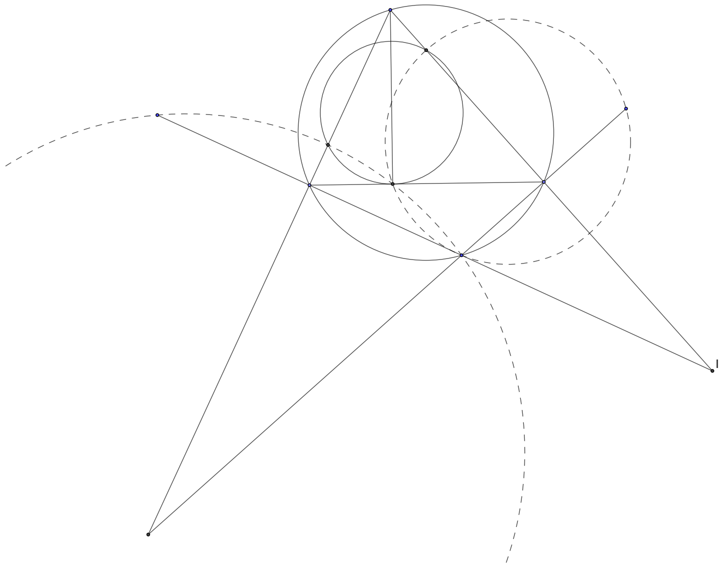

Figure 1.19: Construction

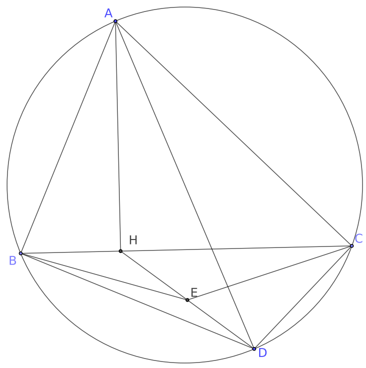

Figure 1.20: Lemma

□

Problem 1.3.27 (ISL 2014 G7) **:** Let ABC be a triangle with circumcircle Ω and incentre I. Let the line passing through I and perpendicular to CI intersect the segment BC and the arc BC (not containing A) of Ω at points U and V, respectively. Let the line passing through U and parallel to AI intersect AV at X, and let the line passing through V and parallel to AI intersect AB at Y. Let W and Z be the midpoints of AX and BC, respectively. Prove that if the points I, X, and Y are collinear, then the points I, W, and Z are also collinear.

Solution. Draw a nice diagram, and use the parallel property to find circles.

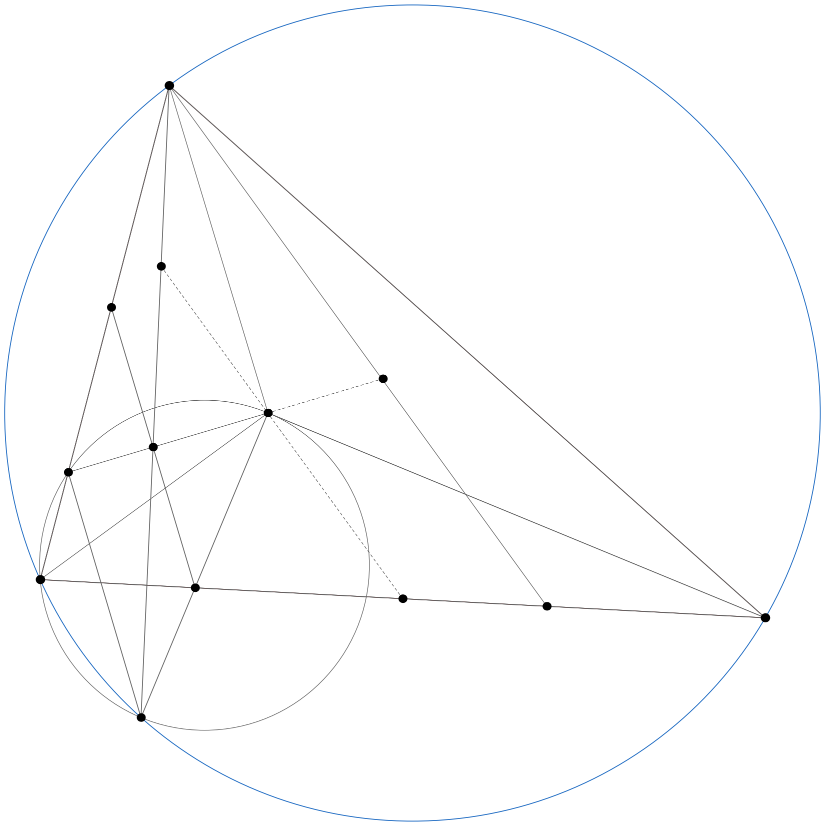

Figure 1.21: ISL 2014 G7

□

Problem 1.3.28 (ISL 2015 G6) **:** Let ABC be an acute triangle with $AB > AC$. Let Γ be its cirumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = 90^\circ$ and let K be the point on Γ such that $\angle HKQ = 90^\circ$. Assume that the points A, B, C, K and Q are all different and lie on Γ in this order.

Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

Solution. Draw the tangent line, and find angles.

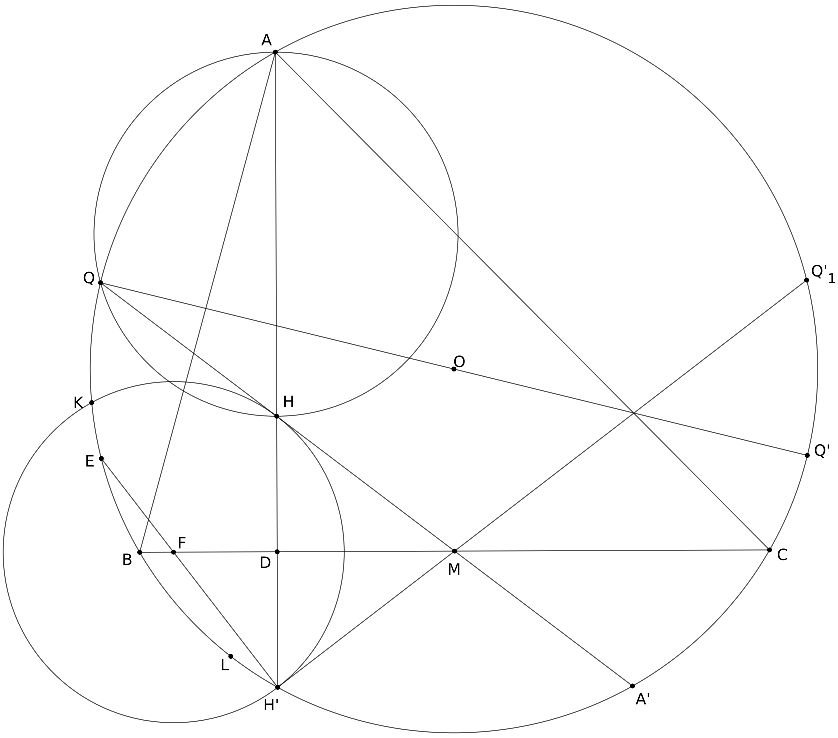

Figure 1.22: ISL 2015 G6

□

Problem 1.3.29 (ISL 2015 G5) **:** Let ABC be a triangle with $CA \neq CB$. Let D, F, and G be the midpoints of the sides AB, AC, and BC respectively. A circle Γ passing through C and tangent to AB at D meets the segments AF and BG at H and I, respectively. The points H' and I' are symmetric to H and I about F and G, respectively. The line $H'I'$ meets CD and FG at Q and M, respectively. The line CM meets Γ again at P. Prove that $CQ = QP$.

Solution. Don't depend on the figure too much, find facts using facts, not figure.

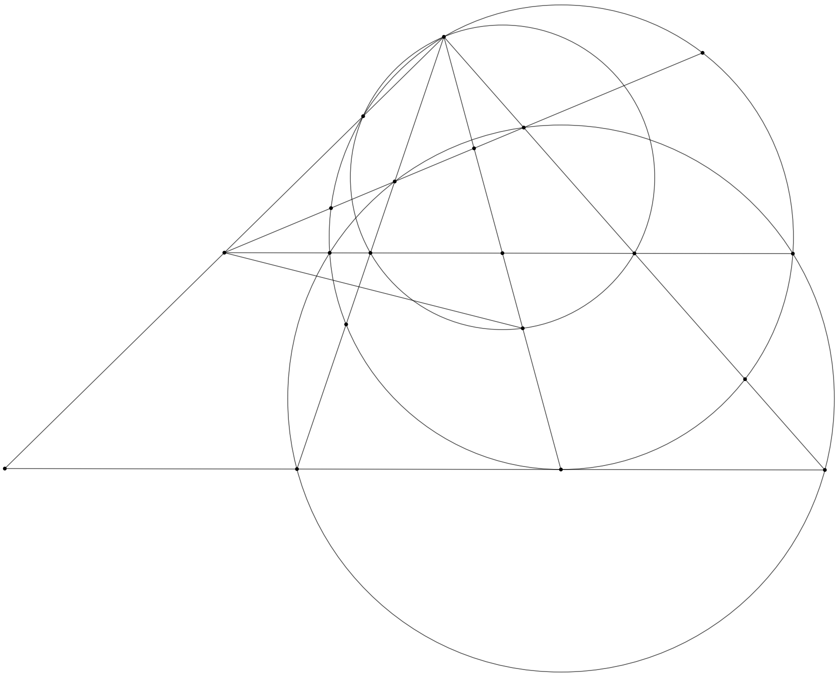

Figure 1.23: ISL 2015 G5

□

Problem 1.3.30 (ISL 2010 G5) **:** Let $ABCDE$ be a convex pentagon such that $BC \parallel AE$, $AB = BC + AE$, and $\angle ABC = \angle CDE$. Let M be the midpoint of CE, and let O be the circumcenter of triangle BCD. Given that $\angle DMO = 90^\circ$, prove that $2\angle BDA = \angle CDE$.

Solution. First try to construct the point. Do this the long way, then find a easier way that includes B, C, not B, A to do that. Then try to translate what 90 degree condition into angles, and take midpoints, since we have midpoints involved.

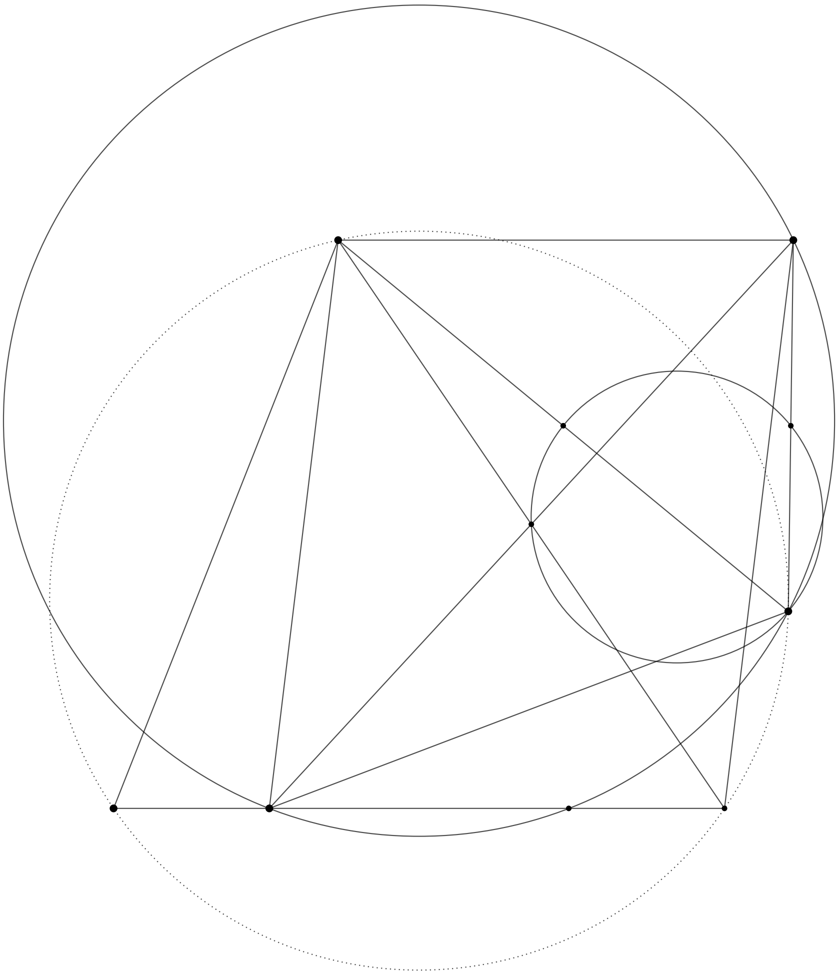

Figure 1.24: ISL 2010 G5

□

Problem 1.3.31 (IGO 2019 A5) **:** Let points A, B and C lie on the parabola Δ such that the point H, orthocenter of triangle ABC, coincides with the focus of parabola Δ. Prove that by changing the position of points A, B and C on Δ so that the orthocenter remain at H, inradius of triangle ABC remains unchanged.

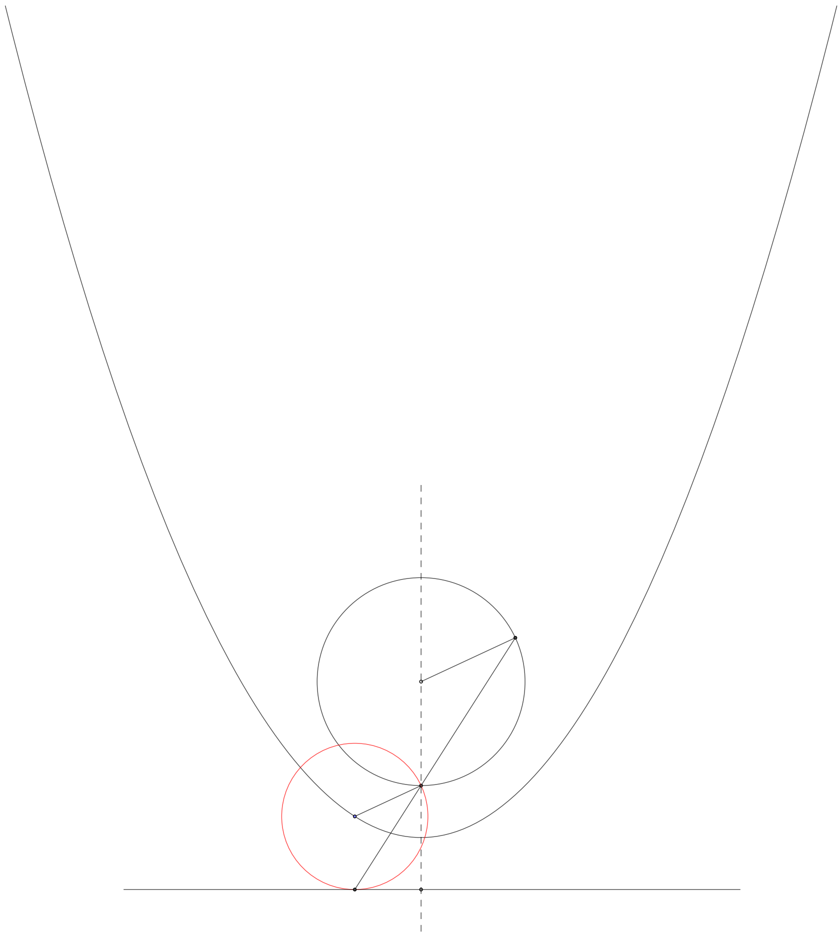

Figure 1.25: IGO 2019 A5

Solution. I think the idea for inversion should have been pretty natural after finding that the incircle is fixed. □

Problem 1.3.32 (Iran 3rd Round 2015 P5) **:** Let ABC be a triangle with orthocenter H and circumcenter O. Let R be the radius of circumcircle of $\triangle ABC$. Let A', B', C' be the points on $\overrightarrow{AH}, \overrightarrow{BH}, \overrightarrow{CH}$ respectively such that $AH.AA' = R^2, BH.BB' = R^2, CH.CC' = R^2$. Prove that O is incenter of $\triangle A'B'C'$.

Solution. The condition easily leads to a nice construction of the points. It should be trivial to figure that the construction is really important. Also, noticing a similarity among the triangles is really important. □

1.3.3 The line parallel to BC

Definition (Let the line parallel to BC through O meet AB, AC at D, E. Let K be the midpoint of AH, M be the midpoint of BC. F be the feet of A-altitude on BC and let H' be the reflection of H on F. Let O' be the circumcenter of KBC.) —

Lemma 1.3.2 — $\angle DKC = \angle EKB = 90°$

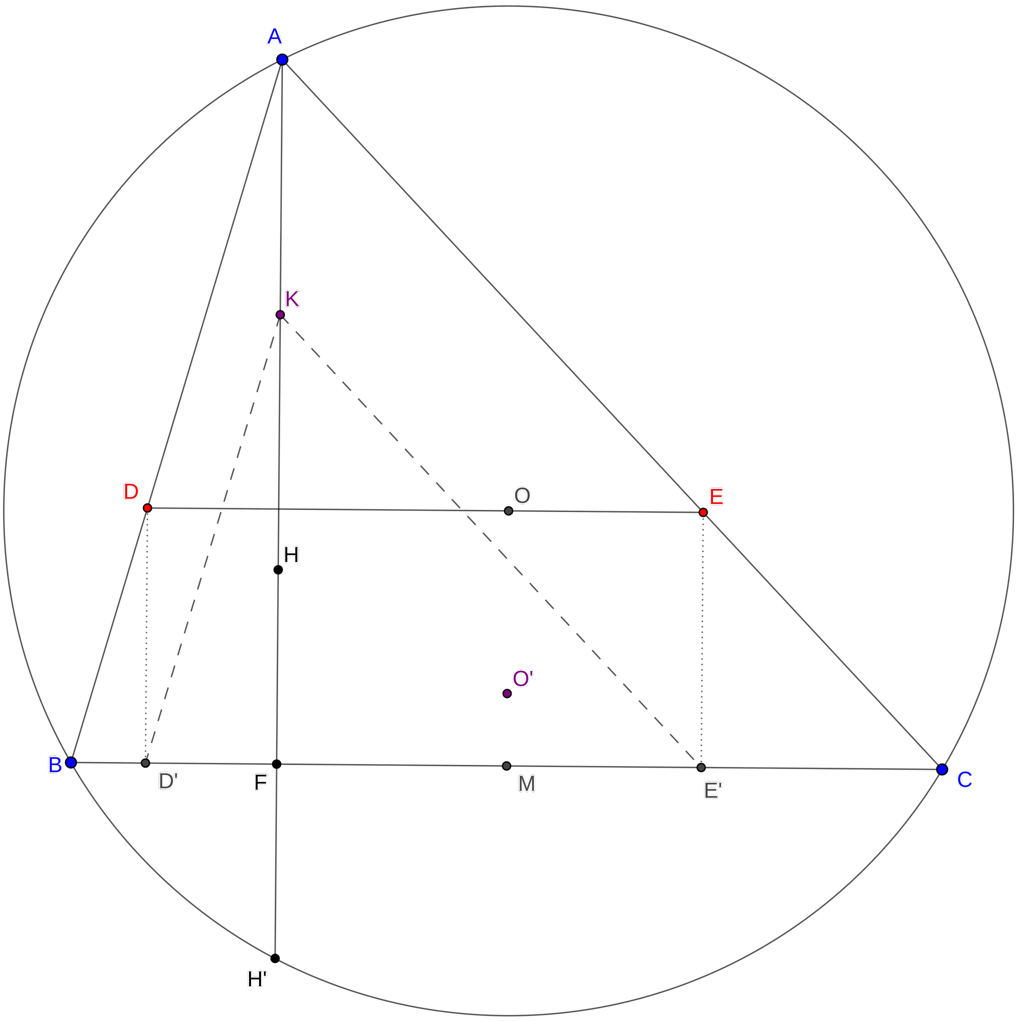

Figure 1.26

Lemma 1.3.3 — *CD, BE, OH', AM, KO' are concurrent. (by lemma)*

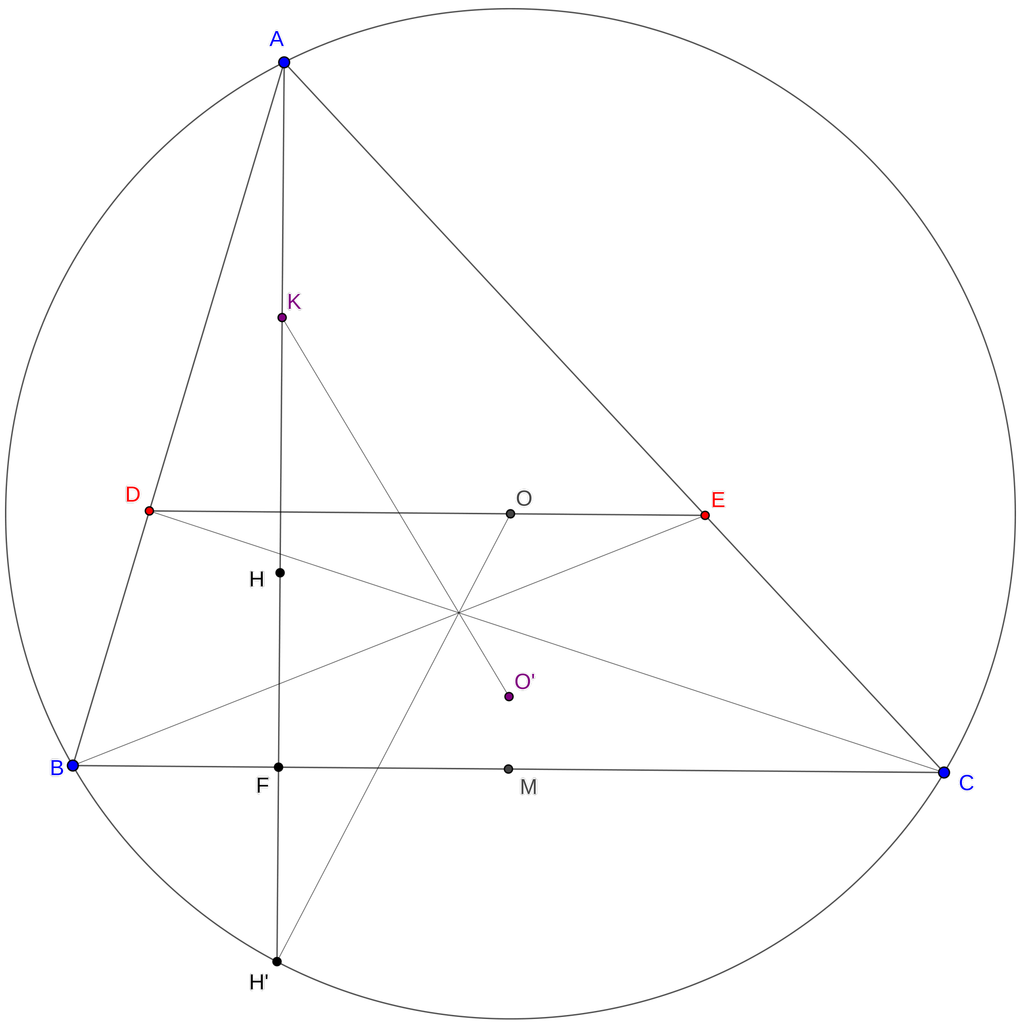

Figure 1.27

Problem 1.3.33 (InfinityDots MO Problem 3) **:** Let $\triangle ABC$ be an acute triangle with circumcenter O and orthocenter H. The line through O parallel to BC intersect AB at D and AC at E. X is the midpoint of AH. Prove that the circumcircles of $\triangle BDX$ and $\triangle CEX$ intersect again at a point on line AO.

Solution. Just using lemma to get another pair of circle where we can apply radical axis arguments. □

Solution. Noticing that the resulting point is the isogonal conjugate of a well defined point, □

Lemma 1.3.4 — *L et P, Q be on AB, AC resp. such that $PQ \parallel BC$. And let A' be such that $A' \in \odot ABC, AA' \parallel BC$. Let $CP \cap BQ = X$, and let the perpendicular bisector of BC meet PQ at Y. Prove that A', X, Y are collinear.*

Solution. No angles... Do Lengths... □

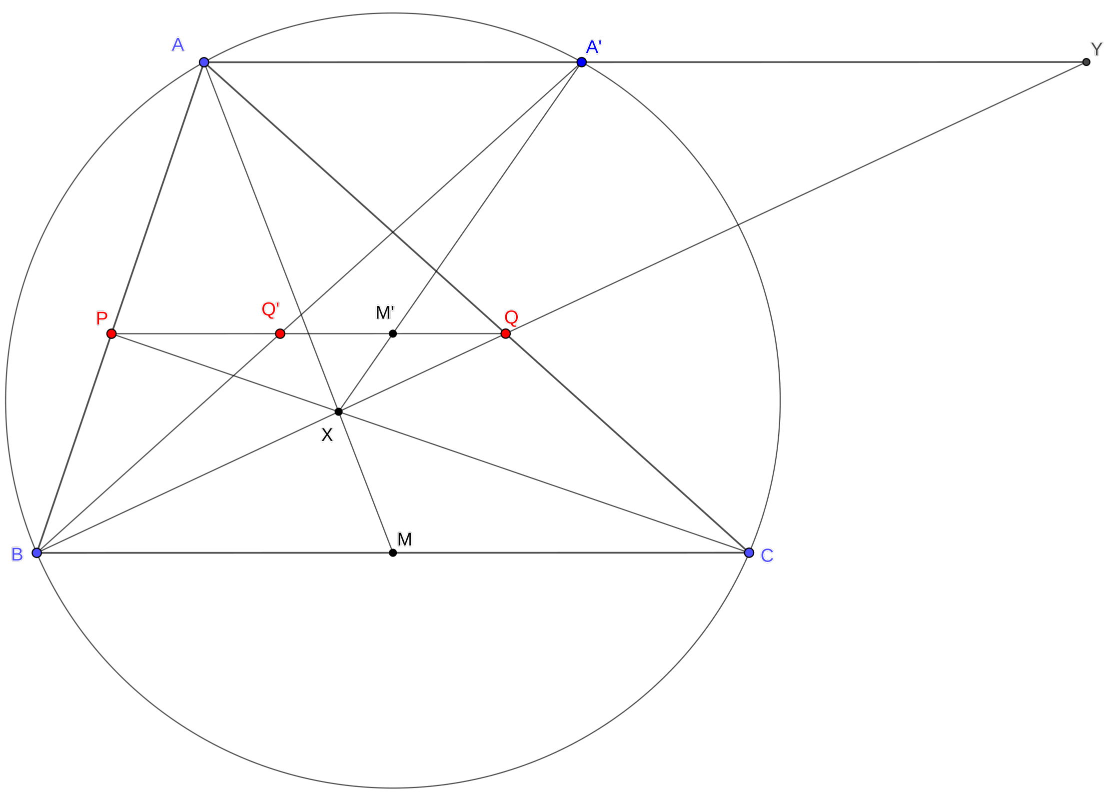

Figure 1.28

Problem 1.3.34 (ARO 2018 P11.4) **:** $P \in AB$, $Q \in AC$, $PQ \parallel BC$, $BQ \cap CP = X$. A' is the reflection of A wrt BC. $A'X \cap \odot APQ = Y$. Prove that $\odot BYC$ is tangent to $\odot APQ$.

Solution. Of co it can be solved using angle chase, lemma makes it almost trivial. □

Problem 1.3.35 (buratinogigle) **:** Let (O) be a circle and E, F are two points inside (O). $(K), (L)$ are two circles passing though E, F and tangent internally to (O) at A, D, respectively. AE, AF cut (O) again at B, C, respectively. BF cuts CE at G. Prove that reflection of A though EF lies on line DG.

Rephrasing the problem as such: In the setup of this lemma, let $A'X \cap \odot ABC = Z$, then $\odot PQZ$ is tangent to $\odot ABC$.

Solution. Simple angle chase. □

Solution. Another solution to this is by taking D as a phantom point. □

Solution. Another solution is with cross ratios □

1.3.4 Simson Line and Stuffs

Lemma 1.3.5 (Simson Line Parallel) — *Let P be a point on the circumcircle, let P' be the reflection of P on BC and let $PP' \cap \Omega = D$, and let l_p be the Simson line of P. Prove that $l_p \parallel AD \parallel HP'$.*

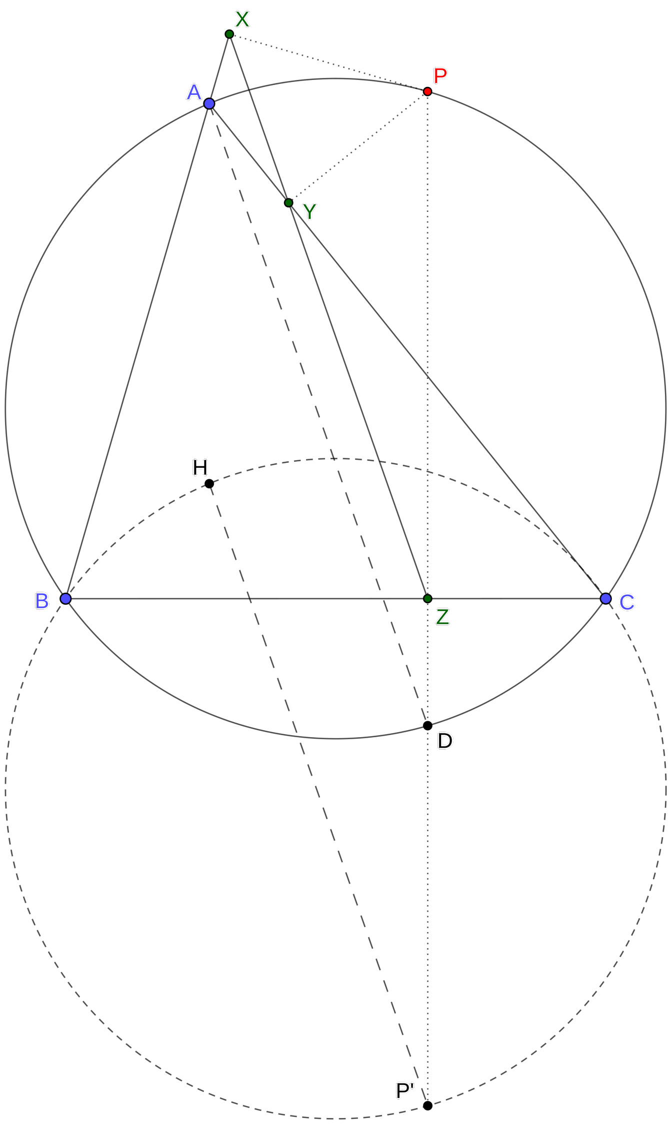

Figure 1.29: The dotted lines are parallel

Lemma 1.3.6 (Simson Line Angle) — *Given triangle ABC and its circumcircle (O). Let E, F be two arbitrary points on (O). Then the angle between the Simson lines of two points E and F is half the measure of the arc EF.*

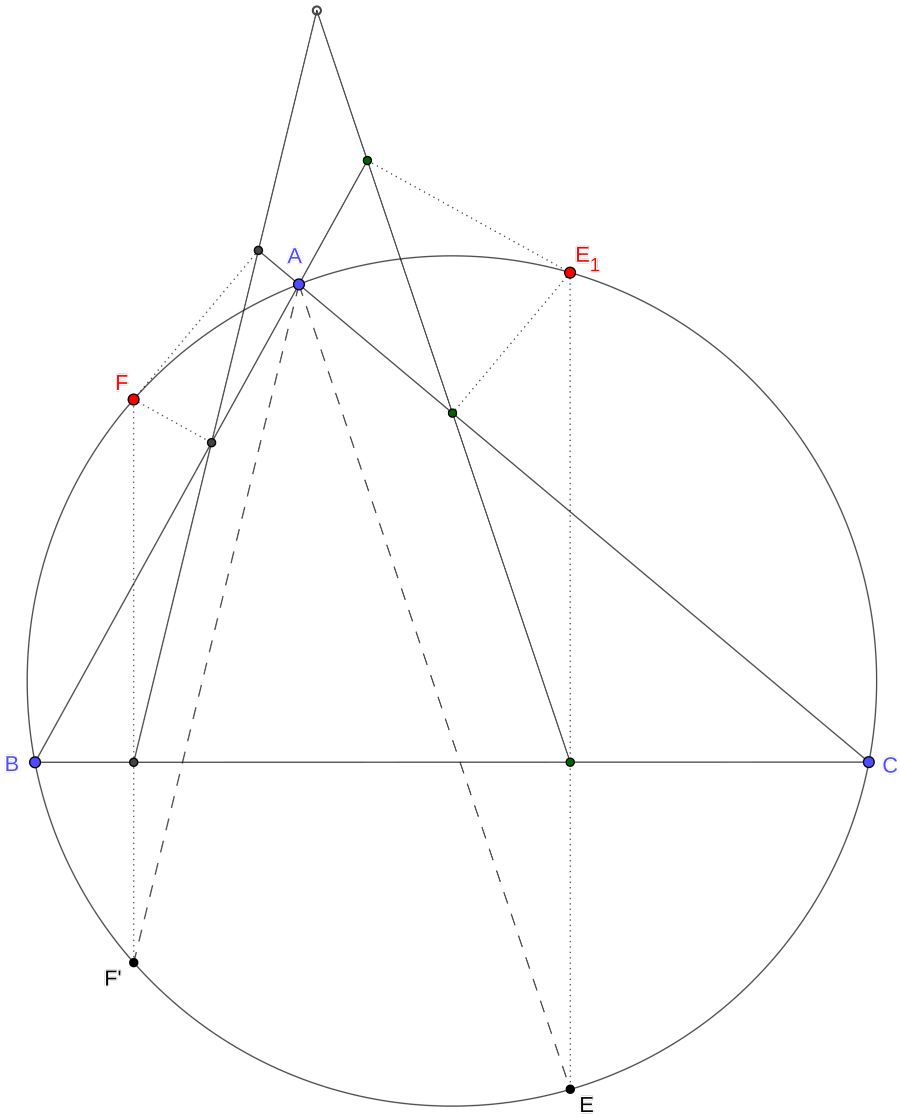

Figure 1.30

1.3.5 Euler Line

Theorem 1.3.7 (Perspectivity Line with Orthic triangle is perpendicular to Euler line) — *Let DEF be the orthic triangle. Then $BC \cap EF, CA \cap FD, AB \cap ED$ are collinear, and the line is perpendicular to the Euler line. In fact this line is the radical axis of the Circumcircle and the NinePoint circle*

Lemma 1.3.8 — *DEF is orthic triangle of ABC, XYZ is the orthic triangle of DEF. Prove that the perspective point of ABC and XYZ lies on the Euler line of ABC*

Solution. Thinking the stuff wrt to the incircle and using cross ratio. □

1.4 Cevian and Circumcevian Triangles

1.4.1 Circumcevian Triangle

Theorem 1.4.1 (Hagge's circles) — *Let P be a point on the plane of $\triangle ABC$, let Ω be the circumcircle. Let A_1, B_1, C_1 be the intersections of AP, BP, CP with Ω for the second time. Let A_2, B_2, C_2 be the reflections of A_1, B_1, C_1 wrt BC, CA, AB. Prove that H, A_2, B_2, C_2 lie on a circle. This circle is called the* ***P*-Hagge's Circle**.

Solution. Either using the dual of Hagge's Circle, or using the reflection points of A, B, C wrt the isogonal conjugate of P. And using Lemma 1.1 to finish. □

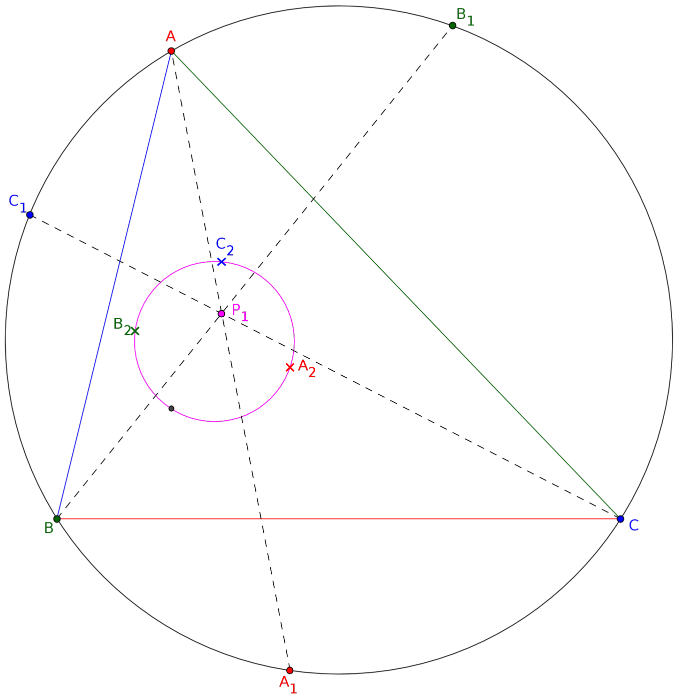

Figure 1.31: P-Hagge Circle

Corollary 1.4.1.1 — $\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2$.

Solution. Straightforward use of Lemma 1.2. □

Corollary 1.4.1.2 — *If AH, BH, CH meet $\odot A_2B_2C_2H$ at A_3, B_3, C_3, then A_2A_3, B_2B_3, C_2B_3 meet at P.*

Solution. Simple angle chase and similarity transformation. □

Corollary 1.4.1.3 — *If I is the incenter of $\odot A_2B_2C_2$, K is the reflection of H over I, AK, BK, CK meet $\odot A_2B_2C_2$ at A_4, B_4, C_4, then A_4A_3, B_4B_3, C_4B_3 are concurrent.*

Solution. Simple angle chasing and trig-ceva. □

Problem 1.4.1 (China TST D2P2, Dual of the Hagge's Circle theorem) **:** Let ω be the circumcircle of $\triangle ABC$. P is an interior point of $\triangle ABC$. A_1, B_1, C_1 are the intersections of AP, BP, CP respectively and A_2, B_2, C_2 are the symmetrical points of A_1, B_1, C_1 with respect to the midpoints of side BC, CA, AB. Show that the circumcircle of $\triangle A_2B_2C_2$ passes through the orthocenter of $\triangle ABC$. Further proof that if this circle's center is O_1, then $HOPO_1$ is a parallelogram.

Solution. Construct Parallelograms. You have to prove two angles are equal. Reflection the smaller trig wrt one of the midpoints. □

Problem 1.4.2 (China TST 2011, Quiz 2, D2, P1) **:** Let AA', BB', CC' be three diameters of the circumcircle of an acute triangle ABC. Let P be an arbitrary point in the interior of $\triangle ABC$, and let D, E, F be the orthogonal projection of P on BC, CA, AB, respectively. Let X be the point such that D is the midpoint of $A'X$, let Y be the point such that E is the midpoint of $B'Y$, and similarly let Z be the point such that F is the midpoint of $C'Z$. Prove that triangle XYZ is similar to triangle ABC.

Solution. A straightforward application of Lemma 6.1 using the O-Hagge's Circle. □

1.4.2 Cevian Triangle

Lemma 1.4.2 (Isogonal Conjugate Lemma) — *Let a circle ω meet the sides of triangle ABC at A_1, A_2; B_1, B_2; C_1, C_2. Let P_1, P_2 be the miquel points of ABC wrt $A_1B_1C_1, A_2B_2C_2$ resp. Then P_1, P_2 are isogonal conjugates.*

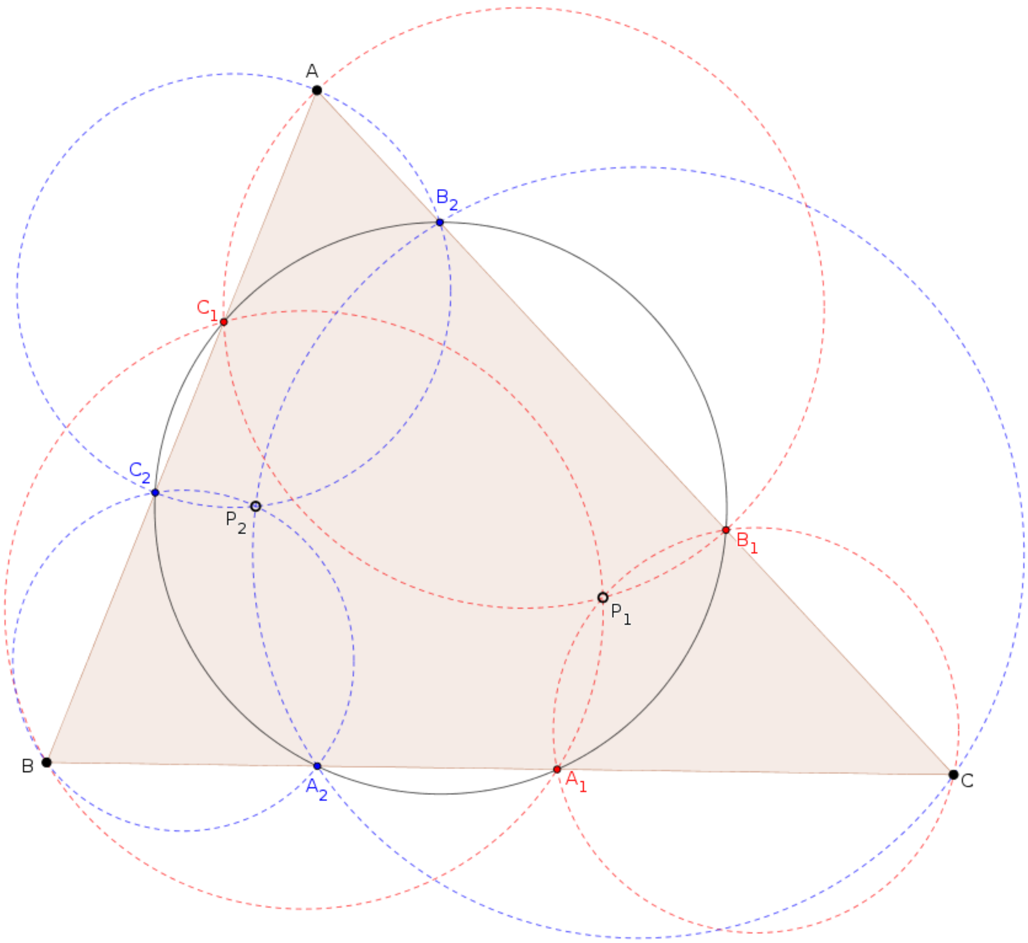

Figure 1.32: The two round points are isogonal conjugates.

Theorem 1.4.3 (Terquem's Cevian Theorem) — *Let a circle ω meet the sides of triangle ABC at A_1, A_2; B_1, B_2; C_1, C_2. If AA_1, BB_1, CC_1 are concurrent, then so are AA_2, BB_2, CC_2*

.

Theorem 1.4.4 (Mannheim's Theorem) — *Let ABC be a triangle, and let L, M, N be points on BC, CA, AB respectively. Let A', B', C' be points on $(AMN), (BNL), (CLM)$, and denote $K \equiv AA' \cap BB'$. Then if $K \in CC'$, A', B', C', K are concyclic.*

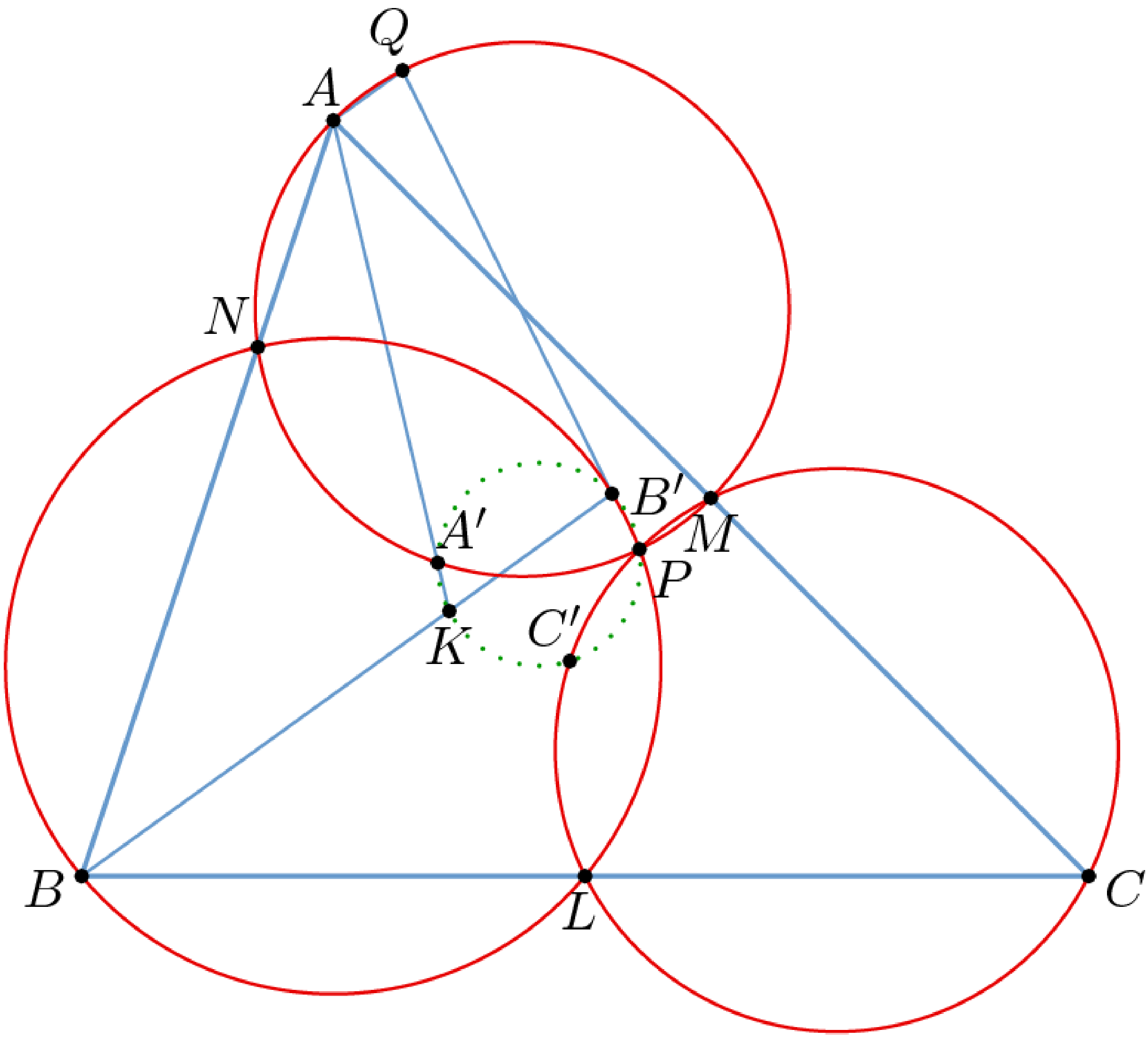

Figure 1.33: Mannheim's Theorem

Theorem 1.4.5 (Mannheim's Theorem's Converse) — *Let ABC be a triangle, and let L, M, N be points on BC, CA, AB respectively. Let A', B', C' be points on $(AMN), (BNL), (CLM)$, and denote $K \equiv AA' \cap BB'$. Then if A', B', C', K are concyclic, $C' \in CK$.*

Theorem 1.4.6 (Brocard Points) — *Brocard Points are points inside a triangle such that*

$$\angle PAB = \angle PBC = \angle PCA = \omega$$

and

$$\angle QCB = \angle QBA = \angle QAC = \omega.$$

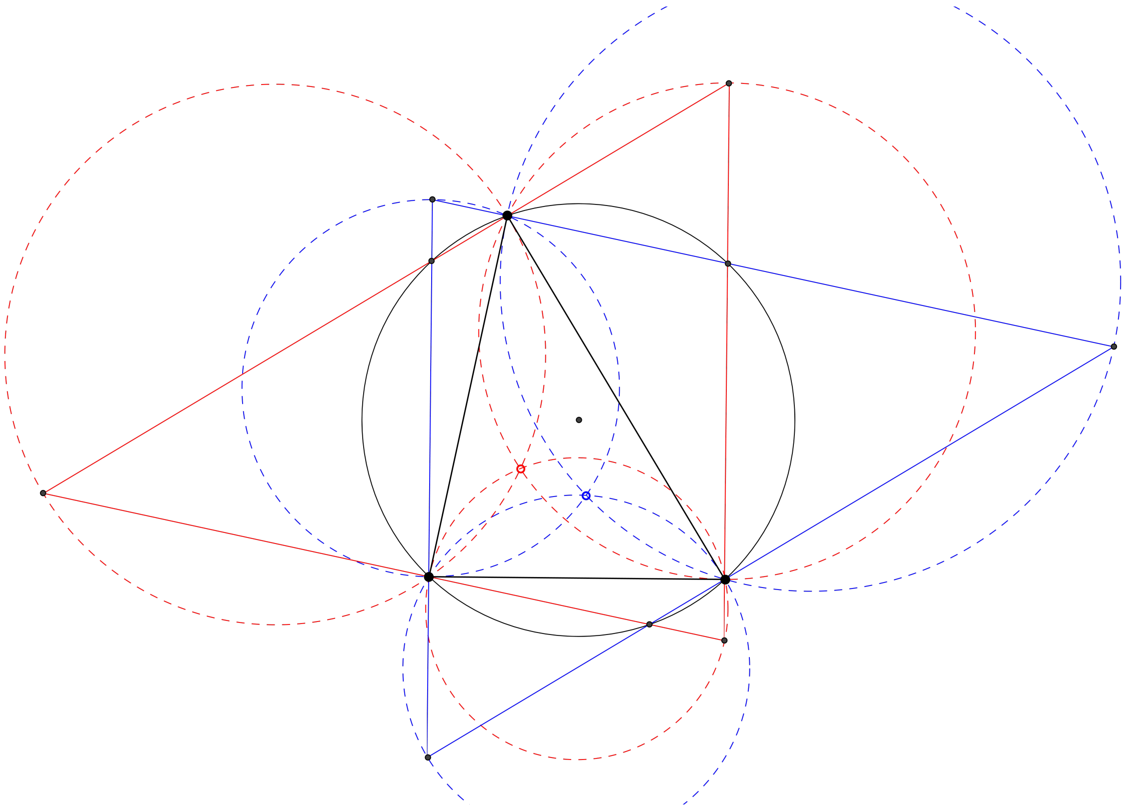

Figure 1.34: Brocard Points

Problem 1.4.3 (Rioplatense Olympiad 2013 Problem 6) **:** Let ABC be an acute-angled scalene triangle, with centroid G and orthocenter H. The circle with diameter AH cuts the circumcircle of BHC at A', distinct from H. Analogously define B', C'. Prove that A', B', C', G are concyclic.

Problem 1.4.4 (Iran 3rd Round Training 2016) **:** ABC is an acute triangle and H, O are its orthocenter and circumcenter respectively. If AO, BO, CO intersect BH, CH, AH at X, Y, Z respectively,then prove that H, X, Y, Z lie on a circle

Solution. Using Brocard Point □

Solution. Using Mannheim's Theorem □

Theorem 1.4.7 (Jacobi's Theorem) — *Suppose that D, E, F are points such that AE, AF are isogonal wrt $\angle BAC$. Similarly with D, E, F. Then AD, BE, CF are concurrent.*

1.5 Incenter–Excenter Lemma stuff

Definition (Incenter and Co.) — Let $\triangle ABC$ be an ordinary triangle, I is its incenter, D, E, F are the touch points of the incenter with BC, CA, AB and D', E', F' are the reflections of D, E, F wrt I.

Let the I_a, I_b, I_c excircles touch BC, CA, AB at D_1, E_1, F_1.

Let M_a, M_b, M_c be the midpoints of the smaller arcs BC, CA, AB, and M_A, M_B, M_C be the midpoints of the major arcs BC, CA, AB. M are the midpoint of BC.

Let (I_a) touch BC, CA, AB at D_A, E_A, F_A. So, $D_A \equiv D_1$.

Let A' be the antipode of A wrt $\odot ABC$.

Call EF, 'A-tangent line', and DE, DF similarly. And call E_AF_A 'A_A-tangent line.

Lemma 1.5.1 — *$A'I, \odot ABC, \odot AEIF$ are concurrent at Y_A. And Y_A, D, M_a are collinear.*

Lemma 1.5.2 — *$DD_H \perp EF$, then D_H, I, A' are collinear.*

Lemma 1.5.3 — *Let X be any point on BC, and let I_1, I_2 be incenters of $\triangle ABX, \triangle ACX$. Then $\square XI_1I_2D$ are cyclic. And the other common tangent of $\odot I_1$ and $\odot I_2$ goes through D.*

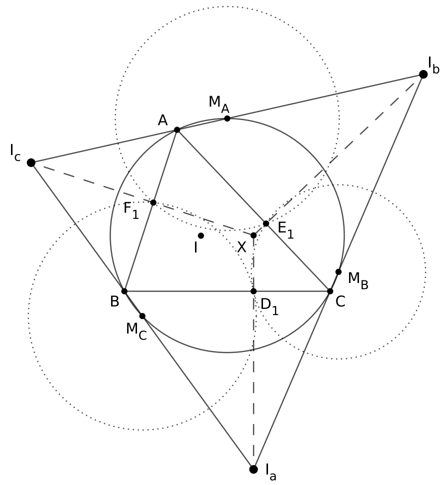

Figure 1.35: Lemma 1.5.4 Excenter Toucpoints are equidistance from the Bigger Arc-midpoint

Lemma 1.5.4 (Arc Midpoint as Centers) —

$$M_AE_1 = M_AF_1$$

$$M_BF_1 = M_BD_1$$

$$M_CD_1 = M_CE_1$$

Lemma 1.5.5 (Incircle Touchpoint and Cevian) — *Let a cevian be AX and let I_1, I_2 be the incirlces of $\triangle ABX, \triangle ACX$. Then D, I_1, I_2, X are concyclic.*

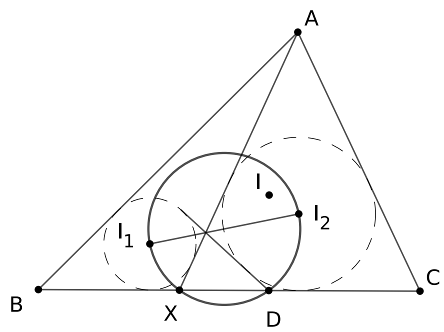

Figure 1.36: Lemma 1.5.5

Lemma 1.5.6 (Apollonius Circle and Incenter) — *Let ω_a be the circle that goes through B, C and is tangent to (I) at X. Then XD', EF, BC are concurrent and X, D, I_a are collinear. The same properties is held if the roles of incenter and excenter are swapped.*

- *The circle BXC is tangent to (I)*
- *X lies on the Apollonius Circle of $(B, C; D, G)$.*
- *XD bisects $\angle BXC$.*

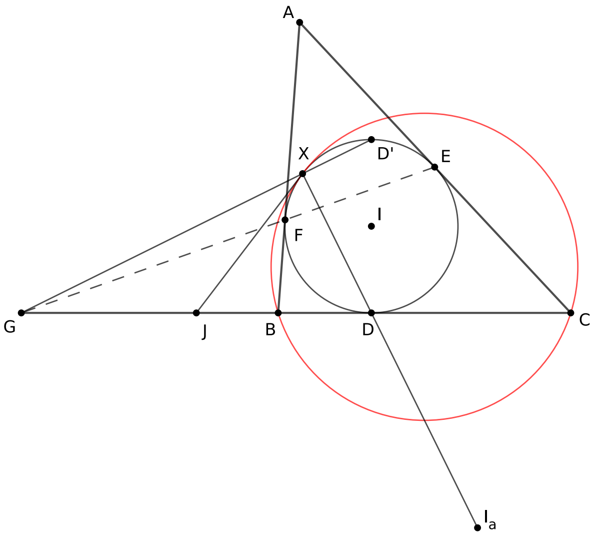

Figure 1.37: Circle through BC tangent to incircle

Solution. Pole-Polar. ISL 2002 G7 □

Theorem 1.5.7 (Paul Yui Theorem) — *B-tangent line, C_A-tangent line, and AH are concurrent.*

Lemma 1.5.8 (Concurrent Lines in Incenter) — *Let $AD \cap \odot(I) = G, AD' \cap \odot(I) = H$. Let the line through D' parallel to BC meet AB, AC at B', C'. Then AM, EF, GH, DD', BC', CB' are concurrent.*

Lemma 1.5.9 (Midline Concurrency with Incircle Touchpoints) — *AI, B, B_A-tangent lines and C-mid-line are concurrent. And, if the concurrency point is X, then $CS \perp AI$*

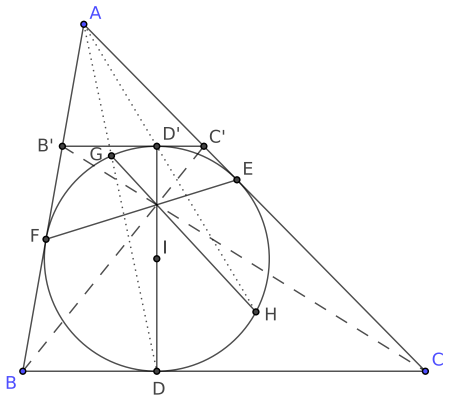

Figure 1.38: Lemma 1.5.8 The lines are concurrent.

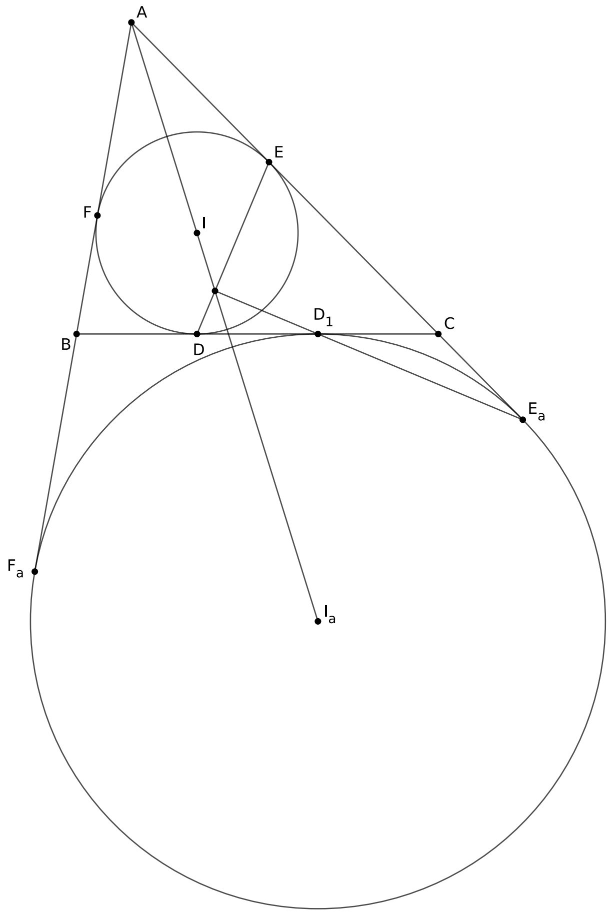

Figure 1.39: Lemma 1.5.9 AI, B, I_A-tangent lines and C-mid-line are concurrent.

Corollary 1.5.9.1 — *Let triangle ABC, incircle (I), the A - excircle (I_a) touches BC at M. IM intersects (I_a) at the second point X. Similarly, we get Y, Z. Prove that AX, BY, CZ are concurrent.*
Extension, by buratinogigle: Triangle ABC and XYZ are homothetic with center I is incenter of ABC. Excircles touches BC, CA, AB at D, E, F. XD, YE, FZ meets excircles again at U, V, W. Prove that AU, BV, CW are concurrent.

Definition (Isodynamic Points) — Let ABC be a triangle, and let the angle bisectors of $\angle A$ meet BC at X, Y. Call ω_a the circumcircle of $\triangle AXY$. Define ω_b, ω_c similarly. The first and second isodynamic points are the points where the three circles $\omega_a, \omega_b, \omega_c$ meet. I.e. these two points are the intersections of the three Apollonius circles. These two points satisfy the following relations:

1. $$PA \sin A = PB \sin B = PC \sin C$$

2. They are the isogonal congugates of the Fermat Points, and they lie on the 'Brocard Axis'

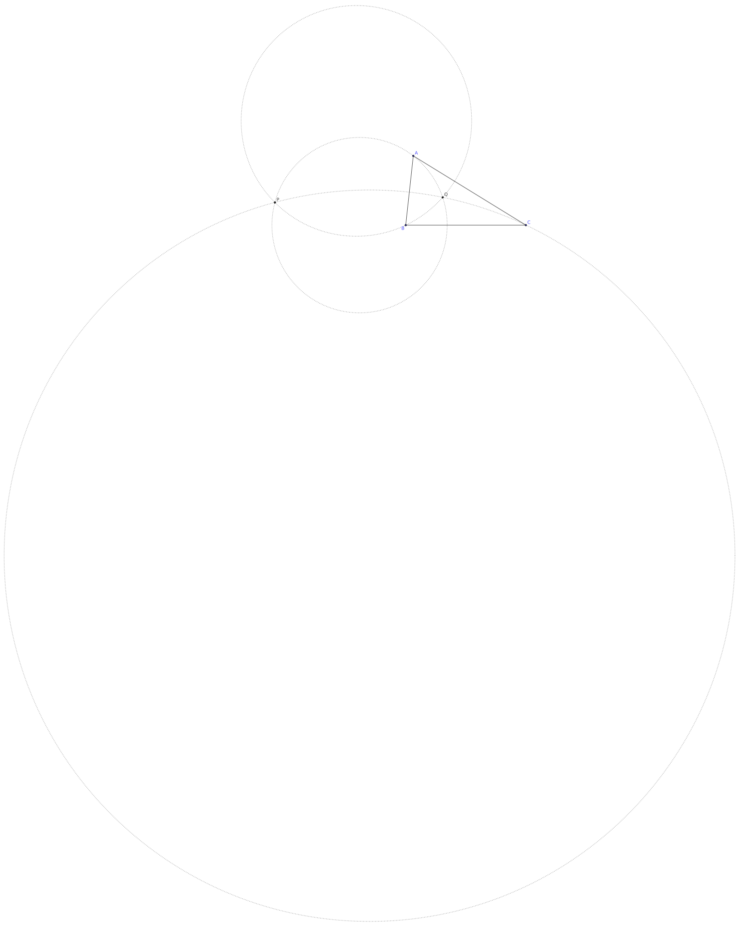

Figure 1.40

Theorem 1.5.10 (Pedal Triangles of Isodynamic Points) — *Prove that the pedal triangles of the isodynamic points are equilateral triangles. Also, Inverting around the Isodynamic Points trasnform $\triangle ABC$ into an equilateral triangle.*

Problem 1.5.1 (China TST 2018 T1P3) **:** Circle ω is tangent to sides AB,AC of triangle ABC at D,E respectively, such that $D \neq B$, $E \neq C$ and $BD + CE < BC$. F,G lies on BC such that

$BF = BD$, $CG = CE$. Let DG and EF meet at K. L lies on minor arc DE of ω, such that the tangent of L to ω is parallel to BC. Prove that the incenter of $\triangle ABC$ lies on KL.

Solution. Using Lemma 1.3.1, in the touch triangle of ω. □

Problem 1.5.2 (Vietnamese TST 2018 P6.a) **:** Triangle ABC circumscribed (O) has A-excircle (I_a) that touches AB, BC, AC at F, D, E, resp. M is the midpoint of BC. Circle with diameter MI_a cuts DE, DF at K, H. Prove that (BDK), (CDH) have an intersecting point on (I_a).

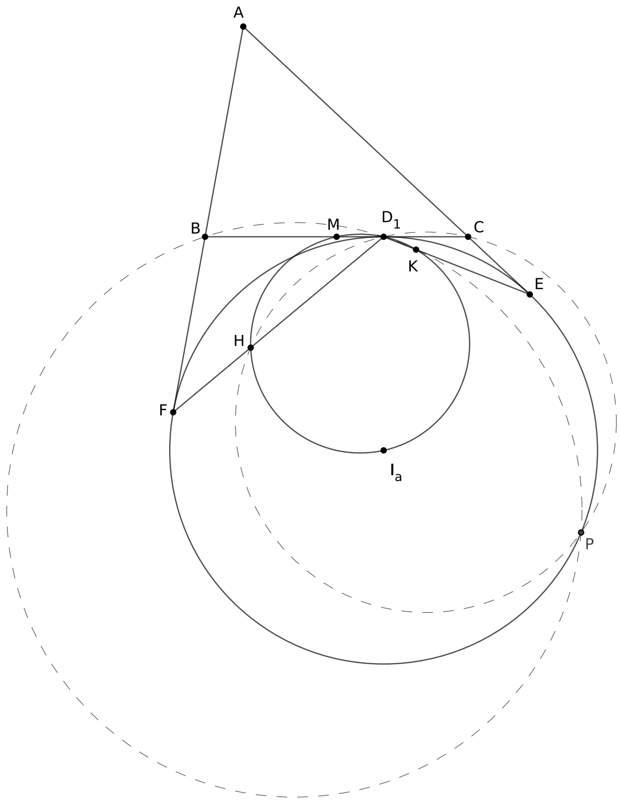

Figure 1.41

Solution [Inversion]***.*** Inverting around point D, we have:

Lemma— MD is a line, I_a is an arbitrary point such that $DI_a \perp MD$. l is the perpendicular bisector of DI_a. F, E are arbitrary points on l. $B = I_aF \cap MD, C = I_aE \cap MD$ $H = FD \cap MI_a, K = DE \cap MI_a$. Then BK, CH, l are concurrent.

Proof. It is straightforward using Puppus's Theorem on lines BDC and HI_aK. □

□

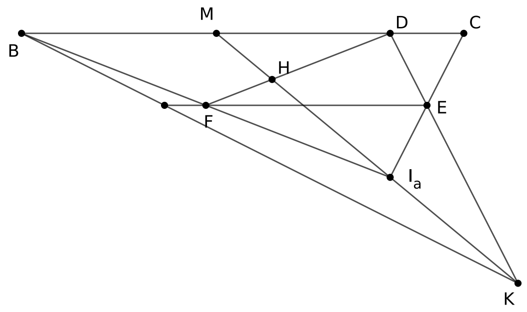

Figure 1.42: After inverting around D

Solution [Synthetic: Length Chase]***.***

Lemma— Let G, H, B', C' be defined the same way in Lemma 3.2. Prove that F lies on the radical axis of $\odot D'GI, D'C'H$. By extension prove that B lies on the radical axis of $\odot D'B'I, D'C'H$

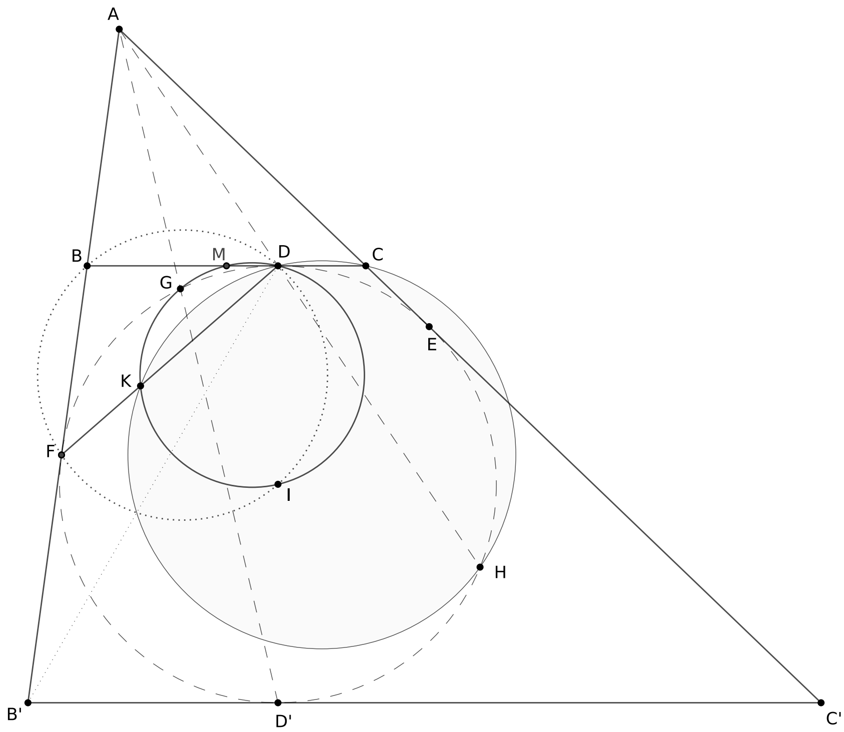

Figure 1.43: Vietnamese TST 2018 P6.a

We prove the first part, and the second part follows using spiral similarity.
Suppose $K \in FD \cap \odot KDI$. Due to spiral similarity on $\odot KDI, \odot(I)$, we have $\triangle GFK \sim \triangle GD'I$. Which implies:

$$\frac{FK}{GF} = \frac{ID}{GD'} \implies FK = ID\frac{GF}{GD'}$$

Now, if $KDCE$ is to be cyclic, we need to have $\triangle HFK \sim \triangle HDC$. So we need,

$$\frac{FK}{HF} = \frac{DC}{HD} \implies FK = DC\frac{HF}{HD}$$

Combining two equations:

$$\frac{GF}{GD'} \cdot \frac{ID}{DC} = \frac{HF}{HD}$$

Now, using Ptolemy's theorem in $\square FDEH$, we have,

$$FD \cdot EH + DE \cdot FH = DH \cdot EF$$
$$EH \cdot \frac{FD}{FH} + DE = EF \cdot \frac{DH}{FH}$$
$$2\frac{DE}{EF} = \frac{DH}{FH}$$

Similarly from $\square FGED'$ we get,

$$2\frac{D'E}{EF} = \frac{GD'}{FG}$$

Combining these two equations gives us the desired result. □

Generalization 1.5.2.1 (Vietnamese TST 2018 P6.a Generalization) **:** Let ABC be a triangle. The points D, E, F are on the lines BC, CA, AB respectively. The circles (AEF), (CFD), (CDE) have a common point P. A circle (K) passes through P, D meet DE, DF again at Q, R respectively. Prove that the circles (DBQ), (DCR) and (DEF) are coaxial.

Solution [Inversion]**.** Invert around D, and use Pappu's Theorem as in Problem 1.5.2. □

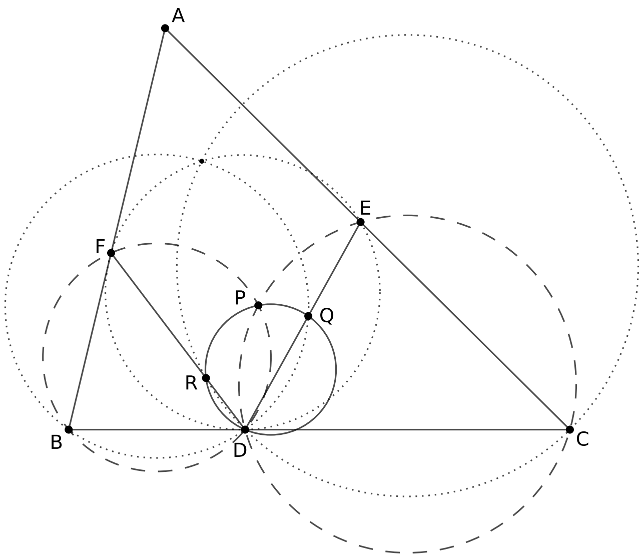

Figure 1.44: Vietnamese TST 2018 P6.a Generalization

Remark. The synthetic solution of Problem 1.5.2 can't be reproduced here maybe because here we don't have A, P, D collinear, and we can't have harmonic quadrilaterals either.

Theorem 1.5.11 (Poncelet's Porism) — *Poncelet's porism (sometimes referred to as Poncelet's closure theorem) states that whenever a polygon is inscribed in one conic section and circumscribes another one, the polygon must be part of an infinite family of polygons that are all inscribed in and circumscribe the same two conics.*

Problem 1.5.3 (IMO 2013 P3) **:** Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.

Solution. Straightforward use of **??** □

Problem 1.5.4 (buratinogigle's proposed probs for Arab Saudi team 2015) **:** Let ABC be acute triangle with $AB < AC$ inscribed circle (O). Bisector of $\angle BAC$ cuts (O) again at D. E is reflection of B through AD. DE cuts BC at F. Let (K) be circumcircle of triangle BEF. BD, EA cut (K) again at M, N, reps. Prove that $\angle BMN = \angle KFM$.

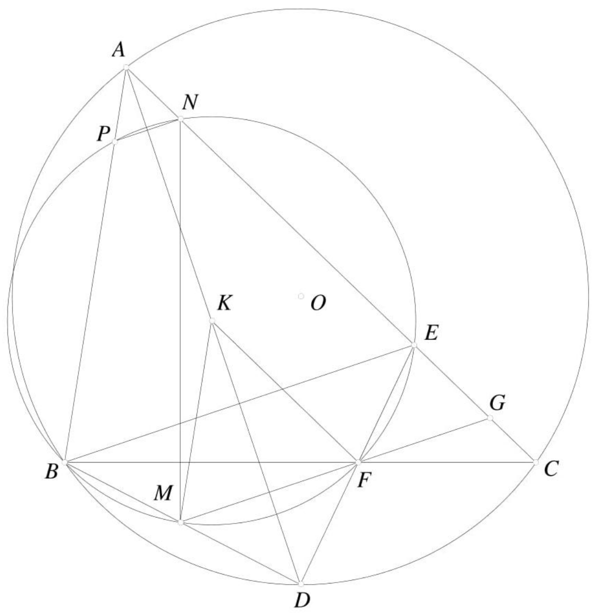

Figure 1.45

Problem 1.5.5 (USAMO 1999 P6) **:** Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles.

Problem 1.5.6 (Serbia 2018 P1) **:** Let $\triangle ABC$ be a triangle with incenter I. Points P and Q are chosen on segments BI and CI such that $2\angle PAQ = \angle BAC$. If D is the touch point of incircle and side BC prove that $\angle PDQ = 90$.

Solution. Straightforward Trig application. □

Problem 1.5.7 (Iran TST T2P5) **:** Let ω be the circumcircle of isosceles triangle ABC ($AB = AC$). Points P and Q lie on ω and BC respectively such that $AP = AQ$ .AP and BC intersect at R. Prove that the tangents from B and C to the incircle of $\triangle AQR$ (different from BC) are concurrent on ω.

Problem 1.5.8 () **:** Let a point P inside of $\triangle ABC$ be such that the following condition is satisfied

$$\frac{AP + BP}{AB} = \frac{BP + CP}{BC} = \frac{CP + AP}{CA}$$

Lines AP, BP, CP intesect the circumcirle again at A', B', C'. Prove that ABC and A', B', C' have the same incircle.

Solution. After finiding the point P, we get a lot of ideas.

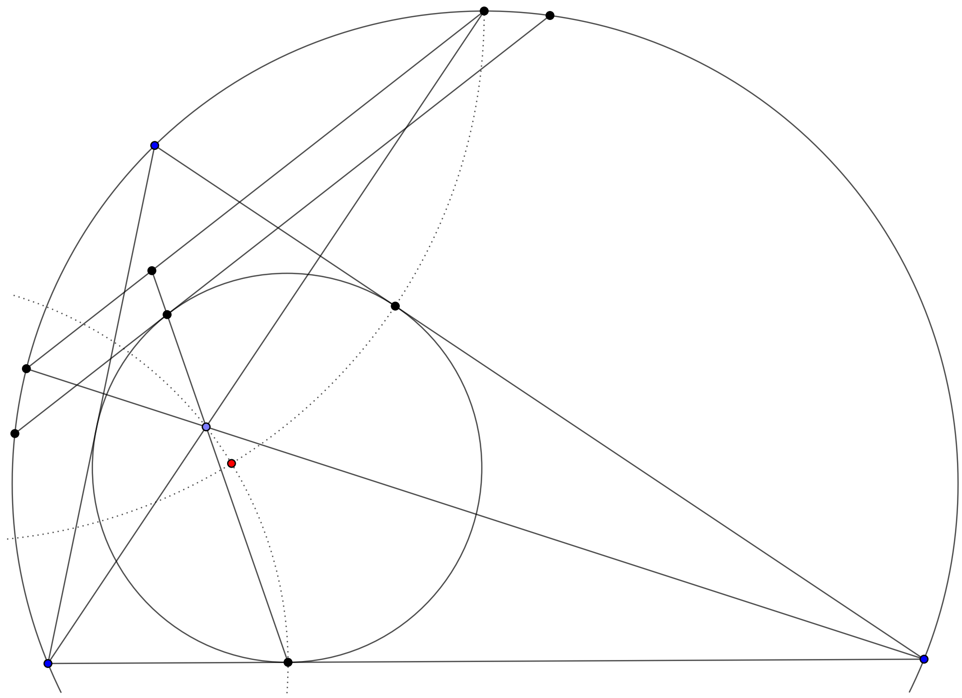

Figure 1.46: two lines are parallel

□

Problem 1.5.9 (Iran TST 2018 P3) **:** In triangle ABC let M be the midpoint of BC. Let ω be a circle inside of ABC and is tangent to AB, AC at E, F, respectively. The tangents from M to ω meet ω at P, Q such that P and B lie on the same side of AM. Let $X \equiv PM \cap BF$ and $Y \equiv QM \cap CE$. If $2PM = BC$ prove that XY is tangent to ω.

Solution. Work backwards □

Problem 1.5.10 (Iran TST 2018 P4) **:** Let ABC be a triangle ($\angle A \neq 90°$). BE, CF are the altitudes of the triangle. The bisector of $\angle A$ intersects EF, BC at M, N. Let P be a point such that $MP \perp EF$ and $NP \perp BC$. Prove that AP passes through the midpoint of BC.

Problem 1.5.11 (APMO 2018 P1) **:** Let H be the orthocenter of the triangle ABC. Let M and N be the midpoints of the sides AB and AC, respectively. Assume that H lies inside the quadrilateral $BMNC$ and that the circumcircles of triangles BMH and CNH are tangent to each other. The line through H parallel to BC intersects the circumcircles of the triangles BMH and CNH in the points K and L, respectively. Let F be the intersection point of MK and NL and let J be the incenter of triangle MHN. Prove that $FJ = FA$.

Problem 1.5.12 (ISL 2006 G6) **:** Circles w_1 and w_2 with centres O_1 and O_2 are externally tangent at point D and internally tangent to a circle w at points E and F respectively. Line t is the common tangent of w_1 and w_2 at D. Let AB be the diameter of w perpendicular to t, so that A, E, O_1 are on the same side of t. Prove that lines AO_1, BO_2, EF and t are concurrent.

Solution. This □

Lemma 1.5.12 (Tangential Quadrilateral Incenters) — *Let $ABCD$ be a tangential quatrilateral. Let I_1, I_2 be the incenters of $\triangle ABD, \triangle BCD$. Then $(I_1), (I_2)$ is tangent to BD at the same point.*

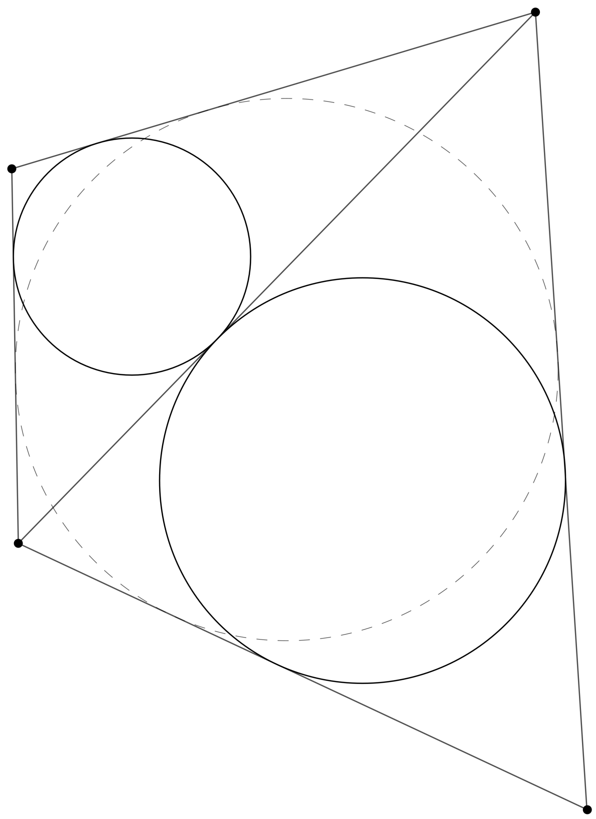

Figure 1.47

Problem 1.5.13 (Four Incenters in a Tangential Quadrilateral) **:** Let $ABCD$ be a quadrilateral. Denote by X the point of intersection of the lines AC and BD. Let I_1, I_2, I_3, I_4 be the centers of the incircles of the triangles XAB, XBC, XCD, XDA, respectively. Prove that the quadrilateral $I_1I_2I_3I_4$ has a circumscribed circle if and only if the quadrilateral $ABCD$ has an inscribed circle.

Solution. There is a lot going on in this figure, firstly, the J_1, J_2 and M, then K, then $\angle I_4ME = \angle I_3ME$. Connecting them with the lemma.

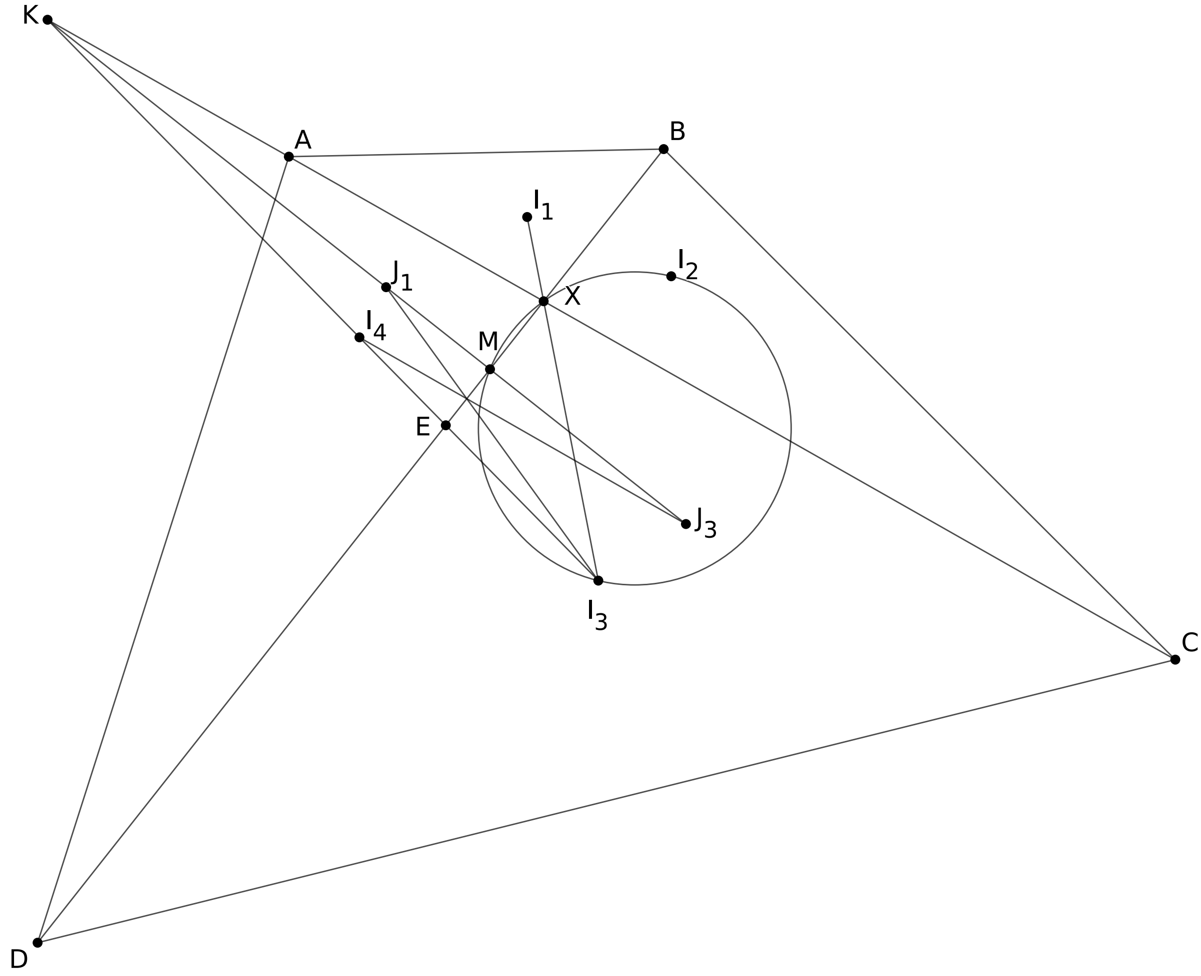

Figure 1.48

□

Problem 1.5.14 (Geodip) **:** Let G be the centeroid. Dilate $\odot I$ from G with constant -2 to get I'. Then I' is tangent to the circumcircle.

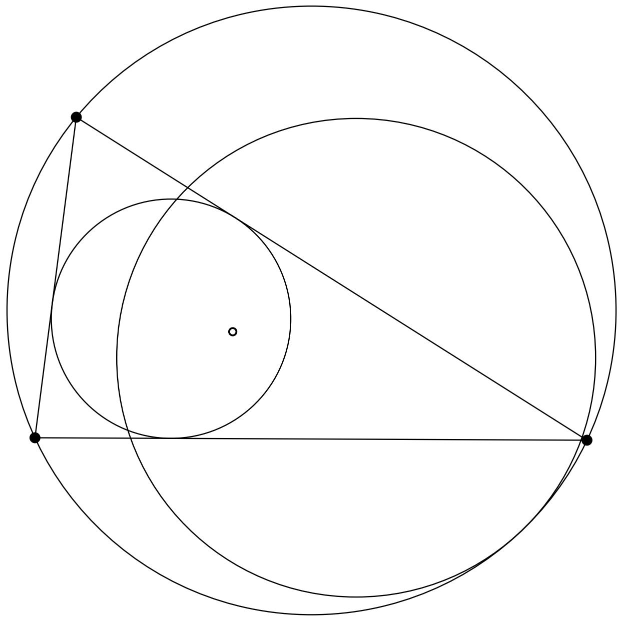

Figure 1.49

Theorem 1.5.13 (Fuhrmann Circle) — *Let X', Y', Z' be the midpoints of the arcs not containing A, B, C of $\odot ABC$. Let X, Y, Z be the reflections of these points on the sides. Then $\odot XYZ$ is called the* **Fuhrmann Circle**. *The orthocenter H and the nagel point N lies on this circle, and HN is a diameter of this circle.*
Furthermore, AH, BH, CH cut the circle for the second time at a distance $2r$ from the vertices.

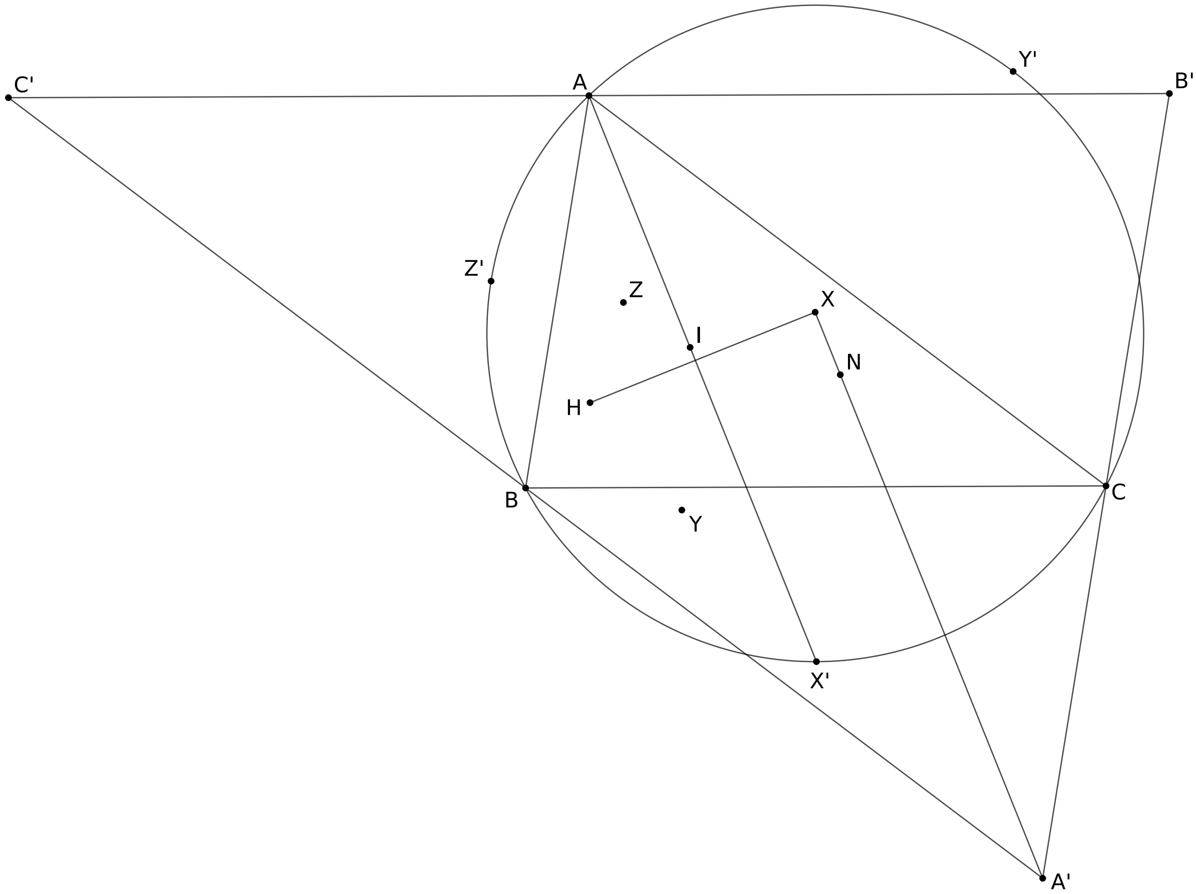

Figure 1.50: Fuhrmann Circle

Problem 1.5.15 (Iran TST 2008 P12) **:** In the acute-angled triangle ABC, D is the intersection of the altitude passing through A with BC and I_a is the excenter of the triangle with respect to A. K is a point on the extension of AB from B, for which $\angle AKI_a = 90^\circ + \frac{3}{4}\angle C$. I_aK intersects the extension of AD at L. Prove that DI_a bisects the angle $\angle AI_aB$ iff $AL = 2R$. (R is the circumradius of ABC)

Solution. □

1.6 Conjugates

1.6.1 Isogonal Conjugate

Theorem 1.6.1 (Isogonal Line Lemma) — *Let AP, AQ are isogonal lines with respect to $\angle BAC$. Let $BP \cap CQ = F$ and $BQ \cap CP = E$. Then AE, AF are isogonal lines with respect to $\angle BAC$.*

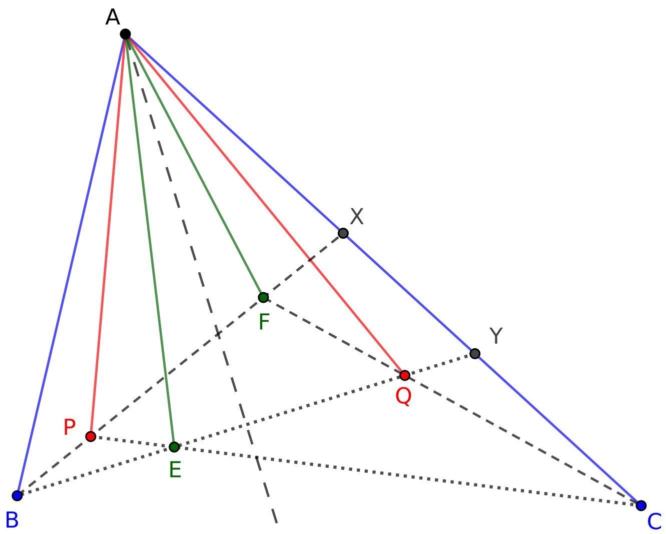

Figure 1.51

Proof.

$$A(B, F; P, X) = (B, F; P, X) = C(B, Q; E, X) = (B, Q; E, X) = (X, E; Q, B)$$

So if we define a projective transformation that swaps isogonal lines wrt $\angle BAC$, we see AE, AF are conjugates of each other. □

Problem 1.6.1 (India Postals 2015 Set 2) **:** Let $ABCD$ be a convex quadrilateral. In the triangle ABC let I and J be the incenter and the excenter opposite the vertex A, respectively. In the triangle ACD let K and L be the incenter and the excenter opposite the vertex A, respectively. Show that the lines IL and JK, and the bisector of the angle BCD are concurrent.

Solution. Using Theorem 1.6.1 □

Lemma 1.6.2 — *Let ω_1, ω_2 be two circles such that ω_1 passes through A, B and is tangent to AC at A. ω_2 is defined similarly by swapping B with C. $\omega_1 \cap \omega_2 = X$.*
Let γ_1, γ_2 be two circles such that γ_1 passes through A, B and is tangent to BC at B. γ_2 is defined similarly by swapping B with C. $\gamma_1 \cap \gamma_2 = Y$.
Then X, Y are isogonal conjugates wrt $\triangle ABC$.

Lemma 1.6.3 (Isogonality in quadrilateral) — *For a point X, its isogonal conjugate wrt a quadrilateral $ABCD$ exists iff*

$$\measuredangle BXA + \measuredangle DXC = 180^\circ$$

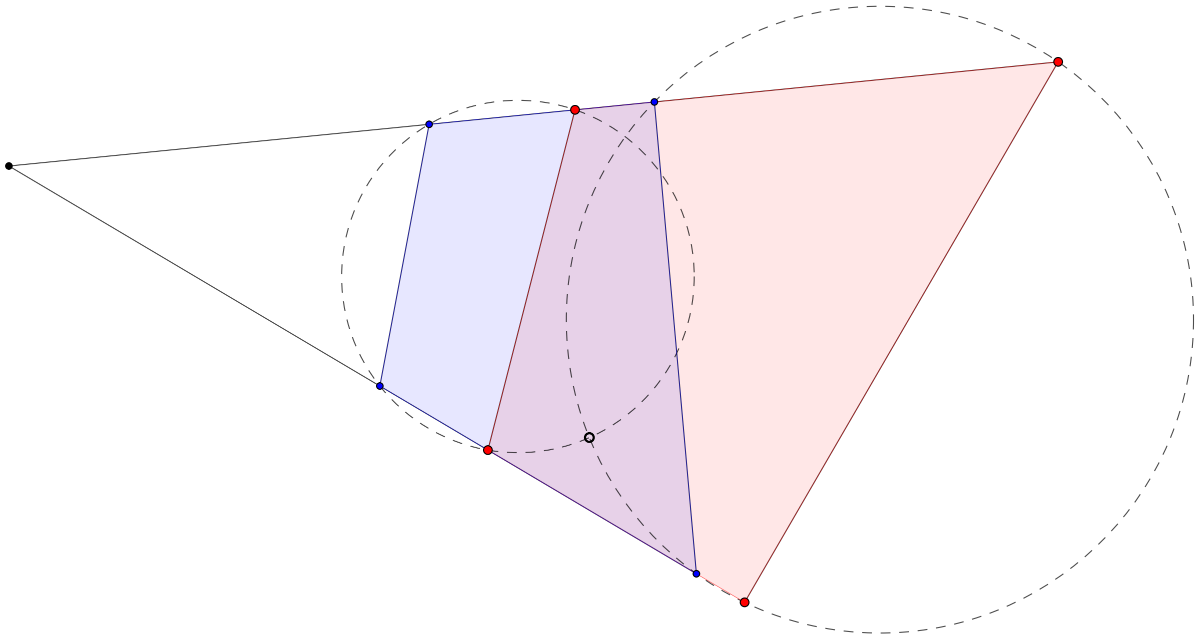

Figure 1.52: Isogonality in quadrilateral

Solution. Draw the cirles, look for similarity. □

Lemma 1.6.4 (Ratio) — *Given a $\triangle ABC$ with isogonal conjugate P, Q. Let AP, AQ cut the circumcircle of $\triangle ABC$ again at U, V, respectively and let $D \equiv AP \cap BC$. Then*

$$\frac{AQ}{QV} = \frac{PD}{DU}$$

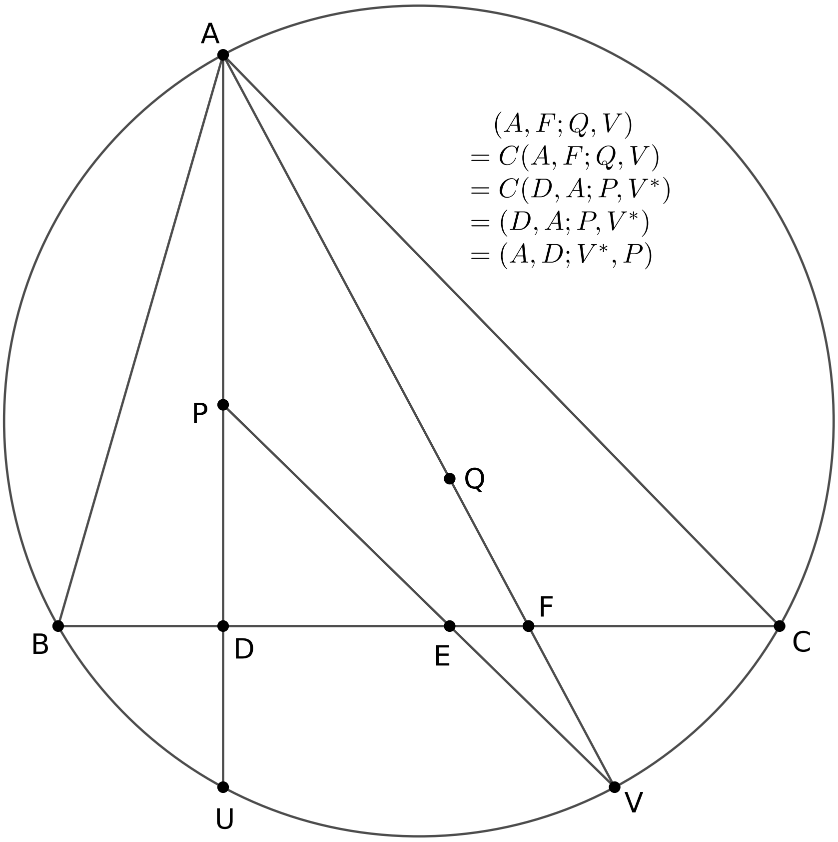

Figure 1.53

1.6.1.1 Symmedians

Definition (Symmedians) — In $\triangle ABC$, let T_a, T_b, T_c be the meet points of the tangents at A, B, C. Let $\triangle N_aN_bN_c$ be the cevian triangle of AT_a, BT_b, CT_c. Let S be the symmedian point of $\triangle ABC$. Let M_a, M_b, M_c be the midpoints of BC, CA, AB.

Lemma 1.6.5 (Most Important Symmedian Property) — *Let the circles tangent to AC, AB at A and passes through B, C respectively meet at T' for the second time. Let $AT_a \cap \odot ABC = A'$. Let the tangents to $\odot ABC$ at A, A' meet BC at T. Prove that, A, T', T_a, and T, T', O are collinear.*

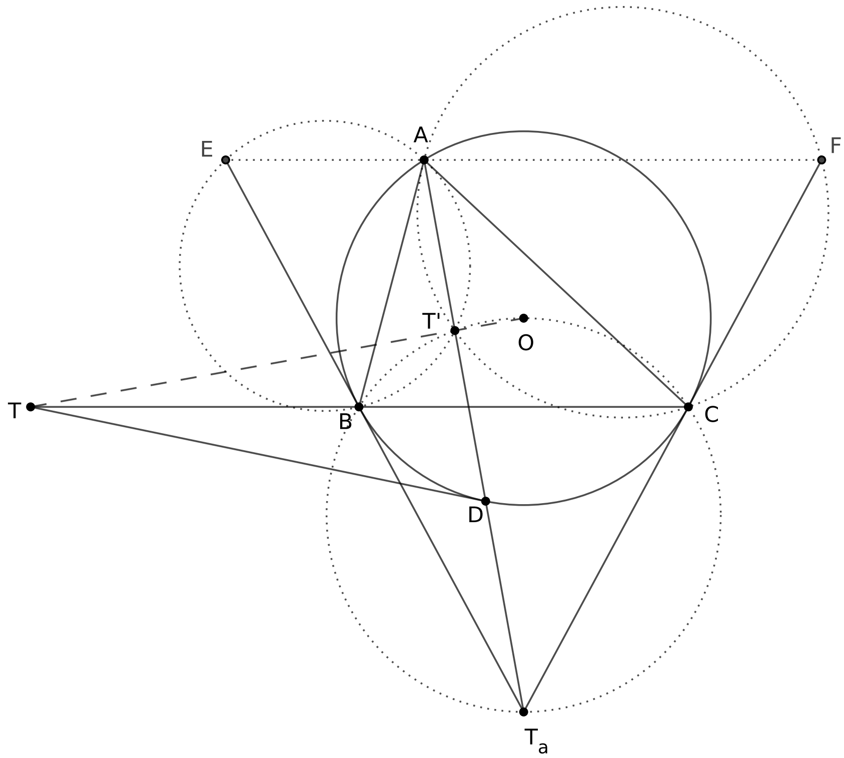

Figure 1.54: T' is quite special!

Problem 1.6.2 (USAMO 2008 P2) **:** Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of BC, CA, and AB, respectively. Let the perpendicular bisectors of AB and AC intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.

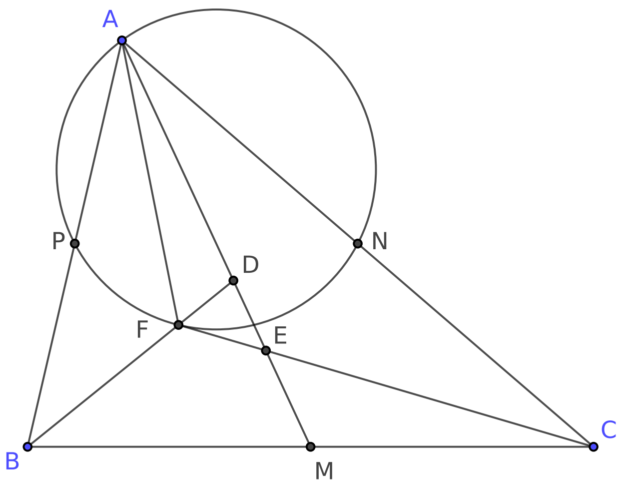

Figure 1.55: USAMO 2008 P2

Solution [Phantom Point]**.** First assume $F \in BD$, and $F = T'$ (Where T' comes from Lemma 1.6.5, and prove that $F \in CE$.) $\square$

Solution [Isogonal Conjugate]**.** Construct the isogonal conjugate of F, which is the intersection of the circles touching BC and passing through A, B and A, C. $\square$

Solution. Using Theorem 1.6.1 by taking the reflections of B, C over D, F $\square$

Problem 1.6.3 (IRAN TST 2015 Day 3, P3) **:** AH is the altitude of triangle ABC and H' is the reflection of H trough the midpoint of BC. If the tangent lines to the circumcircle of ABC at B and C, intersect each other at X and the perpendicular line to XH' at H', intersects AB and AC at Y and Z respectively, prove that $\angle ZXC = \angle YXB$.

Problem 1.6.4 (Two Symmedian Points) **:** Let E, F be the feet of B, C-altitudes. Let K, K_A be the symmedian points of $\triangle ABC, \triangle AEF$. Prove that $KK_A \perp BC$, $KK_A \cap BC = P$ and $KK_A = KP$

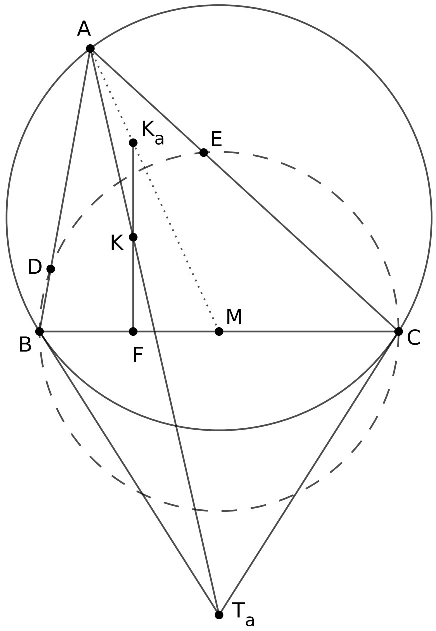

Figure 1.56: $KK_A \perp BC$

1.6.2 Isotonic Conjugate

Theorem 1.6.6 (Isotonic Lemma) — *Let M be the midpoint of BC, and PQ such that Q is the reflection of P on M. Two points Q, R on AP, AQ, $BQ \cap CR = X$, $BR \cap CQ = Y$. Then AX, AY are isotonic wrt BC.*

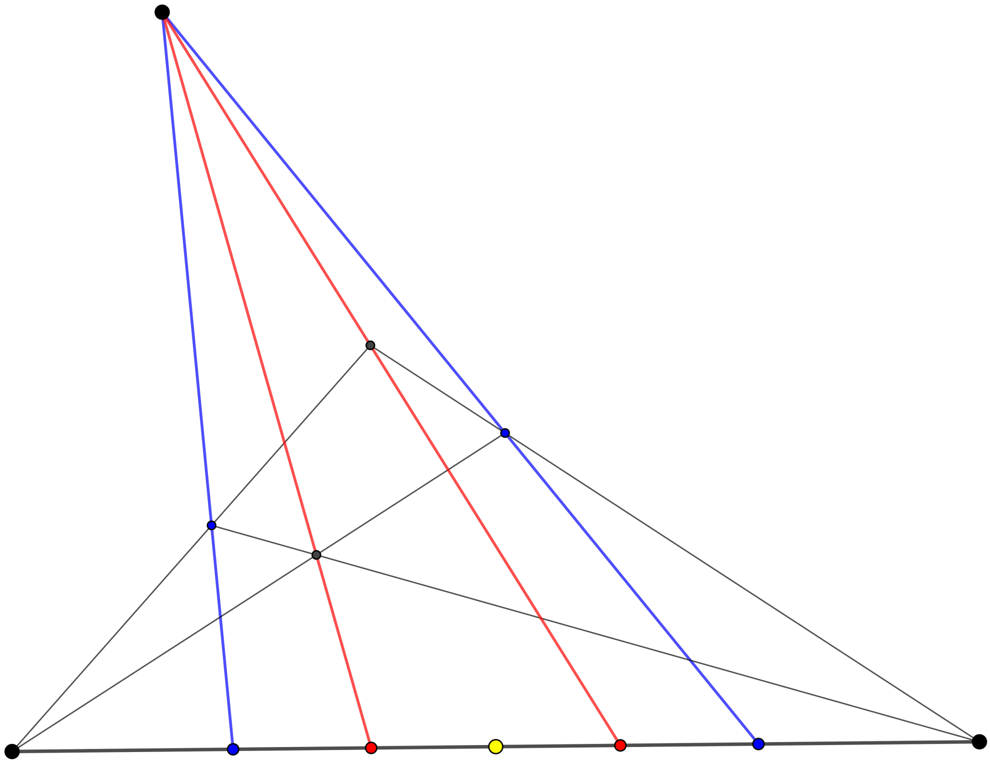

Figure 1.57

Problem 1.6.5 (IGO 2014 S5) **:** Two points P and Q lying on side BC of triangle ABC and their distance from the midpoint of BC are equal.The perpendiculars from P and Q to BC intersect AC and AB at E and F,respectively.M is point of intersection PF and EQ.If H_1 and H_2 be the orthocenters of triangles BFP and CEQ, respectively, prove that $AM \perp H_1H_2$.

Solution. We first show that the slope of H_1H_2 is fixed, and then show that AM is fixed where we use isotonic lemma, and finally show that these two lines are perpendicular. $\square$

1.6.3 Reflection

Lemma 1.6.7 (Homothety and Reflection) — *Let two oppositely oriented congruent triangles be $\triangle ABC, \triangle DEF$. Prove that the midpoints of AD, BE, CF are collinear.*

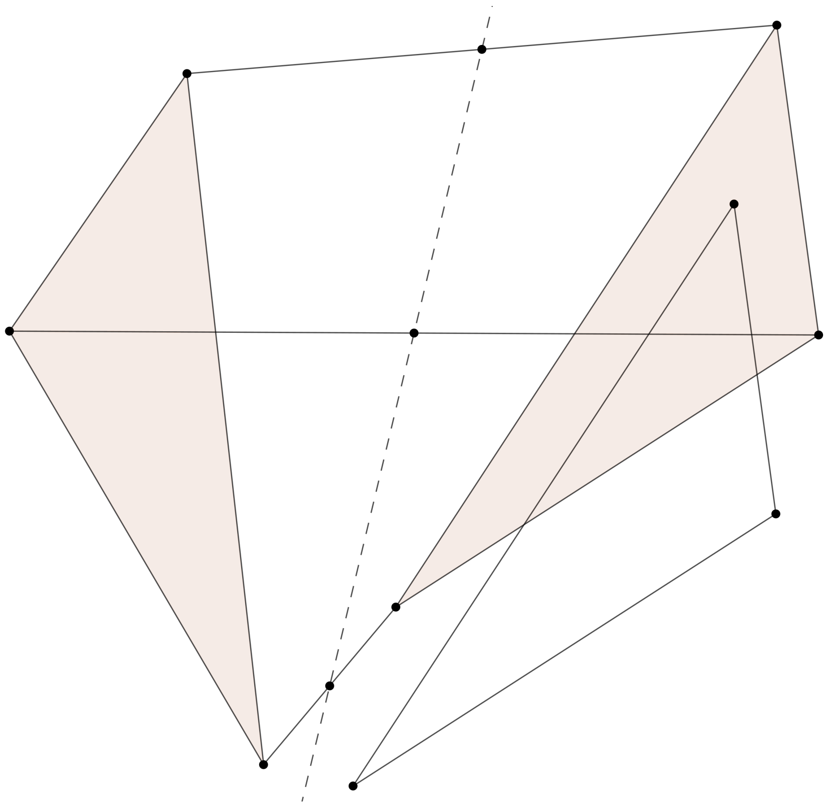

Figure 1.58: Oppositely oriented congruent triangles

Problem 1.6.6 (Autumn Tournament, 2012) **:** Let two oppositely oriented equilateral triangles be $\triangle ABC, \triangle DEF$. What is the least possible value of $\max(AD, BE, CF)$?

1.7 Mixtilinear–Curvilinear–Normal In-Excricles

Definition (Mixtilinear Circle) — Let $\triangle ABC$ be an ordinary triangle, I is its incenter, D is the touch points of the incenter with BC. Let ω be the mixtilinear incircle. Let it touch CA, AB at E, F. Furthermore, let $\omega \cap \odot ABC \equiv T$. Let M_a, M_b, M_c be the midpoints of the smaller arcs BC, CA, AB, and M_A, M_B, M_C be the midpoints of the major arcs BC, CA, AB.

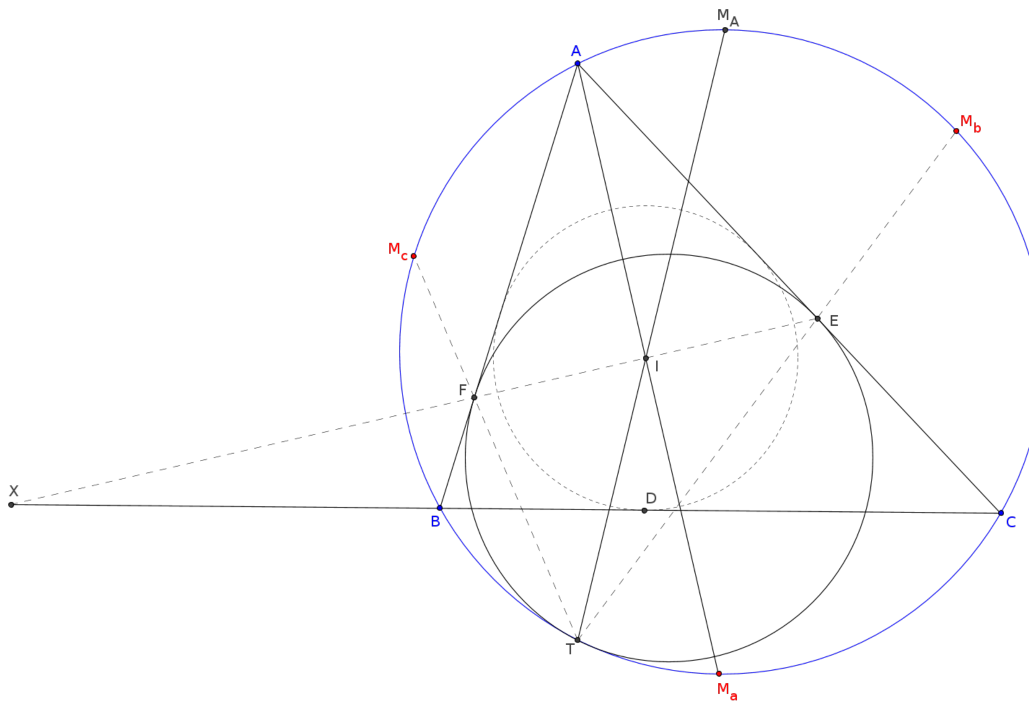

Figure 1.59: Mixtilinear Incircle: Construction

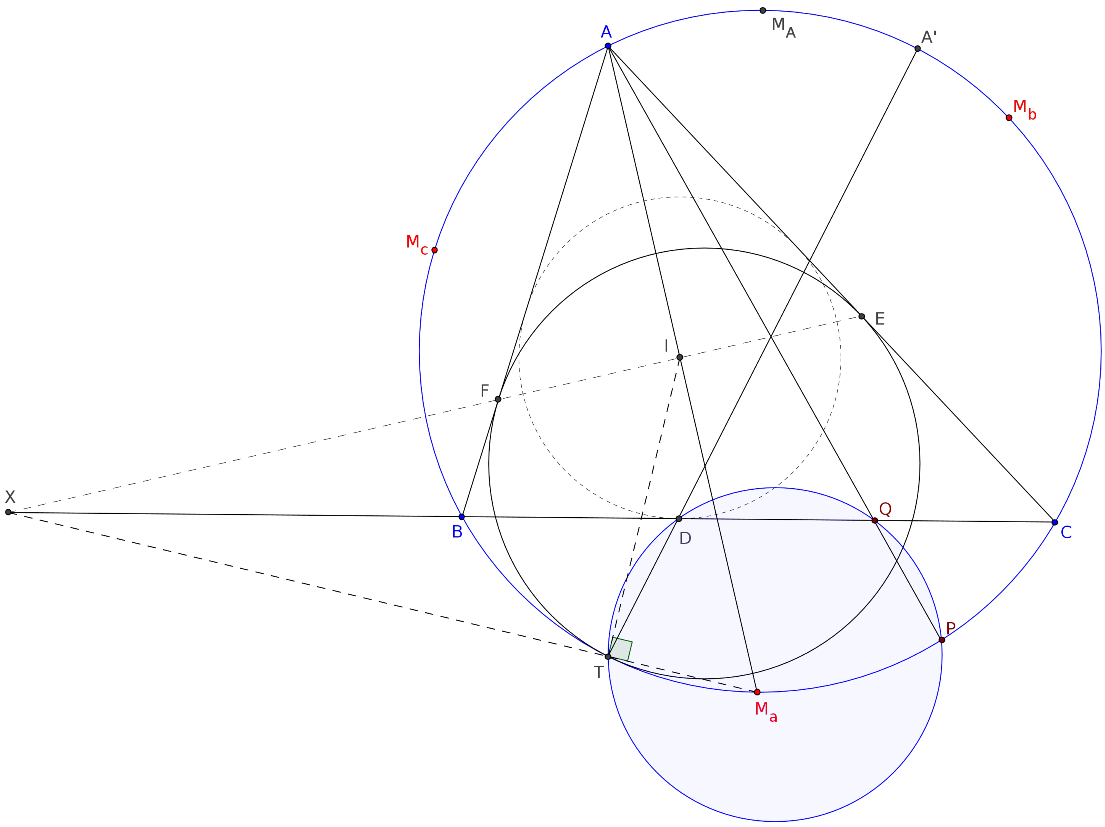

Figure 1.60: Mixtilinear Incircle: Circlicity Lemmas

Solution. List of small proofs

1. E, I, F are collinear. Consider the circle $TI'EC$ and do some angle chasing.
2. T, I, M_A are collinear. Consider the circle $TIEC$ and apply reim's theorem.

□

Lemma 1.7.1 — $\frac{TM_c}{M_cA} = \frac{TM_b}{M_bA}$, *in other words, the bundle* $(A, T; M_b, M_c)$ *is harmonic. And* TA *is a symmedian of* $\triangle TM_cM_b$.

Lemma 1.7.2 — *Let* X *be a variable point on the arc* AB, *and let* O_1 *and* O_2 *be the incenters of the triangles* CAX *and* CBX. *Then* X, O_1, O_2 *and* T *lie on a circle.*

Solution. Using similarity and this lemma. □

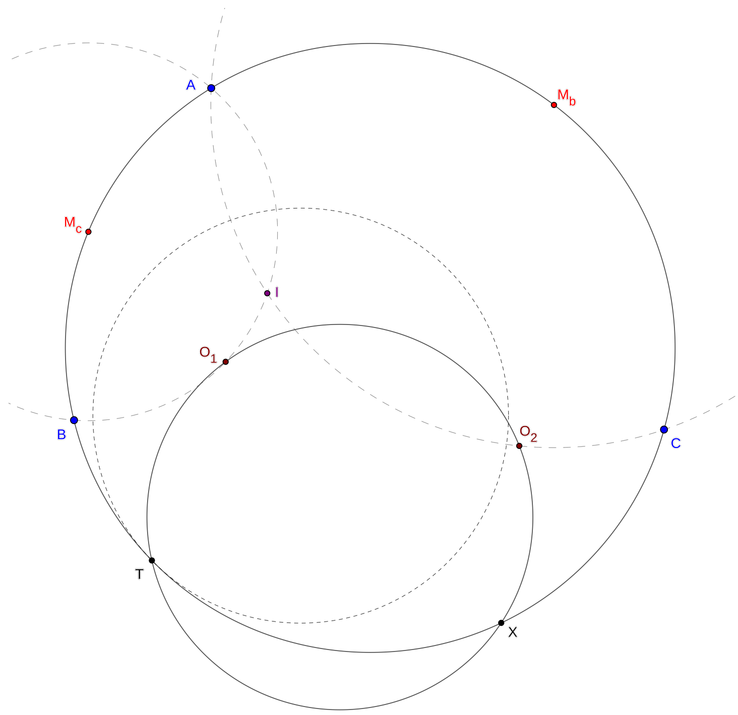

Figure 1.61: The two incenters are cyclic with T, X

Problem 1.7.1 (ISL 1999 G8) **:** Given a triangle ABC. The points A, B, C divide the circumcircle Ω of the triangle ABC into three arcs BC, CA, AB. Let X be a variable point on the arc AB, and let O_1 and O_2 be the incenters of the triangles CAX and CBX. Prove that the circumcircle of the triangle XO_1O_2 intersects the circle Ω in a fixed point.

Solution. This is actually this lemma. □

Problem 1.7.2 (AoPS1) **:** Let $ABCD$ be a quadrilateral inscribed in a circle, such that the inradius of $\triangle ABC$ and ACD are the same. Let T be the touchpoint of A-mixtilinear incircle of the triangle ABD with $\odot ABCD$. Let I_1, I_2 be the incenters of the triangles ABC, ACD respectively. Show that I_1I_2 and the tangents of A, T wrt $\odot ABCD$ are concurrent.

Solution. The main problem is how to relate the two mixtilinear touchpoints to the two incenters. But with our mixtilinear lemma, we can do that easily. □

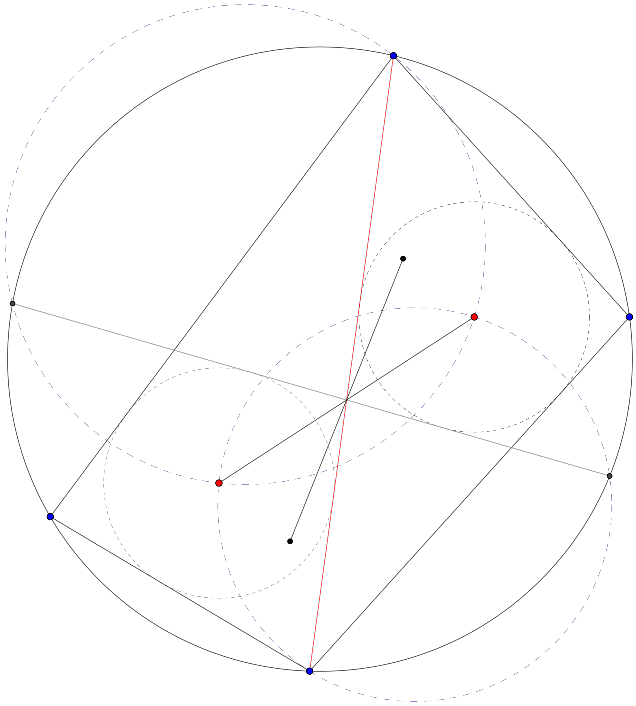

Figure 1.62

Problem 1.7.3 (Generalization of Mixtilinear Incirlce) **:** Consider triangle ABC and let M, N are midpoints of arcs AB, AC. Let E, F on AB, AC such that $EF \parallel MN$. Let EM, FN meet (ABC) second time at P, Q. Consider two intersection points E', F' of $(EFPQ)$ with AB, AC different from E, F. Then $EF' \cap E'F$ is the incenter of ABC.

Problem 1.7.4 () **:** Let the B-mixtilinear and C-mixtilinear circles touch BC at X_B, X_C respectively. Then X_B, X_C, T, M_a lie on a circle

Problem 1.7.5 (Taiwan TST 2014 T3P3) **:** Let M be any point on the circumcircle of $\triangle ABC$. Suppose the tangents from M to the incircle meet BC at two points X_1 and X_2. Prove that T, M, X_1, X_2 lie on a circle.

Problem 1.7.6 (Archer - EChen M1P3) **:** Let the incenter touch BC at D. Let $AI \cap BC = E$, $AI \cap \odot ABC = F$. Let $\odot DEF \cap \odot ABC = X$, $\odot DEF \cap \odot(I_a) = S_1, S_2$. Prove that AX goes through either S_1 or S_2.

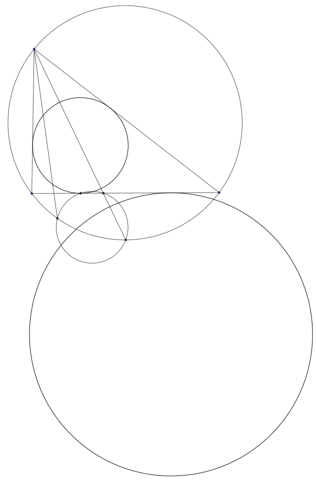

Figure 1.63

Definition (Curvilinear Incircle) — Let $ABCD$ be a cyclic quadrilateral. AC meets BD at X. We call the circle that touches AX, BX and the circumcircle from the inside a curvilinear incircle.

Lemma 1.7.3 — *Let the previously defined curvilinear incircle touch AX, BX at P, Q resp. And let the incircle of $\triangle ABD$ be I. Then P, Q, I are collinear.*

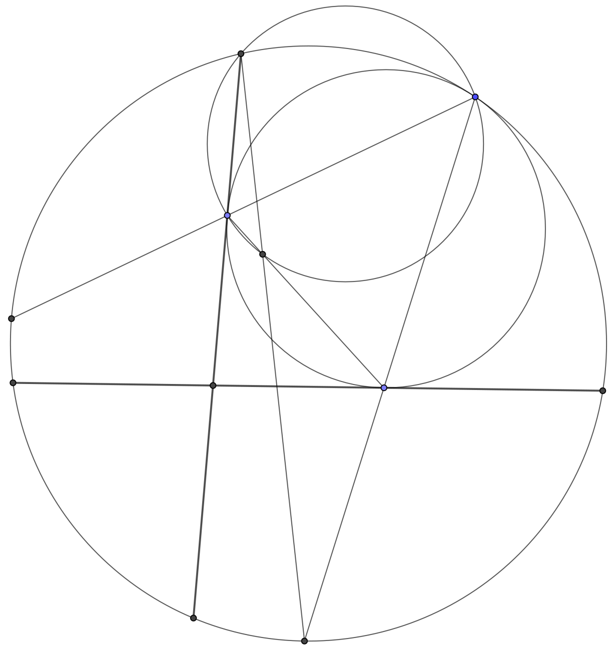

Figure 1.64

Solution. Notice that this is similar to the circles $TIEC$ and $TIFB$ in the mixtilinear circle figures. □

Theorem 1.7.4 (Sawayama and Thebault's theorem) — *Through the vertex A of a triangle ABC, a straight line AD is drawn, cutting the side BC at D. I is the center of the incircle of triangle ABC. Let P be the center of the circle which touches DC, DA at E, F, and the circumcircle of ABC, and let Q be the center of a further circle which touches DB, DA in G, H and the circumcircle of ABC. Then P, I and Q are collinear*

1.8 Circles and Radical Axises

Problem 1.8.1 () **:** In $\triangle ABC$, H is the orthocenter, and AD, BE are arbitrary cevians. Let ω_1, ω_2 denote the circles with diameters AD, BE resp. HD, HE meet ω_1, ω_2 again at F, G. DE meet ω_1, ω_2 again at P_1, P_2. FG meet ω_1, ω_2 again at Q_1, Q_2. P_1H, P_2H meet ω_1, ω_2 at R_1, R_2 and Q_1H, Q_2H meet ω_1, ω_2 at S_1, S_2. $P_1Q_1 \cap P_2Q_2 \equiv X$ and $R_1S_1 \cap R_2S_2 \equiv Y$. Prove that X, Y, H are collinear.

Solution. Too much info... □

Lemma 1.8.1 (Pseudo Miquel's Theorem) **—** *In a $\triangle ABC$ let E, F be points on AC, AB and D be a point on $\odot(ABC)$. Let $X = \odot(BFD) \cap \odot(CED)$ then E, F, X are collinear.*

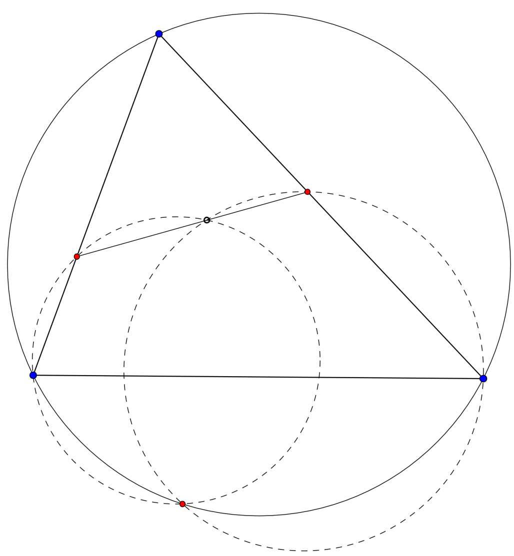

Figure 1.65: Notice the collinearity

Problem 1.8.2 (buratinogigle's proposed problems for Arab Saudi team 2015) **:** Let ABC be a triangle and (K) is a circle that touches segments CA, AB at E, F, reps. M, N lie on (K) such that BM, CN are tangent to (K). G, H are symmetric of A through E, F. The circle passes through G and touches to (K) at N that cuts CA again at S. The circle passes through H and touches (K) at M that cuts AB again at T. Prove that the line passes through K and perpendicular to ST always passes through a fixed point when (K) changes.

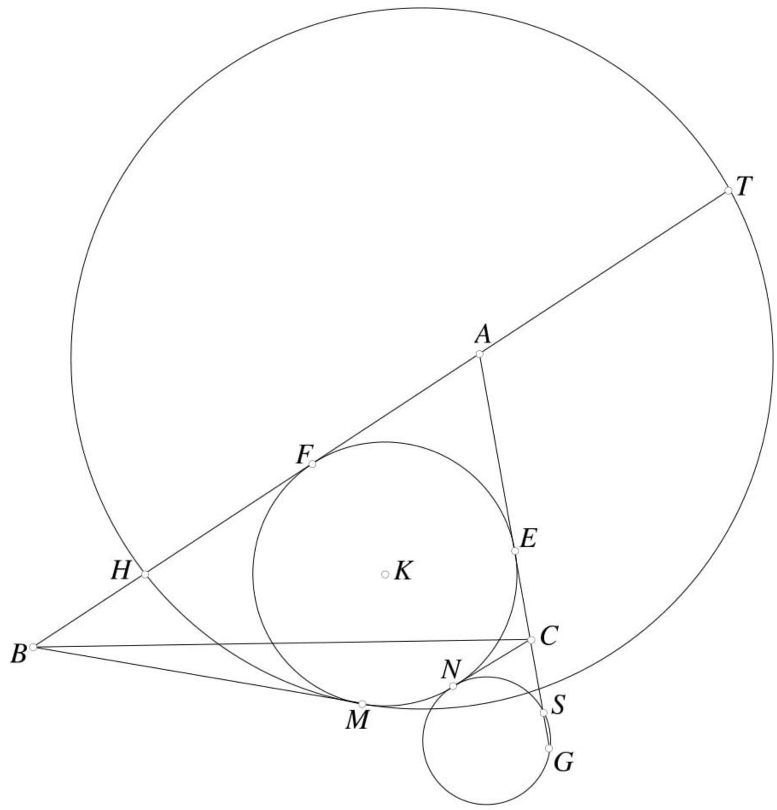

Figure 1.66

Problem 1.8.3 (ISL 2002 G8) **:** Let two circles S_1 and S_2 meet at the points A and B. A line through A meets S_1 again at C and S_2 again at D. Let M, N, K be three points on the line segments CD, BC, BD respectively, with MN parallel to BD and MK parallel to BC. Let E and F be points on those arcs BC of S_1 and BD of S_2 respectively that do not contain A. Given that EN is perpendicular to BC and FK is perpendicular to BD prove that $\angle EMF = 90°$.

Solution. When one single property can produce a lot others, and we need to prove this property, assume the property to be true and work backwards. □

Problem 1.8.4 (APMO 1999 P3) **:** Let Γ_1 and Γ_2 be two circles intersecting at P and Q. The common tangent, closer to P, of Γ_1 and Γ_2 touches Γ_1 at A and Γ_2 at B. The tangent of Γ_1 at P meets Γ_2 at C, which is different from P, and the extension of AP meets BC at R. Prove that the circumcircle of triangle PQR is tangent to BP and BR.

Problem 1.8.5 (USA TST 2019 P1) **:** Let ABC be a triangle and let M and N denote the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Let X be a point such that $\overline{AX}$ is tangent to the circumcircle of triangle ABC. Denote by ω_B the circle through M and B tangent to $\overline{MX}$, and by ω_C the circle through N and C tangent to $\overline{NX}$. Show that ω_B and ω_C intersect on line BC.

Solution [Spiral Similarity]**.** Let $\omega_C \cap BC = P$. If we extend NP to meet AB at R, we get $XANR$ cyclic. Similarly, if $\odot XAM \cap AC = Q$, then we have to prove $QM \cap NR = P$.

Suppose $QM \cap NR = P'$. Then by spiral similarity, X takes $Q \to M$ and $N \to R$. It also takes $Q \to N$ and $M \to R$. So $XMP'R$ is cyclic. We now show that $XMPR$ is also cyclic, which will prove $P = P'$.

Let $T = \odot ABC \cap \odot XAN$. By spiral similarity, T takes $R \to B$ and $N \to C$. It also takes $R \to N$ and $B \to C$, which means $RBPT$ is cyclic.

By spiral similarity, we have, $\triangle TXA \sim \triangle TNC$, $\triangle TXN \sim \triangle TAC$, $\triangle TBA \sim \triangle TPN$ Which implies,

$$\frac{XN}{TN} = \frac{AC}{TC}, \frac{XA}{TA} = \frac{NC}{TC}$$
$$\implies \frac{XN}{XA} = 2\frac{TN}{TA}$$

And so,

$$\frac{AB}{TA} = \frac{NP}{TN} \implies \frac{2AM}{NP} = \frac{TA}{TN} = \frac{XA}{XN}2$$
$$\implies \frac{AM}{NP} = \frac{XA}{XN}$$

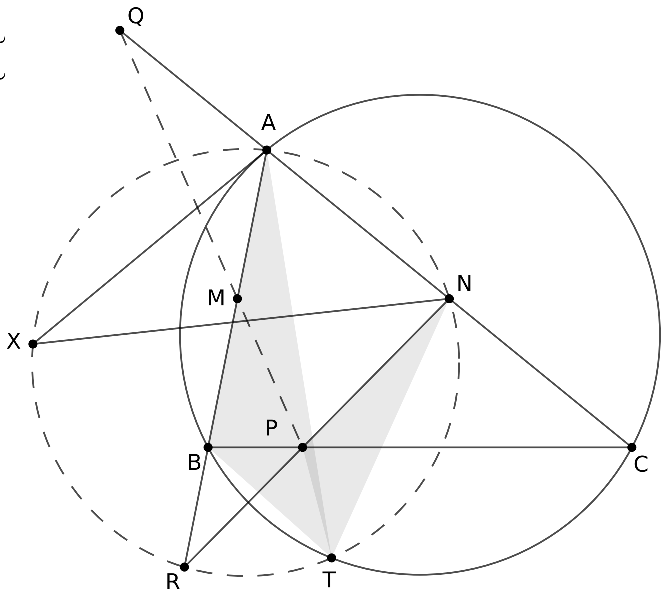

Figure 1.67

Which means $\triangle XAM \sim \triangle XNP$ since $\angle XAM = \angle XNP$, which concludes the proof. $\square$

Solution [Clever Observation]**.** Reflect A over X to A'. Draw the circle with center X with radius XA. Call it ω. Let $P = \omega \cap \odot ABC$. Let $Q = A'B \cap \omega$.

We will show that M, P, B, Q are cyclic, and XM is tangent to the circle.

First, we have $AQ \perp A'B$. So $MB = MQ$. Now,

$$\begin{aligned}\measuredangle MPQ &= \measuredangle APQ - \measuredangle APM \\ &= \measuredangle AA'Q - \measuredangle ANM \\ &= \measuredangle AXM - \measuredangle XAM \\ &= \measuredangle AMX \\ &= \measuredangle MBQ\end{aligned}$$

So M, Q, P, B is cyclic. Also since $MQ = MB$, and $XM \parallel BQ$, XM is tangent to $\odot MPBQ$, and $\odot MPBQ = \omega_B$.

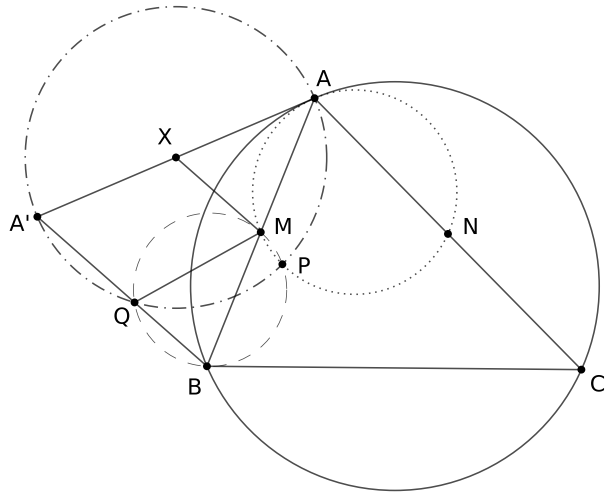

Figure 1.68

Similarly ω_C passes through P, and by Miquel's theorem, their intersection lies on BC. $\square$

Problem 1.8.6 (USA TST 2019 P1 parallel problem) **:** Pick a point X such that AX is parallel to BC. Let M, N be the midpoints of AB, AC. Let w_b be the circle passing through M and B tangent to (AXB) and define w_c similarly. Show that w_b, w_c intersect on (AMN).

Solution. Doing a $\sqrt{\frac{bc}{2}}$ inversion in Problem 1.8 one ends up with this parallel problem. $\square$

Problem 1.8.7 (Sharygin 2010 P3) **:** Points A', B', C' lie on sides BC, CA, AB of triangle ABC. for a point X one has $\angle AXB = \angle A'C'B' + \angle ACB$ and $\angle BXC = \angle B'A'C' + \angle BAC$. Prove that the quadrilateral $XA'BC'$ is cyclic.

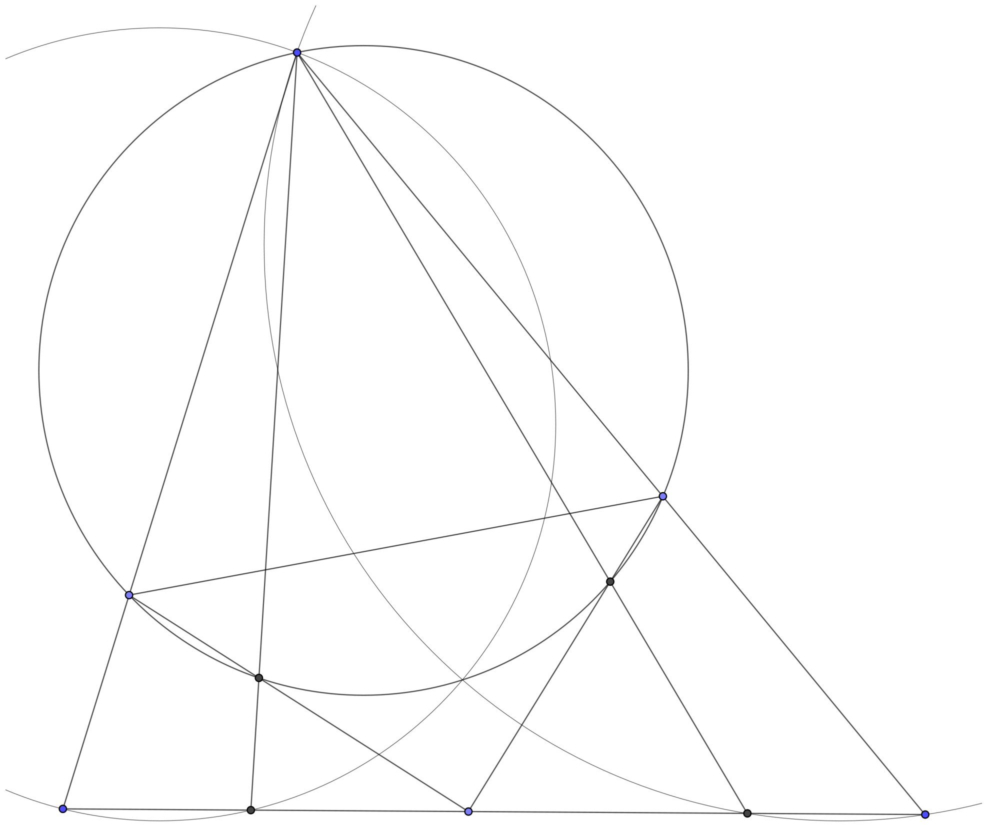

Figure 1.69

Problem 1.8.8 (IMO 2018 P6) **:** A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. Point X lies inside $ABCD$ so that

$$\angle XAB = \angle XCD \quad \text{and} \quad \angle XBC = \angle XDA.$$

Prove that $\angle BXA + \angle DXC = 180°$.

Proof. Let $P = AB \cap CD$, $Q = AD \cap BC$

From the first condition, we get that $\dfrac{AB}{BC} = \dfrac{AD}{DC}$, implying that the angle bisectors of $\angle DAB, \angle DCB$ meet on BD.

And from the second condition, we have $X = \odot QBD \cap \odot PAC$

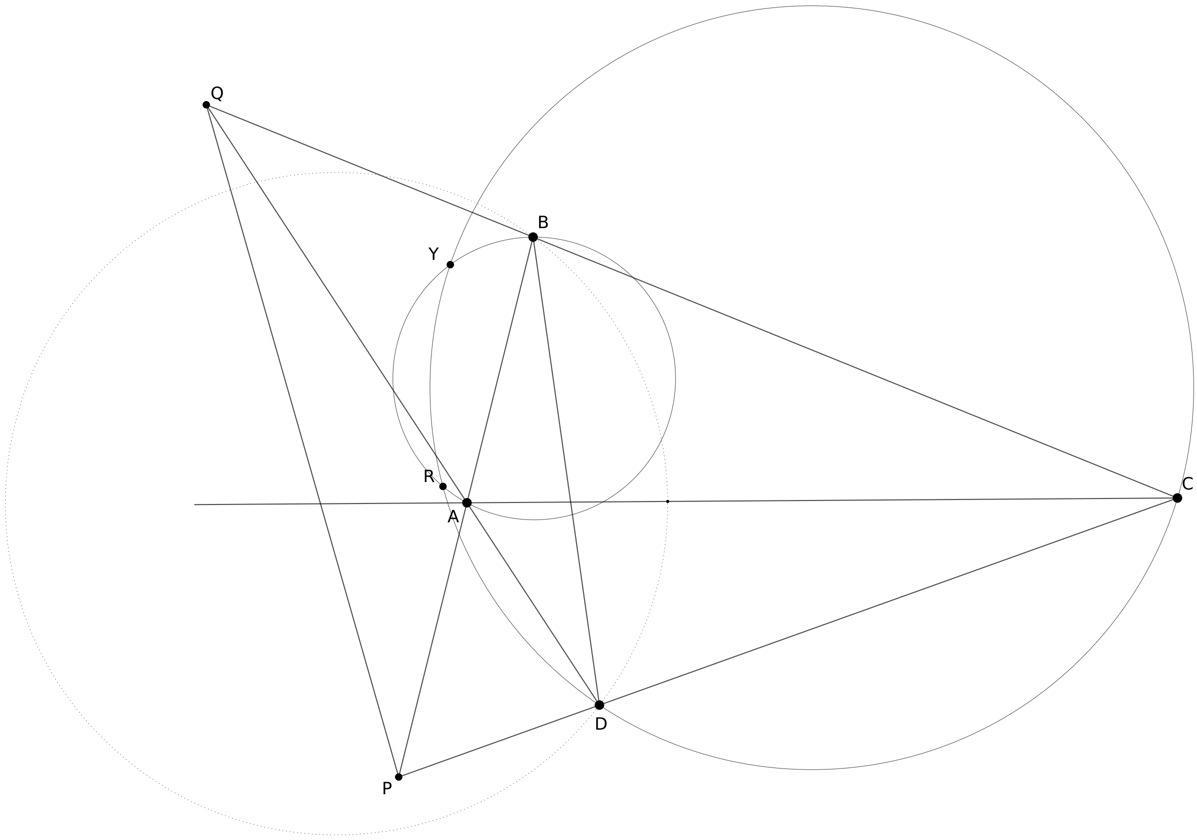

Figure 1.70: IMO 2018 P6, Simple Angle-Chase proof.

Let us define the point R such that AR, CR are isogonal to AC wrt to $\angle DAB, \angle DCB$ respectively. In $\triangle RAC$, we have, the bisectors of $\angle RAC, \angle RCA$ meet on the line BRD, meaning that RB bisects $\angle ARC$.

Let $\odot ARM \cap \odot DRC = Y$. We have,

$$\begin{aligned} \measuredangle AYC &= \measuredangle AYR + \measuredangle RYC \\ &= \measuredangle ABR + \measuredangle RDP \\ &= \measuredangle BPD \\ \implies \square PAYC &\text{ is cyclic.} \end{aligned}$$

And,

$$\begin{aligned}
\measuredangle BYD &= \measuredangle BYR + \measuredangle RYD \\
&= \measuredangle BAR + \measuredangle RCD \\
&= \measuredangle CAD + \measuredangle BCA \\
&= \measuredangle CQD \\
\implies \square QBYD &\text{ is cyclic.}
\end{aligned}$$

So, $Y \equiv X$. So,

$$\measuredangle BYA + \measuredangle DYC = \measuredangle BRA + \measuredangle DRC = \measuredangle BRA + \measuredangle ARD = 180^\circ$$

$\square$

Problem 1.8.9 (Sharygin 2010) **:** In $\triangle ABC$, let AL_a, AM_a be the external and internal bisectors of $\angle A$ with L_a, M_a lying on BC. Let ω_a be the reflection of the circumcircle of $\triangle AL_aM_a$wrt the midpoint of BC. Let ω_a be defined similarly. Prove that ω_a, ω_b are tangent to each other iff $\triangle ABC$ is a right-angled triangle.

1.9 Complete Quadrilateral + Spiral Similarity

Lemma 1.9.1 — *Three lines, l_a, l_b, l_c, origin at point P. Two circles ω_1, ω_2 passing through P meet the lines at A_1, B_1, C_1; A_2, B_2, C_2 resp. Let A_3 be the reflection of A_2 on A_1. Define B_3, C_3 similarly. Then $PA_3B_3C_3$ are concyclic.*

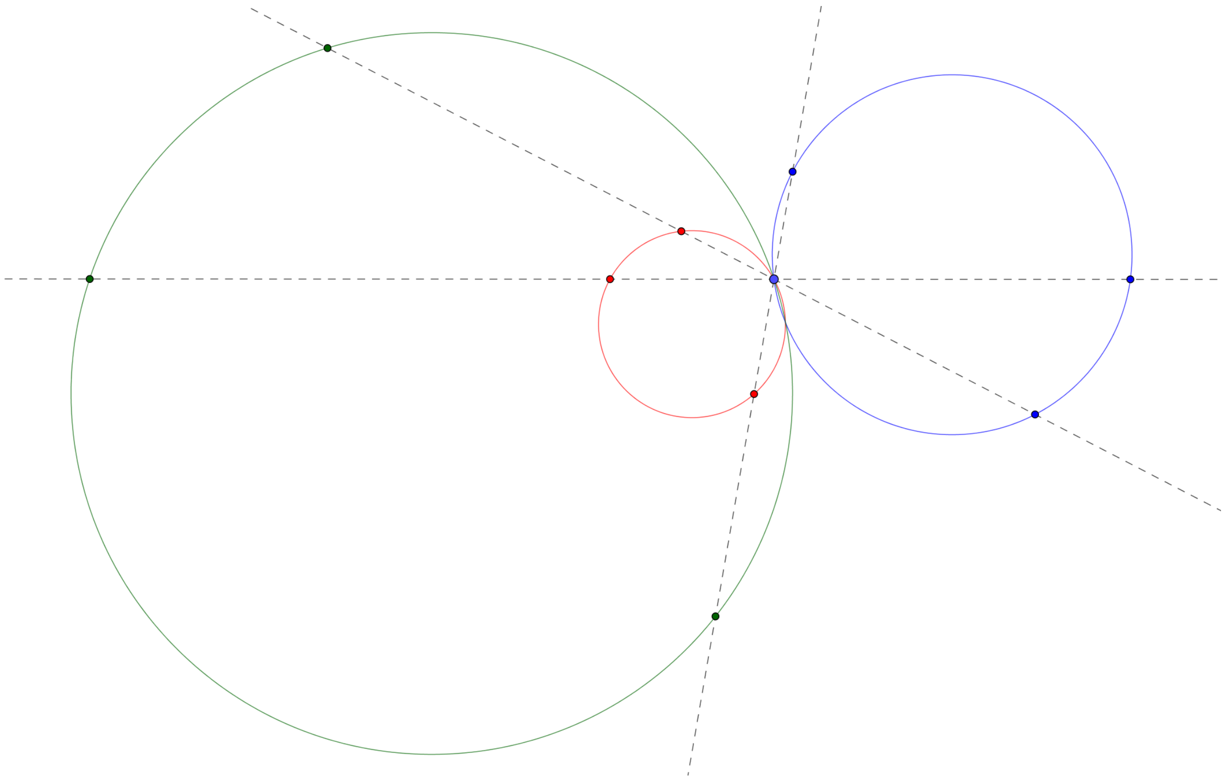

Figure 1.71: Spiral Similarity Lemma 1: the Blue points have been reflected wrt to the Red points to get the Green points

Problem 1.9.1 (ISL 2009 G4) **:** Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G and H.

Solution. This problem generalizes to this □

Problem 1.9.2 (All Russian 2014 Grade 10 Day 1 P4) **:** Given a triangle ABC with $AB > BC$, let Ω be the circumcircle. Let M, N lie on the sides AB, BC respectively, such that $AM = CN$. Let K be the intersection of MN and AC. Let P be the incenter of the triangle AMK and Q be the

K-excenter of the triangle CNK. If R is midpoint of the arc ABC of Ω then prove that $RP = RQ$.

Lemma 1.9.2 — *Let E and F be the intersections of opposite sides of a convex quadrilateral $ABCD$. The two diagonals meet at P. Let M be the foot of the perpendicular from P to EF. Show that $\angle BMC = \angle AMD$. And PM is the bisector of angles $\angle AMC, \angle BMD$.*

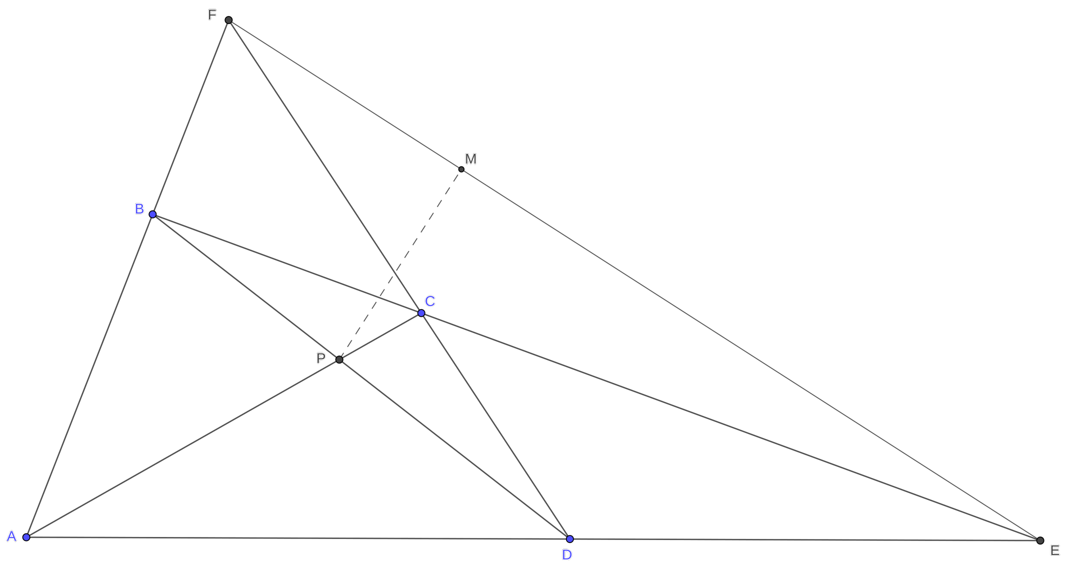

Figure 1.72

Theorem 1.9.3 (Newton-Gauss Line) — *Among the points A, B, C, D no three are collinear. The lines AB and CD intersect at E, and BC and DA intersect at F. Prove that either the circles with diameters AC, BD, EF pass through two common points, or no two of them have any common point.*
The previous can be stated differently: The midpoints of AC, BD, EF are collinear and this line is called "Newton-Gauss Line".

Solution. Either by E.R.I.Q. Lemma, Length Chase, or configurations like Varignon Parallelogram □

Problem 1.9.3 () **:** Let $ABCD$ incribed (O) and a point so-called M .Call $X,\ Y,\ Z,\ T,\ U,\ V$are the projection of M onto $AB,\ BC,\ CD,\ DA,\ CA,\ BD$ respectively. Call $I,\ J,\ H$ the midpoints of

XZ, UV, YT respectively. Prove that N, P, Q are collinear.

Solution. Divide the problem in cases, and prove the easiest case first. □

Lemma 1.9.4 — *In a cyclic quadrilateral $ABCD$, $AC \cap BD = P$, $AD \cap BC = Q$, $AB \cap CD = R$. S, T are the midpoints of PQ, PR. And a point X is on ST. Prove that the power of X wrt $ABCD$ is XP^2.*

Solution. Using polar argument wrt P □

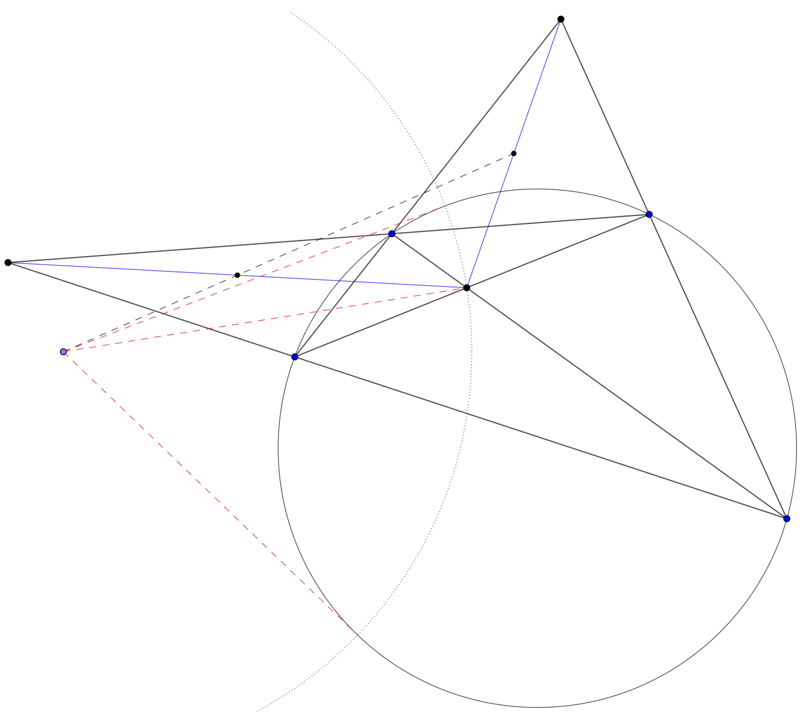

Figure 1.73

Problem 1.9.4 (USA TST 2000 P2) **:** Let $ABCD$ be a cyclic quadrilateral and let E and F be the feet of perpendiculars from the intersection of diagonals AC and BD to AB and CD, respectively. Prove that EF is perpendicular to the line through the midpoints of AD and BC.

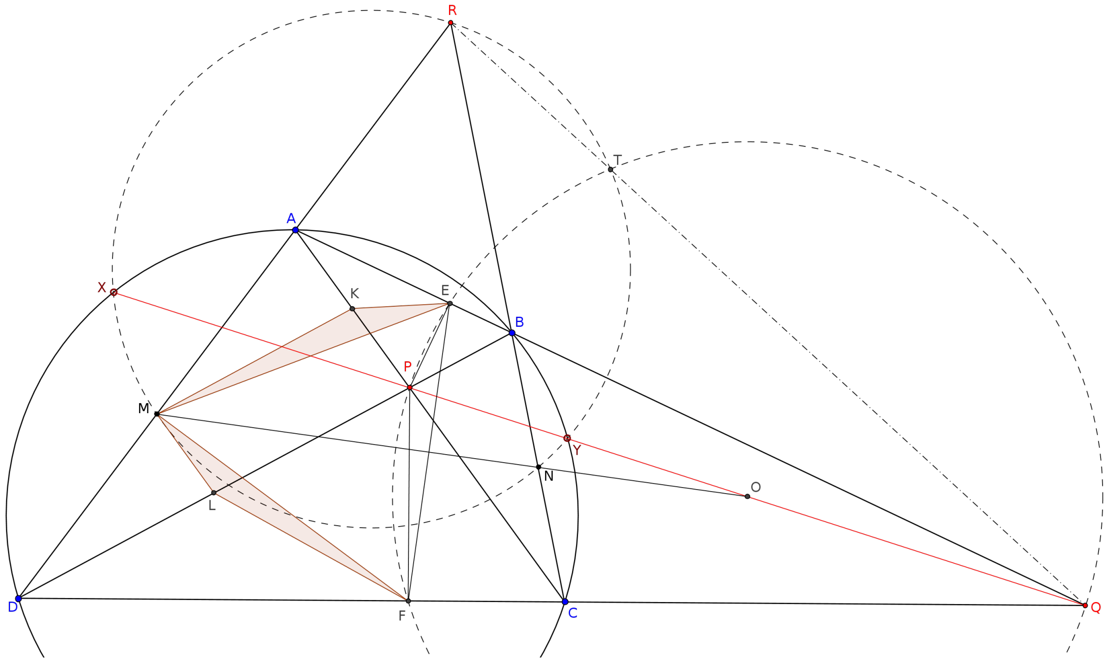

Figure 1.74: USA TST 2000 P2

Solution. First solution is using some properties of the complete quad and angle bash the angle $\measuredangle(MN, EF)$ □

Solution. Second solution is to notice the two brow triangles and proving them congruent. □

Problem 1.9.5 () **:** Let 2 equal circle $(O_1), (O_2)$ meet each other at P, Q. O be the midpoint of PQ. 2 line through P meet the circles at A, B, C, D, $(A, C \in (O_1);\ B, D \in (O_2))$. M, N be midpoint of AD, BC. Prove that M, N, O are collinear.

Problem 1.9.6 (AoPS) **:** In $\triangle ADE$ a circle with center O, passes through A, D meets AE, ED respectively at B, C, $BD \cap AC = G$, line OG meets $\odot ADE$ at P. Prove that $\triangle PBD, \triangle PAC$ has the same incenter (preferably without using inversion).

Problem 1.9.7 (Archer - EChen M1P2) **:** Let a circle ω centered at A meet BC at D, E, such that B, D, E, C all lie on BC in that order. Let ω meet $\odot ABC$ at F, G such that A, F, B, C, G lie on the circle in that order. Let $\odot BFD \cap AB = K$, $\odot CGE \cap AC = L$. Prove that FK, GL, AO are concurrent.

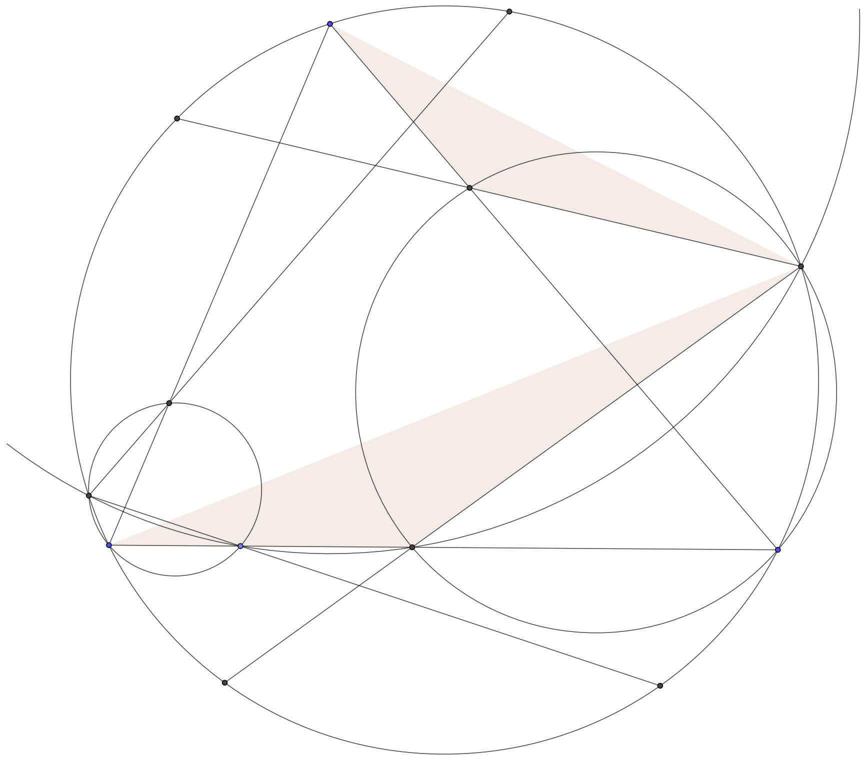

Figure 1.75

Problem 1.9.8 (Sharygin 2012 P22) **:** A circle ω with center I is inscribed into a segment of the disk, formed by an arc and a chord AB. Point M is the midpoint of this arc AB, and point N is the midpoint of the complementary arc. The tangents from N touch ω in points C and D. The opposite sidelines AC and BD of quadrilateral $ABCD$ meet in point X, and the diagonals of $ABCD$ meet in point Y. Prove that points X, Y, I and M are collinear.

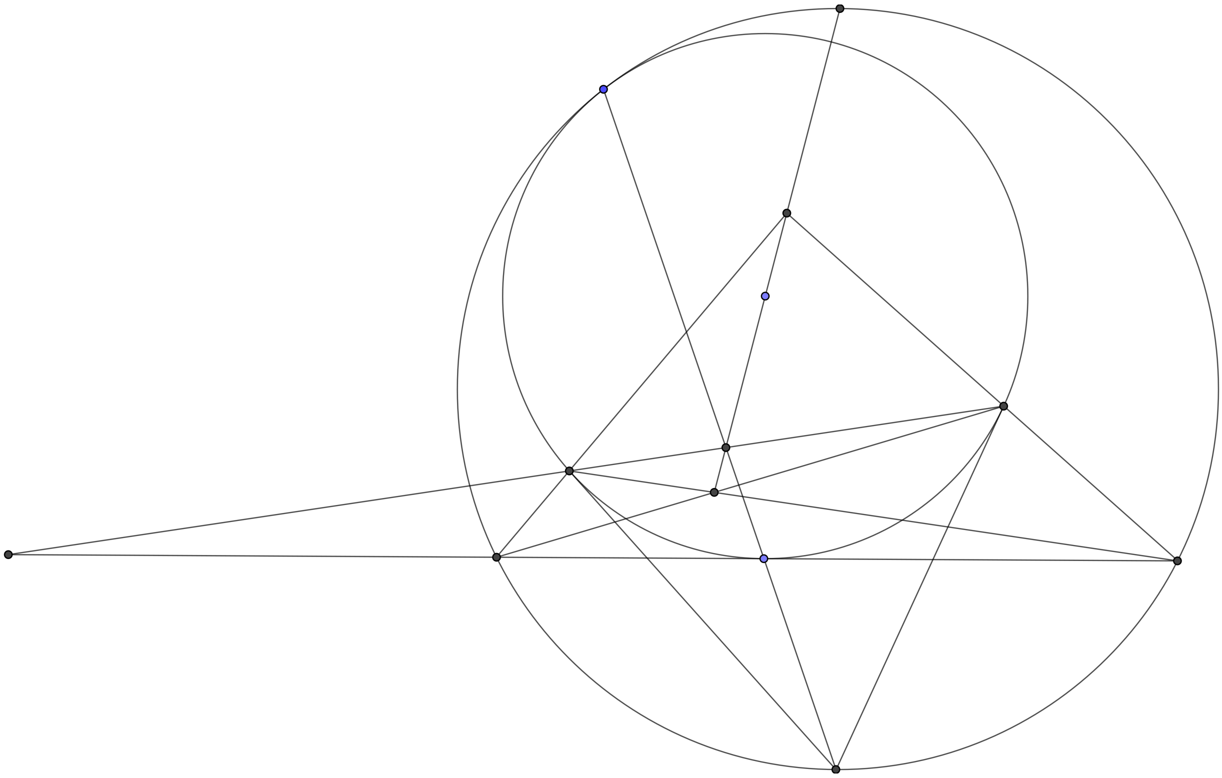

Figure 1.76

Solution. La Hire □

Problem 1.9.9 (Sharygin 2012 P21) **:** Two perpendicular lines pass through the orthocenter of an acute-angled triangle. The sidelines of the triangle cut on each of these lines two segments: one lying inside the triangle and another one lying outside it. Prove that the product of two internal segments is equal to the product of two external segments.

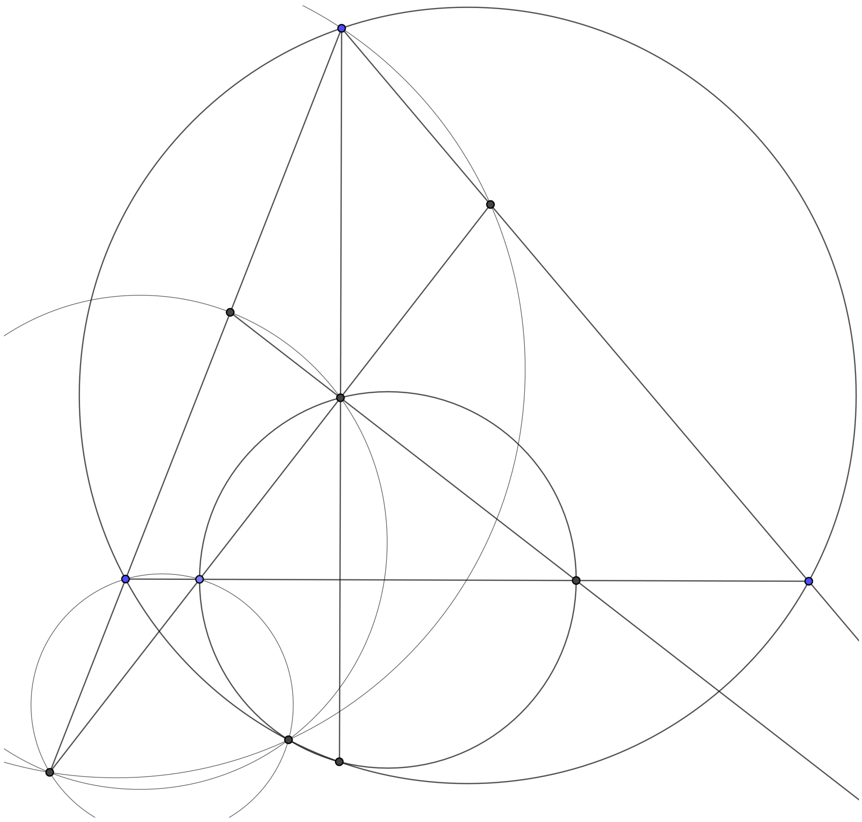

Figure 1.77

Solution. Spiral Similarity ☐

Problem 1.9.10 (Iran TST 2004 P4) **:** Let M, M' be two conjugates point in triangle ABC (in the sense that $\angle MAB = \angle M'AC, \dots$). Let P, Q, R, P', Q', R' be foots of perpendiculars from M and M' to BC, CA, AB. Let $E = QR \cap Q'R'$, $F = RP \cap R'P'$ and $G = PQ \cap P'Q'$. Prove that the lines AG, BF, CE are parallel.

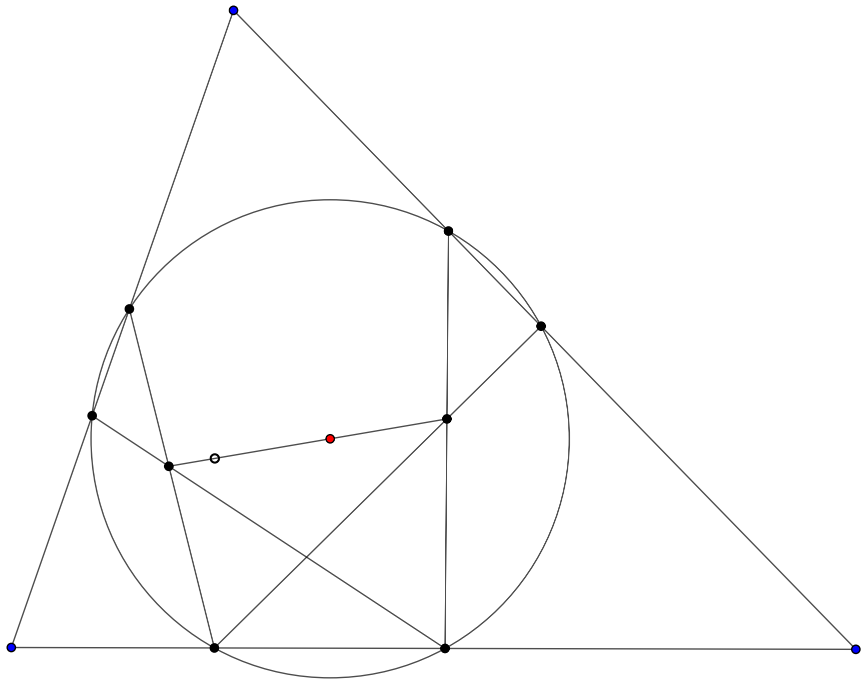

Figure 1.78: The points are collinear, by Zhao Lemmas

Problem 1.9.11 (Iran TST 2018 D2P6) **:** Consider quadrilateral $ABCD$ inscribed in circle ω. $P \equiv AC \cap BD$. E, F lie on sides AB, CD respectively such that $A\hat{P}E = D\hat{P}F$. Circles ω_1, ω_2 are tangent to ω at X, Y respectively and also both tangent to the circumcircle of $\triangle PEF$ at P. Prove that:

$$\frac{EX}{EY} = \frac{FX}{FY}$$

1.10 Projective Geometry

- Cross Ratio - Zarathustra Brady
- Desargues' Involution Theorem - MarkBcc168

1.10.1 Definitions

Definition (Projective Plane) — The *projective plane* $\mathbb{P}^2$ is a *set of lines* passing through an observation point O in three dimensional space. A *porjective line* is a plane passing through O, and a *projective point* is a line passing throught O.

Definition (Coordinates in Projective Planes) — A point in a projective plane $\mathbb{P}^2$ has coordinates $(p : q : r)$. If $r = 0$, we say that the point is an *infinite point*. Every line in $\mathbb{P}^2$ can also be described with $(p : q : r)$ in a sense that this line (which is a plane passing through O) has the equation

$$pa + qb + rc = 0$$

Definition (Projection) — We can define *projection* of $\mathbb{P}^2$ on some plane A^2 not passing through O (for simplicity we will take the plane $z = 1$) by associating

$$P = (p : q : r) \in \mathbb{P}^2 \to P' = \left(\frac{p}{r} : \frac{q}{r} : 1\right) \in A^2$$

If $r = 0$, then we say P' is an infinite point with slope $\frac{q}{p}$.

A projective line l with coordinates $p : q : r$ gets associated to a line $l \in A^2$ likewise. The line at infinity is the line associated with the projective line passing through O and parallel to A^2.

Definition (Projective Line and Inversive Plane) — Every point in a *projective line* $\mathbb{P}^1$ has a coordinate $(s : t)$ which corresponds to the ordinary point $x = \frac{s}{t}$. The point at infinity will have $t = 0$. If we let s, t be complex numbers then the projective line is called the *inversive plane*.

Definition (Cross Ratio) — If four points A, B, C, D lie on a line, their *cross ratio* is defined as

$$(A, B; C, D) = \frac{AC}{BC} \div \frac{AD}{BD}$$

If four lines l_1, l_2, l_3, l_4 pass through a point, then their cross ratio is

$$(l_1, l_2; l_3, l_4) = \frac{\sin \angle l_1 l_3}{\sin \angle l_2 l_3} \div \frac{\sin \angle l_1 l_4}{\sin \angle l_2 l_4}$$

If four points on the inversive plane has the complex coordinate a, b, c, d, then the cross ratio is defined by

$$(A, B; C, D) = \frac{a-c}{b-c} \div \frac{a-d}{c-d}$$

Definition (Möbius Transformation) — A *Möbius Transformation* is defined by a transformation f_M of the inversive plane by a two by two matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with non zero determiant as followed:

$$(s : t) \in \mathbb{P}^1 \to (sa + tb \ : \ sc + td)$$

Which is in ordinary coordinates:

$$f_M(z) = \frac{az + b}{cz + d}$$

A Möbius transformation can be thought as a *matrix transformation* of the projective line (which can be thought as a plane) and then projection on an ordinary line.

Definition (Harmonic Conjugate Map) — For any points A, B on $\mathbb{P}^1$, we define

$$h_{A,B}(C) = D \text{ if } (A, B; C, D) = -1$$

A harmonic cojugate is a Möbius transformation. And a Möbius transformation that is also an *involution*, i.e. that has $f(f(x)) = x$, is a harmonic conjugate.

Definition (Circle Points) — The circle points are the points in $\mathbb{P}^2$ with coordinates $\alpha = (1 : i : 0)$ and $\alpha' = (-i : 1 : 0)$. These two points are both infinite and imaginary. And *every circle* passes through these two points.x

Definition (Coharmonic Points) — Three pairs of points $\{A, A'\}, \{C, C'\}, \{E, E'\}$ on the same line are called *coharmonic points* iff there exists a pair of points $\{M, N\}$ on the line such that

$$(M, N; A, B) = (M, N; C, D) = (M, N; E, F) = -1$$

Definition (Involution) — If there exists a point X on l such that for the Möbius Transformation $f : l \to l$, such that $f(f(X)) = X$, then f is an *involution*.

Theorem 1.10.1 (Properties of Coharmonic Points) — *If A, B, C, A', B', C' lie on a line, no three the same and $A \neq X$, then the following are equivalent:*

1. *$\{A, A'\}, \{B, B'\}, \{C, C'\}$ are coharmonic.*
2. *There is a Möbius Transformation with $f(A) = A', f(B) = B', f(C) = C'$ which is an involution.*
3. *$(A, A'; B, C) = (A', A; B', C')$*
4. *$\dfrac{AC'}{C'B}\dfrac{BA'}{A'C}\dfrac{CB'}{B'A} = -1$*
5. *$(A, A'; C, C') = (A, A'; C, B)\ (A, A'; C, B')$*

Theorem 1.10.2 (Invertible Function on a line) — *If f is an* invertible function *from a line to itself that is defined by some geometric procedure that has no* configuration mess, *then f preserves cross ratio, and is a Möbius Transformation. Similarly, an invertible Möbius Transformation is an involution on the line.*

1.10.2 Cross Ratio

Theorem 1.10.3 (Pappus's Hexagon Theorem) — *Let A, B, C be on a line, and let D, E, F be on another line. Let $X = AE \cap BD, Y = BF \cap CE, Z = CD \cap AF$. Then X, Y, Z are on a line.*

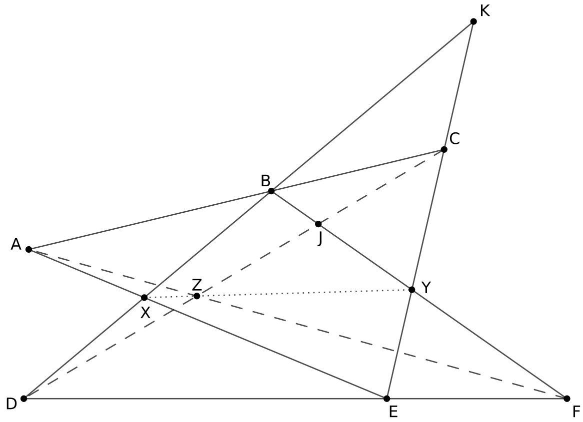

Figure 1.79

Proof. Let $CD \cap BF = J, DE \cap BD = K$. We have,

$$\begin{aligned} &(D, Z;\ J, C) \\ \overset{F}{=}\ &(AB \cap FD, A;\ B, C) \\ \overset{C}{=}\ &(D, X;\ B, K) \\ \overset{Y}{=}\ &(D, XY \cap DC;\ J, C) \end{aligned}$$

□

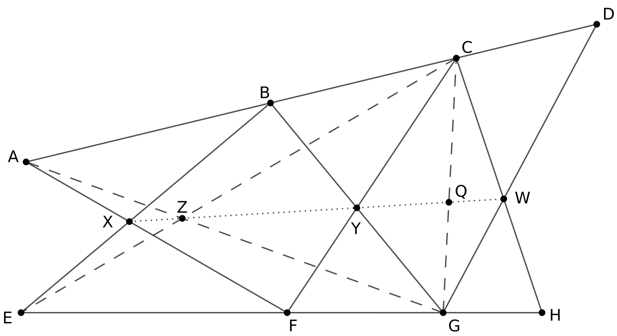

Figure 1.80: $(A, B; C, D) = (E, F; G, H)$

Theorem 1.10.4 (Cross Ratio Equality) — *Let A, B, C, D be on a line, and let E, F, G, H be on another line. Let $X = AF \cap BE, Y = BG \cap CF, Z = CH \cap DG$. Then X, Y, Z are on a line if and only if $(A, B; C, D) = (E, F; G, H)$.*

1.10.3 Involution

Theorem 1.10.5 (Involution on a line) — *An involution on a line l is an inversion around some point on l.*

Theorem 1.10.6 (Involution on a conic) — *Let $f : \mathcal{C} \to \mathcal{C}$ be an involution. Let $f(X) = X'$ for all $X \in \mathcal{C}$. Then all XX' pass through a fixed point P.*

Solution [Polar line, Pascal]. Let l be the polar line of P wrt $\mathcal{C}$. Let l' be the line parallel to l passing through P. Let $l' \cap \mathcal{C} = \{X, X'\}$. We show that, if $A, B \in \mathcal{C}$, and A', B' are the second intersection of AP, BP with $\mathcal{C}$, then, $\{A, A'\}, \{B, B'\}$ and $\{X, X'\}$ are coharmonic points wrt $\mathcal{C}$, that is, for a point $Q \in \mathcal{C}$,

$$(QX, QX'; QA, QB) = (QX', QX; QA', QB')$$

Let $XA, XB \cap l = A_1, B_1$, and $XA', XB' \cap l = A_1', B_1'$. Since the line $(XA \cap X'A', XA' \cap X'A)$ is the polar of $XX' \cap AA'$, X', A', A_1' are collinear. Similarly for X', B', B_1'. Let T be the polar point of XX'.

We have, $T, XB \cap X'A, XA \cap X'B$ collinear by Pascal's theorem on hexagon $XXABX'X'$. Let, $T' = (T, XB \cap X'A, XA \cap X'B) \cap l'$. We have,

$$\frac{XT'}{T'X} = \frac{B_1T}{TA_1'} = \frac{A_1T}{TB_1'} \implies \frac{TB_1}{TA_1} = \frac{TA_1'}{TB_1'}$$

Now, we have

$$\begin{aligned} X(X, X'; A, B) &= (T, \infty; A_1, B_1) \\ &= \frac{TA_1}{TB_1} = \frac{TB_1'}{TA_1'} \qquad = (\infty, T; A_1', B_1') \\ &= X(X', X; A', B') \end{aligned}$$

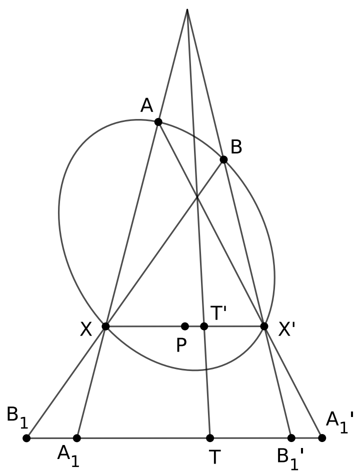

Figure 1.81

Which concludes the proof. □

Solution [Inversion]. First project the conic to a circle, then invert $\mathcal{C} \to l$ across a point P on $\mathcal{C}$. The goal is to show that for every conjugate pair $X, X' \in \mathcal{C}$ and their image after inversion $X_1, X_1' \in l$, $\odot PX_1X_1'$ passes through a fixed point.

By Theorem 1.10.5, we know that there is a point K on l that inverts X_1 to X_1'. So the circles PX_1X_1' have radical axis PK. Which concludes the proof. □

Theorem 1.10.1
Three Conic Law Let A, B, C, D be any four points, no three on a line. Let l be a line passing through at most one of them. For a point P on l, define $f(P) = P'$, where P' is the second intersection of l with the conic passing through A, B, C, D, P'. Then f is an involution.

Then for any three points $X, Y, Z \in l$, $\{X, f(X)\}$, $\{Y, f(Y)\}$, $\{Z, f(Z)\}$ are coharmonic, i.e. $\{X, f(X)\}$ are conjugate pairs of an involution on l,

Theorem 1.10.7 (Desargues' Involution Theorem) — *Let $ABCD$ be a quadrilateral, let a conic $\mathcal{C}$ pass through A, B, C, D. And let a line l intersect $(AB, CD), (AD, BC), (AC, BD), \mathcal{C}$ at $(X_1, X_2), (Y_1, Y_2), (Z_1, Z_2), (W_1, W_2)$. Then*

$$\{X_1, X_2\}, \{Y_1, Y_2\}, \{Z_1, Z_2\}, \{W_1, W_2\}$$

are coharmonic points i.e. they are reciprocal pairs of some involution on l.

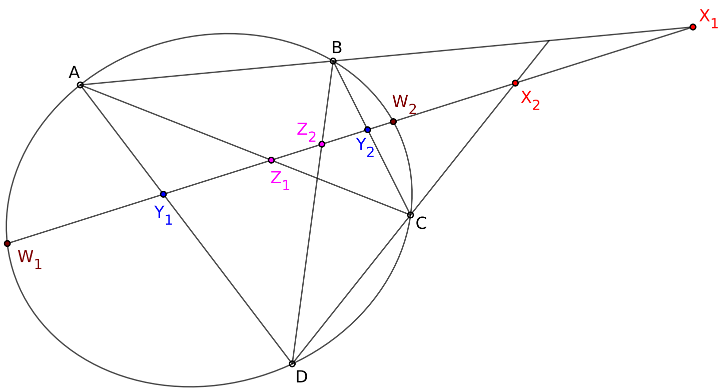

Figure 1.82: Desargues' Involution Theorem

Proof. Apply the *Three Conic Law* on l with points A, B, C, D. □

Theorem 1.10.8 (Degenerate Desargues' Involution: 2 Points) — *Let A, B, be two points on a conic $\mathcal{C}$, let a line l meet $AB, \mathcal{C}$ and the tangents at A, B to $\mathcal{C}$ at $X, (W_1, W_2), (Y_1, Y_2)$. Then $(X, X), (W_1, W_2), (Y_1, Y_2)$ are reciprocal pairs of an involution on l.*

1.10.4 Inversion

TelvCohl's $\sqrt{bc}$ inversion problem collection

Lemma 1.10.9 — *WRT a circle ω with center O the polar of a point A can be constructed as the radical axis of ω and the circle with diameter OA.*

Lemma 1.10.10 — $\measuredangle(a, b) = \measuredangle AOB$

Theorem 1.10.11 (Pascal's Theorem for Octagons: A special case) — *Let $ABCDA'B'C'D'$ be a octagon inscribed in a conic section. If the points:*

$$AD \cap BC,\ AC' \cap BB',\ AD' \cap CA',\ BD' \cap DA',\ DB' \cap CC'$$

are collinear, then so are the points

$$A'D' \cap B'C',\ A'C \cap B'B,\ A'D \cap C'A,\ B'D \cap D'A,\ D'B \cap C'C$$

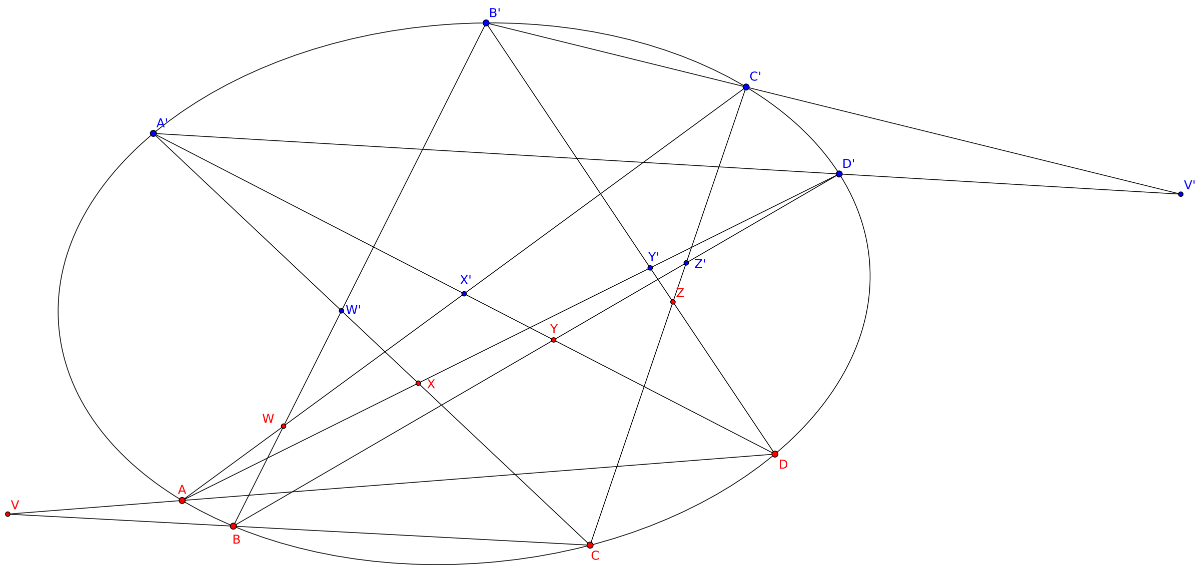

Figure 1.83: If the small Red points are collinear, then the Blue ones are too.

Theorem 1.10.12 (Inscribed Conic in Pascal's theorem) — *$A_1A_2A_3A_4A_5A_6$ be a hexagon inscribed in a conic section. Then the hexagon formed by*

$$A_1A_3 \cap A_2A_6,\ A_2A_4 \cap A_1A_3,\ A_2A_4 \cap A_3A_5,\ A_3A_5 \cap A_4A_6,\ A_5A_1 \cap A_4A_6,\ A_1A_5 \cap A_2A_6$$

has an inscribed conic section.

Problem 1.10.1 () **:** Let $ABCD$ have an incircle (I). Let (I) meet AB, BC, CD, DA at M, N, P, Q. Let K, L be the circumcenters of AMN, APQ. $KL \cap BD = R$, $AI \cap MQ = J$. Prove that $RA = RJ$.

Problem 1.10.2 () **:** Let the A mixtilinear incircle (O) of $\triangle ABC$ meet $\odot ABC, AC, AB$ at P, E, F. Let M be the BC arc midpoint. Let $\mathcal{H}$ be the conic that goes through E, F, O, P, M meet $\odot ABC$ at X, Y. Prove that AA, XY, EF are concurrent.

Problem 1.10.3 (Iran 3rd Round G4) **:** Let ABC be a triangle with incenter I. Let K be the midpoint of AI and $BI \cap \odot(\triangle ABC) = M, CI \cap \odot(\triangle ABC) = N$. points P, Q lie on AM, AN respectively such that $\angle ABK = \angle PBC, \angle ACK = \angle QCB$. Prove that P, Q, I are collinear.

Solution. Since we are dealing with collinearity, and usually we use harmonic bundle in these cases to show collinearity. But in this problem, there is no harmonic bundle. So we use cross ratio... □

Generalization 1.10.3.1 (Iran 3rd Round G4 Generalized version) **:** Let ABC be a triangle inscribed in circle (O) and P, Q are two isogonal conjugate points. PB, PC cut (O) again at M, N. QA cuts MN at K. L is isogonal conjugate of K. LB, LC cut AM, AN at S, T, resp. Prove that S, Q, T are collinear.

Lemma 1.10.13 — *Too long, can't explain, loot at the figure. The dotted lines go through that concurrency point.*

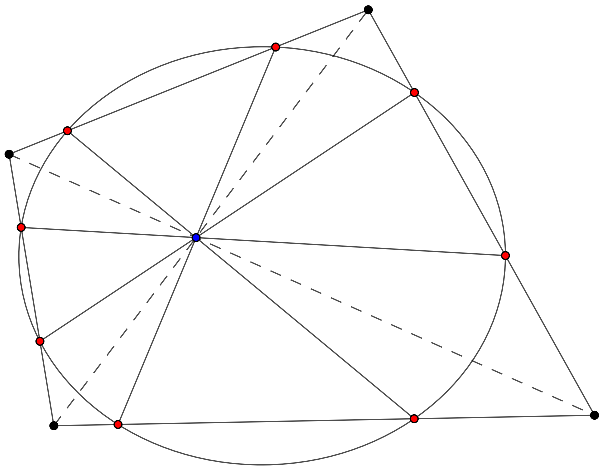

Figure 1.84: Everything concurrs

Lemma 1.10.14 (Construction of Involution Center on Line) — *Given a line l, four points A, B, A', B' such that A, A' and B, B' are two conjugate pairs of some involution i.e. some inversion on l, find the center O of inversion.*

1.10.5 Problems

Problem 1.10.4 (Dunno) **:** Let E, F be on the lines AC, AB of $\triangle ABC$. Let P be a point on EF. Let Q be the intersection of the lines through E, F and parallel to BP and CP respectively. Prove that, as P moves along EF, Q moves along a line.

Solution [Cross Ratio]**.** Let $X \in BF, Y \in CE$ such that $\dfrac{BX}{XF} = \dfrac{CE}{EA}, \dfrac{CY}{YA} = \dfrac{BF}{FA}$. With trivial calculation, we have $XY \parallel BC$ We show that Q, X, Y are collinear. For that we will show $FU \parallel EV$ where $U = BP \cap XY, V = CP \cap XY$. And by reverse Pappu's theorem on FPE, UQV, we will have U, Q, V collinear.

Let $K, L = BP, CP \cap XY$. Also, $S = BC \cap EF$. Then we have,

$$(S, P; F, E) \stackrel{B}{=} (\infty, U; X, K) \stackrel{C}{=} (\infty, V; L, Y)$$

$$\implies \frac{XU}{UK} = \frac{LV}{VY}$$

But we have,

$$\frac{FX}{XB} = \frac{FL}{LC} = \frac{AE}{EC}$$

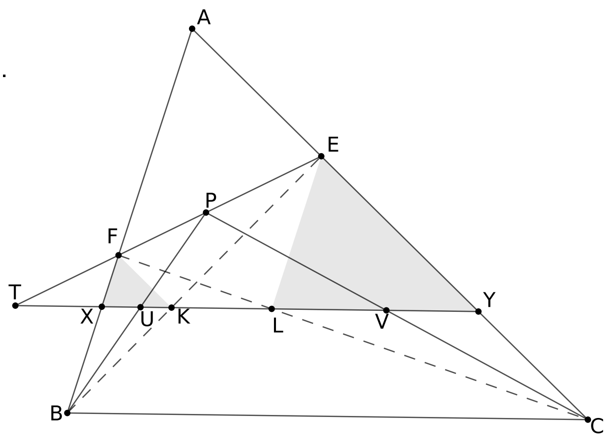

Figure 1.85

So, $FX \parallel EL$, and similarly, $FK \parallel EY$. So by similarity, we have $FU \parallel EV$. $\square$

Lemma 1.10.15 (Conic through orthocenter and vertices) — *Let $ABCD$ be a quadrilateral. Let G, H be the orthocenters of $\triangle ABC$ and $\triangle DBC$. Then A, B, C, D, G, H all lie on a conic.*

Proof. Since $line(AC \cap BD, BG \cap CH)$ is perpendicular to BC, we have $AG \cap DH, AC \cap BD, BG \cap CH$ collinear. So by reverse Pascal's theorem, A, D, G, H, B, C lie on a conic. $\square$

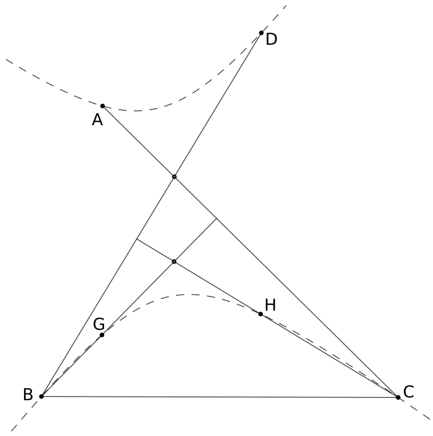

Figure 1.86

Lemma 1.10.16 (Orthogonal Hyperbola) — *Let H be the orthocenter of ABC. Let $\mathcal{C}$ be a conic through A, B, C, H. If XYZ is a triangle with vertices in $\mathcal{C}$, then the orthocenter W of XYZ lies also on $\mathcal{C}$. Also, the asymptotes of $\mathcal{C}$ are orthogonal.*

Solution. As in Lemma 1.10.15, the orthocenter ot XBC lies on $\mathcal{C}$ too. So the orthocenter of XYC lies on $\mathcal{C}$ and so does the orthocenter of XYZ.

Now we show that asymptotes of $\mathcal{C}$ are orthogonal.

Let the two infinity points on $\mathcal{C}$ be s, t. Consider the triangle Ast. Let its orthocenter be B. Then we have, $tB \perp sA$. But tB is parallel to the asymptote through t and sA is parallel to the asymptote through s. So the asymptotes themselves are orthogonal.

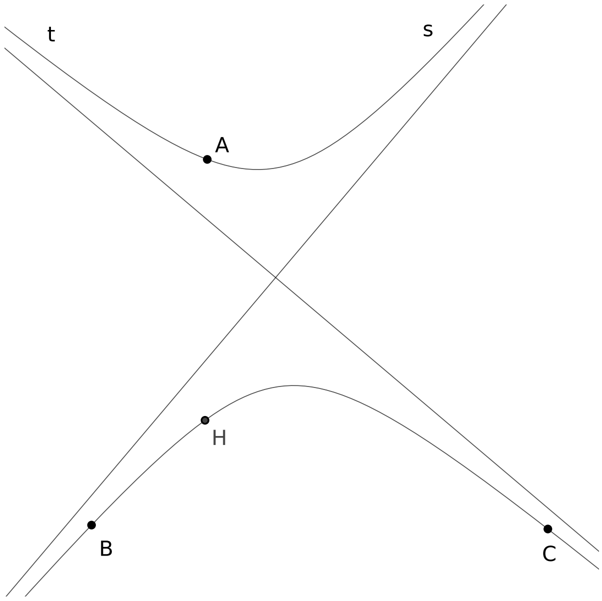

Figure 1.87: Hyperbola through A, B, C, H

□

1.10.6 Projective Constructions

Construction 1 (Second Intersection of Line with Conic) — Given four points A, B, C, D, no three collinear, and a point P on a line l passing through at most one of the four points, construct the point $P' \in l$ such that A, B, C, D, P, P' line on the same conic.

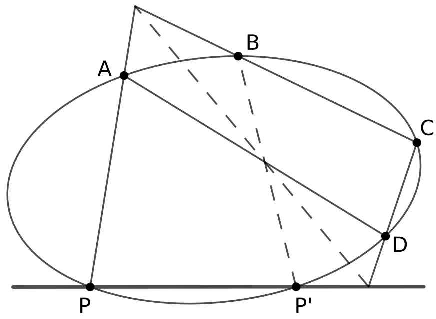

Figure 1.88

Solution. Let $AP \cap BC = X$, $l \cap CD = Y$, $XY \cap AD = Z$. Then by Pascal's Hexagrummum Mysticum Theorem, we have, $P' = BZ \cap l$ □

Construction 2 (Conic touching conic) — Given a conic $\mathcal{C}$, and two points A, B on it, and C inside of it. Construct the conic $\mathcal{H}$ that is tangent to $\mathcal{C}$ at A, B and passes through C.

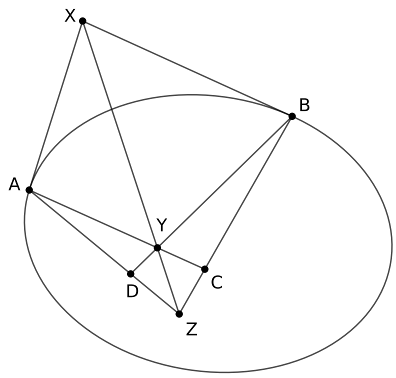

Figure 1.89

Solution. Draw the two tangest at A, B which meet at X. Take an arbitrary line passing through X that intersects AC, BC at Y, Z. Take $D = BY \cap AZ$. Then D lies on $\mathcal{H}$ by Pascal. Construct another point E similarly and draw the conic. □

Construction 3 (Inconic of a quadrilateral) — Given a convex quadrilateral $ABCD$. $P = AC \cap BD$, $S \in AD, T \in BC$ such that S, P, T are collinear. Construct the conic that touches AB, CD, and also touches AD, BC at S, T respectively.

Solution [the_Construction]**.** Draw the polar line l of P wrt to the quadrilateral. Let $Z = BC \cap l$. Let $ZS \cap AB = U$, $ZT \cap CD = V$. Then $SSUUTTVV$ is our desired conic. □

Proof. If $U, V \in CD, AB$ such that UV passes through P, and if the conic passing through U, V and tangent to AD, BC at S, T intersects CD at U' again, then $SV, U'T, DB$ are concurrent. So to show our construction works, we just need to prove that U, V, P are collinear.

Since Pascal's theorem works on $SVBTUD$, we know S, V, B, T, U, D lie on a conic $\mathcal{H}$ and l is the pole of P wrt $\mathcal{H}$. Now, applying Pascal's theorem on $TDVUBS$, and quadrilateral theorem on $BTUD$ and $BVSD$, we have, $ST \cap UV \in AC$, which is P. So we are done. □

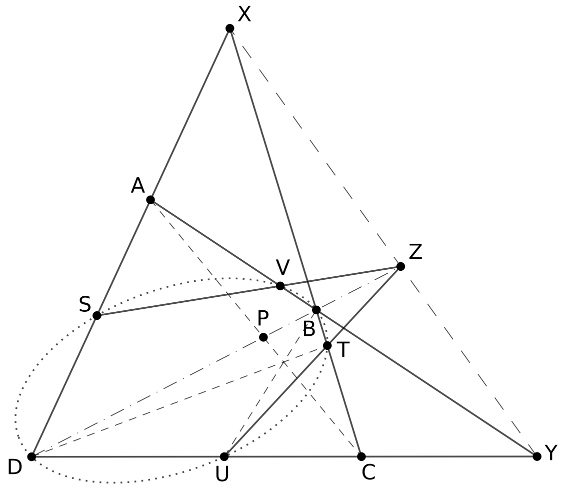

Figure 1.90

Construction 4 (Sharygin Olympiad 2010) — A conic $\mathcal{C}$ passing through the vertices of $\triangle ABC$ is drawn, and three points A', B', C' on its sides BC, CA, AB are chosen. Then the original triangle is erased. Prove that the original triangle can be constructed iff AA', BB', CC' are concurrent.

Solution [the_Construction]. Draw $B'C'$. It intersects the circle at X_1, X_2. Draw the conic $\mathcal{H}$ that is tangent to $\mathcal{C}$ at X_1, X_2 and passes through A'. Then BC is tangent to $\mathcal{H}$ at A'. □

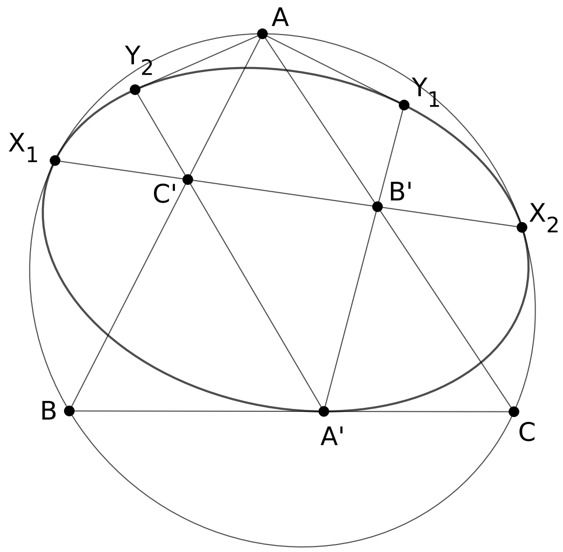

Figure 1.91

Proof. The only if part is easy to prove. Becase if AA', BB', CC' aren't concurrent, then we can get multiple triangles ABC. So suppose that they are concurrent.

Now we define some intersetion points.

$$\begin{array}{llll} W_1 & = & BB' & \cap \ \mathcal{C} \\ S & = & X_1X_1 & \cap \ AW_1 \\ T & = & X_1B & \cap \ AX_2 \\ U & = & X_1X_1 & \cap \ BC \\ V & = & X_2X_2 & \cap \ BC \\ R & = & X_2X_2 & \cap \ AW_1 \\ Y_1 & = & A'B' & \cap \ SR \end{array}$$

T, S, B' are collinear by Pascal's theorem on $BX_1X_1X_2AW_1$. T, B', V are similarly collinear for $AX_2X_2X_1BC$. And similarly R, B', U are collinear.

We will prove that $\mathcal{H}$ is an inconic of $SRVU$ that goes through A', X_1, X_2.

For a point X on UV, define $f : UV \to UV$ such that $f(X)$ is the second intersection of the conic $X_1X_1X_2X_2X$ ($X_1X_1 = SU, X_2X_2 = RV$) with UV. f is an involution by **??**.

Suppose A_1 is the intersection with the inconic of $SRUV$ through X_1, X_2 and UV. Let $A_2 = X_1X_2 \cap UV$. Then $f(A_1) = A_1, f(A_2) = A_2, f(B) = C$.

Which means, $A(B, C; A_1, A_2) = -1$. Which means $A_1 = A'$. So, $X_1X_2A'X_2X_2$ is an inconic of $SRVU$, just as we wanted.

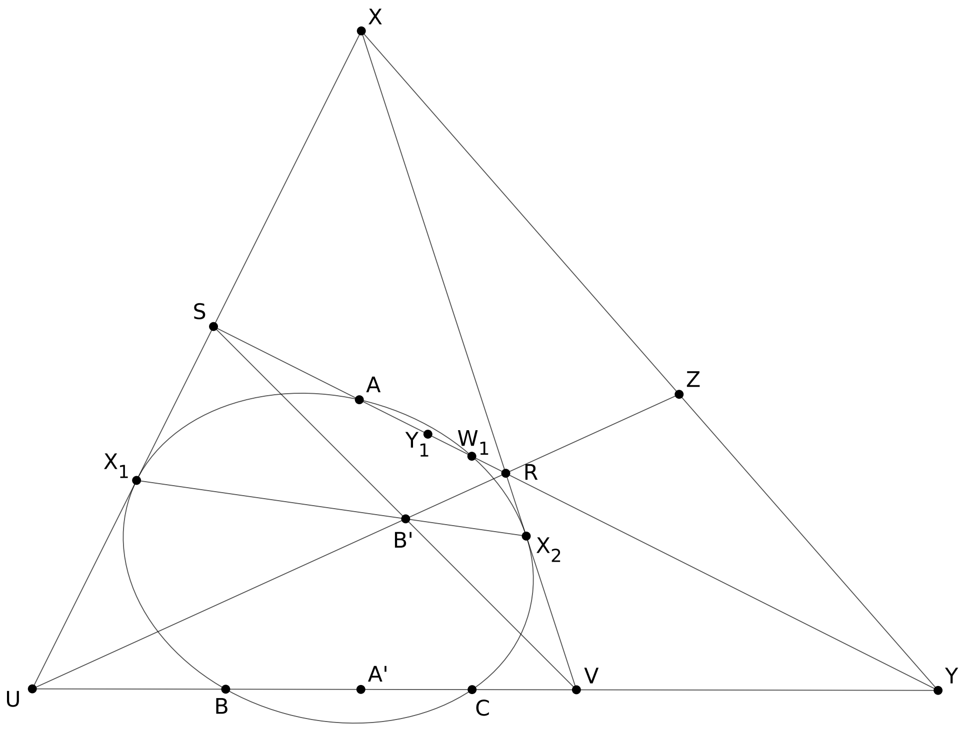

Figure 1.92

□

1.11 Parallelogram Stuff

Theorem 1.11.1 (Maximality of the Area of a Cyclic Quadrilateral) — *Among all quadrilaterals with given side lengths, the cyclic one has maximal area.*

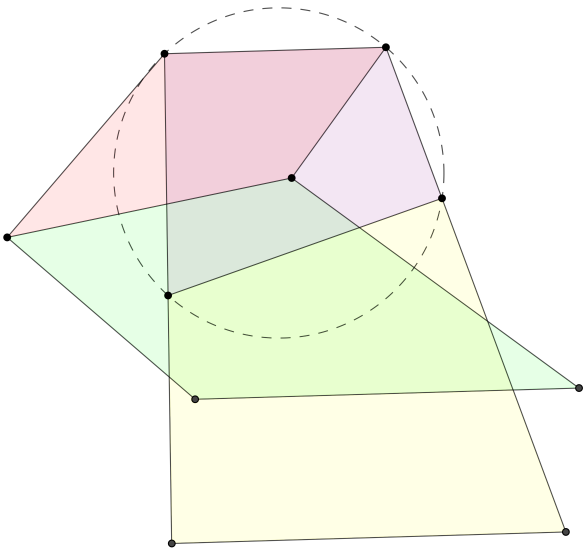

Figure 1.93: The cyclic quad has the maximal area

Problem 1.11.1 (IOM 2017 P1) **:** Let $ABCD$ be a parallelogram in which angle at B is obtuse and $AD > AB$. Points K and L on AC such that $\angle ADL = \angle KBA$(the points A, K, C, L are all different, with K between A and L). The line BK intersects the circumcircle ω of ABC at points B and E, and the line EL intersects ω at points E and F. Prove that $BF \parallel AC$.

Simplify: Make the diagram easier to draw.

Problem 1.11.2 (USA TST 2006 P6) **:** Let ABC be a triangle. Triangles PAB and QAC are constructed outside of triangle ABC such that $AP = AB$ and $AQ = AC$ and $\angle BAP = \angle CAQ$. Segments BQ and CP meet at R. Let O be the circumcenter of triangle BCR. Prove that $AO \perp PQ$.

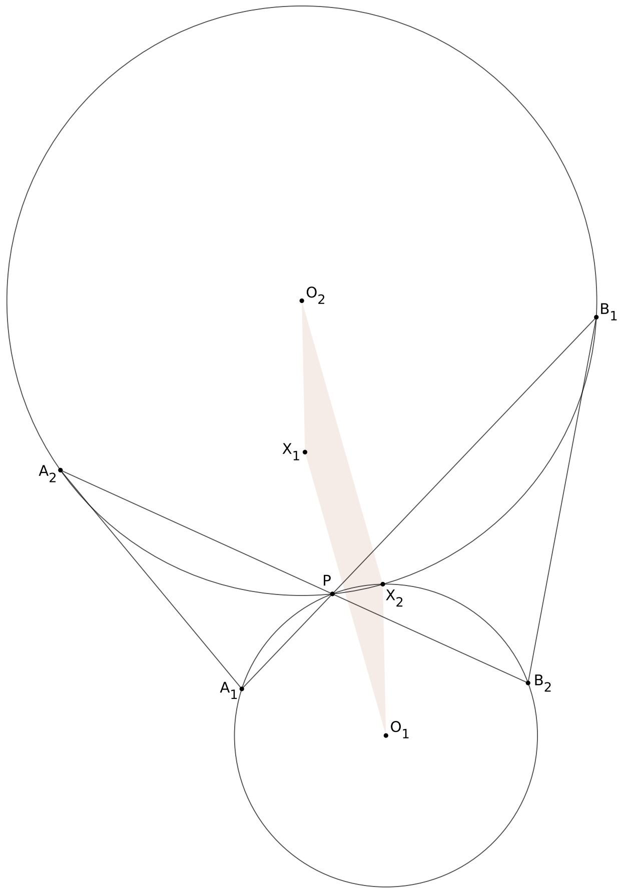

Figure 1.94: USA TST 2006 P6, That is a prallelogram

1.12 Length Relations

Lemma 1.12.1 (E.R.I.Q. (Equal Ration in Quadrilateral) Lemma) — *Let* $A_1, B_1, C_1;\ A_2, B_2, C_2$ *be two sets of collinear points such that*

$$\frac{A_1B_1}{B_1C_1} = \frac{A_2B_2}{B_2C_2} = k$$

. *Let points* A, B, C *be on* A_1A_2, B_1B_2, C_1, C_2 *such that:*

$$\frac{A_1A}{A_2A} = \frac{B_1B}{B_2B} = \frac{C_1C}{C_2C}$$

Then we have,

$$A, B, C \text{ are collinear and, } \frac{AB}{BC} = k$$

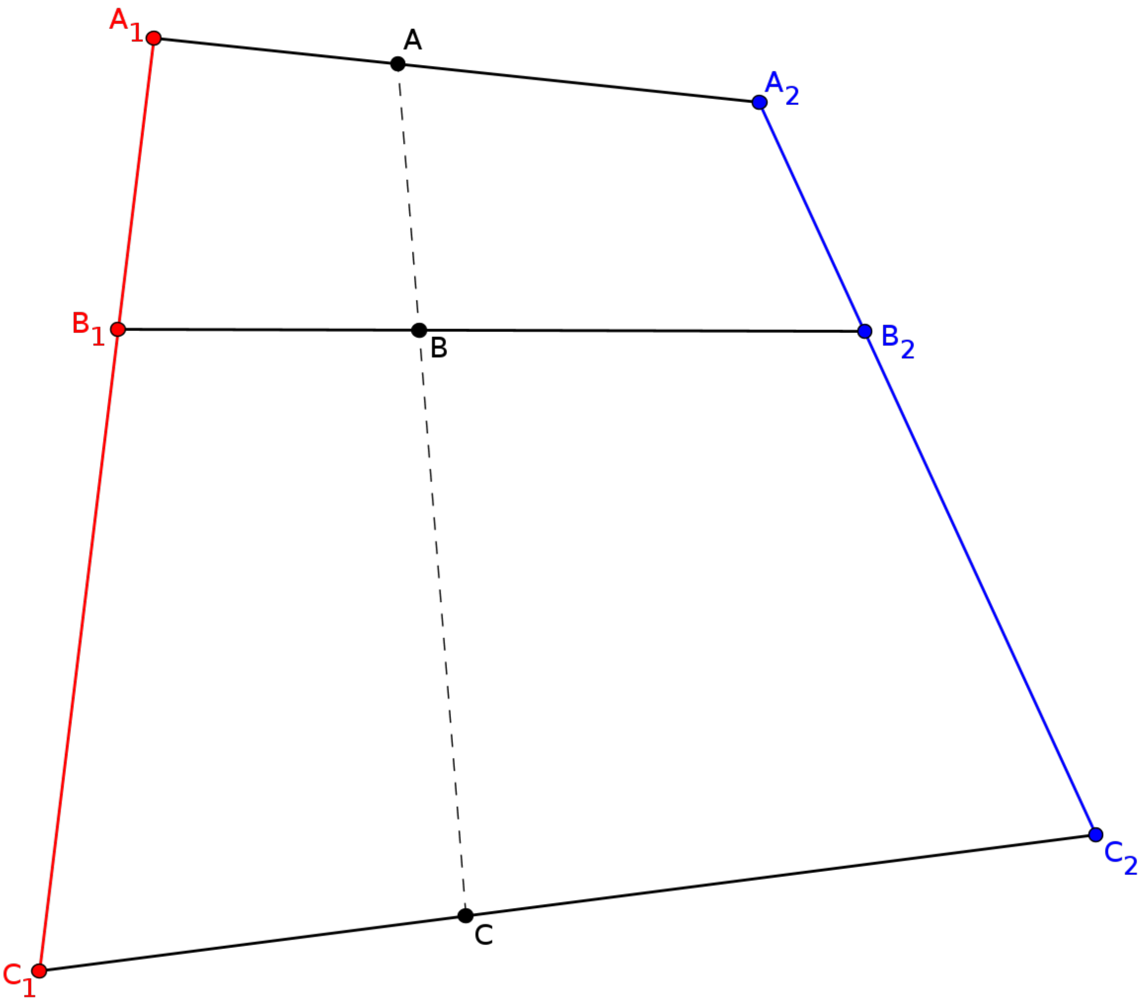

Figure 1.95: E.R.I.Q. Lemma

Solution. A great use of this problem is in proving some midpoints collinear. Line in Newton-Gauss Line and some other such problems (1, 2, 3) where it is asked to proved that some midpoints are collinear. □

Lemma 1.12.2 (Steiner's Isogonal Cevian Lemma) — *In $\triangle ABC$, AA_1, AA_2 are two isogonal* *cevians, with $A_1, A_2 \in BC$. Then we have*

$$\frac{BA_1}{A_1C} \times \frac{BA_2}{A_2C} = \frac{BA^2}{AC^2}$$

Theorem 1.12.3 () — *Let P_1, P_2 be two isogonal conjugates wrt $\triangle ABC$. Then if the Pedal triangle of P_1 is homological wrt to $\triangle ABC$ then so is the Pedal triangle of P_2.*

Theorem 1.12.4 (Erdos-Mordell Theorem (Forum Geometricorum Volume 1 (2001) 7-8)) — *If from a point O inside a given $\triangle ABC$ perpendiculars OD, OE, OF are drawn to its sides, then $OA + OB + OC \geq 2(OD + OE + OF)$. Equality holds if and only if $\triangle ABC$ is equilateral.*

Apparently nothing is needed except "Ptolemy's Theorem". Think of a way to connect OA with OE, OF and the sides of the triangle. As it is the most natural to use AB, AC, we have to deal with BE, CF too. And dealing with lengths is the easiest when we have similar triangles. So we do some construction.

Problem 1.12.1 (ISL 2011 G7) **:** Let $ABCDEF$ be a convex hexagon all of whose sides are tangent to a circle ω with centre O. Suppose that the circumcircle of triangle ACE is concentric with ω. Let J be the foot of the perpendicular from B to CD. Suppose that the perpendicular from B to DF intersects the line EO at a point K. Let L be the foot of the perpendicular from K to DE. Prove that $DJ = DL$.

Solution. There are a LOT of equal lenths, equal angles, and we have a perpendicularity lemma working as well. Why don't we try cosine :0 □

1.13 Pedal Triangles

Definition (Pedal Triangles) — Let P be an arbitrary point, let $\triangle A_1B_1C_1$ be its pedal triangle wrt $\triangle ABC$. Let A', B', C' and A_0, B_0, C_0 be the feet of the altitudes and the midpoints of $\triangle ABC$.

$$B_1C_1 \cap B_0C_0 = A_2,\ C_1A_1 \cap C_0A_0 = B_2,\ A_1B_1 \cap A_0B_0 = C_2$$
$$B'C' \cap B_0C_0 = A_3,\ C'A' \cap C_0A_0 = B_3,\ A'B' \cap A_0B_0 = C_3$$

Theorem 1.13.1 (Fontene's First Theorem) — *A_1A_2, B_1B_2, C_1C_2 are concurrent at the intersection of $\odot A_1B_1C_1$ and $\odot A_0B_0C_0$*

Lemma 1.13.2 — *$A'A_3, B'B_3, C'C_3$ and A_0A_3, B_0B_3, C_0C_3 concur at the nine point circle of $\triangle ABC$.*

Theorem 1.13.3 (Fontene's Second Theorem) — *Let the concurrency point in the first theorem be Q. Then, if the line OP is fixed and P moves along that line, Q will stay fixed.*

The previous result leads to another beautiful result:

Lemma 1.13.4 — *Suppose a varying point P is chosen on the Euler Line of $\triangle ABC$. Then the pedal circle of P wrt $\triangle ABC$ intersects the 9p circle at a fixed point which is the Euler Reflection Point of the median triangle.*

1.14 Pending Problems

Problem 1.14.1 () **:** In $\triangle ABC$, I is the incenter, D is the touch point of the incenter with BC. $AD \cap \odot ABC \equiv X$. The tangents line from X to $\odot I$ meet $\odot ABC$ at Y, Z. Prove that YZ, BC and the tangent at A to $\odot ABC$ concur.

Problem 1.14.2 (IRAN TST 2017 Day 1, P3) **:** In triangle ABC let I_a be the A-excenter. Let ω be an arbitrary circle that passes through A, I_a and intersects the extensions of sides AB, AC (extended from B, C) at X, Y respectively. Let S, T be points on segments I_aB, I_aC respectively such that $\angle AXI_a = \angle BTI_a$ and $\angle AYI_a = \angle CSI_a$. Lines BT, CS intersect at K. Lines KI_a, TS intersect at Z. Prove that X, Y, Z are collinear.

Problem 1.14.3 (IRAN TST 2015 Day 3, P2) **:** In triangle ABC(with incenter I) let the line parallel to BC from A intersect circumcircle of $\triangle ABC$ at A_1 let $AI \cap BC = D$ and E is tangency point of incircle with BC let $EA_1 \cap \odot(\triangle ADE) = T$ prove that $AI = TI$.

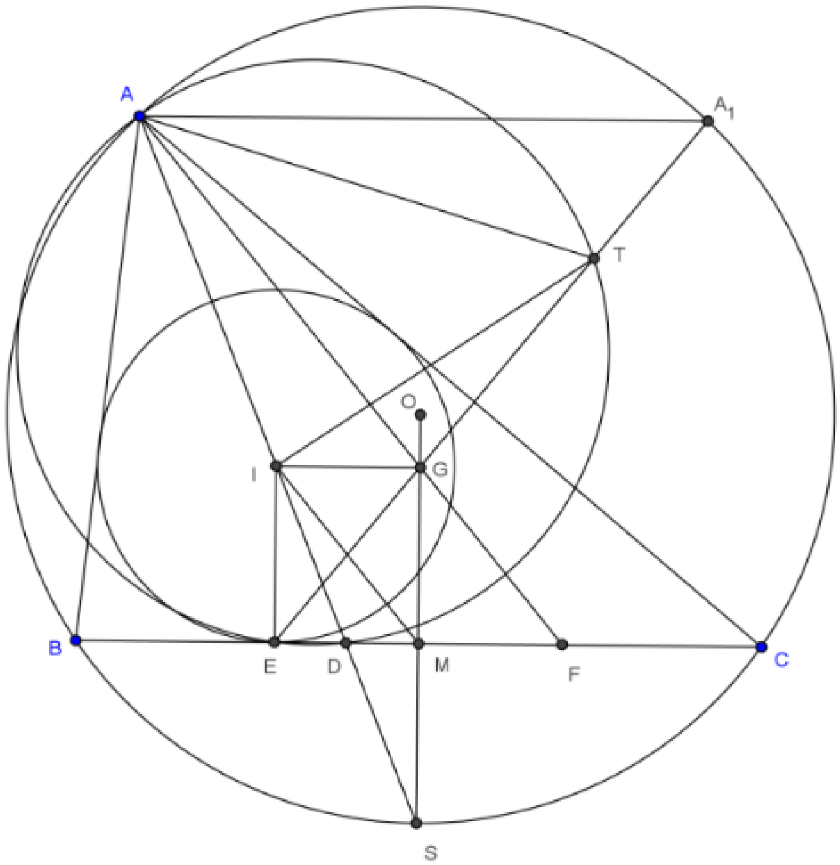

Figure 1.96: IRAN TST 2015 Day 3, P2

Problem 1.14.4 (Generalization of Iran TST 2017 P5) **:** Let ABC be triangle and the points P, Q lies on the side BC s.t B, C, P, Q are all different. The circumcircles of triangles ABP and ACQ intersect again at G. AG intersects BC at M. The circumcircle of triangle APQ intersects AB, AC again at E, F , respectively. EP and FQ intersect at T. The lines through M and parallel to AB, AC, intersect EP, FQ at X, Y, respectively. Prove that the circumcircles of triangle TXY and APQ are tangent to each other.

Problem 1.14.5 (ARMO 2013 Grade 11 Day 2 P4) **:** Let ω be the incircle of the triangle ABC and with center I. Let Γ be the circumcircle of the triangle AIB. Circles ω and Γ intersect at the point X and Y. Let Z be the intersection of the common tangents of the circles ω and Γ. Show that the circumcircle of the triangle XYZ is tangent to the circumcircle of the triangle ABC.

Problem 1.14.6 (AoPS) **:** Let ABC be a triangle with incircle (I) and A-excircle (I_a). $(I),(I_a)$ are tangent to BC at D,P, respectively. Let $(I_1),(I_2)$ be the incircle of triangles APC,APB, respectively, $(J_1),(J_2)$ be the reflections of $(I_1),(I_2)$ wrt midpoints of AC,AB. Prove that AD is the radical axis of (J_1) and (J_2).

Problem 1.14.7 (AoPS) **:** Let ABC be a $A-$right-angled triangle and $MNPQ$ a square inscribed into it, with M,N onto BC in order $B-M-N-C$, and P,Q onto CA,AB respectively. Let $R=BP\cap QM, S=CQ\cap PN$. Prove that $AR=AS$ and RS is perpendicular to the $A-$inner angle bisector of $\triangle ABC$.

Problem 1.14.8 (AoPS) **:** P is an arbitrary point on the plane of $\triangle ABC$ and let $\triangle A'B'C'$ be the cevian triangle of P WRT $\triangle ABC$. The circles $\odot(ABB')$ and $\odot(ACC')$ meet at A,X. Similarly, define the points Y and Z WRT B and C. Prove that the lines AX,BY,CZ concur at the isogonal conjugate of the complement of P WRT $\triangle ABC$.

Problem 1.14.9 (AoPS) **:** Given are $\triangle ABC$, L is Lemoine point,L_a, L_b, L_c are three Lemoine point of triangles LBC,LCA,LAB prove that AL_a,BL_b,CL_a are concurrent!

A question: What is the locus of point P such that AL_a,BL_b,CL_a are concurrent with L_a,L_b,L_c are three 'Lemoine points' of triangles PBC,PCA,PAB?

Problem 1.14.10 (AoPS) **:** Let ABC be a triangle inscribed circle (O). Let (O') be the circle wich is tangent to the circle (O) and the sides CA,AB at D and E,F, respectively. The line BC intersects the tangent line at A of (O), EF and AO' at T,S and L, respectively. The circle (O) intersects AS again at K. Prove that the circumcenter of triangle AKL lies on the circumcircle of triangle ADT.

Problem 1.14.11 () **:** Let P and Q be isogonal conjugates of each other. Let $\triangle XYZ,\triangle KLM$ be the pedal triangles of P and Q wrt $\triangle ABC$. (X,K lie on BC; Y,L lie on CA; Z,M lie on AB) Prove that YM,ZL,PQ are concurrent.

Problem 1.14.12 (2nd Olympiad of Metropolises) **:** Let $ABCDEF$ be a convex hexagon which has an inscribed circle and a circumscribed circle. Denote by ω_A, ω_B, ω_C, ω_D, ω_E, and ω_F the inscribed circles of the triangles FAB,ABC,BCD,CDE,DEF,and EFA,respectively. Let l_{AB} be the external common tangent of ω_A and ω_B other than the line AB; lines $l_{BC},l_{CD},l_{DE},l_{EF}$, and l_{FA} are analogously defined. Let A_1 be the intersection point of the lines l_{FA} and l_{AB}; B_1 be the intersection point of the lines l_{AB} and l_{BC}; points C_1,D_1,E_1, and $F1$ are analogously defined. Suppose that $A_1B_1C_1D_1E_1F_1$ is a convex hexagon. Show that its diagonals A_1D_1, B_1E_1, and C_1F_1

meet at a single point.

Problem 1.14.13 (ISL 2016 G6) **:** Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC < 90^\circ$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at E and F respectively, and meet each other at point P. Let M be the midpoint of AC and let ω be the circumcircle of triangle BPD. Segments BM and DM intersect ω again at X and Y respectively. Denote by Q the intersection point of lines XE and YF. Prove that $PQ \perp AC$.

1.15 Problems

Problem 1.15.1 (IRAN 3rd Round 2016 P2) **:** Let ABC be an arbitrary triangle. Let E, F be two points on AB, AC respectively such that their distance to the midpoint of BC is equal. Let P be the second intersection of the triangles ABC, AEF circumcircles . The tangents from E, F to the circumcircle of AEF intersect each other at K. Prove that : $\angle KPA = 90$

Problem 1.15.2 (IRAN 2nd Round 2016 P6) **:** Let ABC be a triangle and X be a point on its circumcircle. Q, P lie on a line BC such that $XQ \perp AC, XP \perp AB$. Let Y be the circumcenter of $\triangle XQP$. Prove that ABC is equilateral triangle if and if only Y moves on a circle when X varies on the circumcircle of ABC

Problem 1.15.3 (AoPS) **:** Consider ABC with orthic triangle $A'B'C'$, let $AA' \cap B'C' = E$ and E' be reflection of E wrt BC. Let M be midpoint of BC and O be circumcenter of $E'B'C'$. Let M' be projection of O on BC and N be the intersection of a perpendicular to $B'C'$ through E with BC. Prove that $MM' = 1/4MN$.

Problem 1.15.4 (IRAN 3rd Round 2010 D3, P5) **:** In a triangle ABC, I is the incenter. D is the reflection of A to I. the incircle is tangent to BC at point E. DE cuts IG at P (G is centroid). M is the midpoint of BC. Prove that $AP||DM$ and $AP = 2DM$.

Problem 1.15.5 (IRAN 3rd Round 2011 G5) **:** Given triangle ABC, D is the foot of the external angle bisector of A, I its incenter and I_a its A-excenter. Perpendicular from I to DI_a intersects the circumcircle of triangle in A'. Define B' and C' similarly. Prove that AA', BB' and CC' are concurrent.

Problem 1.15.6 (AoPS3) **:** I is the incenter of ABC, $PI, QI \perp BC$, PA, QA intersect BC at DE. Prove: $IADE$ is on a circle.

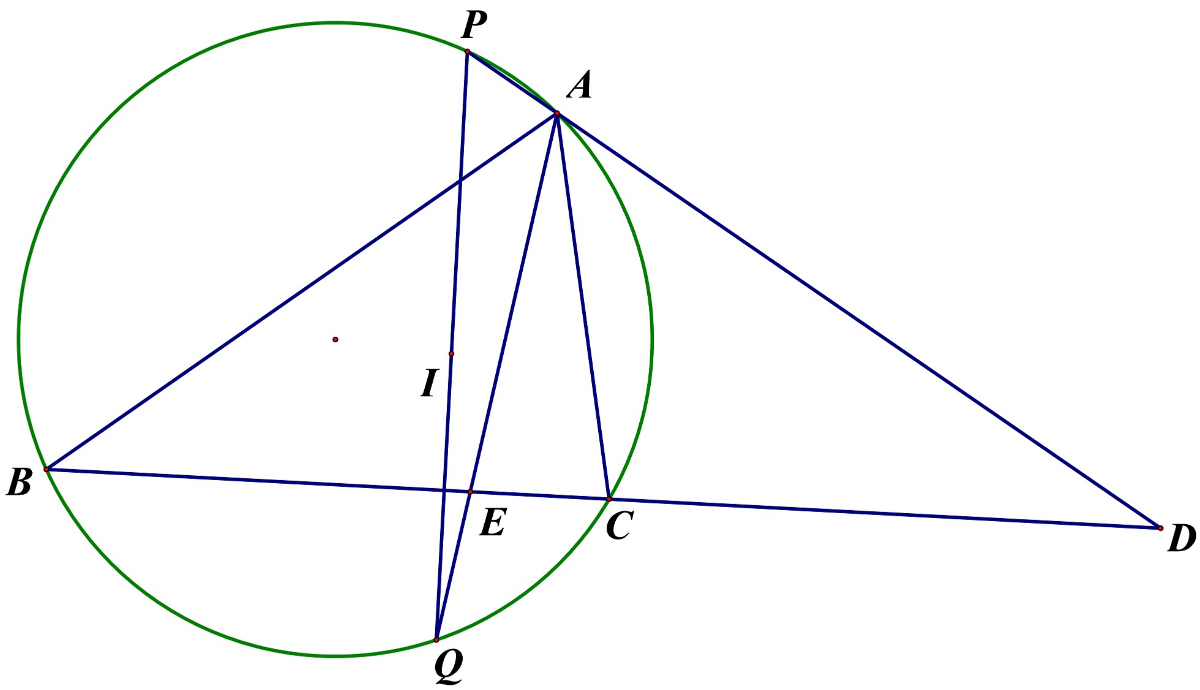

Figure 1.97: AoPS3

Problem 1.15.7 (AoPS4) **:** Given a triangle ABC, the incircle (I) touch BC, CA, AB at D, E, F respectively. Let AA_1, BB_1, CC_1 be $A, B, C - altitude$ respectively. Let N be the orthocenter of the triangle AEF. Prove that N is the incenter of AB_1C_1

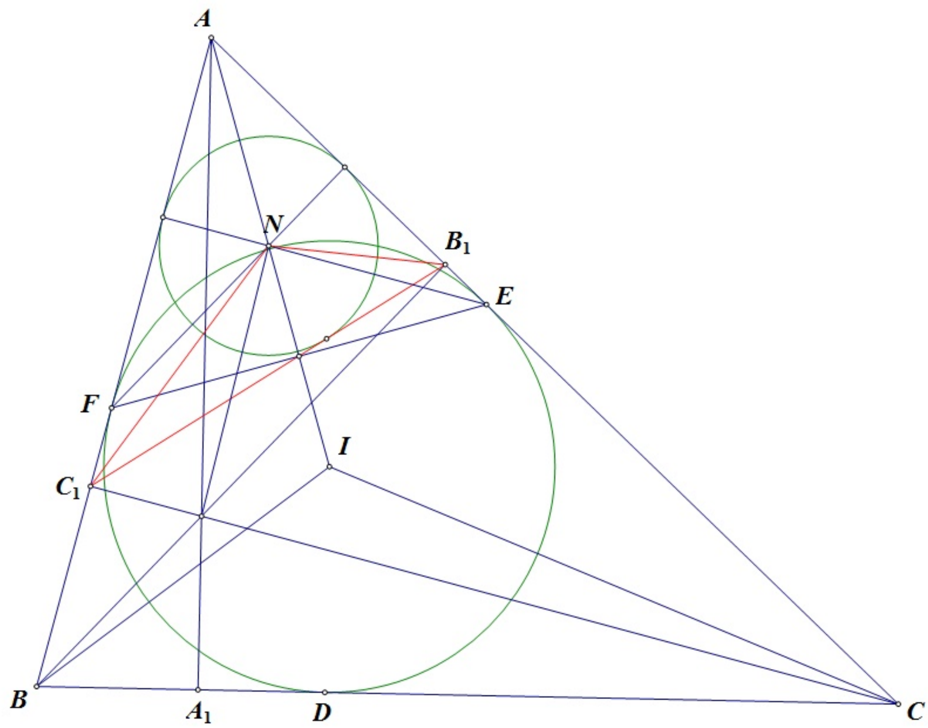

Figure 1.98: AoPS4

Problem 1.15.8 (IRAN TST 2015 Day 2, P3) **:** $ABCD$ is a circumscribed and inscribed quadrilateral. O is the circumcenter of the quadrilateral. E, F and S are the intersections of AB, CD; AD, BC and AC, BD respectively. E' and F' are points on AD and AB such that $\angle AEE' = \angle E'ED$ and $\angle AFF' = \angle F'FB$. X and Y are points on OE' and OF' such that $\frac{XA}{XD} = \frac{EA}{ED}$ and $\frac{YA}{YB} = \frac{FA}{FB}$. M is the midpoint of arc BD of (O) which contains A. Prove that the circumcircles of triangles OXY and OAM are coaxial with the circle with diameter OS.

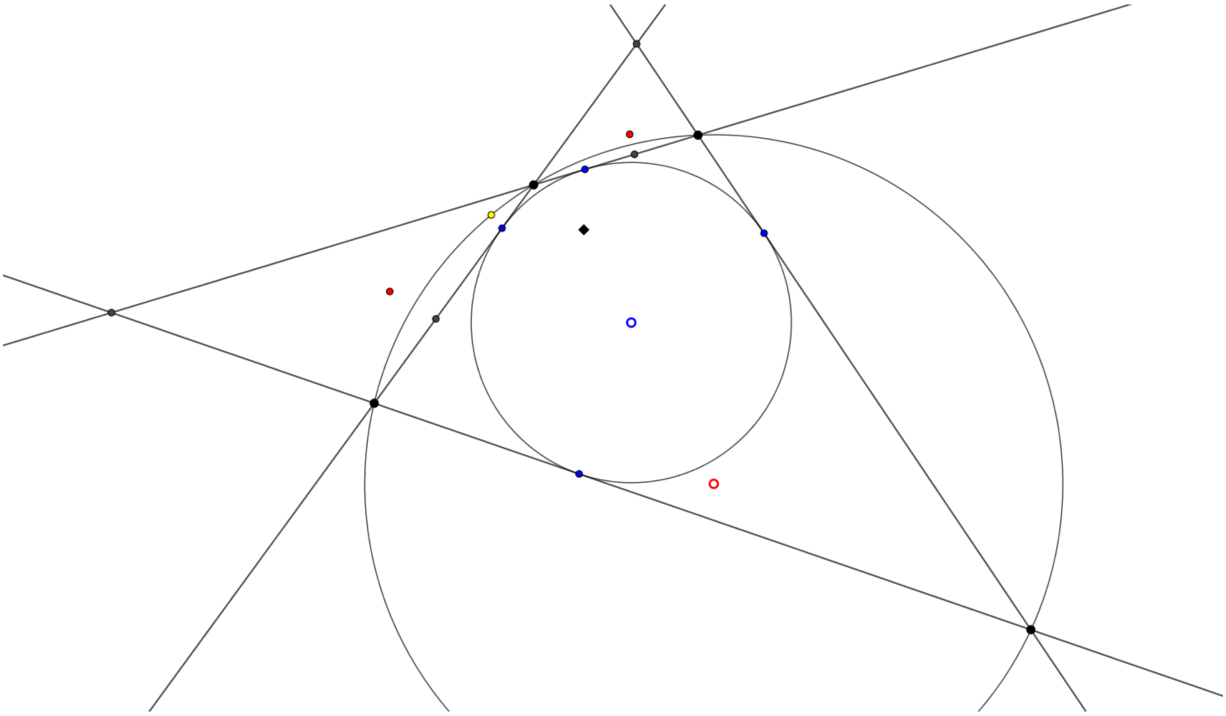

Figure 1.99: Actual Prob

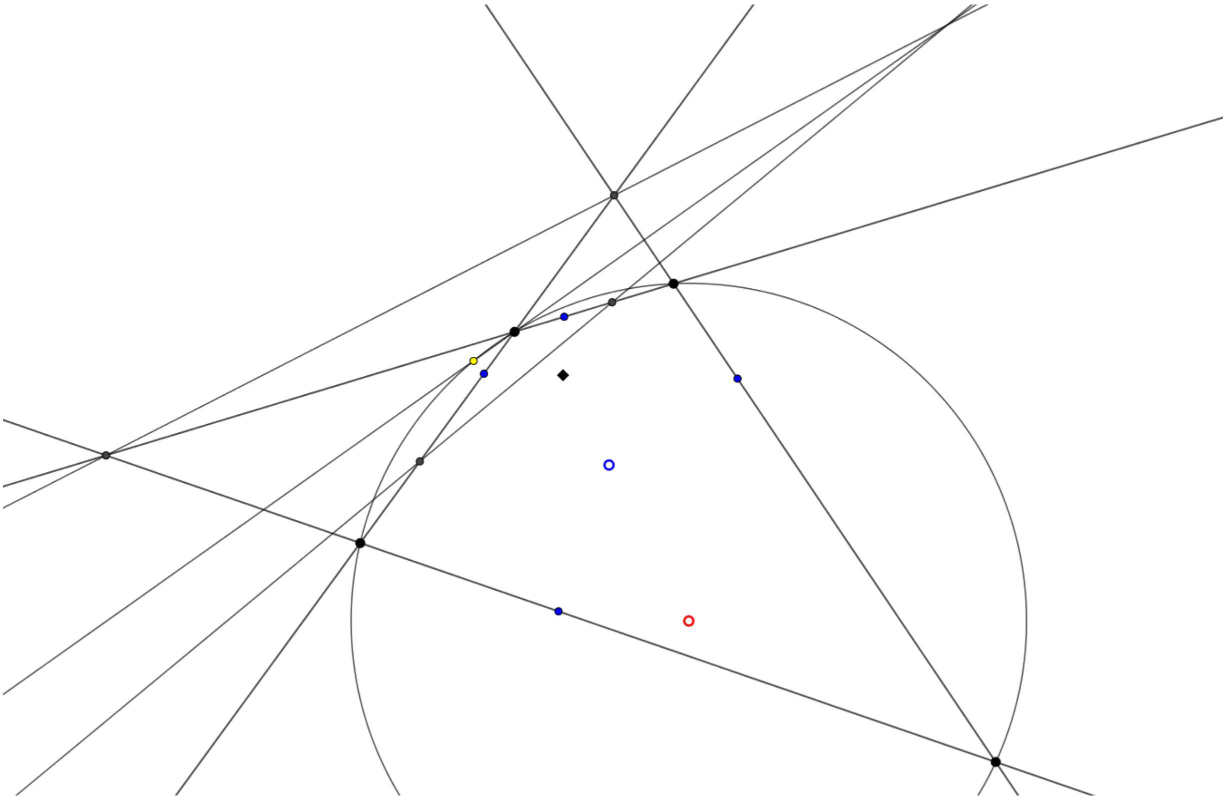

Figure 1.100: Inverted

Problem 1.15.9 (USA TST 2017 P2) **:** Let ABC be an acute scalene triangle with circumcenter O, and let T be on line BC such that $\angle TAO = 90^\circ$. The circle with diameter $\overline{AT}$ intersects the circumcircle of $\triangle BOC$ at two points A_1 and A_2, where $OA_1 < OA_2$. Points B_1, B_2, C_1, C_2 are defined analogously.

1. Prove that $\overline{AA_1}$, $\overline{BB_1}$, $\overline{CC_1}$ are concurrent.
2. Prove that $\overline{AA_2}$, $\overline{BB_2}$, $\overline{CC_2}$ are concurrent on the Euler line of triangle ABC.

Problem 1.15.10 (AoPS2) **:** Let ABC be a triangle with circumcenter O and altitude AH. AO meets BC at M and meets the circle (BOC) again at N. P is the midpoint of MN. K is the projection of P on line AH. Prove that the circle (K, KH) is tangent to the circle (BOC).

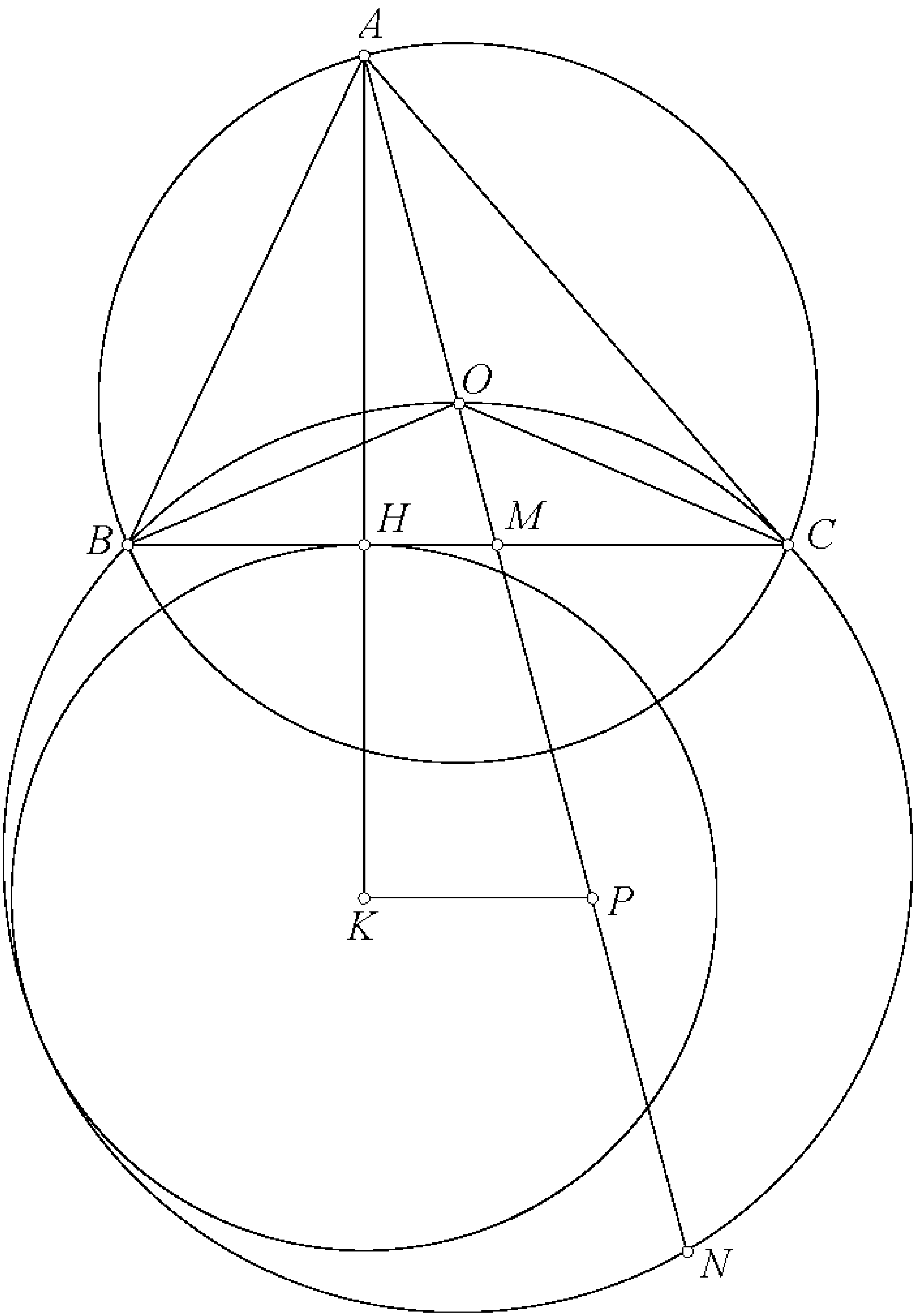

Figure 1.101: AoPS2

| ***Solution.*** Inversion all the way... □

Problem 1.15.11 (AoPS5) **:** Let ABC be a triangle inscribed in (O) and P be a point. Call P' be the isogonal conjugate point of P. Let A' be the second intersection of AP' and (O). Denote by M the intersection of BC and $A'P$. Prove that $P'M \parallel AP$.

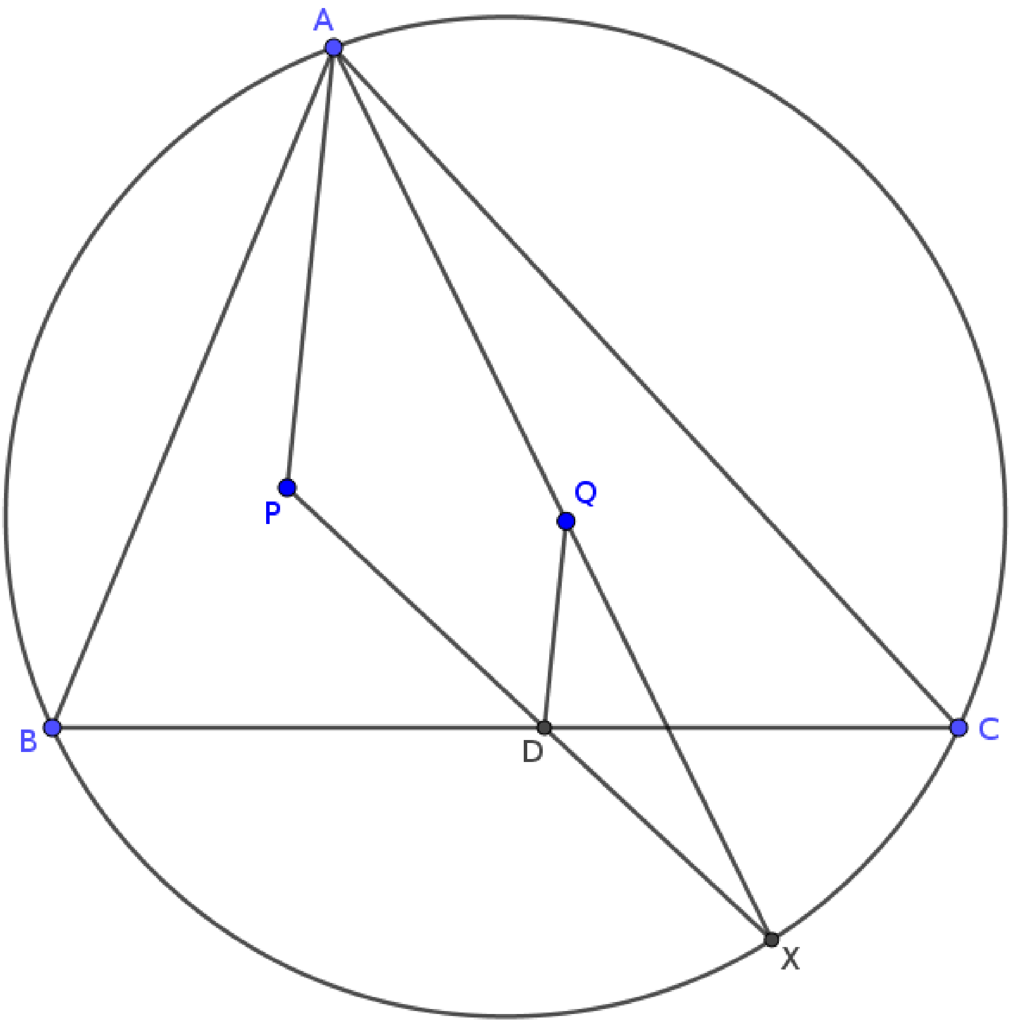

Figure 1.102: AoPS5

Problem 1.15.12 (AoPS) **:** I is the incenter of a non-isosceles triangle $\triangle ABC$. If the incircle touches BC, CA, AB at A_1, B_1, C_1 respectively, prove that the circumcentres of the triangles $\triangle AIA_1$, $\triangle BIB_1$, $\triangle CIC_1$ are collinear.

Problem 1.15.13 (AoPS) **:** Given $\triangle ABC$ and a point P inside. AP cuts BC at M. Let M', A' be the reflection of M, A in the perpendicular bisector of BC. $A'P$ cuts the perpendicular bisector of BC at N. Let Q be the isogonal conjugate of P in triangle ABC. Prove that $QM' \parallel AN$.

Problem 1.15.14 (IRAN 3rd Round 2016 G6) **:** Given triangle $\triangle ABC$ and let D, E, F be the foot of angle bisectors of A, B, C ,respectively. M, N lie on EF such that $AM = AN$. Let H be the foot of A-altitude on BC.
Points K, L lie on EF such that triangles $\triangle AKL, \triangle HMN$ are correspondingly similar (with the given order of vertices's) such that $AK \nparallel HM$ and $AK \nparallel HN$. Show that: $DK = DL$.

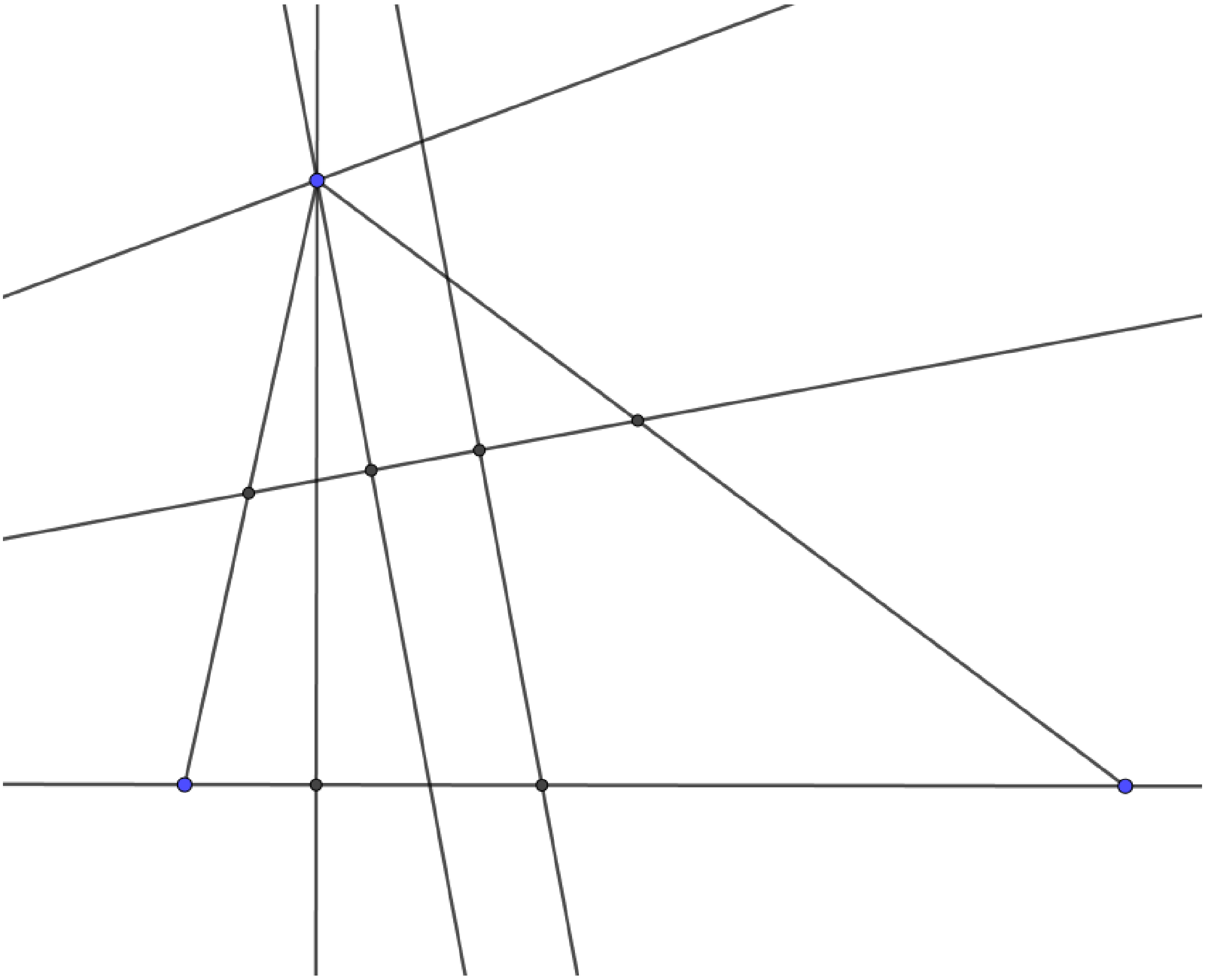

Figure 1.103: IRAN 3rd Round 2016 G6

Problem 1.15.15 (Iran TST 2017 T3 P6) **:** In triangle ABC let O and H be the circumcenter and the orthocenter. The point P is the reflection of A with respect to OH. Assume that P is not on the same side of BC as A. Points E, F lie on AB, AC respectively such that $BE = PC$, $CF = PB$. Let K be the intersection point of AP, OH. Prove that $\angle EKF = 90°$.

Spiral Similarity (points on AB, AC with some properties)

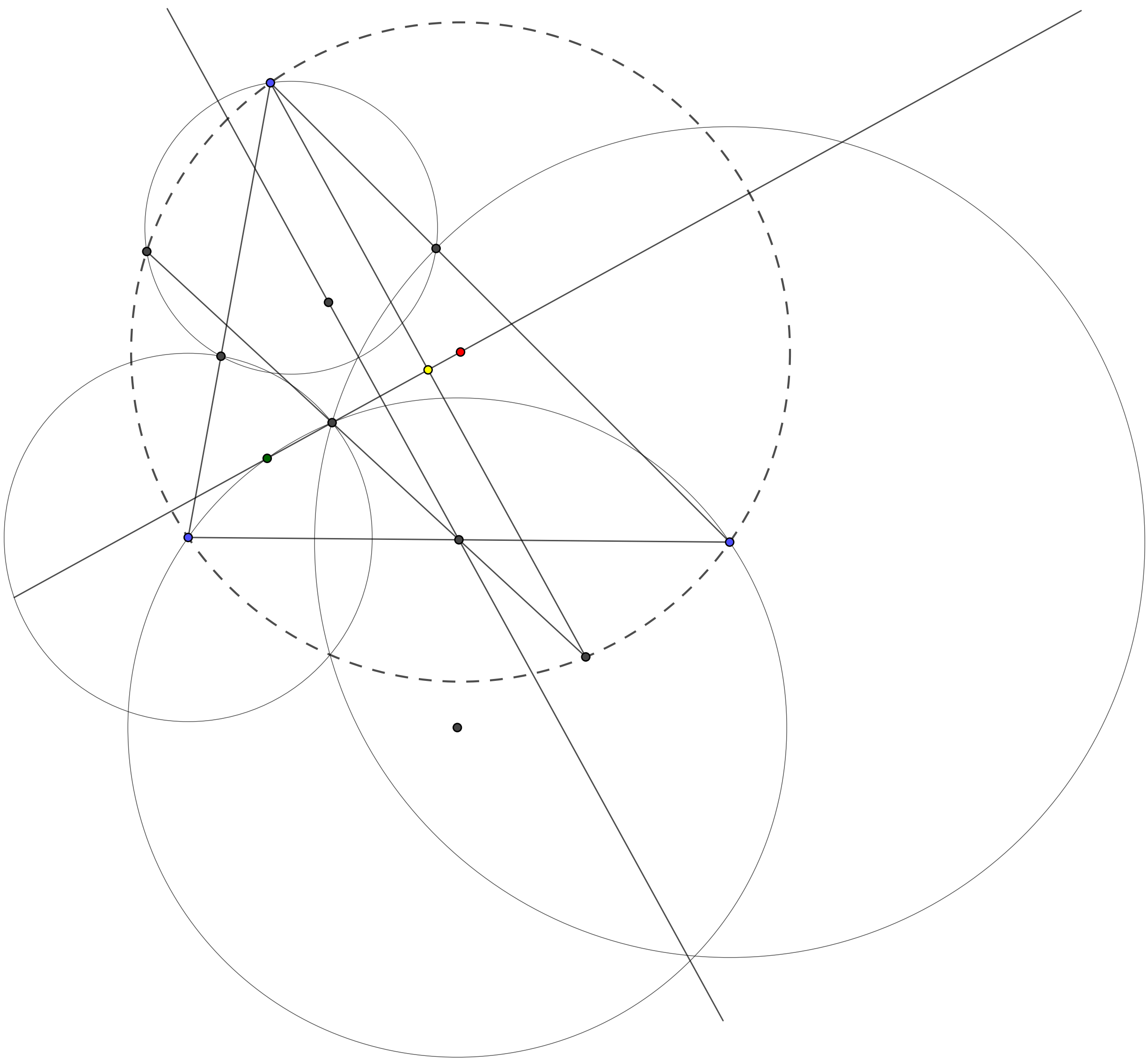

Figure 1.104: Iran TST 2017 T3 P6

Problem 1.15.16 (IRAN 3rd Round 2010 D3, P6) **:** In a triangle ABC, $\angle C = 45°$. AD is the altitude of the triangle. X is on AD such that $\angle XBC = 90 - \angle B$ (X is inside of the triangle). AD and CX cut the circumcircle of ABC in M and N respectively. Ff the tangent to $\odot ABC$ at M cuts AN at P, prove that P, B and O are collinear.

Cross-Ratio

Problem 1.15.17 (Iran TST 2014 T1P6) **:** I is the incenter of triangle ABC. perpendicular from

I to AI meet AB and AC at B' and C' respectively. Suppose that B'' and C'' are points on half-line BC and CB such that $BB'' = BA$ and $CC'' = CA$. Suppose that the second intersection of circumcircles of $AB'B''$ and $AC'C''$ is T. Prove that the circumcenter of AIT is on the BC.

projective, inversion

> ***Solution.*** Too many collinearity, need to prove concurrency, what else can come into mind except projective approach. □

> ***Solution.*** Too many incenter related things, $\sqrt{bc}$-inversion :o □

Problem 1.15.18 (APMO 2014 P5) **:** Circles ω and Ω meet at points A and B. Let M be the midpoint of the arc AB of circle ω (M lies inside Ω). A chord MP of circle ω intersects Ω at Q (Q lies inside ω). Let ℓ_P be the tangent line to ω at P, and let ℓ_Q be the tangent line to Ω at Q. Prove that the circumcircle of the triangle formed by the lines ℓ_P, ℓ_Q and AB is tangent to Ω.

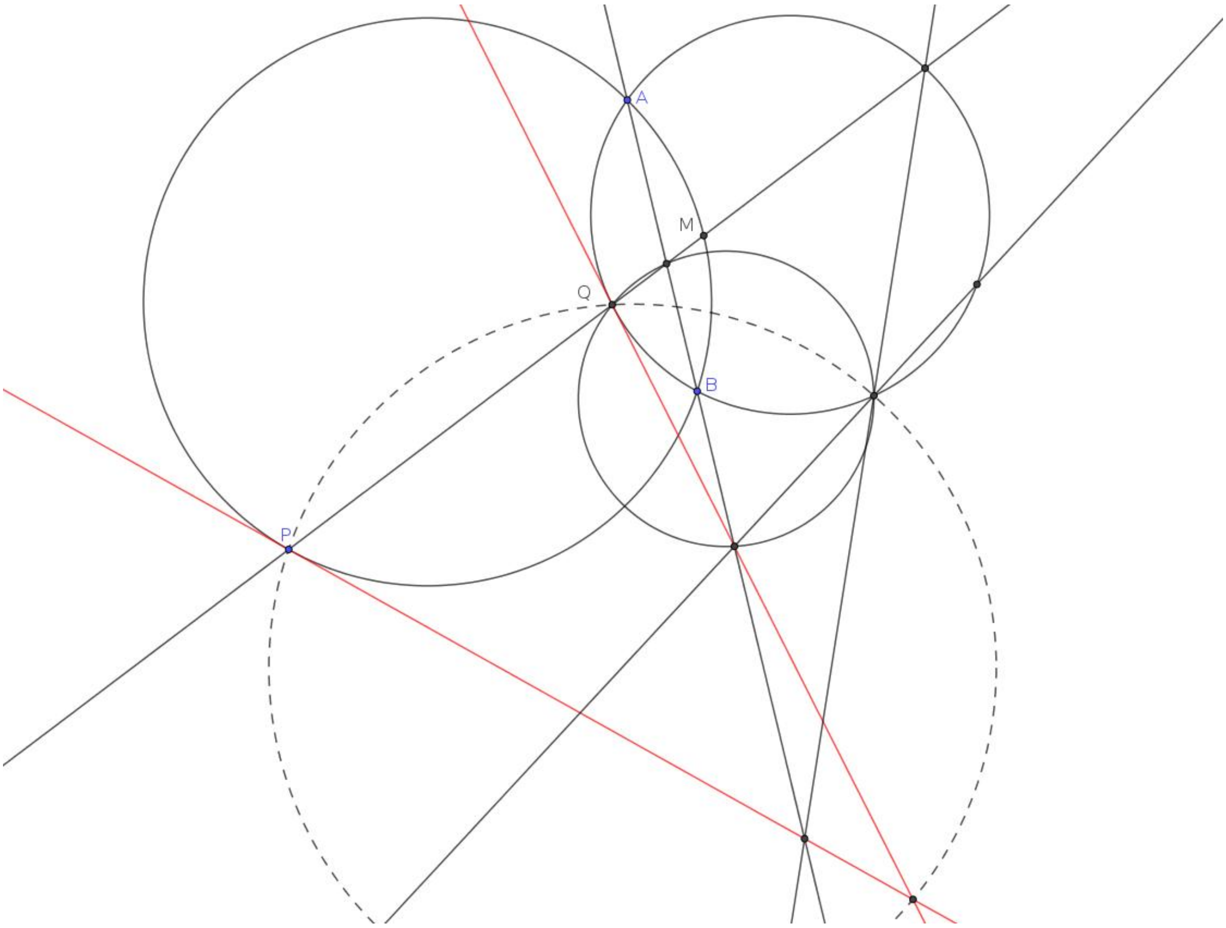

Figure 1.105: APMO 2014 P5

Problem 1.15.19 () **:** Let ABC be a triangle, D, E, F are the feet of the altitudes, $DF \cap BE \equiv P, DE \cap CF \equiv Q$. Prove that the perpendicular from A to PQ goes through the reflection of O on BC.

projective

Solution. Projective approach. □

Problem 1.15.20 (RMM 2018 P6) **:** Fix a circle Γ, a line ℓ to tangent Γ, and another circle Ω disjoint from ℓ such that Γ and Ω lie on opposite sides of ℓ. The tangents to Γ from a variable point X on Ω meet ℓ at Y and Z. Prove that, as X varies over Ω, the circumcircle of XYZ is tangent to two fixed circles.

inversion

Solution. Too many circles, plus tangency, what else other than inversion? After the inversion the problem turns into a pretty obvious work-around problem. □

Problem 1.15.21 (AoPS6) **:** Let O and I be the circumcenter and incenter of ΔABC. Draw circle ω so that $B, C \in \omega$ and ω touches (I) internally at P. AI intersects BC at X. Tangent at X to (I) which is different from BC, intersects tangent at P to (I) at S. $SA \cap (O) = T \neq A$. Prove that $\angle ATI = 90°$

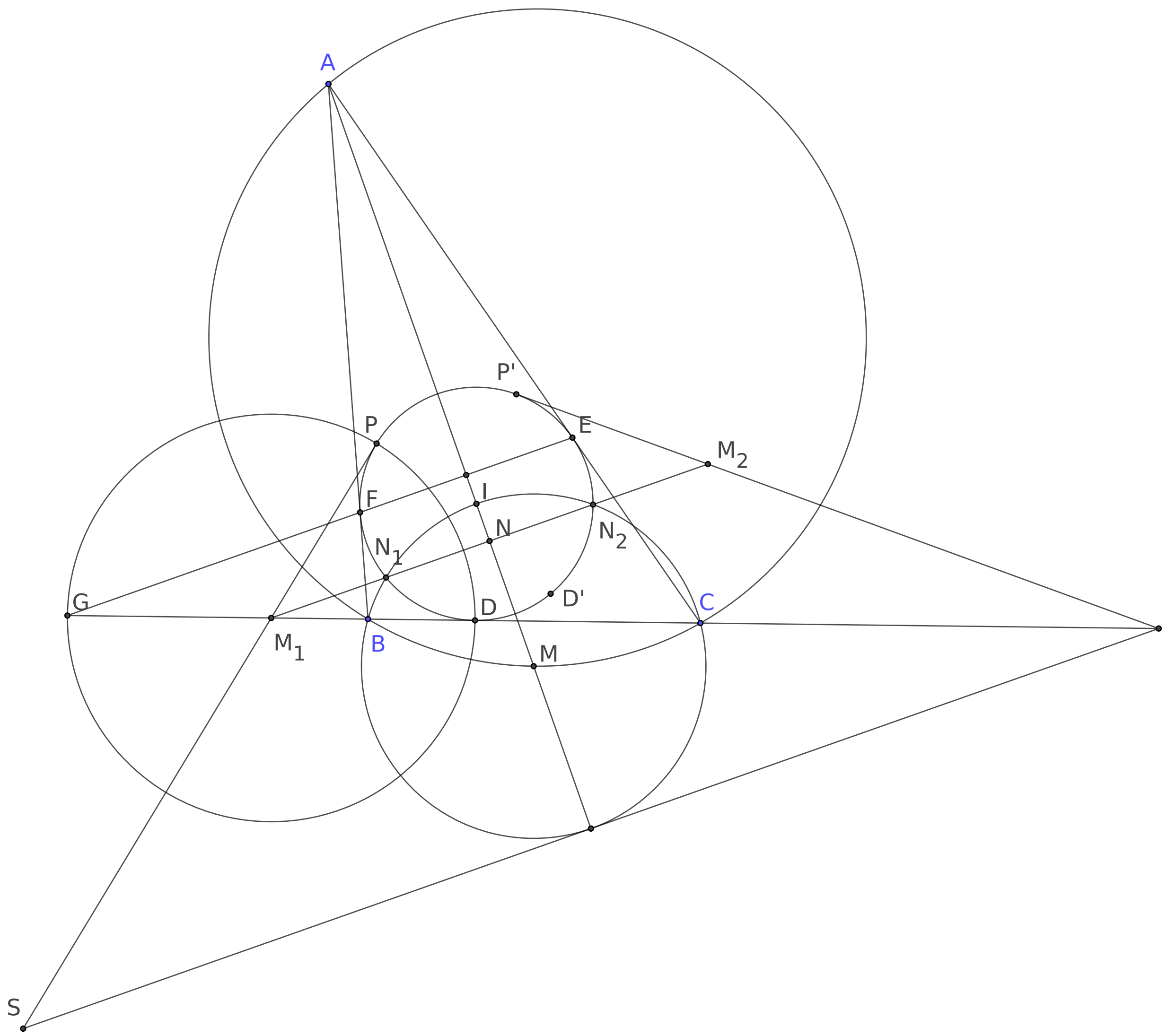

Figure 1.106: Solution 1

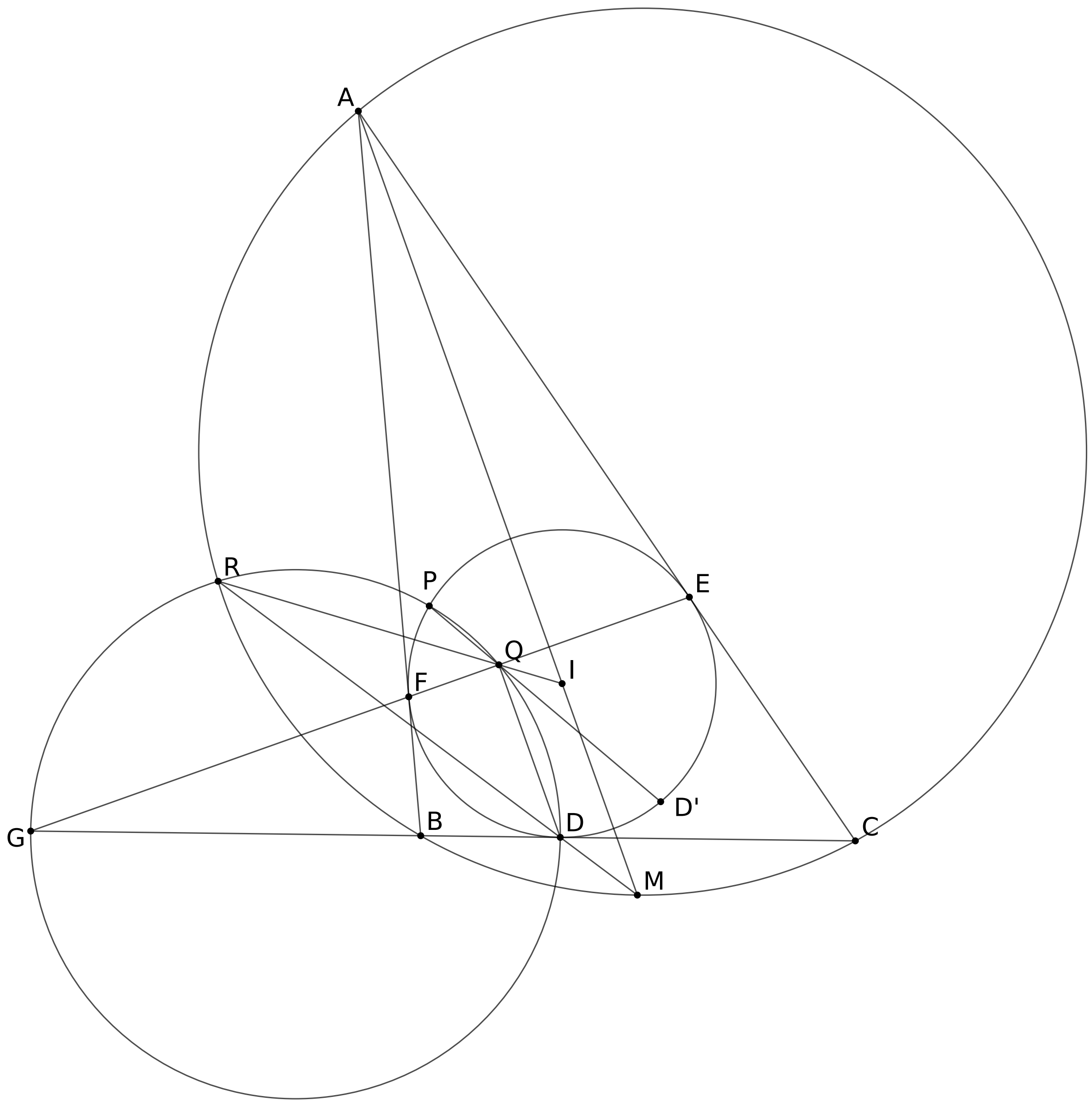

Figure 1.107: Solution 2

Problem 1.15.22 (AoPS7) **:** Let ABC be a triangle with incenter I and circumcircle Γ. Let the line through I perpendicular to AI meet AB at E and AC at F. Let the circumcircles of triangles AIB and AIC intersect the circumcircle of triangle AEF ω again at points M and N, and let ω intersect Γ again at Q. Prove that AQ, MN, and BC are concurrent.

Problem 1.15.23 (AoPS) **:** Given a circle (O) with center O and A, B are 2 fixed points on (O). E lies on AB. C, D are on (O) and CD pass through E. P lies on the ray DA, Q lies on the ray DB such that E is the midpoint of PQ. Prove that the circle passing through C and touch PQ at E also pass through the midpoint of AB

Problem 1.15.24 (WenWuGuangHua Mathematics Workshop) **:** O_B, O_C are the B and C mixtilinear centers respectively. (O_B) touches BC, AB at X_B, Y_B respectively, and $X_BY_B \cap O_BO_C$ at Z_B. Define X_C, Y_C, Z_C similarly. Prove that if $BZ_C \cap CZ_B = T$, then AT is the A-angle bisector.

Problem 1.15.25 (All Russia 1999 P9.3) **:** A triangle ABC is inscribed in a circle S. Let A_0 and C_0 be the midpoints of the arcs BC and AB on S, not containing the opposite vertex, respectively. The circle S_1 centered at A_0 is tangent to BC, and the circle S_2 centered at C_0 is tangent to AB. Prove that the incenter I of $\triangle ABC$ lies on a common tangent to S_1 and S_2.

Problem 1.15.26 (All Russia 2000 P11.7) **:** A quadrilateral $ABCD$ is circumscribed about a circle ω. The lines AB and CD meet at O. A circle ω_1 is tangent to side BC at K and to the extensions of sides AB and CD, and a circle ω_2 is tangent to side AD at L and to the extensions of sides AB and CD. Suppose that points O, K, L lie on a line. Prove that the midpoints of BC and AD and the center of ω also lie on a line.

Problem 1.15.27 (All Russia 2000 P9.3) **:** Let O be the center of the circumcircle ω of an acute-angle triangle ABC. A circle ω_1 with center K passes through A, O, C and intersects AB at M and BC at N. Point L is symmetric to K with respect to line NM. Prove that $BL \perp AC$.

Problem 1.15.28 (WenWuGuangHua Mathematics Workshop) **:**

1. AD, BE, CF are concurrent cevians. Angle bisectors of $\angle ADB$ and $\angle AEB$ meet at C_0. Again the angle bisectors of $\angle ADC$ and $\angle AFC$ meet at B_0. And bisectors of $\angle BEC$ and $\angle BFC$ meet at A_0. Prove that AA_0, BB_0, CC_0 are concurrent.
2. Angle bisectors of $\angle AEB$ and $\angle AFC$ meet at D_0, of $\angle BFC$ and BDA meet at E_0, and of $\angle CEB$ and $\angle CDA$ meet at F_0. Prove that DD_0, EE_0, FF_0 are concurrent.

Solution. As this problem is purely made up with lines, we can do a projective transformation to simplify the problem. And as there are perpendicularity at D, E, F, we make D, E, F the feet of the altitudes of $\triangle ABC$. Then the angle bisector properties get replaced by simpler properties wrt DEF. □

Problem 1.15.29 (WenWuGuangHua Mathematics Workshop) **:** Generalization: Let AD, BE, CF

be any cevians concurrent at T. $AD \cap EF = A'$, $BE \cap DF = B'$, $CF \cap DE = C'$, $B'A' \cap AC = X$, $B'A' \cap BC = Y$, $C'X \cap EF = Z$. Prove that T, Y, Z are collinear.

Problem 1.15.30 (AoPS) **:** On circumcircle of triangle ABC, T and K are midpoints of arcs BC and BAC respectively . And E is foot of altitude from C on AB . Point P is on extension of AK such that PE is perpendicular to ET . Prove that $PC = CK$.

Problem 1.15.31 (USJMO 2018 P3) **:** Let $ABCD$ be a quadrilateral inscribed in circle ω with $\overline{AC} \perp \overline{BD}$. Let E and F be the reflections of D over lines BA and BC, respectively, and let P be the intersection of lines BD and EF. Suppose that the circumcircle of $\triangle EPD$ meets ω at D and Q, and the circumcircle of $\triangle FPD$ meets ω at D and R. Show that $EQ = FR$.

Problem 1.15.32 (All Russia 2002 P11.6) **:** The diagonals AC and BD of a cyclic quadrilateral $ABCD$ meet at O. The circumcircles of triangles AOB and COD intersect again at K. Point L is such that the triangles BLC and AKD are similar and equally oriented. Prove that if the quadrilateral $BLCK$ is convex, then it has an incircle.

Problem 1.15.33 (WenWuGuangHua Mathematics Workshop) **:** Let O_B, O_C be the B, C mixtilinear excircles. O meet CA, CB at X_C, Y_C and O_B meet BA, BC at X_B, Y_B. Let I_C be the C-excircle. $I_C Y_B$ meet $O_B O_C$ at T. Prove that $BT \perp O_B O_C$

Solution. From what we have to prove, we find two circles, from where we get another circle. This circle suggests that we try power of point. $\square$

Problem 1.15.34 (Iran TST 2018 T1P3) **:** In triangle ABC let M be the midpoint of BC. Let ω be a circle inside of ABC and is tangent to AB, AC at E, F, respectively. The tangents from M to ω meet ω at P, Q such that P and B lie on the same side of AM. Let $X \equiv PM \cap BF$ and $Y \equiv QM \cap CE$. If $2PM = BC$ prove that XY is tangent to ω.

Problem 1.15.35 (Iran TST 2018 T1P4) **:** Let ABC be a triangle ($\angle A \neq 90°$). BE, CF are the altitudes of the triangle. The bisector of $\angle A$ intersects EF, BC at M, N. Let P be a point such that $MP \perp EF$ and $NP \perp BC$. Prove that AP passes through the midpoint of BC.

Solution. :'3 kala para na T_T $\square$

Problem 1.15.36 (Iran TST 2018 T3P6) **:** Consider quadrilateral $ABCD$ inscribed in circle ω.

$AC \cap BD = P$. E, F lie on sides AB, CD, respectively such that $\angle APE = \angle DPF$. Circles ω_1, ω_2 are tangent to ω at X, Y respectively and also both tangent to the circumcircle of PEF at P. Prove that:

$$\frac{EX}{EY} = \frac{FX}{FY}$$

Solution. fucking beautiful. □

Problem 1.15.37 (ISL 2006 G6) **:** Circles ω_1 and ω_2 with centres O_1 and O_2 are externally tangent at point D and internally tangent to a circle ω at points E and F respectively. Line t is the common tangent of ω_1 and ω_2 at D. Let AB be the diameter of ω perpendicular to t, so that A, E, O_1 are on the same side of t. Prove that lines AO_1, BO_2, EF and t are concurrent.

Problem 1.15.38 (ISL 2006 G7) **:** In a triangle ABC, let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB, respectively, and T_a, T_b, T_c be the midpoints of the arcs BC, CA, AB of the circumcircle of ABC, not containing the vertices's A, B, C, respectively. For $i \in a, b, c$, let w_i be the circle with M_iT_i as diameter. Let p_i be the common external common tangent to the circles w_j and w_k (for all $i, j, k = a, b, c$) such that w_i lies on the opposite side of p_i than w_j and w_k do.

Prove that the lines p_a, p_b, p_c form a triangle similar to ABC and find the ratio of similitude

Problem 1.15.39 (ISL 2006 G9) **:** Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC, respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2, respectively ($A_2 \neq A, B_2 \neq B, C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

Solution. In this type of "Miquel's Point and the intersections of the circumcircles" related problems, it is useful to think about the second intersections of the lines joining the first intersections and the Miquel's Point with the main circle. □

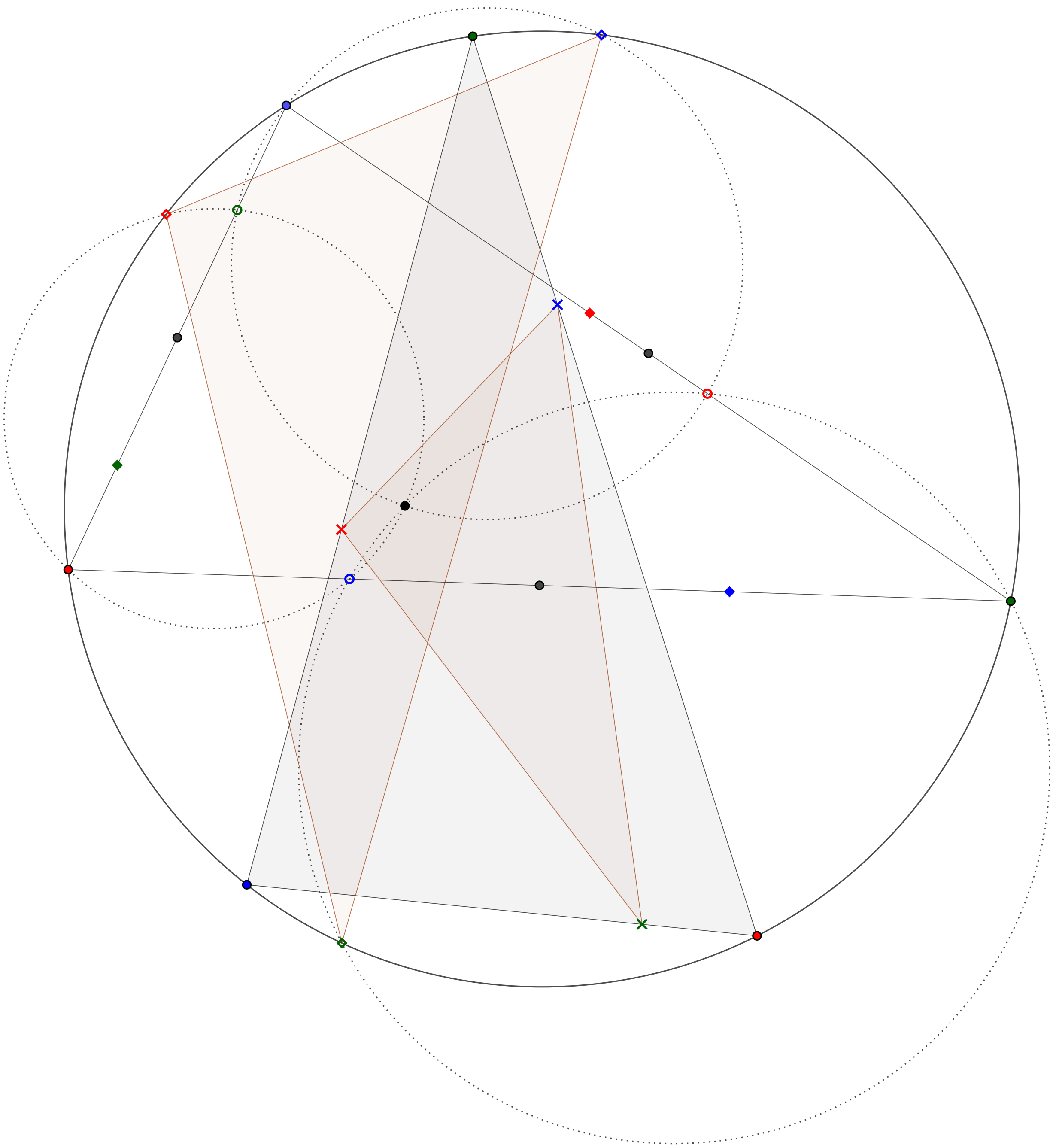

Figure 1.108: IMO Shortlist G9

Problem 1.15.40 (Iran TST 2017 P5) **:** In triangle ABC, arbitrary points P, Q lie on side BC such that $BP = CQ$ and P lies between B, Q. The circumcircle of triangle APQ intersects sides AB, AC at E, F respectively. The point T is the intersection of EP, FQ. Two lines passing through the midpoint of BC and parallel to AB and AC, intersect EP and FQ at points X, Y respectively.

Prove that the circumcircle of triangle TXY and triangle APQ are tangent to each other.

Problem 1.15.41 () **:** Let X be the touchpoint of the incircle with BC and let AX meet $\cdot ABC$ at D. The tangents from D to the incircle meet $\cdot ABC$ at E, F. Prove that the tangent to the circumcircle at A, EF and BC are concurrent.

Problem 1.15.42 (ISL 2012 G8) **:** Let ABC be a triangle with circumcircle ω and ℓ a line without common points with ω. Denote by P the foot of the perpendicular from the center of ω to ℓ. The side-lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove that the circumcircles of the triangles AXP, BYP and CZP have a common point different from P or are mutually tangent at P.

Solution. Using Cross ratio and Desergaus's Involution Theorem. □

Problem 1.15.43 () **:** Suppose an involution on a line l sending X, Y, Z to X', Y', Z'. Let l_x, l_y, l_z be three lines passing through X, Y, Z respectively. And let $X_0 = l_y \cap l_z,\ Y_0 = l_x \cap l_z,\ Z_0 = l_x \cap l_y$. Then X_0X', Y_0Y', Z_0Z' are concurrent.

Problem 1.15.44 (USAMO 2018 P5) **:** In convex cyclic quadrilateral $ABCD$, we know that lines AC and BD intersect at E, lines AB and CD intersect at F, and lines BC and DA intersect at G. Suppose that the circumcircle of $\triangle ABE$ intersects line CB at B and P, and the circumcircle of $\triangle ADE$ intersects line CD at D and Q, where C, B, P, G and C, Q, D, F are collinear in that order. Prove that if lines FP and GQ intersect at M, then $\angle MAC = 90°$.

Problem 1.15.45 (Japan MO 2017 P3) **:** Let ABC be an acute-angled triangle with the circumcenter O. Let D, E and F be the feet of the altitudes from A, B and C, respectively, and let M be the midpoint of BC. AD and EF meet at X, AO and BC meet at Y, and let Z be the midpoint of XY. Prove that A, Z, M are collinear.

Problem 1.15.46 (ISL 2002 G1) **:** Let B be a point on a circle S_1, and let A be a point distinct from B on the tangent at B to S_1. Let C be a point not on S_1 such that the line segment AC meets S_1 at two distinct points. Let S_2 be the circle touching AC at C and touching S_1 at a point D on the opposite side of AC from B. Prove that the circumcenter of triangle BCD lies on the circumcircle of triangle ABC.

Problem 1.15.47 (ISL 2002 G2) **:** Let ABC be a triangle for which there exists an interior point

F such that $\angle AFB = \angle BFC = \angle CFA$. Let the lines BF and CF meet the sides AC and AB at D and E respectively. Prove that

$$AB + AC \geq 4DE.$$

Solution. Pari nai. □

Problem 1.15.48 (ISL 2002 G3) **:** The circle S has center O, and BC is a diameter of S. Let A be a point of S such that $\angle AOB < 120°$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and at F. Prove that I is the incenter of the triangle CEF.

Problem 1.15.49 (ISL 2002 G4) **:** Circles S_1 and S_2 intersect at points P and Q. Distinct points A_1 and B_1 (not at P or Q) are selected on S_1. The lines A_1P and B_1P meet S_2 again at A_2 and B_2 respectively, and the lines A_1B_1 and A_2B_2 meet at C. Prove that, as A_1 and B_1 vary, the circumcentres of triangles A_1A_2C all lie on one fixed circle.

Problem 1.15.50 (ISL 2002 G7) **:** The incircle Ω of the acute-angled triangle ABC is tangent to its side BC at a point K. Let AD be an altitude of triangle ABC, and let M be the midpoint of the segment AD. If N is the common point of the circle Ω and the line KM (distinct from K), then prove that the incircle Ω and the circumcircle of triangle BCN are tangent to each other at the point N.

Problem 1.15.51 (Japan MO 2017 P3) **:** Let ABC be an acute-angled triangle with the circumcenter O. Let D, E and F be the feet of the altitudes from A, B and C, respectively, and let M be the midpoint of BC. AD and EF meet at X, AO and BC meet at Y, and let Z be the midpoint of XY. Prove that A, Z, M are collinear.

Problem 1.15.52 (India TST) **:** ABC triangle, D, E, F touchpoints, M midpoint of BC, K orthocenter of $\triangle AIC$, prove that $MI \perp KD$

Problem 1.15.53 (ISL 2009 G3) **:** Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelogram. Prove that $GR = GS$.

Solution. Point Circle, distance same means Power same wrt point circles. □

Problem 1.15.54 (ARO 2018 P11.6) **:** Three diagonals of a regular n-gon prism intersect at an interior point O. Show that O is the center of the prism.

(The diagonal of the prism is a segment joining two vertices's not lying on the same face of the prism.)

Problem 1.15.55 (ISL 2011 G4) **:** Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.

Problem 1.15.56 () **:** Given 3 circle, construct another circle that is tangent to these three circles.

Solution. A trick to remember: decreasing the radius's of some circles doesn't effect much. □

Problem 1.15.57 () **:** Let $ABCD$ be a convex quadrilateral, let $AD \cap BC = P$. Let O, O'; H, H' be the circumcentres and orthocenter of $\triangle PCD, \triangle PAB$. $\odot DOC$ is tangent to $\odot AD'B$, if and only if $\odot DHC$ is tangent to $\odot AH'B$

Problem 1.15.58 (Iran MO 3rd round 2017 mid-terms Geometry P3) **:** Let ABC be an acute-angle triangle. Suppose that M be the midpoint of BC and H be the orthocenter of ABC. Let $F \equiv BH \cap AC$ and $E \equiv CH \cap AB$. Suppose that X be a point on EF such that $\angle XMH = \angle HAM$ and A, X are in the distinct side of MH. Prove that AH bisects MX.

1.16 Research Stuffs for later

Problem 1.16.1 (AoPS) **:** Let ABC be a triangle with incenter I. L_a, L_b, L_c are symmedian points of triangles IBC, ICA, IAB. Let X, Y, Z be the reflections of I through L_a, L_b, L_c.

- Prove that AX, BY, CZ and OI are concurrent.
- Let I_a, I_b, I_c be the excenters of ABC. Prove that I_aX, I_bY, I_cZ are concurrent at a point P and isogonal conjugate of P with respect to triangle $I_aI_bI_c$ lies on Euler line of ABC.

Problem 1.16.2 (buratinogigle Tough P1) **:** Let ABC be a triangle inscribed in circle (O) with A-excircle (J). Circle passing through A, B touches (J) at M. Circle passing through A, C touches (J) at N. BM cuts CN at P. Prove that AP passes through tangent point of A-mixtilinear incircle with (O).