18.404/6.840 Lecture 4

Last time:

- Finite automata → regular expressions
- Proving languages aren't regular
- Context free grammars

Finite automata -> capability are extremely limited

Today: (Sipser §2.2)

- Context free grammars (CFGs) – definition
- Context free languages (CFLs)
- Pushdown automata (PDA)
- Converting CFGs to PDAs

Context Free Grammars (CFGs)

G_1

S → 0S1

S → R

R → ε

Shorthand:

S → 0S1 | R

R → ε

Recall that a CFG has terminals, variables, and rules.

Grammars generate strings

1. Write down start variable
2. Replace any variable according to a rule
   Repeat until only terminals remain
3. Result is the generated string   You have generated a string that's in the language of the grammar
4. $L(G)$ is the language of all generated strings
5. We call $L(G)$ a Context Free Language.

The grammar's language is going to be a language over strings whose alphabet are the terminal symbols

Terminal symbols = input alphabet for the finite automata (in some sense)

Example of G_1 generating a string

Tree of substitutions "parse tree"   S   S   Resulting string

$\in L(G_1)$

$$L(G_1) = \{0^k 1^k \mid k \geq 0\}$$

CFG – Formal Definition

Defn: A Context Free Grammar (CFG) G is a 4-tuple (V, Σ, R, S)

Why called "context free"?
You can replace the variable independent of its context in the intermediate string

- V finite set of variables
- Σ finite set of terminal symbols
- R finite set of rules (rule form: $V \to (V \cup \Sigma)^*$)
- S start variable

For $u, v \in (V \cup \Sigma)^*$ write

1) $u \overset{\text{yields}}{\Rightarrow} v$ if can go from u to v with one substitution step in G
2) $u \overset{*}{\Rightarrow} v$ if can go from u to v with some number of substitution steps in G

$u \Rightarrow u_1 \Rightarrow u_2 \Rightarrow \cdots \Rightarrow u_k = v$ is called a derivation of v from u.

If $u = S$ then it is a derivation of v.

$L(G) = \{w \mid w \in \Sigma^* \text{ and } S \overset{*}{\Rightarrow} w\}$

Defn: A is a Context Free Language (CFL) if $A = L(G)$ for some CFG G.

Check-in 4.1

Which of these are valid CFGs?

C_1: B → 0B1 | ε
B1 → 1B
0B → 0B

illegal

C_2: S → 0S | S1
R → RR

a) C_1 only
b) C_2 only Correct
c) Both C_1 and C_2
d) Neither

Although you're always going to be stuck with the variable

That doesn't violate the definition of a context-free grammar

This is a context-free grammar whose language happens to be the empty language

Check-in 4.1

CFG – Example

One application of context-free grammars is to describe the syntax of programming languages

The grammar can be used to automatically generate the part of the compiler of that programming language

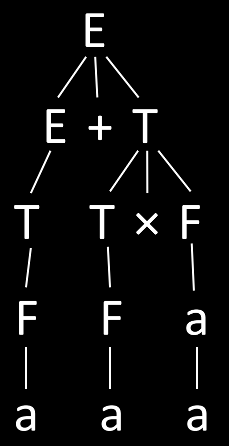
So called parser -> figure out the meaning
The meaning is embedded within the structure of the parse tree

G_2

E → E+T | T
T → T×F | F
F → (E) | a

$V =$ {E, T, F}
$\Sigma =$ {+, ×, (,), a}
$R =$ the 6 rules above
$S =$ E

Parse tree



Resulting string

E
E+T
T+T×F
F+F×a
a+a×a $\in L(G_2)$

Generates a+a×a, (a+a)×a, a, a+a+a, etc.

The times is going to be done before the plus

Observe that the parse tree contains additional information, such as the precedence of × over + .

If a string has two different parse trees then it is derived ambiguously and we say that the grammar is <u>ambiguous</u>.

Check-in 4.2

Check-in 4.2

How many reasonable distinct meanings does the following English sentence have?

The boy saw the girl with the mirror.

(a) 1

(b) 2

(c) 3 or more

Ambiguity

G_2

E → E+T | T

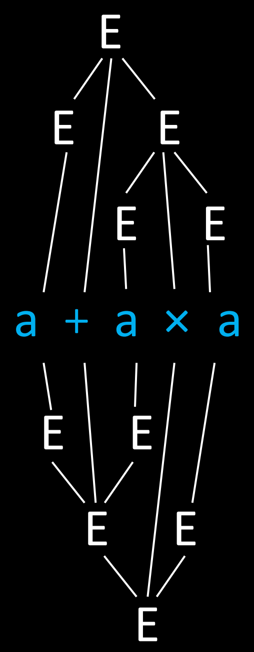
T → T×F | F

F → (E) | a

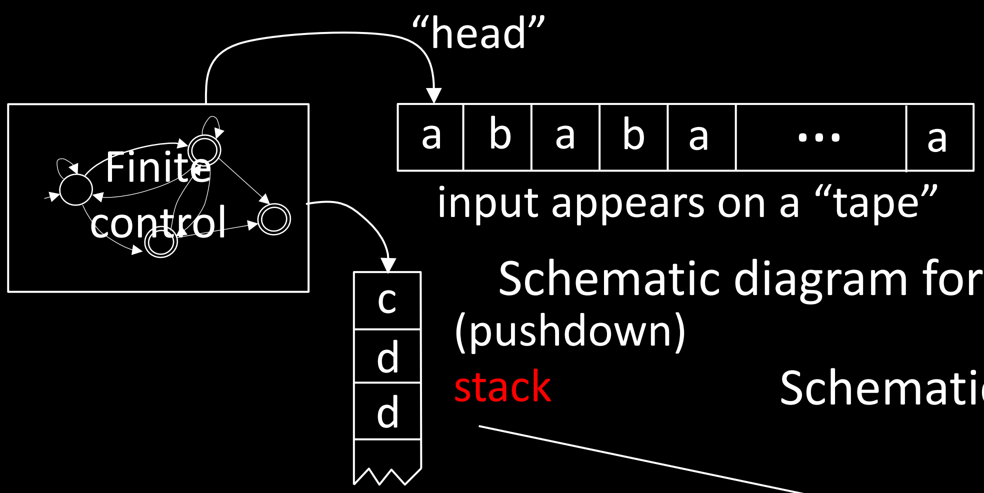
G_3

E → E+E | E×E | (E) | a

Both G_2 and G_3 recognize the same language, i.e., $L(G_2) = L(G_3)$.
However G_2 is an unambiguous CFG and G_3 is ambiguous.



Pushdown Automata (PDA)



Schematic diagram for DFA or NFA

Schematic diagram for PDA

The part of the limitation for a finite automaton was that we had a limit amount of memory So we are not able to do some simple things like counting

Using its stack as an unbounded memory

Operates like an NFA except can write-add or read-remove symbols from the top of stack.

push pop

Example: PDA for $D = \{0^k 1^k | \ k \geq 0\}$

1) Read 0s from input, push onto stack until read 1.
2) Read 1s from input, while popping 0s from stack.
3) Enter accept state if stack is empty. (note: acceptance only at end of input)

PDA – Formal Definition

Defn: A Pushdown Automaton (PDA) is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$

Σ input alphabet

Γ stack alphabet

δ: $\mathrm{Q}\times\Sigma_\varepsilon\times\Gamma_\varepsilon \rightarrow \mathcal{P}(Q\times\Gamma_\varepsilon)$

$\delta(q, \mathrm{a}, \mathrm{c}) = \{(r_1, \mathrm{d}),\ (r_2, \mathrm{e})\}$

Accept if some thread is in the accept state at the end of the input string.

Example: delta in state q, reading an input symbol a and popping a c from the top of the stack
In this case you might have two possibilities you end up going to: r1 or r2 with (r1: pushing a d onto this top of the stack) (r2: pushing a e onto this top of the stack)

Example: PDA for $B = \{ww^{\mathcal{R}} |\ w \in \{0,1\}^* \}$ Sample input:

0	1	1	1	1	0

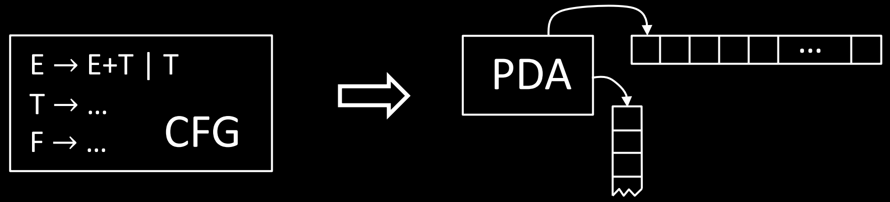
1) Read and push input symbols.
   Nondeterministically either repeat or go to (2).
2) Read input symbols and pop stack symbols, compare.
   If ever $\neq$ then thread rejects.
3) Enter accept state if stack is empty. (do in "software")

The nondeterministic forks replicate the stack.
Each sides of the fork go independently in their merry way
This language requires nondeterminism.
Our PDA model is nondeterministic.

Converting CFGs to PDAs

Theorem: If A is a CFL then some PDA recognizes A

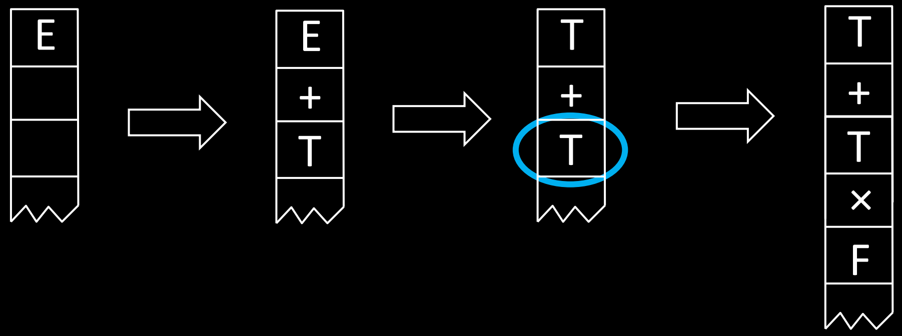
Proof: Convert A's CFG to a PDA



G_2 E → E+T | T
T → T×F | F
F → (E) | a

IDEA: PDA begins with starting variable and guesses substitutions.
It keeps intermediate generated strings on stack. When done, compare with input.

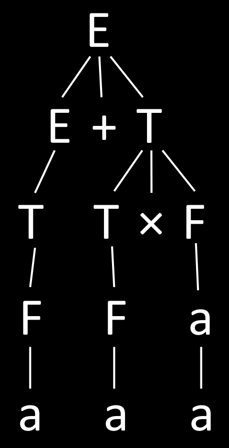
It has only terminal strings on the stack



Input:

a	+	a	×	a

E
E+T
T+T×F
F+F×a
a+a×a



Problem! Access below the top of stack is cheating!

Instead, only substitute variables when on the top of stack.

If a terminal is on the top of stack, pop it and compare with input. Reject if ≠.

They're all going to rise up to the top(nonterminal)

Converting CFGs to PDAs (contd)

G_2

E → E+T | T

T → T×F | F

F → (E) | a

Theorem: If A is a CFL then some PDA recognizes A

Proof construction: Convert the CFG for A to the following PDA.

1) Push the start symbol on the stack.
2) If the top of stack is

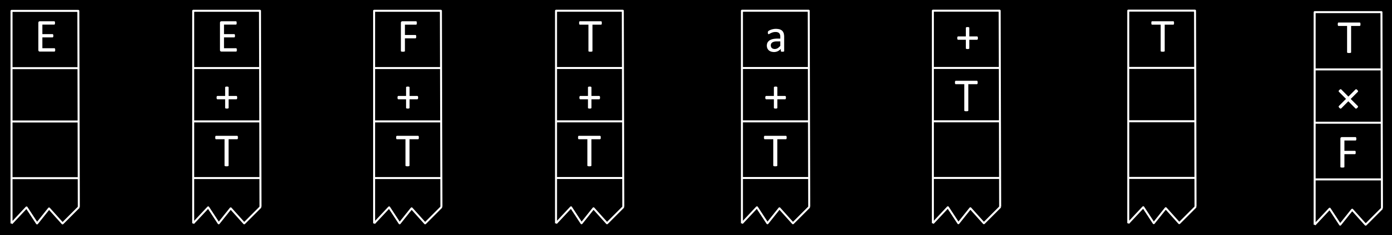
   Always do substitution at the top

   Variable: replace with right hand side of rule (nondet choice).

   Terminal: pop it and match with next input symbol.
3) If the stack is empty, *accept.*

Example:

a	+	a	×	a



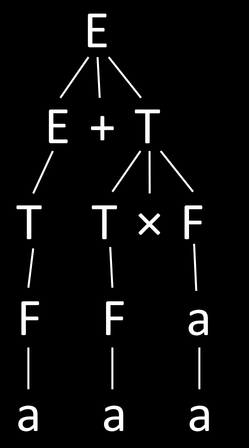
E

E+T

T+T×F

F+F×a

a+a×a



Equivalence of CFGs and PDAs

If and only if

Theorem: A is a CFL iff* some PDA recognizes A

$\longleftrightarrow$ Done.

In book. You are responsible for knowing it is true, but not for knowing the proof.

* "iff" = "if an only if" means the implication goes both ways.
So we need to prove both directions: forward (→) and reverse (←).

Every regular language can be done by a dfa or nfa,
which is really just a push-down automata that never uses its stack

Check-in 4.3

Is every Regular Language also a Context Free Language?

(a) Yes

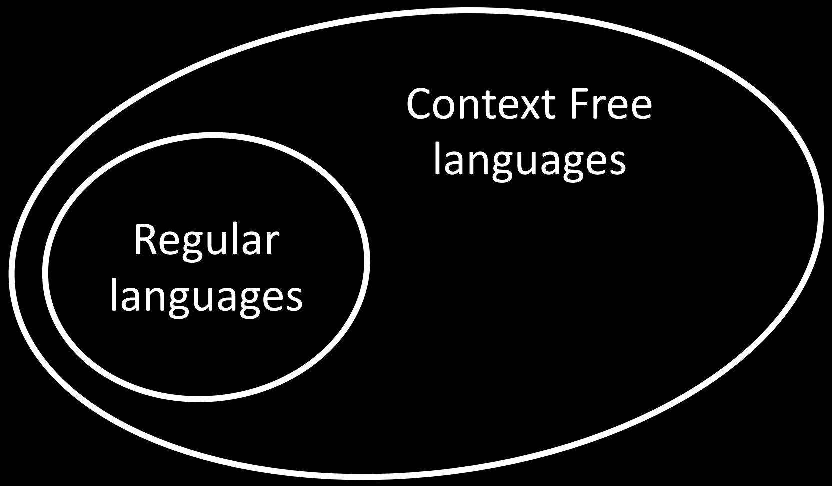
(b) No

(c) Not sure

Check-in 4.3

Recap

	Recognizer	Generator
Regular language	DFA or NFA	Regular expression
Context Free language	PDA	Context Free Grammar



Quick review of today

1. Defined Context Free Grammars (CFGs) and Context Free Languages (CFLs)
2. Defined Pushdown Automata(PDAs)
3. Gave conversion of CFGs to PDAs.