COMP S264F Discrete Mathematics
Tutorial 11: Conditional Probability, Random Variables – Suggested Solution

Question 1.
Let E be the event that the family has two boys, and let F be the event that the family has at least one boy. It follows that $E = \{BB\}$ and $F = \{BB, BG, GB\}$. Thus, $E \cap F = \{BB\}$.
As the four possibilities are equally likely, it follows that $p(F) = \frac{3}{4}$ and $p(E \cap F) = \frac{1}{4}$.
Therefore,

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)} = \frac{1/4}{3/4} = \frac{1}{3} \; .$$

Question 2.
Let B be the event that the first child is a boy, and let G be the event that the last two children are girls. We need to calculate $p(B \cup G)$, which by the principle of inclusion-exclusion, is

$$\begin{aligned} p(B \cup G) &= p(B) + p(G) - p(B \cap G) \\ &= p(B) + p(G) - p(B) \cdot p(G) \quad \text{(as } B \text{ and } G \text{ are independent events)} \end{aligned}$$

(a) As a boy and a girl are equally likely, the probability of a boy and the probability of a girl are both $\frac{1}{2}$.
Then, $p(B) = \frac{1}{2}$ and $p(G) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.
Thus, $p(B \cup G) = \frac{1}{2} + \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{4} = \frac{4+2-1}{8} = \frac{5}{8}$.

(b) As the probability of a boy is 0.51, the probability of a girl is $1 - 0.51 = 0.49$.
Then, $p(B) = 0.51$ and $p(G) = 0.49 \cdot 0.49 = 0.2401$.
Thus, $p(B \cup G) = 0.51 + 0.2401 - 0.51 \cdot 0.2401 = 0.627649$.

(c) Plugging in $i = 1, 2, 3, 4, 5$, the probability of having boys on the successive births are $0.50, 0.49, 0.48, 0.47, 0.46$.
Then, $p(B) = 0.50$ and $p(G) = (1 - 0.47) \cdot (1 - 0.46) = 0.53 \cdot 0.54 = 0.2862$.
Thus, $p(B \cup G) = 0.50 + 0.2862 - 0.50 \cdot 0.2862 = 0.6431$.

Question 3.
Let E be the event that at least two of the n people have the same birthday.
Then, $\overline{E}$ is the event that all the n people have distinct birthday.
As the birthday of each person can be one of the 366 days, the size of the sample space is 366^n.
If all the n people have distinct birthday, then these birthdays is a n-permutation of the 366 days.
Thus, $|\overline{E}| = P(366, n)$.
Therefore,

$$p(\overline{E}) = \frac{P(366, n)}{366^n} = \frac{366!}{(366-n)! \cdot 366^n} \qquad \text{and} \qquad p(E) = 1 - \frac{366!}{(366-n)! \cdot 366^n} \; .$$

With the help of our Python program in the Jupyter notebook "T11.ipynb", we can find that

- for $n = 22$, $p(E) \approx 0.475$, and
- for $n = 23$, $p(E) \approx 0.506$.

Therefore, the minimum number of people needed so that the probability that at least two people have the same birthday is greater than $\frac{1}{2}$ is 23.

Question 4.
Let X be the random variable that equals the number of people who receive the correct hat from the checker. Let X_i be the random variable with $X_i = 1$ if the i-th person receives the correct hat and $X_i = 0$ otherwise. Then,

$$X = X_1 + X_2 + \cdots + X_n \; .$$

As it is equally likely that the checker returns any of the hats to a person, for all i, $p(X_i = 1) = \frac{1}{n}$. Thus,

$$\begin{aligned} E(X_i) &= 1 \cdot p(X_i = 1) + 0 \cdot p(X_i = 0) \\ &= 1 \cdot \frac{1}{n} + 0 = \frac{1}{n} \ . \end{aligned}$$

By the linearity of expectations, we have

$$E(X) = E(X_1) + E(X_2) + \cdots + E(X_n) = n \cdot \frac{1}{n} = 1 \ .$$

Therefore, the average number of people who receive the correct hat is exactly 1.

Question 5.
Let X be the random variable equal to the number of inversions in the permutation.
Let $I_{i,j}$ be the random variable on the set of all permutations of the first n positive integers with $I_{i,j} = 1$ if (i, j) is an inversion of the permutation and $I_{i,j} = 0$ otherwise. Then,

$$X = \sum_{1 \le i < j \le n} I_{i,j} \ .$$

Note that for any permutation where i precedes j, we can switch the positions of i and j to obtain a permutation where j precedes i.
As a permutation will only be related to another unique permutation, there is a bijection from the set of permutations where i precedes j to the set of permutations where j precedes i.
It follows that the number of permutations where i precedes j is equal to the number of permutations where j precedes i.
Therefore, for any $1 \le i < j \le n$, $p(I_{i,j} = 1) = \frac{1}{2}$. Then,

$$\begin{aligned} E(I_{i,j}) &= 1 \cdot p(I_{i,j} = 1) + 0 \cdot p(I_{i,j} = 0) \\ &= 1 \cdot \frac{1}{2} + 0 = \frac{1}{2} \ . \end{aligned}$$

As the number of pairs i and j where $i < j$ is $C(n, 2)$, by the linearity of expectations,

$$\begin{aligned} E(X) &= \sum_{1 \le i < j \le n} E(I_{i,j}) \\ &= C(n, 2) \cdot \frac{1}{2} \\ &= \frac{n(n-1)}{2} \cdot \frac{1}{2} \\ &= \frac{n(n-1)}{4} \ . \end{aligned}$$

Therefore, the expected number of inversions in the permutation is $\frac{n(n-1)}{4}$.

Question 6.
There are 4 possible outcomes from clipping a coin twice: $\{HH, HT, TH, TT\}$.
Thus, $p(X = 2) = \frac{1}{4}$, $p(X = 1) = \frac{2}{4} = \frac{1}{2}$, $p(X = 0) = \frac{1}{4}$. Then,

$$\begin{aligned} E(X) &= 2 \cdot p(X = 2) + 1 \cdot p(X = 1) + 0 \cdot p(X = 0) \\ &= 2 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 0 = 1 \ . \end{aligned}$$

Similarly, we can show that $E(Y) = 1$.

Now, consider the value of XY for the 4 outcomes:

- HH: $X = 2$ and $Y = 0$, so $XY = 0$.
- HT: $X = 1$ and $Y = 1$, so $XY = 1$.
- TH: $X = 1$ and $Y = 1$, so $XY = 1$.
- TT: $X = 0$ and $Y = 2$, so $XY = 0$.

Thus,

$$E(XY) = \frac{1}{4} \cdot (0 + 1 + 1 + 0) = \frac{1}{2} \ .$$

As $E(X) \cdot E(Y) = 1 \cdot 1 = 1 \neq \frac{1}{2}$, we have

$$E(XY) \neq E(X) \cdot E(Y) \ .$$

Therefore, X and Y are not independent random variables.