COMP S264F Unit 3: Set Theory

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Overview

- Set notations
- Equality & Subset
- Venn diagrams
- Finite sets
- Set operators
  - Union, Intersection, Difference, Complement
- Set identities
- Power sets
- Cartesian products

Sets

- A set is a group of distinct objects.
- To describe a set, we can
  - list all its elements (enclosed by a pair of braces); or
  - specify the properties of the objects in the set.

- Examples:
  - Let A be the set of all Computing students born in 1999.
  - Let X be the set of {A, B, C, D}.
  - Let $\mathbb{Z}$ be the set of all integers.
  - Let $\mathbb{N}$ (natural numbers) = {0, 1, 2, 3, …}.
  - EVEN = $\{ x \mid x \in \mathbb{Z}$ and $(x \bmod 2 = 0)\}$.

{ x | … } is the set builder notation, read as *the set of all x such that …*

~Python's list comprehension:

- `EVEN = [x for x in Z if x mod 2 == 0]`.

Basic terminology

- The objects in a set are also called the elements or members of the set.
- The notation "**$x \in A$**" is defined to be the statement "x is an element of A", "x is in A", or "A contains x".
  - "**$x \notin A$**" means "x is not in A".
- The set that contains no element is called the empty set, or the null set, denoted by $\varnothing$.
- The universal set is the set containing all objects under consideration.

Equality & Subset

- Two sets A and B are equal, denoted by **$A = B$**, if and only if <u>they have the same elements</u> (i.e., $\forall x\ (x \in A \Leftrightarrow x \in B)$).

- E.g., $\{1, 3, 5\} = \{3, 5, 1\}$? $\varnothing = \{\varnothing\}$?

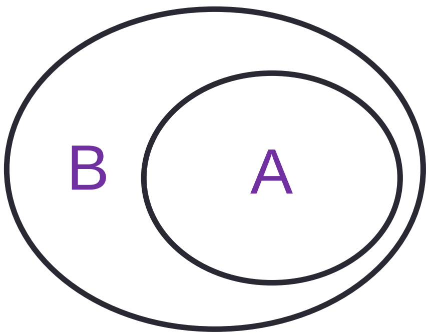
- A set A is said to be a subset of a set B, denoted by **$A \subseteq B$**, if and only if <u>every element of A is also an element of B</u> (i.e., $\forall x\ (x \in A \Rightarrow x \in B)$).

- E.g., $\{1, 2\} \subseteq \{3, 1, 2\} \subseteq \mathbb{N} \subseteq \mathbb{Z}$

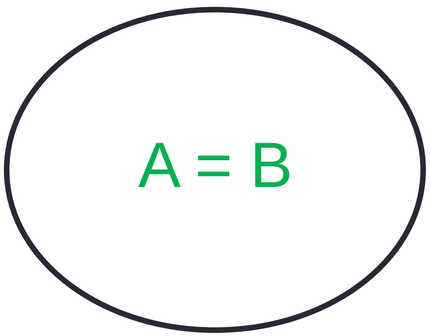
Venn diagrams

Let A and B be sets.

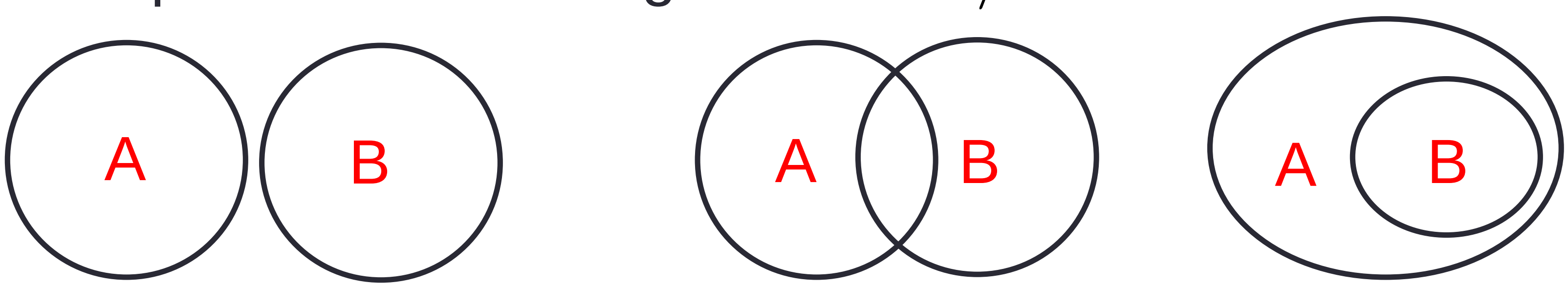
- If A and B are represented as <u>regions in the plane</u>, their <u>relationships</u> can be represented by a Venn diagram.
- E.g., $A \subseteq B$:



$A = B$:



- Three possible Venn diagrams for $A \not\subseteq B$:



Equality: Example 1

How to prove **A = B**?

- Show that $\mathbf{A} \subseteq \mathbf{B}$ and also $\mathbf{B} \subseteq \mathbf{A}$
- I.e. $\forall x\, (x \in A \Rightarrow x \in B)$ **and** $\forall x\, (x \in B \Rightarrow x \in A)$.

Example 1: Let $A = \{\, n \mid n = 2p \text{ for some } p \in \mathbb{Z} \,\}$, and let $B = \{m \mid m = 2q - 2 \text{ for some } q \in \mathbb{Z}\}$. Is A = B?

- Before proving a theorem, it is always good to consider some **small examples** of it.

Python:

```
Z = list(range(-10,10))
A = [2*p for p in Z]
B = [2*q-2 for q in Z]
print("Z:", Z)
print("A:", A)
print("B:", B)
```

Output:

```
Z: [-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
A: [-20, -18, -16, -14, -12, -10, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10, 12, 14, 16, 18]
B: [-22, -20, -18, -16, -14, -12, -10, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10, 12, 14, 16]
```

Equality: Example 1 (cont')

How to prove **A = B**?

- Show that **A** $\subseteq$ **B** and also **B** $\subseteq$ **A**
- I.e. $\forall x\ (x \in A \Rightarrow x \in B)$ **and** $\forall x\ (x \in B \Rightarrow x \in A)$.

Example 1: Let A = { n | n = 2p for some $p \in \mathbb{Z}$ }, and let B = {m | m = 2q - 2 for some $q \in \mathbb{Z}$}. Is A = B?

Proof.

Let $x \in A$. Then, there is an integer p such that x = 2p = 2(p+1) - 2.

As p+1 is also an integer, $x \in B$.

Let $y \in B$. Then, there is an integer q such that y = 2q - 2 = 2(q-1).

As q-1 is also an integer, $y \in A$.

Therefore, A = B.

Equality: Example 2

<u>Example 2</u>: Let A = { n | n = 2p for some $p \in \mathbb{Z}$ }, and
let C = { k | k = 3r + 1 for some $r \in \mathbb{Z}$}. Is A = C?

Python:

```
Z = list(range(-10,10))
A = [2*p for p in Z]
C = [3*r+1 for r in Z]
print("Z:", Z)
print("A:", A)
print("C:", C)
```

Output:

```
Z: [-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
A: [-20, -18, -16, -14, -12, -10, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10, 12, 14, 16, 18]
C: [-29, -26, -23, -20, -17, -14, -11, -8, -5, -2, 1, 4, 7, 10, 13, 16, 19, 22, 25, 28]
```

Proof. No, we can find a ***counterexample***, as follows:

C contains k = 3(**2**) + 1 = 7, i.e., **$7 \in C$**.

To check whether $7 \in A$, we set $7 = 2p$.

Then, $p = 7/2 = 3.5$, which is not an integer.

Thus, **$7 \notin A$** $\Rightarrow$ $A \neq C$.

Subsets

Let A = {1, 2, 3}.

True or false?

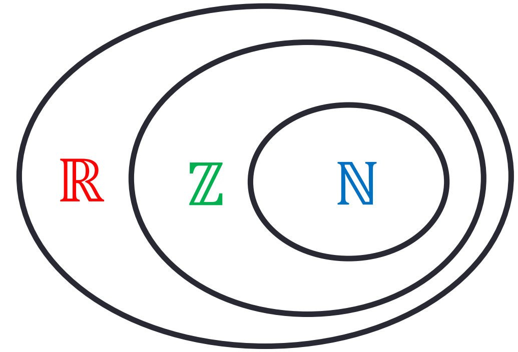
- $\varnothing \subseteq A$
- $\varnothing \subseteq \varnothing$
- $\varnothing \subseteq S$ for all sets S.

Let A and B be any two sets. A is a proper subset of B, denoted by **$A \subset B$**, if and only if **$A \neq B$** and **$A \subseteq B$**.

True or false?

- $\varnothing \subset \varnothing$

Subsets: Example



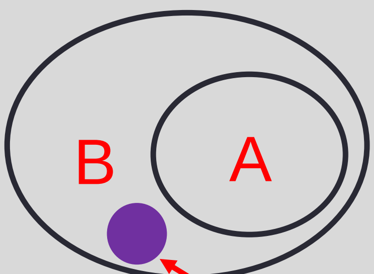
- The set of real numbers is denoted by $\mathbb{R}$.
- E.g., $1.33 \in \mathbb{R}$, $\pi \in \mathbb{R}$, $-2 \in \mathbb{R}$, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{R}$.

Example: Let $A = \{x \in \mathbb{R} \mid x > 4\}$, and let $B = \{x \in \mathbb{R} \mid x^2 > 1\}$.
Prove that $\mathbf{A} \subset \mathbf{B}$.

Python:

```
Z = list(range(-10,10))
A = [x for x in Z if x > 4]
B = [x for x in Z if x**2 > 1]
print("Z:", Z)
print("A:", A)
print("B:", B)
```

x ** 2 means x^2



Output:

```
Z: [-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
A: [5, 6, 7, 8, 9]
B: [-10, -9, -8, -7, -6, -5, -4, -3, -2, 2, 3, 4, 5, 6, 7, 8, 9]
```

Subsets: Example

Example: Let $A = \{x \in \mathbb{R} \mid x > 4\}$, and let $B = \{x \in \mathbb{R} \mid x^2 > 1\}$.
Prove that $\mathbf{A} \subset \mathbf{B}$.

Proof.

First, we show that $\mathbf{A} \subseteq \mathbf{B}$. For every $x \in A$,

$$
\begin{aligned}
x > 4 \;\Rightarrow\; x^2 &> 4^2 \qquad (\text{as } x > 0)\\
&= 16\\
&> 1\\
\Rightarrow\; x &\in B.
\end{aligned}
$$

Then, we show that $\mathbf{A} \neq \mathbf{B}$. Consider $x = -2$.

$$x^2 = (-2)^2 = 4 > 1 \;\Rightarrow\; \mathbf{-2 \in B}\,.$$

But, $x = -2 \leq 4 \;\Rightarrow\; \mathbf{-2 \notin A}$.

It follows that $\mathbf{A} \subset \mathbf{B}$.

Finite sets

Let S be a set containing exactly $n \geq 0$ elements.

- We say that S is a **finite** set.
- The **cardinality** of S, denoted by **|S|**, is the number of elements in S, i.e., n.

An **infinite** set is a set that is not finite.

Questions: finite or infinite?

- {A, B, C}
- The set of all Computing students who were born in 1999.
- $\mathbb{N} = \{0, 1, 2, \ldots\}$
- PRIME = the set of prime numbers

What is the cardinality of $\emptyset$? $\{\emptyset\}$? $\{\emptyset, \{\emptyset\}\}$?

Set Operators

Let A and B be two sets.

- The union of A and B, denoted by **$A \cup B$**, is the set $\{x \mid x \in A$ **or** $x \in B\}$.

E.g., $\{1, 2\} \cup \{3, 2\} = \{1, 2, 3\}$

- The intersection of A and B, denoted by **$A \cap B$**, is the set $\{x \mid x \in A$ **and** $x \in B\}$.

E.g., $\{1, 2\} \cap \{3, 2\} = \{2\}$

- The difference of A and B, denoted by **$A - B$**, is the set $\{x \mid x \in A$ **and** $x \notin B\}$.

E.g., $\{1, 2\} - \{3, 2\} = \{1\}$

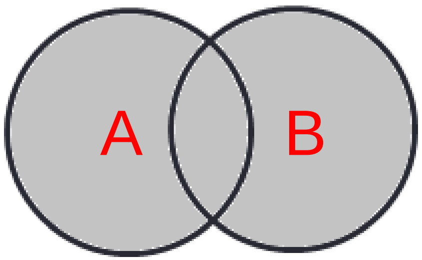
- The complement of a set A (with respect to a fixed universal set U), denoted by $\overline{A}$, is the set U - A.

E.g., $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 4\}$. Then, $\overline{A} = \{2, 3, 5\}$.

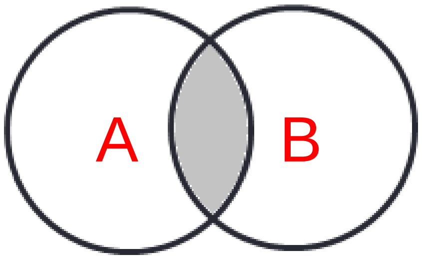
Venn diagram of Set Operators

Let A and B be two sets.

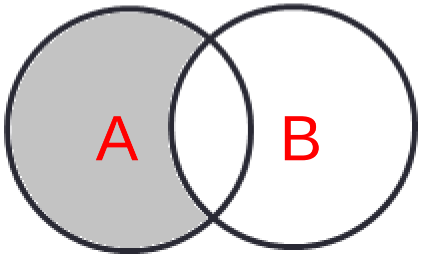
- **A ∪ B:**



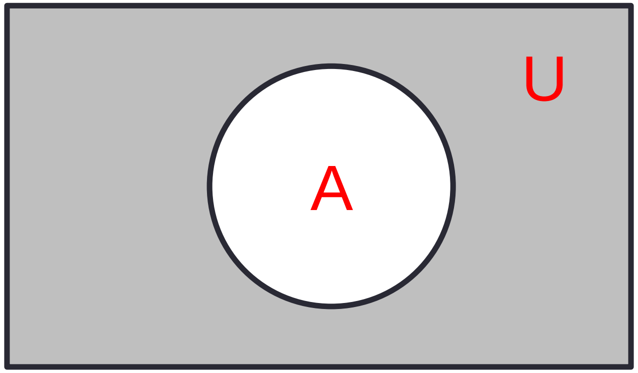
- **A ∩ B:**



- **A - B:**



- $\overline{A}$ = U - A**:**



Set Identities - 1

Identity	*Name*
$A \cup \varnothing = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{\overline{A}} = A$	Complementation law

Set Identities - 2

Identity	*Name*
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{B} \cap \overline{A}$ $\overline{A \cap B} = \overline{B} \cup \overline{A}$	De Morgan's laws

Sets in Python

- Sets are enclosed in braces { }.
- Lists care about order and repetition; while sets only care about membership.
  - In [1], the set has repeated item (i.e., 1) and is thus invalid.
  - In [2], the repeated item is removed automatically.
- `in` operator is for set membership.
- `set` function converts a list to a set.
- `sorted` function converts a set to a list.

```
In [1]: my_set = {1,3,5,1}

In [2]: my_set
Out[2]: {1, 3, 5}

In [3]: 2 in my_set
Out[3]: False

In [4]: 3 in my_set
Out[4]: True

In [5]: set(['c','b','a'])
Out[5]: {'a', 'b', 'c'}

In [6]: sorted(my_set)
Out[6]: [1, 3, 5]
```

Subset & Superset in Python

- $A \subseteq B$ can be checked using `issubset` function or `<=` .
- $B \supseteq A$ (i.e., B is a **superset** of A, or $A \subseteq B$) can be checked using `issuperset` function or `>=`.

- $A \subset B$ (A is a proper subset of B) can be checked using A < B.
- $B \supset A$ (B is a proper superset of A) can be checked using B > A.

- N.B. Set does not care about order, e.g., see [12].

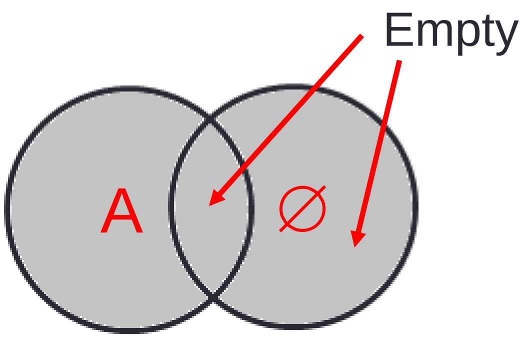
```
In [7]: {2,4}.issubset({2,4,6})
Out[7]: True

In [8]: {2,4} <= {2,4,6}
Out[8]: True

In [9]: {2,4,6}.issuperset({2,4})
Out[9]: True

In [10]: {2,4,6} >= {2,4}
Out[10]: True

In [11]: {2,4,6} < {2,4,6}
Out[11]: False

In [12]: {2,4} < {6,4,2}
Out[12]: True
```

Set operators in Python

- Set union can be done by `union` function or `|` .
- Set intersection can be done by `intersection` function or `&` .
- Set difference can be done by `difference` function or `-` .

```
In [13]: {1,2,3}.union({3,4,5})
Out[13]: {1, 2, 3, 4, 5}

In [14]: {1,2,3} | {3,4,5}
Out[14]: {1, 2, 3, 4, 5}

In [15]: {1,2,3}.intersection({3,4,5})
Out[15]: {3}

In [16]: {1,2,3} & {3,4,5}
Out[16]: {3}

In [17]: {1,2,3}.difference({3,4,5})
Out[17]: {1, 2}

In [18]: {1,2,3} - {3,4,5}
Out[18]: {1, 2}
```

Proving Set Identities



- $A \cup \varnothing = A$

Python:

```
A = set({1,2,3,4,5})
B = set()
print("A | B =", A | B)
```

Output:

```
A | B = {1, 2, 3, 4, 5}
```

Proof.

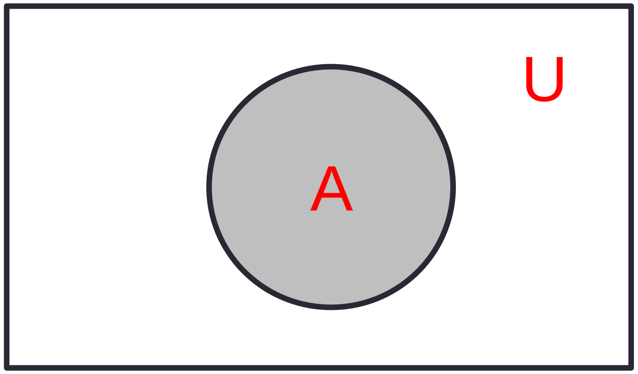
It is obvious that $A \subseteq A \cup \varnothing$.

If $x \in A \cup \varnothing$, then $(x \in A)$ or $(x \in \varnothing) \Rightarrow (x \in A)$ or false $\Rightarrow x \in A$.

Thus, $A \cup \varnothing \subseteq A$.

Therefore, $A \cup \varnothing = A$.

Proving Set Identities (cont')



- $A \cap U = A$

Python:

```
U = set({1,2,3,4,5,6,7,8,9,10})
A = set({1,2,3,4,5})
print("A & U =", A & U)
```

Output:

```
A & U = {1, 2, 3, 4, 5}
```

Proof.

It is obvious that $A \cap U \subseteq A$.

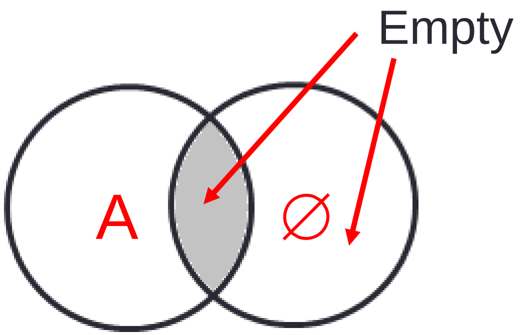
If $x \in A$, then $x \in A \Rightarrow (x \in A)$ and $(x \in U) \Rightarrow x \in A \cap U$.

Thus, $A \subseteq A \cap U$.

Therefore, $A \cap U = A$.

Proving Set Identities (cont’)

- $A \cap \emptyset = \emptyset$



Python:

```
A = set({1,2,3,4,5})
B = set()
print("A & B =", A & B)
```

Output:

```
A & B = set()
```

Proof.

It is obvious that $A \cap \emptyset \subseteq \emptyset$.

As $\emptyset \subseteq X$ for any set X. Thus, $\emptyset \subseteq A \cap \emptyset$.

Therefore, $A \cap \emptyset = \emptyset$.

Another useful set identity

- $A - B = A \cap \overline{B}$

Proof.

$x \in A - B$

$\Leftrightarrow$ ($x \in A$) and ($x \notin B$)

$\Leftrightarrow$ ($x \in A$) and ($x \in \overline{B}$).

$\Leftrightarrow$ $x \in A \cap \overline{B}$.

Therefore, $A - B = A \cap \overline{B}$.

Proving Subset: Example 1

Theorem 1. $A \cap B \subseteq B$.

Proof.

We need to prove $\forall x\ (x \in A \cap B\ ) \Rightarrow x \in B$.

Assume $x \in A \cap B$.

Then, $x \in A$ and $x \in B$ (Definition of $\cap$)

$\Rightarrow\ x \in B$ (p and $q \Rightarrow p$)

Proving Subset: Example 2

Theorem 2. If $A \subset B$, then $A \cap B \subset B$.

Proof.

By Theorem 1, $A \cap B \subseteq B$.

Thus, it *remains* to prove:

If $A \subset B$, then $\exists$**x ((x$\in$ B) $\wedge$ (x $\notin$ A $\cap$ B))**.

Assume $A \subset B$.

Then, $\exists x\,(x \in B \wedge x \notin A)$ (Definition of $\cap$)

$\Rightarrow \exists x\,(x \in B \wedge ((x \notin A) \vee (x \notin B)))$ $(p \Rightarrow p \vee q)$

$\Rightarrow \exists x\,(x \in B \wedge (x \notin A \cap B))$ $(x \notin A \cap B \Leftrightarrow x \notin A$ or $x \notin B)$

Proving Subset: Example 3

Theorem 3. $A \subseteq B$ if and only if $A \cap B = A$.

Proof.

We first prove that if $A \subseteq B$, then $A \cap B = A$.

Assume $A \subseteq B$.

It is obvious that $A \cap B \subseteq A$.

If $x \in A$, then $(x \in A)$ and $(x \in B)$ (as $A \subseteq B$) $\Rightarrow$ $x \in A \cap B$.

Thus, $A \subseteq A \cap B$.

Therefore, $A \cap B = A$.

Proving Subset: Example 3 (con't)

Theorem 3. $A \subseteq B$ if and only if $A \cap B = A$.

Proof (cont').

Next, we prove that if $A \cap B = A$, then $A \subseteq B$.

Assume $A \cap B = A$.

$x \in A$

$\Rightarrow x \in A \cap B$ (as $A \cap B = A$)

$\Rightarrow (x \in A)$ and $(x \in B)$

$\Rightarrow x \in B$

Thus, $A \subseteq B$.

The theorem follows.

Power Sets

- The power set of a set S, denoted by P(S), is the set of all subsets of S.
- That means, for any element x of P(S), $x \subseteq S$.

E.g.,

- P({0, 1, 2}) = {∅, {0}, {1}, {2}, {0,1}, {0,2}, {1,2}, {0,1,2}}

- |{0, 1, 2}| = 3; |P({0, 1, 2})| = 8.

- What is the power set of the empty set?

- In general, if |S| = n, then |P(S)| = ?

Size of power set

Theorem. For any set S with n elements, $|P(S)| = 2^n$.

Proof. We can prove by induction on the number of elements.

Base case. When n = 1, let S = {a}. Then P(S) = {∅, {a}}.
Thus, $|P(S)| = 2 = 2^1 = 2^n$.

Inductive step. Assume that if a set S' contains k elements for some k, then $|P(S')| = 2^k$.

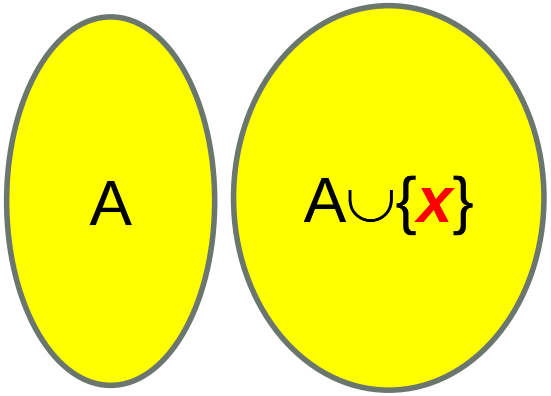
Consider the case where |S| = k + 1.

Let ***x*** be an element of S and denote **S'** = S - {***x***}.

Consider each subset A of S'.

- A is also a subset of S.
- A ∪ {***x***} gives another distinct subset of S.

Thus, S has twice as many subsets as S', and
$|P(S)| = 2 \times |P(S')| = 2 \times 2^k = 2^{k+1}$.



Generating power set in Python

- Python does not support a set of sets.
- We will use list instead to generate items in a power set.

```
def P(s):
    if len(s) == 0:
        return [[]]
    else:
        a = s[0]
        subsets = P(s[1:])
        newsubsets = []
        for subset in subsets:
            newsubsets.append([a]+subset)
        return subsets + newsubsets
```

-
```
a = P([1,2,3,4])
a.sort()
a.sort(key=len)
print(a)
```

```
[[], [1], [2], [3], [4], [1, 2],
 [1, 3], [1, 4], [2, 3], [2, 4],
 [3, 4], [1, 2, 3], [1, 2, 4],
 [1, 3, 4], [2, 3, 4], [1, 2, 3, 4]]
```

Power Sets: Example

Prove that $P(A \cap B) = P(A) \cap P(B)$.

Proof.

$x \in P(A) \cap P(B)$

$\Leftrightarrow \; x \in P(A)$ and $x \in P(B)$

$\Leftrightarrow \; x \subseteq A$ and $x \subseteq B$

$\Leftrightarrow \; x \subseteq A \cap B$

$\Leftrightarrow \; x \in P(A \cap B)$

Therefore, $P(A \cap B) = P(A) \cap P(B)$.

Cartesian Products

- Ordering is important: An ordered ***n-tuple*** $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as the first element, a_2 as its second element, …, and a_n as the n-th element.

NB. The ***set*** $\{a_1, a_2, \ldots, a_n\}$ does not carry any ordering.

- Let A and B be sets. The Cartesian product of A and B, denoted by A x B, is the set of all 2-tuples (a, b) where $a \in A$ and $b \in B$.

  E.g., {a,b} x {c,d} = { (a,c), (a,d), (b,c), (b,d) }.

- The Cartesian product of the sets A1, A2, …, An, denoted by A1 x A2 x … x An, is the set

  $\{ (a_1, a_2, \ldots, a_n) \mid a_1 \in A1 \text{ and } a_2 \in A2 \text{ and } \ldots \text{ and } a_n \in An\}$

Cartesian Products: Questions

Assume that A contains *n* elements and B contains *m* elements.

- How many elements does A x B contain?
- A x B = B x A ?
- Suppose A x B is equal to the empty set.
  What can you conclude?

Cartesian Products: Questions (cont')

Let A be a set of n elements.

- How many elements does P(A) x P(A) contain?
  By definition, $P(A) \times P(A) = \{ (X, Y) \mid X \subseteq A, Y \subseteq A \}$.

- How many elements are in the set
  $\{ (X, Y) \mid X \subseteq A,\ Y \subseteq A,\ X \cap Y = \varnothing \}$?