COMP S264F Discrete Mathematics
Tutorial 3: Methods of Proof – Suggested Solution

Question 1.

(a) For any two consecutive integers, one must be odd and another must be even.
Let $2m$ and $2n+1$ be the two consecutive integers for some integers m and n.
Then, their product is $2m(2n+1) = 2[m(2n+1)]$, which is even.

(b) This is a bicoditional statement. We need to prove for 2 cases.

Case 1: x **is odd** $\implies x^2+2x+1$ **is even** .

Let $x = 2n+1$ for some integer n.

$$\begin{aligned}\text{Then, } x^2+2x+1 &= (2n+1)^2 + 2(2n+1) + 1 \\ &= (4n^2+4n+1) + (4n+2) + 1 \\ &= 4n^2+8n+4 \\ &= 2(2n^2+4n+2) \quad \text{which is even.}\end{aligned}$$

Case 2: x **is even (i.e. not odd)** $\implies x^2+2x+1$ **is odd (i.e. not even)** .

Let $x = 2n$ for some integer n.

$$\begin{aligned}\text{Then, } x^2+2x+1 &= (2n)^2 + 2(2n) + 1 \\ &= 4n^2+4n+1 \\ &= 2(2n^2+2n)+1 \quad \text{which is not even.}\end{aligned}$$

Question 2.

(a) Let x, y be the two positive real numbers.
Assume that $x \leq 10$ and $y \leq 10$.
Then, $xy \leq 10 \cdot 10 = 100$.
Therefore, if $xy > 100$, then at least one of x and y is greater than 10.

(b) Assume that one of them is odd and another is even.
Let $2x$ and $2y+1$ be the two integers.
Then, $2x + (2y+1) = 2(x+y)+1$, which is odd.
Therefore, the statement follows.

Question 3. The errors occur in steps 3, 4, 5 and 6. From step 2, for any element c in the domain, there can be two possible cases that

Case 1. $P(c)$ is true and $Q(c)$ is false, or

Case 2. $P(c)$ is false and $Q(c)$ is true.

Thus, due to Case 2, we can only say in step 3 that c is an element in the domain such that $P(c)$ is true, but cannot say that c is any element in the domain. Then, we cannot conclude in step 4 that $P(x)$ is true for all element x in the domain.

Similarly, due to Case 1, we can only say in step 5 that c is an element in the domain such that $Q(c)$ is true, but cannot say that c is any element in the domain. Then, we cannot conclude in step 6 that $Q(x)$ is true for all element x in the domain.

Take this example for your easier understanding. Consider the domain $\{a, b\}$. Assume $P(a)$ and $Q(b)$ are true while $P(b)$ and $Q(a)$ are false. We can find that step 1 and step 2 are valid. However, step 3 and step 4 are invalid because $P(b)$ is false. Similarly, step 5 and step 6 are also invalid because $Q(a)$ is false. Hence, step 7 that relies on step 4 and step 6 is also invalid.

Question 4. There are two possible cases:

Case 1: n is odd. Let $n = 2k + 1$ for some integer k.

$$\begin{aligned} \text{Then, } n^2 - n + 3 &= (2k+1)^2 - (2k+1) + 3 \\ &= 4k^2 + 4k + 1 - 2k - 1 + 3 \\ &= 4k^2 + 2k + 3 \\ &= 2(2k^2 + 2k + 1) + 1 \quad \text{which is odd.} \end{aligned}$$

Case 2: n is even. Let $n = 2k$ for some integer k.

$$\begin{aligned} \text{Then, } n^2 - n + 3 &= (2k)^2 - (2k) + 3 \\ &= 4k^2 - 2k + 3 \\ &= 2(2k^2 - k + 1) + 1 \quad \text{which is odd.} \end{aligned}$$

Question 5.

(a) We need to consider the two cases that "x is odd" is true or false:

Case 1: x is odd. Let $x = 2n + 1$ for some integer n.

$$\begin{aligned} \text{Then, } x^2 + 6x + 9 &= (2n+1)^2 + 6(2n+1) + 9 \\ &= (4n^2 + 4n + 1) + (12n + 6) + 9 \\ &= 4n^2 + 16n + 16 \\ &= 2(2n^2 + 8n + 8) \quad \text{which is even.} \end{aligned}$$

Case 2: x is not odd, i.e., even. Let $x = 2n$ for some integer n.

$$\begin{aligned} \text{Then, } x^2 + 6x + 9 &= (2n)^2 + 6(2n) + 9 \\ &= 4n^2 + 12n + (8 + 1) \\ &= 2(2n^2 + 6n + 4) + 1 \quad \text{which is not even.} \end{aligned}$$

(b) We prove the biconditional statement in the two directions.

(i) We first prove (xy is odd) $\leftarrow$ (x and y are both odd).
Assume x and y are both odd. Then, $x = 2n + 1$ and $y = 2m + 1$ for some integers n, m.

$$\begin{aligned} xy &= (2n+1)(2m+1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \quad \text{which is odd.} \end{aligned}$$

(ii) Next, we prove (xy is odd) $\rightarrow$ (x and y are both odd), which is equivalent to

$$\neg(x \text{ and } y \text{ are both odd}) \rightarrow \neg(xy \text{ is odd})$$

Thus, we consider the following two cases.

Case 1: Both x and y are not odd. Let $x = 2n$ and $y = 2m$ for some integers n, m.

$$\begin{aligned} \text{Then, } xy &= (2n)(2m) \\ &= 2(2mn) \quad \text{which is not odd.} \end{aligned}$$

Case 2: One of x and y is not odd. Let x and y be $2n + 1$ and $2m$ for some integers n, m.

$$\begin{aligned} \text{Then, } xy &= (2n+1)(2m) \\ &= 2(2mn + m) \quad \text{which is not odd.} \end{aligned}$$

Therefore, (xy is odd) $\leftrightarrow$ (x and y are both odd).

Question 6.

(a) Suppose, for the sake of contradiction, $a^2 - 4b - 2 = 0$, i.e., $a^2 = 4b + 2$.
Therefore, $a^2 \bmod 4 = 2$.
We consider the two cases that a is odd or even.

Case 1: a **is odd** Then a^2 is odd, which contradicts that $a^2 = 4b + 2 = 2(2b + 1)$ is even.

Case 2: a **is even.** Let $a = 2k$ for some integer k.

Then, $a^2 = (2k)^2 = 4k^2$, which is divisible by 4, which contradicts that $a^2 \bmod 4 = 2$.

(b) Suppose, for the sake of contradiction, that a is rational, ab is irrational, and b is also rational.
Therefore, $a = \frac{x_1}{y_1}$ and $b = \frac{x_2}{y_2}$ for some integers $x_1, x_2, y_1 > 0, y_2 > 0$.
Then, $ab = \frac{x_1 x_2}{y_1 y_2}$ is rational, which contradicts that ab is irrational.

Question 7.

(a) **Base case.** When $n = 1$, $3^n + 1 = 3^1 + 1 = 4$ which is divisible by 2.

Inductive step. Assume that $3^k + 1 = 2m$ for some positive integer k and m.

When $n = k + 1$,
$$\begin{aligned} 3^n + 1 &= 3^{k+1} + 1 \\ &= 3 \cdot (3^k) + 1 \\ &= 3 \cdot (3^k + 1 - 1) + 1 \\ &= 3 \cdot (3^k + 1) - 3 + 1 \\ &= 3 \cdot 2m - 2 \\ &= 2(3m - 1) \quad \text{which is also divisible by 2} \end{aligned}$$

By the principle of mathematical induction, for any positive integer n, $3^n + 1$ is divisible by 2.

(b) **Base case.** When $n = 1$, L.H.S. $= 1 = 1^2 = n^2 =$ R.H.S.

Inductive step. Assume that $1 + 3 + 5 + \cdots + (2k - 1) = k^2$ for some positive integer k.

When $n = k + 1$,
$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1) &= k^2 + (2k + 1) \\ &= (k + 1)^2 \\ &= n^2 \end{aligned}$$

By the principle of mathematical induction, for any positive integer n, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.