COMP S264F Discrete Mathematics
Tutorial 10: Discrete Probability – Suggested Solution

Question 1.
Let H and T be the outcomes "head" and "tail" of a coin flip.
The sample space contains all the sequences of length 6 of H's and T's, so its size is $2^6 = 64$.
The event only contains the sequence $HHHHHH$, so the probability is $\frac{1}{64}$.

Question 2.
The experiment is selecting a 5-card poker hand from 52 cards, so the size of the sample space is $C(52,5)$.

(a) As the poker hand does not contain the queen of hearts, all cards must be from the remaining 51 cards. Thus, the size of the event is $C(51,5)$ and the probability is $\frac{C(51,5)}{C(52,5)} = \frac{47}{52}$.

(b) As the question completely specifies the poker hand, the size of this event is 1. Thus, the probability is $\frac{1}{C(52,5)}$.

(c) Let E be the event that the poker hand contains at least one ace.
Then, $\overline{E}$ is the event that the poker hand does not contain any ace.
For event $\overline{E}$, all cards must be from the $52 - 4 = 48$ non-ace cards, so $p(\overline{E}) = \frac{C(48,5)}{C(52,5)}$.
Therefore, $p(E) = 1 - p(\overline{E}) = 1 - \frac{C(48,5)}{C(52,5)}$.

(d) We need to compute the number of possible 5-card poker hands containing two pairs. We can specify the poker hand, as follows:

- Step 1: Choose two kinds (e.g., kings and five) the pairs will be.
- Step 2: For each chosen kind, choose 2 card from the 4 suits (e.g., hearts, clubs).
- Step 3: Choose the fifth card from the remaining $13 - 2$ kinds.

The number of possible choices for Step 1 is $C(13,2) = 78$.
The number of possible choices for Step 2 is $C(4,2) \times C(4,2) = 6 \times 6 = 36$.
The number of possible choices for Step 3 is $(13 - 2) \times 4 = 44$.
By product rule, the number of possible poker hands is $78 \times 36 \times 44 = 123,552$.
Therefore, the probability is $\frac{123,552}{C(52,5)}$.

Question 3. There are two different solutions by considering two different experiments.
Solution 1:
The experiment is randomly selecting 6 numbers for a ticket, and the lottery commission has fixed its 6 numbers in advance.
Thus, the size of the sample space is $C(56,6)$.
A winning ticket can be selected, as follows:

- Step 1: Select 1 number from the 6 numbers selected by the lottery commission.
- Step 2: Select the remaining 5 numbers from the $56 - 6 = 50$ numbers not selected by the lottery commission.

The number of possible choices for Step 1 is 6.
The number of possible choices for Step 2 is $C(50,5)$.
By product rule, the number of possible winning ticket is $6 \cdot C(50,5)$.
Therefore, the probability of a winning ticket is $\frac{6 \cdot C(50,5)}{C(56,6)}$.

Solution 2:
The experiment is randomly selecting 6 numbers by the lottery commission, and the 6 numbers on your ticket are fixed in advance.
Thus, the size of the sample space is $C(56, 6)$.
To make your ticket a winning ticket, the lottery commission can select the 6 numbers, as follows:

- Step 1: Select 1 number from the 6 numbers on your ticket.
- Step 2: Select the remaining 5 numbers from the $56 - 6 = 50$ numbers not on your ticket.

The number of possible choices for Step 1 is 6.
The number of possible choices for Step 2 is $C(50, 5)$.
By product rule, the number of possible choices for the lottery commission to make your ticket winning is $6 \cdot C(50, 5)$.
Therefore, the probability of a winning ticket is $\dfrac{6 \cdot C(50,5)}{C(56,6)}$.

Question 4.
The experiment is randomly selecting the winners for first, second, and third prizes.
Thus, the size of the sample space is $P(100, 3) = 100 \cdot 99 \cdot 98$.
We can obtain an outcome where Tom wins one of the prizes, as follows:

- Step 1: Choose one of the 3 prizes for Tom.
- Step 2: Choose 2 other persons to win the remaining 2 prizes.

The number of possible choices for Step 1 is 3.
The number of possible choices for Step 2 is $P(100 - 1, 2) = P(99, 2) = 99 \cdot 98$.
By product rule, the number of possible outcomes is $3 \cdot 99 \cdot 98$.
Therefore, the probability is $\dfrac{3 \cdot 99 \cdot 98}{100 \cdot 99 \cdot 98} = \dfrac{3}{100}$.

Question 5.
Let B be the event that the first child is a boy, and let G be the event that the last two children are girls.
We need to calculate $p(B \cup G)$, which by the principle of inclusion-exclusion, is

$$p(B \cup G) = p(B) + p(G) - p(B \cap G) \ .$$

We can compute each term, as follows:

- $p(B) = \dfrac{1}{2}$
- $p(G) = \dfrac{1}{2 \times 2} = \dfrac{1}{4}$
- $p(B \cap G) = \dfrac{1}{2 \times 2 \times 2} = \dfrac{1}{8}$

Therefore,

$$\begin{aligned} p(B \cup G) &= p(B) + p(G) - p(B \cap G) \\ &= \frac{1}{2} + \frac{1}{4} - \frac{1}{8} \\ &= \frac{4 + 2 - 1}{8} = \frac{5}{8} \ . \end{aligned}$$