COMP S264F Discrete Mathematics
Tutorial 7: Functions (2) – Suggested Solution

Question 1.

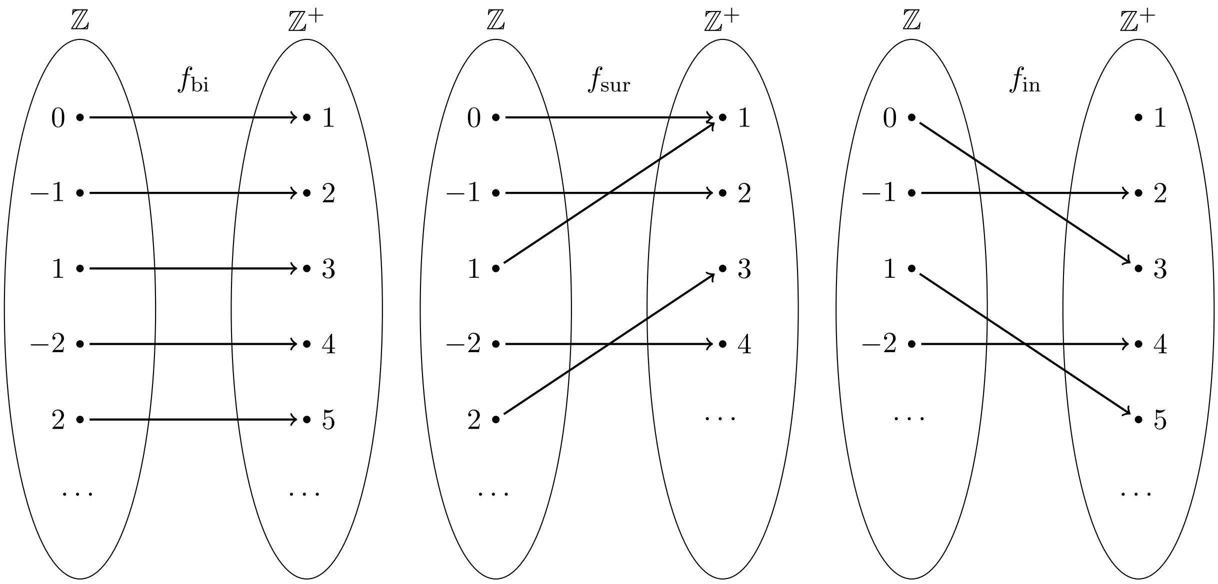
(a) No. When $x = -2$, $f(-2) = \sqrt{-2} \notin \mathbb{R}$.

(b) Yes. As x^2 is non-negative, $\sqrt{x^2+1}$ is a real number.

(c) No. When $x = 0$, $f(0) = \sqrt{0^2+1} = \pm 1$ which has two values.

(d) No. When $x = 4$, $f(4) = \dfrac{1}{4^2-16} = \dfrac{1}{0}$ which is undefined.

Question 2. To deal with this type of question, it is better to think of a bijective function first. An arrow diagram will be helpful for your thinking. First, try to draw the arrow diagram and define a bijective function f_{bi}. Then, we can have the solution of (c). Next, we can adapt the arrow diagram such that it matches (a) and (b), respectively. Here is one of the possible solutions:



(a) $f(x) = f_{\text{sur}}(x) = \begin{cases} -2x & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ 2x-1 & \text{if } x > 0 \end{cases}$

(b) $f(x) = f_{\text{in}}(x) = \begin{cases} -2x & \text{if } x < 0 \\ 2x+3 & \text{if } x \geq 0 \end{cases}$

(c) $f(x) = f_{\text{bi}}(x) = \begin{cases} -2x & \text{if } x < 0 \\ 2x+1 & \text{if } x \geq 0 \end{cases}$

Question 3.

(a) $A = \{1, 2, 3, 4\}$
$B = \{6, 7, 8, 9\}$
$|A| = |B| = 4$

(b) Define $f : A \to B$ such that $f(x) = x + 1$.

$$\text{Let } x, y \in A.\ f(x) = f(y) \implies x + 1 = y + 1 \implies x = y \implies f \text{ is injective.}$$

$$\text{For any } b \in B,\ b = f(a) \implies b = a + 1 \implies a = b - 1$$

Since b is even, $a = b - 1$ is odd. We have $a \in A$, so f is surjective.
Hence, f is bijective $\implies$ A and B have the same cardinality.

(c) Define $f : A \to B$ such that $f(x) = 3x$.

$$\text{Let } x, y \in A.\ f(x) = f(y) \implies 3x = 3y \implies x = y \implies f \text{ is injective.}$$

$$\text{For any } b \in B,\ b = f(a) \implies b = 3a \implies a = \frac{b}{3}$$

Since $b = 3k$ for some integer k, we have $a = k \in \mathbb{Z}$, i.e., $a \in A$, so f is surjective.
Hence, f is bijective $\implies$ A and B have the same cardinality.

(d) Define $f : A \to B$ such that $f(x) = 3x + 2$.

$$\text{Let } x, y \in A.\ f(x) = f(y) \implies 3x + 2 = 3y + 2 \implies x = y \implies f \text{ is injective.}$$

$$\text{For any } b \in B,\ b = f(a) \implies b = 3a + 2 \implies b - 2 = 3a \implies a = \frac{b-2}{3}$$

$\because 2 < b < 5$

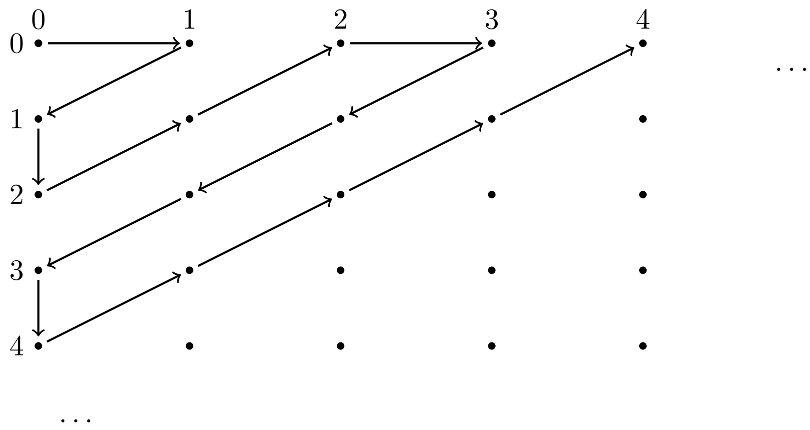
$$\therefore a = \frac{b-2}{3} > \frac{2-2}{3} = 0 \quad \text{and} \quad a = \frac{b-2}{3} < \frac{5-2}{3} = 1$$

As $a = \dfrac{b-2}{3} \in \mathbb{R}$ and $0 < a < 1$, we have $a \in A$ and thus f is surjective.

Hence, f is bijective $\implies$ A and B have the same cardinality.

Question 4.

(a) Consider the following grid with a path $(0,0),(0,1),(1,0),(2,0),(1,1),(0,2),(0,3),(1,2),\dots$.



We can define a bijective function $f:\mathbb{N}\to\mathbb{N}\times\mathbb{N}$ such that $f(i)$ is the $(i+1)$-th node in the path of the grid, implying that $\mathbb{N}\times\mathbb{N}$ is countable.

(b) (i) First, assume both A_0 and A_1 are countable.
By (a), as f is a bijective function from $\mathbb{N}$ to $\mathbb{N}\times\mathbb{N}$. To show that $A_0\times A_1$ is countable, it suffices to show that there is a bijective function $g:\mathbb{N}\times\mathbb{N}\to A_0\times A_1$.
As A_0 and A_1 are countable, there are two bijective functions $p:\mathbb{N}\to A_0$ and $q:\mathbb{N}\to A_1$.
Therefore, we can define a bijective function $g:\mathbb{N}\times\mathbb{N}\to A_0\times A_1$ such that

$$g(x,y)=(p(x),q(y)) \ .$$

It follows that $A_0\times A_1$ is countable.

(ii) Next, assume $A_0\times A_1$ is countable.
Let k be some element in A_1.
Then, $A_0\times\{k\}\subseteq A_0\times A_1$ is also countable, which implies A_0 is countable.
Similarly, A_1 is countable by the same argument.

In conclusion, the Cartesian product $A_0\times A_1$ is countable if and only if both A_0, A_1 are countable.

(c) We prove that statement by mathematical induction on n.

Base case. The base case is proven in (b).

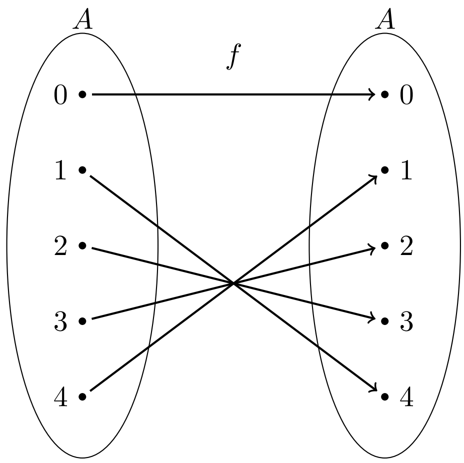
Inductive step. Assume that for some integers $k\geq 1$, $A_0\times A_1\times\cdots\times A_k$ is countable if and only if $A_0,A_1,\dots,A_k$ are all countable.
Consider any set A_{k+1}. Then,

$$\begin{aligned}
&A_0,A_1,\dots,A_{k+1} \text{ are countable}\\
\Leftrightarrow\ &(A_0\times A_1\times\cdots\times A_k \text{ is countable})\wedge(A_{k+1}\text{ is countable}) &&\text{(by the induction hypothesis)}\\
\Leftrightarrow\ &(A_0\times A_1\times\cdots\times A_k)\times A_{k+1}\text{ is countable} &&\text{(by the result of (b))}
\end{aligned}$$

(d) Let $C=\{(x,y)\mid x,y\in\mathbb{Z}\text{ and }y\neq 0\text{ and }x,y\text{ are relatively prime}\}$.
By definition, there is a bijective function $f:C\to\mathbb{Q}$ such that for any $(x,y)\in C$, $f(x,y)=\dfrac{x}{y}$.
To show that $\mathbb{Q}$ is countable, it suffices to show that C is countable.
Note that $C\subseteq\mathbb{Z}\times\mathbb{Z}$. As $\mathbb{Z}$ is countable, by (b), $\mathbb{Z}\times\mathbb{Z}$ is countable and thus C is countable.

Question 5.

(a) Yes. Since the cardinality of A is small, we can prove it by exhaustion. By drawing the arrow diagram of f, we can find that f is bijective.



(b) No. Consider $x = 1.5$ and $y = 2.5$.
$L.H.S. = \lceil 1.5 + 2.5 \rceil = \lceil 4 \rceil = 4$.
$R.H.S. = \lceil 1.5 \rceil + \lceil 2.5 \rceil = 2 + 3 = 5 \neq L.H.S.$.

(c) Yes. Let $n = 2k + 1$ for some integer k.

$$\begin{aligned}\frac{n^2}{4} &= \frac{(2k+1)^2}{4} \\ &= \frac{4k^2+4k+1}{4} \\ &= k^2 + k + \frac{1}{4}\end{aligned}$$

Since $k^2 + k$ is an integer, $L.H.S. = \left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor k^2 + k + \frac{1}{4} \right\rfloor = k^2 + k + \left\lfloor \frac{1}{4} \right\rfloor = k^2 + k$.

$$\begin{aligned}R.H.S. = \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) &= \left(\frac{(2k+1)-1}{2}\right)\left(\frac{(2k+1)+1}{2}\right) \\ &= \frac{2k(2k+2)}{4} \\ &= \frac{4k^2+4k}{4} \\ &= k^2 + k \\ &= L.H.S.\end{aligned}$$