COMP S264F Discrete Mathematics
Specimen – Suggested Solution

Question 1 (10 marks).

(a) The following is the truth table of $p \to (q \to \neg p)$:

p	q	$\neg p$	$q \to \neg p$	$p \to (q \to \neg p)$
T	T	F	F	F
T	F	F	T	T
F	T	T	T	T
F	F	T	T	T

(b)
$$\begin{aligned}(p \to \neg q) \vee (p \wedge q) &\equiv (\neg p \vee \neg q) \vee (p \wedge q) && \text{(as } a \to b \equiv \neg a \vee b)\\ &\equiv \neg(p \wedge q) \vee (p \wedge q) && \text{(by De Morgan's law)}\\ &\equiv T && \text{(as } \neg a \vee a \equiv T)\end{aligned}$$

(c) No, when p and q are both true, by (a), $p \to (q \to \neg p) \equiv F$, but by (b), $(p \to \neg q) \vee (p \wedge q) \equiv T$.

(d)
$$\begin{aligned}\neg\forall x\ (P(x) \to Q(x)) &\equiv \exists x\ \neg(P(x) \to Q(x))\\ &\equiv \exists x\ \neg(\neg P(x) \vee Q(x)) && \text{(as } a \to b \equiv \neg a \vee b)\\ &\equiv \exists x\ (P(x) \wedge \neg Q(x)) && \text{(by De Morgan's law)}\end{aligned}$$

Question 2 (10 marks).
Suppose, for the sake of contradiction, that there is a real number x that x^2 is irrational but x is rational.
Then, we can let $x = \dfrac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$.
Thus,

$$x^2 = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2}\ .$$

Note that $p, q \in \mathbb{Z}$ implies $p^2, q^2 \in \mathbb{Z}$.
Therefore, x^2 is rational, which contradicts that x^2 is irrational.

Question 3 (15 marks).

(a)

$$f \circ g(x) = f(g(x)) = \frac{1}{g(x) - 2} = \frac{1}{\left(\frac{1}{x} + 2\right) - 2} = \frac{1}{\frac{1}{x}} = x$$

$$g \circ f(x) = g(f(x)) = \frac{1}{f(x)} + 2 = \frac{1}{\frac{1}{x-2}} + 2 = x - 2 + 2 = x$$

(b) If $x = 2$ which is in $\mathbb{R}$, then $f(x) = \dfrac{1}{0}$ is undefined. Thus, X cannot be $\mathbb{R}$.
Similarly, if $y = 0$ which is in $\mathbb{R}$, then $g(y) = \dfrac{1}{0} + 2$ is also undefined. Thus, Y cannot be $\mathbb{R}$.

(c) (i) We first show that f is injective. Let $x, y \in X$. Then,

$$f(x) = f(y) \implies \frac{1}{x-2} = \frac{1}{y-2} \implies x - 2 = y - 2 \implies x = y$$

Therefore, f is injective.
Next, we show that f is surjective. Consider any $b \in Y = \mathbb{R} - \{0\}$. Then,

$$\begin{aligned} b = f(a) &\implies b = \frac{1}{a-2} \\ &\implies a - 2 = \frac{1}{b} \qquad (\text{as } b \neq 0) \\ &\implies a = \frac{1}{b} + 2\ . \end{aligned}$$

Note that $\frac{1}{b} \neq 0$ implies $a \neq 2$. Thus, $a \in \mathbb{R} - \{2\} = X$ and hence f is surjective.
Since f is both injective and surjective, f is bijective.

(ii) Yes, f^{-1} exists because f is bijective.
By (a), $f \circ g(x) = x$, so $f^{-1}(x) = g(x) = \frac{1}{x} + 2$.

Question 4 (5 marks).
The smallest number of students in a class is $7 \times 3 + 1 = 22$.
Consider any 22 students.
We divide them into groups according to the day of week of their birthdays, i.e., {Sunday, Monday, ...}.
There are 22 students and 7 groups. By pigeonhole principle, one of the groups will contain at least $\left\lceil \frac{22}{7} \right\rceil = 4$ students. These 4 students were born on the same day of the week.
The smallest number is 22, because when there are 21 students in a class, it is possible for each of the days of the week, there are only 3 students born on that day.

Question 5 (10 marks).

(a) The number of ways to draw 8 notes of any of the four types is $C(8+4-1, 8) = C(11, 8) = 165$.

(b) Since one note must be \$20, the problem becomes finding the number of ways to draw 7 notes from the pool of 4 types of notes. Therefore, the number of ways to draw 8 notes if at least one \$20 note has to be drawn is $C(7+4-1, 7) = C(10, 7) = 120$.

(c) By (a), the number of ways to draw 8 notes of any of the four types is $C(8+4-1, 8) = C(11, 8) = 165$.
The number of ways to draw 8 notes of exactly one type is 4.
Therefore, the number of ways to draw 8 notes if notes of at least two types must be drawn is $165 - 4 = 161$.

Question 6 (10 marks).
We can form a committee of 6 members with a chairperson from 15 people, as follows:

Method 1:
Step 1: Select 6 people to the committee.
Step 2: Select one of the these 6 people to be the chairperson.
Step 1 has $C(15, 6)$ choices, and Step 2 has 6 choices.
By product rule, the number of ways to form the committee is $6 \cdot C(15, 6)$.
Method 2:
Step 1: Select one person to be the chairperson.
Step 2: Select the other 5 members from the remaining 14 people.
Step 1 has 15 choices, and Step 2 has $C(14, 5)$ choices.
By product rule, the number of ways to form the committee is $15 \cdot C(14, 5)$.

As any committee satifying the requirement can be formed by both methods,

$$6 \cdot C(15, 6) = 15 \cdot C(14, 5)\ .$$

Question 7 (10 marks).
Base case: When $n = 4$, $n! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ and $2^n = 2^4 = 16 < 24 = n!$
Induction hypothesis: Assume for some positive integer $k \geq 4$, $2^k < k!$
Inductive step: When $n = k + 1$,

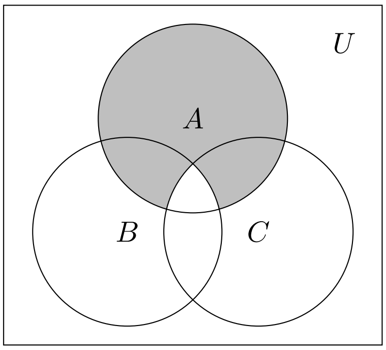
$$\begin{aligned} 2^n &= 2^{k+1} = 2 \cdot 2^k \\ &< 2 \cdot k! \qquad \text{(by the induction hypothesis)} \\ &< (k+1) \cdot k! = (k+1)! \qquad \text{(as } k \geq 4 \text{, so } k+1 > 2) \end{aligned}$$

Question 8 (10 marks).

(a) Let U be the universal set.

$$\begin{aligned} (A \cap \overline{B}) \cup (A \cap \overline{C}) &= A \cap (\overline{B} \cup \overline{C}) \qquad \text{(by distributive law)} \\ &= A \cap \overline{(B \cap C)} \qquad \text{(by De Morgan's law)} \\ &= A \cap (U - (B \cap C)) \end{aligned}$$

Its Venn diagram is shown, as follows:



(b)
$$\begin{aligned} x \in (A - B) - C &\implies (x \in A - B) \wedge (x \notin C) \\ &\implies (x \in A) \wedge (x \notin C) \\ &\implies x \in A - C \end{aligned}$$
Therefore, $(A - B) - C \subseteq A - C$.

Question 9 (10 marks).
Let A be the event that A attends the lecture, and let B be the event that B attends the lecture.
Then, $p(A) = \frac{1}{4}$ and $p(B) = \frac{2}{5}$.
The probability that at least one of them attends the lecture is

$$\begin{aligned}
p(A \cup B) &= p(A) + p(B) - p(A \cap B) && \text{(by the principle of inclusion-exclusion)} \\
&= p(A) + p(B) - p(A) \cdot p(B) && \text{(as } A \text{ and } B \text{ are independent events)} \\
&= \frac{1}{4} + \frac{2}{5} - \frac{1}{4} \cdot \frac{2}{5} \\
&= \frac{5 + 8 - 2}{20} = \frac{11}{20}
\end{aligned}$$

Alternative solution:
The probability that at least one of them attends the lecture is

$$\begin{aligned}
p(A \cup B) &= 1 - p(\overline{A \cup B}) \\
&= 1 - p(\overline{A} \cap \overline{B}) && \text{(by De Morgan's law)} \\
&= 1 - p(\overline{A}) \cdot p(\overline{B}) && \text{(as } A \text{ and } B \text{ are independent events)} \\
&= 1 - (1 - p(A)) \cdot (1 - p(B)) \\
&= 1 - (1 - \frac{1}{4}) \cdot (1 - \frac{2}{5}) \\
&= 1 - \frac{3 \cdot 3}{20} = \frac{11}{20}
\end{aligned}$$

Question 10 (10 marks).

(a) $$\begin{aligned}
p(M \cap B \mid B) &= \frac{p(M \cap B)}{p(B)} \\
&= \frac{10\%}{15\%} = \frac{2}{3}
\end{aligned}$$

(b) $$\begin{aligned}
p(\overline{M} \cap B \mid \overline{M}) &= \frac{p(\overline{M} \cap B)}{p(\overline{M})} \\
&= \frac{p(B) - p(B \cap M)}{1 - p(M)} \\
&= \frac{15\% - 10\%}{1 - 25\%} \\
&= \frac{5}{75} = \frac{1}{15}
\end{aligned}$$

(c) $$\begin{aligned}
p(M \cup B) &= p(M) + p(B) - p(M \cap B) && \text{(by the principle of inclusion-exclusion)} \\
&= 25\% + 15\% - 10\% \\
&= 30\% = \frac{3}{10}
\end{aligned}$$

(d) $$\begin{aligned}
p(\overline{M} \cap \overline{B}) &= p(\overline{M \cup B}) && \text{(by De Morgan's law)} \\
&= 1 - p(M \cup B) \\
&= 1 - \frac{3}{10} = \frac{7}{10}
\end{aligned}$$