COMP S264F Discrete Mathematics
Tutorial 11: Conditional Probability, Random Variables

Question 1. Consider a family with two children. Assume that each of the possibilities BB, BG, GB, and GG is equally likely, where B represents a boy and G represents a girl. (Note that BG represents a family with an older boy and a younger girl while GB represents a family with an older girl and a younger boy.) What is the conditional probability that the family has two boys, given they have at least one boy?

Question 2. Find the probability that the first child of a family with five children is a boy or that the last two children of the family are girls, if the sexes of children are independent and if

(a) a boy and a girl are equally likely.

(b) the probability of a boy is 0.51.

(c) the probability that the i-th child is a boy is $0.51 - \frac{i}{100}$.

Question 3. (Birthday Problem) Suppose there are n people in a room. We assume that the birthdays of the people in the room are independent, and that each birthday is equally likely, and that there are 366 days in the year. (In reality, more people are born on some days of the year than others, such as days nine months after some holidays including New Year's Eve, and only leap years have 366 days.) What is the minimum number n of people who need to be in the room so that the probaility that at least two of them have the same birthday is greater than $\frac{1}{2}$?

Hint: You may use a Python program to help finding n.

Question 4. A new employee checks the hats of n people at a restaurant, forgetting to put claim check numbers on the hats. When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats. What is the expected number of hats that are returned correctly?

Question 5. The ordered pair (i, j) is called an *inversion* in a permutation of the first n positive integers if $i < j$ but j precedes i in the permutation. For instance, there are six inversions in the permutation $(3, 5, 1, 4, 2)$; these inversions are

$$(1, 3), (1, 5), (2, 3), (2, 4), (2, 5), (4, 5) \ .$$

What is the expected number of inversions in the permutation?

Hint: Let X be the random variable equal to the number of inversions in the permutation. Let $I_{i,j}$ be the random variable on the set of all permutations of the first n positive integers with $I_{i,j} = 1$ if (i, j) is an inversion of the permutation and $I_{i,j} = 0$ otherwise. Then,

$$X = \sum_{1 \le i < j \le n} I_{i,j} \ .$$

Question 6. Let X and Y be random variables that count the number of heads and the number of tails when a coin is flipped twice. Are X and Y independent random variables?