# COMPS264F

# Discrete Mathmatics

## Assignment

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## Question 1 (10 marks)

(a) $\neg \exists x \forall y P(x,y)$

$= \forall x \neg \forall y P(x,y) = \forall x \exists y \neg P(x,y)$

(b) $\neg \exists y (\forall x \exists z P(x,y,z) \vee \exists x \forall z Q(x,y,z))$.

$= \forall y \neg (\forall x \exists z P(x,y,z) \vee \exists x \forall z Q(x,y,z))$.

$= \forall y (\exists x \forall z \neg P(x,y,z) \vee \forall x \exists z \neg Q(x,y,z))$.

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## Question 2 (15 marks).

(a) $N = \{ \emptyset, \{\emptyset,\{\emptyset\}\}, \{\emptyset,\{\emptyset,\{\emptyset\}\}\}, \{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset\}\}\}\}, \{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset\}\}\}\}\} \}$

(b) False, because $N$ contains $\emptyset$, which cardinality is 0, that means contains no element.

(c) Suppose $N$ is a finite set, let's get this expression:

"if $N$ contains the set $X$, then $N$ contains the set $X \cup \{X\}$".

Since there are limited $n$ elements $\{a_1, a_2, \dots, a_n\} = N$. we can consider that the $\{a_i, \{a_i\}\}$ must in $N$. which contradicts that $N$ is a infinite set. because if we gave the cardinality $n$ of $N$, there will always be $N_{n+1}$ set there.

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(d). Because $N$ is an infinite set, which means the cardinality is very big, let's make it as $\aleph_0$. but any element in $N$, except $\emptyset$, only contains two elements, which means the cardinality is always 2. So. $N \notin N$, because the cardinary of elements in $N$: $2 <$ the cardinaty of $N$.

Question 3 ( 15 marks)

prove :

$S = \{ s_1, s_2, \dots, s_n, \dots \}$ if $S$ and $T$ have the same cardinality.

$T = \{ t_1, t_2, \dots, t_n, \dots \}$ Which means we can define a bijection $t_n = g(s_n)$ between set $S$ and $T$.

which means for $t_n$, elements in $T$, there will be an only corresponding $s_n$.

$\because$ $t_n$ is a mapping between $S$ and $S$.

Let's consider a mapping $t$ between $S$ and $S$, which mapping all elements in $S$ into a specific element $S_i$ in $S$. if the cardinality of $S$ is $\aleph_1$, then the corresponding $t$ is also $\aleph_1$, because for every element in $S$, we can define this mapping $t$.

but there are more elements of $T$. for example, we can define another mapping $t_j$, which mapping $(\aleph_1 - 1)$ value into a specific element in $S$ $(S_j)$. and another value $S_k$ is mapped into another element in $S$. (like $S_p$). and so on ... .

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So, In conclusion. we can always find more mapping $t_n$. between set $S$ and $S$. than the cardinality of $S$.

which prove that $S$ and $T$ do not have the same cardinality.

Question 4 ( 10 marks )

Let's build 12 basketball players $\{ a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ a_8\ a_9\ a_{10}\ a_{11}\ a_{12} . \}$

and divide them into 10 sets. $\{ A_0\ A_1\ A_2\ A_3\ A_4\ A_5\ A_6\ A_7\ A_8\ A_9 . \}$.

From $A_0 \sim A_9$. each set contains three consecutive players.

we can know that. the total uniform numbers of 12 players is $1+2+3+\dots+12 = 78$.

So the average uniform number each player is equal to : $\frac{78}{12} = 6.5$.

In $A_0 \sim A_9$. $\{a_3, a_4, \dots, a_{10}\}$ were count 3 times each of them ; So there are $3 \times 8 \times 6.5 = 156$ uniform numbers

$\{a_2, a_{11}\}$ were count twice ; So there are $2 \times 2 \times 6.5 = 26$ uniform numbers.

$\{a_1, a_{12}\}$ were only count once. So there are $1 \times 2 \times 6.5 = 13$ uniform numbers.

(Question 4 Cont'd)

So, there are total $156+26+13=195$ uniform numbers.
which need to be placed into size $(A_0, A_1, \cdots, A_9) = 10$ sets.

Thus, according to the generalized Pigeonhole principle:

there is at least one set $A_t$. containing at least $\lceil \frac{195}{10} \rceil = \lceil 19.5 \rceil = 20$ uniform numbers.

Since every $A_t$ is the set of three consecutive players.
which ensure our proof: "Some three consecutive players have the sum of their numbers at least 20".

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Question 5 (10 marks).

Consider the following two ways to arrange the ways in a n-person wine tour:

Method 1:

Step 1: we choose a driver from n person. which has n different ways.

Step 2: For the remaining $(n-1)$ person. each of them has 4 different choices. (3 alcoholic, 1 non-al)
which has $4^{n-1}$ different ways.

Thus, the ways to arrange a n-person wine tour is equal to: $4^{n-1} \times n = n \cdot 4^{n-1}$.

Method 2:

Let $k$ be an integer such that $0 \le k \le n$.

Step 1: we choose $k$ person. who will have the alcoholic menu.
there are. $C(n,k)$ ways to choose and each of the $k$ people have 3 different menu.
$\therefore$ there are $C(n,k) \cdot 3^k$ ways

Step 2: for the remaining $(n-k)$ person, each of them only have one choice — no-alcoholic menu.
So there are $(n-k) \cdot 1$. ways.

For a particular $k$, the number of ways to arrange this tour is $C(n,k) \cdot 3^k (n-k)$.
Therefore. the total number of ways to arrange the tour is $\sum_{k=0}^{n} C(n,k) \cdot 3^k \cdot (n-k)$.

The wine tour can be arranged by both methods. So:

$$n \cdot 4^{n-1} = \sum_{k=0}^{n} C(n,k) \cdot 3^k (n-k).$$

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Question 6 ( 10 marks )

Base case. when $n=2$:

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \leq P(E_1) + P(E_2).$$

Induction step.

Assume that

$$P(E_1 \cup E_2 \cup \cdots \cup E_k) \leq \sum_{i=1}^{k} P(E_i).$$ for some positive integer $k$.

when $n=k+1$.

$$P(E_1 \cup E_2 \cup \cdots \cup E_k \cup E_{k+1}) = \sum_{i=1}^{k} P(E_i) - P(E_1 \cap E_2 \cap \cdots \cap E_k \cap E_{k+1}) + P(E_{k+1})$$

$$\leq \sum_{i=1}^{k+1} P(E_i).$$ ( when $P(E_1 \cap E_2 \cap \cdots \cap E_k \cap E_{k+1}) = 0$, equal ).

By the principle of mathematical induction.

for all events $E_1, E_2, \cdots, E_n$. $P(E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_n) \leq \sum_{i=1}^{n} P(E_i)$

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Question 7 ( 15 marks )

(a) when $r=5$. Let's guess another envelop:

$P(\text{grater than } r) = \frac{5}{11-1} = \frac{1}{2} = P(\text{less than } r)$. this is the example of 50% chance winning.

when $r<5$. Let's guess another envelop:

$P(\text{win} / \text{grater than } r) > \frac{6}{11-1} = \frac{3}{5}$ ( when $r=4$ ).

this is the example of more than 50% chance of winning.

(b). (i) if the $X$ is between the two envelope numbers.
the chance of wining is 100%.

(ii). $P>0$, means. the case $a<x<b$ can occur. acctually. it can occur.
for example: if $a=4$ and $b=5$. and $X=4.5$ . etc.

(iii) ① if $X$ is between the two envelop numbers. $P_{win}=100\%$.

② if $X$ is less than the smallest envelop number. $P_{win}=0\%$.

③ if $X$ is bigger than the largest envelop number. $P_{win}=50\%$. ↓ next page will discuss detail

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(Question 7 Cont'd)

(b)(iii)

① $P\left(\begin{matrix} a<x<b \\ b<x<a \end{matrix}\right) = 1 - P\left(\begin{matrix} a<x<b \\ b<x<a \end{matrix}\right) - P\left(\begin{matrix} a<b<x \\ b<a<x \end{matrix}\right) = 50\%$ $\therefore P(\text{win})_{\text{mid}} = 50\% \times 100\% = 50\%$. ×

② $P\left(\begin{matrix} x<a<b \\ x<b<a \end{matrix}\right) = \dfrac{9+8+7+6+5+4+3+2+1}{2\times 9\times 10} = \dfrac{45}{180} = \dfrac{1}{4} = 25\%$ $\therefore P(\text{win})_{\text{Less}} = 25\% \times 0\% = 0\%$.

③ $P\left(\begin{matrix} a<b<x \\ b<a<x \end{matrix}\right) = \dfrac{9+8+7+6+5+4+3+2+1}{2\times 9\times 10} = 25\%$. $\therefore P(\text{win})_{\text{bigger}} = 25\% \times 50\% = 12.5\%$.

$\therefore P(\text{win}) = P(\text{win})_{\text{mid}} + P(\text{win})_{\text{Less}} + P(\text{win})_{\text{bigger}} = 50\% + 0\% + 12.5\% = 62.5\%$

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This shows that this strategy has a better than 50% chance of winning.

Question 8 (15 marks)

Let the value of dice is D

(a) $E(X) = 2\cdot P(X=2) + 4\cdot P(X=4) + P(X=6)\cdot 6 + P(X=8)\cdot 8 + P(X=10)\cdot 10 + P(X=12)\cdot 12$.

$= 2\cdot P(D=1, X=2) + 4\cdot P(D=2, X=4) + P(D=3, X=6)\cdot 6$

$= \frac{1}{6}\times 2 + \frac{1}{6}\times 4 + \frac{1}{6}\times 6 + \frac{1}{6}\times 8 + \frac{1}{6}\times 10 + \frac{1}{6}\times 12 = \frac{1}{6}\times 42 = 7$. ✓

(b). $E(Y) = P(D\text{odd}, Y=1)\cdot 1 + P(D\text{even}, Y=3)\cdot 3$

$= \frac{1}{2}\times 1 + \frac{1}{2}\times 3 = 2$. ✓

(c) $Z(1) = X(1) + Y(1) = 2+3 = 5$

$Z(2) = X(2) + Y(2) = 4+1 = 5$

$Z(3) = X(3) + Y(3) = 6+3 = 9$.

$Z(4) = X(4) + Y(4) = 8+1 = 9$.

$Z(5) = X(5) + Y(5) = 10+3 = 13$.

$Z(6) = X(6) + Y(6) = 12+1 = 13$.

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$\therefore E(Z) = E(X) + E(Y) = (5+5+9+9+13+13)\times\frac{1}{6} = 54\times\frac{1}{6} = 9$. ✓

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