COMP S264F Discrete Mathematics
Tutorial 1: Logic (1) – Suggested Solution

Question 1. Note that p is true, q is false, and r is true.

(a) $$\begin{aligned} p \wedge q \to r &\equiv (T \wedge F) \to T \\ &\equiv F \to T \\ &\equiv T \end{aligned}$$

(b) $$\begin{aligned} p \vee q \to \neg r &\equiv (T \vee F) \to \neg T \\ &\equiv T \to F \\ &\equiv F \end{aligned}$$

(c) $$\begin{aligned} p \wedge (q \to r) &\equiv T \wedge (F \to T) \\ &\equiv T \wedge T \\ &\equiv T \end{aligned}$$

(d) $$\begin{aligned} p \leftrightarrow (q \to r) &\equiv T \leftrightarrow (F \to T) \\ &\equiv T \leftrightarrow T \\ &\equiv T \end{aligned}$$

Question 2.

(a) Truth table of $p \wedge q \to r$:

p	q	r	$p \wedge q$	$p \wedge q \to r$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

(b) Truth table of $p \vee q \to \neg r$:

p	q	r	$\neg r$	$p \vee q$	$p \vee q \to \neg r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

(c) Truth table of $p \wedge (q \to r)$:

p	q	r	$q \to r$	$p \wedge (q \to r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	F
F	T	F	F	F
F	F	T	T	F
F	F	F	T	F

(d) Truth table of $p \leftrightarrow (q \to r)$:

p	q	r	$q \to r$	$p \leftrightarrow (q \to r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	F
F	T	F	F	T
F	F	T	T	F
F	F	F	T	F

Question 3.

Solution 1: Using truth table

Truth table can be used to show logical equivalences of propositions. The truth tables of the four propositions are shown below:

p	q	$\neg p$	$\neg q$	*Implication* $p \to q$	*Converse* $q \to p$	*Contrapositive* $\neg q \to \neg p$	*Inverse* $\neg p \to \neg q$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	T
F	T	T	F	T	F	T	F
F	F	T	T	T	T	T	T

By identifying the identical columns in the truth table, we can conclude that

- *Implication* $\equiv$ *Contrapositive*
- *Converse* $\equiv$ *Inverse*

Solution 2: Using the propositions

- *Implication:* $p \to q \;\equiv\; \neg p \vee q$
- *Converse:* $q \to p \;\equiv\; \neg q \vee p \equiv p \vee \neg q$
- *Contrapositive:* $\neg q \to \neg p \;\equiv\; q \vee \neg p \;\equiv\; \neg p \vee q \;\equiv$ *Implication*
- *Inverse:* $\neg p \to \neg q \;\equiv\; p \vee \neg q \;\equiv$ *Converse*

Question 4.

(a) $$\begin{aligned} \neg(\neg p \wedge q) \wedge (p \vee q) &\equiv (p \vee \neg q) \wedge (p \vee q) && \text{(by De Morgan's law)} \\ &\equiv p \vee (\neg q \wedge q) && \text{(by the distributive law)} \\ &\equiv p \vee F \\ &\equiv p \end{aligned}$$

Thus, the logical equivalence is **true**.

(b) $$\begin{aligned} (p \wedge \neg q) \to (q \to \neg r) &\equiv (p \wedge \neg q) \to (\neg q \vee \neg r) && \text{(as } a \to b \equiv \neg a \vee b\text{)} \\ &\equiv \neg(p \wedge \neg q) \vee (\neg q \vee \neg r) && \text{(as } a \to b \equiv \neg a \vee b\text{)} \\ &\equiv (\neg p \vee q) \vee (\neg q \vee \neg r) && \text{(by De Morgan's law)} \\ &\equiv (q \vee \neg q) \vee \neg p \vee \neg r \\ &\equiv T \vee \neg p \vee \neg r \\ &\equiv T \end{aligned}$$

However, when $p = T$, $q = F$, $r = T$, we have $(\neg p \vee q) \vee \neg r \equiv F$.

Thus, the logical equivalence is **false**.

Question 5.

Solution 1: Using truth table

(a) The following truth table shows that $p \otimes p$ is not a tautology ($p \otimes p$ is a contradiction instead).

p	$p \otimes p$
T	F
F	F

(b) The following truth table shows that $p \otimes \neg p$ is a tautology.

p	$\neg p$	$p \otimes \neg p$
T	F	T
F	T	T

(c) The following truth table shows that $[(p \to q) \wedge \neg q] \to \neg p$ is a tautology.

p	q	$\neg p$	$\neg q$	$p \to q$	$(p \to q) \wedge \neg q$	$[(p \to q) \wedge \neg q] \to \neg p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

(d) The following truth table shows that $[p \wedge (p \to q)] \to q$ is a tautology.

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

(e) The following truth table shows that $[(p \vee q) \wedge \neg p] \to q$ is a tautology.

p	q	$\neg p$	$p \vee q$	$(p \vee q) \wedge \neg p$	$[(p \vee q) \wedge \neg p] \to q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

Solution 2: Simplifying propositions

(a)
$$\begin{aligned} p \otimes p &\equiv (p \wedge \neg p) \vee (\neg p \wedge p) && (\text{as } a \otimes b \equiv (a \wedge \neg b) \vee (\neg a \wedge b)) \\ &\equiv F \vee F && (\text{as } a \wedge \neg a \equiv F) \\ &\equiv F \end{aligned}$$

Thus, the proposition is not a tautology (it is a contradiction instead).

(b)
$$\begin{aligned} p \otimes \neg p &\equiv (p \wedge \neg(\neg p)) \vee (\neg p \wedge \neg p) && (\text{as } a \otimes b \equiv (a \wedge \neg b) \vee (\neg a \wedge b)) \\ &\equiv (p \wedge p) \vee (\neg p \wedge \neg p) && (\text{as } \neg(\neg a) \equiv a) \\ &\equiv p \vee \neg p && (\text{as } a \wedge a \equiv a) \\ &\equiv T && (\text{as } a \vee \neg a \equiv T) \end{aligned}$$

Thus, the proposition is a tautology.

(c)
$$\begin{aligned} [(p \to q) \wedge \neg q] \to \neg p &\equiv \neg[(p \to q) \wedge \neg q] \vee \neg p && (\text{as } a \to b \equiv \neg a \vee b) \\ &\equiv [\neg(p \to q) \vee q] \vee \neg p && (\text{by De Morgan's law}) \\ &\equiv \neg(p \to q) \vee (p \to q) && (\text{as } \neg p \vee q \equiv p \to q) \\ &\equiv T \end{aligned}$$

Thus, the proposition is a tautology.

(d)
$$\begin{aligned} [p \wedge (p \to q)] \to q &\equiv \neg[p \wedge (p \to q)] \vee q && (\text{as } a \to b \equiv \neg a \vee b) \\ &\equiv [\neg p \vee \neg(p \to q)] \vee q && (\text{by De Morgan's law}) \\ &\equiv \neg(p \to q) \vee (p \to q) && (\text{as } \neg p \vee q \equiv p \to q) \\ &\equiv T \end{aligned}$$

Thus, the proposition is a tautology.

(e)
$$\begin{aligned} [(p \vee q) \wedge \neg p] \to q &\equiv \neg[(p \vee q) \wedge \neg p] \vee q && (\text{as } a \to b \equiv \neg a \vee b) \\ &\equiv [\neg(p \vee q) \vee p] \vee q && (\text{by De Morgan's law}) \\ &\equiv \neg(p \vee q) \vee (p \vee q) \\ &\equiv T \end{aligned}$$

Thus, the proposition is a tautology.

Question 6. Let *vowel* and *even* be the statements "a card has a vowel on a side" and "a card has an even number on a side". Then, the statement equals $vowel \to even$, and its truth table is:

vowel	*even*	*vowel* $\to$ *even*
T	T	T
T	F	F
F	T	T
F	F	T

Thus, the only case to falsify the statement is that a card has a vowel and an odd number on the two sides.

Card `|A|`**:** Since `|A|` is a vowel, we need to turn over the card to check the number on the other side. If the number is even, then the statement is true; otherwise (the number is odd), the statement is false.

Card `|B|`**:** Since `|B|` is not a vowel, the statement must be true.

Card `|4|`**:** Since `|4|` is an even number, the statement must be true.

Card `|7|`**:** Since `|7|` is an odd number, we need to turn over the card to check the letter on the other side. If the letter is a vowel, the statement is false; otherwise (the letter is not a vowel), the statement is true.

Therefore, we need to turn over cards `|A|` and `|7|`.