Comps 264f

Online

Examination

22. January 2021 (Fri)

14:00 ~ 16:00

Name: Jiawei Wang

ID: S12395870

Question 1 (10 marks)

(a) $\neg((p \to q) \vee q)$

$= \neg((\neg p \vee q) \vee q)$ $\quad (a \to b \equiv \neg a \vee b)$

$= \neg(\neg p \vee q)$ $\quad (a \vee a = a)$.

$= p \wedge \neg q$ $\quad$ (De Morgan's law).

(b) $\neg \exists x (((\forall y\, P(x,y)) \to Q(x)) \vee Q(x))$

$= \forall x (\neg((\forall y\, P(x,y)) \to Q(x)) \wedge \neg Q(x))$ $\quad$ (De Morgan's law)

$= \forall x (\neg((\exists y \neg P(x,y)) \vee Q(x)) \wedge \neg Q(x))$ $\quad (a \to b \equiv \neg a \vee b)$

$= \forall x ((\forall y\, P(x,y)) \wedge \neg Q(x) \wedge \neg Q(x))$ $\quad (a \wedge a = a)$

$= \forall x ((\forall y\, P(x,y)) \wedge \neg Q(x))$

Question 2 (10 marks)

Proof:

Base case:

Suppose when $n=1$, $4^{n+1}+5^{2n-1}=4^2+5=21$ is divisible by 21.

Induction hypothesis:

Assume for some positive integer k, $4^{k+1}+5^{2k-1}$ is divisible by 21.

$\Rightarrow 4^{k+1}+5^{2k-1}=21A$; $A \in N$

Inductive step:

When $n=k+1$.

$$\begin{aligned} 4^{n+1}+5^{2n-1} = 4^{k+2}+5^{2k+1} &= 4\cdot 4^{k+1}+25\cdot 5^{2k-1} \\ &= 4\cdot 4^{k+1}+(21+4)\cdot 5^{2k-1} \\ &= 4(4^{k+1}+5^{2k-1})+21\cdot 5^{2k-1} \\ &= 4\cdot 21A+21\cdot 5^{2k-1} \\ &= 21(4A+5^{2k-1}) \end{aligned}$$

which is divisible by 21.

So, $S(k+1)$ is true whenever $S(k)$ is true.

Therefore, $4^{n+1}+5^{2n-1}$ is divisiby by 21. (when n is a positive integer).

Question 3 (10 marks)

(a) $\overline{(\bar{A}-B)} \cap (A \cup C)$

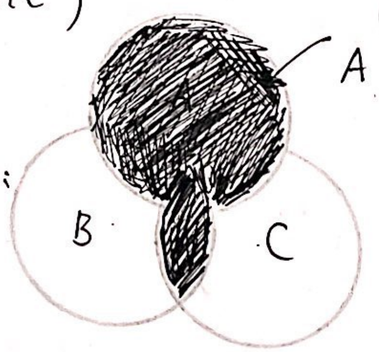
$= \overline{(\bar{A} \cap \bar{B})} \cap (A \cup C)$ $\quad (A-B = A \cap \bar{B})$

$= (A \cup B) \cap (A \cup C)$ $\quad$ (De Morgan's law)

$= A \cup (B \cap C)$ $\quad$ (distribution law)

(b)

The Venn diagram:



Question 4 (10 marks)

(a) Yes, f is one to one.

Let $X, Y \in R$ that $f(X) = f(Y)$

$\Rightarrow 4X+3 = 4Y+3 \Rightarrow X = Y \Rightarrow f$ is one to one

(b) Yes, f is onto.

For any $b \in R$ that $b = f(a)$ $(a \in R)$

$\Rightarrow b = 4a+3 \Rightarrow a = \frac{b-3}{4} \in R$

since for every b, there exist a different $a = \frac{b-3}{4}$

$\Rightarrow f$ is onto

(c) Yes, they have the same cardinality.

since f is both one to one and onto. $\Rightarrow f$ is a bijection

$\Rightarrow$ for any element in R, there exist an only corresponding element number in $\{4x+3 \mid x \in R\}$. So they have the same cardinality.

Question 5 (10 marks)

Consider the following two ways to from a k-member Commitee from n boys and m girls. with at least one girl and one boy..

Method 1:

Step 1: select k person as member from $n+m$ students. which has $C(n+m, k)$ ways.

Step 2: delete the ways that only contain boys and girls

Thus, a number of ways form a Commitee is:

$$C(n+m, k) - C(n, k) - C(m, k)$$

Question 5 Cont'd

Method 2:

Since for both boys and girls, the number of them to join the committee is from 1 to $k-1$:

Let i from 1 to $k-1$.

Step 1: we select i boys from n boys, which has $C(n,i)$ ways.

Step 2: We select $k-i$ girls from m girls, which has $C(m,k-i)$ ways.

For a particular $1 \le i \le k-1$, the number of ways to form a committee is $C(n,i) \cdot C(m,k-i)$

Therefore, the total number of ways is $\sum_{i=1}^{k-1} C(n,i) \cdot C(m,k-i)$

Any committee can be formed by both methods, so

$$C(n+m,k) - C(n,k) - C(m,k) = \sum_{i=1}^{k-1} C(n,i) \cdot C(m,k-i).$$

Question 6 (15 marks)

Let q to be number are multiples of 2 or 3 or 11.

$\therefore \bar{q}$ to be number are not multiples of 2, 3 nor 11.

$q = 1000 - \bar{q}$

There are 500 even integers (divisible by 2) from 1 to 1000. $[1000 \div 2 = 500]$

There are $\lfloor 1000 \div 3 \rfloor = 333$ integers is divisible by 3.

There are $\lfloor 1000 \div 11 \rfloor = 90$ integers is divisible by 11

There are $\lfloor 1000 \div 6 \rfloor = 166$ integers is both divisible by 2 and 3.

There are $\lfloor 1000 \div 22 \rfloor = 45$ integers is both divisible by 2 and 11

There are $\lfloor 1000 \div 33 \rfloor = 30$ integers is both divisible by 3 and 11.

There are $\lfloor 1000 \div 66 \rfloor = 15$ integers is divisible by 2, 3, 11.

So there are $\bar{q} = 1000 - 333 - 500 - 90 + 166 + 45 + 30 + 15 = 333$ (numbers)

Question 7 (10 marks)

(a). this problem can be treat as
assiging 15 balls to 3 bucket X, Y, Z.

∴ the total ways is $C(15+2, 2) = 136$

(b) step 1: we assign 3 balls in X, 4 balls in Y and 5 balls in Z.
step 2: we can assign the remaining $15-3-4-5=3$ balls as we want.
∴ the total ways is $C(3+2, 2) = 3$.

Question 8 (15 marks)

(a). The experiment is randomly select 3 numbers from 1 to 20.
the lottery commission has fixed its 7 numbers or 13 numbers in advance.

(b). the sample space is contain all the experiment.
which is $C(20, 7)$.

(c). $$P(\text{win}) = \frac{C(20,3)\cdot C(20,4) + C(20,3)\cdot C(20,13)}{C(20,7)}.$$

Way 1: select 3 numbers from 20 numbers, which has $C(20,3)$ ways.
then select 4 numbers not like these there numbers.

Way 2: select 3 numbers. from 20.
the select 13 number do not like these numbers

Question 9 (10 marks)

(a) Because filp a coin only can get two outcomes: head and tail.

since $P(H)$ is $\frac{1}{4}$, $P(T) = 1 - P(H) = \frac{3}{4}$.

(b) Because H and T are independent events. $P(H \cap T) = P(H) \cdot P(T)$.

So P(getting two tails)

$= C_3^2 \times P(H) \cdot P(T) \cdot P(T) = 3 \times \frac{1}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{27}{64}$

(c) $P(1) = C_3^1 \times P(T) \cdot P(H) \cdot P(H) = 3 \times \frac{3}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{9}{64}$.

from (b). $P(2) = \frac{27}{64}$.

$P(3) = P(T) \cdot P(T) \cdot P(T) = \frac{27}{64}$

$P(0) = P(H) P(H) \cdot P(H) = \frac{1}{64}$

$E(T) = P(1) \cdot 1 + P(2) \cdot 2 + P(3) \cdot 3 + P(0) \cdot 0$

$= \frac{9}{64} + \frac{27}{64} \times 2 + \frac{27}{64} \times 3 = \frac{144}{64} = 2.25$.

End of ASM