COMP S264F Discrete Mathematics
Tutorial 6: Functions (1) – Suggested Solution

Question 1.

(a)

	f	g	$f \circ g$
Domain	$B = \{1, 2, 3\}$	$A = \{p, q, s\}$	$A = \{p, q, s\}$
Codomain	$C = \{\alpha, \beta, \gamma\}$	$B = \{1, 2, 3\}$	$C = \{\alpha, \beta, \gamma\}$
Range	$\{\alpha, \beta, \gamma\}$	$\{1, 3\}$	$\{\alpha, \beta\}$

(b) γ

(c) β

(d) p, s

Question 2.

(a) Let $y = f(x) = 4x + 2$

$$\begin{aligned} y &= 4x + 2 \\ y - 2 &= 4x \\ x &= \frac{y-2}{4} \\ f^{-1}(y) &= \frac{y-2}{4} \end{aligned}$$

(b) Let $y = f(x) = 3 + \frac{1}{x}$

$$\begin{aligned} y &= 3 + \frac{1}{x} \\ yx &= 3x + 1 \\ x(y-3) &= 1 \\ x &= \frac{1}{y-3} \\ f^{-1}(y) &= \frac{1}{y-3} \end{aligned}$$

Question 3.

(a)
$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(\frac{x}{2}) \\ &= 3(\frac{x}{2}) + 1 \\ &= \frac{3x+2}{2} \end{aligned}$$

(b)
$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g(3x+1) \\ &= \frac{3x+1}{2} \end{aligned}$$

Question 4.

(a) **Injective.** Yes.

Let $x, y \in \mathbb{Z}$.

$$f(x) = f(y) \implies -x = -y \implies x = y$$

Surjective. Yes.

$$\text{For any } b \in \mathbb{Z},\ b = f(a) \implies b = -a \implies a = -b \implies a \in \mathbb{Z}$$

Bijective. Yes, because f is *injective* and *surjective.*

(b) **Injective.** No.

Let $x = 2$ and $y = -2$.

Then $f(x) = |2| = 2$ and $f(y) = |-2| = 2$.

Therefore, $x \neq y \implies f(x) \neq f(y)$ is false.

Surjective. No.

Let $b = -2$.

$b = f(a) \implies |a| = -2$

However, $|a|$ must be positive.

Therefore, there does not exist any $a \in \mathbb{R}$ such that $f(a) = b$.

Bijective. No, because f is neither *injective* nor *surjective.*

(c) **Injective.** Yes.

Let $x, y \in \mathbb{R}$.

$$f(x) = f(y) \implies 6x - 9 = 6y - 9 \implies x = y$$

Surjective. No.

$$\text{For any } b \in \mathbb{Z},\ b = f(a) \implies b = 6a - 9 \implies a = \frac{b+9}{6}$$

However, when $b = 0$, $a = 1.5 \notin \mathbb{Z}$.

Bijective. No, because f is not *surjective.*

(d) **Injective.** Yes.

Let $x, y \in \mathbb{R}$.

$$f(x) = f(y) \implies 2x^3 - 4 = 2y^3 - 4 \implies x^3 = y^3 \implies x = y$$

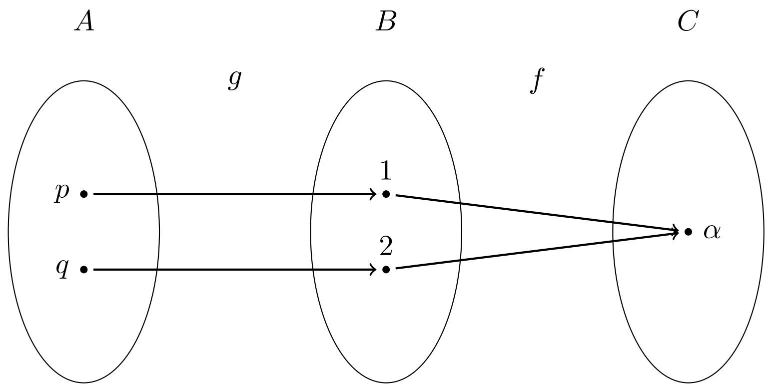
Surjective. Yes.

$$\text{For any } b \in \mathbb{Z},\ b = f(a) \implies b = 2a^3 - 4 \implies a^3 = \frac{b+4}{2} \implies a = \sqrt[3]{\frac{b+4}{2}} \implies a \in \mathbb{R}$$

Bijective. Yes, because f is *injective* and *surjective.*

Question 5.

(a) False. Consider the following arrow diagram for the functions f and g.



It is obvious that g is injective.
However, we can find a counterexample $(f \circ g)(p) = (f \circ g)(q) = \alpha$ to show that $f \circ g$ is not injective.

(b) True. Let $x \in A$ and $y \in A$ such that
$$g(x) = g(y) \implies f(g(x)) = f(g(y)) \implies (f \circ g)(x) = (f \circ g)(y)$$
Since $f \circ g$ is injective, $x = y$.
Hence, g is also injective.

(c) True. Assume $c \in C$.
Since f is surjective, there exists $b \in B$ such that $f(b) = c$.
Since g is surjective, there exists $a \in A$ such that $g(a) = b$.
Then $f(g(a)) = f(b) = c$. Therefore, $f \circ g$ is surjective.