COMP S264F Discrete Mathematics
Online Examination

22 January 2021 (Fri)
14:00 - 16:00

This open-book exam paper contains **NINE** questions. Please answer **ALL** of them.

You are required to hand-write your answers on blank papers, take photos on them using your smartphone, and convert them to a PDF file for submission to the submission page in the OLE. Note that computer-typed answers will not be accepted.

Please write down your name and student ID on the first page, and submit a backup copy to Keith's OUHK Google mail account lklee@study.ouhk.edu.hk with email title "COMPS264F exam answers".

Question 1 (10 marks).

(a) Simplify the proposition $\neg((p \rightarrow q) \vee q)$. [5]

(b) Rewrite the following proposition such that negations appear only within predicates:
(For example, $\neg\ \exists x\ \exists y\ P(x, y)$ should be rewritten as $\forall x\ \forall y\ \neg P(x, y)$.) [5]

$$\neg\ \exists x\ (\ (\ (\forall y\ P(x, y)) \rightarrow Q(x)\ )\ \vee\ Q(x)\ )$$

Question 2 (10 marks). Use mathematical induction to prove that for any positive integer n, $4^{n+1} + 5^{2n-1}$ is divisible by 21.

Question 3 (10 marks). Let A, B, C be sets. Consider the set

$$\overline{(\overline{A} - B)} \cap (A \cup C)$$

(a) Simplify the given set. [5]

(b) Draw the Venn diagram and shade the given set. [5]

Question 4 (10 marks). Consider the following function.

$$f : \mathbb{R} \rightarrow \mathbb{R} \text{ such that } f(x) = 4x + 3$$

(a) Determine whether the function f is one-to-one. Justify your answer. [4]

(b) Determine whether the function f is onto. Justify your answer. [4]

(c) Do the two sets $\mathbb{R}$ and $\{4x + 3 \mid x \in \mathbb{R}\}$ have the same cardinality? Justify your answer. [2]

Question 5 (10 marks). Give a combinatorial argument to prove that

$$C(n + m, k) - C(n, k) - C(m, k) = \sum_{i=1}^{k-1} C(n, i) \cdot C(m, k - i)\ ,$$

where n, m, k are positive integers, and $k < n, k < m$.
You may consider the scenario that you need to form a committee from n boys and m girls with some constraints. Note that a non-combinatorial proof will receive 0 marks.

Question 6 (15 marks). Find the number of integers between 1 and 1000 (inclusive) that are multiples of 2 or 3 or 11.

Question 7 (10 marks). Find the number of non-negative integer solutions to the equation $x + y + z = 15$ if

(a) there are no additional restrictions on x, y, z. [4]

(b) $x \geq 3, y \geq 4, z \geq 5$. [6]

Question 8 (15 marks). Consider the following lottery game. One can buy a ticket with any 3 distinct numbers chosen from 1 to 20. On the lottery day, the lottery commission will first randomly choose 7 numbers from 1 to 20, denoted as the set A. After that, the lottery commission will randomly choose 7 numbers from the remaining 13 numbers, denoted as the set B.

To win the grand prize, the 3 numbers in the ticket must all appear in set A, or all appear in set B.
(In the followings, you may express your answer in terms of $C(n, r)$ or $P(n, r)$; exact value is not required.)

(a) What is an outcome of the lottery game? Justify your answer. [4]

(b) What is the size of the sample space? Justify your answer. [5]

(c) Consider a ticket with $\{1, 2, 3\}$ chosen, what is the probability for it to be a winning ticket? Justify your answer. [6]

Question 9 (10 marks). Given a biased coin for which the probability of getting a head (H) is $\frac{1}{4}$, suppose the coin is flipped three times.
(In the following, please express your answer as a fraction of integers.)

(a) What is the probability of getting a tail (T) in a single coin flip? Justify your answer. [1]

(b) What is the probability of getting two tails in the three coin flips? Justify your answer. [3]

(c) What is the expected number of tails in the three coin flips? Justify your answer. [6]

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