COMP S264F Discrete Mathematics
Tutorial 9: Combinatorics – Suggested Solution

Question 1.

(a) The number of ways to order any 6 distinct dishes from the menu with 8 dishes is $C(8,6) = 28$.

(b) We can first order 2 chicken dishes, then 2 seafood dishes and finally 1 vegetarian dish:

- The number of ways to order 2 chicken dishes is $C(3,2) = 3$.
- The number of ways to order 2 seafood dishes is $C(3,2) = 3$.
- The number of ways to order 1 vegetarian dish is $C(2,1) = 2$.

By product rule, the number of ways to order the required dishes is $3 \times 3 \times 2 = 18$.

(c) *Solution 1:*
There are three possible cases:

- The number of ways to order 2 chicken dishes, 1 seafood dish and 1 vegetarian dish is $C(3,2) \times C(3,1) \times C(2,1) = 3 \times 3 \times 2 = 18$.
- The number of ways to order 1 chicken dish, 2 seafood dishes and 1 vegetarian dish is $C(3,1) \times C(3,2) \times C(2,1) = 3 \times 3 \times 2 = 18$.
- The number of ways to order 1 chicken dish, 1 seafood dish and 2 vegetarian dishes is $C(3,1) \times C(3,1) \times C(2,2) = 3 \times 3 \times 1 = 9$.

By the sum rule, the number of ways to order the required dishes is $18 + 18 + 9 = 45$.

Solution 2:
Let A, B, C be the sets of 4-dish orders without any chicken, seafood, vegetarian dish, respectively.
The number of ways to order the required dishes is $|\overline{A} \cap \overline{B} \cap \overline{C}| = |\overline{A \cup B \cup C}| = |U| - |A \cup B \cup C|$.
By the principle of inclusion-exclusion,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \ .$$

We can compute each of the above terms, as follows:

- $|U| = C(3+3+2, 4) = C(8,4) = 70$.
- $|A| = C(3+2, 4) = C(5,4) = 5$.
- $|B| = C(3+2, 4) = C(5,4) = 5$.
- $|C| = C(3+3, 4) = C(6,4) = 15$.
- $A \cap B$ is the set of 4-dish orders without any chicken and seafood dish. As it is impossible to order 4 distinct dishes from 2 vegetarian dishes, $|A \cap B| = 0$.
  Similarly, $|A \cap C| = |B \cap C| = |A \cap B \cap C| = 0$.

Thus, the number of ways to order the required dishes is $|U| - |A \cup B \cup C| = |U| - |A| - |B| - |C| = 70 - 5 - 5 - 15 = 45$.

Question 2.

(a) In the 6-letter string, there are two occurrences of “E”. Hence, there are $\dfrac{6!}{2!} = 360$ ways to order the letters.

(b) As the two E’s must be consecutive, we consider “EE” as one unit and the remaining four letters as four distinct units. Hence, there are 5 distinct units to permute and thus $5! = 120$ ways to order the letters.

(c) There are 7 possible lengths of the formed string:

- Length 0: The number of ways to form an empty string is 1.
- Length 1: There are 5 distinct letters (E, T, H, A, N), so the number of ways to form the string is $C(5,1) = 5$.
- Length 2: There are two cases:
  - The string contains at most one E. We can first select 2 letters from the 5 distinct letters and then permute the 2 letters. The number of ways to form the string is $C(5,2) \times 2! = 10 \times 2 = 20$.
  - The string is simply two E's. The number of ways to form the string is 1.
- Length 3: There are two cases:
  - The string contains at most one E. We can first select 3 letters from the 5 distinct letters and then permute the 3 letters. Thus, the number of ways to form the string is $C(5,3) \times 3! = 10 \times 6 = 60$.
  - The string contains two E's. We can select the remaining letter from the other 4 distinct letters (T, H, A, N) and then permute the 3 letters (where 2 of them are E's).
    Thus, the number of ways to form the string is $C(4,1) \times \dfrac{3!}{2!} = 4 \times 3 = 12$.
- Length 4: There are two cases:
  - The string contains at most one E. We can first select 4 letters from the 5 distinct letters and then permute the 4 letters. Thus, the number of ways to form the string is $C(5,4) \times 4! = 5 \times 24 = 120$.
  - The string contains two E's. We can select the remaining 2 letters from the other 4 distinct letters (T, H, A, N) and then permute the 4 letters (where 2 of them are E's).
    Thus, the number of ways to form the string is $C(4,2) \times \dfrac{4!}{2!} = 6 \times 12 = 72$.
- Length 5: There are two cases:
  - The string contains at most one E. We can first select 5 letters from the 5 distinct letters and then permute the 5 letters. Thus, the number of ways to form the string is $C(5,5) \times 5! = 1 \times 120 = 120$.
  - The string contains two E's. We can select the remaining 3 letters from the other 4 distinct letters (T, H, A, N) and then permute the 5 letters (where 2 of them are E's).
    Thus, the number of ways to form the string is $C(4,3) \times \dfrac{5!}{2!} = 4 \times 60 = 240$.
- Length 6: By (a), the number of ways to form the string is $\dfrac{6!}{2!} = 360$.

By sum rule, the total number of ways to form the string using some or all of the letters is $1 + 5 + (20 + 1) + (60 + 12) + (120 + 72) + (120 + 240) + 360 = 1011$.

(d) The formed string must be in the form $x_1 \text{N} x_2 \text{A} x_3$ where x_1, x_2, x_3 are three strings and they are formed together by the remaining 4 letters E, T, H, E.
Therefore, the number of ways to form the string is equal to the number of ways to form x_1, x_2, x_3 from the 4 letters E, T, H, E, which can be done in two steps:

1. Assign the 4 letters $*$ to the three strings x_1, x_2, x_3.
   E.g., $**\,|\,*\,|*$, i.e., x_1 is a 2-letter string, x_2 and x_3 are 1-letter strings.
   The number of ways to do Step 1 is $C(4+3-1, 4) = C(6,4) = 15$.
2. Permute the 4 letters, where 2 of them are E's.
   E.g., EE|T|H, ET|H|E, EH|T|E, ...
   I.e., EENTAH, ETNHAE, EHNTAE, ...
   The number of ways to do Step 2 is $\dfrac{4!}{2!} = 12$.

By product rule, the number of ways to form the string is $C(4+3-1, 4) \cdot \dfrac{4!}{2!} = 15 \cdot 12 = 180$.

Question 3.

(a) This is equivalent to selecting a 10-combination with repetition from 3 balls (red, blue, green). By setting $n = 10, r = 3$, there are $C(n + r - 1, r) = C(10 + 3 - 1, 10) = 66$ ways of selection.

(b) Let A be the set of ways to select 10 balls with at least one red ball.
Then the number of ways to select 10 balls without any red ball is $|\overline{A}|$.
By setting $n = 10$ and $r = 2$, $|\overline{A}| = C(n + r - 1, r) = C(10 + 2 - 1, 10) = 11$.
By (a), $|U| = 66$.
Therefore, $|A| = |U| - |\overline{A}| = 66 - 11 = 55$.

(c) If exactly one blue ball must be selected, than the remaining 9 balls must be red or green.
There are $C(9 + 2 - 1, 9) = 10$ ways of selection.

(d) There are two possible cases:

- No green ball is selected: The number of selections is $C(10 + 2 - 1, 10) = 11$.
- One green ball is selected: The number of selections is $C(9 + 2 - 1, 9) = 10$.

By sum rule, there are $11 + 10 = 21$ ways of selection.

(e) Let (r, g, b) denote the selection with r red balls, g green balls and b blue balls.
As $g = 2r$, there are only 4 possible selections with $r = 0, 1, 2, 3$, i.e., $(0, 0, 10), (1, 2, 7), (2, 4, 4), (3, 6, 1)$.

Question 4.

(a) The problem is equivalent to assigning 18 balls to 3 buckets x, y, z.
Thus, the number of solutions is $C(18 + 3 - 1, 18) = 190$.

(b) The problem is equivalent to assigning 18 balls to 3 buckets x, y, z, where x has at least 3 balls, y has at least 2 balls, and z has at least 1 ball. Therefore, this is equivalent to first assigning the $3 + 2 + 1 = 6$ balls to x, y, z and then assigning $18 - 6 = 12$ balls to the 3 buckets.
By product rule, the number of solutions is $1 \times C(12 + 3 - 1, 12) = 91$.

(c) Let A, B, C be the set of solutions where $x \geq 7$, $y \geq 8$, and $z \geq 9$, respectively.
Then, the number of solutions with $x < 7, y < 8, z < 9$ is

$$|\overline{A} \cap \overline{B} \cap \overline{C}| = |\overline{A \cup B \cup C}| = |U| - |A \cup B \cup C| \ .$$

By the principle of inclusion-exclusion,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \ .$$

We can compute each of the above terms, as follows:

- By (a), $|U| = 190$.
- We assign 7 balls to x and then assign the remaining $18 - 7 = 11$ balls to x, y, z. Thus, $|A| = C(11 + 3 - 1, 11) = 78$.
- We assign 8 balls to y and then assign the remaining $18 - 8 = 10$ balls to x, y, z. Thus, $|B| = C(10 + 3 - 1, 10) = 66$.
- We assign 9 balls to z and then assign the remaining $18 - 9 = 9$ balls to x, y, z. Thus, $|C| = C(9 + 3 - 1, 9) = 55$.
- We assign 7 balls to x and 8 balls to y, and then assign the remaining $18 - 7 - 8 = 3$ balls to x, y, z. Thus, $|A \cap B| = C(3 + 3 - 1, 3) = 10$.
- We assign 7 balls to x and 9 balls to z, and then assign the remaining $18 - 7 - 9 = 2$ balls to x, y, z. Thus, $|A \cap C| = C(2 + 3 - 1, 2) = 6$.

- We assign 8 balls to y and 9 balls to z, and then assign the remaining $18 - 8 - 9 = 1$ ball to x, y, z. Thus, $|B \cap C| = C(1 + 3 - 1, 1) = 3$.
- As $7+8+9 = 24 > 18$, it is impossible to have a solution with $x \geq 7, y \geq 8, z \geq 9$, i.e., $|A \cap B \cap C| = 0$.

Therefore, the number of solutions with $x < 7, y < 8, z < 9$ is

$$190 - (78 + 66 + 55 - 10 - 6 - 3 + 0) = 10 \ .$$

Question 5. Consider the following two ways to form a binary string using r 0's and $n + 1$ 1's.

Method 1:
Step 1: Select r of the $n + 1 + r$ positions of the binary string to be 0's.
Step 2: Assign 1's to the remaining positions.
Thus, the number of ways to form the binary string is $C(n + r + 1, r)$.

Method 2: We design Method 2 according to the left-hand side of the identity.
Let k be an integer such that $0 \leq k \leq r$.
Step 1: Assign k 0's and n 1's to the first $n + k$ positions of the binary string.
Step 2: Assign the remaining 1 to position $n + k + 1$.
Step 3: Assign the remaining $(r - k)$ 0's to positions $n + k + 2$ to $n + r + 1$.
Note that position $n + k + 1$ contains the last 1 in the binary string.
For a particular k, the number of ways to form the binary string is $C(n + k, k)$.
Therefore, the total number of ways to form the binary string is

$$\sum_{k=0}^{r} C(n + k, k) \ .$$

Any binary string can be formed by both methods, so

$$\sum_{k=0}^{r} C(n + k, k) = C(n + r + 1, r) \ .$$