COMP S265F Design and Analysis of Algorithms
Lab 12: Finite Automata and Regular Expressions

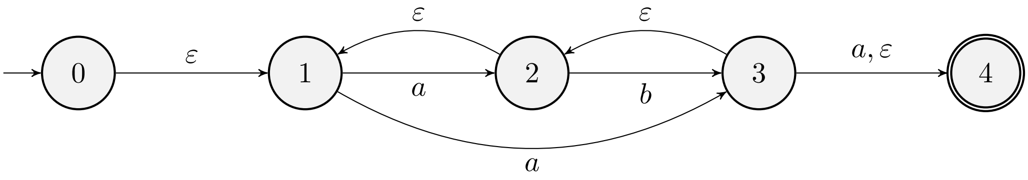
1 From NFA with ε moves to DFA

Given a NFA M_ε with ε moves, we can construct a DFA M_D using the subset construction technique, where each state of the DFA corresponds to a subset of states in the NFA. A challenge is that state transition may happen by some ε moves.

1.1 Lambda closure of a state

Given the NFA transition function f_ε, we can define the *lambda closure* $\lambda(s)$ of a state s to be the set of states which can be reached from s by zero or more ε moves (without consuming any input). More precisely, for any state s in M_ε, $s \in \lambda(s)$; if $p \in \lambda(s)$ and there is an ε move from p to q, then $q \in \lambda(s)$.

For example, the following NFA with ε moves has the transition table f_ε below:



f_ε	s	a	b	ε	$\lambda(s)$
start	0	$\emptyset$	$\emptyset$	$\{1\}$	
	1	$\{2,3\}$	$\emptyset$	$\emptyset$	
	2	$\emptyset$	$\{3\}$	$\{1\}$	
	3	$\{4\}$	$\emptyset$	$\{2,4\}$	
final	4	$\emptyset$	$\emptyset$	$\emptyset$	

Then, we can fill in the lambda closures of states $0, 1, 2, 3, 4$, as follows:

1. At a state s without any ε moves (i.e., $f_\varepsilon(s, \varepsilon) = \emptyset$), we can only stay there, so $\lambda(s) = \{s\}$.
   In the example, $\lambda(1) = \{1\},\ \lambda(4) = \{4\}$.
2. At a state s' with ε moves to a state s in the previous step, $\lambda(s')$ includes s' and the lambda closure $\lambda(s)$.
   In the example, $\lambda(0) = \{0\} \cup \lambda(1) = \{0, 1\},\quad \lambda(2) = \{2\} \cup \lambda(1) = \{1, 2\}$.
3. The other steps continue similarly.
   In the example, $\lambda(3) = \{3\} \cup \lambda(2) \cup \lambda(4) = \{1, 2, 3, 4\}$.

The completed transition table f_ε with lambda closures is:

f_ε	s	a	b	ε	$\lambda(s)$
start	0	$\emptyset$	$\emptyset$	$\{1\}$	$\{0,1\}$
	1	$\{2,3\}$	$\emptyset$	$\emptyset$	$\{1\}$
	2	$\emptyset$	$\{3\}$	$\{1\}$	$\{1,2\}$
	3	$\{4\}$	$\emptyset$	$\{2,4\}$	$\{1,2,3,4\}$
final	4	$\emptyset$	$\emptyset$	$\emptyset$	$\{4\}$

We also extend the definition of lambda closure for a set of states:

$$\lambda(\{s_1, s_2, \ldots, s_n\}) = \lambda(s_1) \cup \lambda(s_2) \cup \cdots \cup \lambda(s_n)\ .$$

For example, $\lambda(\{2, 3\}) = \lambda(2) \cup \lambda(3) = \{1, 2, 3, 4\}$.

1.2 Construction steps

The construction of the transition function f_D of the DFA can be done, as follows:

Step 1. The DFA start state is $\lambda(s)$, where s is the NFA start state. It is also a DFA final state if $\lambda(s)$ contains any NFA final state.

Step 2. If $\{s_1, s_2, \ldots, s_n\}$ is a DFA state and $a \in \Sigma$, then construct a next DFA state in either way below:

- **Closure of union:**

$$f_D(\{s_1, s_2, \ldots, s_n\}, a) = \lambda(f_\varepsilon(s_1, a) \cup f_\varepsilon(s_2, a) \cup \cdots \cup f_\varepsilon(s_n, a)) \text{ , or}$$

- **Union of closure:**

$$f_D(\{s_1, s_2, \ldots, s_n\}, a) = \lambda(f_\varepsilon(s_1, a)) \cup \lambda(f_\varepsilon(s_2, a)) \cup \cdots \cup \lambda(f_\varepsilon(s_n, a)) \text{ .}$$

Step 3. If the next states in Step 2 are new DFA states, add them to the DFA table.

A new DFA state is a final state if its set contains a NFA final state.

Repeat Step 2 for these new DFA states.

Continuing on the previous example, we complete the DFA transition table f_D row by row, as follows:

- Step 1:

  start state $= \lambda(0) = \{0, 1\}$

- Step 2 on state $\{0, 1\}$:

  $$\begin{aligned} f_D(\{0,1\}, a) &= \lambda(f_\varepsilon(0, a)) \cup \lambda(f_\varepsilon(1, a)) \quad \text{(union of closure)} \\ &= \lambda(\emptyset) \cup \lambda(\{2, 3\}) \\ &= \lambda(2) \cup \lambda(3) = \{1, 2\} \cup \{1, 2, 3, 4\} = \{1, 2, 3, 4\} \end{aligned}$$

  Therefore, we have the following table:

	f_D	a	b
start	$\{0, 1\}$	$\{1, 2, 3, 4\}$	$\emptyset$

- Step 3:

  There are two new states: $\{1, 2, 3, 4\}$ and $\emptyset$.

  $\{1, 2, 3, 4\}$ is also a DFA final state, as 4 is a NFA final state.

- Step 2 on state $\{1, 2, 3, 4\}$ (and on state $\emptyset$, which is trivial):

  $$\begin{aligned} f_D(\{1,2,3,4\}, a) &= \lambda(f_\varepsilon(1, a) \cup f_\varepsilon(2, a) \cup f_\varepsilon(3, a) \cup f_\varepsilon(4, a)) \quad \text{(closure of union)} \\ &= \lambda(\emptyset \cup \{2, 3\} \cup \emptyset \cup \{4\}) = \lambda(\{2, 3, 4\}) \\ &= \lambda(2) \cup \lambda(3) \cup \lambda(4) = \{1, 2\} \cup \{1, 2, 3, 4\} \cup \{4\} = \{1, 2, 3, 4\} \end{aligned}$$

  $$\begin{aligned} f_D(\{1,2,3,4\}, b) &= \lambda(f_\varepsilon(1, b) \cup f_\varepsilon(2, b) \cup f_\varepsilon(3, b) \cup f_\varepsilon(4, b)) \quad \text{(closure of union)} \\ &= \lambda(\{3\}) \\ &= \{1, 2, 3, 4\} \end{aligned}$$

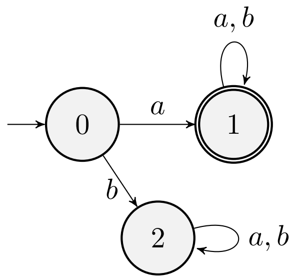
  Therefore, we have the following table:

	f_D	a	b
start	$\{0, 1\}$	$\{1, 2, 3, 4\}$	$\emptyset$
final	$\{1, 2, 3, 4\}$	$\{1, 2, 3, 4\}$	$\{1, 2, 3, 4\}$
	$\emptyset$	$\emptyset$	$\emptyset$

1.3 Renaming the states

For clarity of presentation, we can assign a unique name to each DFA state. In the previous example, the renamed transition table f_D and the resultant DFA are shown below:

f_D	s	a	b
start	0	1	2
final	1	1	1
	2	2	2



1.4 Reducing the number of states

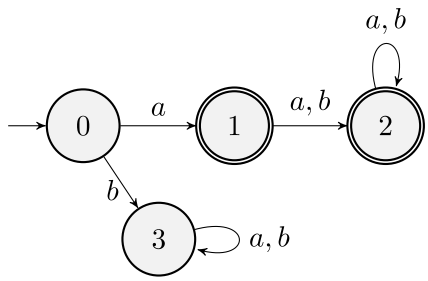
Sometimes, we can easily observe unnecessary states in our solution and can reduce the number of states:

> If two states are **both final** or **both non-final**, and their transitions for all inputs are exactly the same, these two states can be merged as one state.

Note that if one of these two states is a start state, the merged state is also a start state.

Consider the following example:

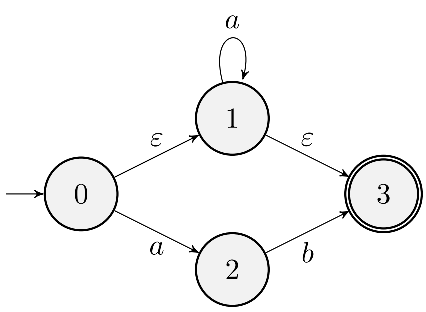
f_D	s	a	b
start	0	1	3
final	1	2	2
final	2	2	2
	3	3	3



We can merge states 1 and 2 to a merged state (named state 1), and rename state 3 to state 2. Then the resultant DFA is the same as the DFA in Section 1.3.

1.5 Example

Example 1. Transform the following NFA with ε moves to a DFA:



Solution. The transition table f_ε for the NFA with lambda closures is:

f_ε	s	a	b	ε	$\lambda(s)$
start	0	$\{2\}$	$\emptyset$	$\{1\}$	$\{0, 1, 3\}$
	1	$\{1\}$	$\emptyset$	$\{3\}$	$\{1, 3\}$
	2	$\emptyset$	$\{3\}$	$\emptyset$	$\{2\}$
final	3	$\emptyset$	$\emptyset$	$\emptyset$	$\{3\}$

Now, we construct the transition table f_D of the DFA, as follows:

- start state $= \lambda(0) = \{0,1,3\}$ (also final state as it contains NFA final state 3)
- $$\begin{aligned} f_D(\{0,1,3\},a) &= \lambda(f_\varepsilon(0,a) \cup f_\varepsilon(1,a) \cup f_\varepsilon(3,a)) \quad \text{(closure of union)} \\ &= \lambda(\{1,2\}) \\ &= \{1,2,3\} \end{aligned}$$
  $$\begin{aligned} f_D(\{0,1,3\},b) &= \lambda(f_\varepsilon(0,b) \cup f_\varepsilon(1,b) \cup f_\varepsilon(3,b)) \quad \text{(closure of union)} \\ &= \lambda(\emptyset) \\ &= \emptyset \end{aligned}$$
  Therefore, we have the following table:

	f_D	a	b
start, final	$\{0,1,3\}$	$\{1,2,3\}$	$\emptyset$

- There are two new states: $\{1,2,3\}$ (final state as it contains NFA final state 3) and $\emptyset$.
- $$\begin{aligned} f_D(\{1,2,3\},a) &= \lambda(f_\varepsilon(1,a) \cup f_\varepsilon(2,a) \cup f_\varepsilon(3,a)) \quad \text{(closure of union)} \\ &= \lambda(\{1\}) \\ &= \{1,3\} \end{aligned}$$
  $$\begin{aligned} f_D(\{1,2,3\},b) &= \lambda(f_\varepsilon(1,b) \cup f_\varepsilon(2,b) \cup f_\varepsilon(3,b)) \quad \text{(closure of union)} \\ &= \lambda(\{3\}) \\ &= \{3\} \end{aligned}$$
  Therefore, we have the following table:

	f_D	a	b
start, final	$\{0,1,3\}$	$\{1,2,3\}$	$\emptyset$
final	$\{1,2,3\}$	$\{1,3\}$	$\{3\}$
	$\emptyset$	$\emptyset$	$\emptyset$

- There are two new final states (as they contain NFA final state 3): $\{1,3\}$ and $\{3\}$.
- $$\begin{aligned} f_D(\{1,3\},a) &= \lambda(f_\varepsilon(1,a) \cup f_\varepsilon(3,a)) \quad \text{(closure of union)} \\ &= \lambda(\{1\}) \\ &= \{1,3\} \end{aligned}$$
  $$\begin{aligned} f_D(\{1,3\},b) &= \lambda(f_\varepsilon(1,b) \cup f_\varepsilon(3,b)) \quad \text{(closure of union)} \\ &= \lambda(\emptyset) \\ &= \emptyset \end{aligned}$$
  Therefore, we have the following table:

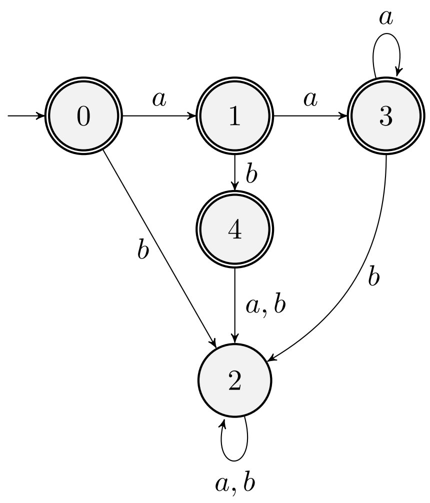
	f_D	a	b
start, final	$\{0,1,3\}$	$\{1,2,3\}$	$\emptyset$
final	$\{1,2,3\}$	$\{1,3\}$	$\{3\}$
	$\emptyset$	$\emptyset$	$\emptyset$
final	$\{1,3\}$	$\{1,3\}$	$\emptyset$
final	$\{3\}$	$\emptyset$	$\emptyset$

- We rename the five states to $0,1,2,3,4$:

	f_D	a	b
start, final	0	1	2
final	1	3	4
	2	2	2
final	3	3	2
final	4	2	2

- States 2 and 4 have the same transitions for all inputs, but state 2 is non-final while state 4 is final. Therefore, we cannot merge these two states.

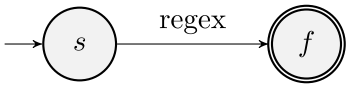
  As a result, we obtain the following DFA:



2 From regular expression to NFA with ε moves

2.1 The four rules

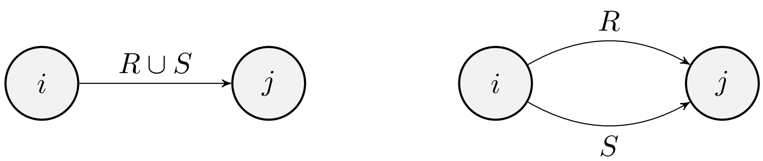
Given a regular expression, we can construct the corresponding NFA with ε moves by starting with a start state, a single final state, and an edge labeled with the given regular expression (regex):



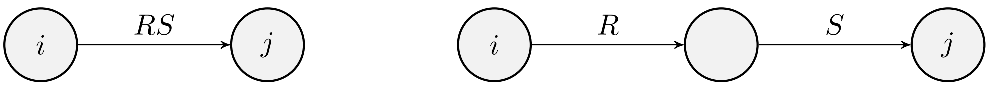
Then, we can repeatedly apply the following rules until all edges are labeled with an input symbol or ε:

Rule 1. If an edge is labeled with $\emptyset$, then remove the edge.

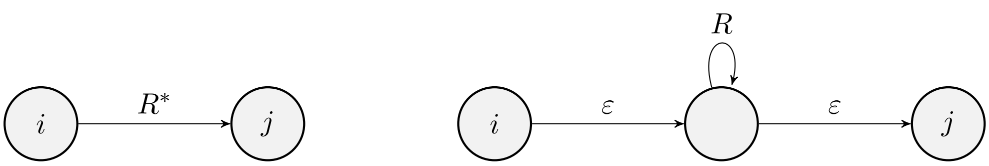
Rule 2. Transform the diagram on the left to that on the right:



Rule 3. Transform the diagram on the left to that on the right:



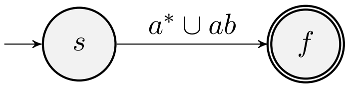
Rule 4. Transform the diagram on the left to that on the right:

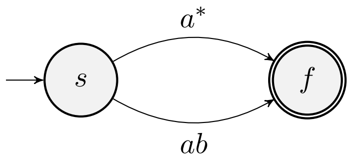


2.2 Example

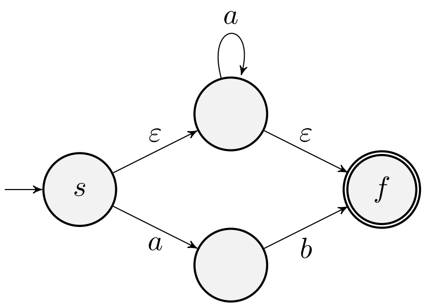
Example 2. Construct an NFA with ε moves for the regular expression $a^* \cup ab$.

Solution. We start with the diagram on the left. Next, we can apply Rule 2 to obtain the diagram on the right:



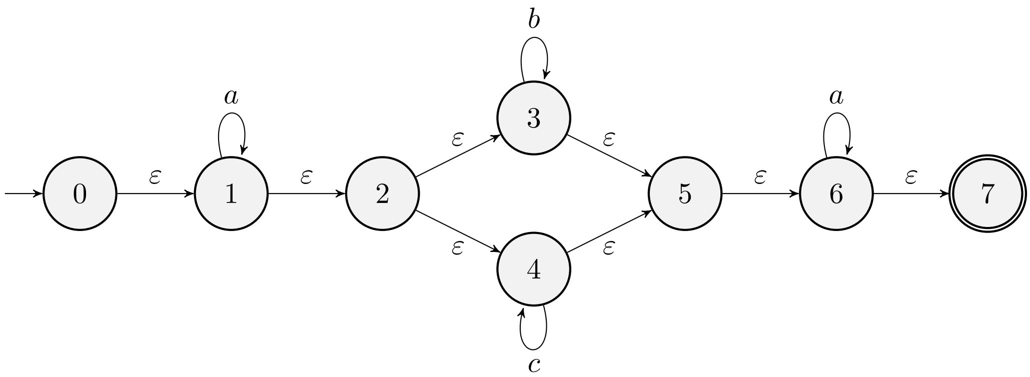


Then, we apply Rule 4 on the upper edge, and Rule 3 on the lower edge to obtain the following answer:



3 Exercises

Question 1. Let $\Sigma = \{a, b, c\}$. Consider the following NFA with ε moves:



(i) Write down the transition table of the above NFA including the lambda closure of the states.

(ii) Transform the above NFA to a DFA. Write down the transition table of the DFA.

(iii) Reduce the number of states of the DFA in (ii), and write down the new transition table.

(iv) Draw the diagram of the DFA in (iii).

Question 2. A *regular* language is a language that can be described by a regular expression. Using the pumping lemma to prove that the following languages are not regular:

(i) $L = \{a^n b c^n \mid n \in \mathbb{N}\}$

(ii) $M = \{a^m b^n \mid m, n \in \mathbb{N} \text{ and } |m - n| < 10\}$

Question 3. Let $\Sigma = \{a, b, c\}$. Construct a NFA with ε moves for the regular expression

$$a^*(b^* \cup c^*)a^*$$