COMPS264F

Discrete Mathmatics

Assignment

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Question 1 (10 marks)

(a) $\neg \exists x \forall y P(x,y)$

$= \forall x \neg \forall y P(x,y) = \forall x \exists y \neg P(x,y)$

(b) $\neg \exists y (\forall x \exists z P(x,y,z) \vee \exists x \forall z Q(x,y,z))$.

$= \forall y \neg (\forall x \exists z P(x,y,z) \vee \exists x \forall z Q(x,y,z))$.

$= \forall y (\exists x \forall z \neg P(x,y,z) \vee \forall x \exists z \neg Q(x,y,z))$.

Question 2 (15 marks).

(a) $N = \{\emptyset, \{\emptyset,\{\emptyset\}\}, \{\emptyset,\{\emptyset,\{\emptyset\}\}\}, \{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset\}\}\}\}, \{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset,\{\emptyset\}\}\}\}\}\}$

(b) False, because N contains $\emptyset$, which cardinality is 0, that means contains no element.

(c) Suppose N is a finite set, let's get this expression:

"if N contains the set X, then N contains the set $X \cup \{X\}$".

Since there are limited n elements $\{a_1, a_2, \dots, a_n\} = N$. we can consider that the $\{a_i, \{a_i\}\}$ must in N. which contradicts that N is a infinite set. because if we gave the cardinality n of N, there will always be No_{n+1} set there.

(d). Because N is an infinite set, which means the cardinality is very big, let's make it as $\aleph_0$. but any element in N, except $\emptyset$, only contains two elements, which means the cardinality is always 2.

So. $N \notin N$, because the cardinary of elements in N: $2 <$ the cardinaty of N.

Question 3 (15 marks)

prove :

$S = \{ s_1, s_2, \cdots, s_n, \cdots \}$ if S and T have the same cardinality.

$T = \{ t_1, t_2, \cdots, t_n, \cdots \}$ Which means we can define a bijection $t_n = g(s_n)$ between set S and T.

Which means for t_n, elements in T, there will be an only corresponding s_n.

$\because$ t_n is a mapping between S and S.

Let's consider a mapping t between S and S, which mapping all elements in S into a specific element s_i in S. if the cardinality of S is $\aleph_1$, then the corresponding t is also $\aleph_1$, because for every element in S, we can define this mapping t.

but there are more elements of T. for example, we can define another mapping t_j, which mapping $(\aleph_1 - 1)$ value into a specific element in S (s_j). and another value s_k is mapped into another element in S. (like s_p). and so on ....

So, In conclusion. we can always find more mapping t_n. between set S and S. than the cardinality of S. which prove that S and T do not have the same cardinality.

Question 4 (10 marks)

Let's build 12 basketball players $\{ a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ a_8\ a_9\ a_{10}\ a_{11}\ a_{12}. \}$ and divide them into 10 sets. $\{ A_0\ A_1\ A_2\ A_3\ A_4\ A_5\ A_6\ A_7\ A_8\ A_9. \}$.
From $A_0 \sim A_9$. each set contains three consecutive players.

we can know that. the total uniform numbers of 12 players is $1+2+3+\cdots+12 = 78$.
So the average uniform number each player is equal to : $\frac{78}{12} = 6.5$.

In $A_0 \sim A_9$. $\{a_3, a_4, \cdots, a_{10}\}$ were count 3 times each of them: So there are $3 \times 8 \times 6.5 = 156$ uniform numbers
$\{a_2, a_{11}\}$ were count twice: So there are $2 \times 2 \times 6.5 = 26$ uniform numbers.
$\{a_1, a_{12}\}$ were only count once. So there are $1 \times 2 \times 6.5 = 13$ uniform numbers.

(Question4 Cont'd)

So, there are total $156 + 26 + 13 = 195$ uniform numbers.
which need to be placed into $size(A_0, A_1, \cdots, A_9) = 10$ sets.

Thus, according to the generalized Pigeonhole principle:

there is at least one set A_t. containing at least $\lceil \frac{195}{10} \rceil = \lceil 19.5 \rceil = 20$ uniform numbers.

Since every A_t is the set of three consecutive players.
which ensure our proof: "Some three consecutive players have the sum of their numbers at least 20".

Question 5 (10 marks).

Consider the following two ways to arrange the ways in a n-person wine tour:

Method 1:

Step 1: we choose a driver from n person. which has n different ways.

Step 2: For the remaining $(n-1)$ person. each of them has 4 different choices. (3 alcoholic, 1 non-al)
which has 4^{n-1} different ways.

Thus, the ways to arrange a n-person wine tour is equal to: $4^{n-1} \times n = n \cdot 4^{n-1}$.

Method 2:

Let k be an integer such that $0 \le k \le n$.

Step 1: we choose k person who will have the alcoholic menu.
There are. $C(n,k)$ ways to choose and each of the k people have 3 different menu.
$\therefore$ there are $C(n,k) \cdot 3^k$ ways

Step 2: for the remaining $(n-k)$ person, each of them only have one choice — no-alcoholic menu.
So there are $(n-k) \cdot 1$. ways.

For a particular k, the number of ways to arrange this tour is $C(n,k) \cdot 3^k (n-k)$.
Therefore. the total number of ways to arrange the tour is $\sum_{k=0}^{n} C(n,k) \cdot 3^k \cdot (n-k)$.

The wine tour can be arranged by both methods. So:

$$n \cdot 4^{n-1} = \sum_{k=0}^{n} C(n,k) \cdot 3^k (n-k).$$

Question 6 (10 marks)

Base case. when $n=2$:

$$P(E_1 \cup E_2) = P(E_1)+P(E_2)-P(E_1 \cap E_2) \leq P(E_1)+P(E_2).$$

Induction step.

Assume that

$$P(E_1 \cup E_2 \cup \cdots \cup E_k) \leq \sum_{i=1}^{k} P(E_i).$$ for some positive integer k.

when $n=k+1$.

$$P(E_1 \cup E_2 \cup \cdots \cup E_k \cup E_{k+1}) = \sum_{i=1}^{k} P(E_i) - P(E_1 \cap E_2 \cap \cdots \cap E_k \cap E_{k+1}) + P(E_{k+1})$$

$$\leq \sum_{i=1}^{k+1} P(E_i).$$ (when $P(E_1 \cap E_2 \cap \cdots \cap E_k \cap E_{k+1})=0$, equal).

By the principle of mathematical induction.

for all events $E_1, E_2, \cdots, E_n$. $P(E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_n) \leq \sum_{i=1}^{n} P(E_i)$

Question 7 (15 marks)

(a) when $r=5$. Let's guess another envelop :

$P(\text{grater than } r) = \frac{5}{11-1} = \frac{1}{2} = P(\text{less than } r)$. this is the example of 50% chance winning.

when $r<5$. Let's guess another envelop :

$P(\text{win grater than } r) > \frac{6}{11-1} = \frac{3}{5}$ (when $r=4$).

this is the example of more than 50% chance of winning.

(b). (i) if the X is between the two envelope numbers.
the chance of wining is 100%.

(ii). $P>0$, means. the case $a<x<b$ can occur. actually. it can occur.
for example: if $a=4$ and $b=5$. and $X=4.5$. etc.

(iii) ① if X is between the two envelop numbers. $P_{win}=100\%$.

② if X is less than the smallest envelop number. $P_{win}=0\%$.

③ if X is bigger than the largest envelop number. $P_{win}=50\%$. ↓ next page will discuss detail

(Question 7 Cont'd)

(b)(iii)

① $P\left({a<x<b \atop b<x<a}\right) = 1 - P\left({a<x<b \atop b<x<a}\right) - P\left({a<b<x \atop b<a<x}\right) = 50\% \quad \therefore P(\text{win})_{\text{mid}} = 50\% \times 100\% = 50\%.$

② $P\left({x<a<b \atop x<b<a}\right) = \dfrac{9+8+7+6+5+4+3+2+1}{2\times 9\times 10} = \dfrac{45}{180} = \dfrac{1}{4} = 25\% \quad \therefore P(\text{win})_{\text{Less}} = 25\% \times 0\% = 0\%.$

③ $P\left({a<b<x \atop b<a<x}\right) = \dfrac{9+8+7+6+5+4+3+2+1}{2\times 9\times 10} = 25\%. \quad \therefore P(\text{win})_{\text{bigger}} = 25\% \times 50\% = 12.5\%.$

$\therefore P(\text{win}) = P(\text{win})_{\text{mid}} + P(\text{win})_{\text{Less}} + P(\text{win})_{\text{bigger}} = 50\% + 0\% + 12.5\% = 62.5\%$

This shows that this strategy has a better than 50% chance of winning.

Question 8 (15 marks)

Let the value of dice is D

(a) $E(X) = 2\cdot P(X=2) + 4\cdot P(X=4) + P(X=6)\cdot 6 + P(X=8)\cdot 8 + P(X=10)\cdot 10 + P(X=12)\cdot 12.$

$= 2\cdot P(D=1, X=2) + 4\cdot P(D=2, X=4) + P(D=3, X=6)\cdot 6.$

$= \frac{1}{6}\times 2 + \frac{1}{6}\times 4 + \frac{1}{6}\times 6 + \frac{1}{6}\times 8 + \frac{1}{6}\times 10 + \frac{1}{6}\times 12 = \frac{1}{6}\times 42 = 7.$

(b). $E(Y) = P(D\text{odd}, Y=1)\cdot 1 + P(D\text{even}, Y=3)\times 3$

$= \frac{1}{2}\times 1 + \frac{1}{2}\times 3 = 2.$

(c) $Z(1) = X(1) + Y(1) = 2+3 = 5$

$Z(2) = X(2) + Y(2) = 4+1 = 5$

$Z(3) = X(3) + Y(3) = 6+3 = 9.$

$Z(4) = X(4) + Y(4) = 8+1 = 9.$

$Z(5) = X(5) + Y(5) = 10+3 = 13.$

$Z(6) = X(6) + Y(6) = 12+1 = 13.$

$\therefore E(Z) = E(X) + E(Y) = (5+5+9+9+13+13)\times\frac{1}{6} = 54\times\frac{1}{6} = 9.$