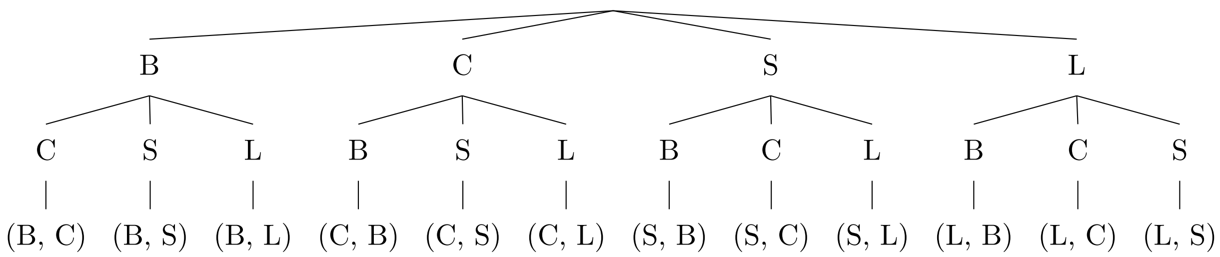
COMP S264F Discrete Mathematics
Tutorial 8: Basics of Counting – Suggested Solution

Question 1. We use B, C, S, L to represent Brown, Cony, Sally, and Leonard, respectively. The following decision tree shows all the possible selections in the format (President, Secretary):



(a) As the president must be Brown, the secretary will be one of the remaining 3 members, i.e., there are 3 selections in which Brown is president: (B, C), (B, S), (B, L).

(b) As Leonard is excluded, the president is one of the other 3 members, and then the secretary is one of the remaining 2 members. There are $3 \times 2 = 6$ selections in which Leonard is not an official.

(c) By similar argument in (a), there will be 3 selections in which Cony is the secretary. By similar argument in (b), there are 6 selections in which Cony is not an official.
As these two selections are disjoint, by sum rule, there are $3 + 6 = 9$ selections to fulfill the requirement.

(d) There will be 3 selections that Sally is the president and another 3 selections that she is the secretary. However, we need to exclude the case that Cony is the president and Sally is the secretary, i.e., (C, S). By sum rule, there are $3 + 3 - 1 = 5$ selections to fulfill the requirement.

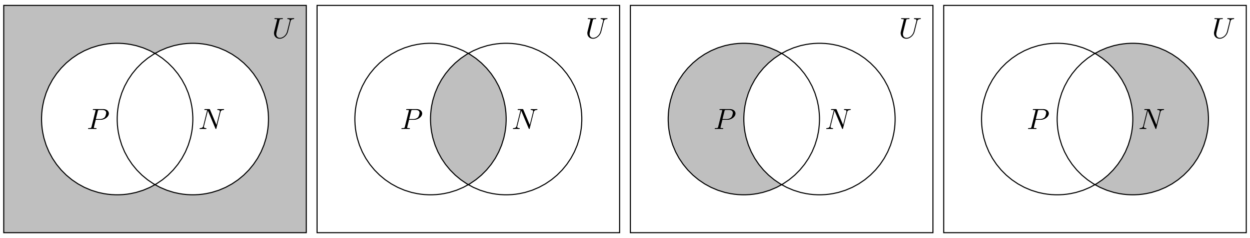
Question 2.

(a) There are 4 routes to go from home to Kwun Tong and 3 routes from Kwun Tong to OUHK.
By product rule, there will be $4 \times 3 = 12$ routes from home to OUHK via Kwun Tong.

(b) There are 4 routes from home to Kwun Tong, 3 routes from Kwun Tong to OUHK, 3 routes from OUHK to Kwun Tong, and 4 from Kwun Tong to home.
By product rule, there will be $4 \times 3 \times 3 \times 4 = 144$ routes for the round trip.

(c) Edward can travel 4 routes from home to Kwun Tong and 3 routes from Kwun Tong to OUHK.
For the return, he can only travel $3 - 1 = 2$ routes from OUHK to Kwun Tong, and $4 - 1 = 3$ routes from Kwun Tong to home.
By product rule, there will be $4 \times 3 \times 2 \times 3 = 72$ routes to fulfill the requirement.

Question 3. Let P and N denote the sets of students owning PS5 and NS, respectively, and let U denote the universal set of students. We know that $|U| = 70, |P| = 41, |N| = 26$ and $|P \cup N| = 53$. By translating the sub-questions into Venn diagrams, we need to find the cardinality of the shaded area.



(a) (b) (c) (d)

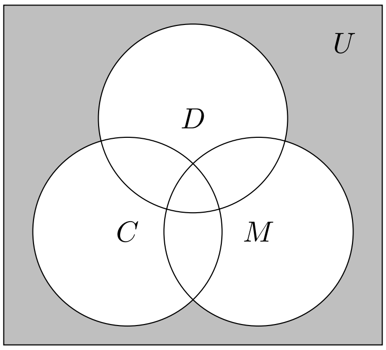
(a) $|\overline{P \cup N}| = |U| - |P \cup N| = 70 - 53 = 17$

(b) $|P \cap N| = |P| + |N| - |P \cup N| = 41 + 26 - 53 = 14$

(c) $|P - N| = |P| - |P \cap N| = 41 - 14 = 27$

(d) $|N - P| = |N| - |P \cap N| = 26 - 14 = 12$

Question 4. Let U denote the universal set of computing students. Let C, M and D denote the set of students studying cryptography, machine learning and database, respectively. We know that $|U| = 191, |C| = 65, |M| = 76, |D| = 63, |C \cap M| = 36, |C \cap D| = 20, |M \cap D| = 18$ and $|C \cap M \cap D| = 10$. By translating the question into a Venn diagram, we need to find the cardinality of the shaded area.



$$
\begin{aligned}
|\overline{C \cup M \cup D}| &= |U| - |C \cup M \cup D| \\
&= |U| - (|C| + |M| + |D| - |C \cap M| - |C \cap D| - |M \cap D| + |C \cap M \cap D|) \\
&= 191 - (65 + 76 + 63 - 36 - 20 - 18 + 10) \\
&= 191 - 140 \\
&= 51
\end{aligned}
$$

Question 5. Let U be the universal set of integers between 1 and 10000 (inclusive). Let X_k be the set of integers in U that are multiples of k. By the principle of inclusion-exclusion, we need to find

$$|X_3 \cup X_5 \cup X_{11}| = |X_3| + |X_5| + |X_{11}| - |X_3 \cap X_5| - |X_3 \cap X_{11}| - |X_5 \cap X_{11}| + |X_3 \cap X_5 \cap X_{11}| \ .$$

Let's consider X_3 which is the set of integers between 1 and 10000 (inclusive) that are multiples of 3. As a multiple of 3 occurs in every 3 consecutive integers, we have

$$|X_3| = \left\lfloor \frac{10000}{3} \right\rfloor = 3333 \ .$$

Similarly, we can show that $|X_5| = 2000$ and $|X_{11}| = 909$.

$X_3 \cap X_5$ is the set of integers in U that are multiples of both 3 and 5. As the least common multiple (LCM) of 3 and 5 is 15, $X_3 \cap X_5$ is equivalent to the set of integers in U that are multiples of 15, and thus

$$|X_3 \cap X_5| = \left\lfloor \frac{10000}{15} \right\rfloor = 666 \ .$$

Similarly, the LCM of 3 and 11 is 33 and thus $|X_3 \cap X_{11}| = 303$; the LCM of 5 and 11 is 55 and thus $|X_5 \cap X_{11}| = 181$; the LCM of 3, 5 and 11 is 165 and thus $|X_3 \cap X_5 \cap X_{11}| = 60$.

Therefore, the number of integers between 1 and 10000 (inclusive) that are multiples of 3 or 5 or 11 is

$$|X_3 \cup X_5 \cup X_{11}| = 3333 + 2000 + 909 - 666 - 303 - 181 + 60 = 5152 \ .$$

Question 6.

(a) By the pigeonhole principle, when taking 5 courses offered by 3 schools, there must be $\left\lceil \frac{5}{3} \right\rceil = 2$ courses offered by the same school.

(b) Suppose, for the sake of contradiction, that there is a school that the student does not take any course from it. The student can take at most 2 courses from each of the remaining 2 schools, which sums up to at most 4 courses, contradicting that the student will take 5 courses.

Question 7.

(a) By the pigeonhole principle, if N cards are selected from 4 suits, there must be a suit containing at least $\left\lceil \frac{N}{4} \right\rceil$ cards. The smallest integer N that makes $\left\lceil \frac{N}{4} \right\rceil = 3$ is $N = 2 \times 4 + 1 = 9$, so 9 cards suffice.

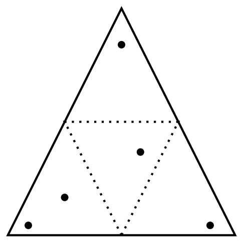
(b) In the worst case, we can select all the clubs, diamonds, and spades, $13 \times 3 = 39$ cards in total, before we select a single heart. Therefore, we need to select $39 + 3 = 42$ cards to get three hearts.

Question 8. Let a_i denote the position of the ith available item. We will show that $a_i - a_j = 9$ for some i and j. Consider two sets of numbers

$$A = \{a_1, a_2, \dots, a_{50}\} \text{ and } B = \{a_1 + 9, a_2 + 9, \dots, a_{50} + 9\} \text{ where } 1 \le a_i \le 89\ .$$

There are 100 numbers in total and their possible values are from 1 to $89 + 9 = 98$.
By the pigeonhole principle, there exists an integer $1 \le x \le 98$ such that $\left\lceil \frac{100}{98} \right\rceil = 2$ numbers equal to x.
All a_i's are distinct and thus $(a_i + 9)$'s are also distinct, so there exist i, j such that $a_i = a_j + 9 = x$, which implies $a_i - a_j = 9$, as desired.

Question 9. Partition the triangle, by connecting the midpoints of its sides, into 4 equilateral triangles with sides of length 0.5 cm, as shown in the figure. The maximum distance between any two points in any of the small triangles is 0.5 cm, i.e. any two points in a small triangle must have a distance at most 0.5 cm.



By the pigeonhole principle, if N points are in the big triangle, at least one small triangle contains at least $\left\lceil \frac{N}{4} \right\rceil$ points. The smallest integer N to make $\left\lceil \frac{N}{4} \right\rceil = 2$ is $N = 1 \times 4 + 1 = 5$, so at least 5 points should be placed in the big triangle such that there exist two points whose distance is at most 0.5 cm.

Question 10. Let x be the number of available IPv4 addresses, and let x_A, x_B and x_C denote the number of Class A, B, C addresses available, respectively. By sum rule, $x = x_A + x_B + x_C$.

- To find x_A, note that there are $2^7 - 1 = 127$ Class A netids, excluding the netid 1111111. For each netid, there are $2^{24} - 2 = 16,777,214$ hostids, excluding the hostids consisting of all 0s and all 1s. Thus, $x_A = 127 \times 16,777,214 = 2,130,706,178$.
- To find x_B, note that there are $2^{14} = 16,384$ Class B netids. For each Class B netid, there are $2^{16} - 2 = 65,534$ hostids, excluding the hostids consisting of all 0s and all 1s. Thus, $x_B = 1,073,709,056$.
- To find x_C, note that there are $2^{21} = 2,097,152$ Class C netids. For each Class C netid, there are $2^8 - 2 = 254$ hostids, excluding the hostids consisting of all 0s and all 1s. Thus, $x_C = 532,676,608$.

We conclude that the total number of IPv4 addresses available is $x = x_A + x_B + x_C = 2,130,706,178 + 1,073,709,056 + 532,676,608 = 3,737,091,842$.

Question 11. There are a total of $26+26+10+6=68$ allowable characters for the password selections. Let P be the total number of possible passwords of the system. Let P_k be the number of possible passwords of length k.
By sum rule, $P = P_8 + P_9 + P_{10} + P_{11} + P_{12}$.

(a) Since we do not have extra restriction on the choice of character, $P_k = 68^k$. Then, $P = P_8 + P_9 + P_{10} + P_{11} + P_{12} = 68^8 + 68^9 + 68^{10} + 68^{11} + 68^{12} = 9{,}920{,}671{,}339{,}261{,}325{,}541{,}376 \approx 9.9 \times 10^{21}$.

(b) To find P_k, it is easier to find the number of strings of all allowable characters, and subtract from this the number of strings with no special characters. Hence, $P_k = 68^k - 62^k$. Thus, $P = P_8+P_9+P_{10}+P_{11}+P_{12} = 6{,}641{,}514{,}961{,}387{,}068{,}437{,}760$.

(c) For the system in (a), the longest time needed $= \dfrac{9{,}920{,}671{,}339{,}261{,}325{,}541{,}376}{10^9} \approx 314{,}373$ years.

For the system in (b), the longest time needed $= \dfrac{6{,}641{,}514{,}961{,}387{,}068{,}437{,}760}{10^9} \approx 210{,}461$ years.

Question 12.

(a) By the product rule, there are $8 \times 2 \times 10 = 160$ area codes with format *NYX*. Similarly, by the product rule, there are $8 \times 8 \times 10 = 640$ office codes with format *NNX* and $10^4 = 1000$ station codes with format *XXXX*.
Consequently, applying the product rule again, under the old plan, there are $160 \times 640 \times 10{,}000 = 1{,}024{,}000{,}000$ different numbers available.

(b) Similar to (a), there are $8 \times 10 \times 10 = 800$ area codes, $8 \times 10 \times 10 = 800$ office codes and $10^4 = 1000$ station codes under the new plan.
Consequently, applying the product rule again, under the new plan, there are $800 \times 800 \times 10{,}000 = 6{,}400{,}000{,}000$ different numbers available.

(c) The number of different phone numbers in the form *NXX-XXXX* is $800 \times 1000 = 8{,}000{,}000$.
Hence, by the pigeonhole principle, among the 25 million phones, at least $\left\lceil \dfrac{25{,}000{,}000}{8{,}000{,}000} \right\rceil = 4$ of them must have identical phone numbers.
Hence, 4 area codes are required to provide $4 \times 8{,}000{,}000 = 32{,}000{,}000$ different telephone numbers, which ensures that all 10-digit numbers of the 25 million phones are different.