COMP S264F Discrete Mathematics
Tutorial 7: Functions (2)

Question 1. Determine whether f is a function. Give a counterexample if your answer is *no*.

(a) $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \sqrt{x}$

(b) $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \sqrt{x^2 + 1}$

(c) $f : \mathbb{Z} \to \mathbb{R}$ such that $f(x) = \pm\sqrt{x^2 + 1}$

(d) $f : \mathbb{Z} \to \mathbb{R}$ such that $f(x) = \dfrac{1}{x^2 - 16}$

Question 2. Consider a function $f : \mathbb{Z} \to \mathbb{Z}^+$. Give an example of f which is

(a) surjective but not injective.

(b) injective but not surjective.

(c) bijective.

Question 3. Prove that A and B have the same cardinality in each of the followings.

(a) $A = \{n \in \mathbb{Z} \mid 0 < n < 5\}$
$B = \{n \in \mathbb{Z} \mid 5 < n < 10\}$

(b) $A = \{n \mid n \text{ is odd integer}\}$
$B = \{n \mid n \text{ is even integer}\}$

(c) $A = \mathbb{Z}$
$B = \{3n \mid n \in \mathbb{Z}\}$

(d) $A = \{x \in \mathbb{R} \mid 0 < x < 1\}$
$B = \{x \in \mathbb{R} \mid 2 < x < 5\}$

Question 4. Let A_i be a non-empty set for any integer i. Prove the followings.

(a) The Cartesian product $\mathbb{N} \times \mathbb{N}$ is countable.

(b) The Cartesian product $A_0 \times A_1$ is countable if and only if both A_0 and A_1 are countable.

(c) The Cartesian product $A_0 \times A_1 \times \cdots \times A_n$ is countable if and only if $A_0, A_1, \ldots, A_n$ are all countable.

(d) $\mathbb{Q} = \left\{ \dfrac{x}{y} \mid x, y \in \mathbb{Z} \text{ and } y \neq 0 \text{ and } x, y \text{ are relatively prime} \right\}$ is countable.

Question 5. Prove or disprove the followings.

(a) If $A = \{0, 1, 2, 3, 4\}$ and $f : A \to A$, $f(x) = 4x \bmod 5$ is bijective.

(b) If $x, y \in \mathbb{R}$, $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$.

(c) If n is an odd integer, $\left\lfloor \dfrac{n^2}{4} \right\rfloor = \left(\dfrac{n-1}{2} \right) \left(\dfrac{n+1}{2} \right)$.