COMP S264F Unit 4: Functions

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Overview

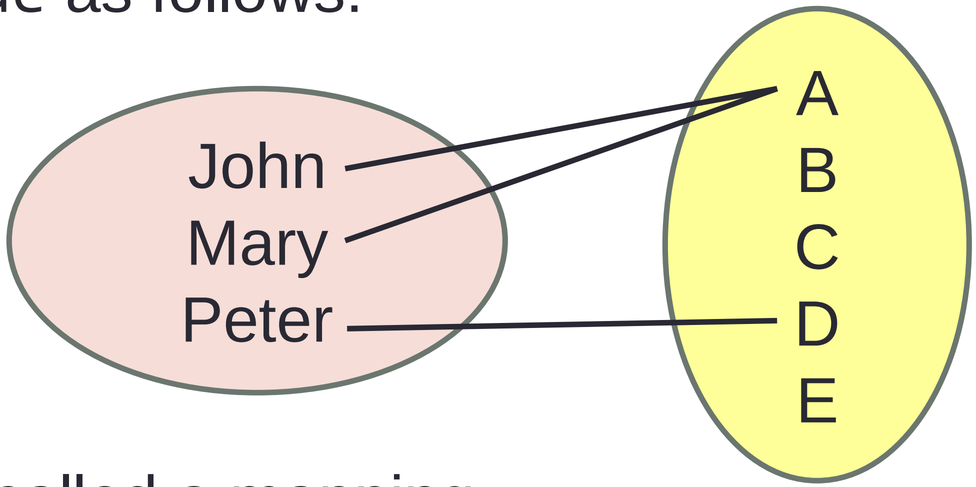
- What is a function?
- Domain, image, range
- One-to-one, Onto, Bijective
- Inverse function
- Composite function
- Plotting function in Python
- Non-functions
- Some useful functions:
  - floor, ceiling, exponential, log, mod
- Cardinality of infinite sets: Countable / Uncountable
- Functions with more than 1 argument

Functions

- Let A and B be sets. A function ***f*** from A to B is an *assignment* of exactly one element of B to *each* element of A.
- We write ***f(a) = b*** if ***b*** is the element of B assigned to the element ***a*** of A.

Example: Let A = {John, Mary, Peter}. Let B = {A,B,C,D,E}. Define a function `Grade` as follows:

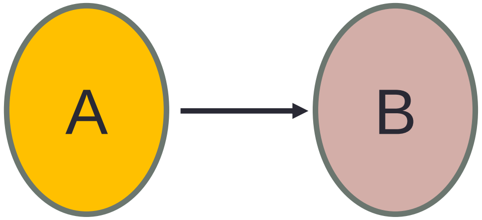
`Grade`(John) = A
`Grade`(Mary) = A
`Grade`(Peter) = D



NB. A function is also called a mapping.

Basic terminology

Consider a function ***f*** from A to B.



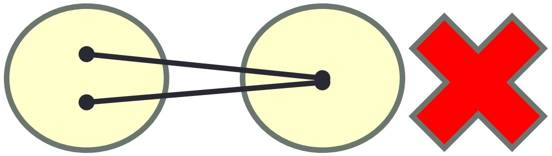
- A is the <u>domain</u> of ***f***.
- B is the codomain of ***f***.
- We write ***f*** : $A \to B$
- If ***f***(a) = b, b is the <u>image</u> of a.
- The ***range*** of ***f*** is the set comprising the images of elements of A.
  I.e., $\{ b \mid b \in B \text{ and } (\exists a\ f(a) = b) \}$.

Example: Let f_1 be the function from $\mathbb{Z}$ to $\mathbb{Z}$ such that, for every $x \in \mathbb{Z}$, $f_1(x) = x^2$.

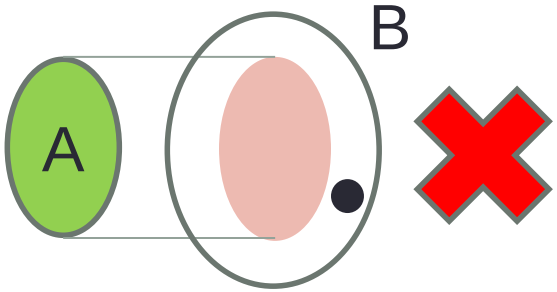
Domain = $\mathbb{Z}$. Range = $\mathbb{Z}$?

One-to-one, Onto, Bijective

- A function ***f*** is said to be <u>one-to-one</u> (injective) if, for every distinct elements x, y in the domain, ***f***(x) ≠ ***f***(y).



- A function ***f*** from A to B is said to be <u>onto</u> (surjective) if, for every element b of codomain B, there exists an element a of A such that f(a) = b.
  - I.e., the range of ***f*** is exactly B.



- A one-to-one and onto function is also called a <u>one-to-one correspondence</u>, or a <u>bijective</u> function, or a bijection.

Example 1

Consider a function $f_2: \mathbb{Z} \rightarrow \mathbb{N}$ such that,

for any $a \in \mathbb{Z}$, $f_2(a) = a^2$.

- Is f_2 one-to-one?
- Is f_2 onto?

- If f_2 is one-to-one (injective), then for any $x, y \in \mathbb{Z}$,
  - $x \neq y \Rightarrow f_2(x) \neq f_2(y)$
  - In other words, $f_2(x) = f_2(y) \Rightarrow x = y$

- If f_2 is onto (surjective), then for any $b \in \mathbb{N}$,

  there is $a \in \mathbb{Z}$ such that $f_2(a) = b$.

Example 1 (cont')

Consider a function f_2: $\mathbb{Z} \to \mathbb{N}$ such that,
for any $a \in \mathbb{Z}$, $f_2(a) = a^2$.

- Is f_2 one-to-one?

No. Let x = -2 and y = 2.
Then, $f(x) = (-2)^2 = 4$ and $f(y) = 2^2 = 4$.
Therefore, $x \neq y \Rightarrow f_2(x) \neq f_2(y)$ is false.

- Is f_2 onto?

No. Let b = 2.
$b = f_2(a) \Rightarrow 2 = a^2$
$\Rightarrow a = \sqrt{2}$ or $-\sqrt{2}$
Therefore, there does not exist $a \in \mathbb{Z}$ such that $f_2(a) = b$.

Example 2

Consider a function $f_3: \mathbb{Z} \rightarrow \mathbb{Z}$ such that,

for any $a \in \mathbb{Z}$, $f_3(a) = a-1$.

- Is f_3 one-to-one?

Yes. Let x and y such that $f_3(x) = f_3(y)$.

$\Rightarrow x-1 = y-1$

$\Rightarrow x = y$

- Is f_3 onto?

Yes. For any $b \in \mathbb{Z}$,

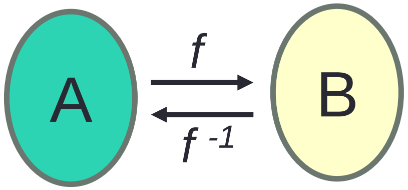
$b = f_3(a) \Rightarrow a-1 = b$

$\Rightarrow a = b+1$

$\Rightarrow a \in \mathbb{Z}$

- Therefore, f_3 is bijective.

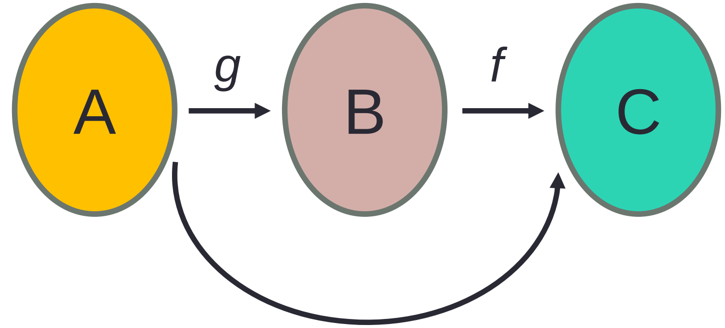
Inverse functions



- If ***f*** is a **bijection** (*one-to-one* and *onto* function) from the set A to the set B, then we can define the **inverse** of ***f*** (denoted by f^{-1}) as follows:
  - f^{-1} is a function from B to A.
  - For every element b of B, $f^{-1}(b) = a$ if and only if $f(a) = b$.

1. Is ***f*** a bijection?
2. Recall that $f_3(a) = a+1$ for any $a \in \mathbb{Z}$. What is the inverse of f_3?
3. If ***f*** is <u>not</u> one-to-one, f^{-1} may not be well-defined. Why?
4. What happens if ***f*** is <u>not</u> an onto function?

Composite functions



Consider two functions

$g : A \to B$ and $f : B \to C$.

The composition of f and g, denoted by $f \circ g$, is a function from A to C, defined as follows.

For any $a \in A$, $f \circ g\,(a) = f(\,g\,(a)\,)$.

Note that $g \circ f$ may not be well-defined.

Example: For $x \in \mathbb{R}$, $f(x) = 3x + 2$ and $g(x) = x^2 + x$.

- $f \circ g\,(x) = ?$
- $g \circ f\,(x) = ?$

- What is $f_3^{-1} \circ f_3$? $f_3 \circ f_3^{-1}$?

Visualizing a function in Python

- We can use the matplotlib and numpy packages in Python.

```
import numpy as np
import matplotlib.pyplot as plt
```

- NumPy is the fundamental package for scientific computing with Python, which provides powerful array objects *ndarray* and functions.
- E.g.,

```
x1 = np.arange(0, 6, 0.1)
x2 = np.linspace(0, 10, 100)
```

- **x1** is a ndarray from 0 to 6 (exclusive) with a step 0.1, i.e., [0, 0.1, 0.2, …]
- **x2** is a ndrray of size 100, evenly spaced over the interval [0, 10].

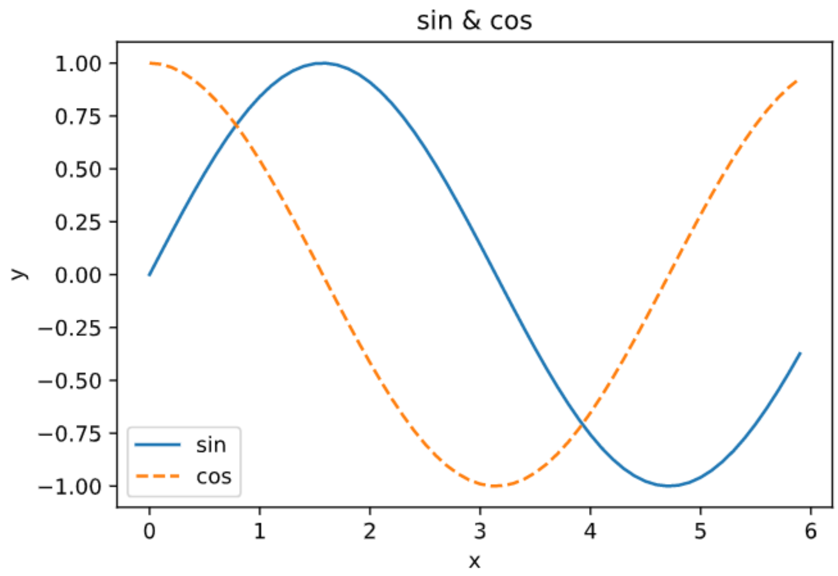
- Matplotlib is a data visualization library built on NumPy arrays.
- E.g.,

```
plt.plot(x, np.sin(x))
plt.show()
```

- y = sin(x) is plotted using points (x[0], sin(x[0])), (x[1], sin(x[1])), …

- We can plot both y = sin(x) and y = cos(x), as follows:

```
x = np.arange(0, 6, 0.1) # array [0, 0.1, ..., 5.9]

plt.plot(x, np.sin(x), label="sin") # solid line for sin
plt.plot(x, np.cos(x), linestyle="--",
                       label="cos") # dashed line for cos
plt.xlabel("x")
plt.ylabel("y")
plt.title("sin & cos")
plt.legend()
plt.show() # display the figure
```



Non-functions

- If ***f*(*a*) = *b*** has more than one value or no value of ***b*** for a particular ***a***, then ***f*** is a *non-function* (i.e., not a function).
  - ***f*** is just a *relation* that relates ***a*** to ***b***.

Function or non-function?

1. $f\colon \mathbb{R} \to \mathbb{R}$ such that $x = (f(x))^2$.
   Non-function. When $x = 4$, $f(x) = 2 \text{ or } -2$.

2. $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = \sqrt{x}$.
   Non-function. $f(x)$ is undefined if $x < 0$.

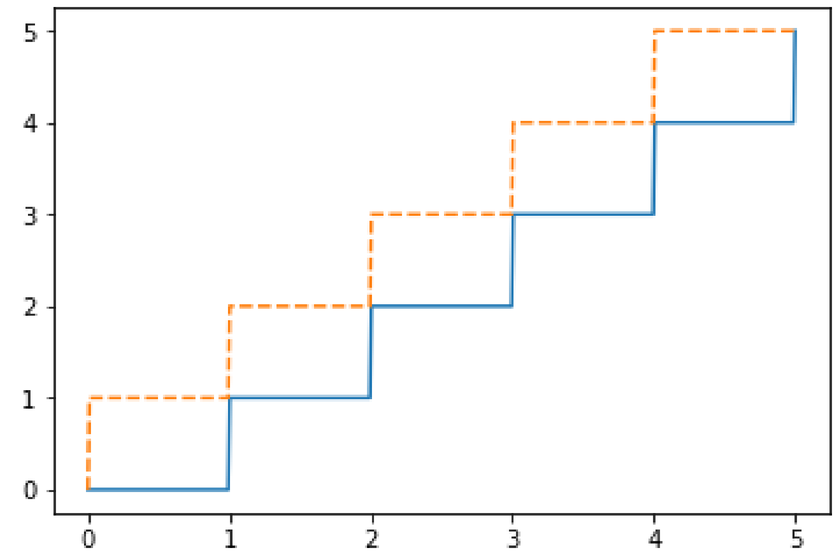
3. $f\colon \mathbb{R}^+ \to \mathbb{R}$ such that $f(x) = \sqrt{x}$.
   Function. $f(x)$ has exactly one value for all $x \in \mathbb{R}^+$.

The floor and ceiling functions

- **Floor function:** $f(x) = \lfloor x \rfloor$, where $f(x) \in \mathbb{Z}$ and $f(x) \leq x < f(x) + 1$
- E.g., $\lfloor 2.3 \rfloor = 2$
  $\lfloor -2.3 \rfloor = -3$

- **Ceiling function**: $f(x) = \lceil x \rceil$, where $f(x) \in \mathbb{Z}$ and $f(x) - 1 < x \leq f(x)$
- E.g., $\lceil 2.3 \rceil = 3$
  $\lceil -2.3 \rceil = -2$

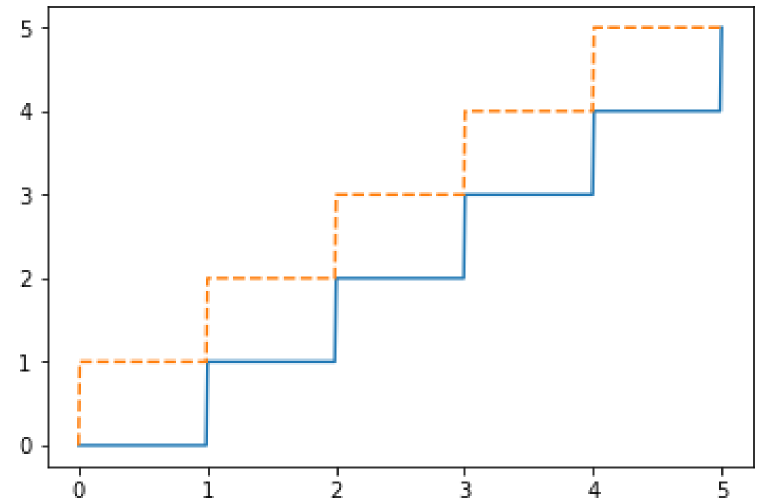
```
x = np.linspace(0, 5, 500)

plt.plot(x, np.floor(x), '-')
plt.plot(x, np.ceil(x), '--')

plt.show()
```



Floor and ceiling properties

Floor	Ceiling
$\lfloor x+1 \rfloor = \lfloor x \rfloor + 1$	$\lceil x+1 \rceil = \lceil x \rceil + 1$
$\lfloor x-1 \rfloor = \lfloor x \rfloor - 1$	$\lceil x-1 \rceil = \lceil x \rceil - 1$
$\lceil x \rceil = \lfloor x \rfloor$ if and only if $x \in \mathbb{Z}$	
$\lceil x \rceil = \lfloor x \rfloor + 1$ if and only if $x \notin \mathbb{Z}$	

- Is the inverse of the floor function well-defined?
- Is the inverse of the ceiling function well-defined?



Exponential Functions

- $f(x) = \boldsymbol{b}^x$ is the exponential function for the base $\boldsymbol{b}$, where $\boldsymbol{b} \neq 1$ and $\boldsymbol{b} > 0$.
- $f : \mathbb{R} \to \mathbb{R}^+$
- E.g., when the base $\boldsymbol{b} = 2$, $f(x) = \boldsymbol{2}^x$.

Properties: Let a, b $\in \mathbb{R}^+$ s.t. $a \neq 1$, $b \neq 1$, and let x, y $\in \mathbb{R}$.

- $a^x \times a^y = a^{x+y}$
- $\dfrac{a^x}{a^y} = a^{x-y}$
- $(a^x)^y = a^{xy}$
- $a^0 = 1$
- $(ab)^x = a^x b^x$
- $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$
- $a^x = a^y \text{ if and only if } x = y$
- $\text{For } x \neq 0,\ a^x = b^x \text{ if and only if } a = b$

Logarithmic functions (log functions)

- The ***inverse of an exponential function*** is called a logarithmic function.
- For $b > 0$ and $b \neq 1$,

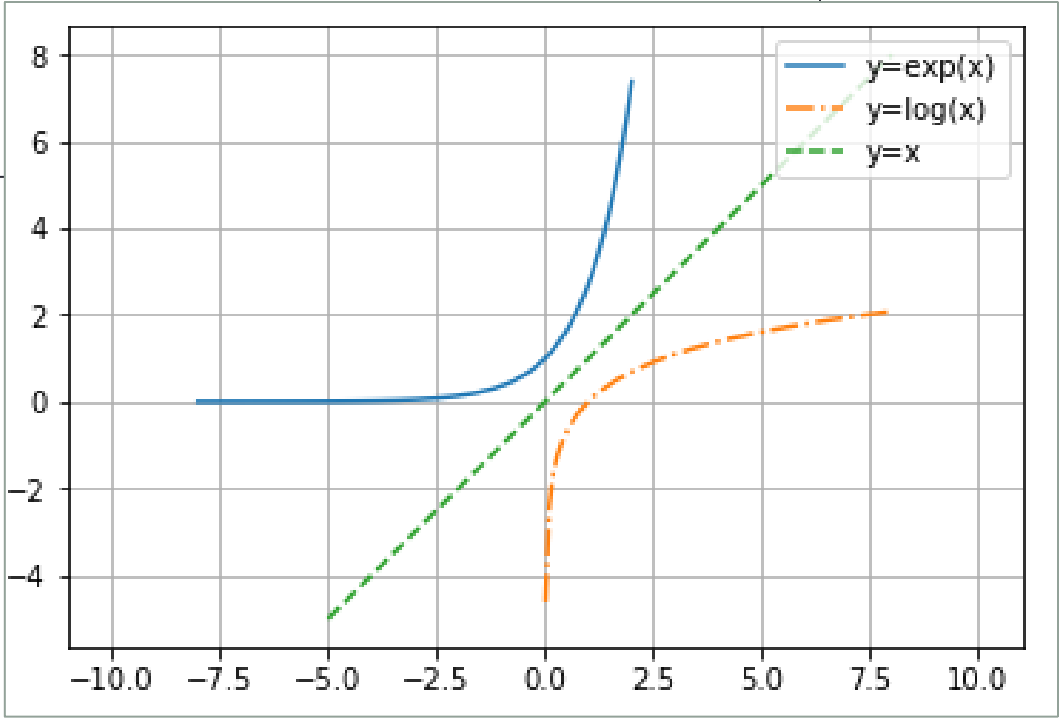
  $\mathbf{f(x)} = \log_b x$ is equivalent to $x = b^{\mathbf{f(x)}}$.

- $\log_{\mathbf{b}} x$ is read as *log to the base **b** of x*.
- $\mathbf{f} : \mathbb{R}^+ \to \mathbb{R}$

- $\boldsymbol{ln}\ x = \boldsymbol{log_e}\ x$ (natural log; e = 2.7182… is the Euler's number)
- $\boldsymbol{lg}\ x = \boldsymbol{log_2}\ x$ (binary log)
- $\boldsymbol{log}\ x$ may equal $\boldsymbol{log_{10}}\ x$ or $\boldsymbol{log_2}\ x$ depending on context.

- We can plot e^x and **ln x**, as follows:

```
x1 = np.linspace(-8, 2, 100)
x2 = np.linspace(0.01, 8, 100)
x3 = np.linspace(-5, 8, 100)

plt.plot(x1, np.exp(x1), '-', label='y=exp(x)')
plt.plot(x2, np.log(x2), '-.', label='y=log(x)')
plt.plot(x3, x3, '--', label='y=x')

plt.legend(loc='upper right')
plt.axis('equal')
plt.grid(True)
plt.show()
```



Log function properties

- $\log_b(b^x) = x$

Proof. Let $y = \log_b(b^x)$. Then,

$b^y = b^x$ *(by definition of log)*

$\Rightarrow y = x$ *(by properties of exponential function)*

$\Rightarrow \log_b(b^x) = x$

- $\log_b(x^y) = y \log_b x$

Proof. Let $p = \log_b x$. Then,

$x = b^p$ *(by definition of log)*

$\Rightarrow x^y = (b^p)^y = b^{py}$

$\Rightarrow \log_b(x^y) = py$ *(by definition of log)*

$\Rightarrow \log_b(x^y) = y \log_b x$

Change of base in log function

- $\log_a x = \dfrac{\log_b x}{\log_b a}$

Proof. Let $p = \log_a x,\ q = \log_b x,\ r = \log_b a.$

By definition of log,

$$
\begin{aligned}
x = a^p, x = b^q, a = b^r \implies & a^p = b^q \\
\implies & (b^r)^p = b^q \\
\implies & b^{rp} = b^q \\
\implies & rp = q \\
\implies & \log_b a \cdot \log_a x = \log_b x \\
\implies & \log_a x = \frac{\log_b x}{\log_b a}
\end{aligned}
$$

Let a, b $\in \mathbb{R}^+$ s.t. a $\neq$ 1, b $\neq$ 1, and let x, y, p, q $\in \mathbb{R}$.

- $a^x \times a^y = a^{x+y}$
- $\dfrac{a^x}{a^y} = a^{x-y}$
- $(a^x)^y = a^{xy}$
- $a^0 = 1$
- $(ab)^x = a^x b^x$
- $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$
- $a^x = a^y$ if and only if $x = y$
- For $x \neq 0$, $a^x = b^x$ if and only if $a = b$

- $\log_a(pq) = \log_a p + \log_a q$
- $\log_a\left(\dfrac{p}{q}\right) = \log_a p - \log_a q$
- $\log_a 1 = 0$
- $\log_a a^x = x$
- $\log_a p^y = y \log_a p$
- $\log_a x = \log_a y$ if and only if $x = y$

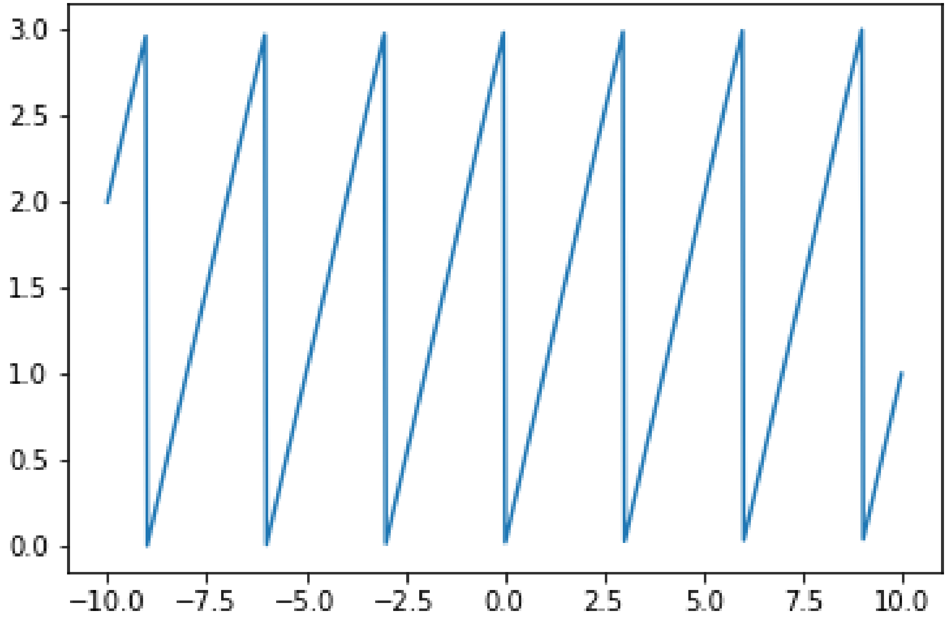
Modulo functions (mod functions)

- $f(x) = x \bmod m$, where $m > 0$, is the modulo function which is the remainder of the division of x by m.
- Note that

$$f(x) = x - m \lfloor x/m \rfloor$$

- We can plot $x \bmod 3$, as follows:

```
x = np.linspace(-10, 10, 500)
plt.plot(x, np.mod(x,3))
plt.show()
```



Mod function properties

- $x \bmod m = y \bmod m \Leftrightarrow (x - y) \bmod m = 0$
- $(x + y) \bmod m = ((x \bmod m) + (y \bmod m)) \bmod m$
- $(x \times y) \bmod m = ((x \bmod m) \times (y \bmod m)) \bmod m$
- Let *a* $\in \mathbb{R}$ such that *a* and *m* are relatively prime. Then,

  $ax \bmod m = ay \bmod m \Longrightarrow x \bmod m = y \bmod m$

Cardinality of infinite sets

- The cardinality of a finite set is the number of elements in the set.
- What about infinite sets? No exact number.
- Two sets A and B are said to have the *same cardinality* if and only if there is a bijection from A to B.
- Infinite sets are classified as countable or uncountable.

- Definition. A set is said to be *countable* if it
  - is finite; or
  - has the same cardinality as the set of natural numbers (i.e., $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$).
- A set that is not countable is called *uncountable*.

Countable or not?

- The set of all (OUHK) Computing students who were admitted in the year 2019.

- $\mathbb{N}$ = {0, 1, 2, …}

- {1, 3, 5, …} (I.e., the set of all odd positive integers)
  Consider the following bijection: f(0) = 1, f(1) = 3, …
  In general, f(i) = 2i + 1

- $\mathbb{Z}$ = the set of all integers (including –ve integers)

Countable or not?

$\mathbb{Z}$ = the set of all integers (including –ve integers)

- Consider the following bijection:
  f(0) = 0, f(1) = 1, f(2) = -1, f(3) = 2, f(4) = -2, …

- In general,

$$f(i) = \begin{cases} (i+1)/2 & \text{if } i \text{ is odd} \\ -i/2 & \text{if } i \text{ is even} \end{cases}$$

- Therefore, $\mathbb{Z}$ is countable.

Countable or not: the set of all prime integers

- In general, let A be a countable set and let B be a subset of A. Is B countable?   Yes.

- Assume B is infinite.
- Since A is countable, there exists a bijection $\boldsymbol{f}$ from $\mathbb{N}$ to A.
- The elements of A can be enumerated (written down) in the order of

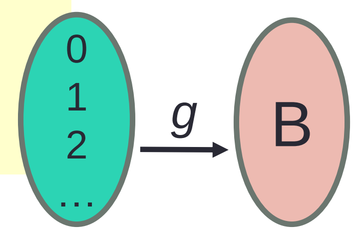
$$\boldsymbol{f}(0), \boldsymbol{f}(1), \boldsymbol{f}(2), \boldsymbol{f}(3), \boldsymbol{f}(4), \boldsymbol{f}(5), \boldsymbol{f}(6), \boldsymbol{f}(7), \boldsymbol{f}(8), \ldots$$

- Assume B is infinite.
- Since A is countable, there exists a bijection ***f*** from $\mathbb{N}$ to A.
- The elements of A can be enumerated (written down) in the order of

$$\boldsymbol{f}(0), \boldsymbol{f}(1), \boldsymbol{f}(2), \boldsymbol{f}(3), \boldsymbol{f}(4), \boldsymbol{f}(5), \boldsymbol{f}(6), \boldsymbol{f}(7), \boldsymbol{f}(8), \ldots$$

- Define a function ***g*** from $\mathbb{N}$ to B as follows:

$$\begin{array}{ccccccccc} \boldsymbol{f}(0), & \boldsymbol{f}(1), & \boldsymbol{f}(2), & \boldsymbol{f}(3), & \boldsymbol{f}(4), & \boldsymbol{f}(5), & \boldsymbol{f}(6), & \boldsymbol{f}(7), & \boldsymbol{f}(8), \ldots \\ \notin B & \in B & \notin B & \notin B & \in B & \notin B & \in B & \in B & \notin B \\ & \boldsymbol{g}(0), & & & \boldsymbol{g}(1), & & \boldsymbol{g}(2), & \boldsymbol{g}(3), \ldots & \end{array}$$



- Precisely, let $a_{-1} = -1$. For any integer $i \geq 0$,
  - let a_i be the smallest integer > a_{i-1} such that $f(a_i) \in B$; and
  - define $g(i) = f(a_i)$.
- E.g., when $i = 2$, $a_i = 6$.

Claim: g is bijective.

Lemma 1. g is *one-to-one*.

Proof.

Consider any $i, j \in \mathbb{N}$.

$g(i) = g(j)$

$\Rightarrow f(a_i) = f(a_j)$ (as $g(i) = f(a_i)$ and $g(j) = f(a_j)$)

$\Rightarrow a_i = a_j$ (as f is one-to-one)

$\Rightarrow$ **$i = j$** (as a_i's are all distinct)

Therefore, g is one-to-one.

Claim: g is bijective.

Lemma 2. g is *onto*.

Proof.

Consider **any element $x \in B$**.

As $B \subseteq A$, $x \in A$.

Because f is onto, there exists $i \in \mathbb{N}$ such that $f(i) = x$.

Let k be the number of elements in $\{ f(0), f(1), \ldots, f(i-1) \} \cap B$.

$$\boxed{\begin{array}{c} f(0), f(1), f(2), f(3), \ldots, f(i-1), \\ \\ g(0), g(1), \ldots, g(k-1), \end{array}} \begin{array}{l} f(i), f(i+1), \ldots \\ \in B \\ g(k) \end{array}$$

Note that $i \geq k \geq 0$, and $k \in \mathbb{N}$.

As $x \in B$, by definition of g, $g(k) = f(i) = x$.

Theorem 3. g is bijective.

Proof.
By Lemma 1, g is one-to-one.
By Lemma 2, g is onto.
Therefore, g is bijiective.

- Let A be a countable set and let B be a subset of A.
- Is B countable?   Yes, we have formally proven it.

- Is the set of all prime integers countable?

Countable or not: the set of real numbers

Theorem 4. Let R be the set of all real numbers ***a*** such that $0 \leq \boldsymbol{a} < 1$. Then R is uncountable.

Proof. Suppose R is countable.

Then, there is a bijective function f from $\mathbb{N}$ to R.

We construct a real number **x** < 1 as follows: …

We can show that

- **x** = f(i) for some integer i; and
- **x** $\neq$ f(i) for all integers i.

Then a contradiction occurs. (I.e., if R is countable, then "**x** = f(i) for some integer i" and "**x** $\neq$ f(i) for all integers i".)

Therefore, R is uncountable.

What is x?

- Recall that f is a bijective function from $\mathbb{N}$ to R.
  We can enumerate elements of R in order, e.g.,
- $f(0) = 0.111$
- $f(1) = 0.33333\ldots$
- $f(2) = 0.5$
- …

$f(0) =$	0.	1	1	1	
$f(1) =$	0.	3	3	3	…
$f(2) =$	0.	5	0	0	
…					
$\mathbf{x} =$	0.	2	4	1	…

- $\mathbf{x} = 0.241\ldots \quad \Rightarrow \quad$ Such $\mathbf{x} \neq f(\mathrm{i})$ for all integers i.

What is x?

Notation: For any real number $y \in R$, for any integer $i \geq 0$, let y_i be the (i+1)-th digit after the decimal point.

E.g., Suppose y = 0.101. Then $y_0 = 1$; $y_1 = 0$; $y_2 = 1$; $y_3 = y_4 = \ldots = 0$

Recall that f is a bijective function from $\mathbb{N}$ to R.
We can enumerate elements of R in the order of

$$f(0), f(1), f(2), f(3), \ldots$$

Define a real number **x** in R such that for all **i** ≥ 0, $\mathbf{x}_i \neq f(i)_i$.

- Obviously, $\mathbf{x} \neq f(i)$ for all integers i.
- On the other hand, **x** is in R and f is bijective; thus, there exists an integer i such that $\mathbf{x} = f(i)$.

Countable or not: the set of real numbers

Corollary 5. The set of real numbers is uncountable.

Proof.

By Theorem 3, if the set of real numbers is countable, then R (which is a subset of real numbers) is also countable.

As Theorem 4 shows that R is uncountable, the set of real numbers is also uncountable. *(Modus tollens)*

Functions with more than 1 argument

- If **f**(x) = y, then x is called an argument of **f**, and y is called a value of **f**.
- If the domain of **f** is the Cartesian product $A_1 \times A_2 \times \ldots \times A_n$, then **f** has n arguments.
- **f**$(x_1, x_2, \ldots, x_n)$ denotes the value at $(x_1, x_2, \ldots, x_n)$, where $x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n$.

- **Example:** Let f: $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ such that f(x, y) = x+y.