COMP S264F Discrete Mathematics
Assignment – Suggested Solution

Question 1 (10 marks).

(a) $\neg\, \exists x\ \forall y\ P(x,y) \equiv \forall x\ \neg\, \forall y\ P(x,y)$
$\equiv \forall x\ \exists y\ \neg P(x,y)$

(b)
$$\begin{aligned}
&\neg\, \exists y\ (\forall x\ \exists z\ P(x,y,z)\ \vee\ \exists x\ \forall z\ Q(x,y,z)) \\
\equiv\ &\forall y\ \neg\, (\forall x\ \exists z\ P(x,y,z)\ \vee\ \exists x\ \forall z\ Q(x,y,z)) \\
\equiv\ &\forall y\ (\neg\, \forall x\ \exists z\ P(x,y,z)\ \wedge\ \neg\, \exists x\ \forall z\ Q(x,y,z)) \qquad \text{(by De Morgan's law)} \\
\equiv\ &\forall y\ (\exists x\ \neg\, \exists z\ P(x,y,z)\ \wedge\ \forall x\ \neg\, \forall z\ Q(x,y,z)) \\
\equiv\ &\forall y\ (\exists x\ \forall z\ \neg P(x,y,z)\ \wedge\ \forall x\ \exists z\ \neg Q(x,y,z))
\end{aligned}$$

Question 2 (15 marks).

(a) Five example elements of N:

- $\varnothing$
- $\{\varnothing\}$
- $\{\varnothing, \{\varnothing\}\}$
- $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$
- $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}\}$

(b) True. We can prove that the i-th element of N derived from $\varnothing$ has i elements using mathematical induction, as follows:
Base case: When $i = 1$, the 1st element derived from $\varnothing$ is $\{\varnothing\}$, which has 1 element.
Induction step: Let x be the k-th element of N derived from $\varnothing$ for **some** positive integer k. Assume x has k elements.
Then, the $(k+1)$-th element of N derived from $\varnothing$ is $x \cup \{x\}$, which has $|x| + 1 = k + 1$ elements, as $|x| = k$ by the induction hypothesis.
By the principle of mathematical induction, the i-th element of N derived from $\varnothing$ has i elements.
Together with the fact that $\varnothing$ has 0 elements, all elements in N are finite sets and have distinct number of elements.

(c) Suppose, for the sake of contradiction, that N is a finite set.
By (b), each element of N is a finite set, i.e., its cardinality is a non-negative integer.
Let y be the element with the maximum cardinality among all elements of N.
By definition, as N contains y, N also contains the set $y \cup \{y\}$.
The cardinality of $y \cup \{y\}$ is $|y| + 1 > |y|$, contradicting that y is the element with the maximum cardinality.
Therefore, N is an infinite set.

(d) By (b), every element of N has a finite number of elements.
But, by (c), N is an infinite set (containing infinite number of elements).
Therefore, N cannot be an element of N, i.e., $N \notin N$.

Question 3 (15 marks).
Suppose, for the sake of contradiction, S and T have the same cardinality.
Then, there exists a bijective function $g : S \to T$.
For any element $x \in S$, let $f_x = g(x)$.

Now, we construct a function $h : S \to S$, as follows:
For any element $x \in S$, $h(x)$ equals to an element in S that is different to $f_x(x)$.

To illustrate the idea, let $S = \{s_1, s_2, s_3, \dots\}$. Then, $T = \{f_{s_1}, f_{s_2}, f_{s_3}, \dots\}$. We also let $s_{i,j} = f_{s_i}(s_j)$.

x	s_1	s_2	s_3	$\cdots$
$f_{s_1}(x)$	$s_{1,1}$	$s_{1,2}$	$s_{1,3}$	$\cdots$
$f_{s_2}(x)$	$s_{2,1}$	$s_{2,2}$	$s_{2,3}$	$\cdots$
$f_{s_3}(x)$	$s_{3,1}$	$s_{3,2}$	$s_{3,3}$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$h(x)$	$\neq s_{1,1}$	$\neq s_{2,2}$	$\neq s_{3,3}$	$\cdots$

Note that h is a function from S to S, so $h \in T$ and thus there exists $k \in S$ such that $h = f_k$.
On the other hand, the construction of h ensures that $h \neq f_k$ for any $k \in S$, resulting a contradiction.
It follows that S and T do not have the same cardinality.

Question 4 (10 marks). The sum of all players' numbers is

$$\sum_{i=1}^{12} i = \frac{12 \cdot (1+12)}{2} = 6 \cdot 13 = 78 \ .$$

Approach 1: We can apply pigeonhole principle. Given an arrangement of the players, we can group every three consecutive players and form 4 groups.
By the pigeonhole principle, at least one group has a sum of their numbers at least

$$\left\lceil \frac{78}{4} \right\rceil = \lceil 19.5 \rceil = 20 \ .$$

Approach 2: We can prove by contradiction. Suppose, for the sake of contradiction, any three consecutive players have the sum of their numbers less than 20, i.e., at most 19.
Therefore, the sum of all the 12 players' numbers is at most

$$\frac{12}{3} \cdot 19 = 4 \cdot 19 = 76 \ ,$$

contradicting that the total sum is 78.

Question 5 (10 marks). We can assign the driver and the menus to the n people, as follows:
Method 1 (L.H.S.):
Step 1: Select a person to be the driver.
Step 2: Each of the other $n-1$ people chooses one of the 4 menus (3 alcoholic menus and 1 non-alcoholic menu).
Step 1 has n choices, and Step 2 has 4^{n-1} choices.
By product rule, the number of ways to assign the driver and the menus is $n \cdot 4^{n-1}$.

Method 2 (R.H.S.):
Let k be the number of people choosing an alcoholic menu.
Step 1: Select k people to choose the alcoholic menus.
Step 2: Each of these k people chooses one of the 3 alcoholic menus.
Step 3: Select one of the remaining $n-k$ people as the driver, and the rest choose the non-alcoholic menus.
For a particular k, Step 1 has $C(n,k)$ choices, Step 2 has 3^k choices, and Step 3 has $n-k$ choices. By product rule, the total number of choices is $C(n,k) \cdot 3^k \cdot (n-k)$.
Therefore, the total number of ways to assign the driver and the menus is

$$\sum_{k=0}^{n} C(n,k) \cdot 3^k \cdot (n-k) \ ,$$

completing the combinatorial proof.

Question 6 (10 marks).
Base case: When $n = 1$, $p(E_1 \cup E_2 \cup \cdots \cup E_n) = p(E_1) = \sum_{i=1}^{n} p(E_i)$.
Induction step: Assume for some positive integer k, $p(E_1 \cup E_2 \cup \cdots \cup E_k) \le \sum_{i=1}^{k} p(E_i)$.
When $n = k+1$,

$$\begin{aligned} p(E_1 \cup E_2 \cup \cdots \cup E_{k+1}) &= p((E_1 \cup E_2 \cup \cdots \cup E_k) \cup E_{k+1}) \\ &\le p(E_1 \cup E_2 \cup \cdots \cup E_k) + p(E_{k+1}) \qquad (\text{as } p(A \cup B) \le p(A) + p(B)) \\ &\le \sum_{i=1}^{k} p(E_i) + p(E_{k+1}) \qquad (\text{by the induction hypothesis}) \\ &= \sum_{i=1}^{k+1} p(E_i) \ . \end{aligned}$$

By the principle of mathematical induction, for any positive integer n, $p(E_1 \cup E_2 \cup \cdots \cup E_n) \le \sum_{i=1}^{n} p(E_i)$.

Question 7 (15 marks). Let *win* be the winning event.

(a)
- Exactly 50% chance of winning: the two numbers are 0 and 1.
  There are 50% chance that $r = 0$ and another 50% chance that $r = 1$.
  If $r = 0$, the other number 1 is chosen as the larger number, which is a win case.
  If $r = 1$, the other number 0 is chosen as the larger number, which is a loss case.
  Therefore, $$\begin{aligned} p(win) &= p(win \cap (r=0)) + p(win \cap (r=1)) \\ &= p(win \mid r=0) \cdot p(r=0) + p(win \mid r=1) \cdot p(r=1) \\ &= 1 \cdot 0.5 + 0 \cdot 0.5 = 0.5 \end{aligned}$$
- More than 50% chance of winning: the two numbers are 0 and 10.
  There are 50% chance that $r = 0$ and another 50% chance that $r = 10$.
  If $r = 0$, the other number 10 is chosen as the larger number, which is a win case.
  If $r = 10$, the number 10 is chosen as the larger number, which is also a win case.
  Therefore, $$\begin{aligned} p(win) &= p(win \cap (r=0)) + p(win \cap (r=10)) \\ &= p(win \mid r=0) \cdot p(r=0) + p(win \mid r=10) \cdot p(r=10) \\ &= 1 \cdot 0.5 + 1 \cdot 0.5 = 1 > 0.5 \end{aligned}$$

(b) Let a and b be the two envelope numbers such that $a < b$.

(i) Consider the case that $a < x < b$.
There are 50% chance that $r = a$ and another 50% chance that $r = b$.
If $r = a$, as $a < x$, the other number b is chosen as the larger number, which is a win case.
If $r = b$, as $b > x$, the number b is chosen as the larger number, which is also a win case.
$$\begin{aligned} & p(win \mid a < x < b) \\ = \ & p(win \cap (r=a) \mid a < x < b) + p(win \cap (r=b) \mid a < x < b) \\ = \ & p(win \mid (r=a) \cap (a < x < b)) \cdot p(r=a) + p(win \mid (r=b) \cap (a < x < b)) \cdot p(r=b) \\ = \ & 1 \cdot 0.5 + 1 \cdot 0.5 = 1 \ . \end{aligned}$$
Therefore, the chance of winning is 100%.

(ii) We can set $x = a + 0.5$.
As a, b are integers and $a < b$, we have $a \le b - 1 \le 10 - 1 = 9$.
Note that a is an integer in $[0, 9]$, $a + 0.5$ must be in the set $\{0.5, 1.5, \cdots, 9.5\}$.
Therefore, $x = a + 0.5 > a$ and $x = a + 0.5 \le (b-1) + 0.5 = b - 0.5 < b$.
It is always possible to choose x such that $a < x < b$, i.e., $p > 0$.

(iii) Consider the case that $x \notin [a, b]$. Then, there are two subcases $x < a < b$ or $a < b < x$.
When $x < a < b$, we win if and only if $r = b$, so the chance of winning is 50%.
When $a < b < x$, we win if and only if $r = a$, so the chance of winning is also 50%.
Thus, $p(win \mid \neg(a < x < b)) = 0.5$

$$
\begin{aligned}
\text{Then,} \quad p(win) &= p(win \cap (a < x < b)) + p(win \cap \neg(a < x < b)) \\
&= p(win \mid a < x < b) \cdot p(a < x < b) + p(win \mid \neg(a < x < b) \cdot p(\neg(a < x < b)) \\
&= 1 \cdot p + 0.5 \cdot (1 - p) \qquad \text{(by (i) and let } p = p(a < x < b)) \\
&= 0.5p + 0.5 \\
&> 0.5 \qquad (\text{as } p > 0 \text{ by (ii)})
\end{aligned}
$$

Question 8 (15 marks). Let $S = \{1, 2, 3, 4, 5, 6\}$ be the sample space of tossing the dice.

(a)
$$
\begin{aligned}
E(X) &= \sum_{s \in S} p(s) \cdot X(s) \\
&= \sum_{s=1}^{6} \frac{1}{6} \cdot 2s \qquad (\text{as } p(s) = \frac{1}{6}) \\
&= \frac{1}{3} \sum_{s=1}^{6} s = \frac{1}{3} \cdot \frac{6 \cdot (1+6)}{2} = 7
\end{aligned}
$$

(b) Note that $Y(s) = 1$ for $s \in \{1, 3, 5\}$, and $Y(s) = 3$ for $s \in \{2, 4, 6\}$.
$$
\begin{aligned}
\text{Then,} \quad E(Y) &= p(Y = 1) \cdot 1 + p(Y = 3) \cdot 3 \\
&= \frac{3}{6} \cdot 1 + \frac{3}{6} \cdot 3 \\
&= \frac{4}{2} = 2
\end{aligned}
$$

(c) Consider the value of $Z(y)$ for each outcome $y \in S$:

- $y = 1$: $Z(y) = X(y) + Y(y) = 2 \cdot 1 + 1 = 3$
- $y = 2$: $Z(y) = X(y) + Y(y) = 2 \cdot 2 + 3 = 7$
- $y = 3$: $Z(y) = X(y) + Y(y) = 2 \cdot 3 + 1 = 7$
- $y = 4$: $Z(y) = X(y) + Y(y) = 2 \cdot 4 + 3 = 11$
- $y = 5$: $Z(y) = X(y) + Y(y) = 2 \cdot 5 + 1 = 11$
- $y = 6$: $Z(y) = X(y) + Y(y) = 2 \cdot 6 + 3 = 15$

$$
\begin{aligned}
\text{Then,} \quad E(Z) &= \sum_{y \in S} p(y) \cdot Z(y) \\
&= \sum_{y=1}^{6} \frac{1}{6} \cdot Z(y) \qquad (\text{as } p(y) = \frac{1}{6}) \\
&= \frac{1}{6} \cdot (3 + 7 + 7 + 11 + 11 + 15) = \frac{54}{6} = 9
\end{aligned}
$$