Variational Principle for the Forge Equation in Curved Spacetime

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1. Geometric Framework

Let $(\mathcal{M}, g_{\mu\nu})$ be a 4-dimensional Lorentzian manifold with metric signature $(-,+,+,+)$. Let $\rho(x)$ be a scalar field representing the pulse density, and $u^\mu(x)$ a future-directed null 4-velocity field, satisfying

$$g_{\mu\nu}u^\mu u^\nu = 0. \tag{1}$$

Let $\sigma(x)$ be a scalar source term describing localized emission.

2. Action and Lagrangian Density

We define the action functional:

$$S[\rho, u^\mu, \lambda] = \int_{\mathcal{M}} d^4x\, \sqrt{-g}\, \left(-u^\mu \nabla_\mu \rho + \sigma \ln \rho - \lambda\, g_{\mu\nu}u^\mu u^\nu\right), \tag{2}$$

where ∇_μ is the covariant derivative compatible with $g_{\mu\nu}$, and $\lambda(x)$ is a Lagrange multiplier enforcing the null condition.

3. Variational Derivation

Variation with respect to ρ

We compute:

$$\delta S = \int d^4x\, \sqrt{-g} \left(-u^\mu \nabla_\mu \delta\rho + \frac{\sigma}{\rho}\delta\rho\right).$$

Using integration by parts and neglecting boundary terms:

$$\delta S = \int d^4x\, \sqrt{-g} \left(\nabla_\mu u^\mu + \frac{\sigma}{\rho}\right)\delta\rho.$$

Thus the Euler–Lagrange equation is:

$$\nabla_\mu(\rho u^\mu) = \sigma. \tag{3}$$

Variation with respect to u^μ

$$\delta S = \int d^4x \sqrt{-g} \left(-\delta u^\mu \nabla_\mu \rho - 2\lambda \, g_{\mu\nu} u^\nu \delta u^\mu \right),$$

yielding the equation of motion:

$$\nabla_\mu \rho + 2\lambda \, g_{\mu\nu} u^\nu = 0. \tag{4}$$

Variation with respect to λ

$$\delta S = \int d^4x \sqrt{-g} \, (-\delta\lambda)(u^\mu u_\mu) \quad \Rightarrow \quad u^\mu u_\mu = 0.$$

4. Summary

The system of equations derived from the variational principle is:

$$\nabla_\mu(\rho u^\mu) = \sigma, \tag{1}$$

$$\nabla_\mu \rho + 2\lambda \, g_{\mu\nu} u^\nu = 0, \tag{2}$$

$$g_{\mu\nu} u^\mu u^\nu = 0. \tag{3}$$

These equations describe a pulse-density field propagating along null geodesics, governed by local emission rates $\sigma(x)$. The formalism is compatible with general relativity and may serve as a basis for a discrete-event reinterpretation of light propagation.