Graph theory homework, task 2

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1 Triangle inequality

Let's assume that $dist(x, z) > dist(x, y) + dist(y, z)$. So, dist(x, z) is the shortest path between x and z. However, we can find a shorter path between these vertices: $dist(x, z) = dist(x, y) + dist(y, z)$. It's a contradiction. Consequently, for any connected graph $dist(x, z) \leq dist(x, y) + dist(y, z)$.

2 Tree

G is a connected graph. Let's consider 2 cases.

- $|E| < |V| - 1$. Then some vertices will not have edges. So, the graph will not be connected. It is a contradiction.

- $|E| > |V| - 1$. G is a connected graph. So, $|V| - 1$ edges should connect all vertices. There is no cycles in a graph because with $|V| - 1$ we can create only a one path between all $|V|$ vertices. So, if we add more than $|V| - 1$ edges, we will create a new path. Two different paths between vertices would create a cycle. We would get a cyclic graph.

Consequently, a connected graph with only $|V| - 1$ edges would be acyclic, would be a tree.

3 Whitney

Let be $\varkappa(G) = n$, $\lambda(G) = m$, $\delta(G) = q$.

- Let's assume that $n > q$. But if some vertex has q as a minimum degree, we can just remove q vertices to disconnect graph. So, $\varkappa(G)$ equals at least q. We came to contradiction. Consequently, $n \leq q$.

- Let's assume that $m > q$. But if some vertex has at least q edges, we can just remove q edges to disconnect graph. So, $\lambda(G)$ equals at least q. We came to contradiction. Consequently, $m \leq q$

In conclusion, $\varkappa(G) \leq \lambda(G) \leq \delta(G)$

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