Task 2

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1 Triangle inequality

Let's assume that $dist(x, z) > dist(x, y) + dist(y, z)$. So, dist(x, z) is the shortest path between x and z. However, we can find a shorter path between these vertices: $dist(x, z) = dist(x, y) + dist(y, z)$. It's a contradiction. Consequently, for any connected graph $dist(x, z) \leq dist(x, y) + dist(y, z)$.

2 Tree

G is a connected graph. Let's consider 2 cases.

- $|E| < |V| - 1$.
  Then some vertices will not have edges. So, the graph will not be connected. It is a contradiction.
- $|E| > |V| - 1$.
  G is a connected graph. So, $|V| - 1$ edges should connect all vertices. There is no cycles in a graph because with $|V| - 1$ we can create only a one path between all $|V|$ vertices. So, if we add more than $|V| - 1$ edges, we will create a new path. Two different paths between vertices would create a cycle. We would get a cyclic graph.

Consequently, a connected graph with only $|V| - 1$ edges would be acyclic, would be a tree.

3 Whitney

Let be $\varkappa(G) = n$, $\lambda(G) = m$, $\delta(G) = q$.

- Let's assume that $n > q$.
  But if some vertex has q as a minimum degree, we can just remove q vertices to disconnect graph. So, $\varkappa(G)$ equals at least q. We came to contradiction. Consequently, $n \leq q$.

- Let's assume that $m > q$.
  But if some vertex has at least q edges, we can just remove q edges to disconnect graph. So, $\lambda(G)$ equals at least q. We came to contradiction. Consequently, $m \leq q$

In conclusion, $\varkappa(G) \leq \lambda(G) \leq \delta(G)$

4 Chartrand

Let's prove by induction. It's a correct statement for graphs that consist 1, 2 or 3 vertices. Assume that this statement is correct for a graph with n vertices. Let's prove that this statement also is correct for a graph with (n+1) vertices.
A new vertex of a graph must have $|V/2|$ or more edges. Besides, only removing all new edges can disconnect a graph because it has already been connected by previous edges. So, if we add a new vertex, we won't be able to get better answer for $\lambda(G)$.Consequently, $\delta(G) = \lambda(G)$ for a graph with (n+1) vertices.

5 Menger

Let's assume that the size of the minimum vertex cut does not equal to the maximum number of pairwise internally vertex-disjoint paths between some vertices v and u. Consider 2 cases.

- The size of the minimum vertex cut is less than the maximum number of paths.
  So, the number of vertices we must delete to disconnect a graph is less than the number of paths. But paths are pairwise internally vertex-disjoint. Consequently, to "destroy" all paths between v and u we must remove at least a vertex from every path. We came to contradiction. The size of vertex cut couldn't be less than the maximum number of paths.

- The size of the minimum vertex cut is larger than the maximum number of paths.
  Removing one vertex from every pairwise internally vertex-disjoint paths is enough to "destroy" all paths between v and u and disconnect a graph. So, vertex cut is not minimum. We came to contradiction.

In conclusion, the size of the minimum vertex cut equal to the maximum number of pairwise internally vertex-disjoint paths between some vertices v and u.

6 Harary

A block of a block graph always has two vertex-disjoint paths between vertices (definition of 2-vertex-connectivity). Assume that some vertices of a block are not connected. But it couldn't be true because they have a path between them. So, they have same vertex cut. They should be connected.
Consequently, all of vertices of a block of a block graph are connected. It is a clique.