Robust Multistage

11374049

Angus

Original problem : (1)

$$\min_{x,y,I} E\left[\sum_{t=1}^{T} C_t \cdot x_t(\xi_1,\dots,\xi_{t-1}) + y_{t+1}(\xi_1,\dots,\xi_t)\right]$$

Production quantity   Holding / Backorder cost (Auxiliary variable)

$$\text{s.t.} \quad I_{t+1}(\xi_1,\dots,\xi_t) = I_t(\xi_1,\dots,\xi_{t-1}) + x_t(\xi_1,\dots,\xi_{t-1}) - \xi_t \qquad \forall t \in [T]$$

Current initial inventory   Current demand

$$y_{t+1}(\xi_1,\dots,\xi_t) \ge h_t \cdot I_{t+1}(\xi_1,\dots,\xi_t) \qquad \forall t \in [T]$$

$$y_{t+1}(\xi_1,\dots,\xi_t) \ge -b_t \cdot I_{t+1}(\xi_1,\dots,\xi_t) \qquad \forall t \in [T]$$

Auxiliary constraint for auxiliary variable y.

$$0 \le x_t(\xi_1,\dots,\xi_{t-1}) \le \overline{x}_t \qquad \forall t \in [T] \rightarrow \text{Capacity constraint.}$$

Reformulate as robust optimization : (2)

$$\min_{x,y,I} \frac{1}{N}\sum_{j=1}^{N} \sup_{\zeta \in \mathcal{U}_N^j} \sum_{t=1}^{T}\left[C_t \cdot x_t(\zeta_1,\dots,\zeta_{t-1}) + y_{t+1}(\zeta_1,\dots,\zeta_t)\right]$$

$$\text{s.t.} \quad I_{t+1}(\zeta_1,\dots,\zeta_t) = I_t(\zeta_1,\dots,\zeta_{t-1}) + x_t(\zeta_1,\dots,\zeta_{t-1}) - \zeta_t \qquad \forall \zeta \in \cup_{j=1}^{N}\mathcal{U}_N^j,\ t \in [T]$$

$I_{t+1}(\zeta_1,\dots,\zeta_t)$ will not be independent to j, and so is $y_{t+1}(\zeta_1,\dots,\zeta_t)$.

$$y_{t+1}(\zeta_1,\dots,\zeta_t) \ge h_t \cdot I_{t+1}(\zeta_1,\dots,\zeta_t) \qquad \forall \zeta \in \cup_{j=1}^{N}\mathcal{U}_N^j,\ t \in [T]$$

$$y_{t+1}(\zeta_1,\dots,\zeta_t) \ge -b_t \cdot I_{t+1}(\zeta_1,\dots,\zeta_t) \qquad \forall \zeta \in \cup_{j=1}^{N}\mathcal{U}_N^j,\ t \in [T]$$

$$0 \le x_t(\zeta_1,\dots,\zeta_{t-1}) \le \overline{x}_t \qquad \forall \zeta \in \cup_{j=1}^{N}\mathcal{U}_N^j,\ t \in [T]$$

Linear decision rule for production quantity (x) :

$$x_t(\zeta_1,\dots,\zeta_{t-1}) := x_{t,0} + \sum_{s=1}^{t-1} X_{t,s}\cdot\zeta_s$$

Decision on stage t will consider uncertainty until stage $t-1$.

⇒ $t=1$: $x_{1,0}$ ※ No need to consider uncertainty.

⇒ $t=2$: $x_{2,0} + (X_{2,1}\,\zeta_1)$ ※ Consider uncertainty until stage $2-1=1$.

⇒ $t=3$: $x_{3,0} + (X_{3,1}\cdot\zeta_1) + (X_{3,2}\cdot\zeta_2)$ ※ Consider uncertainty until stage $3-1=2$. $(1, 2)$

Linear decision rule for involved cost (y) :

$$y_{t+1}(\zeta_1,\dots,\zeta_t) := y_{t+1,0} + \sum_{s=1}^{t}(Y_{t+1,s}\cdot\zeta_s)$$

Decision on stage $t+1$ will consider uncertainty until stage t

⇒ $t=1$: $y_{2,0} + (Y_{2,1}\,\zeta_1)$

⇒ $t=2$: $y_{3,0} + (Y_{3,1}\cdot\zeta_1) + (Y_{3,2}\cdot\zeta_2)$

⇒ $t=3$: $y_{4,0} + (Y_{4,1}\cdot\zeta_1) + (Y_{4,2}\cdot\zeta_2) + (Y_{4,3}\cdot\zeta_3)$

Linear decision rule for inventory (I) :

$$I_{t+1}(\zeta_1,\dots,\zeta_t) := i_{t+1,0} + \sum_{s=1}^{t}(I_{t+1,s}\,\zeta_s) \; ; \; I_t(\zeta_1,\dots,\zeta_{t-1}) := i_{t,0} + \sum_{s=1}^{t-1}(I_{t,s}\cdot\zeta_s)$$

⇒ $t=1$: $i_{2,0} + (I_{2,1}\cdot\zeta_1)$; $i_{1,0}$ + None

⇒ $t=2$: $i_{3,0} + (I_{3,1}\,\zeta_1) + (I_{3,2}\cdot\zeta_2)$; $i_{2,0} + (I_{2,1}\cdot\zeta_1)$

⇒ $t=3$: $i_{4,0} + (I_{4,1}\cdot\zeta_1) + (I_{4,2}\,\zeta_2) + (I_{4,3}\cdot\zeta_3)$; $i_{3,0} + (I_{3,1}\cdot\zeta_1) + (I_{3,2}\cdot\zeta_2)$

✡ y and I are not independent to j :

$$y_{t+1,0} + \sum_{s=1}^{t}(Y_{t+1,s}\cdot\zeta_s) \longrightarrow y^j_{t+1,0} + \sum_{s=1}^{t}(Y^j_{t+1,s}\cdot\zeta_s)$$

$$i_{t+1,0} + \sum_{s=1}^{t}(I_{t+1,s}\cdot\zeta_s) \longrightarrow i^j_{t+1,0} + \sum_{s=1}^{t}(I^j_{t+1,s}\cdot\zeta_s)$$

Use linear decision rule rewrite problem :

$$\min_{x,y,I} \frac{1}{N}\sum_{j=1}^{N} \sup_{\zeta\in\mathcal{U}_N^j} \sum_{t=1}^{T}\Big\{ C_t\cdot\Big[x_{t,0}+\sum_{s=1}^{t-1}(X_{t,s}\cdot\zeta_s)\Big] + \Big[y_{t+1,0}^j+\sum_{s=1}^{t}(Y_{t+1,s}^j\cdot\zeta_s)\Big]$$

$$\text{s.t.}\ \Big[i_{t+1,0}^j+\sum_{s=1}^{t}(I_{t+1,s}^j\cdot\zeta_s)\Big] \ge \Big[i_{t,0}^j+\sum_{s=1}^{t-1}(I_{t,s}^j\cdot\zeta_s)\Big] + \Big[x_{t,0}+\sum_{s=1}^{t-1}(X_{t,s}\cdot\zeta_s)\Big] - \zeta_t^j \qquad \forall\ t\in[T],\ j\in[N],\ \zeta\in\mathcal{U}_N^j$$

$$\Big[i_{t+1,0}^j+\sum_{s=1}^{t}(I_{t+1,s}^j\cdot\zeta_s)\Big] \le \Big[i_{t,0}^j+\sum_{s=1}^{t-1}(I_{t,s}^j\cdot\zeta_s)\Big] + \Big[x_{t,0}+\sum_{s=1}^{t-1}(X_{t,s}\cdot\zeta_s)\Big] - \zeta_t^j \qquad \forall\ t\in[T],\ j\in[N],\ \zeta\in\mathcal{U}_N^j$$

$$\Big[y_{t+1,0}^j+\sum_{s=1}^{t}(Y_{t+1,s}^j\cdot\zeta_s)\Big] \ge h_t\cdot\Big\{\Big[i_{t,0}^j+\sum_{s=1}^{t-1}(I_{t,s}^j\cdot\zeta_s)\Big] + \Big[x_{t,0}+\sum_{s=1}^{t-1}(X_{t,s}\cdot\zeta_s)\Big] - \zeta_t^j\Big\} \qquad \forall\ t\in[T],\ j\in[N],\ \zeta\in\mathcal{U}_N^j$$

$$\Big[y_{t+1,0}^j+\sum_{s=1}^{t}(Y_{t+1,s}^j\cdot\zeta_s)\Big] \ge -b_t\cdot\Big\{\Big[i_{t,0}^j+\sum_{s=1}^{t-1}(I_{t,s}^j\cdot\zeta_s)\Big] + \Big[x_{t,0}+\sum_{s=1}^{t-1}(X_{t,s}\cdot\zeta_s)\Big] - \zeta_t^j\Big\} \qquad \forall\ t\in[T],\ j\in[N],\ \zeta\in\mathcal{U}_N^j$$

$$0 \le \Big[x_{t,0}+\sum_{s=1}^{t-1}(X_{t,s}\cdot\zeta_s)\Big] \le \overline{x}_t \qquad \forall\ t\in[T],\ j\in[N],\ \zeta\in\mathcal{U}_N^j$$

Introduce epigraph variables . $V_j^t \cdot r_j^t \cdot W_j^t$

$$\min_{x,y,I} \frac{1}{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\big[C_t\cdot V_j^t + r_j^{t+1}\big]$$

$$\text{s.t.}\ V_j^t \ge x_{t,0}+\sum_{s=1}^{t-1}(X_{t,s}\cdot\zeta_s) \qquad \forall\ t\in[T],\ j\in[N],\ \zeta\in\mathcal{U}_N^j$$

$$r_j^t \ge y_{t,0}^j+\sum_{s=1}^{t-1}(Y_{t,s}^j\cdot\zeta_s) \qquad \forall\ t\in[T],\ j\in[N],\ \zeta\in\mathcal{U}_N^j$$

$$W_j^t \ge i_{t,0}^j+\sum_{s=1}^{t-1}(I_{t,s}^j\cdot\zeta_s) \qquad \forall\ t\in[T],\ j\in[N],\ \zeta\in\mathcal{U}_N^j$$

Semi-infinite : use dual method.

$$W_j^{t+1} \ge W_j^t + V_j^t - \zeta_t^j \qquad \forall\ t\in[T],\ j\in[N],\ \zeta\in\mathcal{U}_N^j$$

$$W_j^{t+1} \le W_j^t + V_j^t - \zeta_t^j \qquad \forall\ t\in[T],\ j\in[N],\ \zeta\in\mathcal{U}_N^j$$

$$r_j^{t+1} \ge h_t\cdot\big\{W_j^t + V_j^t - \zeta_t^j\big\} \qquad \forall\ t\in[T],\ j\in[N],\ \zeta\in\mathcal{U}_N^j$$

$$r_j^{t+1} \ge -b_t\cdot\big\{W_j^t + V_j^t - \zeta_t^j\big\} \qquad \forall\ t\in[T],\ j\in[N],\ \zeta\in\mathcal{U}_N^j$$

Semi-infinite : use upper and lower bound.

$$0 \le V_j^t \le \overline{x}_t \qquad \forall\ t\in[T],\ j\in[N]$$

① $\max_{\zeta\in\mathcal{U}_N^j} \sum_{s=1}^{t-1}(X_{t,s})\cdot\zeta_s \le V_j^t - x_{t,0}$ $\quad \forall\ j\in[N],\ t\in[T]$

$\Rightarrow \sum_{s=1}^{t-1}(\alpha_{t,s}\cdot u_s^j - \beta_{t,s}\cdot l_s^j) \le V_j^t - x_{t,0}$ $\quad \forall j\in[N],\ t\in[T]$

$\alpha_{t,s}-\beta_{t,s} = X_{t,s}$ $\quad \forall t\in[T],\ s\in[t-1]$

② $\max_{\zeta\in\mathcal{U}_N^j} \sum_{s=1}^{t-1}(Y_{t,s}^j)\cdot\zeta_s \le r_j^t - y_{t,0}^j$ $\quad \forall j\in[N],\ t\in[T+1]$

$\Rightarrow \sum_{s=1}^{t-1}(\gamma_{t,s}^j\cdot u_s^j - \delta_{t,s}^j\cdot l_s^j) \le r_j^t - y_{t,0}^j$ $\quad \forall j\in[N],\ t\in[T+1]$

$\gamma_{t,s}^j - \delta_{t,s}^j = Y_{t,s}^j$ $\quad \forall j\in[N],\ t\in[T+1],\ s\in[t-1]$

③ $\max_{\zeta\in\mathcal{U}_N^j} \sum_{s=1}^{t-1}(I_{t,s}^j)\,\zeta_s \le W_j^t - i_{t,0}^j$ $\quad \forall j\in[N],\ t\in[T+1]$

$\Rightarrow \sum_{s=1}^{t-1}(\varepsilon_{t,s}^j\cdot u_s^j - \theta_{t,s}^j\cdot l_s^j) \le W_j^t - i_{t,0}^j$ $\quad \forall j\in[N],\ t\in[T+1]$

$\varepsilon_{t,s}^j - \theta_{t,s}^j = I_{t,s}^j$ $\quad \forall j\in[N],\ t\in[T+1],\ s\in[t-1]$

④ Lower bound of inventory : maximize demand

$\Rightarrow$ use $\overline{\zeta}_t^j := \max_{\zeta\in\mathcal{U}_N^j} \zeta_t$

⑤ Upper bound of inventory : minimize demand

$\Rightarrow$ use $\underline{\zeta}_t^j := \min_{\zeta\in\mathcal{U}_N^j} \zeta_t$

⑥ Maximize holding : minimize demand

$\Rightarrow$ use $\underline{\zeta}_t^j$

⑦ Maximize backorder : maximize demand

$\Rightarrow$ use $\overline{\zeta}_t^j$

Eliminate semi-infinite constraints.

$$
\begin{array}{lll}
\min\limits_{x,y,I} & \frac{1}{N}\sum\limits_{j=1}^{N}\sum\limits_{t=1}^{T}\left[c_t \cdot v_j^t + r_j^{t+1}\right] & \\
\text{s.t.} & v_j^t \geq x_{t,0} + \sum\limits_{s=1}^{t-1}\left(\alpha_{t,s}\cdot u_s^j - \beta_{t,s}\cdot l_s^j\right) & \forall\, t\in[T],\, j\in[N] \\
& r_j^t \geq y_{t,0}^j + \sum\limits_{s=1}^{t-1}\left(\gamma_{t,s}^j\cdot u_s^j - \delta_{t,s}^j\cdot l_s^j\right) & \forall\, t\in[T],\, j\in[N] \\
& w_j^t \geq i_{t,0}^j + \sum\limits_{s=1}^{t-1}\left(\varepsilon_{t,s}^j\cdot u_s^j - \theta_{t,s}^j\cdot l_s^j\right) & \forall\, t\in[T],\, j\in[N] \\
& w_j^{t+1} \geq w_j^t + v_j^t - \overline{\zeta_t^j} & \forall\, t\in[T],\, j\in[N] \\
& w_j^{t+1} \leq w_j^t + v_j^t - \underline{\zeta_t^j} & \forall\, t\in[T],\, j\in[N] \\
& r_j^{t+1} \geq h_t\cdot\left\{w_j^t + v_j^t - \underline{\zeta_t^j}\right\} & \forall\, t\in[T],\, j\in[N] \\
& r_j^{t+1} \geq -b_t\cdot\left\{w_j^t + v_j^t - \overline{\zeta_t^j}\right\} & \forall\, t\in[T],\, j\in[N] \\
& 0 \leq v_j^t \leq \overline{x}_t & \forall\, t\in[T],\, j\in[N] \\
& \alpha_{t,s} - \beta_{t,s} = X_{t,s} & \forall\, t\in[T],\, s\in[t-1] \\
& \gamma_{t,s}^j - \delta_{t,s}^j = Y_{t,s}^j & \forall\, j\in[N],\, t\in[T+1],\, s\in[t-1] \\
& \varepsilon_{t,s}^j - \theta_{t,s}^j = I_{t,s}^j & \forall\, j\in[N],\, t\in[T+1],\, s\in[t-1]
\end{array}
$$

Remove epigraph variables:

$$
\begin{array}{lll}
\min\limits_{x,y,I} & \frac{1}{N}\sum\limits_{j=1}^{N}\sum\limits_{t=1}^{T}\left\{c_t\left[x_{t,0} + \sum\limits_{s=1}^{t-1}\left(\alpha_{t,s}\cdot u_s^j - \beta_{t,s}\cdot l_s^j\right)\right] + \left[y_{t+1,0}^j + \sum\limits_{s=1}^{t}\left(\gamma_{t+1,s}^j\cdot u_s^j - \delta_{t+1,s}^j\cdot l_s^j\right)\right]\right\} & \\
\text{s.t.} & \left[i_{t+1,0}^j + \sum\limits_{s=1}^{t}\left(\varepsilon_{t+1,s}^j\, u_s^j - \theta_{t+1,s}^j\cdot l_s^j\right)\right] \geq \left[i_{t,0}^j + \sum\limits_{s=1}^{t-1}\left(\varepsilon_{t,s}^j\cdot u_s^j - \theta_{t,s}^j\cdot l_s^j\right)\right] + \left[x_{t,0} + \sum\limits_{s=1}^{t-1}\left(\alpha_{t,s}\cdot u_s^j - \beta_{t,s}\cdot l_s^j\right)\right] - \overline{\zeta_t^j} & \forall\, t\in[T],\, j\in[N] \\
& \left[i_{t+1,0}^j + \sum\limits_{s=1}^{t}\left(\varepsilon_{t+1,s}^j\, u_s^j - \theta_{t+1,s}^j\cdot l_s^j\right)\right] \leq \left[i_{t,0}^j + \sum\limits_{s=1}^{t-1}\left(\varepsilon_{t,s}^j\cdot u_s^j - \theta_{t,s}^j\cdot l_s^j\right)\right] + \left[x_{t,0} + \sum\limits_{s=1}^{t-1}\left(\alpha_{t,s}\cdot u_s^j - \beta_{t,s}\cdot l_s^j\right)\right] - \underline{\zeta_t^j} & \forall\, t\in[T],\, j\in[N] \\
& \left[y_{t+1,0}^j + \sum\limits_{s=1}^{t}\left(\gamma_{t+1,s}^j\cdot u_s^j - \delta_{t+1,s}^j\cdot l_s^j\right)\right] \geq h_t\cdot\left\{\left[i_{t,0}^j + \sum\limits_{s=1}^{t-1}\left(\varepsilon_{t,s}^j\cdot u_s^j - \theta_{t,s}^j\cdot l_s^j\right)\right] + \left[x_{t,0} + \sum\limits_{s=1}^{t-1}\left(\alpha_{t,s}\cdot u_s^j - \beta_{t,s}\cdot l_s^j\right)\right] - \underline{\zeta_t^j}\right\} & \forall\, t\in[T],\, j\in[N] \\
& \left[y_{t+1,0}^j + \sum\limits_{s=1}^{t}\left(\gamma_{t+1,s}^j\cdot u_s^j - \delta_{t+1,s}^j\cdot l_s^j\right)\right] \geq -b_t\cdot\left\{\left[i_{t,0}^j + \sum\limits_{s=1}^{t-1}\left(\varepsilon_{t,s}^j\cdot u_s^j - \theta_{t,s}^j\cdot l_s^j\right)\right] + \left[x_{t,0} + \sum\limits_{s=1}^{t-1}\left(\alpha_{t,s}\cdot u_s^j - \beta_{t,s}\cdot l_s^j\right)\right] - \overline{\zeta_t^j}\right\} & \forall\, t\in[T],\, j\in[N] \\
& 0 \leq \left[x_{t,0} + \sum\limits_{s=1}^{t-1}\left(\alpha_{t,s}\cdot u_s^j - \beta_{t,s}\cdot l_s^j\right)\right] \leq \overline{x}_t & \forall\, t\in[T],\, j\in[N] \\
& \alpha_{t,s} - \beta_{t,s} = X_{t,s} & \forall\, t\in[T],\, s\in[t-1] \\
& \gamma_{t,s}^j - \delta_{t,s}^j = Y_{t,s}^j & \forall\, j\in[N],\, t\in[T+1],\, s\in[t-1] \\
& \varepsilon_{t,s}^j - \theta_{t,s}^j = I_{t,s}^j & \forall\, j\in[N],\, t\in[T+1],\, s\in[t-1]
\end{array}
$$