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A Data-Driven Approach to Multistage Stochastic Linear Optimization

Dimitris Bertsimas,[a,*] Shimrit Shtern,[b] Bradley Sturt[c]

[a] Operations Research Center, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139; [b] The Faculty of Industrial Engineering and Management, Technion–Israel Institute of Technology, Haifa 3200003, Israel; [c] Department of Information and Decision Sciences, University of Illinois at Chicago, Chicago, Illinois 60607
*Corresponding author

Contact: dbertsim@mit.edu, https://orcid.org/0000-0002-1985-1003 (DB); shimrits@technion.ac.il, https://orcid.org/0000-0001-8398-7178 (SS); bsturt@uic.edu, https://orcid.org/0000-0002-1053-0780 (BS)

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Abstract. We propose a new data-driven approach for addressing multistage stochastic linear optimization problems with unknown distributions. The approach consists of solving a robust optimization problem that is constructed from sample paths of the underlying stochastic process. We provide asymptotic bounds on the gap between the optimal costs of the robust optimization problem and the underlying stochastic problem as more sample paths are obtained, and we characterize cases in which this gap is equal to zero. To the best of our knowledge, this is the first sample path approach for multistage stochastic linear optimization that offers asymptotic optimality guarantees when uncertainty is arbitrarily correlated across time. Finally, we develop approximation algorithms for the proposed approach by extending techniques from the robust optimization literature and demonstrate their practical value through numerical experiments on stylized data-driven inventory management problems.

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1. Introduction

In the traditional formulation of linear optimization, one makes a decision that minimizes a known objective function and satisfies a known set of constraints. Linear optimization has, by all measures, succeeded as a framework for modeling and solving numerous real-world problems. However, in many practical applications, the objective function and constraints are unknown at the time of decision making. To incorporate uncertainty into the linear optimization framework, Dantzig (1955) proposed partitioning the decision variables across multiple stages, which are made sequentially as more uncertain parameters are revealed. This formulation is known today as multistage stochastic linear optimization, which has become an integral modeling paradigm in many applications (e.g., supply chain management, energy planning, finance) and remains a focus of the stochastic optimization community (Shapiro et al. 2009, Birge and Louveaux 2011).

In practice, decision makers increasingly have access to historical data, which can provide valuable insight into future uncertainty. For example, consider a manufacturer that sells short life cycle products. The manufacturer does not know a joint probability distribution of the demand over a new product's life cycle but has access to historical demand trajectories over the life cycle of similar products. Another example is energy planning, where operators must coordinate and commit to production levels throughout a day; the output of wind turbines is subject to uncertain weather conditions, and data on historical daily wind patterns are increasingly available. Other examples include portfolio management, where historical asset returns over time are available to investors, and transportation planning, where data come in the form of historical ride usage of transit and ride-sharing systems over the course of a day. Such historical data provide significant potential for operators to better understand how uncertainty unfolds through time, which can in turn be used for better planning.

When the underlying probability distribution is unknown, data-driven approaches to multistage stochastic linear optimization traditionally follow a two-step procedure. The historical data are first fit to a parametric model (e.g., an autoregressive moving average process), and decisions are then obtained by solving a multistage stochastic linear optimization problem using the estimated distribution. The estimation step is considered essential, as techniques for solving multistage stochastic linear optimization (e.g.,

scenario tree discretization) generally require knowledge of the correlation structure of uncertainty across time; see Shapiro et al. (2009, section 5.8). A fundamental difficulty in this approach is choosing a parametric model that will accurately estimate the underlying correlation structure and lead to good decisions.

Nonparametric data-driven approaches to multistage stochastic linear optimization where uncertainty is correlated across time are surprisingly scarce. Pflug and Pichler (2016) propose a nonparametric estimate-then-optimize approach based on applying a kernel density estimator to the historical data, which enjoys asymptotic optimality guarantees under a variety of strong technical conditions. Hanasusanto and Kuhn (2013) present another nonparametric approach, wherein the conditional distributions in stochastic dynamic programming (DP) are estimated using kernel regression. Krokhmal and Uryasev (2007) discuss nonparametric path-grouping heuristics for constructing scenario trees from historical data. In the case of multistage stochastic linear optimization, to the best of our knowledge, there are no previous nonparametric data-driven approaches that are asymptotically optimal in the presence of time-dependent correlations. Moreover, in the absence of additional assumptions on the estimated distribution or on the problem setting, multistage stochastic linear optimization problems are notorious for being computationally demanding.

The main contribution of this paper is a new data-driven approach for multistage stochastic linear optimization that can be asymptotically optimal even when uncertainty is arbitrarily correlated across time. In other words, we propose a data-driven approach for addressing multistage stochastic linear optimization with unknown distributions that (i) does not require any parametric modeling assumptions on the correlation structure of the underlying probability distribution and (ii) converges to the underlying multistage stochastic linear optimization problem under certain conditions as the size of the data set tends to infinity. Such asymptotic optimality guarantees are of practical importance, as they ensure that the approach offers a near-optimal approximation of the underlying stochastic problem in the presence of big data.

Our approach for multistage stochastic linear optimization is based on robust optimization. Specifically, given sample paths of the underlying stochastic process, the proposed approach consists of constructing and solving a multistage robust linear optimization problem with multiple uncertainty sets. The main result of this paper (Theorem 1) establishes, under certain assumptions, that the optimal cost of this robust optimization problem converges nearly to that of the stochastic problem as the number of sample paths tends to infinity. Although this robust optimization problem is computationally demanding to solve exactly, we provide evidence that it can be tractably approximated to reasonable accuracy by leveraging approximation techniques from the robust optimization literature. To the best of our knowledge, there was no similar work in the literature that addressed multistage stochastic linear optimization by solving a sequence of robust optimization problems.

The paper is organized as follows. Section 2 introduces multistage stochastic linear optimization in a data-driven setting. Section 3 presents the new data-driven approach to multistage stochastic linear optimization. Section 4 states the main asymptotic optimality guarantees. Section 5 presents two examples of approximation algorithms by leveraging techniques from robust optimization. Section 6 discusses implications of our asymptotic optimality guarantees in the context of Wasserstein-based distributionally robust optimization. Sections 7 and 8 demonstrate the practical value of the proposed methodologies in computational experiments. Section 9 offers concluding thoughts. All technical proofs are relegated to the online appendix.

1.1. Related Literature

Originating with Soyster (1973) and Ben-Tal and Nemirovski (1999), robust optimization has been widely studied as a general framework for decision making under uncertainty, in which "optimal" decisions are those that perform best under the worst-case parameter realization from an "uncertainty set." Beginning with the seminal work of Ben-Tal et al. (2004), robust optimization has been viewed with particular success as a computationally tractable framework for addressing multistage problems. Indeed, by restricting the space of decision rules, a stream of literature showed that multistage robust linear optimization problems can be solved in polynomial time by using duality-based reformulations or cutting-plane methods. For a modern overview of decision rule approximations, we refer the reader to Ben-Tal et al. (2009), Bertsimas et al. (2011a), Delage and Iancu (2015), and Georghiou et al. (2018). A variety of non-decision rule approaches to solving multistage robust optimization have been proposed as well, such as Zeng and Zhao (2013), Zhen et al. (2018), Xu and Burer (2018), and Georghiou et al. (2019).

Despite its computational tractability for multistage problems, a central critique of traditional robust optimization is that it does not aspire to find solutions that perform well on average. Several works have aimed to quantify the quality of solutions to multistage robust linear optimization problems from the perspective of multistage stochastic linear optimization (Chen et al. 2007, Bertsimas and Goyal 2010, Bertsimas et al. 2011b). By and large, it is fair to say

that multistage robust linear optimization is viewed today as a distinct framework from multistage stochastic linear optimization, aiming to find solutions with good worst-case performance as opposed to good average performance.

Providing a potential trade-off between the stochastic and robust frameworks, distributionally robust optimization has recently received significant attention. First proposed by Scarf (1958), distributionally robust optimization models the uncertain parameters with a probability distribution, but the distribution is presumed to be unknown and contained in an ambiguity set of distributions. Even though single-stage stochastic optimization is generally intractable, the introduction of ambiguity can surprisingly emit tractable reformulations (Delage and Ye 2010, Wiesemann et al. 2014). Consequently, the extension of distributionally robust optimization to multistage decision making is an active area of research (see Bertsimas et al. 2019 for multistage distributionally robust linear optimization with moment-based ambiguity sets).

There has been a proliferation of data-driven constructions of ambiguity sets that offer various probabilistic performance guarantees, including those based on the p-Wasserstein distance for $p \in [1, \infty)$ (Pflug and Wozabal 2007, Esfahani and Kuhn 2018), phi-divergences (Ben-Tal et al. 2013, Bayraksan and Love 2015, Van Parys et al. 2021), and statistical hypothesis tests (Bertsimas et al. 2018). Many of these data-driven approaches have since been applied to the particular case of two-stage distributionally robust linear optimization, including Jiang and Guan (2018) for phi-divergence and Hanasusanto and Kuhn (2018) for p-Wasserstein ambiguity sets when $p \in [1, \infty)$. To the best of our knowledge, no previous work has demonstrated whether such distributionally robust approaches, if extended to solve multistage stochastic linear optimization (with three or more stages) directly from data, retain their asymptotic optimality guarantees.

In contrast to the literature, our motivation for robust optimization in this paper is not to find solutions that perform well on the worst-case realization in an uncertainty set, are risk averse, or have finite-sample probabilistic guarantees. Rather, our proposed approach to multistage stochastic linear optimization adds robustness to the historical data as a tool to avoid overfitting as the number of data points tends to infinity. In this spirit, our work is perhaps closest related to several papers in the context of machine learning (Xu et al. 2012, Shafieezadeh-Abadeh et al. 2019), which showed that adding robustness to historical data can be used to develop machine learning methods that have nonparametric performance guarantees when the solution space (of classification or regression models) is not finite dimensional. To the best of our knowledge, this paper is the first to apply this use of robust optimization in the context of multistage stochastic linear optimization to achieve asymptotic optimality without restricting the space of decision rules.

As far as we are aware, our data-driven approach of averaging over multiple uncertainty sets is novel in the context of multistage stochastic linear optimization, and its asymptotic optimality guarantees do not follow from existing literature. Xu et al. (2012) considered averaging over multiple uncertainty sets to establish convergence guarantees for predictive machine learning methods, drawing connections with distributionally robust optimization and kernel density estimation. Their convergence results require that the objective function is continuous, the underlying distribution is continuous, and there are no constraints on the support. Absent strong assumptions on the problem setting and on the space of decision rules (which in general, can be discontinuous), these properties do not hold in multistage problems. Erdoğan and Iyengar (2006) provide feasibility guarantees on robust constraints over unions of uncertainty sets with the goal of approximating ambiguous chance constraints using the Prohorov metric. Their probabilistic guarantees require that the constraint functions have a finite VC dimension (Erdoğan and Iyengar 2006, theorem 5), an assumption that does not hold in general for two- or multistage problems (Erdoğan and Iyengar 2007). In this paper, we instead establish general asymptotic optimality guarantees for the proposed data-driven approach for multistage stochastic linear optimization by developing new bounds for distributionally robust optimization with the 1-Wasserstein ambiguity set and connections with nonparametric support estimation (Devroye and Wise 1980).

Under a particular construction of the uncertainty sets, we show that the proposed data-driven approach to multistage stochastic linear optimization can also be interpreted as distributionally robust optimization using the ∞-Wasserstein ambiguity set (see Section 6). However, the asymptotic optimality guarantees in our paper do not make use of this interpretation, as there were surprisingly few previous convergence results for this ambiguity set, even in single-stage settings. Indeed, when an underlying distribution is unbounded, the ∞-Wasserstein distance between an empirical distribution and true distribution is always infinite (Givens and Shortt 1984) and thus, does not converge to zero as more data are obtained. Therefore, it is not possible to develop measure concentration guarantees for the ∞-Wasserstein distance (akin to those of Fournier and Guillin 2015) that hold in general for light-tailed but unbounded probability distributions. Consequently, the proof techniques used by Esfahani and Kuhn (2018, theorem 3.6) to establish

convergence guarantees for the 1-Wasserstein ambiguity set do not appear to extend to the ∞-Wasserstein ambiguity set. As a by-product of the results in this paper, we obtain asymptotic optimality guarantees for distributionally robust optimization with the ∞-Wasserstein ambiguity set under the same mild probabilistic assumptions as Esfahani and Kuhn (2018) for the first time.

1.2. Notation

We denote the real numbers by $\mathbb{R}$, the nonnegative real numbers by $\mathbb{R}_+$, and the integers by $\mathbb{Z}$. Lowercase and uppercase bold letters refer to vectors and matrices. We assume throughout that $\|\cdot\|$ refers to an ℓ_p-norm in $\mathbb{R}^d$, such as $\|\mathbf{v}\|_1 = \sum_{i=1}^d |v_i|$ or $\|\mathbf{v}\|_\infty = \max_{i\in[d]} |v_i|$. We let $\emptyset$ denote the empty set, int$(\cdot)$ be the interior of a set, and $[K]$ be shorthand for the set of consecutive integers $\{1,\ldots,K\}$. Throughout the paper, we let $\boldsymbol{\xi} := (\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_T) \in \mathbb{R}^d$ denote a stochastic process with a joint probability distribution $\mathbb{P}$ and assume that $\widehat{\boldsymbol{\xi}}^1,\ldots,\widehat{\boldsymbol{\xi}}^N$ are independent and identically distributed (i.i.d.) samples from that distribution. Let $\mathbb{P}^N := \mathbb{P}\times\cdots\times\mathbb{P}$ denote the N-fold probability distribution over the historical data. We let $S \subseteq \mathbb{R}^d$ denote the support of $\mathbb{P}$: that is, the smallest closed set where $\mathbb{P}(\boldsymbol{\xi}\in S) = 1$. The extended real numbers are defined as $\overline{\mathbb{R}} := \mathbb{R}\cup\{-\infty,\infty\}$, and we adopt the convention that $\infty - \infty = \infty$. The expectation of a measurable function $f : \mathbb{R}^d \to \mathbb{R}$ applied to the stochastic process is denoted by $\mathbb{E}[f(\boldsymbol{\xi})] = \mathbb{E}_{\mathbb{P}}[f(\boldsymbol{\xi})] = \mathbb{E}_{\mathbb{P}}[\max\{f(\boldsymbol{\xi}),0\}] - \mathbb{E}_{\mathbb{P}}[\max\{-f(\boldsymbol{\xi}),0\}]$. Finally, for any set $\mathcal{Z}\subseteq\mathbb{R}^d$, we let $\mathcal{P}(\mathcal{Z})$ denote the set of all probability distributions on $\mathbb{R}^d$, which satisfy $\mathbb{Q}(\boldsymbol{\xi}\in\mathcal{Z}) \equiv \mathbb{E}_{\mathbb{Q}}[\mathbb{I}\{\boldsymbol{\xi}\in\mathcal{Z}\}] = 1$.

2. Problem Setting

We consider multistage stochastic linear optimization problems with $T \geq 1$ stages. The uncertain parameters observed over the time horizon are represented by a stochastic process $\boldsymbol{\xi} := (\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_T) \in \mathbb{R}^d$ with an underlying joint probability distribution, where $\boldsymbol{\xi}_t \in \mathbb{R}^{d_t}$ is a random variable that is observed immediately after the decision in stage t is selected. We assume throughout that the random variables $\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_T$ may be correlated. A decision rule $\mathbf{x} := (\mathbf{x}_1,\ldots,\mathbf{x}_T)$ is a collection of policies that specify what decision to make in each stage based on the information observed up to that point. More precisely, a policy in each stage is a measurable function of the form $\mathbf{x}_t : \mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_{t-1}} \to \mathbb{R}^{n_t-p_t}\times\mathbb{Z}^{p_t}$. We use the shorthand notation $\mathcal{X}$ to denote the space of all decision rules.

In multistage stochastic linear optimization, our goal is to find a decision rule that minimizes a linear cost function in expectation while satisfying a system of linear inequalities almost surely. These problems are represented by

$$\begin{aligned} \underset{\mathbf{x}\in\mathcal{X}}{\text{minimize}} \quad & \mathbb{E}\left[\sum_{t=1}^T \mathbf{c}_t(\boldsymbol{\xi})\cdot\mathbf{x}_t(\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_{t-1})\right] \\ \text{subject to} \quad & \sum_{t=1}^T \mathbf{A}_t(\boldsymbol{\xi})\mathbf{x}_t(\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_{t-1}) \leq \mathbf{b}(\boldsymbol{\xi}) \quad \text{a.s.} \end{aligned} \tag{1}$$

Following standard convention, we assume that the problem parameters $\mathbf{c}_1(\boldsymbol{\xi}) \in \mathbb{R}^{n_1},\ldots,\mathbf{c}_T(\boldsymbol{\xi}) \in \mathbb{R}^{n_T}$, $\mathbf{A}_1(\boldsymbol{\xi}) \in \mathbb{R}^{m\times n_1},\ldots,\mathbf{A}_T(\boldsymbol{\xi}) \in \mathbb{R}^{m\times n_T}$, and $\mathbf{b}(\boldsymbol{\xi}) \in \mathbb{R}^m$ are affine functions of the stochastic process.

In this paper, we assume that the underlying joint probability distribution of the stochastic process is unknown. Instead, our information comes from historical data of the form

$$\widehat{\boldsymbol{\xi}}^j \equiv (\widehat{\boldsymbol{\xi}}_1^j,\ldots,\widehat{\boldsymbol{\xi}}_T^j), \qquad j = 1,\ldots,N.$$

We refer to each of these trajectories as a sample path of the stochastic process. This setting corresponds to many real-life applications. For example, consider managing the inventory of a new short life cycle product, in which production decisions must be made over the product's life cycle. In this case, each sample path represents the historical sales data observed over the life cycle of a comparable product. Further examples are readily found in energy planning and finance, among many others. We assume that $\widehat{\boldsymbol{\xi}}^1,\ldots,\widehat{\boldsymbol{\xi}}^N$ are i.i.d. realizations of the stochastic process $\boldsymbol{\xi} \equiv (\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_T)$. Our goal in this paper is a general purpose, nonparametric sample path approach for solving Problem (1) in practical computation times.

We will also assume that the support of the stochastic process is unknown. For example, in inventory management, an upper bound on the demand, if one exists, is generally unknown. On the other hand, we often have partial knowledge on the underlying support. For example, when the stochastic process captures the demand for a new product or the energy produced by a wind turbine, it is often the case that the uncertainty will be nonnegative. To allow any partial knowledge on the support to be incorporated, we assume knowledge of a convex superset $\Xi \subseteq \mathbb{R}^d$ of the support of the underlying joint distribution: that is, $\mathbb{P}(\boldsymbol{\xi}\in\Xi) = 1$.

3. A Robust Approach to Multistage Stochastic Linear Optimization

We now present the proposed data-driven approach, based on robust optimization, for solving multistage stochastic linear optimization. First, we construct an uncertainty set $\mathcal{U}_N^j \subseteq \Xi$ around each sample path, consisting of realizations $\boldsymbol{\zeta} \equiv (\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\zeta}_T)$, which are slight perturbations of $\widehat{\boldsymbol{\xi}}^j \equiv (\widehat{\boldsymbol{\xi}}_1^j,\ldots,\widehat{\boldsymbol{\xi}}_T^j)$. Then, we optimize

for decision rules by averaging over the worst-case costs from each uncertainty set and require that the decision rule is feasible for all realizations in all of the uncertainty sets. Formally, the proposed approach is the following:

$$\begin{aligned} \underset{\mathbf{x}\in\mathcal{X}}{\text{minimize}} \quad & \frac{1}{N}\sum_{j=1}^{N}\sup_{\boldsymbol{\zeta}\in\mathcal{U}_N^j}\sum_{t=1}^{T}\mathbf{c}_t(\boldsymbol{\zeta})\cdot\mathbf{x}_t(\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\zeta}_{t-1}) \\ \text{subject to} \quad & \sum_{t=1}^{T}\mathbf{A}_t(\boldsymbol{\zeta})\mathbf{x}_t(\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\zeta}_{t-1})\le\mathbf{b}(\boldsymbol{\zeta}) \quad \forall\,\boldsymbol{\zeta}\in\textstyle\bigcup_{j=1}^{N}\mathcal{U}_N^j. \end{aligned} \tag{2}$$

In contrast to traditional robust optimization, Problem (2) involves averaging over multiple uncertainty sets. Thus, the explicit goal here is to obtain decision rules that perform well on average while simultaneously not overfitting the historical data. We note that Problem (2) only requires that the decision rules are feasible for the realizations in the uncertainty sets. These feasibility requirements are justified when the overlapping uncertainty sets encompass the variability of future realizations of the uncertainty; see Section 4.

Of the various possible constructions of the uncertainty sets, our investigation shall henceforth be focused on uncertainty sets constructed as balls of the form

$$\mathcal{U}_N^j := \left\{\boldsymbol{\zeta}\equiv(\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\zeta}_T)\in\Xi:\ \|\boldsymbol{\zeta}-\widehat{\boldsymbol{\xi}}^j\|\le\epsilon_N\right\},$$

where $\epsilon_N\ge 0$ is a parameter that controls the size of the uncertainty sets. The parameter is indexed by N to allow for the size of the uncertainty sets to change as more data are obtained. The rationale for this particular uncertainty set is threefold. First, it is conceptually simple, requiring only a single parameter to estimate both the expectation in the objective and the support of the distribution in the constraints. Second, under appropriate choice of the robustness parameter, we will show that Problem (2) with these uncertainty sets provides a near-optimal approximation of Problem (1) in the presence of big data (see Section 4). Finally, the uncertainty sets are of similar structure, which can be exploited to obtain tractable reformulations (see Section 5).

Our approach, in a nutshell, uses robust optimization as a tool for solving multistage stochastic linear optimization directly from data. More specifically, we obtain decision rules and estimate the optimal cost of Problem (1) by solving Problem (2). We refer to the proposed data-driven approach for solving multistage stochastic linear optimization problems as *sample* or *sample path* robust optimization. As mentioned previously, the purpose of robustness is to ensure that resulting decision rules do not overfit the historical sample paths. To illustrate this role performed by robustness, we consider the following example.

Example 1. Consider a supplier that aims to satisfy uncertain demand over two phases at minimal cost. The supplier selects an initial production quantity at \$1 per unit after observing preorders and produces additional units at \$2 per unit after the regular orders are received. To determine the optimal production levels, we wish to solve

$$\begin{aligned} \text{minimize} \quad & \mathbb{E}[x_2(\xi_1)+2x_3(\xi_1,\xi_2)] \\ \text{subject to} \quad & x_2(\xi_1)+x_3(\xi_1,\xi_2)\ge\xi_1+\xi_2 \quad \text{a.s.} \\ & x_2(\xi_1),\ x_3(\xi_1,\xi_2)\ge 0 \quad \text{a.s.} \end{aligned} \tag{3}$$

The outputs of the optimization problem are decision rules, $x_2:\mathbb{R}\to\mathbb{R}$ and $x_3:\mathbb{R}^2\to\mathbb{R}$, which specify what production levels to choose as a function of the demands observed up to that point. The joint probability distribution of the demand process $(\xi_1,\xi_2)\in\mathbb{R}^2$ is unknown, and the supplier's knowledge comes from historical demand realizations of past products, denoted by $\left(\widehat{\xi}_1^1,\widehat{\xi}_2^1\right),\ldots,\left(\widehat{\xi}_1^N,\widehat{\xi}_2^N\right)$. For the sake of illustration, suppose we attempted to approximate Problem (3) by choosing the decision rules that perform best when averaging over the historical data without any robustness. Such a sample average approach amounts to solving

$$\begin{aligned} \text{minimize} \quad & \frac{1}{N}\sum_{j=1}^{N}\left(x_2\left(\widehat{\xi}_1^j\right)+2x_3\left(\widehat{\xi}_1^j,\widehat{\xi}_2^j\right)\right) \\ \text{subject to} \quad & x_2\left(\widehat{\xi}_1^j\right)+x_3\left(\widehat{\xi}_1^j,\widehat{\xi}_2^j\right)\ge\widehat{\xi}_1^j+\widehat{\xi}_2^j \qquad \forall\, j\in[N] \\ & x_2\left(\widehat{\xi}_1^j\right),\ x_3\left(\widehat{\xi}_1^j,\widehat{\xi}_2^j\right)\ge 0 \qquad \forall\, j\in[N]. \end{aligned}$$

Suppose that the random variable ξ_1 for preorders has a continuous distribution. In that case, it immediately follows that $\widehat{\xi}_1^1\ne\cdots\ne\widehat{\xi}_1^N$ almost surely, and thus, an optimal decision rule for the optimization problem is

$$x_2(\xi_1)=\begin{cases}\widehat{\xi}_1^j+\widehat{\xi}_2^j, & \text{if } \xi_1=\widehat{\xi}_1^j \text{ for } j\in[N],\\ 0, & \text{otherwise;}\end{cases} \qquad x_3(\xi_1,\xi_2)=0.$$

Unfortunately, these decision rules are *nonsensical* with respect to Problem (3). Indeed, the decision rules will not result in feasible decisions for the true stochastic problem with probability one. Moreover, the optimal cost of the optimization problem will converge almost surely to $\mathbb{E}[\xi_1+\xi_2]$ as the number of sample paths N tends to infinity, which can in general be far from that of the stochastic problem. Clearly,

such a sample average approach results in overfitting, even in big data settings, and thus, provides an unsuitable approximation of Problem (3).

In the following section, we show that the asymptotic overfitting phenomenon illustrated in the example is eliminated by adding robustness to the historical data via Problem (2).

4. Asymptotic Optimality

In this section, we provide asymptotic bounds on the gap between the optimal cost of our robust optimization approach, Problem (2), and the optimal cost of the multistage stochastic linear optimization problem, Problem (1), as the number of sample paths grows to infinity. From a practical standpoint, the bounds suggest that our robust optimization problem can provide a reasonable approximation of the multistage stochastic linear optimization problem in the presence of big data. Moreover, we show that our bounds collapse in particular examples of multistage stochastic linear optimization, in which case our robust optimization approach is guaranteed to be asymptotically optimal. These results can be viewed as significant because of the generality of multistage stochastic optimization problems considered in this paper.

In Section 4.1, we describe the assumptions used in the subsequent convergence results. In Section 4.2, we present the main result of this paper (Theorem 1), which establishes asymptotic lower and bound bounds on our proposed data-driven approach. In Section 4.3, we interpret Theorem 1 through several examples. In Section 4.4, we present asymptotic feasibility guarantees.

4.1. Assumptions

We begin by introducing our assumptions, which will be used for establishing asymptotic optimality guarantees. First, we will assume that the joint probability distribution of the stochastic process satisfies the following light-tail assumption.

Assumption 1. *There exists a constant* $a > 1$ *such that* $b := \mathbb{E}\left[\exp(\|\boldsymbol{\xi}\|^a)\right] < \infty$.

For example, this assumption is satisfied if the stochastic process has a multivariate Gaussian distribution and is not satisfied if the stochastic process has a multivariate exponential distribution. Importantly, Assumption 1 does not require any parametric assumptions on the correlation structure of the random variables across stages, and we do not assume that the coefficient $a > 1$ is known.

Second, we will assume that the robustness parameter ϵ_N is chosen to be strictly positive and decreases to zero as more data are obtained at the following rate.

Assumption 2. *There exists a constant* $\kappa > 0$ *such that* $\epsilon_N := \kappa N^{-\frac{1}{\max\{3,d+1\}}}$.

In a nutshell, Assumption 2 provides a theoretical requirement on how to choose the robustness parameter to ensure that Problem (2) will not overfit the historical data (see Example 1 from Section 3). The rate also provides practical guidance on how the robustness parameter can be updated as more data are obtained. We note that, for many of the following results, the robustness parameter can decrease to zero at a faster rate; nonetheless, we shall impose Assumption 2 for all our results for simplicity.

Finally, our convergence guarantees for Problem (2) do not require any restrictions on the space of decision rules. Our analysis will only require the following assumption on the problem structure.

Assumption 3. *There exists a* $L \geq 0$ *such that, for all* $N \in \mathbb{N}$, *the optimal cost of Problem* (2) *would not change if we added the following constraints*:

$$\sup_{\boldsymbol{\zeta} \in \cup_{j=1}^{N} \mathcal{U}_N^j} \|\mathbf{x}_t(\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_{t-1})\| \leq \sup_{\boldsymbol{\zeta} \in \cup_{j=1}^{N} \mathcal{U}_N^j} L(1 + \|\boldsymbol{\zeta}\|) \quad \forall\, t \in [T].$$

This assumption says that there always exists a near-optimal decision rule to Problem (2) where the decisions that result from realizations in uncertainty sets are bounded by the largest realization in the uncertainty sets. Moreover, this is a mild assumption that we find can be easily verified in many practical examples. In Online Appendix A, we show that every example presented in this paper satisfies this assumption.

4.2. Main Result

We now present the main result of this paper (Theorem 1), which establishes asymptotic lower and upper bounds on the optimal cost of Problem (2). For notational convenience, let J^* be the optimal cost of Problem (1), $\widehat{J}_N$ be the optimal cost of Problem (2), and $S \subseteq \Xi$ be the support of the underlying joint probability distribution of the stochastic process.

First, let $\underline{J}$ be defined as the maximal optimal cost of any chance-constrained variant of the multistage stochastic linear optimization problem:

$$\underline{J} := \lim_{\rho \downarrow 0} \; \underset{\mathbf{x} \in \mathcal{X},\, \tilde{S} \subseteq \Xi}{\text{minimize}} \; \mathbb{E}\left[\sum_{t=1}^{T} \mathbf{c}_t(\boldsymbol{\xi}) \cdot \mathbf{x}_t(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{t-1}) \mathbb{I}\{\boldsymbol{\xi} \in \tilde{S}\}\right]$$
$$\text{subject to} \sum_{t=1}^{T} \mathbf{A}_t(\boldsymbol{\zeta}) \mathbf{x}_t(\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_{t-1}) \leq \mathbf{b}(\boldsymbol{\zeta}) \quad \forall\, \boldsymbol{\zeta} \in \tilde{S}$$
$$\mathbb{P}\left(\boldsymbol{\xi} \in \tilde{S}\right) \geq 1 - \rho.$$

We observe that the limit must exist, as the optimal cost of the chance-constrained optimization problem is monotone in ρ. We also observe that $\underline{J}$ is always a lower bound on J^* because for every $\rho > 0$, adding the constraint $\mathbb{P}(\boldsymbol{\xi} \in \tilde{S}) = 1$ on the decision variable $\tilde{S}$ to the chance-constrained optimization problem would increase its optimal cost to J^*.[1]

Second, let $\overline{J}$ be the optimal cost of the multistage stochastic linear optimization problem with an additional restriction that the decision rules are feasible on an expanded support:

$$\overline{J} := \lim_{\rho \downarrow 0} \quad \underset{\mathbf{x} \in \mathcal{X}}{\text{minimize}} \ \overline{\mathbb{E}}\left[\sum_{t=1}^{T} \mathbf{c}_t(\boldsymbol{\xi}) \cdot \mathbf{x}_t(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{t-1})\right]$$
$$\text{subject to} \sum_{t=1}^{T} \mathbf{A}_t(\boldsymbol{\zeta}) \mathbf{x}_t(\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_{t-1}) \leq \mathbf{b}(\boldsymbol{\zeta})$$
$$\forall\, \boldsymbol{\zeta} \in \Xi : \operatorname{dist}(\boldsymbol{\zeta}, S) \leq \rho.$$

We remark that the limit as ρ tends down to zero must exist as well because the optimal cost of the optimization problem with expanded support is monotone in ρ. Note also that the expectation in the objective function has been replaced with $\overline{\mathbb{E}}[\cdot]$, which we define here as the local upper semicontinuous envelope of an expectation (i.e., $\overline{\mathbb{E}}[f(\boldsymbol{\xi})] := \lim_{\epsilon \to 0} \mathbb{E}[\sup_{\boldsymbol{\zeta} \in \Xi : \|\boldsymbol{\zeta} - \boldsymbol{\xi}\| \leq \epsilon} f(\boldsymbol{\zeta})]$). We similarly observe that $\overline{J}$ is an upper bound on J^* because the optimization problem involves additional constraints and an upper envelope of the objective function.

Our main result is the following.

Theorem 1. *Suppose Assumptions 1–3 hold. Then, $\mathbb{P}^\infty$-almost surely, we have*

$$\underline{J} \leq \liminf_{N \to \infty} \widehat{J}_N \leq \limsup_{N \to \infty} \widehat{J}_N \leq \overline{J}.$$

Proof. See Online Appendix C. □

Note that Theorem 1 holds in very general cases; for example, it does not require boundedness on the decisions or random variables, requires no parametric assumptions on the correlations across stages, and holds when the decisions contain both continuous and integer components. Moreover, these asymptotic bounds for Problem (2) do not necessitate imposing any restrictions on the space of decision rules. To the best of our knowledge, such nonparametric convergence guarantees for a sample path approach to multistage stochastic linear optimization are the first of their kind when uncertainty is correlated across time.

Our proof of Theorem 1 is based on a new uniform convergence result (Theorem 2), which establishes a general relationship for arbitrary functions between the in-sample worst-case cost and the expected out-of-sample cost over the uncertainty sets. We state this theorem because of its independent interest.

Theorem 2. *If Assumptions 1 and 2 hold, then there exists a $\overline{N} \in \mathbb{N}$, $\mathbb{P}^\infty$-almost surely, such that*

$$\mathbb{E}\left[f(\boldsymbol{\xi}) \mathbb{I}\left\{\boldsymbol{\xi} \in \bigcup_{j=1}^{N} \mathcal{U}_N^j\right\}\right] \leq \frac{1}{N} \sum_{j=1}^{N} \sup_{\boldsymbol{\zeta} \in \mathcal{U}_N^j} f(\boldsymbol{\zeta}) + M_N \sup_{\boldsymbol{\zeta} \in \bigcup_{j=1}^{N} \mathcal{U}_N^j} |f(\boldsymbol{\zeta})|$$

for all $N \geq \overline{N}$ and all measurable functions $f : \mathbb{R}^d \to \mathbb{R}$, where $M_N := N^{-\frac{1}{(d+1)(d+2)}} \log N$.

Proof. See Online Appendix D. □

We note that our proofs of Theorems 1 and 2 also utilize a feasibility guarantee (Theorem 3), which can be found Section 4.4.

4.3. Examples Where $\overline{J} - \underline{J}$ Is Zero or Strictly Positive

Theorem 1 establishes asymptotic lower and upper bounds on the optimal cost of Problem (2). We next show that the lower and upper bounds can be equal, $\underline{J} = \overline{J}$, in which case the optimal cost of Problem (2) provably converges to the optimal cost of Problem (1). We first show that these bounds can be equal by revisiting the stochastic inventory management problem from Example 1.

Proposition 1. *For Problem (3), $\underline{J} = J^*$. If there is an optimal $x_2^* : \mathbb{R} \to \mathbb{R}$ for Problem (3), which is continuous, then $\overline{J} = J^*$.*

Proof. See Online Appendix B. □

The proof of this proposition holds for any underlying probability distribution, which satisfies Assumption 1. In combination with Theorem 1, the proposition shows that adding robustness to the historical data provably overcomes the asymptotic overfitting phenomenon discussed in Section 3.

More generally, the equality of the lower bound $\underline{J}$ and upper bound $\overline{J}$ can be established for much broader classes of problems than Example 1. In Online Appendix B, we provide sufficient conditions for the lower and upper bounds to be equal and demonstrate that these sufficient conditions for asymptotic optimality can be satisfied in the real-world applications studied in Sections 7 and 8.

Unless restrictions are placed on the space of multistage stochastic linear optimization problems, we show next that the lower and upper bounds can have a nonzero gap. In the following, we present three examples that provide intuition on the situations in which this gap may be strictly positive. Our first example presents a problem in which the lower bound $\underline{J}$ is equal to J^* but is strictly less than the upper bound $\overline{J}$.

Example 2. Consider the single-stage stochastic problem

$$\begin{array}{ll}\underset{x_1\in\mathbb{Z}}{\text{minimize}} & x_1 \\ \text{subject to} & x_1 \geq \xi_1 \quad \text{a.s.},\end{array}$$

where the random variable-ξ_1 is governed by the probability distribution $\mathbb{P}(\xi_1 > \alpha) = (1-\alpha)^k$ for fixed $k > 0$ and $\Xi = [0,2]$. We observe that the support of the random variable is $S = [0,1]$, and thus, the optimal cost of the stochastic problem is $J^* = 1$. We similarly observe that the lower bound is $\underline{J} = 1$ and that the upper bound, because of the integrality of the first-stage decision, is $\overline{J} = 2$. If $\epsilon_N = N^{-\frac{1}{3}}$, then we prove in Online Appendix E that the bounds in Theorem 1 are tight under different choices of k (Table 1).

This example shows that gaps can arise between the lower and upper bounds when mild changes in the support of the underlying probability distribution lead to significant changes in the optimal cost of Problem (1). Moreover, this example illustrates that each of the inequalities in Theorem 1 can hold with equality or strict inequality when the feasibility of decisions depends on random variables that have not yet been realized.

Our second example presents a problem in which the upper bound $\overline{J}$ is equal to J^* but is strictly greater than the lower bound $\underline{J}$. This example deals with the special case in which any chance-constrained version of a stochastic problem leads to a decision that is infeasible for the true stochastic problem.

Example 3. Consider the single-stage stochastic problem

$$\begin{array}{ll}\underset{x_1\in\mathbb{R}^2}{\text{minimize}} & x_{12} \\ \text{subject to} & \xi_1(1-x_{12}) \leq x_{11} \quad \text{a.s.}, \\ & 0 \leq x_{12} \leq 1,\end{array}$$

where $\xi_1 \sim \text{Gaussian}(0,1)$ and $\Xi = \mathbb{R}$. The constraints are satisfied only if $x_{12} = 1$, and so, the optimal cost of the stochastic problem is $J^* = 1$. Because there is no expectation in the objective and Ξ equals the true support, we also observe that $\overline{J} = 1$. However, we readily observe that there is always a feasible solution to the sample robust optimization problem (Problem (2)), where $x_{12} = 0$, and therefore, $\underline{J} = \widehat{J}_N = 0$ for all $N \in \mathbb{N}$.

Our third and final example demonstrates the necessity of the upper semicontinuous envelope $\overline{\mathbb{E}}[\cdot]$ in the definition of the upper bound.

Example 4. Consider the two-stage stochastic problem

$$\begin{array}{ll}\underset{x_2:\mathbb{R}\to\mathbb{Z}}{\text{minimize}} & \mathbb{E}[x_2(\xi_1)] \\ \text{subject to} & x_2(\xi_1) \geq \xi_1 \quad \text{a.s.},\end{array}$$

Table 1. Results for Example 2

Range of k	Result
$k \in (0,3)$	$\mathbb{P}^\infty\left(\underline{J} < \liminf_{N\to\infty} \widehat{J}_N = \limsup_{N\to\infty} \widehat{J}_N = \overline{J}\right) = 1$
$k = 3$	$\mathbb{P}^\infty\left(\underline{J} = \liminf_{N\to\infty} \widehat{J}_N < \limsup_{N\to\infty} \widehat{J}_N = \overline{J}\right) = 1$
$k \in (3,\infty)$	$\mathbb{P}^\infty\left(\underline{J} = \liminf_{N\to\infty} \widehat{J}_N = \limsup_{N\to\infty} \widehat{J}_N < \overline{J}\right) = 1$

where $\theta \sim \text{Bernoulli}(0.5)$ and $\psi \sim \text{Uniform}(0,1)$ are independent random variables, $\xi_1 = \theta\psi$, and $\Xi = [0,1]$. An optimal decision rule $x_2^* : \mathbb{R} \to \mathbb{Z}$ to the stochastic problem is given by $x_2^*(\xi_1) = 0$ for all $\xi_1 \leq 0$ and $x_2^*(\xi_1) = 1$ for all $\xi_1 > 0$, which imply that $J^* = \frac{1}{2}$. It follows from similar reasoning that $\underline{J} = \frac{1}{2}$. Because Ξ equals the support of the random variable, the only difference between the stochastic problem and the upper bound is that the latter optimizes over the local upper semicontinuous envelope, and we observe that $\lim_{N\to\infty} \widehat{J}_N = \overline{J} = \overline{\mathbb{E}}[x_2^*(\xi_1)] = 1$.

In each of the examples, we observe that the bounds in Theorem 1 are tight, in the sense that the optimal cost of Problem (2) converges either to the lower bound or to the upper bound. This provides some indication that the bounds in Theorem 1 offer an accurate depiction of how Problem (2) can behave in the asymptotic regime. On the other hand, the examples that illustrate a nonzero gap seem to require intricate construction, and future work may identify subclasses of Problem (1) where the equality of the bounds can be ensured.

4.4. Feasibility Guarantees

We conclude Section 4 by discussing out-of-sample feasibility guarantees for decision rules obtained from Problem (2). Recall that Problem (2) finds decision rules that are feasible for each realization in the uncertainty sets. However, one cannot guarantee that these decision rules will be feasible for realizations outside of the uncertainty sets. Thus, a pertinent question is whether a decision rule obtained from approximately solving Problem (2) is feasible with high probability. To address the question of feasibility, we leverage classic results from detection theory.

Let $S_N := \cup_{j=1}^N \mathcal{U}_N^j$ be shorthand for the union of the uncertainty sets. We say that a decision rule is S_N feasible if

$$\sum_{t=1}^{T} \mathbf{A}_t(\boldsymbol{\zeta})\mathbf{x}_t(\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\zeta}_{t-1}) \leq \mathbf{b}(\boldsymbol{\zeta}) \quad \forall\, \boldsymbol{\zeta} \in S_N.$$

In other words, the set of feasible decision rules to Problem (2) is exactly those that are S_N feasible. Our subsequent analysis utilizes the following (seemingly

tautological) observation; for any decision rule that is S_N feasible,

$$\mathbb{P}\left(\sum_{t=1}^{T}\mathbf{A}_t(\boldsymbol{\xi})\mathbf{x}_t(\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_{t-1})\le\mathbf{b}(\boldsymbol{\xi})\right)\ge\mathbb{P}(\boldsymbol{\xi}\in S_N),$$

where $\mathbb{P}(\boldsymbol{\xi}\in S_N)$ is shorthand for $\mathbb{P}(\boldsymbol{\xi}\in S_N\mid\widehat{\boldsymbol{\xi}}^1,\ldots,\widehat{\boldsymbol{\xi}}^N)$. Indeed, this inequality follows from the fact that a decision rule that is S_N feasible is definitionally feasible for all realizations $\boldsymbol{\zeta}\in S_N$, and thus, the probability of feasibility is at least the probability that $\boldsymbol{\xi}\in S_N$.

We have thus transformed the analysis of feasible decision rules for Problem (2) to the problem of analyzing the performance of S_N as an estimate for the support S of a stochastic process. Interestingly, this nonparametric estimator for the support of a joint probability distribution has been widely studied in the statistics literature, with perhaps the earliest results coming from Devroye and Wise (1980) in detection theory. Since then, the performance of S_N as a nonparametric estimate of S has been studied with applications in cluster analysis and image recognition (Korostelev and Tsybakov 1993, Schölkopf et al. 2001). Leveraging this connection between stochastic optimization and support estimation, we obtain the following guarantee on feasibility.

Theorem 3. *Suppose Assumptions* 1 and 2 *hold. Then,* $\mathbb{P}^\infty$*-almost surely we have*

$$\lim_{N\to\infty}\left(\frac{N^{\frac{1}{d+1}}}{(\log N)^{d+1}}\right)\mathbb{P}(\boldsymbol{\xi}\notin S_N)=0.$$

Proof. See Online Appendix F. □

Intuitively speaking, Theorem 3 provides a guarantee that *any* feasible decision rule to Problem (2) will be feasible with high probability on future data when the number of sample paths is large. To illustrate why robustness is indeed necessary to achieve such feasibility guarantees, we recall from Example 1 that decision rules may prohibitively overfit the data and be infeasible with probability one if the robustness parameter ϵ_N is set to zero.

5. Approximation Techniques

In the previous section, we developed theoretical guarantees that demonstrated that Problem (2) provides a good approximation of multistage stochastic linear optimization when the number of sample paths is large. In this section, we demonstrate that Problem (2) can be addressed using approximation techniques from the field of robust optimization. Specifically, we show that two decision rule approximation schemes from robust optimization, *linear decision rules* and *finite adaptability*, can be extended to obtain approximations of Problem (2). In particular, we present a novel duality argument (Theorem 4), which allows the computational cost of these techniques to scale efficiently in the number of sample paths. The computational tractability and out-of-sample performance of these approximation schemes are illustrated via numerical experiments in Sections 7 and 8.

5.1. Linear Decision Rules

Generally speaking, multistage optimization problems are computationally demanding because of optimizing over an unrestricted space of decision rules. To overcome this challenge, a common approximation technique in robust optimization is to restrict the space of decision rules to a space that can more easily be optimized. As described in Section 1.1, the success of robust optimization as a modeling framework for addressing real-world multistage problems is often attributed to the computational tractability of such decision rule approximations. This section extends one such decision rule scheme, known as linear decision rules, to approximately solve Problem (2) and illustrates its computational tractability in big data settings.

Specifically, we consider approximating Problem (2) by restricting its decision rules to those of the form

$$\mathbf{x}_t(\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\zeta}_{t-1})=\mathbf{x}_{t,0}+\sum_{s=1}^{t-1}\mathbf{X}_{t,s}\boldsymbol{\zeta}_s.$$

Thus, rather than optimizing over the space of all possible decision rules (functions), we instead optimize over a finite collection of decision variables that parameterize a linear decision rule. For the setting where $\mathbf{c}_t(\boldsymbol{\xi})$ and $\mathbf{A}_t(\boldsymbol{\xi})$ do not depend on the uncertain parameters and all decision variables are continuous, the resulting linear decision rule approximation of Problem (2) is given by

$$\begin{aligned}\text{minimize}\quad & \frac{1}{N}\sum_{j=1}^{N}\sup_{\boldsymbol{\zeta}\in\mathcal{U}_N^j}\sum_{t=1}^{T}\mathbf{c}_t\cdot\left(\mathbf{x}_{t,0}+\sum_{s=1}^{t-1}\mathbf{X}_{t,s}\boldsymbol{\zeta}_s\right)\\ \text{subject to}\quad & \sum_{t=1}^{T}\mathbf{A}_t\left(\mathbf{x}_{t,0}+\sum_{s=1}^{t-1}\mathbf{X}_{t,s}\boldsymbol{\zeta}_s\right)\le\mathbf{b}(\boldsymbol{\zeta})\quad\forall\,\boldsymbol{\zeta}\in\cup_{j=1}^{N}\mathcal{U}_N^j,\end{aligned}\tag{4}$$

where the decision variables are $\mathbf{x}_{t,0}\in\mathbb{R}^{n_t}$ and $\mathbf{X}_{t,s}\in\mathbb{R}^{n_t\times d_s}$ for all $1\le s<t\le T$ and the affine function $\mathbf{b}(\boldsymbol{\zeta})\in\mathbb{R}^m$ is shorthand for $\mathbf{b}^0+\sum_{t=1}^{T}\mathbf{B}_t\boldsymbol{\zeta}_t$.

Much like linear decision rules in robust optimization, we observe that Problem (4), when feasible, always produces a feasible decision rule for Problem (2) and an upper bound on its optimal cost. Nonetheless, Problem (4) has semi-infinite constraints, which must be eliminated in order for the optimization problem to be solvable by off-the-shelf solvers. A standard

technique from robust optimization for eliminating semiinfinite constraints is to introduce (dual) auxiliary decision variables and constraints for each uncertainty set. Importantly, for Problem (4) to be practically tractable in the presence of big data, the size of an equivalent finite-dimensional optimization problem must scale efficiently in the number of sample paths.

We now show that Problem (4) can be reformulated as a linear optimization with size that scales linearly in the number of sample paths (Theorem 4). The central idea enabling the following reformulation is that the worst-case realizations over the various uncertainty sets are found by optimizing over identical linear functions. Thus, when constructing the robust counterparts for each uncertainty set, we can combine the dual auxiliary decision variables from different uncertainty sets, resulting in a reformulation where the number of auxiliary decision variables is independent of the number of sample paths. To illustrate this reformulation technique, we focus on uncertainty sets that satisfy the following construction.

Assumption 4. *The uncertainty sets have the form* $\mathcal{U}_N^j := \{\boldsymbol{\zeta} \in \mathbb{R}^d : \boldsymbol{\ell}^j \le \boldsymbol{\zeta} \le \mathbf{u}^j\}$.

For example, Assumption 4 holds if we choose the set Ξ to be $\mathbb{R}_+^d$ and use the $\|\cdot\|_\infty$ norm in the uncertainty sets from Section 3. The following illustrates the novel duality technique described.

Theorem 4. *If Assumption 4 holds, then Problem (4) can be reformulated as a linear optimization problem with $O(md)$ auxiliary decision variables and $O(md + mN)$ linear constraints.*

Proof. By introducing epigraph variables $v_1, \ldots, v_N \in \mathbb{R}$, the constraints in Problem (4) can be rewritten as

$$\sum_{t=1}^{T}\left(\sum_{s=t+1}^{T}\mathbf{X}_{s,t}^\top \mathbf{c}_s\right)\cdot\boldsymbol{\zeta}_t \le v_j - \sum_{t=1}^{T}\mathbf{c}_t\cdot\mathbf{x}_{t,0} \qquad \forall\, \boldsymbol{\zeta}\in\mathcal{U}_N^j,\ j\in\{1,\ldots,N\},$$

$$\sum_{t=1}^{T}\left(-\mathbf{B}_t + \sum_{s=t+1}^{T}\mathbf{A}_s\mathbf{X}_{t,s}\right)\boldsymbol{\zeta}_t \le \mathbf{b}^0 - \sum_{t=1}^{T}\mathbf{A}_t\mathbf{x}_{t,0} \qquad \forall\, \boldsymbol{\zeta}\in\mathcal{U}_N^j,\ j\in\{1,\ldots,N\}.$$

We will now reformulate each of these semiinfinite constraints by introducing auxiliary variables. First, we observe that each of the semiinfinite constraints can be rewritten as

$$\max_{\boldsymbol{\zeta}\in\mathcal{U}_N^j}\sum_{t=1}^{T}\mathbf{d}_t\cdot\boldsymbol{\zeta}_t \le \gamma$$

for some vector $\mathbf{d} := (\mathbf{d}_1, \ldots, \mathbf{d}_T) \in \mathbb{R}^d$ and scalar $\gamma \in \mathbb{R}$. Moreover, it follows from strong duality for linear optimization that

$$\max_{\boldsymbol{\zeta}\in\mathcal{U}_N^j}\sum_{t=1}^{T}\mathbf{d}_t\cdot\boldsymbol{\zeta}_t = \begin{cases}\underset{\boldsymbol{\mu}_t,\boldsymbol{\lambda}_t\in\mathbb{R}_+^{d_t}}{\text{minimize}} & \displaystyle\sum_{t=1}^{T}\left(\mathbf{u}_t^j\cdot\boldsymbol{\mu}_t - \boldsymbol{\ell}_t^j\cdot\boldsymbol{\lambda}_t\right)\\ \text{subject to} & \boldsymbol{\mu}_t - \boldsymbol{\lambda}_t = \mathbf{d}_t \quad \forall\, t\in[T],\end{cases}$$

where $\mathbf{u}^j := (\mathbf{u}_1^j, \ldots, \mathbf{u}_T^j) \in \mathbb{R}^d$ and $\boldsymbol{\ell}^j := (\boldsymbol{\ell}_1^j, \ldots, \boldsymbol{\ell}_T^j) \in \mathbb{R}^d$ are the upper and lower bounds that define the uncertainty set. We readily observe that the solutions $\boldsymbol{\mu}_t = [\mathbf{d}_t]_+$ and $\boldsymbol{\lambda}_t = [-\mathbf{d}_t]_+$ are optimal for the optimization problem. Importantly, these optimal solutions to the dual problem are *independent* of the index j. Thus, the semiinfinite constraints in the epigraph formulation of Problem (4) are satisfied if and only if there exists $\boldsymbol{\alpha} := (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_T) \in \mathbb{R}_+^d$ and $\boldsymbol{\beta} := (\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_T) \in \mathbb{R}_+^d$, which satisfy

$$\sum_{t=1}^{T}\left(\boldsymbol{\alpha}_t\cdot\mathbf{u}_t^j - \boldsymbol{\beta}_t\cdot\boldsymbol{\ell}_t^j + \mathbf{c}_t\cdot\mathbf{x}_{t,0}\right) \le v_j \qquad \forall\, j\in[N]$$

$$\boldsymbol{\alpha}_t - \boldsymbol{\beta}_t = \sum_{s=t+1}^{T}\mathbf{X}_{s,t}^\top\mathbf{c}_s \qquad \forall\, t\in[T],$$

and there exists $\mathbf{M} := (\mathbf{M}_1, \ldots, \mathbf{M}_T) \in \mathbb{R}_+^{m\times d}$ and $\boldsymbol{\Lambda} := (\boldsymbol{\Lambda}_1, \ldots, \boldsymbol{\Lambda}_T) \in \mathbb{R}_+^{m\times d}$, which satisfy

$$\sum_{t=1}^{T}\left(\mathbf{M}_t\mathbf{u}_t^j - \boldsymbol{\Lambda}_t\boldsymbol{\ell}_t^j + \mathbf{A}_t\mathbf{x}_{t,0}\right) \le \mathbf{b}^0 \qquad \forall\, j\in[N]$$

$$\mathbf{M}_t - \boldsymbol{\Lambda}_t = -\mathbf{B}_t + \sum_{s=t+1}^{T}\mathbf{A}_s\mathbf{X}_{t,s} \qquad \forall\, t\in[T].$$

Removing the epigraph decision variables, the resulting reformulation of Problem (4) is

$$\begin{aligned}\text{minimize}\quad & \frac{1}{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\left(\boldsymbol{\alpha}_t\cdot\mathbf{u}_t^j - \boldsymbol{\beta}_t\cdot\boldsymbol{\ell}_t^j + \mathbf{c}_t\cdot\mathbf{x}_{t,0}\right)\\ \text{subject to}\quad & \boldsymbol{\alpha}_t - \boldsymbol{\beta}_t = \sum_{s=t+1}^{T}\mathbf{X}_{s,t}^\top\mathbf{c}_s \qquad t\in[T]\\ & \sum_{t=1}^{T}\left(\mathbf{M}_t\mathbf{u}_t^j - \boldsymbol{\Lambda}_t\boldsymbol{\ell}_t^j + \mathbf{A}_t\mathbf{x}_{t,0}\right) \le \mathbf{b}^0 \qquad j\in[N]\\ & \mathbf{M}_t - \boldsymbol{\Lambda}_t = -\mathbf{B}_t + \sum_{s=t+1}^{T}\mathbf{A}_s\mathbf{X}_{t,s} \qquad t\in[T],\end{aligned}$$

where the auxiliary decision variables are $\boldsymbol{\alpha} \equiv (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_T), \boldsymbol{\beta} \equiv (\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_T) \in \mathbb{R}_+^d$ and $\mathbf{M} \equiv (\mathbf{M}_1, \ldots, \mathbf{M}_T), \boldsymbol{\Lambda} \equiv (\boldsymbol{\Lambda}_1, \ldots, \boldsymbol{\Lambda}_T) \in \mathbb{R}_+^{m\times d}$. Thus, the reformulation technique allowed us to decrease the number of auxiliary decision variables from $O(Nmd)$ to $O(md)$. □

Although linear decision rules can sometimes provide a near-optimal approximation of Problem (2) (see Section 8), we do not expect this to be the case in general. Indeed, we recall from Section 4 that Problem (2) can provide a near-optimal approximation of Problem (1), and it has been known from the early literature that linear decision rules generally provide a poor approximation for multistage stochastic linear optimization (see, e.g., Garstka and Wets 1974, section 6).

Nonetheless, we can obtain tighter approximations of Problem (2) by selecting a richer space of decision rules, an abundance of which can be found in the robust optimization literature (see Section 5.2 for an example). Moreover, Problem (2) is also amenable to new approximation schemes that exploit its particular structure; we refer to our companion paper Bertsimas et al. (2022) for such an approximation algorithm for two-stage problems. In all cases and as a result of the convergence guarantees from Section 4, Problem (2) offers an opportunity to extend algorithmic advances from robust optimization to obtain approximations of multistage stochastic linear optimization.

5.2. Finite Adaptability

In this section, we show how to extend the decision rule approximation scheme of finite adaptability from robust optimization (Bertsimas and Caramanis 2010) to obtain tighter approximations of Problem (2). Specifically, finite adaptability partitions the set Ξ into smaller regions and then, optimizes a separate static or linear decision rule in each region. The approach of finite adaptability extends to problems with integer decision variables, and the practitioner can trade off the tightness of their approximations with an increase in computational cost. We show that the duality techniques from the previous section (Theorem 4) readily extend to this richer class of decision rules, and a practical demonstration of finite adaptability is presented in Section 7.

We begin by describing the approximation scheme of finite adaptability from robust optimization. In finite adaptability, one partitions the uncertainty set into different regions and optimizes a separate linear decision rule for each region. Let $P^1,\dots,P^K \subseteq \mathbb{R}^d$ be regions that form a partition of $\Xi \subseteq \mathbb{R}^d$. For each stage t, let $P_t^k \subseteq \mathbb{R}^{d_1+\cdots+d_t}$ be the projection of the region P^k onto the first t stages. Then, we consider approximating Problem (2) by restricting its decision rules to those of the form

$$\mathbf{x}_t(\boldsymbol{\zeta}_1,\dots,\boldsymbol{\zeta}_{t-1}) = \begin{cases} \mathbf{x}_{t,0}^1 + \sum_{s=1}^{t-1} \mathbf{X}_{t,s}^1 \boldsymbol{\zeta}_s, & \text{if } (\boldsymbol{\zeta}_1,\dots,\boldsymbol{\zeta}_{t-1}) \in P_{t-1}^1, \\ \vdots & \\ \mathbf{x}_{t,0}^K + \sum_{s=1}^{t-1} \mathbf{X}_{t,s}^K \boldsymbol{\zeta}_s, & \text{if } (\boldsymbol{\zeta}_1,\dots,\boldsymbol{\zeta}_{t-1}) \in P_{t-1}^K. \end{cases}$$

In contrast to a single linear decision rule, finite adaptability allows for greater degrees of freedom at a greater computational cost. Indeed, for each region P^k, we choose a separate linear decision rule, which is locally optimal for that region. To accommodate integer decision variables, we restrict the corresponding component of each $\mathbf{x}_{t,0}^k$ to be integer and restrict the associated rows of each matrix $\mathbf{X}_{t,s}^k$ to be zero.

A complication of finite adaptability is that we may not have enough information at any intermediary stage to determine which region P^k will contain the entire trajectory. In other words, at the start of stage t, a decision must be chosen after only observing the values of $(\boldsymbol{\zeta}_1,\dots,\boldsymbol{\zeta}_{t-1})$, and there may be two or more regions of the partition for which their projections P_{t-1}^k and $P_{t-1}^{k'}$ are overlapping. Fortunately, the following proposition shows that the aforementioned complication caused by overlapping projections can be resolved by adding constraints of the form $\mathbf{x}_t^k = \mathbf{x}_t^{k'}$ and $\mathbf{X}_{t,s}^k = \mathbf{X}_{t,s}^{k'}$ for every $1 \le s < t$ when the regions P^k and $P^{k'}$ are indistinguishable at stage t.

Proposition 2 (Proposition 4 in Bertsimas and Dunning 2016). *If there exists $\boldsymbol{\zeta} \equiv (\boldsymbol{\zeta}_1,\dots,\boldsymbol{\zeta}_T) \in P^k$ and $\boldsymbol{\zeta}' \equiv (\boldsymbol{\zeta}'_1,\dots,\boldsymbol{\zeta}'_T) \in P^{k'}$ such that $(\boldsymbol{\zeta}_1,\dots,\boldsymbol{\zeta}_{t-1}) = (\boldsymbol{\zeta}'_1,\dots,\boldsymbol{\zeta}'_{t-1})$ and $\boldsymbol{\zeta} \in \operatorname{int}(P^{k'})$ or $\boldsymbol{\zeta}' \in \operatorname{int}(P^k)$ hold, then we must enforce the constraints that $\mathbf{x}_{t,0}^k = \mathbf{x}_{t,0}^{k'}$ and $\mathbf{X}_{t,s}^k = \mathbf{X}_{t,s}^{k'}$ for all $1 \le s < t$ at stage t as the two regions cannot be distinguished with the uncertain parameters realized by that stage. Otherwise, we do not need to enforce any constraints at stage t for this pair.*

For brevity, we let $\mathcal{T}(P^1,\dots,P^K)$ denote the collection of tuples (k,k',t) for which P^k and $P^{k'}$ cannot be distinguished at stage t, which we assume can be tractably computed.

We now extend the approach of finite adaptability to Problem (2). Let $P^1,\dots,P^K$ be a given partition of Ξ, and let the intersections between regions of the partition and uncertainty sets be denoted by $\mathcal{K}_j := \{k \in [K] : \mathcal{U}_N^j \cap P^k \neq \emptyset\}$. For the setting where $\mathbf{c}_t(\boldsymbol{\xi})$ and $\mathbf{A}_t(\boldsymbol{\xi})$ do not depend on the uncertain parameters, the resulting linear decision rule approximation of Problem (2) is given by

$$\begin{aligned} \text{minimize} \quad & \frac{1}{N}\sum_{j=1}^{N} \max_{k\in\mathcal{K}_j} \max_{\boldsymbol{\zeta}\in\mathcal{U}_N^j\cap P^k} \sum_{t=1}^{T} \mathbf{c}_t \cdot \left(\mathbf{x}_{t,0}^k + \sum_{s=1}^{t-1} \mathbf{X}_{t,s}^k \boldsymbol{\zeta}_s\right) \\ \text{subject to} \quad & \sum_{t=1}^{T} \mathbf{A}_t \left(\mathbf{x}_{t,0}^k + \sum_{s=1}^{t-1} \mathbf{X}_{t,s}^k \boldsymbol{\zeta}_s\right) \le \mathbf{b}(\boldsymbol{\zeta}) \\ & \qquad \forall\, \boldsymbol{\zeta} \in \textstyle\bigcup_{j=1}^{N} \mathcal{U}_N^j \cap P^k,\ k \in [K] \\ & \mathbf{x}_t^k = \mathbf{x}_t^{k'},\ \mathbf{X}_{t,s}^k = \mathbf{X}_{t,s}^{k'} \\ & \qquad \forall\, (k,k',t) \in \mathcal{T}(P^1,\dots,P^K),\ 1 \le s < t, \end{aligned} \tag{5}$$

where the decision variables are $\mathbf{x}_{t,0}^k \in \mathbb{R}^{n_t}$ and $\mathbf{X}_{t,s}^k \in \mathbb{R}^{n_t \times d_s}$ for all $1 \le s < t$ and $k \in [K]$.

Speaking intuitively, the approximation gap between Problems (2) and (5) depends on the selection and granularity of the partition. By choosing partitions with a greater number of regions, Problem (5) can produce a tighter approximation of Problem (2),

although this comes with an increase in problem size. For heuristic algorithms for selecting the partitions, we refer the reader to Bertsimas and Dunning (2016) and Postek and Den Hertog (2016). Once the partitions are determined, we obtain a reformulation of Problem (5) by employing the same duality techniques as in Section 5.1. To this end, we will assume that the intersections between the regions of the partition and uncertainty sets take a rectangular form.

Assumption 5. *The intersection between each uncertainty set and region of the partition either has the form* $\mathcal{U}_N^j \cap P^k := \{\boldsymbol{\zeta} \in \mathbb{R}^d : \boldsymbol{\ell}^{jk} \le \boldsymbol{\zeta} \le \mathbf{u}^{jk}\}$ *or is empty.*

We remark that this assumption can be guaranteed under the same conditions as Assumption 4 when the partition's regions are constructed as hyperrectangles. We now show that Problem (5) can be reformulated as a finite-dimensional linear optimization problem, which scales lightly in the number of sample paths N as well as the number of regions K.

Corollary 1. *If Assumption* 5 *holds, then* (5) *can be reformulated by adding at most* $O(N + Kmd)$ *auxiliary continuous decision variables and* $O(m\sum_{j=1}^{N}|\mathcal{K}_j| + Kmd)$ *linear constraints. The reformulation is*

$$\begin{aligned}
&\text{minimize} && \frac{1}{N}\sum_{j=1}^{N} v_j \\
&\text{subject to} && \sum_{t=1}^{T}\left(\mathbf{u}_t^{jk}\cdot\boldsymbol{\alpha}_t^k - \boldsymbol{\ell}_t^{jk}\cdot\boldsymbol{\beta}_t^k + \mathbf{c}_t\cdot\mathbf{x}_{t,0}^k\right) \le v_j \\
&&& \qquad\qquad j\in[N],\ k\in\mathcal{K}_j \\
&&& \boldsymbol{\alpha}_t^k - \boldsymbol{\beta}_t^k = \sum_{s=t+1}^{T}\left(\mathbf{X}_{s,t}^k\right)^{\top}\mathbf{c}_s \quad t\in[T],\ k\in[K] \\
&&& \sum_{t=1}^{T}\left(\mathbf{M}_t^k\mathbf{u}_t^{jk} - \boldsymbol{\Lambda}_t^k\boldsymbol{\ell}_t^{jk} + \mathbf{A}_t\mathbf{x}_{t,0}^k\right) \le \mathbf{b}^0 \\
&&& \qquad\qquad j\in[N],\ k\in\mathcal{K}_j \\
&&& \mathbf{M}_t^k - \boldsymbol{\Lambda}_t^k = -\mathbf{B}_t + \sum_{s=t+1}^{T}\mathbf{A}_s\mathbf{X}_{t,s}^k \\
&&& \qquad\qquad t\in[T],\ k\in[K] \\
&&& \mathbf{x}_t^k = \mathbf{x}_t^{k'},\ \mathbf{X}_{t,s}^k = \mathbf{X}_{t,s}^{k'} \\
&&& \qquad\qquad (k,k',t)\in\mathcal{T}(P^1,\ldots,P^K),\ 1\le s<t,
\end{aligned}$$

where the auxiliary decision variables are $\mathbf{v}\in\mathbb{R}^N$ *as well as* $\boldsymbol{\alpha}^k := (\boldsymbol{\alpha}_1^k,\ldots,\boldsymbol{\alpha}_T^k)$, $\boldsymbol{\beta}^k := (\boldsymbol{\beta}_1^k,\ldots,\boldsymbol{\beta}_T^k)\in\mathbb{R}_+^d$, *and* $\mathbf{M}^k := (\mathbf{M}_1^k,\ldots,\mathbf{M}_T^k)$, $\boldsymbol{\Lambda}^k := (\boldsymbol{\Lambda}_1^k,\ldots,\boldsymbol{\Lambda}_T^k)\in\mathbb{R}_+^{m\times d}$ *for each* $k\in[K]$. *Note that* $\mathbf{b}(\boldsymbol{\zeta}) := \mathbf{b}^0 + \sum_{t=1}^{T}\mathbf{B}_t\boldsymbol{\zeta}_t \in \mathbb{R}^m$.

Proof. The proof follows from similar duality techniques as Theorem 4 and is thus omitted. □

This result suggests that Problem (2) with finite adaptability is scalable in the sense that the size of the resulting reformulation for a given partition $P^1,\ldots,P^K$ scales lightly in the number of sample paths N. Assuming that the partition's regions and uncertainty sets are hyperrectangles, we remark that $\boldsymbol{\ell}^{jk}$, $\mathbf{u}^{jk}$, and $\mathcal{T}(P^1,\ldots,P^K)$ can be obtained efficiently by computing the intersection of each uncertainty set and region of the partition.

6. Relationships with Distributionally Robust Optimization

In the previous sections, we discussed the theoretical underpinnings and computational tractability of Problem (2) as a data-driven approach to multistage stochastic linear optimization. An attractive aspect of the proposed approach is its simplicity, interpretable as a straightforward robustification of historical sample paths. In this section, we explore connections between our data-driven approach to multistage stochastic linear optimization and distributionally robust optimization and discuss implications of our results to the latter.

Our exposition in this section focuses on the following formulation of multistage distributionally robust linear optimization:

$$\begin{aligned}
&\underset{\mathbf{x}\in\mathcal{X}}{\text{minimize}} && \sup_{\mathbb{Q}\in\mathcal{A}_N}\mathbb{E}_{\mathbb{Q}}\left[\sum_{t=1}^{T}\mathbf{c}_t(\boldsymbol{\xi})\cdot\mathbf{x}_t(\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_{t-1})\right] \\
&\text{subject to} && \sum_{t=1}^{T}\mathbf{A}_t(\boldsymbol{\xi})\mathbf{x}_t(\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_{t-1}) \le \mathbf{b}(\boldsymbol{\xi})\ \ \mathbb{Q}\text{-a.s.},\ \forall\,\mathbb{Q}\in\mathcal{A}_N.
\end{aligned} \tag{6}$$

Intuitively speaking, this framework chooses the decision rules that minimize the expected cost with respect to an adversarially chosen probability distribution from an ambiguity set. The requirement that the constraints hold almost surely for every distribution in the ambiguity set ensures that the objective function will evaluate the cost function on realizations of the stochastic process where the decision rules are feasible. Examples of this formulation in multistage and data-driven two-stage problems include Hanasusanto and Kuhn (2018) and Bertsimas et al. (2019).

Our following discussion focuses on ambiguity sets, which are constructed using historical data and Wasserstein-based distances between probability distributions. Given two bounded probability distributions, their ∞-Wasserstein distance is defined as

$$\mathrm{d}_\infty(\mathbb{Q},\mathbb{Q}') := \inf\left\{\Pi\text{-ess sup}\,\|\boldsymbol{\xi}-\boldsymbol{\xi}'\|: \begin{array}{l}\Pi \text{ is a joint distribution of } \boldsymbol{\xi} \text{ and } \boldsymbol{\xi}' \\ \text{with marginals } \mathbb{Q} \text{ and } \mathbb{Q}', \text{ respectively}\end{array}\right\},$$

where the essential supremum of the joint distribution is given by

$$\Pi\text{-ess}\sup_{\Xi\times\Xi}\|\boldsymbol{\xi}-\boldsymbol{\xi}'\| := \inf\{M : \Pi(\|\boldsymbol{\xi}-\boldsymbol{\xi}'\| > M) = 0\}.$$

From an intuitive standpoint, we note, in the case of $d = 1$, that the ∞-Wasserstein distance between two bounded probability distributions can be interpreted as the maximum distance between the quantile functions of the two distributions; see Ramdas et al. (2017). For any $p \in [1,\infty)$, the p-Wasserstein distance between two probability distributions is defined as

$$\mathrm{d}_p(\mathbb{Q},\mathbb{Q}') = \inf\left\{\left(\int_{\Xi\times\Xi}\|\boldsymbol{\xi}-\boldsymbol{\xi}'\|^p d\Pi(\boldsymbol{\xi},\boldsymbol{\xi}')\right)^{\frac{1}{p}} : \begin{array}{l}\Pi \text{ is a joint distribution of } \boldsymbol{\xi} \text{ and } \boldsymbol{\xi}' \\ \text{with marginals } \mathbb{Q} \text{ and } \mathbb{Q}', \text{ respectively}\end{array}\right\}.$$

For technical details on these distances, we refer the reader to Givens and Shortt (1984). For any $p \in [1,\infty]$, let the p-Wasserstein ambiguity set be defined as

$$\mathcal{A}_N = \left\{\mathbb{Q}\in\mathcal{P}(\Xi) : \mathrm{d}_p\left(\mathbb{Q},\widehat{\mathbb{P}}_N\right)\le\epsilon_N\right\},$$

where $\epsilon_N \ge 0$ is a robustness parameter that controls the size of the ambiguity set and $\widehat{\mathbb{P}}_N$ is the empirical probability distribution that assigns equal weight to each of the historical sample paths $\widehat{\boldsymbol{\xi}}^1,\ldots,\widehat{\boldsymbol{\xi}}^N$. We henceforth refer to Problem (6) with the p-Wasserstein ambiguity set as p-WDRO.

As discussed at the end of Section 1.1, there are relatively few previous convergence guarantees for distributionally robust optimization with the ∞-Wasserstein ambiguity set, even for single-stage problems. Indeed, when the underlying distribution is unbounded, the ∞-Wasserstein ambiguity set will never contain the true distribution, even as N tends to infinity because the distance $\mathrm{d}_\infty(\mathbb{P},\widehat{\mathbb{P}}_N)$ from the true to the empirical distribution will always be infinite. Thus, except under stronger assumptions than Assumption 1, the techniques used by Esfahani and Kuhn (2018, theorems 3.5 and 3.6) to establish finite-sample and convergence guarantees for the 1-Wasserstein ambiguity set do not extend to the ∞-Wasserstein ambiguity set. Nonetheless, distributionally robust optimization with the ∞-Wasserstein ambiguity set has recently received interest in the context of regularization and adversarial training in machine learning (Gao et al. 2017, Staib and Jegelka 2017).

The following proposition shows that Problem (2), under the particular construction of uncertainty sets from Section 3, can also be interpreted as Problem (6) with the ∞-Wasserstein ambiguity set.

Proposition 3. *Problem (2) with uncertainty sets of the form*

$$\mathcal{U}_N^j := \left\{\boldsymbol{\zeta}\equiv(\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\zeta}_T)\in\Xi : \|\boldsymbol{\zeta}-\widehat{\boldsymbol{\xi}}^j\|\le\epsilon_N\right\}$$

is equivalent to ∞-WDRO.

Proof. See Online Appendix G. □

Therefore, as a by-product of Theorem 1 from Section 4, we have obtained general convergence guarantees for distributionally robust optimization using the ∞-Wasserstein ambiguity set under mild probabilistic assumptions.

For comparison, we now show that similar asymptotic optimality guarantees for multistage stochastic linear optimization are not obtained by p-WDRO for any $p \in [1,\infty)$. Indeed, the following proposition shows that the constraints induced by such an approach are overly conservative in general.

Proposition 4. *If $p \in [1,\infty)$ and $\epsilon_N > 0$, then a decision rule is feasible for p-WDRO only if*

$$\sum_{t=1}^{T}\mathbf{A}_t(\boldsymbol{\zeta})\mathbf{x}_t(\boldsymbol{\zeta}_1,\ldots,\boldsymbol{\zeta}_{t-1})\le\mathbf{b}(\boldsymbol{\zeta})\quad\forall\,\boldsymbol{\zeta}\in\Xi.$$

Proof. See Online Appendix H. □

As discussed in Section 2, the set Ξ is not necessarily a tight approximation of the true (unknown) support of the stochastic process and may be strictly and significantly larger. Thus, the constraints induced from p-WDRO with $p \in [1,\infty)$ may eliminate optimal or high-quality decision rules for Problem (1). Consequently, p-WDRO with $p \in [1,\infty)$ is not asymptotically optimal for multistage stochastic linear optimization in general. We conclude this section with two further remarks.

Remark 1. If we relaxed the constraints of p-WDRO with $p \in [1,\infty)$ in an attempt to decrease its conservatism, then the resulting decision rules are not guaranteed to be feasible for the stochastic problem. Thus, the finite-sample guarantees provided by Esfahani and Kuhn (2018, equation 2), which served as one of the principle justifications for using p-WDRO, would no longer provide meaningful insight into the true out-of-sample performance of this decision rule.

Remark 2. The conservatism of p-WDRO can lead to suboptimal decisions, even for problems where uncertainty does not impact feasibility, if the true support is not known exactly. For example, consider the problem

$$\begin{array}{ll}\underset{x_2:\mathbb{R}\to\mathbb{R},\, x_3:\mathbb{R}^2\to\mathbb{R}}{\text{minimize}} & \mathbb{E}[x_2(\xi_1)+2x_3(\xi_1,\xi_2)] \\ \text{subject to} & x_2(\xi_1)+x_3(\xi_1,\xi_2)\ge\xi_1+\xi_2 \text{ a.s.} \\ & x_2(\xi_1)+x_3(\xi_1,\xi_2)\ge\xi_1-\xi_2,\end{array}$$

We observe that $x_2(\xi_1) = \xi_1$ and $x_3(\xi_1, \xi_2) = |\xi_2|$ are feasible decision rules, regardless of the underlying probability distribution. Suppose that the probability distribution and support of (ξ_1, ξ_2) are unknown, and our only information comes from historical data. If we approximate this stochastic problem using p-WDRO for any $p \in [1, \infty)$ and linear decision rules, we are tasked with solving

$$\begin{aligned}
&\underset{x_{2,0},\, x_{2,1},\, x_{3,0},\, x_{3,1},\, x_{3,2} \in \mathbb{R}}{\text{minimize}} \\
&\quad \sup_{\mathbb{Q} \in \mathcal{A}_N} \mathbb{E}_{\mathbb{Q}}\left[(x_{2,0} + x_{2,1}\xi_1) + 2(x_{3,0} + x_{3,1}\xi_1 + x_{3,2}\xi_2)\right] \\
&\text{subject to} \\
&\quad (x_{2,0} + x_{2,1}\zeta_1) + (x_{3,0} + x_{3,1}\zeta_1 + x_{3,2}\zeta_2) \geq \zeta_1 + \zeta_2 \;\; \forall\, \boldsymbol{\zeta} \in \mathbb{R}^2 \\
&\quad (x_{2,0} + x_{2,1}\zeta_1) + (x_{3,0} + x_{3,1}\zeta_1 + x_{3,2}\zeta_2) \geq \zeta_1 - \zeta_2 \;\; \forall\, \boldsymbol{\zeta} \in \mathbb{R}^2.
\end{aligned}$$

It follows from identical reasoning as in Bertsimas et al. (2019, section 3) that there are no linear decision rules that are feasible for the optimization problem. In particular, the optimization problem will remain infeasible even if the true support of the random variable happens to be bounded, but the bound is unknown. In contrast, the sample robust optimization approach (Problem (2)) to this example will always have a feasible linear decision rule. A similar example is found in Section 8.

7. Application to a Stochastic Inventory Replenishment Problem

In our first set of numerical experiments, we consider a stochastic inventory replenishment problem in a network with a single warehouse and multiple retailers. Our setting is motivated by the real-world case study of Avrahami et al. (2014), who study a supply chain controlled by a major Israeli publisher that sells a weekly magazine. The goal of the publisher is to find production and replenishment policies for delivering magazines to retailers that minimize their expected weekly operating costs. We perform numerical experiments to assess the practical value of our robust optimization approach in this application in comparison with alternative data-driven approaches from the literature.

7.1. Problem Description

The stochastic inventory replenishment problem faced by the publisher can be modeled as a three-stage stochastic linear optimization problem with *mixed-integer* decision variables and a *multidimensional* stochastic process. Let the number of retailers in the supply chain network be denoted by $R \in \mathbb{N}$. The time horizon of the decision problem faced by the publisher corresponds to one calendar week, beginning on Sunday and ending on Saturday, in which procurement and replenishment decisions are made at the start and middle of the week. The dynamics of the decision problem are summarized as follows:

- (Sunday) At the beginning of the week, the publisher decides the production quantity and the destinations of the weekly magazine. These decisions are captured by the decision variables $Q_{11}, \ldots, Q_{1R} \geq 0$, which represent the number of produced magazines delivered directly to each of the retailers at the start of the week, and the decision variable $Q_{10} \geq 0$, which represents the number of produced magazines sent to a common warehouse. Magazines are produced at a cost of c per unit, and $\sum_{r=0}^{R} Q_{1r}$ is the total number of magazines that are produced for the entire week.
- (Sunday to Tuesday) After the publisher has made the initial production decisions, each retailer $r \in \{1, \ldots, R\}$ uses their inventory of Q_{1r} magazines to satisfy the random customer demand ξ_{1r} from the first half of the week. There is no backlogging for demand that exceeds the available inventory, and the inventory at each retailer at the end of the first half of the week is thus equal to $\max\{0, Q_{1r} - \xi_{1r}\}$. The publisher incurs a cost of b per unit of unmet customer demand at each of the retailers, and the realized demands from the first half of the week $\boldsymbol{\xi}_1 := (\xi_{11}, \ldots, \xi_{1R})$ are observed by the publisher on Tuesday night.
- (Wednesday) At the midweek point, the publisher can replenish the inventories of the retailers using the magazines from the warehouse. These replenishment quantities to each of the retailers are captured by second-stage decision rules, $Q_{21}(\boldsymbol{\xi}_1), \ldots, Q_{2R}(\boldsymbol{\xi}_1) \geq 0$. The replenishment quantities are constrained by the inventory at the warehouse, $\sum_{r=1}^{R} Q_{2r}(\boldsymbol{\xi}_1) \leq Q_{10}$, and the publisher pays a fixed shipping cost $f \geq 0$ for each retailer to which it sends a nonzero replenishment quantity.
- (Wednesday to Saturday) After the publisher has made the replenishment decisions, each retailer $r \in \{1, \ldots, R\}$ uses their inventory of $\max\{0, Q_{1r} - \xi_{1r}\} + Q_{2r}(\boldsymbol{\xi}_1)$ magazines to satisfy the random customer demand ξ_{2r} from the second half of the week. The random demands in the second half of the week at each of the retailers are represented by $\boldsymbol{\xi}_2 := (\xi_{21}, \ldots, \xi_{2R})$.
- (Saturday) At the end of the week, the publisher incurs a holding cost of h per unit and a backlogging cost of b per unit at each of the retailers, and a holding cost of h per unit is also applied to any remaining units at the warehouse.

It follows from the dynamics that the stochastic inventory replenishment problem faced by the publisher

corresponds to a three-stage stochastic nonlinear optimization problem of the form

$$\begin{aligned}
\underset{\mathbf{Q}\geq\mathbf{0},\,\mathbf{z}\in\{0,1\}^R,\,\mathbf{I}}{\text{minimize}}\quad & \mathbb{E}\Bigg[c\Bigg(Q_{10}+\sum_{r=1}^{R}Q_{1r}\Bigg)+h\Bigg(Q_{10}-\sum_{r=1}^{R}Q_{2r}(\boldsymbol{\xi}_1)\Bigg)\\
&+b\Bigg(\sum_{r=1}^{R}\max\{0,\xi_{1r}-Q_{1r}\}\Bigg)\\
&+f\Bigg(\sum_{r=1}^{R}z_r(\boldsymbol{\xi}_1)\Bigg)+b\Bigg(\sum_{r=1}^{R}\max\{0,-I_{3r}(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)\}\Bigg)\\
&+h\Bigg(\sum_{r=1}^{R}\max\{0,I_{3r}(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)\}\Bigg)\Bigg]\\
\text{subject to}\quad & \sum_{r=1}^{R}Q_{2r}(\boldsymbol{\xi}_1)\leq Q_{10} && \text{a.s.}\\
& I_{3r}(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)=\max\{0,Q_{1r}-\xi_{1r}\}\\
&\qquad +Q_{2r}(\boldsymbol{\xi}_1)-\xi_{2r}\quad \forall\, r\in[R], && \text{a.s.}\\
& z_r(\boldsymbol{\xi}_1)\mathcal{M}\geq Q_{2r}(\boldsymbol{\xi}_1)\qquad \forall\, r\in[R], && \text{a.s.},
\end{aligned}$$

where the decision rules $I_{31}(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2),\ldots,I_{3R}(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)$ represent the net inventory at each retailer at the end of the week, the decision rules $z_1(\boldsymbol{\xi}_1),\ldots,z_R(\boldsymbol{\xi}_1)\in\{0,1\}$ represent whether a fixed cost should be applied at each retailer, and $\mathcal{M}$ is a sufficiently large big-M constant. Following similar reasoning as Avrahami et al. (2014, appendix A), we show in Online Appendix J.1 that the optimization problem can be equivalently reformulated as the following three-stage stochastic linear optimization problem:

$$\begin{aligned}
\underset{\mathbf{Q}\geq\mathbf{0},\,\mathbf{z}\in\{0,1\}^R,\,\mathbf{v}}{\text{minimize}}\quad & \mathbb{E}\Bigg[c\Bigg(Q_{10}+\sum_{r=1}^{R}Q_{1r}\Bigg)+hQ_{10}\\
&+\sum_{r=1}^{R}v_r(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)+f\sum_{r=1}^{R}z_r(\boldsymbol{\xi}_1)\Bigg]\\
\text{subject to}\quad & \sum_{r=1}^{R}Q_{2r}(\boldsymbol{\xi}_1)\leq Q_{10} && \text{a.s.}\\
& v_r(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)\geq b(\xi_{2r}+\xi_{1r}-Q_{2r}(\boldsymbol{\xi}_1)-Q_{1r})\\
&\qquad -hQ_{2r}(\boldsymbol{\xi}_1)\qquad \forall\, r\in[R], && \text{a.s.}\\
& v_r(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)\geq h(Q_{1r}-\xi_{1r}-\xi_{2r})\quad \forall\, r\in[R], && \text{a.s.}\\
& v_r(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)\geq b(\xi_{1r}-Q_{1r})-h\xi_{2r}\quad \forall\, r\in[R], && \text{a.s.}\\
& z_r(\boldsymbol{\xi}_1)\mathcal{M}\geq Q_{2r}(\boldsymbol{\xi}_1)\qquad \forall\, r\in[R], && \text{a.s.}
\end{aligned}\tag{7}$$

7.2. Experiments

We compare several approaches for finding decision rules for Problem (7) when the only information on the joint probability distribution of the stochastic process comes from historical data. Specifically, following the discussion of Avrahami et al. (2014), we assume that the publisher has collected historical data consisting of the demands for magazines in past weeks, $\widehat{\boldsymbol{\xi}}^1:=(\widehat{\boldsymbol{\xi}}_1^1,\widehat{\boldsymbol{\xi}}_2^1),\ldots,\widehat{\boldsymbol{\xi}}^N:=(\widehat{\boldsymbol{\xi}}_1^N,\widehat{\boldsymbol{\xi}}_2^N)$, which are independent and identically distributed sample paths of the underlying stochastic process $\boldsymbol{\xi}:=(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)$. We also assume that the random demands are known to be nonnegative almost surely, $\Xi:=\mathbb{R}_+^{2R}$. We compare the following data-driven approaches for finding decision rules for Problem (7).

- *SAA Independence*. Given that the only information on the underlying joint probability distribution comes from sample paths, Avrahami et al. (2014) study a method for obtaining approximate solutions to Problem (7) that assumes that the demands in the first half of the week $\boldsymbol{\xi}_1$ and the second half of the week $\boldsymbol{\xi}_2$ are independent random vectors. The assumption of stage-wise independence is a common simplifying assumption in the stochastic programming literature, which allows for a scenario tree to be constructed directly from historical sample paths. Specifically, under this assumption of stage-wise independence, this approach finds first-stage decisions for Problem (7) by solving the following mixed-integer linear optimization problem constructed from the scenario tree:

$$\begin{aligned}
\underset{\mathbf{Q}\geq\mathbf{0},\,\mathbf{z}\in\{0,1\}^{N\times R},\,\mathbf{v}}{\text{minimize}}\quad & c\Bigg(Q_{10}+\sum_{r=1}^{R}Q_{1r}\Bigg)+hQ_{10}\\
&+\frac{1}{N}\sum_{j=1}^{N}\sum_{r=1}^{R}\Bigg(fz_r^j+\frac{1}{N}\sum_{k=1}^{N}v_r^{jk}\Bigg)\\
\text{subject to}\quad & \sum_{r=1}^{R}Q_{2r}^j\leq Q_{10} && j\in[N]\\
& v_r^{jk}\geq b\Big(\widehat{\xi}_{2r}^k+\widehat{\xi}_{1r}^j-Q_{2r}^j-Q_{1r}\Big)-hQ_{2r}^j\\
& && j,\,k\in[N],\,r\in[R]\\
& v_r^{jk}\geq h\Big(Q_{1r}-\widehat{\xi}_{1r}^j-\widehat{\xi}_{2r}^k\Big) && j,k\in[N],\,r\in[R]\\
& v_r^{jk}\geq b\Big(\widehat{\xi}_{1r}^j-Q_{1r}\Big)-h\widehat{\xi}_{2r}^k && j,k\in[N],\,r\in[R]\\
& z_r^j\mathcal{M}\geq Q_{2r}^j && j\in[N],\,r\in[R].
\end{aligned}\tag{8}$$

We denote the optimal first-stage decisions to the linear optimization problem by $\widehat{Q}_{10}^{\text{SAA}},\widehat{Q}_{11}^{\text{SAA}}\ldots,\widehat{Q}_{1R}^{\text{SAA}}\geq 0$. Given these first-stage decisions and a realization of the demand in the first half of the week,

$\overline{\boldsymbol{\xi}}_1 = (\overline{\xi}_{11}\ldots,\overline{\xi}_{1R})$, the approach obtains second-stage decisions by solving the following mixed-integer linear optimization problem:

$$\begin{aligned}
\underset{\mathbf{Q}_2\geq\mathbf{0},\,\mathbf{z}\in\{0,1\}^R,\,\mathbf{v}}{\text{minimize}} \quad & \sum_{r=1}^{R}\left(fz_r+\frac{1}{N}\sum_{k=1}^{N}v_r^k\right)\\
\text{subject to} \quad & \sum_{r=1}^{R}Q_{2r}\leq \widehat{Q}_{10}^{\mathrm{SAA}}\\
& v_r^k \geq b\left(\widehat{\xi}_{2r}^k+\overline{\xi}_{1r}-Q_{2r}-\widehat{Q}_{1r}^{\mathrm{SAA}}\right)\\
& \qquad -hQ_{2r} && k\in[N],\ r\in[R]\\
& v_r^k \geq h\left(\widehat{Q}_{1r}^{\mathrm{SAA}}-\overline{\xi}_{1r}-\widehat{\xi}_{2r}^k\right) && k\in[N],\ r\in[R]\\
& v_r^k \geq b\left(\overline{\xi}_{1r}-\widehat{Q}_{1r}^{\mathrm{SAA}}\right)-h\widehat{\xi}_{2r}^k && k\in[N],\ r\in[R]\\
& z_r\mathcal{M}\geq Q_{2r} && r\in[R].
\end{aligned}$$

We denote the optimal decisions to the optimization problem by $\widehat{Q}_{21}^{\mathrm{SAA}}(\overline{\boldsymbol{\xi}}_1),\ldots,\widehat{Q}_{2R}^{\mathrm{SAA}}(\overline{\boldsymbol{\xi}}_1)\geq 0$ and $\widehat{z}_1^{\mathrm{SAA}}(\overline{\boldsymbol{\xi}}_1),\ldots,\widehat{z}_R^{\mathrm{SAA}}(\overline{\boldsymbol{\xi}}_1)\in\{0,1\}$.

- *AR Linear*. This approach obtains an approximation of Problem (7) using a parametric "estimate-then-optimize" technique from the literature. Specifically, the approach consists of two steps. In the "estimate" step, the historical data $\widehat{\boldsymbol{\xi}}^1,\ldots,\widehat{\boldsymbol{\xi}}^N$ are fit to a parametric family of joint probability distributions; in the "optimize" step, a scenario tree is constructed by sampling from the estimated joint probability distribution, and decision rules are obtained by solving the corresponding approximation of the multistage stochastic linear optimization problem.

In the estimate step, the scenario tree is constructed by first generating $\tilde{N}$ realizations for the demand in the first half of the week and then, for each such realization $\tilde{\boldsymbol{\xi}}_1^j$, generating a conditional set of demand realizations for the second half of the week $\{\tilde{\boldsymbol{\xi}}_2^{jk}\}_{k\in[\tilde{N}]}$. This generation procedure creates a scenario tree with $\tilde{N}^2$ leaves. Given this scenario tree, decisions are obtained by solving Problem (8), where $\widehat{\xi}_{2r}^k$ is replaced by $\tilde{\xi}_{2r}^{jk}$ in each of the constraints. Following the stochastic programming literature (see, e.g., Löhndorf and Shapiro 2019), we fit the historical data to a general linear model of the form $\boldsymbol{\xi}_1\sim\mathcal{N}(\boldsymbol{\mu}_1,\boldsymbol{\Sigma}_1)$ and $\boldsymbol{\xi}_2\sim\mathcal{N}(\mathbf{B}\left[\begin{smallmatrix}1\\ \boldsymbol{\xi}_1\end{smallmatrix}\right],\boldsymbol{\Sigma}_2)$, and we use the standard maximum likelihood estimators for the parameters $\boldsymbol{\mu}_1$, $\mathbf{B}$, $\boldsymbol{\Sigma}_1$, and $\boldsymbol{\Sigma}_2$ (Finn 1974, section 4.4).

- SRO-FA. Our final method obtains decision rules for Problem (7) using the proposed robust optimization approach. Specifically, given historical sample paths of the form $\widehat{\boldsymbol{\xi}}^1:=(\widehat{\boldsymbol{\xi}}_1^1,\widehat{\boldsymbol{\xi}}_2^1),\ldots,\widehat{\boldsymbol{\xi}}^N:=(\widehat{\boldsymbol{\xi}}_1^N,\widehat{\boldsymbol{\xi}}_2^N)$, we first construct an instance of Problem (2) in which the uncertainty sets from Section 3 are defined with the ℓ_∞-norm. We then approximately solve the robust optimization problem to obtain decision rules for Problem (7). To obtain a tractable approximation of the robust optimization problem, we implement the finite adaptability technique described in Section 5.2.

In our implementation of the finite adaptability, we construct a partition of $\Xi=\mathbb{R}_+^{2R}$, in which the historical demands in the first half of the week, $\widehat{\boldsymbol{\xi}}_1^1,\ldots,\widehat{\boldsymbol{\xi}}_1^N$, each are in their own hyperrectangular region. In other words, we construct a partition of Ξ consisting of regions of the form $P^1:=[\boldsymbol{\ell}^1,\mathbf{u}^1]\times\mathbb{R}_+^R,\ldots,P^N:=[\boldsymbol{\ell}^N,\mathbf{u}^N]\times\mathbb{R}_+^R$, which satisfies the property that $\widehat{\boldsymbol{\xi}}^j\in P^j$ for each sample path $j\in[N]$. This partition is motivated by the desire to obtain a tight approximation of the robust optimization problem; indeed, we observe that the number and granularity of regions will increase with the number of sample paths. Details on the partitioning heuristic used in this section can be found in Online Appendix J.2.

Given a partition of the form described, we approximate the robust optimization problem, Problem (2), by restricting the space of second-stage decision rules to those of the form

$$Q_{2r}(\boldsymbol{\zeta}_1)=\begin{cases}Q_{2r}^1, & \text{if } \boldsymbol{\ell}^1\leq\boldsymbol{\zeta}_1\leq\mathbf{u}^1,\\ \vdots & \\ Q_{2r}^N, & \text{if } \boldsymbol{\ell}^N\leq\boldsymbol{\zeta}_1\leq\mathbf{u}^N;\end{cases}$$

$$z_r(\boldsymbol{\zeta}_1)=\begin{cases}z_r^1, & \text{if } \boldsymbol{\ell}^1\leq\boldsymbol{\zeta}_1\leq\mathbf{u}^1,\\ \vdots & \\ z_r^N, & \text{if } \boldsymbol{\ell}^N\leq\boldsymbol{\zeta}_1\leq\mathbf{u}^N.\end{cases}$$

For notational convenience, let $\mathcal{K}_j:=\{k\in[N]:\mathcal{U}_N^j\cap P^k\neq\emptyset\}$ denote the indices of regions $P^1,\ldots,P^N$ that intersect the uncertainty set $\mathcal{U}_N^j$, and let us define the quantities $\underline{\zeta}_{tr}^{jk}:=\min_{\boldsymbol{\zeta}\in\mathcal{U}_N^j\cap P^k}\zeta_{tr}$ and $\overline{\zeta}_{tr}^{jk}:=\max_{\boldsymbol{\zeta}\in\mathcal{U}_N^j\cap P^k}\zeta_{tr}$ for each period $t\in\{1,2\}$, retailer $r\in\{1,\ldots,R\}$, sample path $j\in\{1,\ldots,N\}$, and region $k\in\mathcal{K}_j$. With this notation, we show in Online Appendix J.3 that the resulting

approximation of Problem (2) using finite adaptability is obtained by solving the following mixed-integer linear optimization problem:

$$\begin{aligned}
\underset{\substack{v^j,\, Q_1 \geq 0,\\ \mathbf{Q}_2^k \geq 0,\, \mathbf{z}^k \in \{0,1\}^R\\ \mathbf{u}^{j,k},\, \mathbf{v}^j \in \mathbb{R}^R}}{\text{minimize}} \quad & c\left(Q_{10} + \sum_{r=1}^{R} Q_{1r}\right) + hQ_{10} \\
& + \frac{1}{N}\sum_{j=1}^{N}\sum_{r=1}^{R} v_r^j \\
\text{subject to} \quad & \sum_{r=1}^{R} Q_{2r}^k \leq Q_{10} && \forall\, k \in [K] \\
& v_r^j \geq \sum_{r=1}^{R}\left(u_r^{j,k} + f z_r^k\right) && \forall\, r \in [R],\, j \in [N],\, k \in \mathcal{K}_j \\
& u_r^{j,k} \geq b\left(\overline{\zeta}_{2r}^{jk} + \overline{\zeta}_{1r}^{jk} - Q_{2r}^k - Q_{1r}\right) - hQ_{2r}^k \\
& && \forall\, r \in [R],\, j \in [N],\, k \in \mathcal{K}_j \\
& u_r^{j,k} \geq h\left(Q_{1r} - \underline{\zeta}_{1r}^{jk} - \underline{\zeta}_{2r}^{jk}\right) \\
& && \forall\, r \in [R],\, j \in [N],\, k \in \mathcal{K}_j \\
& u_r^{j,k} \geq b\left(\overline{\zeta}_{1r}^{jk} - Q_{1r}\right) - h\underline{\zeta}_{2r}^{jk} \\
& && \forall\, r \in [R],\, j \in [N],\, k \in \mathcal{K}_j \\
& z_r^k \mathcal{M} \geq Q_{2r}^k && \forall\, r \in [R],\, k \in [K]. \qquad (9)
\end{aligned}$$

Solving the optimization problem yields the first-stage decisions $\widehat{Q}_{10}^{\text{SRO}}, \widehat{Q}_{11}^{\text{SRO}} \ldots, \widehat{Q}_{1R}^{\text{SRO}} \geq 0$ and decision rules $\widehat{Q}_{21}^{\text{SRO}}(\boldsymbol{\xi}_1), \ldots, \widehat{Q}_{2R}^{\text{SRO}}(\boldsymbol{\xi}_1) \geq 0$ and $\widehat{z}_1^{\text{SRO}}(\boldsymbol{\xi}_1), \ldots, \widehat{z}_R^{\text{SRO}}(\boldsymbol{\xi}_1) \in \{0,1\}$.

To compare the data-driven approaches, we perform a variety of numerical experiments on different numbers of retailers, $R \in \{2,3,4\}$, with cost parameters $c = 0.25$, $h = 0.05$, $b = 0.5$, and in cases with no fixed cost, $f = 0$ as well as when the fixed cost is $f = 0.1$. In the joint probability distribution of the stochastic process, the demand in the first half of the week for each retailer, ξ_{1r}, is independent of the demands in the first half of the week for the other retailers and is generated from a truncated normal distribution with mean 6 and standard deviation 2.5 that is bounded below by zero. The demands in the second half of the week are independent across retailers and follow truncated normal distributions with means $2(6 - \xi_{1r})^2$ and standard deviations 2.5 that are bounded below by zero. This joint probability distribution is chosen because of its relative simplicity and particularly in the setting with multiple retailers, the difficulty of correctly identifying its structure using only limited historical data. The number of retailers in our experiments is motivated by the organizational structure of the Israeli publisher, in which sales representatives are assigned to managing the inventory of up to five retailers (Avrahami et al. 2014, p. 452). In Online Appendix J.4, we provide numerical evidence that the fixed cost of $f = 0.1$ leads to replenishment decision rules that are meaningfully different than those obtained in experiments with no fixed cost.

We compare the aforementioned data-driven methods on training sets of sizes $N \in \{50, 100, 200, 400, 800\}$ when there is no fixed cost ($f = 0$) and $N \in \{50, 71, 100, 141, 200, 282\}$ when there are fixed costs ($f = 0.1$). To obtain statistically meaningful results, for each size N, we generate 50 training data sets of size N and apply each of the data-driven approaches to each training data set. We evaluate the decision rules from each approach on a common testing data set of 10,000 sample paths. For "SRO-FA," we select the robustness parameter using five-fold cross-validation. For "AR Linear," we generate scenario trees of size $\tilde{N} = 50$ for all sizes of training sets N. We found that larger values of $\tilde{N}$ resulted in longer computation times but did not have any discernible impact on the performance of the "AR Linear" approach.

For comparison purposes, we also implement a benchmark method that has perfect knowledge of the true joint probability distribution. In the benchmark, we obtain an estimate of the optimal cost of Problem (7) by constructing a scenario tree from the true joint probability distribution. We construct the scenario tree using the same procedure as described previously, with a size of $\tilde{N} = 800$ for experiments without fixed cost and $\tilde{N} = 200$ for experiments with fixed cost. We repeat this process over 50 replications and report the average of the resulting optimal costs.

7.3. Results

In Figure 1, we show the impact of the robustness parameter on the in-sample and out-of-sample costs of the decision rules obtained by our robust optimization approach. The results demonstrate that a *strictly positive* choice of the robustness parameter is essential in order to obtain the best out-of-sample cost for each N. This is because of the fact that Problem (2) has been approximated with a rich space of decision rules and in particular, a space of decision rules that becomes increasingly flexible as more historical sample paths are obtained. In contrast, when the robustness parameter is set to zero, Figure 1 shows that the in-sample cost of Problem (9) does not appear to converge to the optimal cost of the stochastic problem; these findings are consistent with the discussion of Example 1 in Section 3 and show that approximating Problem (2) with a rich restricted space of decision rules will asymptotically overfit the historical data when the robustness parameter is set to zero. Additional numerical results on the relationship between the robustness parameter and the performance of our robust optimization approach can be found in Online Appendix J.4.

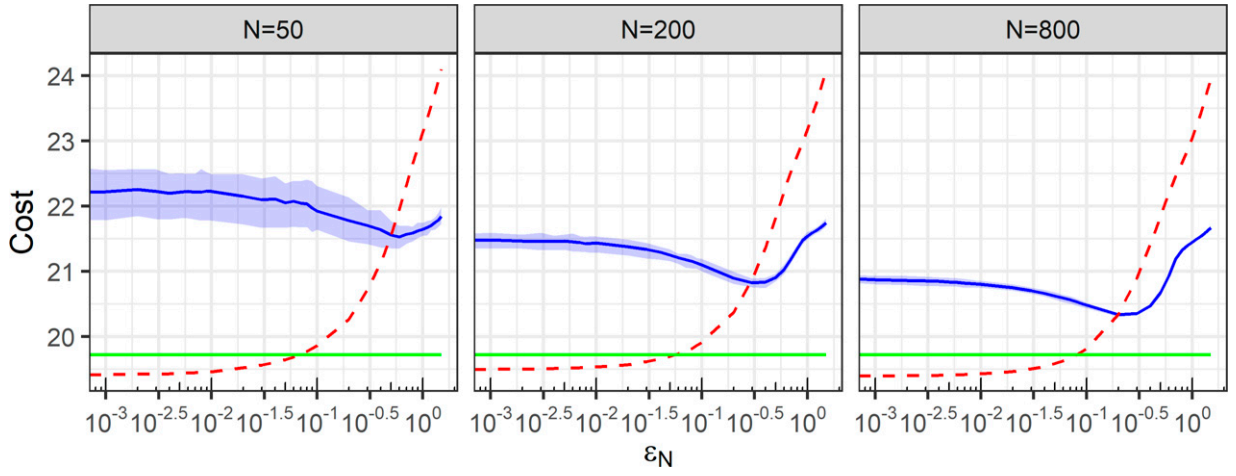
More broadly, we believe that Figure 1 highlights a practical strength of our robust optimization approach. Indeed, the approximation of Problem (2) using piecewise static decision rules did not require nor utilize any information about the structure of optimal decision rules for the underlying stochastic problem. At the same time, despite searching over a rich space of decision rules, Problem (2) with an appropriate choice of the robustness parameter yields decision rules that do not overfit the historical data. This shows that Problem (2) provides an opportunity to find high-quality decision rules for multistage stochastic linear optimization problems, even when (i) the only information on the underlying distribution comes from limited data and (ii) the structure of optimal decision rules for the stochastic problem is complex or unknown.

In Figures 2 and 3, we compare the average out-of-sample costs of the decision rules produced by the various data-driven methods in experiments with no fixed cost ($f = 0$) and with fixed cost ($f = 0.1$). We first observe that the gap between "SRO-FA CV" (where the robustness parameter is chosen through fivefold crossvalidation) and the unrealistic "SRO-FA Best" (where the robustness parameter is chosen post hoc to obtain decision rules with the best average out-of-sample performance) is relatively small across sizes of training sets. This suggests that crossvalidation can indeed be practically effective in choosing the robustness parameter in the robust optimization approach. Moreover, the results of the experiments show that the proposed robust optimization approach ("SRO-FA CV") can find decision rules that significantly outperform those obtained by widely used alternative data-driven approaches ("SAA Independence" and "AR Linear"). In particular, we note that the problem sizes in our experiments are realistic in terms of both the number of retailers R and sizes of training sets N. The results of the experiments thus provide numerical evidence that the robust optimization approach proposed in this paper can be valuable in practice, particularly in challenging applications with mixed-integer decisions and multidimensional stochastic processes.

8. Application to a Multistage Stochastic Inventory Management Problem

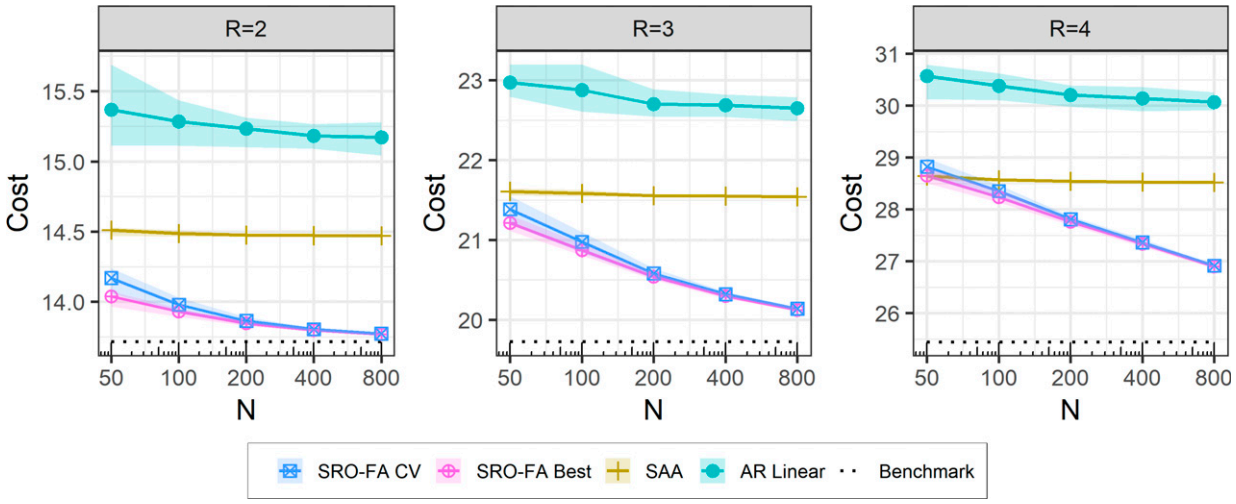
In our second set of experiments, we consider a classic and widely studied stochastic inventory management problem for a single product with an unknown autoregressive demand. Our motivation in this set of experiments is to explore the practical value of Problem (2) in applications where the number of time periods is comparatively large (e.g., $T = 10$) and the number of historical sample paths is relatively small (e.g., $10 \le N \le 100$). In this context, we compare our proposed robust optimization approach with linear decision rules with a variety of alternative data-driven approaches. In contrast to the previous section, where we compared with data-driven approaches that are practically tractable in problems with mixed-integer decisions and short time horizons, this section compares with approaches that are practically tractable in problems with continuous decisions and long time horizons.

Figure 1. Three-Stage Inventory Replenishment Problem: Impact of the Robustness Parameter on SRO-FA for $\mathbf{R} = \mathbf{3}$, No Fixed Cost ($f = 0$)



Notes. The solid blue lines are the average out-of-sample costs of decision rules produced by SRO-FA, and the shaded regions are the 20th and 80th percentiles over the 50 training data sets. The dotted red lines are the average in-sample costs for SRO-FA. The green lines are the benchmark for Problem (7). Results are shown for $N \in \{50, 200, 800\}$.

Figure 2. Three-Stage Inventory Replenishment Problem: Performance of Data-Driven Approaches, No Fixed Cost ($f = 0$)



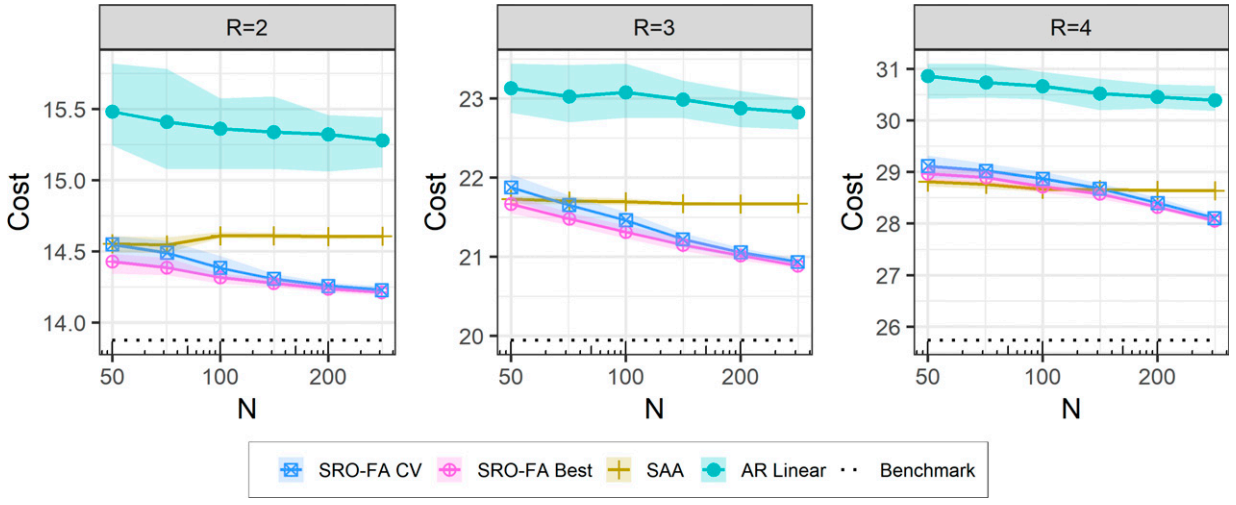
Notes. The solid lines are the average out-of-sample costs of decision rules produced by the various data-driven approaches to Problem (7), and the shaded regions are the 20th and 80th percentiles over the 50 training data sets. The robustness parameter in "SRO-FA CV" is chosen using fivefold crossvalidation, and the robustness parameter in "SRO-FA Best" is chosen optimally with respect to the testing data set.

8.1. Problem Description

We consider an inventory management problem of a single product over a finite planning horizon. At the beginning of each time period $t \in [T]$, we start with $I_t \in \mathbb{R}$ units of product in inventory. We then select a production quantity of $x_t \in [0, \overline{x}_t]$ with zero lead time at a cost of c_t per unit. The product demand $\xi_t \geq 0$ is then revealed, the inventory is updated to $I_{t+1} = I_t + x_t - \xi_t$, and we incur a holding cost of $h_t \max\{I_{t+1}, 0\}$ and a backorder cost of $b_t \max\{-I_{t+1}, 0\}$. We begin with zero units of inventory in the first period. Our goal is to dynamically select the production quantities to minimize the expected total cost over the planning horizon, captured by

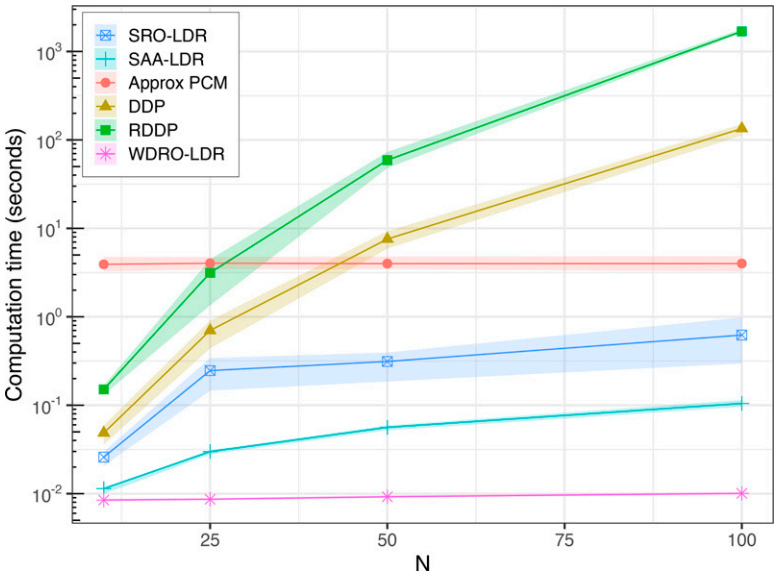
$$
\begin{aligned}
\underset{\mathbf{x},\,\mathbf{I},\,\mathbf{y}}{\text{minimize}} \quad & \mathbb{E}\left[\sum_{t=1}^{T}\left(c_t x_t(\xi_1,\ldots,\xi_{t-1}) + y_{t+1}(\xi_1,\ldots,\xi_t)\right)\right] \\
\text{subject to} \quad & I_{t+1}(\xi_1,\ldots,\xi_t) = I_t(\xi_1,\ldots,\xi_{t-1}) \\
& \quad + x_t(\xi_1,\ldots,\xi_{t-1}) - \xi_t \quad \text{a.s.}, && \forall\, t \in [T] \\
& y_{t+1}(\xi_1,\ldots,\xi_t) \geq h_t I_{t+1}(\xi_1,\ldots,\xi_t) \\
& \text{a.s.}, && \forall\, t \in [T] \\
& y_{t+1}(\xi_1,\ldots,\xi_t) \geq -b_t I_{t+1}(\xi_1,\ldots,\xi_t) \\
& \text{a.s.}, && \forall\, t \in [T] \\
& 0 \leq x_t(\xi_1,\ldots,\xi_{t-1}) \leq \overline{x}_t \quad \text{a.s.}, && \forall\, t \in [T].
\end{aligned} \tag{10}
$$

Figure 3. Three-Stage Inventory Replenishment Problem: Performance of Data-Driven Approaches, Fixed Cost ($f = 0.1$)



Notes. The solid lines are the average out-of-sample costs of decision rules produced by the various data-driven approaches to Problem (7), and the shaded regions are the 20th and 80th percentiles over the 50 training data sets. The robustness parameter in "SRO-FA CV" is chosen using fivefold crossvalidation, and the robustness parameter in "SRO-FA Best" is chosen optimally with respect to the testing data set.

Figure 4. Multistage Stochastic Inventory Management: Computation Times for $\mathbf{T = 10}$, $\boldsymbol{\alpha = 0.25}$



Notes. Computation times for data-driven approaches to the multistage stochastic inventory management problem. Results are shown for $T = 10$ and $\alpha = 0.25$, and similar computation times were observed for other choices of α. The graph shows the mean value of the computation times over 100 training data sets for each value of N.

We consider the setting where the joint probability distribution of the stochastic process $(\xi_1, \ldots, \xi_T) \in \mathbb{R}^T$ is unknown. Our only information on the distribution comes from historical data consisting of demand realizations for past products $(\widehat{\xi}_1^1, \ldots, \widehat{\xi}_T^1), \ldots, (\widehat{\xi}_1^N, \ldots, \widehat{\xi}_T^N)$, which are independent and identically distributed sample paths of the underlying stochastic process, and knowledge that the stochastic process will be contained in $\Xi = \mathbb{R}_+^T$ almost surely.

8.2. Experiments

We perform computational experiments on the following data-driven approaches for obtaining decision rules for Problem (10).

- *SRO-LDR*. This is the proposed data-driven approach for multistage stochastic linear optimization (Problem (2)), where the uncertainty sets are constructed as described in Section 3 with the ℓ_∞-norm. The approach is approximated using linear decision rules (see Section 5.1) and solved using the reformulation developed in Theorem 4. We choose the robustness parameter for each training data set using five-fold cross-validation, where the range of possible values considered in the cross-validation procedure is $\epsilon_N \in \{b \cdot 10^a : a \in \{-2, -1, 0, 1\}, b \in \{1, \ldots, 9\}\}$.
- *SAA-LDR*. This is the same approach as SRO-LDR, except that the robustness parameter is set to zero.
- *Approx PCM*. This is a data-driven extension of the approach developed in Bertsimas et al. (2019). In this approach, decision rules are obtained by solving a multistage distributionally robust optimization problem (Problem (6)), in which $\mathcal{A}_N$ is the set of joint probability distributions with the same mean and covariance as those estimated from the historical data. This distributionally robust optimization problem is solved approximately by restricting to lifted linear decision rules, as described in Bertsimas et al. (2019, section 3).
- *DDP* and *RDDP*. This is the robust data-driven dynamic programming approach proposed by Hanasusanto and Kuhn (2013). The approach estimates cost-to-go functions by applying kernel regression to the historical sample paths. Decisions are obtained from optimizing over the cost-to-go functions, which are evaluated approximately using the algorithm described in Hanasusanto and Kuhn (2013, section 4). Because the algorithm requires both input sample paths and initial state paths, we use half of the training data set as the input sample paths and the other half to generate the state paths via the lifted linear decision rules obtained by Approx PCM. The approach also requires a robustness parameter-γ, which we choose to be either $\gamma = 0$ (DDP) or $\gamma = 10$ (RDDP).
- *WDRO-LDR*. Described in Section 6, this approach obtains decision rules by solving a multistage distributionally robust optimization problem (Problem (6)), in which $\mathcal{A}_N$ is chosen to be the 1-Wasserstein ambiguity set with the ℓ_1-norm. Similarly as SRO-LDR, the distributionally robust optimization problem is approximated using linear decision rules, which are solved

Table 2. Multistage Stochastic Inventory Management: Average Out-of-Sample Cost

			Size of training data set (N)				
T	A	Approach	10	25	50	100	DP
5	0	SRO-LDR	**111.8 (7.2)**	**109.5 (1.3)**	**108.5 (0.7)**	108.0 (0.3)	108
		SAA-LDR	127.3 (12.6)	111.7 (3.1)	108.7 (1.0)	**107.9 (0.3)**	
		Approx PCM	118.5 (2.2)	117.4 (1.0)	117.1 (0.7)	117.0 (0.5)	
		DDP	2,262.7 (363.3)	1,189.6 (854.2)	525.4 (510.8)	205.1 (201.5)	
		RDDP	2,255.5 (393.2)	1,175.8 (856.2)	515.6 (506.0)	202.4 (195.3)	
		WDRO-LDR	2,400.3 (0.0)	2,400.3 (0.0)	2,400.3 (0.0)	2,400.3 (0.0)	
	0.25	SRO-LDR	**113.0 (4.2)**	**110.0 (1.8)**	108.7 (0.8)	108.0 (0.3)	107
		SAA-LDR	127.6 (13.0)	111.7 (3.1)	**108.6 (1.0)**	**108.0 (0.2)**	
		Approx PCM	126.8 (3.8)	125.3 (1.5)	124.8 (0.9)	124.6 (0.8)	
		DDP	2,251.5 (488.8)	1,393.7 (897.8)	679.3 (656.0)	236.9 (240.5)	
		RDDP	2,222.0 (556.5)	1,386.2 (900.7)	670.1 (654.9)	236.2 (237.2)	
		WDRO-LDR	2,400.7 (0.0)	2,400.7 (0.0)	2,400.7 (0.0)	2,400.7 (0.0)	
	0.5	SRO-LDR	**115.5 (5.3)**	**112.0 (4.0)**	110.8 (2.7)	111.7 (2.6)	108
		SAA-LDR	129.5 (13.3)	113.1 (6.8)	**110.7 (2.9)**	**111.6 (2.6)**	
		Approx PCM	136.0 (4.8)	134.0 (1.9)	133.4 (1.2)	133.2 (1.0)	
		DDP	2,263.8 (480.1)	1,563.7 (917.6)	777.5 (787.7)	364.2 (488.1)	
		RDDP	2,253.8 (515.8)	1,532.6 (940.9)	716.8 (758.9)	334.6 (477.1)	
		WDRO-LDR	2,401.2 (0.0)	2,401.2 (0.0)	2,401.2 (0.0)	2,401.2 (0.0)	
10	0	SRO-LDR	**208.9 (1.0)**	**207.5 (0.6)**	**206.8 (0.5)**	**206.2 (0.2)**	206
		SAA-LDR	293.9 (70.1)	212.6 (2.1)	207.8 (1.1)	206.3 (0.4)	
		Approx PCM	215.3 (2.1)	214.5 (0.6)	214.1 (0.6)	214.1 (0.4)	
		DDP	5,211.4 (1131.1)	2,827.9 (1757.5)	1,335.6 (1206.4)	497.5 (550.2)	
		RDDP	5,210.1 (1133.4)	2,820.3 (1758.7)	1,327.6 (1206.0)	500.1 (552.7)	
		WDRO-LDR	5,800.3 (0.0)	5,800.3 (0.0)	5,800.3 (0.0)	5,800.3 (0.0)	
	0.25	SRO-LDR	**210.3 (2.9)**	**207.8 (1.1)**	**206.9 (0.5)**	**206.3 (0.2)**	206
		SAA-LDR	295.1 (70.4)	212.7 (2.1)	207.8 (1.1)	206.3 (0.4)	
		Approx PCM	228.7 (4.5)	226.2 (1.8)	225.7 (1.1)	225.5 (0.9)	
		DDP	5,215.6 (1350.1)	3,214.7 (1984.9)	1,598.0 (1566.5)	440.2 (417.1)	
		RDDP	5,202.0 (1368.7)	3,185.3 (1977.1)	1,593.6 (1566.6)	437.0 (418.0)	
		WDRO-LDR	5,800.2 (0.0)	5,800.5 (0.0)	5,800.5 (0.0)	5,800.5 (0.0)	
	0.5	SRO-LDR	**211.1 (3.9)**	**207.9 (1.0)**	**206.9 (0.6)**	**206.3 (0.2)**	206
		SAA-LDR	297.8 (70.7)	213.0 (2.3)	207.9 (1.1)	206.4 (0.4)	
		Approx PCM	245.3 (7.0)	242.1 (2.7)	241.6 (2.0)	240.9 (1.6)	
		DDP	5,374.8 (1052.7)	3,676.4 (2159.1)	1,960.9 (1878.3)	644.1 (914.6)	
		RDDP	5,313.0 (1173.0)	3,630.9 (2161.4)	1,949.2 (1863.9)	644.0 (913.3)	
		WDRO-LDR	5,800.7 (0.0)	5,800.7 (0.0)	5,800.3 (0.0)	5,800.7 (0.0)	

Notes. Mean (standard deviation) for the out-of-sample cost of decision rules obtained by various data-driven approaches for Problem (10). The robustness parameters in SRO-LDR and WDRO-LDR are chosen using crossvalidation. The column DP presents the dynamic programming approximations of the optimal cost of Problem (10) from See and Sim (2010, tables EC.1 and EC.2), which have an accuracy of ±1%. The minimum mean out-of-sample cost for each setting is in bold.

using a duality-based reformulation provided in Online Appendix I. The robustness parameter is chosen using the same procedure as SRO-LDR.

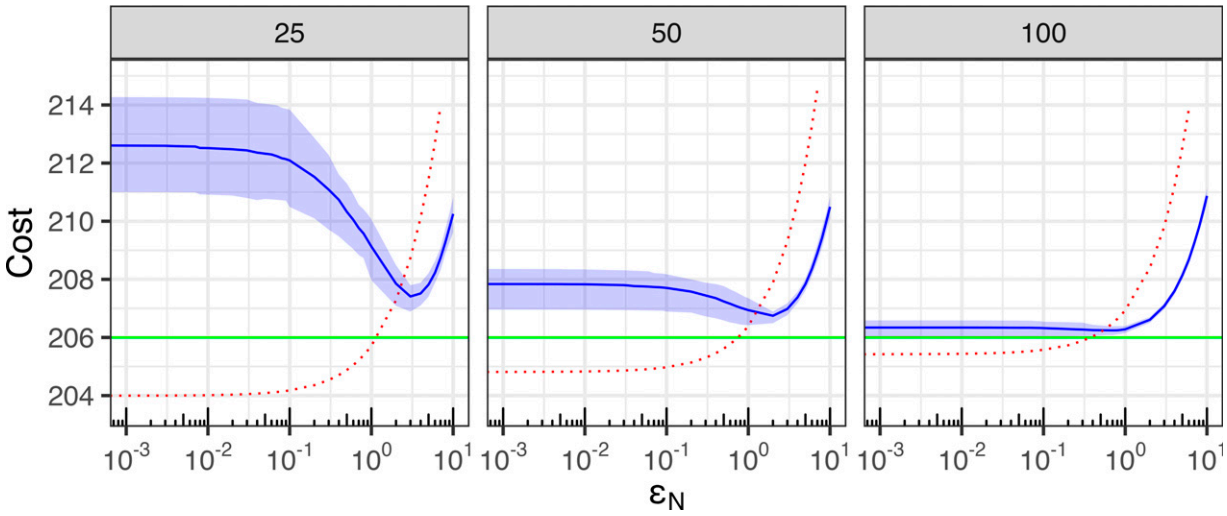
We perform computational simulations using the same parameters and data generation as See and Sim (2010). Specifically, the demand is a nonstationary autoregressive stochastic process of the form $\xi_t = \varsigma_t + \alpha\varsigma_{t-1} + \cdots + \alpha\varsigma_1 + \mu$, where $\varsigma_1, \ldots, \varsigma_T$ are independent random variables distributed uniformly over $[-\overline{\varsigma}, \overline{\varsigma}]$. The parameters of the stochastic process are $\mu = 200$ and $\overline{\varsigma} = 40$ when $T = 5$ and $\mu = 200$ and $\overline{\varsigma} = 20$ when $T = 10$. The capacities and costs are $\overline{x}_t = 260$, $c_t = 0.1$, $h_t = 0.02$ for all $t \in [T]$, $b_t = 0.2$ for all $t \in [T-1]$, and $b_T = 2$.

To compare the data-driven approaches, we take the following steps. For various choices of N, we generate 100 training data sets of size N and obtain decision rules by applying the data-driven approaches to each training data set. The out-of-sample costs of the obtained decision rules are approximated using a common testing data set of 10,000 sample paths.[2] Specifically, for each sample path $(\xi_1^i, \ldots, \xi_T^i) \in \mathbb{R}^T$ in the testing data set, we calculate production quantities $(x_1^{\mathcal{A},i,\ell}, \ldots, x_T^{\mathcal{A},i,\ell}) \in \mathbb{R}^T$ by applying the decision rule obtained from approach $\mathcal{A}$ on the ℓ th training data set. The out-of-sample cost of the decision rule is then approximated as

$$\frac{1}{10,000} \sum_{i=1}^{10,000} \sum_{t=1}^{T} \left(c_t x_t^{\mathcal{A},i,\ell} + \max\left\{ h_t I_{t+1}^{\mathcal{A},i,\ell}, -b_t I_{t+1}^{\mathcal{A},i,\ell} \right\} \right),$$

where the inventory levels $(I_1^{\mathcal{A},i,\ell}, \ldots, I_T^{\mathcal{A},i,\ell})$ are computed from the production quantities $(x_1^{\mathcal{A},i,\ell}, \ldots,$

Figure 5. Multistage Stochastic Inventory Management: Impact of Robustness Parameter on SRO-LDR for $\mathbf{T = 10}$, $\boldsymbol{\alpha = 0.25}$



Notes. The solid blue lines are the average out-of-sample costs of decision rules produced by SRO-LDR, and the shaded regions are the 20th and 80th percentiles over the 100 training data sets. The dotted red lines are the average in-sample costs of SRO-LDR, and the solid green lines are a dynamic programming approximation of the optimal cost of Problem (10) from See and Sim (2010, table EC.2).

$x_T^{A,i,\ell}) \in \mathbb{R}^T$ and the test sample path $(\xi_1^i, \ldots, \xi_T^i) \in \mathbb{R}^T$. All sample paths in the training and testing data sets are drawn independently from the true joint probability distribution.

As discussed earlier, Problem (2) is not guaranteed to find decision rules that are feasible for all realizations in Ξ. Therefore, the linear decision rules obtained by SRO-LDR and SAA-LDR, when applied to sample paths in the testing data set, may result in production quantities that exceed $\overline{x}_1, \ldots, \overline{x}_T$ or are negative. Thus, before computing the out-of-sample costs, we first project each production quantity $x_t^{A,i,\ell}$ onto the interval $[0, \overline{x}_t]$ to ensure it is feasible. We discuss the impact of this projection procedure at the end of the results section.

8.3. Results

In Figure 4 and Table 2, we report the out-of-sample costs and computation times of the various approaches. SRO-LDR produces an out-of-sample cost that outperforms the other approaches, most notably when the size of the training data set is small, and requires less than one second of computation time. We note that the out-of-sample cost of SRO-LDR roughly converges to the DP estimate of the optimal cost of Problem (10), which suggests that linear decision rules provide a good approximation of the optimal production decision rules for this particular stochastic problem. The relationship between the robustness parameter and the in-sample and out-of-sample costs of SRO-LDR is shown in Figure 5.

We briefly reflect on some notable differences between SRO-LDR and the other approaches. First, the results demonstrate that a strictly positive choice of the robustness parameter is not necessary to avoid asymptotic overfitting when Problem (2) is approximated with a fixed, finite-dimensional space of decision rules; indeed, Figure 5 and Table 2 show that SAA-LDR can provide an out-of-sample cost that is similar to that of SRO-LDR for moderate to large training data sets. However, SRO-LDR produces an out-of-sample cost that significantly outperforms SAA-LDR when N is small ($N \in \{10, 25\}$). More generally, this shows that there exist regimes in which a positive choice of the robustness parameter can still provide significant value even when Problem (2) is approximated using linear decision rules. Second, we note that WDRO-LDR consistently produces decision rules with large average out-of-sample cost; this is because of the fact that this approach requires the linear decision rules to satisfy $0 \le x_{t,0} + \sum_{s=1}^{t-1} x_{t,s}\zeta_s \le \overline{x}_t$ for all $(\zeta_1, \ldots, \zeta_T) \in \mathbb{R}_+^T$, which reduces to a static decision rule for the production quantity in each stage. Finally, we remark that the average out-of-sample cost of DDP and RDDP improved significantly with the size of the training data set, but it produced high variability across training data sets and required long computation time.

We recall that Table 2 reports the out-of-sample costs of SRO-LDR and SAA-LDR after their production quantities are projected onto the feasible region (see Section 8.2). In Online Appendix K, we discuss the impact of this projection procedure on the out-of-sample cost. Specifically, we show across the experiments that (i) SRO-LDR produces feasible production quantities for more than 93% of the sample paths in the testing data set and that (ii) the average ℓ_1 distance between the production quantities $(x_1^{A,i,\ell}, \ldots, x_T^{A,i,\ell})$ and the feasible region $[0, \overline{x}_1] \times \cdots \times [0, \overline{x}_T]$ is less than

two units. This shows that SRO-LDR consistently produces feasible or nearly feasible decisions, and thus, the out-of-sample costs of SRO-LDR are unlikely to be an artifact of this projection procedure.

9. Conclusion

In this work, we presented a new data-driven approach, based on robust optimization, for solving multistage stochastic linear optimization problems where uncertainty is correlated across time. We showed that the proposed approach is asymptotically optimal, providing assurance that the approach offers a near-optimal approximation of the underlying stochastic problem in the presence of big data. At the same time, the optimization problem resulting from the proposed approach can be addressed by approximation algorithms and reformulation techniques, which have underpinned the success of multistage robust optimization. The practical value of the proposed approach was illustrated by computational examples inspired by real-world applications, demonstrating that the proposed data-driven approach can produce high-quality decisions in reasonable computation times. Through these contributions, this work provides a step toward helping organizations across domains leverage historical data to make better operational decisions in dynamic environments.

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Endnotes

[1] The definition does not preclude the possibility that $\underline{J}$ is equal to $-\infty$ or ∞. However, we do not expect either of those values to occur outside of pathological cases; see Section 4.3. The same remark applies to the upper bound $\bar{J}$.

[2] For DDP and RDDP, we only evaluated on the first 1,000 sample paths in the testing data set because of the computational cost of optimizing over the cost-to-go functions for each testing sample path.

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