Quantum Galton Boards: A Summary

Angus Crookes

Classical Galton Boards

The Galton Board (GB) is a device capable of simulating different statistical distributions [1]. It consists of a series of pegs arranged in a regular (usually triangular) pattern; a bead dropped from the top then either drops left with probability p, or right with probability $q = 1 - p$, on impact with each peg. For n layers of pegs, a collection of $n+1$ buckets at the bottom of the board collects the beads, which builds up the statistical distribution. A diagram of the most common GB is shown in fig. 1 (a) where the probabilities p and q are identical for every peg. In this case, the distribution of beads in bucket k follows a binomial distribution:

$$\binom{n}{k} p^k (1-p)^{n-k} \tag{1}$$

where k is the number of left drops, $n - k$ the number of right drops, and $\binom{n}{k}$ the number of possible ways of achieving this combination. Notably, when $p = 0.5$ and n is large, the resulting distribution is normal. However, many other distributions can be generated by making each peg have it's own bias (i.e. p_i, q_i for peg i) such as multi-peaked, and exponential distributions.

Quantum Galton Boards

A Quantum Galton Board (QGB) is the quantum analogue of a GB. QGBs use superposition and interference to produce the statistical distributions. Therefore, they can potentially process information faster than a classical GB, offering a more efficient way to model complex random processes. In addition, they are a natural platform to simulate quantum distributions, such as the quantum Hadamard random walk. A simple and efficient method for constructing a QGB was developed in [2] focusing on up to $n = 4$ layers. Here, I summarise this method, and it's extensions.

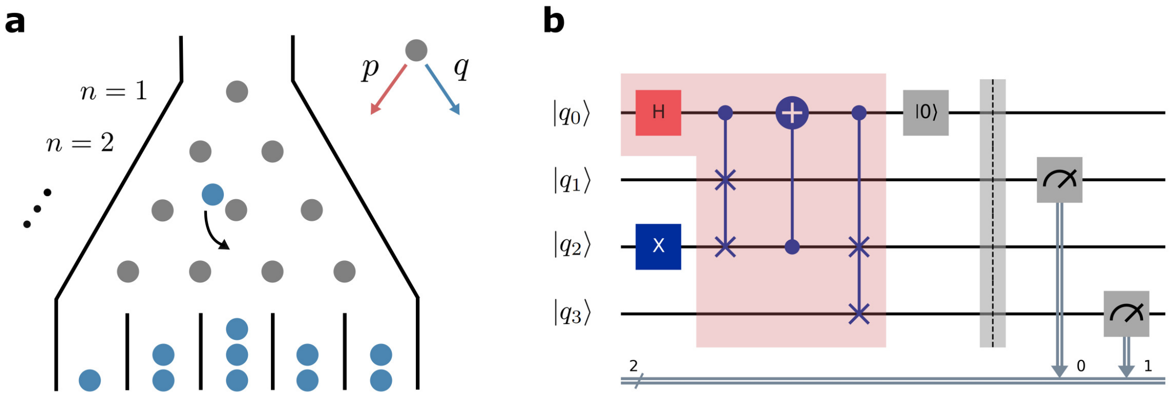

FIG. 1: **a** A schematic of a Galton Board. The number of layers is given by n. The number of buckets is $n+1$. The left drop probability is p. The number of buckets is $n+1$ **b** A quantum circuit for the Quantum Galton Board with $n = 1$ and $p = 0.5$. The highlighted circuit can be thought of as the peg (with or without H).

Single layer boards

The circuit diagram representing a QGB for a single layer with $p = 0.5$ is shown in fig. 1 (b). The top qubit $|q_0\rangle$ acts as an ancilla qubit, while the other qubits $|q_1\rangle \to |q_3\rangle$ represent the column position of the bead, where the state $|1\rangle$ occurs when the bead is present, and $|0\rangle$ when it is not. In particular, the first X gate initialises the bead in the centre of the board. The next part of the code then represents the peg, and moves the beads position to neighbouring qubits with equal probability. This occurs as two steps (1) the H gate puts the ancilla qubit in superposition, and (2) the first $CSWAP$ moves the centre qubit to the right when the ancilla is in the state $|1\rangle$, the $|0\rangle$ part of the ancilla qubit is then inverted with a $CNOT$ gate, and a second $CSWAP$ moves the centre qubit to the left for the remaining part of the superposition. Therefore, before the reset gate:

$$|\psi\rangle = \frac{|1001\rangle + |0011\rangle}{2} \tag{2}$$

where $|\psi\rangle$ is the state of all four qubits. Measuring the states $|q_1\rangle$ and $|q_3\rangle$ gives the probability to find the bead in the bucket to the left and right of the peg. Note that $|q_2\rangle$ is not measured since there is no bucket there.

Multi layer boards and biased pegs

The extension to multiple layers is simple and follows the logic from the single layer case. A circuit diagram for a QGB with two layers is shown in fig. 2 (a). Each peg follows the same structure as before, but with an additional $CNOT$ gate after each peg, except the last in each layer. This CNOT is used to reset the ancilla qubit, and so is not needed for the final peg since this is reset anyway. The distribution produced from $10,000$ shots on an ideal quantum computer is shown in fig. 2 (a) for ten layers. Since we have a constant probability for all pegs, the result is a normal distribution.

Pegs in a classical GB can be biased by changing their shape, or tilting the board. Instead in a QGB they are biased by replacing the Hadamard gate acting on the ancilla qubit with an x rotation $R_x(\theta)$. If one desires a left-drop probability p the required rotation is $\theta = 2\text{arcsin}\left(\sqrt{p}\right)$. Note that the symmetric case $p = 0.5$ is recovered when $\theta = \pi/2$. To individually bias each peg with rotation angle $\theta_1, \theta_2, ...\theta_{n(n+1)/2}$ three further changes must be made: (1) the $CNOT$ resetting the ancilla qubit must be replaced by a full reset gate (2) after this reset the $R_x(\theta_i)$ gate for the next peg must be applied to the ancilla qubit, and (3) to correct for the loss of the $CNOT$ an additional $CNOT$ must be applied for each one lost at the end of the layer, followed by a further reset gate. Further details can be found in my Presentation Deck.

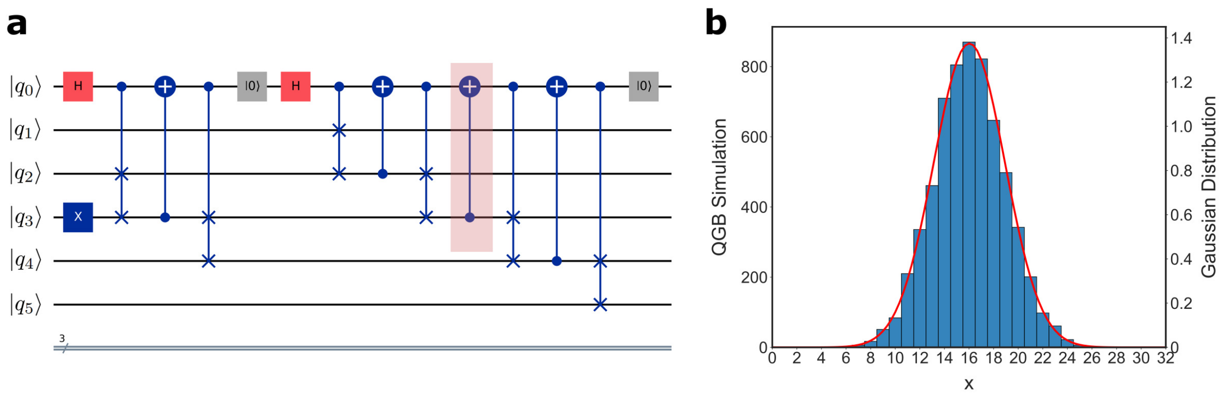

FIG. 2: **a** A Quantum circuit for a two layer ($n = 2$) QGB with $p = 0.5$. The shaded region indicates an additional $CNOT$ gate between two pegs that resets the ancilla qubit. **b** QGB local simulation for $n = 4$. The results have been post-processed as done in [2].

Exponential And Hadamard Distributions

It is possible to generate different types of statistical distribution with small modifications to the QGB. For example, to make an exponential distribution $\sim \lambda e^{-\lambda}$ I use a variant of the QGB in the previous section. The first peg has left drop probability $p = p_0 = 1 - e^{-\lambda/m}$ where m is constant. However, for the next layer of pegs, if the bead went left before, the next peg forces it to keep going left until it reaches the bottom i.e $p = 1$. If the bead went right before then the next peg has the initial probabilities $p = p_0$. The resulting distribution approximates an exponential. The distribution of a Hadamard Quantum Walk [3] can be generated by removing the reset gates and replacing it with a$CNOT$ gate on the ancilla qubit. Depending on the initial state, this generates both skewed and bimodal distributions [4].

[1] Christopher Oshman, "Experiments with a galton board," (2002).
[2] Mark Carney and Ben Varcoe, "Universal statistical simulator," arXiv preprint arXiv:2202.01735 (2022).
[3] Google Quantum AI, "Quantum walk — Cirq," `https://quantumai.google/cirq/experiments/quantum_walks` (2025), accessed: 2025-08-09.
[4] Julia Kempe, "Quantum random walks: an introductory overview," Contemporary Physics **44**, 307–327 (2003).