Simulation and Modelling
Warm-up Problem

Suppose you arrive at a bus stop t minutes after the last bus arrived at the stop. What is the expected time (waiting time) to the next bus arrival?

Recall some notation: if the waiting time random variable (r.v.) is W, then we're looking to find $E(W)$.

Let's assume that the bus arrivals are independent (they aren't in practice!) and let the bus inter-arrival time r.v., X say, have cumulative distribution function (cdf) $F(x)$.

1. Write a simulation of this system that estimates the average waiting time, $E(W)$. Assume that X can be sampled by a magical method called `iat.next()`.

2. The output from such a simulation is an *estimate* of the average waiting time and will be different each time we run the program. Can you remember the magic formula for calculating, say, the 90% confidence interval for $E(W)$? Do the simulation estimates have the right properties for this confidence interval to be exact?

3. Now formulate the model mathematically by expressing the cdf of W, i.e.

   $$F_W(x) = P(W \leq x) = P(X \leq t + x \mid X > t) \ = \ 1 - P(X > t + x \mid X > t)$$

   in terms of $F(x)$.

4. There is a wonderful result that you may not have seen before: for any continuous non-negative r.v. Y:

   $$E(Y) \ = \ \int_0^\infty y f_Y(y)\, dy \ = \ \int_0^\infty 1 - F_Y(y)\, dy$$

   (Can you derive it?!)

   Now plug in the cdf for the following inter-arrival time distributions and work out $E(W)$ using the above result:

   (a) Uniform on $(0, b)$, i.e. $F(x) = x/b$

(b) Exponential, parameter λ, i.e. $F(x) = 1 - e^{-\lambda x}$. Recall that the mean of an exponentially distributed r.v. with parameter λ is $1/\lambda$.

(c) Pareto (a "heavy-tailed" distribution), parameter a, i.e. $F(x) = 1 - x^{-a}$, where $x > 1$

How does the average waiting time vary as t increases? Is this what you might have guessed?!