

343 OPERATIONS RESEARCH

Basic Linear Algebra

Autumn Term - 2016-2017

Part I

Basic Linear Algebra

Matrix

Rectangular array of (real) numbers:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

Example:

$$A = \begin{bmatrix} 1.0 & 5.5 & 6.3 \\ 3.1 & 2.4 & 8.9 \end{bmatrix} \in \mathbb{R}^{2 \times 3}$$

Matrix Equality

Two matrices A and B are *equal*, i.e.,

$$A = B$$

if and only if (iff):

$$a_{ij} = b_{ij}$$

$$\forall i, i = 1, \dots, m$$

$$\forall j, j = 1, \dots, n$$

Vector

Matrix with only *one* column:

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} \in \mathbb{R}^{m \times 1} = \mathbb{R}^m$$

Example:

$$v = \begin{bmatrix} 1.0 \\ 2.5 \end{bmatrix} \in \mathbb{R}^2$$

Scalar Product

The scalar product of two vectors v and u is:

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$v^T u = u^T v = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

Multiplication by a Scalar

For a $m \times n$ Matrix A :

$$3 A = \begin{bmatrix} 3 a_{11} & 3 a_{12} & \dots & 3 a_{1n} \\ 3 a_{21} & 3 a_{22} & \dots & 3 a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 3 a_{m1} & 3 a_{m2} & \dots & 3 a_{mn} \end{bmatrix}$$

Matrix Addition

For two $m \times n$ Matrices A and B :

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

Matrix Transpose

For a $m \times n$ Matrix A , its transpose is

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$$

that is a $n \times m$ Matrix with:

$$(A^T)^T = A$$

Matrix Product

For a Matrix $A(m \times r)$ and a Matrix $B(r \times n)$, their product is the matrix

$$A B = C = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}$$

that is a $m \times n$ Matrix with

$$c_{ij} = (\text{row } i \text{ of } A)^T (\text{column } j \text{ of } B).$$

Matrix Product

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$c_{11} = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 5$$

Matrix Product

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$c_{12} = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = 8$$

Matrix Product

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$c_{21} = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 7$$

Matrix Product

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$c_{22} = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = 11$$

Matrix Product

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$A B = C = \begin{bmatrix} 5 & 8 \\ 7 & 11 \end{bmatrix}$$

Properties: $C = A B$

Associative:

$$A (B C) = (A B) C$$

Distributive:

$$A (B + C) = (A B) + (A C)$$

$$(A + B) C = (A C) + (B C)$$

In general, matrix product is **not** commutative:

$$A B \neq B A$$

Part II

Matrices and Systems of Linear Equations

Linear Equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

m Equations in n Variables

Matrix Notation

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$A x = b$$

Sometimes denoted:

$$\left[\begin{array}{c|c} A & b \end{array} \right]$$

Example

$$\begin{array}{rcrcrcrcrl} x_1 & + & 2x_2 & = & 5 \\ 2x_1 & - & x_2 & = & 0 \end{array}$$

Matrices: $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

Example

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & = & 5 \\ 2x_1 & - & x_2 & = & 0 \end{array}$$

$$\text{Compact: } \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & -1 & 0 \end{array} \right]$$

$$\text{Solution: } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Gauss-Jordan Method

A System of Linear Equations

$$A x = b$$

may have:

- ▶ no solutions;
- ▶ a unique solution;
- ▶ infinitely many solutions.

Commonly solved by the Gauss-Jordan Method, which uses *Elementary Row Operations (ERO)* to progressively simplify the coefficient matrix A .

ERO 1

Multiplying any row by a non-zero scalar $c \neq 0$:

$$(\text{row } i \text{ of } A') = c (\text{row } i \text{ of } A)$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 6 \\ 0 & 1 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 9 & 15 & 18 \\ 0 & 1 & 2 & 3 \end{bmatrix} = A'$$

ERO 2

Multiplying any row by a non-zero scalar $c \neq 0$
and add it to another one ($i \neq j$):

$$(\text{row } j \text{ of } A') = c (\text{row } i \text{ of } A) + (\text{row } j \text{ of } A)$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 6 \\ 0 & 1 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 6 \\ 4 & 13 & 22 & 27 \end{bmatrix} = A'$$

$$4 \begin{bmatrix} 1 & 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 13 & 22 & 27 \end{bmatrix}$$

ERO 3

Interchange any two rows i and j :

$$(\text{row } j \text{ of } A') = (\text{row } i \text{ of } A)$$

$$(\text{row } i \text{ of } A') = (\text{row } j \text{ of } A)$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 6 \\ 0 & 1 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 5 & 6 \\ 1 & 2 & 3 & 4 \end{bmatrix} = A'$$

Solving Equations

Use ERO1 and ERO2 to simplify Equations.

$$x_1 + x_2 = 2$$

$$2x_1 + 4x_2 = 7$$

Solving Equations

Use ERO1 and ERO2 to simplify Equations.

$$\begin{array}{rcl} x_1 + x_2 & = & 2 \\ 2x_1 + 4x_2 & = & 7 \end{array}$$

$$\begin{array}{rcl} x_1 + x_2 & = & 2 \\ 2x_2 & = & 3 \end{array}$$

$$(\text{row 2 of } A') = -2 (\text{row 1 of } A) + (\text{row 2 of } A)$$

Solving Equations

Use ERO1 and ERO2 to simplify Equations.

$$\begin{aligned}x_1 + x_2 &= 2 \\ 2x_1 + 4x_2 &= 7\end{aligned}$$

$$\begin{aligned}x_1 + x_2 &= 2 \\ 2x_2 &= 3\end{aligned}$$

$$\begin{aligned}x_1 + x_2 &= 2 \\ x_2 &= \frac{3}{2}\end{aligned}$$

$$(\text{row 2 of } A') = \frac{1}{2} (\text{row 2 of } A)$$

Solving Equations

Use ERO1 and ERO2 to simplify Equations.

$$\begin{aligned}x_1 + x_2 &= 2 \\ 2x_1 + 4x_2 &= 7\end{aligned}$$

$$\begin{aligned}x_1 + x_2 &= 2 \\ 2x_2 &= 3\end{aligned}$$

$$\begin{aligned}x_1 + x_2 &= 2 \\ x_2 &= \frac{3}{2}\end{aligned}$$

$$\begin{aligned}x_1 &= \frac{1}{2} \\ x_2 &= \frac{3}{2}\end{aligned}$$

$$(\text{row 1 of } A') = -1 (\text{row 2 of } A) + (\text{row 1 of } A)$$

Solving Equations (cont)

In Compact Form:

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 4 & 7 \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 2 & 3 \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \end{array} \right]$$

Basic Principle

If

$$\left[\begin{array}{c|c} A' & b' \end{array} \right]$$

is obtained by ERO1, ER02, and ER03 from

$$\left[\begin{array}{c|c} A & b \end{array} \right]$$

Then:

$$\boxed{A' x = b'} \quad \text{and} \quad \boxed{A x = b}$$

are equivalent.

Example

$$2x_1 + 2x_2 + x_3 = 9$$

$$2x_1 - x_2 + 2x_3 = 6$$

$$x_1 - x_2 + 2x_3 = 5$$

Gauss-Jordan Method:
Solve by systematically applying the ERO

Example (cont)

$$\left[\begin{array}{ccc|c} A & & & b \end{array} \right] = \left[\begin{array}{ccc|c} 2 & 2 & 1 & 9 \\ 2 & -1 & 2 & 6 \\ 1 & -1 & 2 & 5 \end{array} \right]$$

Example (cont)

$$\left[\begin{array}{ccc|c} 2 & 2 & 1 & 9 \\ 2 & -1 & 2 & 6 \\ 1 & -1 & 2 & 5 \end{array} \right]$$

ERO 1

$$(\text{row } 1) = \frac{1}{2}(\text{row } 1)$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & \frac{9}{2} \\ 2 & -1 & 2 & 6 \\ 1 & -1 & 2 & 5 \end{array} \right]$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & \frac{9}{2} \\ \textcolor{red}{2} & -1 & 2 & 6 \\ 1 & -1 & 2 & 5 \end{array} \right]$$

ERO 2

$$(\text{row } 2) = -2 (\text{row } 1) + (\text{row } 2)$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & \frac{9}{2} \\ \color{red}{0} & \color{blue}{-3} & \color{blue}{1} & \color{blue}{-3} \\ 1 & -1 & 2 & 5 \end{array} \right]$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & \frac{9}{2} \\ 0 & -3 & 1 & -3 \\ \textcolor{red}{1} & -1 & 2 & 5 \end{array} \right]$$

ERO 2

$$(\text{row } 3) = -1 (\text{row } 1) + (\text{row } 3)$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & \frac{9}{2} \\ 0 & -3 & 1 & -3 \\ \textcolor{red}{0} & \textcolor{blue}{-2} & \textcolor{blue}{\frac{3}{2}} & \textcolor{blue}{\frac{1}{2}} \end{array} \right]$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & \frac{9}{2} \\ 0 & -3 & 1 & -3 \\ 0 & -2 & \frac{3}{2} & \frac{1}{2} \end{array} \right]$$

ERO 1

$$(\text{row } 2) = -\frac{1}{3} (\text{row } 2)$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & \frac{9}{2} \\ 0 & \color{red}{1} & \color{blue}{-\frac{1}{3}} & \color{blue}{1} \\ 0 & -2 & \frac{3}{2} & \frac{1}{2} \end{array} \right]$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & \color{red}{1} & \frac{1}{2} & \frac{9}{2} \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & -2 & \frac{3}{2} & \frac{1}{2} \end{array} \right]$$

ERO 2

$$(\text{row } 1) = -1 (\text{row } 2) + (\text{row } 1)$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{6} & \frac{7}{2} \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & -2 & \frac{3}{2} & \frac{1}{2} \end{array} \right]$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{6} & \frac{7}{2} \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & -2 & \frac{3}{2} & \frac{1}{2} \end{array} \right]$$

ERO 2

$$(\text{row } 3) = 2(\text{row } 2) + (\text{row } 3)$$

Example (cont)

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{5}{6} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & \frac{5}{6} \end{array} \right] \left| \begin{array}{c} \frac{7}{2} \\ 1 \\ \frac{5}{2} \end{array} \right]$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{6} & \frac{7}{2} \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & 0 & \frac{5}{6} & \frac{5}{2} \end{array} \right]$$

ERO 1

$$(\text{row } 3) = \frac{6}{5} (\text{row } 3)$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{6} & \frac{7}{2} \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & 0 & \textcolor{red}{1} & \textcolor{blue}{3} \end{array} \right]$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{6} & \frac{7}{2} \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

ERO 2

$$(\text{row } 1) = -\frac{5}{6} (\text{row } 3) + (\text{row } 1)$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

ERO 2

$$(\text{row } 2) = \frac{1}{3} (\text{row } 3) + (\text{row } 2)$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

Example (cont)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Example (cont)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

The solution is now straightforward:

$$x_1 = 1, x_2 = 2, \text{ and } x_3 = 3$$

Remark (ER03)

ERO3 can be used to re-arrange rows such that ERO1 and ERO2 can be applied systematically.

$$\begin{array}{rrcrcl} & & 2x_2 & + & x_3 & = & 6 \\ x_1 & + & x_2 & - & x_3 & = & 2 \\ 2x_1 & + & x_2 & + & x_3 & = & 4 \end{array}$$

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & 6 \\ 1 & 1 & -1 & 2 \\ 2 & 1 & 1 & 4 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 2 & 1 & 6 \\ 2 & 1 & 1 & 4 \end{array} \right]$$

Part III

Basic Variables and Solutions

Basic Variables

For any system of linear equations

$$A x = b$$

a variable x_i with:

- ▶ one coefficient a_{ij} equal to 1, and
- ▶ all other coefficients in row i equal to 0

is called a *Basic Variable*.

All other variables are called *Non-Basic Variables*.

Solutions I

Condition I:

$A' x = b'$ has at least one row of the form

$$\left[\begin{array}{cccc|c} 0 & 0 & \dots & 0 & c \end{array} \right]$$

with $c \neq 0$.

Condition I $\implies A x = b$ has **no** solution.

e.g. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{array} \right]$ has no solution!

Solutions II

Suppose Condition I does not hold and the set of non-basic variables is **empty**, $NBV = \emptyset$. Then $A' x = b'$ and hence $A x = b$ has **a unique** solution.

$$\text{e.g. } \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

has a unique solution:

$$x_1 = 1, x_2 = 2, x_3 = 3$$

Solutions III

Suppose Condition I does not hold but the set of non-basic variables is **non-empty**, $\text{NBV} \neq \emptyset$.

Then $A' x = b'$ and $A x = b$ have an **infinite** number of solutions.

In order to obtain these solutions, set each NBV to an arbitrary value. Then solve for the BVs in terms of the NBVs.

Example

$$\begin{array}{rcccccc} x_1 & & + & x_4 & + & x_5 & = & 3 \\ & x_2 & & + & 2x_4 & & = & 2 \\ & & x_3 & & & + & x_5 & = & 1 \end{array}$$

Example

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Example

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right]$$

Condition I does not apply, and:

$$BV = \{x_1, x_2, x_3\} \quad \text{and} \quad NBV = \{x_4, x_5\}$$

Set: $x_4 = a$ and $x_5 = b$, for any value of a and b we get:

$$x_1 = 3 - a - b$$

$$x_2 = 2 - a$$

$$x_3 = 1 - b$$

Gauss-Jordan: Summary

Does $[A \mid b]$ have a row $[0 \ 0 \ \dots \ 0 \mid c]$ with $c \neq 0$?

- ▶ YES: \nexists solution
- ▶ NO: Find BV and NBV. Is $\text{NBV} = \emptyset$?
 - ▶ YES: $\exists!$ solution
 - ▶ NO: \exists solutions (infinite)

Part IV

Linear Combinations

Linear Combinations

Let V be a set of (column) vectors,

$$V = \{v_1, v_2, v_3, \dots, v_k\}$$

all of the same dimension.

A *linear combination* of vectors in V is any vector of the form:

$$v = c_1 v_1 + c_2 v_2 + \dots + c_k v_k.$$

with $c_1, c_2, \dots, c_k \in \mathbb{R}$.

Example

$$V = \left\{ v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

$$2v_1 - v_2 = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$v_1 + 3v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

$$0v_1 + 3v_2 = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

Linear Independence I

Let V be a set of (column) vectors in \mathbb{R}^m

$$V = \{v_1, v_2, v_3, \dots, v_k\}.$$

and the null vector

$$N = \begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix}^T.$$

In order to determine if V is *linearly independent* (l.i.), we try to find a linear combinations of vectors in V which results in N .

Clearly the *trivial linear combination* always works:

$$0v_1 + 0v_2 + \dots + 0v_k = N.$$

Linear Independence II

- ▶ A set of vectors V is *linearly independent* iff the **only** linear combination which gives the null vector is the trivial linear combination.
- ▶ Otherwise the vectors in V are said to be *linearly dependent*.
- ▶ Any V containing N is linearly dependent (for $c \neq 0$)

$$\text{e.g. } c \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = N.$$

Examples

The vectors $e_1 = [1 \ 0]^T$ and $e_2 = [0 \ 1]^T$ are l.i.:

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = N \quad \Rightarrow \quad c_1 = 0 = c_2.$$

The vectors $e_1 = [1 \ 2]^T$ and $e_2 = [2 \ 4]^T$ are l.d.:

$$2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = N.$$

Rank of a Matrix

Take a $m \times n$ matrix A . Consider its column vectors:

$$C = \{c_1, c_2, \dots, c_n\}.$$

The rank of A is the maximal number of linear independent vectors in C .

$$\text{e.g. } \text{rank} \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) = 0 \quad \text{e.g. } \text{rank} \left(\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \right) = 1$$

$$\text{e.g. } \text{rank} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 2$$

Inverse of a Matrix

Let A be an $m \times m$ square matrix.

The $m \times m$ *identity matrix* is given by:

$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

A matrix A^{-1} for which the following holds:

$$A A^{-1} = A^{-1} A = I$$

is called the *inverse* matrix of A .

Inverse of a Matrix II

- ▶ The inverse of a matrix may not exist!
- ▶ An invertible matrix A is called *non-singular*.
- ▶ The inverse of an $m \times m$ matrix **does not** exist if $\text{rank}(A) < m$.

Matrix inversion can be used to solve linear systems:

$$\begin{aligned} A x &= b \\ A^{-1} A x &= A^{-1} b \\ I x &= A^{-1} b \\ x &= A^{-1} b \end{aligned}$$

The Gauss-Jordan process computes A^{-1} indirectly.