A Tiny bit of Revision

Basics...

- Expected value (mean) of rv X:
  - Discrete case: $E(X) = \sum_n nP(X = n)$ for pdf P
  - Continuous case: $E(X) = \int_{-\infty}^{\infty} xf(x)dx$ for density f
- More generally, the k^{th} moment of X is:
  - Discrete case: $E(X^k) = \sum_n n^k P(X = n)$
  - Continuous case: $E(X^k) = \int_{-\infty}^{\infty} x^k f(x)dx$
- The variance of X is $V(X) = E(X^2) - E(X)^2$
- A useful measure of "variation" in the underlying distribution is the squared coefficient of variation:

$$C_X^2 = \frac{V(X)}{E(X)^2}$$

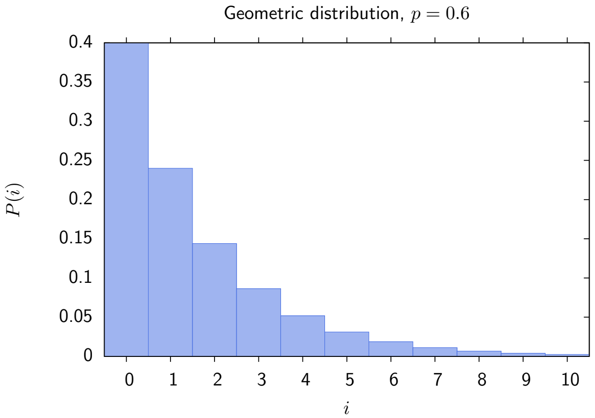
- Small $C_X^2 \Rightarrow$ low variability; Large $C_X^2 \Rightarrow$ high variability

Some discrete probability distributions

Geometric, parameter $0 \leq p \leq 1$

$$P(n) = P(X = n) = (1-p)^n p$$

$$E(X) = \frac{1-p}{p}, \quad V(X) = \frac{1-p}{p^2}, \quad C_X^2 = \frac{1}{1-p}$$

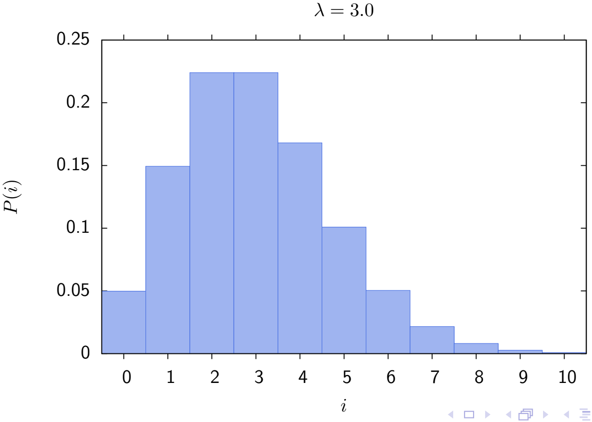


Some discrete probability distributions

Poisson, parameter $\lambda > 0$

$$P(n) = P(X = n) = \frac{\lambda^n e^{-\lambda}}{n!}$$

$$E(X) = \lambda, \;\; V(X) = \lambda, \;\; C_X^2 = \frac{1}{\lambda}$$

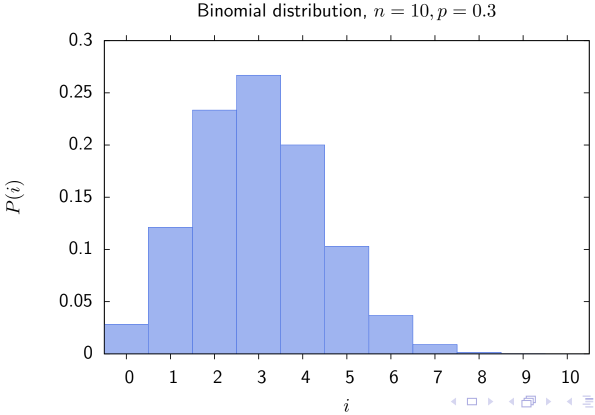


Some discrete probability distributions

Binomial, parameters n, p

$$P(k) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$E(X) = np, \quad V(X) = np(1-p) \quad C_X^2 = \frac{1-p}{np}$$



- Note: For large n and small p the binomial distribution converges on the Poisson distribution with $\lambda = np$:
- If $np = \lambda$ then $p = \lambda/n$, i.e.

$$\begin{aligned} P(k) &= \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \lim_{n\to\infty} \frac{n!}{k!\,(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n} \left(1 - \frac{\lambda}{n}\right)^{-k} \end{aligned}$$

- For large n the term $n!/(n-k)!$ converges to 1 and the $(1-\lambda/n)$ terms converge to $e^{-\lambda}$ and 1 respectively, so

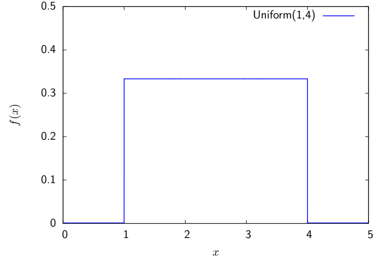
$$P(k) \to \frac{\lambda^k e^{-\lambda}}{k!} \quad \text{as } \; n \to \infty$$

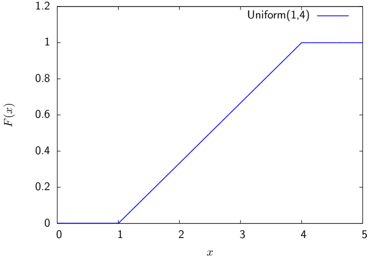
Some continuous probability distributions

Uniform, parameters $-\infty < a < b < \infty$

$$f(x) = \frac{1}{b-a}, \quad F(x) = \frac{x-a}{b-a}, \quad a \le x \le b$$

$$E(X) = \frac{a+b}{2}, \quad V(X) = \frac{1}{12}(b-a)^2, \quad C_X^2 = \frac{(b-a)^2}{3(a+b)^2}$$



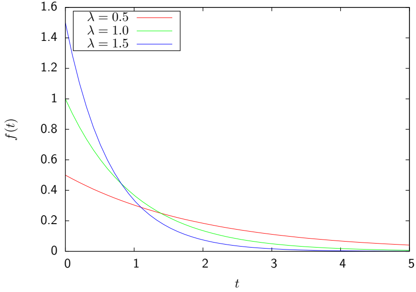


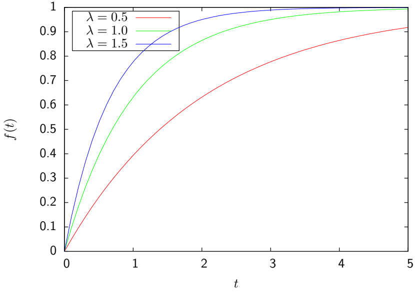
Some continuous probability distributions

Exponential, parameter $\lambda > 0$

$$f(x) = \lambda e^{-\lambda x}, \;\; F(x) = 1 - e^{-\lambda x}, \;\; x \geq 0$$

$$E(X) = \frac{1}{\lambda}, \;\; V(X) = \frac{1}{\lambda^2}, \;\; C_X^2 = 1$$



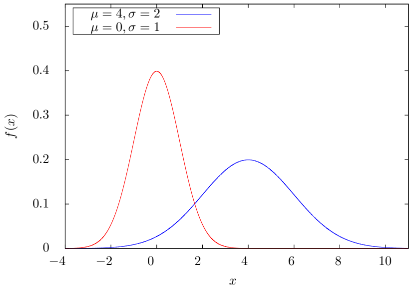


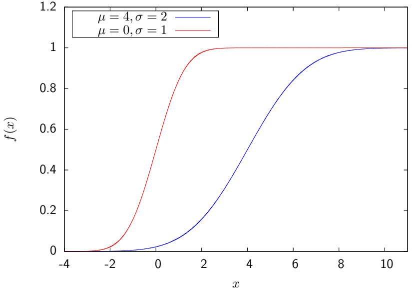
Some continuous probability distributions

Normal, parameters $\mu, \sigma > 0$

$$f(x) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$E(X) = \mu, \;\; V(X) = \sigma^2, \;\; C_X^2 = \frac{\sigma^2}{\mu^2}$$



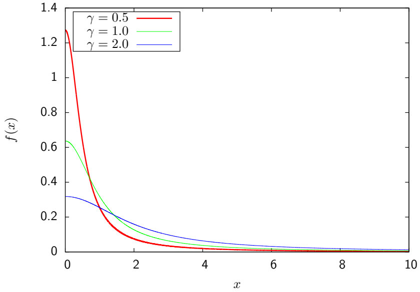


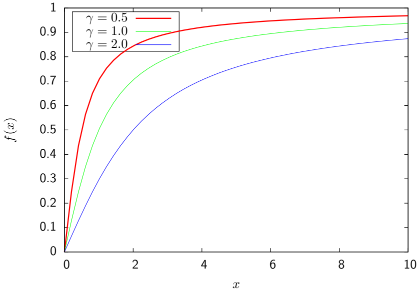
Some continuous probability distributions

(Positive) Cauchy, parameter $\gamma > 0$

$$f(x) = \frac{2}{\pi\gamma\left[1 + \left(\frac{x}{\gamma}\right)^2\right]}, \quad F(x) = \frac{2}{\pi} arctan\left(\frac{x}{\gamma}\right) \quad x \geq 0$$

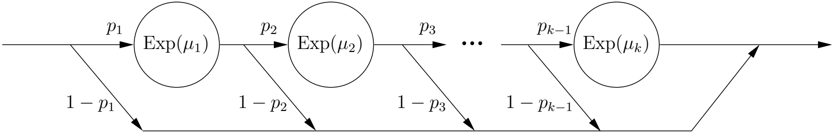
$$E(X) = \infty(!), \quad V(X) = \infty, \quad C_X^2 = \infty$$





Some continuous probability distributions

Coxian (a phase-type distribution, parameters $p_i, \mu_i,\ 0 \le p_i \le 1,\ 1 \le i \le k-1,\ \mu_k$)



- Each circle represents an exponentially-distributed delay
- The density and cdf (omitted) are messy
- **Important fact**: Any distribution can be approximated arbitrarily closely by a Coxian distribution
- Later (if there's time) we'll see an *extraordinary* result in the analysis of queueing networks which depends on this crucial property

Bayes' Theorem

For events A and B,

$$P(A \mid B) = \frac{P(A \,\&\, B)}{P(B)}$$

Note that, by symmetry,

$$P(B \mid A) = \frac{P(A \,\&\, B)}{P(A)}$$

so, equating the $P(A \,\&\, B)$ terms,

$$P(A \mid B)P(B) \;=\; P(B \mid A)P(A)$$

we can derive an alternative formulation of the theorem:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

Miscellaneous

- Law of total probability: The most common formulation assumes event A and discrete rv X:

$$P(A) = \sum_k P(A \mid X = k) \times P(X = k)$$

- Density functions and infinitesimal intervals: for a continuous rv X it doesn't make sense to ask for $P(X = x)$, because this is 0. However...

$$P(x < X \leq x + dx) = f(x)dx$$