Dynamical Systems

Continuous maps of metric spaces

- We work with metric spaces, usually a subset of $\mathbb{R}^n$ with the Euclidean norm.
- A map of metric spaces $F : X \to Y$ is **continuous at** $x \in X$ if it preserves the limits of convergent sequences, i.e., for all sequences $(x_n)_{n \geq 0}$ in X:

$$x_n \to x \Rightarrow F(x_n) \to F(x).$$

- F is **continuous** if it is continuous at all $x \in X$.
- **Examples**:
  - All polynomials, $\sin x$, $\cos x$, e^x are continuous maps.
  - $x \mapsto 1/x : \mathbb{R} \to \mathbb{R}$ is not continuous at $x = 0$ no matter what value we give to $1/0$. Similarly for $\tan x$ at $x = (n + \frac{1}{2})\pi$ for any integer n.
  - The step function $s : \mathbb{R} \to \mathbb{R} : x \mapsto 0$ if $x \leq 0$ and 1 otherwise, is not continuous at 0.
  - Intuitively, a map $\mathbb{R} \to \mathbb{R}$ is continuous iff its graph can be drawn with a pen without leaving the paper.

Continuity and Computability

- Continuity of F is necessary for the computability of F.
- Here is a simple argument for $F : \mathbb{R} \to \mathbb{R}$ to illustrate this.
- An irrational number like π has an infinite decimal expansion and is computable only as the limit of an effective sequence of rationals $(x_n)_{n \geq 0}$ with say $x_0 = 3, x_1 = 3.1, x_2 = 3.14 \cdots$.
- Hence to compute $F(\pi)$ our only hope is to compute $F(x_n)$ for each rational x_n and then take the limit. This requires $F(x_n) \to F(\pi)$ as $n \to \infty$.

Discrete dynamical systems

- A **deterministic discrete dynamical system** $F : X \to X$ is the action of a continuous map F on a metric space (X, d), usually a subset of $\mathbb{R}^n$.
- X is the set of **states** of the system;
  and d measures the distance between states.
- If $x \in X$ is the state at time t, then $F(x)$ is the state at $t + 1$.
- We assume F does not depend on t.
- Here are some key continuous maps giving rise to interesting dynamical systems in $\mathbb{R}^n$:
- Linear maps $\mathbb{R}^n \to \mathbb{R}^n$, eg $x \mapsto ax : \mathbb{R} \to \mathbb{R}$ for any $a \in \mathbb{R}$.
- Quadratic family $F_c : \mathbb{R} \to \mathbb{R} : x \mapsto cx(1 - x)$ for $c \in [1, 4]$.
- We have seen that the behaviour of F_c becomes chaotic at certain values of c.

In Finance

Suppose we deposit $1,000 in a bank at 10% interest. If we leave this money untouched for *n* years, how much money will we have in our account at the end of this period?

Example (Money in the Bank)

$$
\begin{aligned}
x_0 &= 1000, \\
x_1 &= x_0 + 0.1x_0 = 1.1x_0, \\
&\vdots \\
x_n &= x_{n-1} + 0.1x_{n-1} = 1.1x_{n-1}.
\end{aligned}
$$

This linear map is one of the simplest examples of an **iterative process** or discrete dynamical system. $x_n = 1.1x_{n-1}$ is a 1st order **difference equation.** In this case, the function we iterate is $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = 1.1x$.

From Differential Equations to Dynamical Systems

Consider a simple 1d differential equation for $x : \mathbb{R}^+ \to \mathbb{R}$.

- State space $\mathbb{R}$ and continuous time space $\mathbb{R}^+ = [0, \infty)$.

$$\frac{dx}{dt} = \lambda x \qquad \text{with parameter } \lambda \in \mathbb{R}$$

- The general solution: $x(t) = x(0)e^{\lambda t}$ where $x(0)$ is the initial value of x at $t = 0$.
- We have:

$$x(t+1) = x(0)e^{\lambda(t+1)} = x(0)(e^{\lambda+(\lambda t)}) = e^{\lambda}(x(0)e^{\lambda t}) = e^{\lambda}x(t)$$

- Let $F : \mathbb{R} \to \mathbb{R}$ be given by $F(x) = e^{\lambda}x$.
- F gives the **time-one** evolution of the differential equation.
- F gives also a discrete dynamical system and for different values of λ we study the **qualitative behaviour** of the original differential equation.

Iteration

- Given a function $F : X \to X$ and an initial value x_0, what ultimately happens to the sequence of iterates

$$x_0, F(x_0), F(F(x_0)), F(F(F(x_0))), \ldots.$$

- We shall use the notation

$$F^{(2)}(x) = F(F(x)), F^{(3)}(x) = F(F(F(x))), \ldots$$

  For simplicity, when there is no ambiguity, we drop the brackets in the exponent and write

$$F^n(x) := F^{(n)}(x).$$

- Thus our goal is to describe the *asymptotic behaviour* of the iteration of the function F, i.e. the behaviour of $F^n(x_0)$ as $n \to \infty$ for various initial points x_0.

Orbits

Definition

Given $x_0 \in X$, we define the **orbit of x_0 under F** to be the sequence of points

$$x_0 = F^0(x_0), x_1 = F(x_0), x_2 = F^2(x_0), \ldots, x_n = F^n(x_0), \ldots.$$

The point x_0 is called the **initial point** of the orbit.

Example

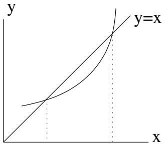
If $F(x) = \sin(x)$, the orbit of $x_0 = 123$ is

$$x_0 = 123, x_1 = -0.4599..., x_2 = -0.4439...,$$

$$x_3 = -0.4294..., \ldots, x_{1000} = -0.0543..., x_{1001} = -0.0543..., \ldots$$

Finite Orbits

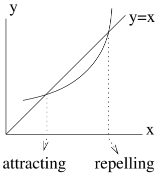
- A **fixed point** is a point x_0 that satisfies $F(x_0) = x_0$.



- **Example:** $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = 4x(1-x)$ has two fixed points at $x = 0$ and $x = 3/4$.
- The point x_0 is **periodic** if $F^n(x_0) = x_0$ for some $n > 0$. The least such n is called the **period** of the orbit. Such an orbit is a repeating sequence of numbers.
- **Example:** $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = -x$ has periodic points of period $n = 2$ for all $x \neq 0$.
- A point x_0 is called **eventually fixed** or **eventually periodic** if x_0 itself is not fixed or periodic, but some point on the orbit of x_0 is fixed or periodic.
- For the map $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = 4x(1-x)$, the point $x = 1$ is eventually fixed since $F(1) = 0$, $F(0) = 0$.

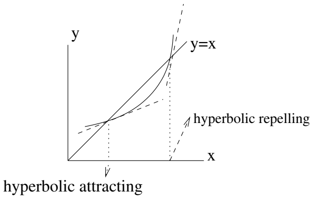
Attracting and Repelling Fixed or Periodic Points

- A fixed point x_0 is **attracting** if the orbit of any nearby point converges to x_0.
- The **basin** of attraction of x_0 is the set of all points whose orbits converge to x_0. The basin can contain points very far from x_0 as well as nearby points.
- **Example:** Take $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = x/2$. Then 0 is an attracting fixed point with basin of attraction $\mathbb{R}$.
- A fixed point x_0 is **repelling** if the orbit of any nearby point runs away from x_0.
- **Example:** Take $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = 2x$. Then 0 is a repelling fixed point.



Attracting/Repelling hyperbolic Fixed/Periodic Points

- If $f : \mathbb{R} \to \mathbb{R}$ has continuous derivative f', then a fixed point x_0 is **attracting** if $|f'(x_0)| < 1$. If $|f'(x_0)| > 1$, then x_0 is **repelling**. In both cases we say x_0 is **hyperbolic**.
- If x_0 is a fixed point of f and $|f'(x_0)| = 1$ then further analysis is required (eg Taylor series expansion near x_0) to determine the type of x_0, which can also be attracting in one direction and repelling in the other.



- If x_0 is a periodic point of period n, then x_0 is **attracting** and **hyperbolic**, if $|(f^n)'(x_0)| < 1$.
- Similarly, x_0 is **repelling** and **hyperbolic**, if $|(f^n)'(x_0)| > 1$.

Graphical Analysis

Given the graph of a function F we plot the orbit of a point x_0.

- First, superimpose the diagonal line $y = x$ on the graph. (The points of intersection are the fixed points of F.)
- Begin at (x_0, x_0) on the diagonal. Draw a vertical line to the graph of F, meeting it at $(x_0, F(x_0))$.
- From this point draw a horizontal line to the diagonal finishing at $(F(x_0), F(x_0))$. This gives us $F(x_0)$, the next point on the orbit of x_0.
- Draw another vertical line to graph of F, intersecting it at $F^2(x_0))$.
- From this point draw a horizontal line to the diagonal meeting it at $(F^2(x_0), F^2(x_0))$.
- This gives us $F^2(x_0)$, the next point on the orbit of x_0.
- Continue this procedure, known as **graphical analysis**. The resulting "staircase" visualises the orbit of x_0.

Graphical analysis of linear maps

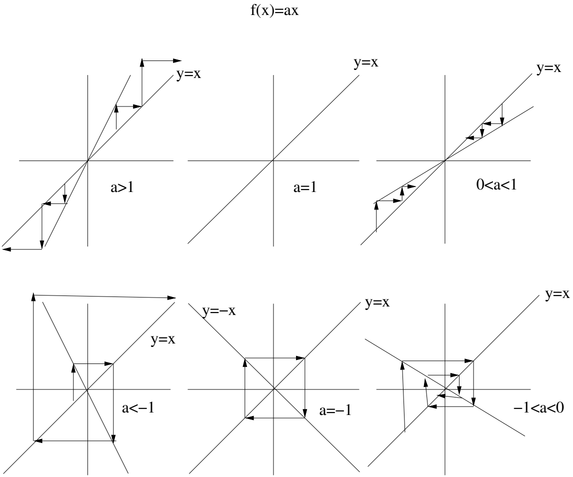
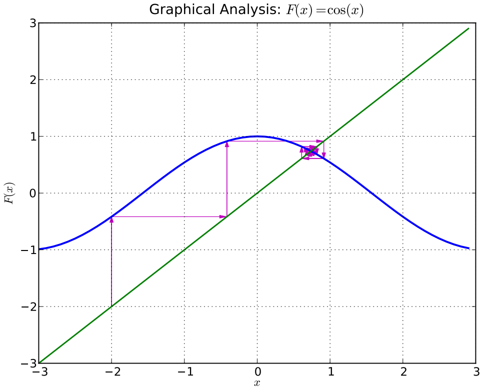


Figure : Graphical analysis of $x \mapsto ax$ for various ranges of $a \in \mathbb{R}$.

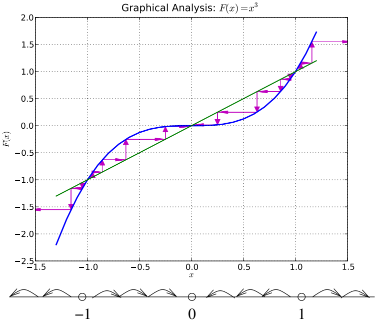
A Non-linear Example: $F(x) = \cos x$

- F has a single fixed point, which is attracting, as depicted.
- What is the basin of attraction of this attracting fixed point?



Phase portrait

- When graphical analysis describes the behaviour of *all* orbits of a dynamical system, we have performed a complete **orbit analysis** providing the **phase portrait** of the system.
- **Example:** Orbit analysis/phase portrait of $x \mapsto x^3$.



- What are the fixed points and the basin of the attracting fixed point?

Phase portraits of linear maps

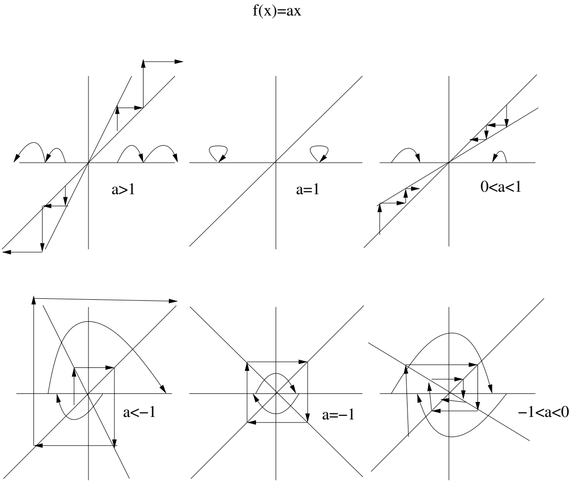


Figure : Graphical analysis of $x \mapsto ax$

Bifurcation

- Consider the one-parameter family of quadratic maps $x \mapsto x^2 + d$ where $d \in \mathbb{R}$.
- For $d > 1/4$, no fixed points and all orbits tend to ∞.
- For $d = 1/4$, a fixed point at $x = 1/2$, the double root of $x^2 + 1/4 = x$.
- This fixed point is locally attracting on the left $x < 1/2$ and repelling on the right $x > 1/2$.
- For d just less than $1/4$, two fixed points $x_1 < x_2$, with x_1 attracting and x_2 repelling.
- The family $x \mapsto x^2 + d$ undergoes **bifurcation** at $d = 1/4$.