Dynamical Systems and Deep Learning

Assessed Coursework 1

You can work in groups of two or individually on your own.
Please submit a hard copy of your solution by 16:00 on 31-10-16.

Given a dynamical system $F : X \to X$ and an attracting fixed point $x_0 \in X$ of F, the **basin of attraction** $B(x_0) \subset X$ of x_0 is defined as the set of points whose orbits tend to x_0, i.e.,

$$B(x_0) = \{y \in X : \lim_{n\to\infty} F^n(y) = x_0\}.$$

Note that if x_0 is an attracting fixed point, then its basin of attraction can contain points which are not in its neighbourhood but whose orbits will **eventually** land in a neighbourhood of x_0.

- Find the fixed points of $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = x^4 - 2x^2$; show that it has two hyperbolic attracting and two hyperbolic repelling fixed points.
  (10 Marks)

- Find the roots of F and its local extrema (the values of local maxima and minima).
  (10 Marks)

- Let a be the greatest fixed point of F and r be its least repellor. Let b, c, d with $d < -1$ and $0 < c < b$ be the three points that are mapped to the repeller r, i.e., $F(b) = F(c) = F(d) = r$. Sketch the graph of F with the points a, b, c, d and r marked on the x-axis.
  (10 Marks)

- Show graphically that there is a strictly increasing sequence $b_0 < b_1 < b_2 < \cdots < b_n < \cdots$ with $b_0 = b$, that is a **backward orbit** for F i.e., $F(b_{n+1}) = b_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n = a$. (You only need to obtain the sequence $b_0 < b_1 < b_2 < \cdots < b_n < \cdots$ graphically without any numerical computation or mathematical analysis.)

Similarly, show graphically (without numerical computation or mathematical analysis) that there is a strictly increasing sequence $c_0 < c_1 < c_2 < \cdots < c_n < \cdots$ with $c_0 = c$, that is a backward orbit for F, i.e., $F(c_{n+1}) = c_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} c_n = a$.
(20 Marks)

- Determine the basins of attraction of each attracting fixed point of F using the two sequences $b_0 < b_1 < b_2 < \cdots < b_n < \cdots$ and $c_0 < c_1 < c_2 < \cdots < c_n < \cdots$ obtained in the previous part.
(30 Marks)

  Hint: What happens to an orbit starting at a point $x_0 \in (c_n, b_n)$ (i.e., $c_n < x_0 < b_n$) for $n \geq 1$? And what happens to an orbit starting from a point in (b_n, c_{n+1}) for $n \geq 0$?

- Determine the complete phase portrait of F on $\mathbb{R}$.
(20 Marks)