Title

Mathematics Extended Essay

Session: May 2025

Candidate Name: Angus Keesbury

Candidate Session Number: 006312-0028

Word Count:

Contents

Contents **i**

1 Introduction **1**

2 The Integrals of Multivariate Calculus **1**

2.1 Integrals of one variable 1

2.2 Integrals of two and three variables 2

2.3 Change of Variables 3

3 The Algebra of Differential Forms **5**

3.1 Properties of forms 5

3.2 Relations between forms 6

3.3 The exterior derivative 7

4 Generalizing The Integral **8**

4.1 The integral of a 1-form over a k-manifold 8

5 Conclusion **9**

Works Referenced **9**

A. A **9**

B. B **9**

1 Introduction

Research Question: How can Areas of Mathematics Such as Exterior Algebra and Differential Topology be Applied in Order to Generalize the Integrals Found in Multivariate Calculus?

The integral is an essential piece of calculus, usually being taught after the derivative and before the infinite series in a first-year course on the subject. Its introduction will usually begin with computing the area under a function using Riemann sums, and then evaluating the limit of these sums as the number of rectangles approaches infinite. It is then generally understood that the integral of a function on a region is the area under the function on that region, and this idea extends nicely to higher dimensions —the double integral is a measure of area, the triple integral a measure of volume.

2 The Integrals of Multivariate Calculus

In this section, an examination of the integrals of multivariate calculus will be provided, with the aim of identifying a list of properties which are shared between them and which must be therefore shared by the general integral. Further, a series of notational and algebraic conventions will be introduced during this section, the latter of which will be formalized in the subsequent section regarding the exterior algebras of differential forms. The reason for doing this is that, bluntly, these conventions will prove themselves to yield more naturally to the sort of abstraction sought by this paper. A heavier focus on the integrands shall be placed than is typical, and, in general, the notations of vector calculus shall be avoided in favor of more discrete terms which allow for patterns to more readily be noticed.

2.1 Integrals of one variable

First, consider the classic definite integral of a function f of 1 variable x on an interval $[a, b] \in \mathbb{R}^2$,

$$\int_{[a,b]} f(x)dx,$$

which shall, for the purposes of this investigation, not be understood in terms of its limit definition[1]. This integral is the type which is the focus of calculus II, and thus is a good place to start when looking for properties. Let it be noted that the notation used places the limits at the bottom, as this will be more useful later, but it is equivalent to writing $\int_a^b f(x)dx$.

The most immediate thought in terms of properties of the definite interval of a one-variable function is the 'fundamental theorem of calculus[2], and therefore is a good starting place for generalization:

Theorem 2.1 (The fundamental theorem of calculus.)**.** *For some function $f : \mathbb{R} \to \mathbb{R}^2$ which is continuously differentiable on $a, b \in \mathbb{R}$,*

$$\int_{[a,b]} \frac{d}{dx} f(x)dx = f(b) - f(a)$$

The implications of this theorem are numerous, such as that the result of integrating a function depends solely upon the value of the function at its endpoints, as well as that

$$\int_{[a,b]} f(x)dx = \int f(b)dx - \int f(a)dx = F(b) - F(a) \tag{2.1}$$

where $F(x)$ is the family of curves satisfying $F'(x) - f(x) = 0$. Further, it immediately follows that

$$\int_{[a,a]} f(x)dx = 0. \tag{2.2}$$

It is then likewise evident that the interval over which the integration is being performed is a 1-dimensional space with an orientation, that is, $[a, b] = -[b, a]$, and as such the integral over this space shall naturally preserve this orientation. This is shown by

$$\int_{[a,b]} f(x)dx = F(b) - F(a) = -F(a) - F(b) = -\int_{[b,a]} f(x)dx, \tag{2.3}$$

2.2 Integrals of two and three variables

When studying calculus, integrals of two and three variables are often considered together, as they are both used to describe surfaces in 3-dimensional space, as well as the fact that their intuitions are almost identical. It is because of the similarities of the two integrals that they shall be of much importance to this investigation, as it shall be a great deal easier to recognize patterns in these objects. The fundamental theorem of calculus in 2.1 is as essential to the study of 1-integrals as the theorems of Green and Stokes are to the study of 2- and 3- dimensional integrals. As such, much of this section shall concern these two theorems and their implications, and they shall arise once more in the later sections as natural results of the generalized integral.

[1]Weisstein, Eric W. "Riemann Integral." From MathWorld–A Wolfram Web Resource. [5]
[2]Note that no proofs shall be provided for any theorems which are standard material up to calculus IV

Integrals of two dimensions. The theorem of Green in the plane.

2.3 Change of Variables

Often, an integral is much easier to compute when a change of basis is induced, and as such it would be greatly beneficial to have an integrand which transforms naturally upon a change of basis. A very common example of change of basis is given in many calculus II courses when computing integrals which would be near impossible to solve by hand; this is is, of course, u- and trigonometric substitutions. However this is much more than a simple mathematical trick for making calculations easier, and in fact there is a great deal of theory that goes into the transformation properties of the integrand.

A perfect example of this is the trigonometric substitution, which transforms the integral into polar coordinates, in other a transformation from Cartesian coordinates to curvilinear coordinates. This is a difficult transformation, however, because one cannot expect a cartesian basis to transform linearly into a curvilinear basis, and as such the transformation of the integrand is more than a simple substitution. Although it cannot be taken as given that there exists a linear relation between the two bases, it may be taken that there exists a linear relation of their differentials, that is, the dx at the end of the integrand. As such, under the transformation

$$\begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned} \tag{2.4}$$

the differential dx should transform according to[3]

$$dx = \frac{dx}{dt} dt \tag{2.5}$$

or for this case

$$dx = -\sin(t) dt. \tag{2.6}$$

The notion of a variable change translates to higher dimensions as well, for instance we shall consider another case of trigonometric substitution, this time into spherical coordinates. The system of spherical coordinates may be described adequately by three variables: the latitude u^1, the longitude u^2, and the radius r. These correspond to the Cartesian system of coordinates according to

$$\begin{aligned} x &= \phi_1(r, u^1, u^2) = r\cos\left(u^1\right)\sin\left(u^2\right) \\ y &= \phi_2(r, u^1, u^2) = r\sin\left(u^1\right)\sin\left(u^2\right) \\ z &= \phi_3(r, u^1, u^2) = r\cos\left(u^2\right) \end{aligned} \tag{2.7}$$

Then the Cartesian volume form $dV = dxdydz$ must also transform accordingly when the basis vectors are switched, however this transformation, like 2.6, is to be nonlinear and depend heavily on the value $\sin\left(u^1\right)$, as is illustrated by 2.7. Here it should be clear that these coefficients should be the Jacobian determinant of the transformation, $\mathcal{D}_{\phi_i}(r, u^1, u^2)$, which describes how exactly the volume of an infinitesimal cube scales under this transformation of basis. The Jacobian matrix is defined to be the partial differentials of the new basis vectors with respect to the old[4], or:

$$\mathcal{J}_{ij} = \frac{\partial \phi_i}{\partial u^j} \tag{2.8}$$

so for 2.7 equation 2.8 becomes

$$\mathcal{J}_{ij} = \begin{pmatrix} \sin\left(u^1\right)\cos\left(u^2\right) & -r\sin\left(u^1\right)\sin\left(u^2\right) & r\cos\left(u^1\right)\cos\left(u^2\right) \\ \sin\left(u^1\right)\sin\left(u^2\right) & r\cos\left(u^1\right)\sin\left(u^2\right) & r\sin\left(u^1\right)\cos\left(u^2\right) \\ \cos\left(u^2\right) & 0 & -r\sin\left(u^2\right) \end{pmatrix} \tag{2.9}$$

and the determinant becomes

$$\mathcal{D} = -r\sin\left(u^1\right); \tag{2.10}$$

Thus, the volume form for spherical coordinates is $-r\sin\left(u^1\right)drdu^1du^2$.

In general, for some diffeomorphism of two subsets of $\mathbb{R}^l$, $\phi : U \to V$, where the bases of U and V are x_i and y_i respectively, the volume form on V is given by $dV = \mathcal{D}_\phi(y_i)dy^I$. One last formalism now stands in the way of a general formula for the change of variables of an integrand, namely, how does the argument of the integrand transform, however this is nearly trivial. When given a function $f : U \to U$ and the function ϕ as defined above, the action of f done on U shall be equivalent to the action of $f \circ \phi$ done on V.

As such the transformation of an integrand, whose general form is that of

$$\omega = \sum_I \psi_I dx^I \tag{2.11}$$

(this notation is introduced formally as a differential form, however for this section they need not be defined as separate from the general integrand), may be defined. This result shall be foundational to the latter sections.

Theorem 2.2 (Change of variables in an integrand)**.** *Consider the integrand ω defined by 2.11 on $U \subset \mathbb{R}^n$ (local coordinates x_i) and the diffeomorphism $\phi : U \to V$ ($V \subset \mathbb{R}^l$ with local coordinates y_i).*

Then ω is defined on V according to

$$\omega = \psi \circ \phi \mathcal{D}_{\phi}(x_i) dy^I$$

3 The Algebra of Differential Forms

In the previous section, the idea of a differential form, an object formed from the different combinations of the differentials of the basis vectors, was introduced as a classification method for multivariate integrands. In this section, this will be built upon, and the algebraic basis for manipulating these differential forms shall be formalized, and by doing so we will lay the groundwork for the generalization of the integral. Note that this section is by far the most algebraically rigorous of this paper, and that many new and interesting topics will be introduced in a relatively short span which will not yield to a detailed investigation of each, as they are not central to the question posed by this paper.

3.1 Properties of forms

The differential forms introduced previously are in fact members of an entire family of algebras which are built up around a vector space V, known as exterior algebras. These algebras are defined by a product, termed the exterior or wedge product, and a space whose basis vectors are the exterior products of the space V. Let $\mathbf{e_i}$ be the basis vectors of the three-dimensional space V, and let $\Lambda^k(V)$ denote the kth exterior power of V. The basis vectors of $\Lambda^k(V)$ are the possible combinations of the basis vectors of V under the exterior product, as such the dimension of $\Lambda^k(V)$ is given by $\binom{3}{k}$ or in general for any $\dim(V) = n$ by $\binom{n}{p}$.

Equally as essential to the definition of the exterior algebra is the aforementioned exterior product, which shall now be developed. The intuition is thus: consider two vectors a and b of arbitrary dimension p, both of which are centered at the origin. There then exists a plane with side lengths $|a|$ and $|b|$, and this plane must be able to be described by some object $\omega(a, b)$ which depends solely on inputs a and b. This object must also preserve the orientation of the plane, such that $\omega(b, a) = -\omega(a, b)$, and must be linear in both a and b, such that $\omega(c_1 a_1 + c_2 a_2, b) = c_1 \omega(a_1, b) + c_2 \omega(a_2, b)$ and $\omega(a, c_1 b_1 + c_2 b_2) = c_1 \omega(a, b_1) + c_2 \omega(a, b_2)$[6]. The object formed by $\omega(a, b)$ shall also clearly reside not in V, but in $\Lambda^2 V$, and as such, the bilinear form ω is called the exterior product of V.

[4]Aris 140 [4]
[5]Aris 137 [3]

It is now that a brief note on terminology is to be made. The objects which reside in $\Lambda^k(V)$ are called *k-forms*, and they take the general form of

$$w = \sum^k f_i(x_1, \ldots, x_k)(x_1 \wedge \cdots \wedge x_k). \tag{3.1}$$

Further, the part of the above summand consisting of the exterior product of k basis vectors is referred to as a *k-blade*. In general, differential forms, which are simply the forms created from blades consisting of the differentials of the basis vectors, behave almost identically to their non-differential counterparts with regard to the properties given above. As such, the purely algebraic manipulation of differential forms is much akin to that of standard forms.

3.2 Relations between forms

Above, the properties of individual forms were examined, including their associativity and commutativity laws. Now an investigation of the properties of the interactions of forms is in order, as this is also essential to understanding how integrands should behave when the integral is generalized. Consider the 1-forms ω and π, whose components are given by

$$\omega = A^i dx^i$$

and

$$\pi = B^i dx^i.$$

Then the natural sum of these terms is defined

$$\omega + \pi = (A^i + B^i)dx^i, \tag{3.2}$$

or for the general case of two p-forms with components $\pi = A^{i_1,\ldots,i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ and $\omega = B^{i_1,\ldots,i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$,

$$\omega + \pi = (A^{i_1,\ldots,i_p} dx^{i_1} + B^{i_1,\ldots,i_p} dx^{i_1}) dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{3.3}$$

While the sum of forms is directly evident in their statements, the wedge of two or more forms, especially of different degree, requires further investigation. Consider the wedge product of two 1-forms

[6] Lovelock and Rund 133. [2]

π and ω whose components are given by $\pi = A_i dx^i$ and $\omega = B_j dx^j$, which is defined to be

$$\pi \wedge \omega = A_i B_j dx^i \wedge dx^j, \tag{3.4}$$

however by the anticommutivity of forms,

$$\pi \wedge \omega = -\omega \wedge \pi \tag{3.5}$$

and as such

$$A_i B_j dx^i \wedge dx^j = -A_j B_i dx^j \wedge dx^j \tag{3.6}$$

$$A_i B_j dx^i \wedge dx^j - A_j B_i dx^j \wedge dx^i = 2A_i B_j dx^i \wedge dx^j \tag{3.7}$$

resulting in

$$\frac{1}{2}(A_i B_j dx^i \wedge dx^j - A_j B_i dx^j \wedge dx^i) = \frac{1}{2}(A_i B_j - A_j B_i) dx^i \wedge dx^j. \tag{3.8}$$

A very similar method may be applied to reach the general formula for a product of a p- and a q-form, however this is beyond the scope of this investigation. As is shown in Lovelock and Rund (135)[7], for a p-form given by $\omega = A_{i_1,\ldots,i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$ and a q-form given by $\pi = B_{j_1,\ldots,j_q} dx^{j_1} \wedge \cdots \wedge dx^{j_q}$, the wedge product $\omega \wedge \pi$ is given by

$$\omega \wedge \pi = A_{i_1,\ldots,i_p} B_{j_1,\ldots,j_q} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_q} = (-1)^{pq} \pi \wedge \omega. \tag{3.9}$$

Thus, the relations between forms have been established. From 3.4 and 3.9, all interactions of forms may be derived.

3.3 The exterior derivative

Before moving on to the generalization of the integral, an important concept from the exterior calculus must be introduced, as it shall prove to be indispenseblable in terms of integrating forms. This is, of course, the exterior derivative of a form, which plays very nicely with the concepts introduced thus far in the section. The intuition under which this is developed is as such: consider the k-form π on an open

[7] Lovelock and Rund 135 [2]

subset $V \subset \mathbb{R}^l$, which may be written generally as

$$\pi = \sum_{i,j} a_{i_j} dx^{i_j} = \sum_I a_I dx^I. \tag{3.10}$$

Denote by $d\pi$ the exterior derivative of π. Because a change of basis into a_I is given by the change of basis formula

$$da_I = \frac{\partial a_I}{\partial x^j} dx^j, \tag{3.11}$$

we define the exterior derivative of a differential form to be

$$d\pi = \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I. \tag{3.12}$$

For instance, consider the 3-form

$$\omega = \sum_{i<j<k} f_{ijk} dx^i \wedge dx^j \wedge dx^k; \tag{3.13}$$

then

$$d\omega = \frac{\partial f}{\partial x^p} dx^p \wedge dx^i \wedge dx^j \wedge dx^k + \frac{\partial f}{\partial x^q} dx^q \wedge dx^i \wedge dx^j \wedge dx^k + \frac{\partial f}{\partial dx^r} dx^r \wedge dx^i \wedge dx^j \wedge dx^k \tag{3.14}$$

4 Generalizing The Integral

In the previous sections, the ideas of integrals and forms have been developed and explored, and this section shall see their combination. In general, it is known that the input of the integral is to consist of a differential form and a region over which this form is integrable. The simplest case of this is the integration of a 1-form over an n-manifold, often considered as taking the integral of the form over the region embedded in $(n+1)$-space, however this is not the most general. In the same way that the wedge product of two forms of different dimension is more complex than that of two forms of the same dimension, so too is the integral of an m-form over a p-manifold when $m < p$. As such it shall be beneficial to first examine the integral of a 1-form over a manifold dimension k.

4.1 The integral of a 1-form over a k-manifold

From section 2.1 it is evident that the integral should preserve orientation, and also that the integral of a form over a region must depend exclusively on the boundaries of that region. We may first consider the regions $V \subset \mathbb{R}^k$ and $U \subset \mathbb{R}^k$ (to distinguish between the two the local coordinates in V shall be denoted by y_i and the local coordinates in U shall be denoted by x_i), the function $h : V \to U$, and the differential form $\omega \in \Lambda(U)$. However, because $\omega \in \Lambda(U)$ and h does not take an input in U (and therefore any of its exterior powers), the form may not immediately be integrated over V, and there must exist some differential 1-form defined exclusively from ω and h which is an element of $\Lambda(U)$.

5 Conclusion

Works Referenced

[1] "§3. Differential Forms". In: Victor Guillemin and Alan Pollack. *Differential Topology*. Providence, R.I: AMS Chelsea Pub, 2010, pp. 162–165. ISBN: 978-0-8218-5193-7.

[2] "5.1 The Exterior (or Wedge) Product of Differential Forms". In: David Lovelock and Hanno Rund. *Tensors, Differential Forms, and Variational Principles*. New York: Dover, 1989, pp. 131–136. ISBN: 978-0-486-65840-7.

[3] "7.12. Proper Transformations". In: Rutherford Aris. *Vectors, Tensors, and the Basic Equations of Fluid Mechanics*. Dover ed. Dover Books on Engineering. New York: Dover Publications, 1989, pp. 136–139. ISBN: 978-0-486-66110-0.

[4] "7.21. Contravariant Vectors". In: Rutherford Aris. *Vectors, Tensors, and the Basic Equations of Fluid Mechanics*. Dover ed. Dover Books on Engineering. New York: Dover Publications, 1989, pp. 140–141. ISBN: 978-0-486-66110-0.

[5] Eric Weisstein. *"Riemann Integral"*. In: *Wolfram MathWorld*. URL: https://mathworld.wolfram.com/RiemannIntegral.html.

A. A

B. B

[7] Guillemin and Pollack 163 [1]