

# PH389 Exam Equation Sheet

## Newtonian Mechanics

$$\mathbf{F} = m\mathbf{a} = \frac{d\mathbf{p}}{dt}$$

Conservative forces

$$\mathbf{F} = -\nabla V$$

## Lagrangian Mechanics

$$\mathcal{L} = T - V$$

$$\frac{\partial \mathcal{L}}{\partial x_i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial x'_i} = 0$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial x'_i} = 0 \leftrightarrow \text{Conserved quantity}$$

Constraints must be **Holonomic**

Free Particle Lagrangian's

$$\mathcal{L} = \frac{m}{2}(x'^2 + y'^2 + z'^2)$$

$$\mathcal{L} = \frac{m}{2}(r'^2 + r\theta'^2)$$

$$\mathcal{L} = \frac{m}{2}(r'^2 + r\theta'^2 + z'^2)$$

$$\mathcal{L} = \frac{m}{2}(r'^2 + r^2(\theta'^2 + \phi'^2 \sin^2 \theta))$$

## Noether's Theorem

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial x'_i} = 0 \leftrightarrow \text{Conserved quantity}$$

If symmetry is **continuous** it can be linearly expanded

$$\tilde{x}_i = x_i + \epsilon_i(x_j)$$

$$I(t) = \sum_i \left. \frac{\partial \mathcal{L}}{\partial x'_i} \epsilon_i(x_j) \right|_t$$

Symmetry	Conserved Quantity
Translation	Linear Momentum
Rotation	Angular Momentum
Time Translation	Energy
QM phase	Charge

## Hamiltonian Mechanics

For a closed system the Lagrangian has no **explicit** time dependence

$$\mathcal{H} = \sum_i \frac{\partial \mathcal{L}}{\partial x'_i} x'_i - \mathcal{L} \quad \mathcal{H} = T + V$$

**Canonical momentum conjugate to  $x_i$**

$$p_{x_i} = \frac{\partial \mathcal{L}}{\partial x'_i} \quad \mathcal{H} = \sum_i p_{x_i} x'_i - \mathcal{L}$$

Equations of motion

$$\frac{dx_i}{dt} = \frac{\partial \mathcal{H}}{\partial p_{x_i}} \quad \frac{dp_{x_i}}{dt} = -\frac{\partial \mathcal{H}}{\partial x_i}, \quad \frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

## Pendulum Dynamics

$$T = \frac{m}{2}l\theta'^2, \quad V = mgl(1 - \cos \theta)$$

Equations of motion

$$\theta'' = -\frac{g}{l} \sin \theta \text{ or when } \theta \ll 1, \quad \theta'' = -\frac{g}{l} \theta$$

## Small angle approximations

$$\sin \theta \approx \theta$$

$$\cos \theta \approx 1 - \frac{\theta^2}{2} \approx 1$$

$$\tan \theta \approx \theta$$

## Legendre Transform

$$g(p) = px - f(x), \text{ where } p = f'(x)$$

## Poisson Brackets

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial x_i} \frac{\partial \mathcal{H}}{\partial p_{x_i}} - \frac{\partial f}{\partial p_{x_i}} \frac{\partial \mathcal{H}}{\partial x_i} + \frac{\partial f}{\partial t} = \{f, \mathcal{H}\} + \frac{\partial f}{\partial t}$$

Useful identities

$$\{f, f\} = 0, \quad \{f, g\} = -\{g, f\}, \quad \{x_i, p_{x_j}\} = \delta_{ij}$$

$$\frac{dx_i}{dt} = \{x_i, \mathcal{H}\}, \quad \frac{dp_{x_i}}{dt} = \{p_{x_i}, \mathcal{H}\}$$

$$\{af + bg, h\} = a\{f, h\} + b\{g, h\}$$

$$\{fg, h\} = g\{f, h\} + f\{g, h\}$$

$$\text{Jacobi: } \{\{A, B\}, H\} + \{\{B, H\}, A\} + \{\{H, A\}, B\} = 0$$

$$\text{Leibniz: } \{\{A, B\}H\} = \{\{A, H\}, B\} + \{A, \{B, H\}\}$$

If  $\{f, \mathcal{H}\} = 0$  and  $f$  has no explicit time dependence:  
 $f$  is conserved

## Quantum Mechanics

Schrödinger Picture

$$i\hbar \frac{d|\psi\rangle}{dt} = \hat{H}|\psi\rangle \Leftrightarrow |\psi(t)\rangle = e^{-i\hat{H}t}|\psi(0)\rangle$$

Heisenberg Picture

$$\langle \hat{A} \rangle = \langle \psi(t) | \hat{A} | \psi(t) \rangle \Leftrightarrow \hat{A}(t) = e^{i\hat{H}t} \hat{A}(0) e^{-i\hat{H}t}$$

$$\frac{d\hat{A}}{dt} = i[\hat{H}, \hat{A}]$$

Ehrenfest's Theorem

$$[\hat{f}, \hat{g}] \leftrightarrow i\hbar\{f, g\}, \quad \langle \hat{A} \rangle \leftrightarrow A(x_i, p_{x_i})$$

Weyl quantisation

$$x^2 p = \frac{1}{3}(\hat{x}^2 \hat{p} + \hat{x} \hat{p} \hat{x} + \hat{p} \hat{x}^2)$$

If  $[\hat{A}, \hat{H}] = 0$  and  $\hat{A}$  has no explicit time dependence:  
 $\langle \hat{A} \rangle$  is conserved

## Group Theory

- 1: **Closure**: For any pair of elements  $X \circ Y$  must also be an element.
- 2: **Associativity**:  $X \circ (Y \circ Z) = (X \circ Y) \circ Z$
- 3: **Identity**: There is an element  $I$  such that  $X \circ I = X$
- 4: **Inverse**: Each element has an inverse,  $X \circ X^{-1} = I$

## Fractals

Fractal generated from shape with  $a$  sides and adding a shape that are a fraction  $\frac{1}{b}$  of original shape. Each side is replaced with  $c$  sides. This iteration now has  $d_n$  total sides.

$$\text{Total length } n=1: L_1 = \frac{d_1}{b}$$

$$\text{Number of new shapes: } S_n = a \cdot c^{n-1}$$

$$\text{Area new shape: } B_n = B_0 / (b \cdot b)^n$$

$$\text{Perimeter: } P_n = a \left(\frac{c}{b}\right)^n$$

$$\text{Total Area: } A_n = B_0 + \sum_{k=1}^n S_k B_k =$$

$$B_0 \left(1 + \frac{a}{c} \sum_{k=1}^n \left(\frac{c}{b \cdot b}\right)^k\right)$$

$$\text{Geometric Sum: } \sum_{k=1}^n r^k = \frac{r(1 - r^n)}{1 - r}$$

$$\text{Box Counting: } D = \frac{\ln N}{\ln 1/\epsilon}, \text{ where } N = c, \epsilon = 1/b$$

## Maps and Chaos

Fixed points:  $x_{n+1} = x_n$

Stability check:  $x_n = x_n + \delta_x \leftrightarrow \delta_{n+1} = Q(r)\delta_n$

Stable if:  $|Q(r)| < 1$

Linear Stability analysis

$$\frac{dx}{dt} = F(x), \quad J(x) = \frac{\partial F}{\partial x}, \text{ Sub in fixed points to } J$$

+ve : Unstable, -ve: Stable

Lyapunov Exponent

$$|\delta \mathbf{x}(t)| \simeq e^{\lambda t} |\delta \mathbf{x}(0)|$$

$$\lambda = \lim_{t \rightarrow \infty} \lim_{|\delta \mathbf{x}(0)| \rightarrow 0} \frac{1}{t} \ln \frac{|\delta \mathbf{x}(t)|}{|\delta \mathbf{x}(0)|}$$

$\lambda > 0$  Chaotic,  $\lambda < 0$  regular

$$E_n = e^{\lambda n} \epsilon, \quad n_{max} = \frac{1}{\lambda} \ln \frac{E}{\epsilon}$$