PH389 Exam Equation Sheet

Newtonian Mechanics

$\boldsymbol{F} = m\boldsymbol{a} = \frac{d\boldsymbol{p}}{dt}$

Conservative forces

$\boldsymbol{F} = -\nabla V$

Lagrangian Mechanics

$\mathcal{L} = T - V$

$\frac{\partial \mathcal{L}}{\partial x_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial x_i'} = 0$

$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial x_i'} = 0 \leftrightarrow$ Conserved quantity

Constraints must be Holonomic

Free Particle Lagrangian's

$\mathcal{L} = \frac{m}{2}(x'^2 + y'^2 + z'^2)$

$\mathcal{L} = \frac{m}{2}(r'^2 + r\theta'^2)$

$\mathcal{L} = \frac{m}{2}(r'^2 + r\theta'^2 + z'^2)$

$\mathcal{L} = \frac{m}{2}(r'^2 + r^2(\theta'^2 + \phi'^2 \sin^2\theta))$

Noether's Theorem

$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial x_i'} = 0 \leftrightarrow$ Conserved quantity

If symmetry is continuous it can be linearly expanded

$\tilde{x}_i = x_i + \epsilon_i(x_j)$

$I(t) = \sum_i \frac{\partial \mathcal{L}}{\partial x_i'}\epsilon_i(x_j)\Big|_t$

Symmetry	Conserved Quantity
Translation	Linear Momentum
Rotation	Angular Momentum
Time Translation	Energy
QM phase	Charge

Hamiltonian Mechanics

For a closed system the Lagrangian has no explicit time dependence

$\mathcal{H} = \sum_i \frac{\partial \mathcal{L}}{\partial x_i'}x_i' - \mathcal{L} \quad \mathcal{H} = T + V$

Canonical momentum conjugate to x_i

$p_{x_i} = \frac{\partial \mathcal{L}}{\partial x_i'} \quad \mathcal{H} = \sum_i p_{x_i}x_i' - \mathcal{L}$

Equations of motion

$$\frac{dx_i}{dt} = \frac{\partial \mathcal{H}}{\partial p_{x_i}} \quad \frac{dp_{x_i}}{dt} = -\frac{\partial \mathcal{H}}{\partial x_i}, \quad \frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

Pendulum Dynamics

$T = \frac{m}{2}l\theta'^2, \quad V = mgl(1 - \cos\theta)$

Equations of motion

$\theta'' = -\frac{g}{l}\sin\theta$ or when $\theta << 1, \quad \theta'' = -\frac{g}{l}\theta$

Small angle approximations

$\sin\theta \approx \theta$

$\cos\theta \approx 1 - \frac{\theta^2}{2} \approx 1$

$\tan\theta \approx \theta$

Legendre Transform

$g(p) = px - f(x)$, where $p = f'(x)$

Poisson Brackets

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial x_i}\frac{\partial \mathcal{H}}{\partial p_{x_i}} - \frac{\partial f}{\partial p_{x_i}}\frac{\partial \mathcal{H}}{\partial x_i} + \frac{\partial f}{\partial t} = \{f, \mathcal{H}\} + \frac{\partial f}{\partial t}$$

Useful identities

$\{f, f\} = 0, \quad \{f, g\} = -\{g, f\}, \quad \{x_i, p_{x_j}\} = \delta_{ij}$

$\frac{dx_i}{dt} = \{x_i, \mathcal{H}\}, \quad \frac{dp_{x_i}}{dt} = \{p_{x_i}, \mathcal{H}\}$

$\{af + bg, h\} = a\{f, h\} + b\{g, h\}$

$\{fg, h\} = g\{f, h\} + f\{g, h\}$

Jacobi: $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$

Leibniz: $\{\{f, g\}h\} = \{\{f, h\}, g\} + \{f, \{g, h\}\}$

If $\{f, \mathcal{H}\} = 0$ and f has no explicit time dependence: f is conserved

Quantum Mechanics

Schrödringer Picture

$i\frac{d|\psi\rangle}{dt} = \hat{H}|\psi\rangle \Leftrightarrow |\psi(t)\rangle = e^{-i\hat{H}t}|\psi(0)\rangle$

Heisenberg Picture

$\langle\hat{A}\rangle = \langle\psi(t)|\hat{A}|\psi(t)\rangle \Leftrightarrow \hat{A}(t) = e^{i\hat{H}t}\hat{A}(0)e^{-i\hat{H}t}$

$\frac{d\hat{A}}{dt} = i[\hat{H}, \hat{A}]$

Ehrenfest's Theorem

$[\hat{f}, \hat{g}] \leftrightarrow i\hbar\{f, g\}, \quad \langle\hat{A}\rangle \leftrightarrow A(x_i, p_{x_i})$

Weyl quantisation

$x^2p = \frac{1}{3}(\hat{x}^2\hat{p} + \hat{x}\hat{p}\hat{x} + \hat{p}\hat{x}^2)$

If $[\hat{A}, \hat{H}] = 0$ and $\hat{A}$ has no explicit time dependence: $\langle\hat{A}\rangle$ is conserved

Group Theory

1: Closure: For any pair of elements $X \circ Y$ must also be an element.

2: Associativity: $X \circ (Y \circ Z) = (X \circ Y) \circ Z$

3: Identity: There is an element I such that $X \circ I = X$

4: Inverse: Each element has an inverse, $X \circ X^{-1} = I$

Fractals

Fractal generated from shape with a sides and adding a shape that are a fraction $\frac{1}{b}$ of original shape. Each side is replaced with c sides. This iteration now has d_n total sides.

Total length n=1: $L_1 = \frac{d_1}{b}$

Number of new shapes: $S_n = a \cdot c^{n-1}$

Area new shape: $B_n = B_0/(b \cdot b)^n$

Perimeter: $P_n = a(\frac{c}{b})^n$

Total Area: $A_n = B_0 + \sum_{k=1}^{n} S_nB_n =$

$B_0(1 + \frac{a}{c}\sum_{k=1}^{n}(\frac{c}{b \cdot b})^k)$

Geometric Sum: $\sum_{k=1}^{n} r^k = \frac{r(1 - r^n)}{1 - r}$

Box Counting: $D = \frac{\ln N}{\ln 1/\epsilon}$, where $N = c$, $\epsilon = 1/b$

Maps and Chaos

Fixed points: $x_{n+1} = x_n$

Stability check: $x_n = x_n + \delta_x \leftrightarrow \delta_{n+1} = Q(r)\delta_n$

Stable if: $|Q(r)| < 1$

Linear Stability analysis

$\frac{dx}{dt} = F(x), \quad J(x) = \frac{\partial F}{\partial x}$, Sub in fixed points to J

+ve : Unstable, −ve: Stable

Lyapunov Exponent

$|\boldsymbol{\delta x}(t)| \simeq e^{\lambda t}|\boldsymbol{\delta x}(0)|$

$$\lambda = \lim_{t\to\infty}\lim_{|\boldsymbol{\delta x}(0)|\to 0}\frac{1}{t}\ln\frac{|\boldsymbol{\delta x}(t)|}{|\boldsymbol{\delta x}(0)|}$$

$\lambda > 0$ Chaotic, $\lambda < 0$ regular

$E_n = e^{\lambda n}\epsilon, \quad n_{max} = \frac{1}{\lambda}\ln\frac{E}{\epsilon}$