# PH389 Exam Cheat Sheet

## Newtonian Mechanics

$$\boldsymbol{F} = m\boldsymbol{a} = \frac{d\boldsymbol{p}}{dt}$$

Conservative forces

$$\boldsymbol{F} = -\nabla V$$

## Lagrangian Mechanics

$$\mathcal{L} = T - V$$
$$\frac{\partial \mathcal{L}}{\partial x_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial x_i'} = 0$$
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial x_i'} = 0 \leftrightarrow \text{Conserved quantity}$$

Constraints must be Holonomic

Free Particle Lagrangian's

$$\mathcal{L} = \frac{m}{2}(x'^2 + y'^2 + z'^2)$$
$$\mathcal{L} = \frac{m}{2}(r'^2 + r\theta'^2)$$
$$\mathcal{L} = \frac{m}{2}(r'^2 + r\theta'^2 + z'^2)$$
$$\mathcal{L} = \frac{m}{2}(r'^2 + r^2(\theta'^2 + \phi'^2 \sin^2\theta))$$

## Noether's Theorem

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial x_i'} = 0 \leftrightarrow \text{Conserved quantity}$$

If symmetry is continuous it can be linearly expanded

$$\tilde{x}_i = x_i + \epsilon_i(x_j)$$
$$I(t) = \sum_i \frac{\partial \mathcal{L}}{\partial x_i'}\epsilon_i(x_j)\bigg|_t$$

| Symmetry | Conserved Quantity |
|---|---|
| Translation | Linear Momentum |
| Rotation | Angular Momentum |
| Time Translation | Energy |
| QM phase | Charge |

## Hamiltonian Mechanics

For a closed system the Lagrangian has no explicit time dependence

$$\mathcal{H} = \sum_i \frac{\partial \mathcal{L}}{\partial x_i'}x_i' - \mathcal{L} \qquad \mathcal{H} = T + V$$

Canonical momentum conjugate to $x_i$

$$p_{x_i} = \frac{\partial \mathcal{L}}{\partial x_i'} \qquad \mathcal{H} = \sum_i p_{x_i} x_i' - \mathcal{L}$$

Equations of motion

$$\frac{dx_i}{dt} = \frac{\partial \mathcal{H}}{\partial p_{x_i}} \qquad \frac{dp_{x_i}}{dt} = -\frac{\partial \mathcal{H}}{\partial x_i}$$

## Pendulum Dynamics

$$T = \frac{m}{2}l\theta'^2, \qquad V = mgl(1 - \cos\theta)$$

Equations of motion

$$\theta'' = -\frac{g}{l}\sin\theta \text{ or when } \theta << 1, \qquad \theta'' = -\frac{g}{l}\theta$$

## Mass on a spring

$$T = \frac{m}{2}x'^2, \qquad V = \frac{1}{2}kx^2$$

Equations of motion

$$mx'' = -kx$$

## Small angle approximations

$$\sin\theta \approx \theta$$
$$\cos\theta \approx 1 - \frac{\theta^2}{2} \approx 1$$
$$\tan\theta \approx \theta$$

## Legendre Transform

$$g(p) = px - f(x), \text{ where } p = f'(x)$$

## Poisson Brackets

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial x_i}\frac{\partial \mathcal{H}}{\partial p_{x_i}} - \frac{\partial f}{\partial p_{x_i}}\frac{\partial \mathcal{H}}{\partial x_i} + \frac{\partial f}{\partial t} = \{f, \mathcal{H}\} + \frac{\partial f}{\partial t}$$

Useful identities

$$\{f, f\} = 0, \quad \{f, g\} = -\{g, f\}, \quad \{x_i, p_{x_j}\} = \delta_{ij}$$
$$\frac{dx_i}{dt} = \{x_i, \mathcal{H}\}, \qquad \frac{dp_{x_i}}{dt} = \{p_{x_i}, \mathcal{H}\}$$

If $\{f, \mathcal{H}\}$ and $f$ has no explicit time dependence: $f$ is conserved

## Quantum Mechanics

Schrödringer Picture

$$i\frac{d|\psi\rangle}{dt} = \hat{H}|\psi\rangle \Leftrightarrow |\psi(t)\rangle = e^{-i\hat{H}t}|\psi(0)\rangle$$

Heisenberg Picture

$$\langle \hat{A}\rangle = \langle\psi(t)|\hat{A}|\psi(t)\rangle \Leftrightarrow \hat{A}(t) = e^{i\hat{H}t}\hat{A}(0)e^{-i\hat{H}t}$$
$$\frac{d\hat{A}}{dt} = i[\hat{H}, \hat{A}]$$

Ehrenfest's Theorem

$$[\hat{f}, \hat{g}] \leftrightarrow i\hbar\{f, g\}, \quad \langle\hat{A}\rangle \leftrightarrow A(x_i, p_{x_i})$$

Weyl quantisation

$$x^2 p = \frac{1}{3}(\hat{x}^2\hat{p} + \hat{x}\hat{p}\hat{x} + \hat{p}\hat{x}^2)$$

## Group Theory

1: Closure: For any pair of elements $X \circ Y$ must also be an element.
2: Associativity: $X \circ (Y \circ Z) = (X \circ Y) \circ Z$
3: Identity: There is an element $I$ such that $X \circ I = X$
4: Inverse: Each element has an inverse, $X \circ X^{-1} = I$

## Fractals

Fractal generated from shape with $a$ sides and adding a shape that are a fraction $\frac{1}{b}$ of original shape. Each side is replaced with $c$ sides. This iteration now has $d_n$ total sides.

Total length n=1: $L_1 = \dfrac{d_1}{b}$

Number of new shapes: $S_n = a \cdot c^{n-1}$

Area new shape: $B_n = B_0/(b \cdot b)^n$

Perimeter: $P_n = a\left(\dfrac{c}{b}\right)^n$

Total Area: $A_n = B_0 + \sum_{k=1}^{n} S_n B_n =$

$$B_0\left(1 + \frac{a}{c}\sum_{k=1}^{n}\left(\frac{c}{b \cdot b}\right)^k\right)$$

Geometric Sum: $\displaystyle\sum_{k=1}^{n} r^k = \frac{r(1 - r^n)}{1 - r}$

Box Counting: $D = \dfrac{\ln N}{\ln 1/\epsilon}$, where $N = c$, $\epsilon = 1/b$

## Maps and Chaos

Fixed points: $x_{n+1} = x_n$

Stability check: $x_n = x_n + \delta_x \leftrightarrow \delta_{n+1} = Q(r)\delta_n$

Stable if: $|Q(r)| < 1$

Linear Stability analysis

$$\frac{dx}{dt} = F(x), \quad J(x) = \frac{\partial F}{\partial x}, \text{ Sub in fixed points to } J$$

+ve : Unstable, −ve: Stable

Lyapunov Exponent

$$|\delta\boldsymbol{x}(t)| \simeq e^{\lambda t}|\delta\boldsymbol{x}(0)|$$

$$\lambda = \lim_{t\to\infty}\lim_{|\delta\boldsymbol{x}(0)|\to 0}\frac{1}{t}\ln\frac{|\delta\boldsymbol{x}(t)|}{|\delta\boldsymbol{x}(0)|}$$

$\lambda > 0$ Chaotic, $\lambda < 0$ regular

$$E_n = e^{\lambda n}\epsilon, \qquad n_{max} = \frac{1}{\lambda}\ln\frac{E}{\epsilon}$$