Introduction to Statistics for Astronomers and Physicists

Section 2b: Fundamentals of Probability II

Dr Angus H Wright

2022-02-09

Section 2: Introduction

Section 2: Probability & Decision Making (Weeks 3-5)

For all aspects of modern science, an understanding of probability is required. We cover a range of topics in probability, from decision theory and the fundamentals of probability theory, to standard probabilistic distributions and their origin. From this module, students will gain an insight into different statistical distributions that govern modern observational sciences, the interpretation of these distributions, and how one accurately models distributions of data in an unbiased manner.

Topics include:

- Decision theory
- Fundamentals of probability
- Statistical distributions and their origins

The Birthday Problem (Question)

How many people do you need to have in a room before there is a more than 50% chance that at least two will share a birthday?

A Game Show (Question)

Suppose you're on a game show, and you're given the choice of three doors:

- Behind one door is a car.
- Behind the other two doors are goats.
- **IMPORTANT:** you want the car.
- You pick a door: say No. 1
- The host of the game show, who knows what's behind every door, opens one of the other doors, say No. 3, which has a goat behind it.
- He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?
- **ACTUALLY IMPORTANT**: In this version of the problem the host *always* opens a door which isn't yours *and* which has a goat.

Image credit: Cepheus

Picking up where we left off

In the last lecture, we discussed a few important ideas that lead to the concept of probability, independence, and 'paradoxes' like the gamblers fallacy. Some of these concepts will be important to our development over this lecture, and so are worth revising before the beginning of this lecture.

Outcomes, Events, Sample Spaces

Assume we have an experiment where we flip a fair coin twice. The **outcomes** of the coin tosses are the possible observations (sets of Heads or Tails) that we make from our experiment. The collection of all outcomes is the **sample space**:

$$\Omega = \{HH, HT, TH, TT\}$$

An **event** is a collection of outcomes, such as the event that I see one head and one tail:

$$\mathcal{E} = \{HT, TH\}.$$

Probability Calculations

We demonstrated that the frequency of observations is related to the probability of each outcome in the limit of large numbers of observations:

$$P(A) = \lim_{N \to \infty} \frac{\#(A)}{N}.$$

This consideration led to the definition of a few probability laws based around individual outcomes $A_i \in \Omega$:

- $0 \leq P(A_i) \leq 1$: Probabilities must be between 0 (impossible) and 1 (guaranteed).
- $\sum_{A_i \in \Omega} P(A_i) = 1$: The sum of all possible outcomes must be 1 (guaranteed).

Independence

We discussed the concept of independence, with particular focus on how independent events can fool your intuition.

- People often underestimate how improbable sequences of independent events are.
- People often reason poorly about independent outcomes because of a subconscious belief that previous outcomes should influence future ones (the gamblers fallacy).

Visualisations of Probabilities and Sets

We presented two methods of visualising probabilities (and sequences of probabilistic events):

- the urn model; and
- the probability tree.

The Urn model

The urn model is a simple way of visualising probabilistic events.

We discussed the urn model last week in the context of the **R** `sample` function, which can be used to simulate random probabilistic events (such as a dice roll or a coin toss).

The Probability Tree

We can visualise the above draws from our urn using the probability tree as well:

```
library(DiagrammeR)
mermaid("
  graph LR
    Start --> S[S, P=0.66]
    Start --> W[W, P=0.33]
    W --> WW[W, P=0.33]
    W --> WS[S, P=0.66]
    S --> SW[W, P=0.33]
    S --> SS[S, P=0.66]
    WW --> WWout[WW, P=0.11]
    WS --> WSout[WS, P=0.22]
    SW --> SWout[SW, P=0.22]
    SS --> SSout[SS, P=0.44]
")
```

Venn Diagrams

One visualisation tool that we didn't look at last week, but which is sometimes very useful to understand, is the Venn diagram.

Thinking again about our urn, we have:

Independence

Two events in an experiment sample space ($A, B \in \Omega$) are independent. How do they appear in our Venn diagram?

- The first plot shows two **disjoint** events. That is: if we observe A, then we cannot observe B. Therefore they cannot be independent, as information about A informs us about B.
- The second plot shows two **independent** events. If we observe A, our probability of observing B is relatively unchanged.

The latter point is a nuanced one, and it is worth understanding. The probability of observing A and B in our Venn diagram is equal, because they cover equal fractions of the sample space. If we observe A, our observation now restricts us to the box containing A only. But A and B are independent, which means that the observation of A can't inform us about B. So the probability of observing B must be relatively unchanged. That is, the intersection of B and A ($A \cap B$) must occupy the same fraction of A as B occupies in Ω.

- $P(A \cap B)/P(A) = P(B)/P(\Omega)$

Remember, though, that

$$P(\Omega) = 1$$

and therefore

- $P(A \cap B) = P(A) \times P(B)$ if A and B are independent.

Independence and Non-independence

Independent events are extremely important in statistics, especially in the context of random variables (which we will discuss later in this section). However non-independent events are also extremely important. These are cases where a subsequent outcome is dependent on the previous results. An obvious example is returning to our **urn**:

Here our draws are independent, and computing the probability of the observed outcome is straightforward (that is, the joint probability of observing a 2 and then an 8):

$$\begin{aligned} P(2 \cap 8) &= P(2) \times P(8) \\ &= \frac{1}{8} \times \frac{1}{8} \\ &= \frac{1}{64} \end{aligned}$$

However, how does this change if we choose not to replace the first ball that we draw?

Conditional Probability

This is where the concept of **conditional probability** becomes relavent. Given a sample space Ω of outcomes and a collection of events, the conditional probability of B, conditioned on A, is the probability that B occurs given that A has definitely occurred

$$P(B|A)$$

With our urn example, if we do not replace the 2 after our first draw, this has fundamentally altered the possible outcomes of the next draw, and therefore changed the probabilities involved. For our second draw, what now want to know is the probability of observing an 8 *given that* we just observed a 2. Said differently, the second draw computes the the probability of observing an 8 *conditioned upon* our prior observation of a 2.

In this example we can compute the conditional proability logically:

$$\begin{aligned} P(2 \cap 8) &= P(2) \times P(8|2) \\ &= \frac{1}{8} \times \frac{1}{7} \\ &= \frac{1}{56}. \end{aligned}$$

This example is a very trivial one, but consider a slightly different calculation.

Suppose we draw two balls from our urn, with replacement. We want to calculate the probability of drawing two balls with a combined value greater than or equal to 10.

The "win" event space is therefore:

$$\begin{aligned} \mathcal{E} = \{&8+2, 8+3, 8+4, 8+5, 8+6, 8+7, 8+8, \\ &7+3, 7+4, 7+5, 7+6, 7+7, 7+8, \\ &6+4, 6+5, 6+6, 6+7, 6+8, \\ &5+5, 5+6, 5+7, 5+8, \\ &4+6, 4+7, 4+8, \\ &3+7, 3+8, \\ &2+8\}. \end{aligned}$$

There are 64 possible ways of drawing 2 balls from a bag of 8 with replacement, which means that we have a 7/16 chance of winning this game. However, suppose now that we **know** that our first draw is an 8. How does this information influence our chance of winning?

If we first observe an 8, there are 7 subsequent draws which will earn us a victory:

$$\mathcal{E}|8 = \{8+2, 8+3, 8+4, 8+5, 8+6, 8+7, 8+8\}$$

Therefore the probability of winning given our first draw is an 8 jumps to $P(\mathcal{E}|8) = 7/8$. What about if we **know** that our first draw is a 2?

$$\mathcal{E}|2 = \{2+8\}$$

And so our probability of winning is a lowly $P(\mathcal{E}|2) = 1/8$. With this simple example we can see how important conditional probability is, and how event probabilities can be wildly influenced by different conditionalisation.

The Probability Tree (again)

The conditional probability can be usefully read-off of a probability tree as well, which (for discrete problems) can be very useful.

Looking again at our urn problem:

```
library(DiagrammeR)
mermaid("
  graph LR
    Start --> S[S, P=0.66]
    Start --> W[W, P=0.33]
    W --> WW[W, P=0.33]
    W --> WS[S, P=0.66]
    S --> SW[W, P=0.33]
    S --> SS[S, P=0.66]
    WW --> WWout[WW, P=0.11]
    WS --> WSout[WS, P=0.22]
    SW --> SWout[SW, P=0.22]
    SS --> SSout[SS, P=0.44]
")
```

we can see that the conditional probability of drawing (e.g.) no striped ball is:

$$P(WW) = 0.33$$

However the probability of observing no striped ball *given* that we observe a white ball with our first draw:

$$P(WW|W) = \frac{P(WW)}{P(WW) + P(WS)}$$

The Birthday Problem (Answer)

How many people do you need to have in a room before there is a more than 50% chance that at least two will share a birthday?

The birthday problem is another example of a statistical paradox; a problem where your intuition will almost certainly have failed you.

To calculate the number of people required to have a 50% chance that two share a birthday is quite simple. First we make some simplifying assumptions:

- There are 365 days in a year
- Birthdays are totally random

Neither of these are true: but the former has a small affect, and the latter makes our answer an *over-estimate* of the number of people required, because birthdays tend to cluster around particular times of the year.

The probability of finding two people with a shared birthday is:

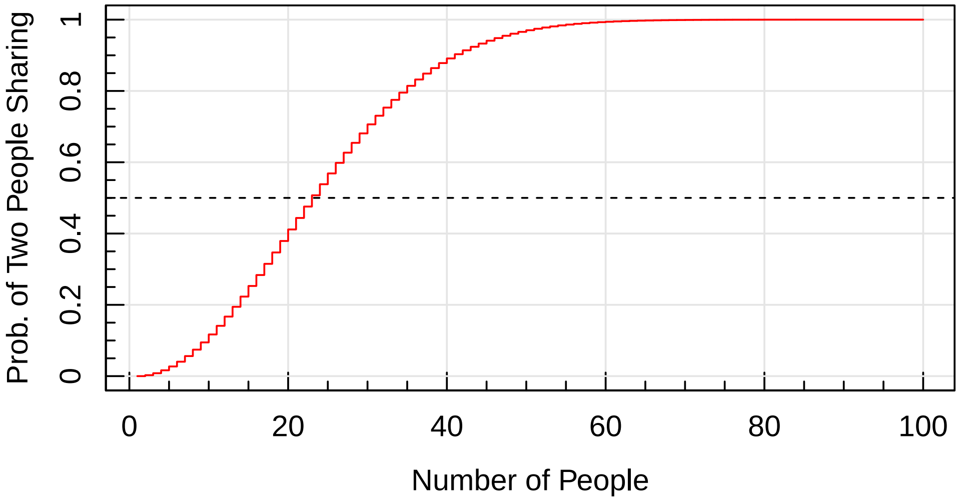
$$P(\text{shared birthday}) = 1 - P(\text{no shared birthday})$$

Calculating the probability of people *not* sharing a birthday is easier, because it's a simple conditional probability. The probability of each additional person not sharing a birthday depends on the number of previous observations.

$$
\begin{aligned}
P(\text{no shared birthday}) =& P(\text{person 1 has a birthday})\times \\
& P(\text{person 2 has a birthday different from person 1})\times \\
& P(\text{person 3 has a birthday different from persons 1 and 2}) \times \ldots \\
=& P(\text{no shared birthday}|\text{0 other birthdays})\times \\
& P(\text{no shared birthday}|\text{1 other birthday})\times \\
& P(\text{no shared birthday}|\text{2 other birthdays}) \times \ldots \\
=& \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \ldots \\
=& \left(\frac{1}{365}\right)^n \times 365 \times 364 \times 363 \times \ldots
\end{aligned}
$$

We can then calculate this probability simply:

```
magplot(1:100,1-(1/365)^(1:100)*cumprod(365:266),type='s',col='red',xlab='Number of People',
      ylab='Prob. of Two People Sharing')
abline(h=0.5,lty=2)
```

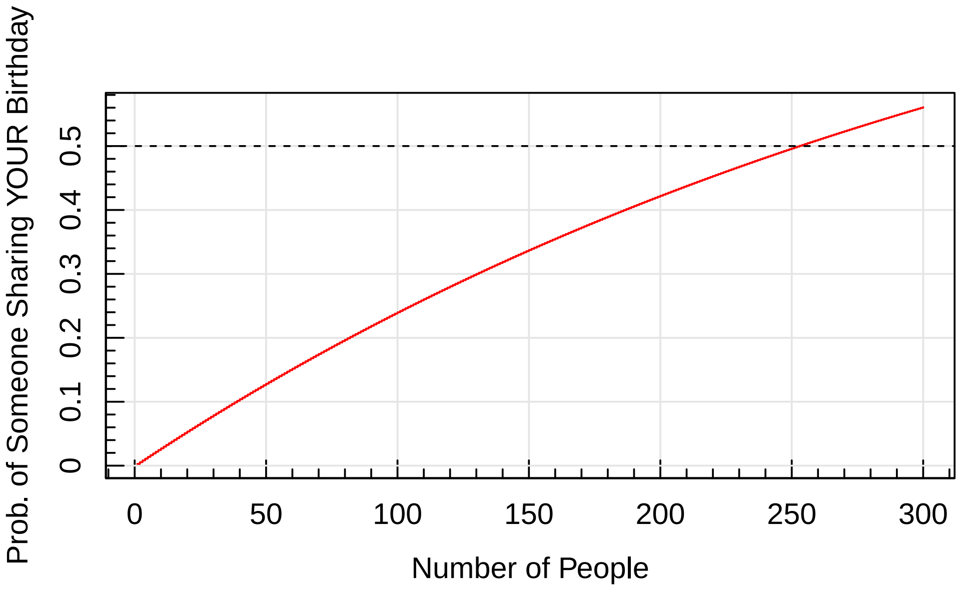


Therefore, there needs to be only 23 people in a group before there is a more-than-50% chance that two will share a birthday. This certainly sounds counter-intuitive, however this is because (again) our intuition isn't well suited to conditional inference.

In this particular case, a common cause of incorrect inference is driven by a conflation between "any two people sharing a birthday" and "someone sharing the same birthday as *me*". The former represents a series of conditional probabilities, whereas the latter is a series of independent probabilities.

The number of people required to be in a room so that the probability of someone sharing *your* birthday is greater than 50% is over 250. Note that in that group, though, there will likely be others who will jointly share a birthday before you!

```
magplot(1:300,1-(364/365)^(1:300),type='s',col='red',xlab='Number of People',
      ylab='Prob. of Someone Sharing YOUR Birthday')
abline(h=0.5,lty=2)
```



Computing Conditional Probability

We now want to derive an expression for the conditional probability $P(B|A)$.

We can start with our venn diagram again:

Our conditionalisation means that we know our probability must reside within A. We want to know $P(B|A)$: the probability that it lies within *both* A and $B \cap A$.

If the outcome lies in A, then it must fall within either $A \cap B$ or $A \cap B^{\rfloor}$. Therefore:

$$P(B|A) + P(B^{\rfloor}|A) = 1$$

Additionally, we can use our link between probability and relative frequency to our advantage. If some outcome $C \cap A$ is k times more likely than $B \cap A$, then $P(C \cap A) = kP(B \cap A)$. But crucially, this must be true regardless of whether A is observed first or not (the order of observation doesn't change the relative positions of items in our venn diagram). So $P(C|A) = kP(B|A)$:

$$P(B|A) \propto P(B \cap A)$$

To determine the coefficient of proportionality (c) we can use the above expressions:

$$\begin{aligned} P(B|A) &= 1 - P(B^{\rfloor}|A) \\ P(B|A) + P(B^{\rfloor}|A) &= 1 \\ P(B|A) + P(B^{\rfloor}|A) &= cP(B \cap A) + cP(B^{\rfloor} \cap A) \\ cP(B \cap A) + cP(B^{\rfloor} \cap A) &= cP(A) = 1 \\ \frac{P(B \cap A)}{P(A)} + \frac{P(B^{\rfloor} \cap A)}{P(A)} &= 1 \end{aligned}$$

and so:

$$P(B|A) = \frac{P(B \cap A)}{P(A)}.$$

The intersection and the conditional probability are therefore very closely related. The probability of B conditioned upon A is the probability of B *and* A, divided by the total probability of A. Said differently;

the intersection probability has range $0 \leq P(B \cap A) \leq P(A)$, while the conditional probability has the range $0 \leq P(B|A) \leq 1$.

This can be a guide as to how to think about the intersection (i.e. $P(B \cap A)$) and the conditional probability. The former provides a probability in the *absence* of any additional information/observations. The conditional probability, however, provides probability based on the *knowledge* that we have already made some observation.

We can use our Venn diagrams again to learn about conditional probability and independence:

We can use this formula to infer the conditional probability $P(B|A)$ in these two scenarios. Clearly on the left the intersection is 0: $P(B \cap A) = 0$. And therefore the conditional probability is also zero. On the right, recall that we asserted that these events were independent, and therefore had the same fractional area of intersection between (A and B) and (B and Ω):

- $P(B \cap A)/P(A) = P(B)$

But notice now that the LHS of this equation is just the conditional probability $P(B|A)$. So:

- $P(B|A) = P(B)$ for independent events

That is, if events are independent, conditionalisation doesn't have any impact (which makes sense!).

Lastly, there is one additional (very important!) observation we can make. Given that the intersection of two probabilities is unchanged under ordering: $P(B \cap A) = P(A \cap B)$, this means that:

$$\begin{aligned} P(B|A) &= \frac{P(B \cap A)}{P(A)} \\ P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ \therefore P(B|A) &= \frac{P(A|B)P(B)}{P(A)} \end{aligned}$$

This turns out to be an extremely valuable relationship known as **Bayes Rule**, and it will form the basis of the *majority* of the second half of this course!

Playing Poker

Let's have a break now and play some poker. We will start with a simple version of the game:

- The game is played with a standard 52 card deck
- You are dealt 5 cards, all face down

What is the probability that you are dealt a royal flush (AKQJ10 in one suit)?

To compute the number of k possible combinations of N possible outcomes *when ordering does not matter* we can use combinatorics:

$$\frac{N!}{k!(N-k)!} = \binom{N}{k}$$

The number of possible royal flush hands (ignoring the order) is 4: one for each suit. The number of possible 5 card hands (ignoring the order) is:

$$\binom{52}{5} = 2\,598\,960$$

Therefore the probability of being dealt a royal flush in our game of poker is:

$$P(\text{royal flush}) = \frac{4}{2\,598\,960} = \frac{1}{649\,740}.$$

Let's say now, though, that you are dealt your five card hand and the fifth card lands face-up. It is a 4 of diamonds.

What is the conditional probability that you have a royal flush ($P(\text{royal flush}|4\diamondsuit)$)

Your sloppy dealer makes the same mistake a second time, but this time the card which lands face-up is the ace of spades. What is the conditional probability that you have a royal flush ($P(\text{royal flush}|A\spadesuit)$)

Let's define some events:

- A is the event that the last card you get is the ace of spades *and* you get a royal flush
- B is the event that the last card you get is the ace of spades

The conditional probability of A given B, from our formula, is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

$P(B)$ is easy to compute; the $A\spadesuit$ is just a normal card in the deck. $P(B) = \frac{1}{52}$. The joint probability of getting the ace of spades and having a royal flush is:

$$\frac{\text{the number of royal flushes where the last card is } A\spadesuit}{\text{total number of 5 card hands}}$$

The numerator is simply the number of ways of drawing the spade royal flush while ensuring that the $A\spadesuit$ is the last card drawn: $4 \times 3 \times 2 \times 1$. The denominator is just the number of available combinations of 5 cards: $52 \times 51 \times 50 \times 49 \times 48$. So:

$$P(\text{royal flush}|A\spadesuit) = \frac{1}{249\,900}.$$

So seeing the Ace makes a big difference, but you still probably wouldn't want to bet the house…

The Prosecutors Fallacy

Reasoning logically with conditional probabilities is difficult.

One prominent logical fallacy that happens with statistics is known as the prosecutors fallacy. This is the implicit assumption that conditional probabilities are reversible.

A prosecutor at court presents evidence $\mathcal{E}$. They argue that the defendent is guilty because the probability of finding the evidence given innocence $P(\mathcal{E}|\mathcal{I})$ is small. But this is totally irrelevant. The real question is what is the probability that the defendent is innocent given the evidence: $P(\mathcal{I}|\mathcal{E})$.

The distinction is relevant, because the conditionalisation can have vastly different outcomes (one can be very large while the other is very small).

We can relate these quantities with our formula from earlier:

$$\begin{aligned} P(\mathcal{I}|\mathcal{E}) &= \frac{P(\mathcal{E}|\mathcal{I})P(\mathcal{I})}{P(\mathcal{E})} \\ &= \frac{P(\mathcal{E}|\mathcal{I})P(\mathcal{I})}{P(\mathcal{E}|\mathcal{I})P(\mathcal{I}) + P(\mathcal{E}|\mathcal{I}^{\lrcorner})P(\mathcal{I}^{\lrcorner})} \end{aligned}$$

The assumed similarity between $P(\mathcal{I}|\mathcal{E})$ and $P(\mathcal{E}|\mathcal{I})$ is clearly violated if $P(\mathcal{I})$ is large, or if $P(\mathcal{E}|\mathcal{I}^{\lrcorner})$ is much smaller than $P(\mathcal{E}|\mathcal{I})$.

This may seem familiar to you from last week, when we discussed the DNA matching problem. In that example, we considered independence between the failures in the database. This leads to another damaging possibility: treating truly conditionally dependent events as independent.

In Summary

Always be careful about your reasoning regarding conditional probability and about independent events. These lead to logical fallacies that can completely negate the accuracy of your work. Always ask yourself two fundamental questions:

- What am I interested in: $P(A|B)$? or $P(B|A)$?
- Am I modelling possibly dependent events as independent?

Consider this trivial example:

- What is the conditional probability of winning the lottery given that you buy a ticket?
- What is the conditional probability of having bought a ticket given that you win the lottery?

The Monty Hall Problem

Suppose you're on a game show, and you're given the choice of three doors:

- Behind one door is a car.
- Behind the other two doors are goats.
- **IMPORTANT:** you want the car.
- You pick a door: say No. 1
- The host of the game show, who knows what's behind every door, opens one of the other doors, say No. 3, which has a goat behind it.
- He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?
- **ACTUALLY IMPORTANT**: In this version of the problem the host *always* opens a door which isn't yours *and* which has a goat.

Image credit: Cepheus

We want to know the conditional probability that there is a car behind door 1 given we now know that there is a goat behind door 3.

$$P(C_1|G_3) = \frac{P(G_3|C_1)P(C_1)}{P(G_3|C_1)P(C_1) + P(G_3|C_2)P(C_2) + P(G_3|C_3)P(C_3)}$$

To work this out, we need to know $P(G_3|C_1)$, $P(G_3|C_2)$, and $P(G_3|C_3)$.

- $P(G_3|C_3) = 0$, because the host chooses from doors with goats behind them.
- $P(G_3|C_1) = \frac{1}{2}$, because the host chooses randomly from doors with goats behind them that are not door one.
- $P(G_3|C_2) = 1$, because there is only one door that (a) has a goat behind it and (b) isn't door one.

We can substitute these values into the formula, and we find:

$$P(C_1|G_3) = \frac{1}{3}.$$

What?!

This means that, given the parameters of the game show, it is in your interest to switch doors. How can this be?

To understand this we can use the tools that we've compiled over the course of the last two lectures. Firstly, this is what a probability tree for this problem looks like, where we only show the branches that produce the relevant outcomes:

This shows that the total conditional probability of the car being behind door 1 is $P = 0.25$, whereas the conditional probability of the car being behind door 2 is $P = 0.5$. Hence our ratio of probabilities being 1/3.

This is not an intuitive result! However, you can begin to understand how the result comes about by considering some extreme examples.

Consider a circumstance where there are 100 doors rather than 3. In this scenario, the game-show host opens not just one "wrong" door, but 98. In this circumstance, the initial probability that you picked the correct answer was 1/100, and now the host *knowingly* discarded 98 wrong doors. Does your intuition tell you that it makes more sense to switch now?

Anomoly detection

The last discussion that we will have in this section on conditional probability is regarding the difficulty of anomaly detection: that is, why it's difficult to reliably detect rare events.

There are many cases in Astronomy and Physics where anomoly detection is interesting/desireable. Discovering new and rare phenomena is an obvious example, be they exotic particles in a detector or exotic transients in the universe.

When discussing accuracy of detection it is worth understanding the different types of result:

```
##           Compare         Reality.Positive        Reality.Negative
## 1  Measured-True             True Positive False Positive (Type I)
## 2 Measured-False False Negative (Type II)            True Negative
```

The "Types" are included because these names are sometimes used for specific types of failures.

The difficulty in anomaly detection arises because, as an event becomes rare, the accuracy of tests required to minimise false positives (Type 1) becomes prohibitively large.

Let's consider two examples: detecting a common event, and detecting a rare event, with an experiment of fixed accuracy.

A common event

A decay process occurs in nature with probability 0.4. You have an experiment that detects this emission with a probability of 0.6, and produces a false positive with probability 0.1. What is the conditional probability that you witness a true decay *and* the experiment produces a positive detection?

```
##               Compare True.Decay No.Decay
## 1 Positive Detection        0.6      0.1
## 2 Negative Detection        0.4      0.9
```

Let $\lceil$ be the event that a true decay occurs, and $\mathcal{P}$ be the event that the experiement produces a positive detection.

$$\begin{aligned}
P(\lceil|\mathcal{P}) &= \frac{P(\mathcal{P}|\lceil)P(\lceil)}{P(\mathcal{P})} \\
&= \frac{P(\mathcal{P}|\lceil)P(\lceil)}{P(\mathcal{P}|\lceil)P(\lceil) + P(\mathcal{P}|\lceil^{\rfloor})P(\lceil^{\rfloor})} \\
&= \frac{0.6 \times 0.4}{0.6 \times 0.4 + 0.1 \times 0.6} \\
&= 0.8
\end{aligned}$$

So your experiment only has to be approximately accurate to produce reliable detections when the event it common.

A rare event

Let's now consider a similar scenario, except the probability of our decay occuring in nature is very small: $P(\lceil) = 0.001$. The experiment, though, has improved to 99.9% accuracy, and only 1% false-positive rate.

```
##               Compare True.Decay No.Decay
## 1 Positive Detection      0.999     0.01
## 2 Negative Detection      0.001     0.99
```

Again we compute our probability that we actually detected the event:

$$
\begin{aligned}
P(\lceil|\mathcal{P}) &= \frac{P(\mathcal{P}|\lceil)P(\lceil)}{P(\mathcal{P})} \\
&= \frac{P(\mathcal{P}|\lceil)P(\lceil)}{P(\mathcal{P}|\lceil)P(\lceil) + P(\mathcal{P}|\lceil^{\rfloor})P(\lceil^{\rfloor})} \\
&= \frac{0.999 \times 0.001}{0.999 \times 0.001 + 0.01 \times 0.999} \\
&= 0.09
\end{aligned}
$$

So despite our experiment becoming much much more accurate, the probability that we make a true detection is less than 10%. Said differently, 10 out of every 11 detections will be false.