The Graphical and Simplex Methods

A Graphical Method

The graphical method is a visual way to solve linear programming problems with two decision variables. It's particularly useful for understanding the feasible region and finding the optimal solution. Here's a step-by-step introduction:
Steps in the Graphical Method:

1. Formulate the Problem: Define the decision variables. Write down the objective function to be maximized or minimized. List all the constraints.

2. Plot the Constraints: Convert each constraint into an equation. Plot these equations on a graph, treating the decision variables as the x and y axes. Identify the feasible region, which is the area where all constraints overlap.

3. Identify the Feasible Region: The feasible region is the common area that satisfies all the constraints. It can be bounded (a closed area) or unbounded (an open area extending infinitely).

4. Plot the Objective Function: Draw the objective function as a line on the same graph. Move this line parallel to itself towards the direction of optimization (maximization or minimization) until it reaches the last point within the feasible region.

5. Find the Optimal Solution: The optimal solution is usually at one of the feasible region's vertices (corner points). Evaluate the objective function at each vertex to find the maximum or minimum value.

A.1 Limitations

The graphical method is a useful tool for solving linear programming problems with two decision variables, but it has several limitations:

- Limited to two variables: The graphical method can only be applied to problems with two decision variables. For problems with three or more variables, it becomes impractical as it is difficult to visualize and plot in higher dimensions

- Approximate Solutions: The graphical method often provides approximate solutions, especially when the feasible region and the optimal point are not clearly defined on the graph. This can lead to less precise results compared to algebraic methods.

- Complexity with Many Constraints: The graphical method can become cumbersome and difficult to manage when there are many constraints. Plotting multiple lines and identifying the feasible region can be challenging and time-consuming.

- Not Suitable for Large-Scale Problems: The graphical method is not applicable to large-scale problems with numerous constraints and variables. In such cases, more advanced methods, such as the simplex algorithm or computational tools, are required.

- Lack of Generalization: The graphical method does not generalize well to more complex linear programming problems. It is primarily a teaching tool and is not used in real-world applications where problems are typically more complex.

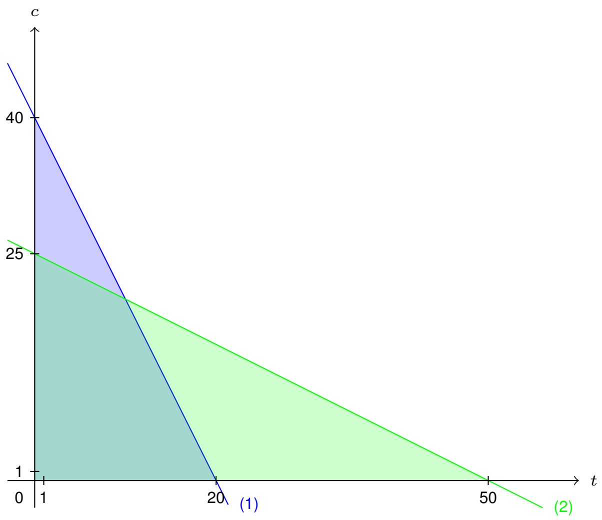
These limitations highlight why the graphical method is mainly used for educational purposes and simple problems, while more sophisticated methods are employed for larger and more complex linear programming problems.

A.2 Example 1: The Carpenter Problem

Suppose a carpenter makes tables and chairs and sells all the tables and chairs he makes in a market. He does not have a steady income and wishes to optimize this situation. So, the carpenter must determine how many tables and chairs he should make to maximize his net income. He knows that the income he receives per table sold is $\$500$ and per chair is $\$300$. The carpenter works 8 hours a day from Monday to Friday and takes 2 hours to make a table and 1 hour to make a chair. Also, he receives 50 units of raw material each week, of which he requires 1 unit for each table and two units for each chair he makes.

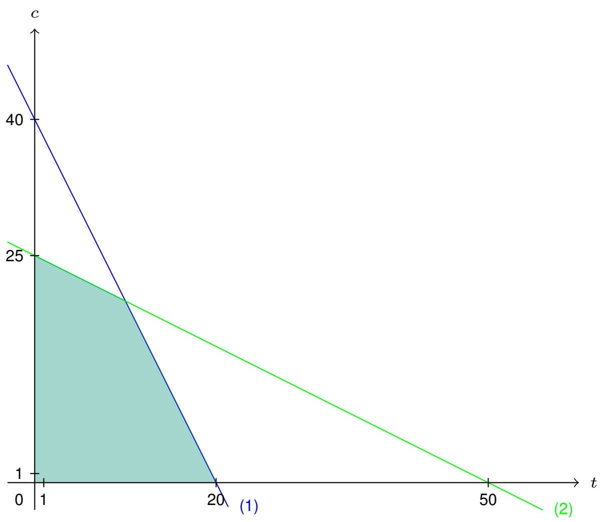
The model: $$\begin{cases} Max & 500t & + & 300c & & \\ s.t. & 2t & + & c & \leq & 40 \\ & t & + & 2c & \leq & 50 \\ & t \geq 0,\ c \geq 0 & & & & \end{cases}$$

Step 1: Plot one by one all constraints on a graph.



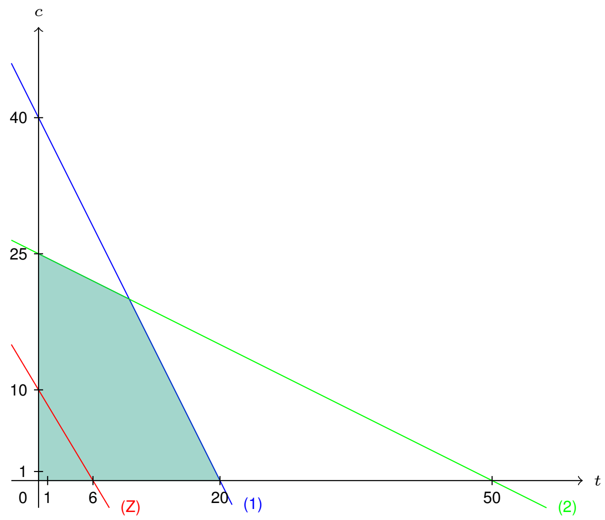
$$\begin{array}{llllll} (1) & 2t & + & c & \leq & 40 \\ (2) & t & + & 2c & \leq & 50 \\ & t \geq 0,\ c \geq 0 \end{array}$$

Step 2: Identify the **feasible region**.



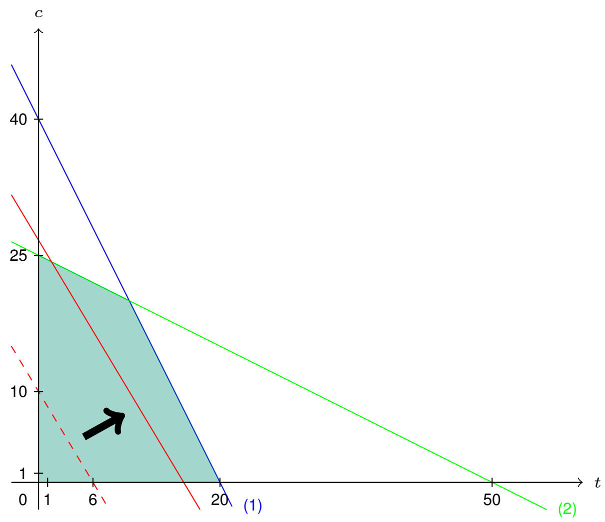
$$\begin{array}{llllll} (1) & 2t & + & c & \leq & 40 \\ (2) & t & + & 2c & \leq & 50 \\ & t \geq 0,\ c \geq 0 \end{array}$$

Step 3: Plot the objective function (**Z line**) with an arbitrary value of z.



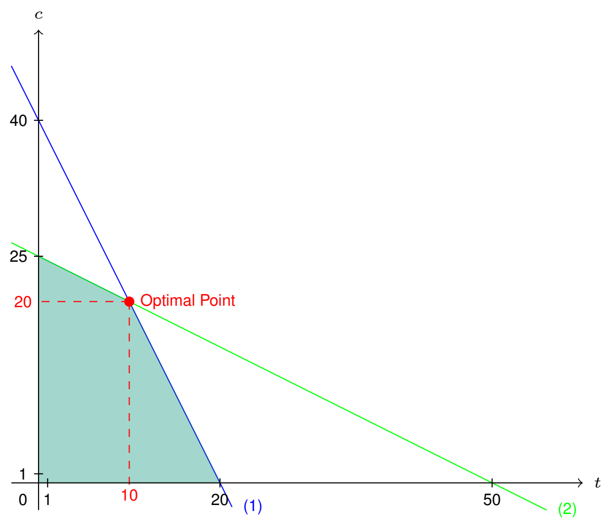
$$
\begin{array}{llllll}
(1) & 2t & + & c & \leq & 40 \\
(2) & t & + & 2c & \leq & 50 \\
 & t \geq 0,\ c \geq 0 & & & & \\
 & & & & & \\
(Z) & 500t & + & 300c & = & z \\
 & (z = 3000) & & & &
\end{array}
$$

Step 4: Find the **Optimal Point** by moving the Z line.



$$
\begin{array}{llllll}
(1) & 2t & + & c & \leq & 40 \\
(2) & t & + & 2c & \leq & 50 \\
 & t \geq 0,\ c \geq 0 & & & & \\
 & & & & & \\
(Z) & 500t & + & 300c & = & z
\end{array}
$$

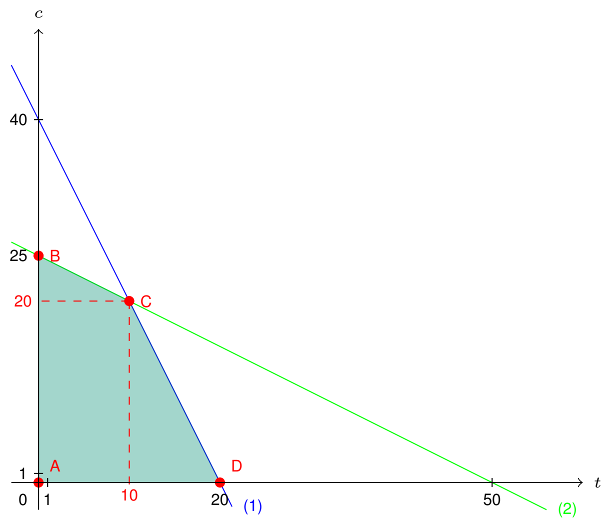
Step 5: Get the coordinates of the Optimal Point and conclude.



$$
\begin{array}{llllll}
(1) & 2t & + & c & \leq & 40 \\
(2) & t & + & 2c & \leq & 50 \\
 & t \geq 0,\ c \geq 0 & & & & \\
 & & & & & \\
(Z) & 500t & + & 300c & = & z
\end{array}
$$

Conclusion: To optimize his profit, the Carpenter should make **10** tables and **20** chairs for a total profit of **$11000** ($z = 500 \times 10 + 300 \times 20 = 11000$)

Remarks: When you plot these constraints on a graph, the feasible region is a polygon with vertices, or extreme points, at (0,0), (20,0), (10,20), and (0,25). These vertices are where the constraint lines intersect, and they represent the potential optimal solutions.
In the graphical method for solving linear programming problems, the optimal solution is typically found at one of the extreme points (vertices) of the feasible region. Indeed, The last point(s), where the objective function line touches the feasible region before leaving it, is always at one of the vertices (extreme points) of the feasible region. This is because the objective function is linear, and the feasible region is a convex polygon. The optimal value of the objective function occurs at these vertices.
In the next example, we will see another method to solve a linear problem with the graphical method, based on this observation.



$$\begin{array}{llllll} (Z) & 500t & + & 300c & = & z \\ (1) & 2t & + & c & \leq & 40 \\ (2) & t & + & 2c & \leq & 50 \\ & t \geq 0, \ c \geq 0 & & & & \end{array}$$

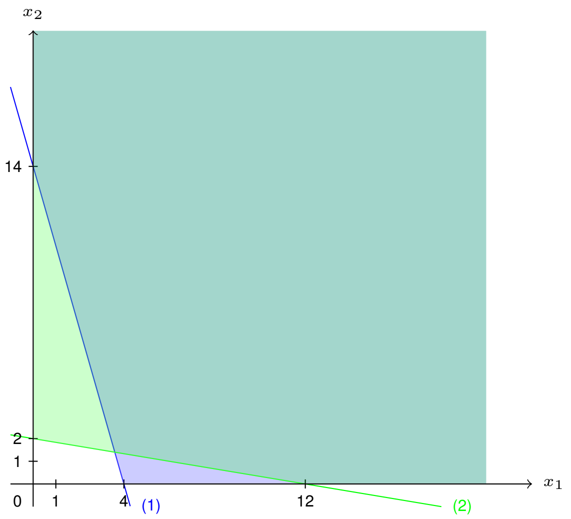
	t	c	z
A	0	0	0
B	0	25	7500
C	10	20	11000
D	20	0	10000

A.3 Example 2: The Dorian Problem

Dorian Auto manufactures luxury cars and trucks. The company believes that its most likely customers are high-income women and high-income men. To reach these groups, Dorian Auto has embarked on an ambitious TV advertising campaign and has decided to purchase 1-minute commercial spots on two types of programs: comedy shows and football games. Each comedy commercial is seen by 7 million women and 2 million men. Each football commercial is seen by 2 million women and 12 million men. A 1-minute comedy show ad costs $50000, and a 1-minute football ad costs $100000. Dorian would like the commercials to be seen by at least 28 million high-income women and at least 24 million high-income men. Use linear programming to determine how Dorian Auto can meet its advertising requirements at a minimum cost.

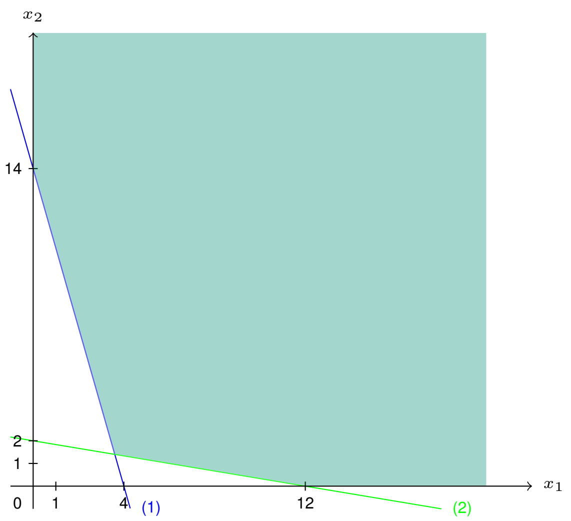
The model: $\left\{ \begin{array}{lllll} Min & 50x_1 & + & 100x_2 & \\ s.t. & 7x_1 & + & 2x_2 & \geq \ 28 \\ & 2x_1 & + & 12x_2 & \geq \ 24 \\ & x_1 \geq 0, \ x_2 \geq 0 & & & \end{array} \right.$

Step 1: Plot one by one all constraints on a graph.



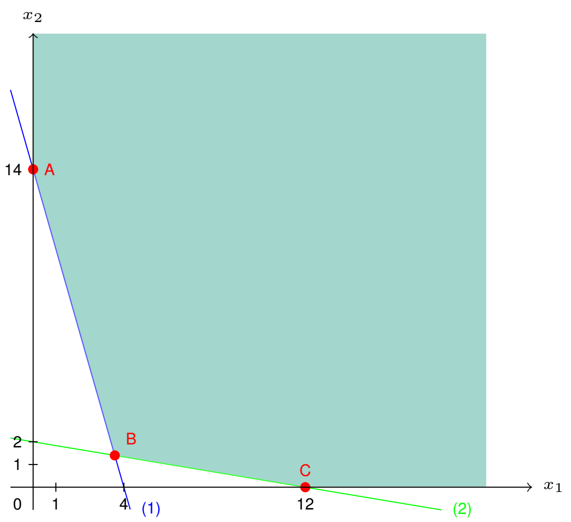
$$\begin{array}{lrcrcr} (1) & 7x_1 & + & 2x_2 & \geq & 28 \\ (2) & 2x_1 & + & 12x_2 & \geq & 24 \\ & \multicolumn{5}{l}{x_1 \geq 0,\ x_2 \geq 0} \end{array}$$

Step 2: Identify the **feasible region**.



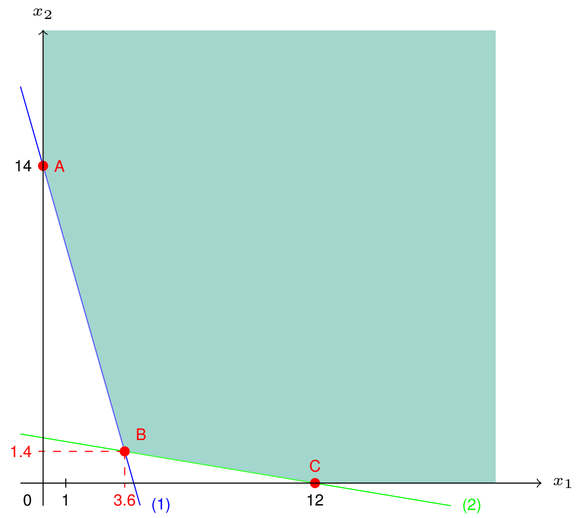
$$\begin{array}{lrcrcr} (1) & 7x_1 & + & 2x_2 & \geq & 28 \\ (2) & 2x_1 & + & 12x_2 & \geq & 24 \\ & \multicolumn{5}{l}{x_1 \geq 0,\ x_2 \geq 0} \end{array}$$

Step 3: Identify the **Extreme Points**.



$$\begin{array}{lrcrcr} (1) & 7x_1 & + & 2x_2 & \geq & 28 \\ (2) & 2x_1 & + & 12x_2 & \geq & 24 \\ & \multicolumn{5}{l}{x_1 \geq 0,\ x_2 \geq 0} \end{array}$$

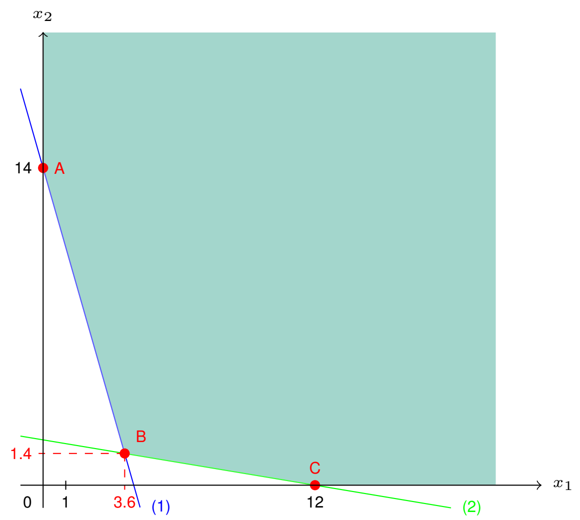
Step 4: Get the coordinates of all **Extreme Points** and compute the value of z.



$$\begin{array}{lrcrcl} (Z) & 50x_1 & + & 100x_2 & = & z \\ (1) & 7x_1 & + & 2x_2 & \geq & 28 \\ (2) & 2x_1 & + & 12x_2 & \geq & 24 \\ & \multicolumn{5}{l}{x_1 \geq 0,\ x_2 \geq 0} \end{array}$$

	x_1	x_2	z
A	0	14	1400
B	3.6	1.4	320
C	12	0	600

Step 5: Identify the **Optimal Point**.



$$\begin{array}{lrcrcl} (Z) & 50x_1 & + & 100x_2 & = & z \\ (1) & 7x_1 & + & 2x_2 & \geq & 28 \\ (2) & 2x_1 & + & 12x_2 & \geq & 24 \\ & \multicolumn{5}{l}{x_1 \geq 0,\ x_2 \geq 0} \end{array}$$

	x_1	x_2	z
A	0	14	1400
B	3.6	1.4	320
C	12	0	600

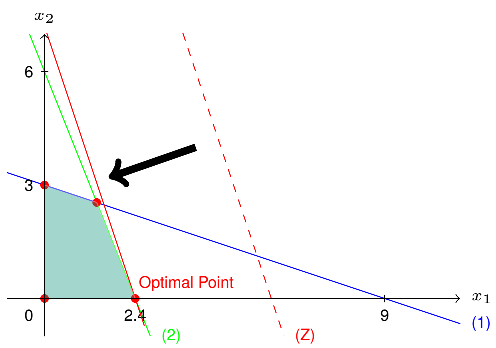
Conclusion: To minimize the costs, Dorian should purchase **3.6** ads on comedy shows and **1.4** ads on football games for a total cost of **$320,000** ($z = 50 \times 3.6 + 100 \times 1.4 = 320$).

B Particular cases

B.1 LP with a Unique Optimal Solution

Consider the following problem and its solution with the graphical method:

$$\begin{array}{llcrcl} Max & 3x_1 & + & 1x_2 & & \\ s.t. & x_1 & + & 3x_2 & \leq & 9 \\ & 5x_1 & + & 2x_2 & \leq & 12 \\ & \multicolumn{5}{l}{x_1 \geq 0,\ x_2 \geq 0} \end{array}$$

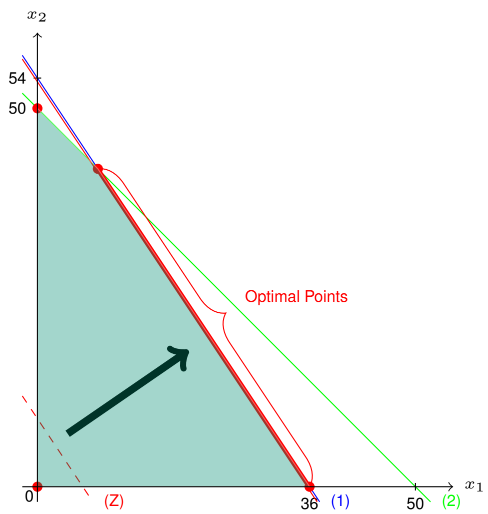


In this case, the feasible region is a polygon, and the objective function will reach its maximum value at one of the vertices of this polygon. If the maximum value occurs at a single vertex, then the solution is unique. An LP problem has a unique optimal solution if there is exactly one point in the feasible region where the objective function reaches its maximum or minimum value.

B.2 LP with Multiple Optimal Solutions

Consider the following problem and its solution with the graphical method:

$$
\begin{array}{lcccccc}
Max & 3x_1 & + & 2x_2 & & \\
s.t. & \frac{1}{4}x_1 & + & \frac{1}{6}x_2 & \leq & 9 \\
 & x_1 & + & x_2 & \leq & 50 \\
 & \multicolumn{5}{l}{x_1 \geq 0,\ x_2 \geq 0}
\end{array}
$$

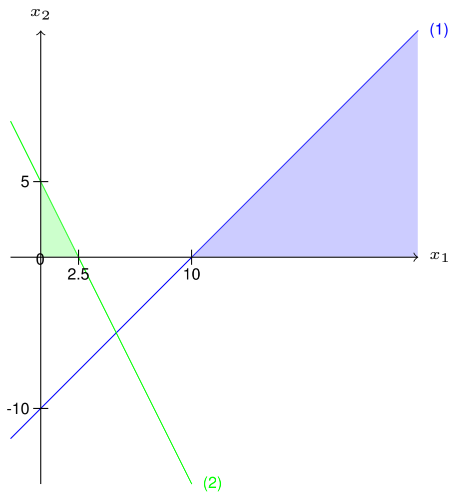


In this case, the feasible region is a polygon, and the objective function reaches its maximum value along the line segment between the points (0.54) and ((36, 0)). Therefore, any point on this line segment is an optimal solution, indicating multiple optimal solutions. A linear programming problem has multiple optimal solutions when there is more than one point in the feasible region where the objective function reaches the same optimal value. This typically occurs when the objective function is parallel to a constraint boundary over a segment of the feasible region.

B.3 Infeasible LP

Consider the following problem and its solution with the graphical method:

$$\begin{array}{llllll} Max & 5x_1 & + & 2x_2 & & \\ s.t. & x_1 & - & x_2 & \geq & 10 \\ & 2x_1 & + & x_2 & \leq & 5 \\ & \multicolumn{5}{l}{x_1 \geq 0,\ x_2 \geq 0} \end{array}$$

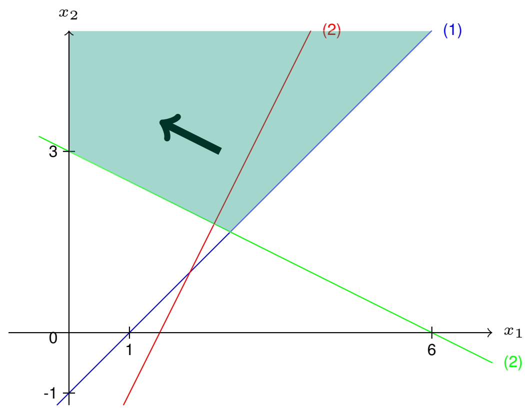


In this case, the constraints are contradictory. There is no point (x, y) with $x \geq 0$ and $y \geq 0$ that can satisfy both constraints simultaneously. Therefore, the feasible region is empty, and the LP problem is infeasible. An infeasible linear programming problem is one where no solution satisfies all the given constraints simultaneously. In other words, the feasible region, which is the set of all points that meet the constraints, is empty.

B.4 Unbounded LP

Consider the following problem and its solution with the graphical method:

$$\begin{array}{llllll} Max & 2x_1 & - & x_2 & & \\ s.t. & x_1 & - & x_2 & \leq & 1 \\ & x_1 & + & 2x_2 & \geq & 6 \\ & \multicolumn{5}{l}{x_1 \geq 0,\ x_2 \geq 0} \end{array}$$



In this case, the feasible region is unbounded because there is no upper limit on the values of x_1 and x_2 that satisfy the constraints. As x_1 and x_2 increase, the value of the objective function can also increase indefinitely. An unbounded linear programming problem is one where the objective function can be increased indefinitely without violating any of the constraints. In other words, there is no finite maximum (or minimum) value for the objective function within the feasible region.

C The simplex Method

C.1 Overview

The simplex method is a widely used algorithm for solving linear programming (LP) problems. It was developed by George Dantzig in 1947 and is particularly effective for problems with many variables and constraints.

The simplex method is an iterative procedure that moves from one feasible solution to another, improving the objective function value at each step, until the optimal solution is reached. It operates on LP problems in standard form, where all constraints are equalities and all variables are non-negative.

Steps of the Simplex Method:

1. Formulate the LP Problem: Write the LP problem in standard form with equality constraints and non-negativity restrictions.

2. Set Up the Initial Simplex Tableau: Create a tableau that includes the coefficients of the objective function, constraints, and slack/excess variables.

3. Identify the Pivot Element: Determine the entering and leaving variables to identify the pivot element. The entering variable is chosen based on the most negative coefficient in the objective function row for a maximization problem (resp. on the most positive coefficient in the objective function row for a minimization problem). The leaving variable is chosen based on the smallest positive ratio of the right-hand side to the pivot column.

4. Perform Pivot Operations: Use row operations to update the tableau, making the pivot element equal to 1 and all other elements in the pivot column equal to 0.

5. Check for Optimality: If there are no negative coefficients in the objective function row in the case of maximization problem(resp. no positive coefficients in the objective function row in the case of minimization problem) , the current solution is optimal. Otherwise, repeat steps 3 and 4.

6. Iterate: Continue the process until the optimal solution is found or it is determined that the problem is unbounded.

In the simplex algorithm, a tableau is a tabular representation of the linear programming (LP) problem that helps in performing the iterative steps of the algorithm. The tableau organizes the coefficients of the objective function, constraints, and variables in a structured manner, making it easier to apply the simplex method systematically.

C.2 **Example: the Carpenter Problem**

The initial model:

$$\begin{array}{llllll} Max\ z = & 500t & + & 300c & & \\ s.t. & 2t & + & c & = & 40 \\ & t & + & 2c & = & 50 \\ & t \geq 0,\ c \geq 0 & & & & \end{array}$$

Since the model does not meet the requirement of the standard form, we convert it:

$$\begin{array}{llllllllll} Max\ z = & 500t & + & 300c & & & & & & \\ s.t. & 2t & + & c & + & s_1 & & & = & 40 \\ & t & + & 2c & & & + & s_2 & = & 50 \\ & t \geq 0,\ c \geq 0,\ s_1 \geq 0,\ s_2 \geq 0 & & & & & & & & \end{array}$$

We then build the initial tableau:

B.V.	t	c	s_1	s_2	rhs
s_1	2	1	1	0	40
s_2	1	2	0	1	50
z	-500	-300	0	0	**0**

Notice that the current feasible solution is: $t = 0,\ c = 0,\ s_1 = 40,\ s_2 = 50$ and the value of the objective function is $z = 0$. In case of maximization: If any coefficient in row z is negative, the current feasible solution is not optimal. Therefore, we have to iterate.
We identify the entering and the leaving variables for the next iteration. In case of maximization: If any coefficient in row z is negative, the current feasible solution is not optimal.

B.V.	t	c	s_1	s_2	rhs
s_1	2	1	1	0	40
s_2	1	2	0	1	50
z	-500	-300	0	0	**0**
	$\uparrow$ *Input*				

In this case, the entering variable is t, with coefficient -500. We compute the ratio to identify the leaving variable. We take care that the denominator in the ratio should be strictly positive (> 0). Otherwise, we cannot compute it. The leaving variable is the winner of the ratio test. Hence, it is the one with the minimum value of $\dfrac{\text{rhs of row}}{\text{coefficient of entering variable in row}}$

B.V.	t	c	s_1	s_2	rhs	ratio	
s_1	2	1	1	0	40	$\frac{40}{2}$	$\rightarrow$ *Output*
s_2	1	2	0	1	50	$\frac{50}{1}$	
z	-500	-300	0	0	**0**		
	$\uparrow$ *Input*						

Then we perform the pivoting operation:

- Update the row of the entering variable: we replace the basic variable s_1 by t and divide each element of the row by the pivot of the row (in this case, we divide by 2).

B.V.	t	c	s_1	s_2	rhs
s_1	2	1	1	0	40
s_2	1	2	0	1	50
z	-500	-300	0	0	**0**

$\rightarrow$

B.V.	t	c	s_1	s_2	rhs
t	1	$1/2$	$1/2$	0	20
s_2					
z					

- Update the remaining rows: We use the row of the entering variable to update the remaining rows.
  - To update row s_2 we must do the following operation:

$$[\text{Old row } s_2] - (\text{pivot in Old row } s_2) \times [\text{new pivot row } t]$$

$$[1\ 2\ 0\ 1\ 50] - 1 \times [1\ \frac{1}{2}\ \frac{1}{2}\ 0\ 20] = [0\ \frac{3}{2}\ -\frac{1}{2}\ 1\ 30]$$

- To update row z we must do the following operation:

$$[\text{Old row } z] - (\text{pivot in Old row } z) \times [\text{new pivot row } t]$$

$$[-500 \;\; -300 \; 0 \; 0 \; 0] - (-500) \times [1 \; \frac{1}{2} \; \frac{1}{2} \; 0 \; 20] = [0 \;\; -50 \; 250 \; 0 \; 10,000]$$

- Update the tableau with the new rows:

B.V.	t	c	s_1	s_2	rhs
s_1	2	1	1	0	40
s_2	1	2	0	1	50
z	-500	-300	0	0	**0**

$\rightarrow$

B.V.	t	c	s_1	s_2	rhs
t	1	$1/2$	$1/2$	0	20
s_2	0	$3/2$	$-1/2$	1	30
z	0	-50	250	0	**10,000**

At the end of the pivoting operation we get the next tableau of the simplex:

B.V.	t	c	s_1	s_2	rhs
t	1	$1/2$	$1/2$	0	20
s_2	0	$3/2$	$-1/2$	1	30
z	0	-50	250	0	**10,000**

The current feasible solution given by this tableau is $t = 20$, $c = 0$, $s_1 = 0$, $s_2 = 30$, and $z = 10,000$ (c and s_2 are not in the basis and take value 0).
We notice that we still have at least one negative coefficient in the z-row, therefore optimality has not been reached and we have to perform another iteration. In this case, the entering variable is c, with coefficient -50 in the z-row, and we recompute the ratio to get the leaving variable (s_2 will be the leaving variable):

B.V.	t	c	s_1	s_2	rhs	ratio	
t	1	$1/2$	$1/2$	0	20	$20/\frac{1}{2}$	
s_2	0	$3/2$	$-1/2$	1	30	$30/\frac{3}{2}$	$\rightarrow$ *Output*
z	0	-50	250	0	**10,000**		
		$\uparrow$ *Input*					

Again, we start by updating the row of the entering variable. We replace the basic variable s_2 by c and divide each element of the row by the pivot of the row (in this case, we divide by $3/2$).

B.V.	t	c	s_1	s_2	rhs
t	1	$1/2$	$1/2$	0	20
s_2	0	$3/2$	$-1/2$	1	30
z	0	-50	250	0	**10,000**

$\rightarrow$

B.V.	t	c	s_1	s_2	rhs
t					
c	0	1	$-1/3$	$2/3$	20
z					

And we update the remaining rows:

$$[\text{Old row } t] - (\text{pivot in Old row } t) \times [\text{new pivot row } c]$$

$$[1 \; \frac{1}{2} \; \frac{1}{2} \; 0 \; 20] - \frac{1}{2} \times [0 \; 1 \; -\frac{1}{3} \; \frac{2}{3} \; 20] = [1 \; 0 \; \frac{2}{3} \; -\frac{1}{3} \; 10]$$

$$[\text{Old row } z] - (\text{pivot in Old row } z) \times [\text{new pivot row } c]$$

$$[0 \ -50 \ 250 \ 0 \ 10,000] - (-50) \times [0 \ 1 \ -\frac{1}{3} \ \frac{2}{3} \ 20] = [0 \ 0 \ \frac{700}{3} \ \frac{100}{3} \ 11,000]$$

B.V.	t	c	s_1	s_2	rhs
t	1	$1/2$	$1/2$	0	20
s_2	0	$3/2$	$-1/2$	1	30
z	0	-50	250	0	**10,000**

$\rightarrow$

B.V.	t	c	s_1	s_2	rhs
t	1	0	$2/3$	$-1/3$	10
c	0	1	$-1/3$	$2/3$	20
z	0	0	$\frac{700}{3}$	$\frac{100}{3}$	11,000

We obtain the third tableau of the simplex algorithm:

B.V.	t	c	s_1	s_2	rhs
t	1	0	$2/3$	$-1/3$	10
c	0	1	$-1/3$	$2/3$	20
z	0	0	$\frac{700}{3}$	$\frac{100}{3}$	11,000

Then, we have finished since all coefficient in the z-row are now positive. The optimal solution has been found and the carpenter must produce 10 tables and 20 chairs to obtain a maximum profit equals to $11,000.

D Retrieve a Solution from a Simplex Tableau

Retrieving a solution from a simplex tableau involves identifying the values of the decision variables and the objective function from the final tableau. Here's a step-by-step explanation:

1. Identify the Basic Variables: In the final tableau, the basic variables are those that correspond to columns with exactly one entry of 1 and all other entries 0. These variables will have non-zero values in the solution.

2. Read the Values of the Basic Variables: The values of the basic variables are found in the right-hand side (rhs) column of the tableau.

3. Set Non-Basic Variables to Zero: The non-basic variables (those not identified as basic variables) are set to zero.

4. Determine the Objective Function Value: The value of the objective function is found in the rhs of the objective function row (the last row of the tableau).

For example, if we have the following last tableau:

	x_1	x_2	s_1	s_2	rhs
s_1	1	0	1	-1	2
x_2	0	1	-1	1	3
z	0	0	1	1	9

The solution to the LP problem is: $x_1 = 0$, $x_2 = 3$, $s_1 = 2$, $s_2 = 0$, and $z = 9$.
In this example, the decision variables x_1 and x_2 are the ones of interest. The optimal solution is $x_1 = 0$, and $x_2 = 3$, and the maximum value of the objective function z is 9. Note that all the tableau generated by the simplex algorithm provides a feasible solution to the problem. However, only the last one gives an optimal solution.

E The Two-Phase Simplex Method

E.1 Overview

The two-phase simplex method is a variant of the simplex algorithm used to solve linear programming (LP) problems, especially when the initial basic feasible solution is not readily apparent. It involves two main phases:

Phase 1: Finding a Feasible Solution

- Formulate an Auxiliary Problem:
  - Introduce artificial variables to convert the constraints into a form that allows for an initial basic feasible solution.
  - Construct an auxiliary objective function to minimize the sum of these artificial variables.
- Set Up the Initial Simplex Tableau: Include the artificial variables in the tableau and set up the initial simplex tableau for the auxiliary problem.
- Solve the Auxiliary Problem:
  - Use the simplex method to minimize the auxiliary objective function.
  - If the minimum value of the auxiliary objective function is zero, a feasible solution to the original problem has been found.
  - If the minimum value is greater than zero, the original problem has no feasible solution.

Phase 2: Optimizing the Original Objective Function

- Remove Artificial Variables: Remove the artificial variables from the tableau, as they are no longer needed.
- Set Up the Initial Tableau for the Original Problem:
  - Use the feasible solution obtained from Phase 1 as the starting point for Phase 2.
  - Set up the initial simplex tableau for the original problem.
- Solve the Original Problem:
  - Use the simplex method to optimize the original objective function.
  - Continue iterating until the optimal solution is found.

E.2 Example the Dorian Problem

The model:

$$\begin{cases} Min\ z = & 50x_1 & + & 100x_2 & & \\ s.t. & 7x_1 & + & 2x_2 & \geq & 28 \\ & 2x_1 & + & 12x_2 & \geq & 24 \\ & x_1 \geq 0,\ x_2 \geq 0 & & & & \end{cases}$$

Since the model does not meet the requirement of the standard form, we convert it:

$$\begin{array}{llllllll} Min\ z = & 50x_1 & + & 100x_2 & & & & \\ s.t. & 7x_1 & + & 2x_2 & - & e_1 & & = 28 \\ & 2x_1 & + & 12x_2 & & & - \ e_2 & = 24 \\ & x_1 \geq 0,\ x_2 \geq 0,\ e_1 \geq 0,\ e_2 \geq 0 & & & & & & \end{array}$$

E.2.1 Phase 1

If a constraint i is an equality or $\geq$ constraint, we add an artificial variable a_i to constraint i. Then we replace the original objective function by $Min\ w$ modeled as the sum of the all artificial variables. Since we will add artificial variables a_1 and a_2, the new objective function is $Min\ w = a_1 + a_2$ and we get the following problem:

$$
\begin{array}{llllllllll}
Min\ w = & a_1 & + & a_2 & & & & & & \\
s.t. & 7x_1 & + & 2x_2 & - e_1 & & + a_1 & & = & 28 \\
 & 2x_1 & + & 12x_2 & & - e_2 & & + a_2 & = & 24 \\
 & \multicolumn{9}{l}{x_1 \geq 0,\ x_2 \geq 0,\ e_1 \geq 0,\ e_2 \geq 0,\ a_1 \geq 0,\ a_2 \geq 0}
\end{array}
$$

This sub-problem of the first phase in the two-phase simplex method is an auxiliary linear programming problem designed to find a feasible solution to the original LP problem. This sub-problem is necessary when the original problem does not have an obvious initial basic feasible solution. It is always a minimization problem, independently of the original problem we want to solve.
We now have to solve this sub-problem with the simplex algorithm. Remind that, because we are minimizing the entering variable is the one with the highest positive value in row w.

Initial Tableau

B.V.	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{a_1}$	$\mathbf{a_2}$	**rhs**
a_1	7	2	-1	0	1	0	28
a_2	2	12	0	-1	0	1	24
w	0	0	0	0	-1	-1	**0**

Notice that a_1 and a_2 are basic variables, so the coefficient of these variables in row w must be equals 0. We have to resolve that before performing any pivoting operations:

$$
\begin{array}{r|rrrrrrr|l}
 & 7 & 2 & -1 & 0 & 1 & 0 & 28 & (Row\ a_1) \\
+ & 2 & 12 & 0 & -1 & 0 & 1 & 24 & (Row\ a_2) \\
 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & (Row\ w) \\
\hline
w = & 9 & 14 & -1 & -1 & 0 & 0 & 52 &
\end{array}
$$

Updated Initial tableau

B.V.	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{a_1}$	$\mathbf{a_2}$	**rhs**
a_1	7	2	-1	0	1	0	28
a_2	2	12	0	-1	0	1	24
w	9	14	-1	-1	0	0	52

As you can observe, the coefficients of all basic variables in row w are now equal to zero. The pivoting operation can start.

B.V.	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{a_1}$	$\mathbf{a_2}$	**rhs**	*ratio*	
a_1	7	2	-1	0	1	0	28	$\frac{28}{2}$	
a_2	2	12	0	-1	0	1	24	$\frac{24}{12}$	$\rightarrow$ *Output*
w	9	14	-1	-1	0	0	52		
		$\uparrow$ *Input*							

First Tableau

B.V.	x_1	x_2	e_1	e_2	a_1	a_2	rhs	*ratio*	
a_1	$20/3$	0	-1	$1/6$	1	$-1/6$	24	$24/\frac{20}{3}$	$\rightarrow$ *Output*
x_2	$1/6$	1	0	$-1/12$	0	$1/12$	2	$2/\frac{1}{6}$	
w	$20/3$	0	-1	$1/6$	0	$-7/6$	24		
	$\uparrow$ *Input*								

Second Tableau

B.V.	x_1	x_2	e_1	e_2	a_1	a_2	rhs
x_1	1	0	$-3/20$	$1/40$	$3/20$	$-1/40$	$18/5$
x_2	0	1	$1/40$	$-7/80$	$-1/40$	$7/80$	$7/5$
w	0	0	0	0	-1	-1	0

Notice that the artificial variables have left the basis and w has reached its optimal value ($w = 0$). Hence, the Phase I of the method finishes here and since $w = 0$, we know that the problem is feasible.

E.2.2 Phase 2

We start by building the initial tableau by removing all nonbasic artificial variables from the last tableau in Phase I, and replacing the last row by the original objective z.
Initial Tableau

B.V.	x_1	x_2	e_1	e_2	a_1	a_2	rhs
x_1	1	0	$-3/20$	$1/40$	0	$-1/40$	$18/5$
x_2	0	1	$1/40$	$-7/80$	0	$7/80$	$7/5$
w	0	0	0	0	-1	-1	0

$\rightarrow$

B.V.	x_1	x_2	e_1	e_2	rhs
x_1	1	0	$-3/20$	$1/40$	$18/5$
x_2	0	1	$1/40$	$-7/80$	$7/5$
z	-50	100	0	0	0

Notice that x_1 and x_2 are basic variables, so the coefficient of these variables in row z must be equals 0. That means, as previously, we have to resolve this situation before starting any pivoting operation:

$$\begin{array}{rrrrrrl}
 & 1 & 0 & -3/20 & 1/40 & 18/5 & (Row\ x_1) \\
+ & 0 & 1 & 1/40 & -7/80 & 7/5 & (Row\ x_2) \\
 & -50 & -100 & 0 & 0 & 0 & (Row\ z) \\
\hline
z = & 0 & 0 & -5 & -65/8 & 320 &
\end{array}$$

Updated Initial Tableau

B.V.	x_1	x_2	e_1	e_2	rhs
x_1	1	0	$-3/20$	$1/40$	$18/5$
x_2	0	1	$1/40$	$-7/80$	$7/5$
z	0	0	-5	$-65/8$	320

We note that all coefficients of nonbasic variables in row z are negative. So, there is no entering variable and we have reach the optimal solution of the problem. The Two-Phase Simplex Method concludes here for this case.

However, this is not always the case. After updating the initial tableau in the second phase, it may be necessary to perform several pivoting operations to reach an optimal solution. In this problem, the first phase found a feasible solution that happened to be optimal. Nonetheless, the first phase could have returned any other extreme point of the feasible region, requiring further iterations in the second phase to find the optimal solution.