Introduction to Combinatorial Optimization

Mathematical Modeling

Abstract

Combinatorial optimization focuses on selecting the best solution from a finite set of possibilities while satisfying given constraints. It has broad applications in operations research, logistics, manufacturing, and computer science. This document introduces key concepts in combinatorial optimization and briefly presents five well-known problems: the shortest path problem, the traveling salesman problem, the facility location problem, the set covering problem, and the flowshop scheduling problem.

1 Introduction

Combinatorial optimization involves solving problems where the solution space consists of discrete elements. The objective is to find an optimal configuration according to the given criteria. These problems often arise in industrial engineering, logistics, scheduling, and telecommunications.

Unlike continuous optimization, where variables take values from a continuous range, combinatorial optimization problems require selecting combinations of discrete choices. Many of these problems are NP-hard, making them computationally challenging.

2 Examples of Combinatorial Optimization Problems

2.1 Shortest Path Problem

The shortest path problem involves finding the minimum-cost path between two nodes in a weighted graph.

- *Input:* A graph $G = (V, E)$ where V is the set of nodes, E is the set of edges, and each edge (i, j) has a non-negative weight w_{ij}.
- *Goal:* Find a path from a source node s to a destination node t that minimizes the total weight.

2.2 Traveling Salesman Problem (TSP)

The TSP is one of the most famous combinatorial optimization problems. Given a set of cities, the task is to find the shortest possible route that visits each city exactly once and returns to the starting city.

- *Input:* A set of cities, each with a known distance to every other city.
- *Goal:* Minimize the total travel distance while visiting each city exactly once.

2.3 Facility Location Problem

The facility location problem (FLP) determines the optimal placement of facilities (e.g., warehouses, factories) to serve a set of customers while minimizing costs.

- *Input:* A set of potential facility locations F and a set of customers C, with costs associated with opening facilities and serving customers.
- *Goal:* Minimize the total cost of facility opening and transportation.

2.4 Set Covering Problem (SCP)

The set covering problem is widely used in facility location, scheduling, and logistics. The goal is to find the minimum number of sets that cover all required elements.

- *Input:* A universal set U and a collection of subsets S_i, each with a cost c_i.
- *Goal:* Select a subset of S_i that covers all elements in U while minimizing the total cost.

2.5 Flowshop Scheduling Problem

The flowshop scheduling problem involves scheduling a set of jobs on multiple machines in the same processing order to minimize the makespan (total completion time).

- *Input:* A set of jobs J and machines M, where each job follows the same sequence of operations.
- *Goal:* Minimize the makespan:

  $$C_{\max} = \min \max_{j \in J} C_{j,m}$$

  where $C_{j,m}$ is the completion time of job j on the last machine m.