Mathematical modelling of IPIS and its state space models

Structure of IPIS in three dimensions is given in Fig. 1 as follows:

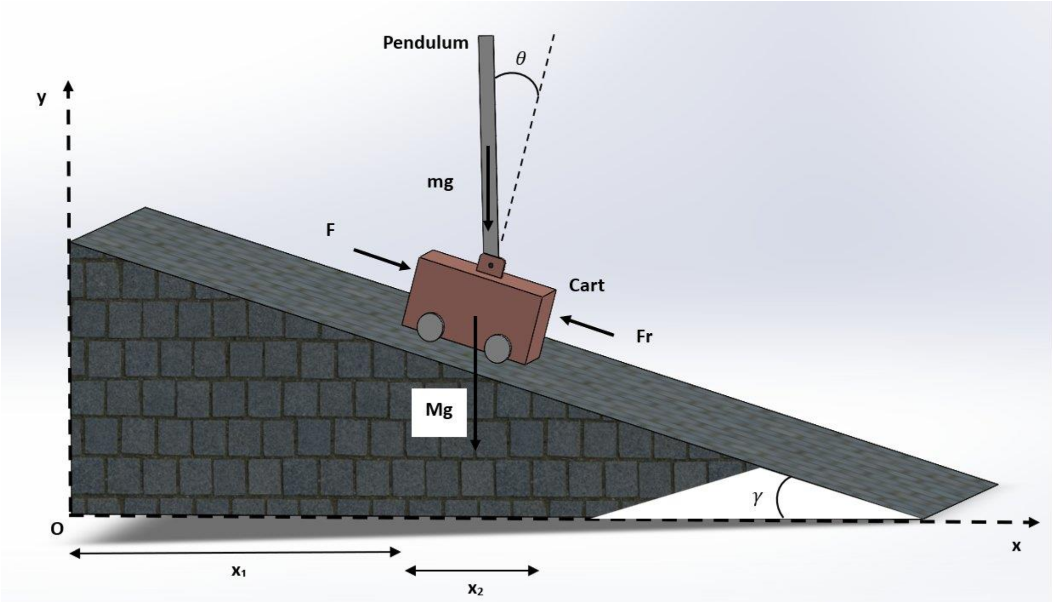

Fig. 1 Structure of IPIS in 3D

Before figuring out the mathematical modelling of IPIS based on Fig. 1, we define the system parameters in Tab. 1 as follows:

Tab. 1 Notation and unit of constant parameters of IPIS

Parameter	***Unit***	***Description***
M	$[kg]$	Mass of cart
m	$[kg]$	Mass of pendulum
l_c	$[meter]$	Length of the cart
b	kgm^2/s	Viscous friction co-efficient of pendulum section
γ	$[rad]$	Angle of inclined plane
L	$[meter]$	Distance from pivot joint to the centre of mass of pendulum
g	$[m/s^2]$	Gravity acceleration
I	$[kgm^2]$	Pendulum moment of inertia
F	$[N]$	External force
x	$[meter]$	Cart position
θ	$[rad]$	Angular of pendulum

According to [1], the mathematical modelling of IPIS is given in Equations as follows:

$$(M+m)\ddot{x} - (ml_c cos\theta)\ddot{\theta} + (bcos\gamma)\dot{x} + (ml_c sin\gamma)\dot{\theta}^2 = Fcos\gamma \quad (1)$$

$$(ml_c cos\theta + ml_c tan\gamma sin\theta)\ddot{x} - (I + mL^2)\ddot{\theta} + mgLsin\theta = 0 \quad (2)$$

Let:

$$A_1 = M + m \quad (4)$$

$$B_1 = -ml_c cos\theta \tag{5}$$
$$C_1 = Fcos\gamma - (bcos\gamma)\dot{x} - (ml_c sin\gamma)\dot{\theta}^2 \tag{6}$$
$$A_2 = ml_c cos\theta + ml_c tan\gamma sin\theta \tag{7}$$
$$B_2 = I + mL^2 \tag{8}$$
$$C_2 = -mgLsin\theta \tag{9}$$
$$x_1 = x, x_2 = \dot{x}, x_3 = \theta, x_4 = \dot{\theta} \tag{10}$$

From equations (1) and (2), we have:

$$\begin{cases} A_1\ddot{x} + B_1\ddot{\theta} = C_1 \\ A_2\ddot{x} + B_2\ddot{\theta} = C_2 \end{cases} \tag{11}$$

$$\Rightarrow \begin{cases} \ddot{x} = \dfrac{C_1B_2 - C_2B_1}{A_1B_2 - A_2B_1} \\ \ddot{\theta} = \dfrac{C_2A_1 - C_1A_2}{A_1B_2 - A_2B_1} \end{cases} \tag{12}$$

$$\Rightarrow \begin{cases} \ddot{x} = \dfrac{-(mL^2+I)(l_c msin\gamma x_4^2 - Fcos\gamma + bx_2cos\gamma) + gl_c^2m^2cos(x_3)}{(M+m)*(mL^2+I) - l_c^2m^2cos(x_3)\big(cos(x_3) + \tan(\gamma)sin(x_3)\big)} \\ \ddot{\theta} = \dfrac{-\Big(l_c m\big(cos(x_3) + tan\gamma sin(x_3)\big)(l_c msin\gamma x_4^2 - Fcos\gamma + bx_2cos\gamma) - gl_c msin(x_3)(M+m)\Big)}{(M+m)*(mL^2+I) - l_c^2m^2cos(x_3)\big(cos(x_3) + \tan(\gamma)sin(x_3)\big)} \end{cases} \tag{13}$$

The nonlinear state space equations of IPIS are described as follows:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = fcn_1 = \dfrac{-(mL^2+I)(l_c msin\gamma x_4^2 - Fcos\gamma + bx_2cos\gamma) + gl_c^2m^2cos(x_3)}{(M+m)*(mL^2+I) - l_c^2m^2cos(x_3)\big(cos(x_3) + \tan(\gamma)sin(x_3)\big)} \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = fcn_2 = \dfrac{-\begin{pmatrix} l_c m\big(cos(x_3) + tan\gamma sin(x_3)\big)(l_c msin\gamma x_4^2 - Fcos\gamma + bx_2cos\gamma) + \\ -gl_c msin(x_3)(M+m) \end{pmatrix}}{(M+m)*(mL^2+I) - l_c^2m^2cos(x_3)\big(cos(x_3) + \tan(\gamma)sin(x_3)\big)} \end{cases} \tag{14}$$

$$\Rightarrow \dot{x} = f(x, F) \tag{15}$$

On the other hand, system (14) can be simplified into a linear system through Taylor expansion as $x_1, x_2 \to 0$, and this simplified form is given by:

$$\dot{x} = (A)x(t) + (B)u(t) \tag{16}$$

Where $u(t) = F(t)$, and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{\partial fcn_1}{\partial x_1} & \dfrac{\partial fcn_1}{\partial x_2} & \dfrac{\partial fcn_1}{\partial x_3} & \dfrac{\partial fcn_1}{\partial x_4} \\ 0 & 0 & 0 & 1 \\ \dfrac{\partial fcn_2}{\partial x_1} & \dfrac{\partial fcn_2}{\partial x_2} & \dfrac{\partial fcn_2}{\partial x_3} & \dfrac{\partial fcn_2}{\partial x_4} \end{bmatrix} \tag{17}$$
$$B = \begin{bmatrix} 0 & \dfrac{\partial fcn_1}{\partial F} & 0 & \dfrac{\partial fcn_2}{\partial F} \end{bmatrix}^T$$