$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \quad & f_0(\mathbf{x}) \\ \text{subject to} \quad & f_i(\mathbf{x}) \le 0,\ \ i = 1,\dots,k \\ & h_j(\mathbf{x}) = a_j^T\mathbf{x} + b_j = 0,\ \ j = 1,\dots,m \end{aligned}$$

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \quad & f_0(\mathbf{x}) \\ \text{subject to} \quad & f_i(\mathbf{x}) \le 0,\ \ i = 1,\dots,k \\ & A\mathbf{x} + b = 0. \end{aligned}$$

Definition

An optimization problem is convex if its objective f_0 is a convex function and the inequality constraints f_i are convex.

Note: the feasible region

$$D = \{x \in \mathbb{R}^n |\ f_i(\mathbf{x}) \le 0,\ A\mathbf{x} + b = 0,\ i = 1,\dots,k\}$$

is convex.

Convex optimization problems

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \quad & f_0(\mathbf{x}) \\ \text{subject to} \quad & f_i(\mathbf{x}) \leq 0, \ i = 1, \ldots, k \\ & h_j(\mathbf{x}) = a_j^T \mathbf{x} + b_j = 0, \ j = 1, \ldots, m \end{aligned}$$

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \quad & f_0(\mathbf{x}) \\ \text{subject to} \quad & f_i(\mathbf{x}) \leq 0, \ i = 1, \ldots, k \\ & A\mathbf{x} + b = 0. \end{aligned}$$

Definition

An optimization problem is convex if its objective f_0 is a convex function and the inequality constraints f_i are convex.

Note: the feasible region

$$D = \{x \in \mathbb{R}^n | \ f_i(\mathbf{x}) \leq 0, \ A\mathbf{x} + b = 0, \ i = 1, \ldots, k\}$$

is convex.
The maximization problem

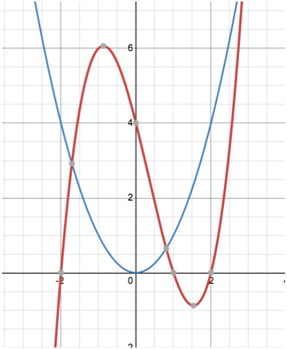
$$\begin{aligned} \max_{\mathbf{x}\in\mathbb{R}^n} \quad & f_0(\mathbf{x}) \\ \text{subject to} \quad & f_i(\mathbf{x}) \leq 0, \ i = 1, \ldots, k \\ & A\mathbf{x} + b = 0. \end{aligned}$$

is convex if f_0 is concave and f_i are convex (using fact $\max f_0 = \min(-f_0)$).

Why should optimization problems be convex?

Theorem

If x^* is a local minimizer of a convex optimization problem, it is also a global minimizer.



Consider the problem (P):

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \quad & f_0(\mathbf{x}) \\ \text{subject to} \quad & f_i(\mathbf{x}) \le 0, \ \ i = 1, \dots, k \\ & h_j(\mathbf{x}) = a_j^T \mathbf{x} + b_j = 0, \ \ j = 1, \dots, m \end{aligned}$$

Lagrangian function

$$\mathcal{L}(\mathbf{x}, \lambda, \mu) = f_0(\mathbf{x}) + \sum_{i=1}^{k} \lambda_i f_i(x) + \sum_{j=1}^{m} \mu_j h_j(\mathbf{x})$$

KKT conditions

1. $\nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \lambda, \mu) = 0$
2. $\lambda_i f_i(\mathbf{x}) = 0$, for $i = 1, \dots, k$
3. $f_i(\mathbf{x}) \le 0$, for $i = 1, \dots, k$, and $h_j(\mathbf{x}) = 0$, for $j = 1, \dots, m$
4. $\lambda_i \ge 0$, for $i = 1, \dots, k$

Theorem

If (x^*, λ^*, μ^*) satisfy the KKT conditions above then x^* is a global minimizer of (P).

Consider the quadratic programming problem, where Q is PSD matrix:

$$\min_{\mathbf{x}\in\mathbb{R}^n} \quad \frac{1}{2}\mathbf{x}^T Q\mathbf{x} + c^T\mathbf{x}$$

Lagrangian function

$$\mathcal{L}(\mathbf{x}) = f_0(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q\mathbf{x} + c^T\mathbf{x}$$

KKT conditions

x^* is global optimal if and only if $\nabla\mathcal{L}(\mathbf{x}) = 0 \Leftrightarrow Q\mathbf{x} + c = 0$

Consider the quadratic programming problem, where Q is PSD matrix:

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \quad & \frac{1}{2}\mathbf{x}^T Q\mathbf{x} + c^T\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = 0 \end{aligned}$$

Lagrangian function

$$\mathcal{L}(\mathbf{x}, \mu) = \frac{1}{2}\mathbf{x}^T Q\mathbf{x} + c^T\mathbf{x} + \mu^T(A\mathbf{x})$$

KKT conditions

1. $\nabla_{\mathbf{x}}\mathcal{L}(\mathbf{x}, \mu) = Q\mathbf{x} + c + A^T\mu = 0$
2. $A\mathbf{x} = 0$

Conditions 1 and 2 can be written as

$$\begin{bmatrix} Q & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mu \end{bmatrix} = \begin{bmatrix} -c \\ 0 \end{bmatrix}$$

$$\begin{array}{ll} \text{minimize} & x^2 + 1 \\ \text{subject to} & (x-2)(x-4) \leq 0, \end{array}$$

Lagrangian function

$$\mathcal{L}(\mathbf{x}, \lambda) = x^2 + 1 + \lambda(x-2)(x-4)$$

KKT conditions

1. $\mathcal{L}'_x = 2x + 2\lambda x - 6\lambda = 0 \Leftrightarrow x + \lambda x - 3\lambda = 0$
2. $\lambda(x-2)(x-4) = 0$
3. $(x-2)(x-4) \leq 0$
4. $\lambda \geq 0$

Solutions: $x^* = 2, \lambda = 2$ satisfy the KKT conditions, and thus $x^* = 2$ is a minimizer.

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^2} \quad & x + 3y \\ \text{subject to} \quad & x^2 + y^2 \leq 1 \end{aligned}$$

$$\min_{\mathbf{x}\in\mathbb{R}^2} \quad x+3y$$
$$\text{subject to} \quad x^2+y^2 \le 1$$

Lagrangian function

$$\mathcal{L}(\mathbf{x},\lambda) = x+3y+\lambda(x^2+y^2-1)$$

KKT conditions

1. $\mathcal{L}'_x = 1+2\lambda x = 0,\ \mathcal{L}'_y = 3+2\lambda y = 0$
2. $\lambda(x^2+y^2-1) = 0$
3. $x^2+y^2-1 \le 0$
4. $\lambda \ge 0$

$$\lambda \ne 0 \Rightarrow x = -1/(2\lambda),\ \ y = -3/(2\lambda) \Rightarrow 1/(4\lambda^2)+9/(4\lambda^2) = 1 \Rightarrow \lambda = \sqrt{10}/2$$

$$\Rightarrow x = -\frac{1}{\sqrt{10}},\ \ y = -\frac{3}{\sqrt{10}}$$

$$\begin{aligned} \max \quad & xy \\ \text{subject to} \quad & x + y^2 \le 2 \\ & x, y \ge 0 \end{aligned}$$

$$\begin{aligned} \min \quad & -xy \\ \text{subject to} \quad & x + y^2 - 2 \le 0 \\ & -x \le 0 \\ & -y \le 0 \end{aligned}$$

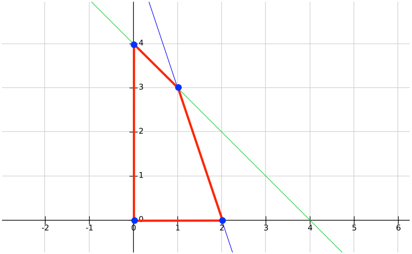
Lagrangian function

$$\mathcal{L}(x, y, \lambda) = -xy + \lambda_1(x + y^2 - 2) - \lambda_2 x - \lambda_3 y$$

KKT conditions

1. $\mathcal{L}'_x = -y + \lambda_1 - \lambda_2 = 0,$
2. $\mathcal{L}'_y = -x + 2\lambda_1 y - \lambda_3 = 0$
3. $\lambda_1(x + y^2 - 2) = 0$
4. $\lambda_2 x = 0$
5. $\lambda_3 y = 0$
6. $x + y^2 \le 2$
7. $x, y, \lambda_1, \lambda_2, \lambda_3 \ge 0$

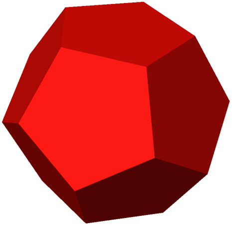
$$
\begin{aligned}
\text{maximize} \quad & g(x,y) = 2x + y \\
\text{subject to} \quad & 3x + y \leq 6 \\
& x + y \leq 4 \\
& x \geq 0 \\
& y \geq 0
\end{aligned}
$$



- The problem is convex
- Question: How can one solve the problem by using KKT conditions?

$$
\begin{aligned}
\text{maximize} \quad & g(x, y) = 4x + 2y + 3z \\
\text{subject to} \quad & x + 2y + 5z \leq 8 \\
& 4x + y + 2z \leq 6 \\
& x \geq 0 \\
& y \geq 0 \\
& z \geq 0
\end{aligned}
$$

- How to solve the problem by using geometric method
- Solve the problem by using KKT conditions?



Standard form

$$
\begin{aligned}
\text{minimize} \quad & c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \\
\text{subject to} \quad & a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1 \\
& a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_2 \\
& \quad \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
& a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m \\
\text{and} \quad & x_1 \geqslant 0,\ x_2 \geqslant 0,\ \ldots,\ x_n \geqslant 0,
\end{aligned}
\tag{1}
$$

Standard form

$$
\begin{aligned}
\text{minimize} \quad & \mathbf{c}^T \mathbf{x} \\
\text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \quad \text{and} \quad \mathbf{x} \geqslant \mathbf{0}.
\end{aligned}
\tag{2}
$$

Non-Standard form with inequality constraints

$$
\begin{array}{lll}
\text{minimize} & c_1x_1 + c_2x_2 + \cdots + c_nx_n & \\
\text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \leqslant b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \leqslant b_2 \\
& \vdots & \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \leqslant b_m \\
\text{and} & x_1 \geqslant 0,\ x_2 \geqslant 0,\ \ldots,\ x_n \geqslant 0, &
\end{array}
$$

Transform into a standard form by adding slack variables

$$
\begin{array}{llll}
\text{minimize} & c_1x_1 + c_2x_2 + \cdots + c_nx_n & & \\
\text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + y_1 & & = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & + y_2 & = b_2 \\
& \vdots & & \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & + y_m & = b_m \\
\text{and} & x_1 \geqslant 0,\ x_2 \geqslant 0,\ \ldots,\ x_n \geqslant 0, & & \\
\text{and} & y_1 \geqslant 0,\ y_2 \geqslant 0,\ \ldots,\ y_m \geqslant 0. & &
\end{array}
$$

Non-Standard form with free variables

$$\begin{aligned} \text{minimize} \quad & x_1 + 3x_2 + 4x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 = 5 \\ & 2x_1 + 3x_2 + x_3 = 6 \\ & x_2, x_3 \geq 0 \end{aligned}$$

Transform into a standard form by adding variables

Write $x_1 = x_1^+ - x_1^-$

$$\begin{aligned} \text{minimize} \quad & x_1^+ - x_1^- + 3x_2 + 4x_3 \\ \text{subject to} \quad & x_1^+ - x_1^- + 2x_2 + x_3 = 5 \\ & 2x_1^+ - 2x_1^- + 3x_2 + x_3 = 6 \\ & x_1^+, x_1^-, x_2, x_3 \geq 0 \end{aligned}$$

System of linear equations

$$Ax = b$$

where $A \in \mathbb{R}_{+}^{m \times n}$, $b \in \mathbb{R}_{+}^{m}$.

Basic solutions

Suppose that $m < n$, let B be any nonsingular $m \times m$ submatrix made up of columns of A. Then, if all $n - m$ components of x not associated with columns of B are set equal to zero, the solution to the resulting set of equations is said to be a basic solution to $Ax = b$ with respect to the basis B. The components of x associated with columns of B are called basic variables.

Constraints of LP

$$Ax = b, x \geq 0 \quad (*)$$

where $A \in \mathbb{R}_+^{m\times n}$, $b \in \mathbb{R}_+^m$; $m < n$.

Basic feasible solutions

- A vector x satisfying (*) is said to be feasible.
- A feasible solution that is also basic is said to be a basic feasible solution (BFS).

Example

Find all BFS to the following system:

$$x_1 + 2x_2 + x_3 = 5$$
$$2x_1 + 3x_2 + x_3 = 6$$
$$x_1, x_2, x_3 \geq 0$$

Standard form

$$\begin{aligned} &\text{minimize} \quad \mathbf{c}^T\mathbf{x} \\ &\text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b} \quad \text{and} \quad \mathbf{x} \geqslant \mathbf{0}. \end{aligned} \tag{2}$$

Fundamental theorem

Given a LP (2):

- if there is a feasible solution, there is a basic feasible solution.
- if there is an optimal feasible solution, there is an optimal basic feasible solution.

Solving LP using fundamental theorem

The theorem reduces the task of solving a linear program to that of searching over basic feasible solutions. Since for a problem having n variables and $m < n$ constraints there are at most

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

basic solutions.

Equivalence of extreme points and basic solutions

Let K be the convex polytope consisting of all feasible solutions x satisfying

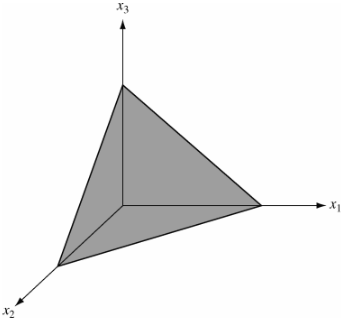
$$Ax = b, \;\; x \geq 0.$$

A feasible solution x is an extreme point of K if and only if x is a BFS.

Example

Consider the following system:

$$x_1 + x_2 + x_3 = 1$$
$$x_1, x_2, x_3 \geq 0$$



Convert the following problems to standard form and solve

a)

$$\begin{aligned} \text{minimize} \quad & x_1 + 3x_2 + 4x_3 \\ \text{subject to} \quad & 2 \leq x_1 + x_2 \leq 3 \\ & 4 \leq x_1 + x_3 \leq 5 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

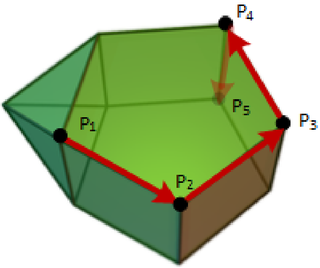
b)

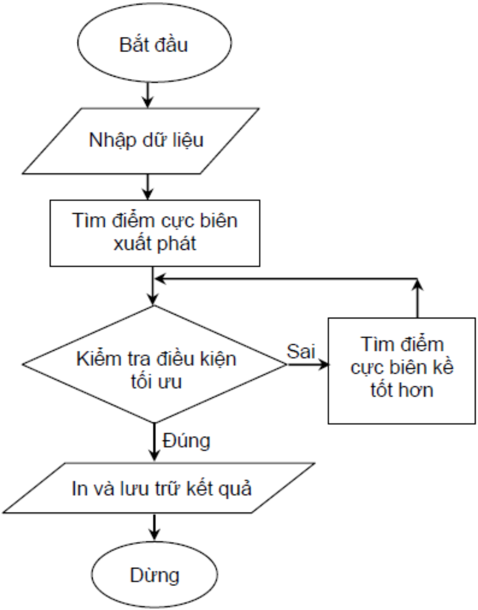
$$\begin{aligned} \text{minimize} \quad & x_1 + x_2 + x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + 3x_3 = 10 \\ & x_1 \geq 1, x_2 \geq 2, x_3 \geq 1 \end{aligned}$$

c)

$$\begin{aligned} \text{minimize} \quad & x_1 + 4x_2 + x_3 \\ \text{subject to} \quad & 2x_1 - 2x_2 + x_3 = 4 \\ & x_1 - x_3 = 1 \\ & x_2, x_3 \geq 0 \end{aligned}$$

George Bernard Dantzig
(November 8, 1914 – May 13, 2005)

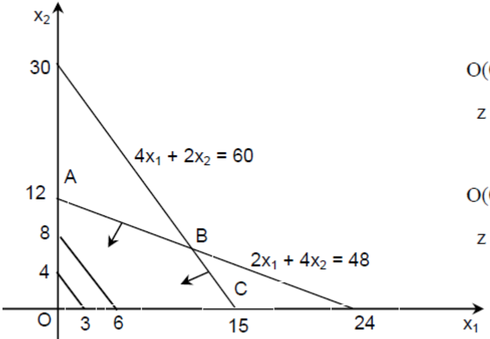



Bắt đầu
Nhập dữ liệu
Tìm điểm cực biên xuất phát
Kiểm tra điều kiện tối ưu
Sai
Tìm điểm cực biên kề tốt hơn
Đúng
In và lưu trữ kết quả
Dừng

Ví dụ 1. Xét BTQHTT: Max $z = 8x_1 + 6x_2$, với các ràng buộc

$$\begin{cases} 4x_1 + 2x_2 \le 60 \\ 2x_1 + 4x_2 \le 48 \\ x_1, x_2 \ge 0. \end{cases}$$

Giải bài toán bằng phương pháp đồ thị?



$O(0, 0) \rightarrow A(0, 12) \rightarrow B(12, 6)$ dừng

$z = 0 \qquad z = 72 \qquad z = 132$

$O(0, 0) \rightarrow C(15, 0) \rightarrow B(12, 6)$ dừng

$z = 0 \qquad z = 120 \qquad z = 132$

$$\begin{aligned} \text{maximize} \quad & 5x_1 + 4x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 8 \\ & 4x_1 + 3x_2 \leq 10 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \text{maximize} \quad & 2x_1 - 4x_2 + 5x_3 \\ \text{subject to} \quad & x_1 + 3x_2 - x_3 \leq 6 \\ & 4x_1 + x_2 + 2x_3 \leq 5 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & 4x_1 + x_2 \\ \text{subject to} \quad & 3x_1 - 5x_2 \leq 15 \\ & -x_1 + 2x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & -2x_1 + 4x_2 - 3x_3 \\ \text{subject to} \quad & 5x_1 - 2x_2 + x_3 \leq 8 \\ & 4x_1 + x_2 + 2x_3 \leq 6 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$