Problem Set 1

Exercise 1. Find the local extrema and saddle points of the following functions:

1. $f(x,y) = \frac{1}{3}x^3 - 3x^2 + \frac{y^2}{4} + xy + 13x - y + 2$
2. $f(x,y) = x^4 + y^4 - 4xy + 2$
3. $f(x,y) = x^3 - 2x^2 + y^2 + z^2 - 2xy + xz - yz + 3z$
4. $f(x,y) = x^3 + y^5 - 3x - 10y + 4$
5. $f(x,y) = \frac{1}{3}x^3 - x - \frac{1}{3}y^3 + y$
6. $f(x,y) = 2x^3 + 4y^2 - 2y^4 - 6x$
7. $f(x,y) = x^2y - ye^z + 2x + z$
8. $f(x,y) = -\frac{1}{2}xy + \frac{2}{x} - \frac{1}{y}$
9. $f(x,y) = (x^2 - y^2)e^{-(x^2+y^2)/2}$
10. $f(x,y) = e^{-(x^2+y^2)}(x^2 + 2y^2)$

Exercise 2. Find global maximum and global minimum of the function:

$$f : [-2,2] \times [-2,2] \to \mathbb{R} \quad \text{given by} \quad f(x,y) = 4xy - 2x^2 - y^4$$

Exercise 3. Determine if each set below is convex.

1. $\{(x,y) \in \mathbb{R}^2_+ \mid x/y \leq 1,\ x,y > 0\}$
2. $\{(x,y) \in \mathbb{R}^2_+ \mid x/y \geq 1,\ x,y > 0\}$
3. $\{(x,y) \in \mathbb{R}^2_+ \mid xy \leq 1,\ x,y > 0\}$
4. $\{(x,y) \in \mathbb{R}^2_+ \mid xy \geq 1,\ x,y > 0\}$

Exercise 4. Prove that if S_1, S_2 are convex, then

$$S_1 - S_2 = \{x - y : x \in S_1, y \in S_2\} \quad \text{is convex.}$$

Exercise 5.

- Let $f : \mathbb{R}^n \to \mathbb{R}$ be function, where $f(x) = a^T x + b$ with $a \in \mathbb{R}^n, b \in \mathbb{R}$. Prove that if $S \in \mathbb{R}^n$ is convex then $f(S) = \{f(x) : x \in S\}$ is convex.
- Prove that $\{x \in \mathbb{R}^m \mid Ax \leq b\}$, where $A \in \mathbb{R}^{n\times m}, b \in \mathbb{R}^n$, is convex.

Exercise 6. Prove that $\text{conv}(A + B) = \text{conv}(A) + \text{conv}(B)$, where $\text{conv}(S)$ denotes the convex hull of a set S.

Exercise 7. Check if the following matrices are positive definite or positive semi-definite?

$$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} -2 & 2 \\ 2 & -4 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}, \begin{bmatrix} -5 & 1 & 1 \\ 1 & -7 & 1 \\ 1 & 1 & -5 \end{bmatrix}.$$