Data Visualization

Lecture 5
Mathamatics Visualization

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Content

Plot a mathmatics graph

- *Introduction to Linear Algebra*

Vector

- *If* n *is a positive integer and* $\mathbb{R}$ *is the set of real numbers, then* $\mathbb{R}^n$ *is the set of all* n*-tuples of real numbers.*
- *A vector* $v \in \mathbb{R}^n$ *is one such* n*-tuple, for example*

$$V = (v_1, v_2) \in (\mathbb{R}, \mathbb{R}) \equiv \mathbb{R}^2$$

Example 1 *The following* 1*-dimensional array*

$$\begin{bmatrix} 907 \\ 220 \\ 625 \\ 502 \end{bmatrix}$$

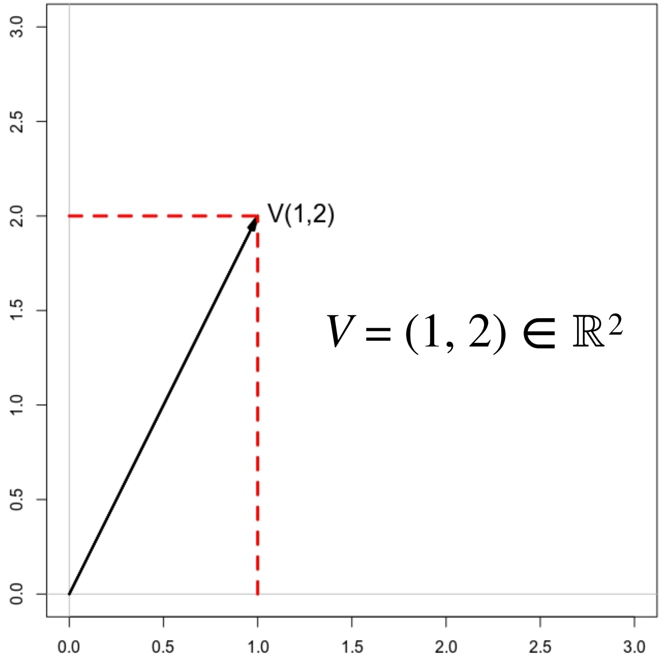
is a 4*-dimensional vector. In* `R`*, we can use c() function to create a vector. For example,*

```
v <- c(907,220,625,502)
```

https://kevinbinz.com/2017/02/20/linear-algebra/

Vector

```
vplot <- function(v){
        library(matlib)
        par(mar=c(3,3,1,1)+.1)
        xlim <- c(0, max(v[1], v[2])+1)
        ylim <- c(0, max(v[1], v[2])+1)
        plot(xlim, ylim, type="n", xlab="X", ylab="Y", asp=1)
        abline(v=0, h=0, col="gray")
        vectors(v, labels=paste0(deparse(substitute(v)), "(", v[1], ",", v[2], ")"), pos.lab=c(4,2))
        lines(c(v[1], v[1]), c(0, v[2]), col="red", lwd=3, lty=2)
        lines(c(0, v[1]), c(v[2], v[2]), col="red", lwd=3, lty=2)
}
V <- c(1, 2)
vplot(V)
```



https://kevinbinz.com/2017/02/20/linear-algebra/

Zero vector or null vector

Definition 4 *A* **zero vector** *or a* **null vector** *is an n-dimensional vector with all zeros as its elements.*

Example 10 *For $n = 5$, the 5-dimensional zero vector is*

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

In R *you can create the 5-dimensional zero vector using the rep() function.*

```
rep(0, 5)
```

The first argument is the value you want to assign as its element and the second argument is the dimension of the vector. The output from R *is as follows:*

```
> rep(0, 5)
[1] 0 0 0 0 0
```

Matrix

A **matrix** $A \in \mathbb{R}^{m \times n}$ is a rectangular array of real numbers with m rows and n columns. For example,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \in \begin{bmatrix} \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} \end{bmatrix} \equiv \mathbb{R}^{3x2}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Matrix

Example 6 *Here, we show column vectors and row vectors of the matrix from Example 4. Suppose we have a* 2×4 *matrix such that*

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

Then its column vectors are

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix},$$

and its row vectors are

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}.$$

```
M <- matrix(c(1,1,1,2,1,3,1,4),nrow=2,ncol=4)
```

Matrix

Example 7 *In this example we show column vectors and row vectors of the matrix from Example 5. Recall we have a* 4×4 *matrix:*

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Then its column vectors are

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

and its row vectors are

$$\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}.$$

Matrix

```
M <- matrix(c(1,1,1,2,1,3,1,4),nrow=2,ncol=4)
```

- Using rbind instead
- Using cbind instead
- Access into cell ij (i is row number and j is column number) with i = 2, j =3
- Subset the matrix into smaller matrix, vectors and columns

Matrix

Definition 3 *Suppose we have an $m \times n$ matrix such that*

$$\begin{bmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,n} \\ x_{2,1} & x_{2,2} & \dots & x_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m,1} & x_{m,2} & \dots & x_{m,n} \end{bmatrix}.$$

Then $m \times 1$ matrices

$$\begin{bmatrix} x_{1,1} \\ x_{2,1} \\ \vdots \\ x_{m,1} \end{bmatrix}, \begin{bmatrix} x_{1,2} \\ x_{2,2} \\ \vdots \\ x_{m,2} \end{bmatrix}, \cdots, \begin{bmatrix} x_{1,n} \\ x_{2,n} \\ \vdots \\ x_{m,n} \end{bmatrix}$$

are called **column vectors** *of the matrix. Also $1 \times n$ matrices such that*

$$\begin{gathered} \begin{bmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,n} \end{bmatrix}, \\ \begin{bmatrix} x_{2,1} & x_{2,2} & \dots & x_{2,n} \end{bmatrix}, \\ \vdots \\ \begin{bmatrix} x_{m,1} & x_{m,2} & \dots & x_{m,n} \end{bmatrix} \end{gathered}$$

are called **row vectors** *of the matrix.*

Matrix

Example 7 *In this example we show column vectors and row vectors of the matrix from Example 5. Recall we have a* 4×4 *matrix:*

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Then its column vectors are

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

and its row vectors are

$$\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}.$$

Matrix

Example 9 *We look at the working example in this section the "Smarket" data set from* `ISLR` *package. Using the head() function you can see the first six observations:*

```
> head(Smarket)
  Year   Lag1   Lag2   Lag3   Lag4   Lag5 Volume  Today Direction
1 2001  0.381 -0.192 -2.624 -1.055  5.010 1.1913  0.959        Up
2 2001  0.959  0.381 -0.192 -2.624 -1.055 1.2965  1.032        Up
3 2001  1.032  0.959  0.381 -0.192 -2.624 1.4112 -0.623      Down
4 2001 -0.623  1.032  0.959  0.381 -0.192 1.2760  0.614        Up
5 2001  0.614 -0.623  1.032  0.959  0.381 1.2057  0.213        Up
6 2001  0.213  0.614 -0.623  1.032  0.959 1.3491  1.392        Up
```

Matrix

Definition 5 *Suppose we have an* $m \times n$ *matrix. If* $m = n$*, then we call this matrix a* **square matrix**.

Example 11 *From Example 5. A* 4×4 *matrix:*

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

is a square matrix since the number of row vectors and the number of column vectors are equal.

Definition 6 *The* **identity matrix**, I_n, *of size* n *is an* $n \times n$ *square matrix such that all elements in the* i*th row and* i*th column equal* 1 *for all* i *from* 1 *to* n*, and otherwise all* 0.

Matrix

Example 12 *For $n = 3$, an identity matrix of size 3 is*

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and for $n = 4$, an identity matrix of size 4 is

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

To create the identity matrix of size n in `R` you can use the diag() function. If you type diag(n), then you can create the identity matrix of size n.

Matrix

Example 12 *For $n = 3$, an identity matrix of size 3 is*

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and for $n = 4$, an identity matrix of size 4 is

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

To create the identity matrix of size n in `R` you can use the diag() function. If you type diag(n), then you can create the identity matrix of size n.

Matrix

Example 13 *We create here the identity matrix of size 3 and the identity matrix of size 4 in* `R` *with the diag() function. If you type*

```
diag(3)
```

then you will see the output as follows:

```
> diag(3)
     [,1] [,2] [,3]
[1,]    1    0    0
[2,]    0    1    0
[3,]    0    0    1
```

Similarly, if you type diag(4), then you will see the following output in `R`*:*

```
> diag(4)
     [,1] [,2] [,3] [,4]
[1,]    1    0    0    0
[2,]    0    1    0    0
[3,]    0    0    1    0
[4,]    0    0    0    1
```

Matrix

Example 15 *Suppose we have a matrix*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Then $a_{11} = 1$, $a_{12} = 2$, $a_{13} = 3$, $a_{21} = 4$, $a_{22} = 5$, $a_{23} = 6$.

Notation 1.2 *Suppose we have an* $m \times n$ *matrix* A *for any positive integers* m *and* n. *Then we notate* A *as*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

Example 16 *Suppose* $m = 2$ *and* $n = 2$. *Then we have*

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Example 17 *Suppose* $m = 2$ *and* $n = 3$. *Then we have*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

Matrix

Definition 7 *Suppose we have an $m \times n$ matrix A such that*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

The **diagonal entries** *of A are entries of A such that*

$$a_{ij} \text{ for all } j = i.$$

Definition 8 *Suppose we have an $m \times n$ matrix A such that*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

If $a_{ij} = 0$ for all $j \neq i$, then we call A a **diagonal matrix**.

Matrix

Example 19 *Suppose*

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}.$$

Then A is a diagonal matrix.

Definition 9 *Suppose we have a square matrix A such that*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}.$$

If $a_{ij} = a_{ji}$ for all $i = 1, \dots, m$ and $j = 1, \dots, m$, then we say A is a **symmetric matrix**.

Example 20 *Any square diagonal matrices are symmetric matrices.*

Example 21 *Suppose we have*

$$A = \begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix}.$$

Then A is a symmetric matrix.

Matrix

Example 22 *Suppose we have*

$$A = \begin{bmatrix} -1 & 2 & 0 & 5 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 12 & -1 \\ 5 & 1 & -1 & 9 \end{bmatrix}.$$

Then A is a symmetric matrix.

Definition 10 *Suppose we have an $m \times n$ matrix A such that*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

If $a_{ij} = 0$ for all $i > j$, then we say A is an **upper triangular matrix**.

Matrix

Example 24 *Suppose we have*

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}.$$

Then A is an upper triangular matrix.

Example 25 *Suppose we have*

$$A = \begin{bmatrix} -1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 12 & -1 \\ 0 & 0 & 0 & 9 \end{bmatrix}.$$

Then A is an upper triangular matrix.

Example 26 *Suppose we have*

$$A = \begin{bmatrix} -1 & 2 & 0 & 5 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 12 & -1 & -3 \\ 0 & 0 & 0 & 9 & -4 \end{bmatrix}.$$

Then A is an upper triangular matrix.

Example 27 *Suppose we have*

$$A = \begin{bmatrix} -1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 12 & -1 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then A is an upper triangular matrix.

Definition 11 *Suppose we have an $m \times n$ matrix A such that*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

If $a_{ij} = 0$ for all $i < j$, then we say A is a **lower triangular matrix**.

Example 28 *Any square diagonal matrices are lower triangular matrices.*

Example 29 *Suppose we have*

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}.$$

Then A is a lower triangular matrix.

Matrix

Example 30 *Suppose we have*

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 5 & 1 & -1 & 9 \end{bmatrix}.$$

Then A is a lower triangular matrix.

Example 31 *Suppose we have*

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 0 \\ 5 & 1 & -1 & 9 & 0 \end{bmatrix}.$$

Then A is a lower triangular matrix.

Example 32 *Suppose we have*

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 5 & 1 & -1 & 9 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Then A is a lower triangular matrix.

System of Linear Equations

To define a system of equations, we will set up a smaller example of the transportation problem described in the previous section. A beer company has two production sites, A and B, and they want to transport their beers to two distributors, C and D. The demand from distributor C is 542 beers per week, and the demand from distributor D is 422 beers per week. The supply from production site A is 475 beers per week, and the supply from production site B is 489 beers per week. Can these sites produce enough beers to satisfy the demands from the distributors?

In this example, we can set up variables x_{ij}, the amount of beer shipped from a production site i to the distributor j. In the transportation problem the easiest thing is to set up the table as follows:

	C	D	total
A	x_{AC}	x_{AD}	475
B	c_{BC}	x_{BD}	489
total	542	422	964

The total supply from production site A is 475 and the total supply from production site B is 489. The total demand from distributor C is 542 and the total demand from distributor D is 422. Therefore we have the following system of equations:

$$
\begin{array}{ccccccccc}
x_{AC} & + & x_{AD} & & & & & = & 475 \\
& & & & x_{BC} & + & x_{BD} & = & 489 \\
x_{AC} & & & + & x_{BC} & & & = & 542 \\
& & x_{AD} & & & + & x_{BD} & = & 422
\end{array}
$$

System of Linear Equations

Definition 13 *A* **system of linear equations** *is a collection of one or more equations with the same set of variables, for example,* $x_1, \ldots, x_n$*:*

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & \ldots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & \ldots & + & a_{mn}x_n & = & b_m \end{array}$$

where $a_{11}, \ldots, a_{1n}, \ldots, a_{mn}$ *are coefficients and* $b_1, \ldots b_m$ *are real numbers.*

Example 33 *The set of linear equations with variables* x_{AC}, x_{AD}, x_{BC} *and* x_{BD} *such that*

$$\begin{array}{ccccccccc} 3x_{AC} & + & x_{AD} & & & & & = & 475 \\ & & & & x_{BC} & + & x_{BD} & = & 489 \\ x_{AC} & & & + & 2x_{BC} & & & = & 542 \\ & & x_{AD} & & & + & x_{BD} & = & 422 \end{array}$$

is a system of linear equations.

In general, a system of linear equations with four variables is difficult to see geometrically. First, we show systems of linear equations with two variables so that we can demonstrate them geometrically.

System of Linear Equations

A **solution** of a system of linear equations with the variables $x_1, \ldots, x_n$ such that

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & \ldots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & \ldots & + & a_{mn}x_n & = & b_m \end{array}$$

is a vector $\begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$ which satisfies all equations such that

$$\begin{array}{ccccccc} a_{11}s_1 & + & a_{12}s_2 & \ldots & + & a_{1n}s_n & = & b_1 \\ a_{21}s_1 & + & a_{22}s_2 & \ldots & + & a_{2n}s_n & = & b_2 \\ \vdots & & & & & & & \vdots \\ a_{m1}s_1 & + & a_{m2}s_2 & \ldots & + & a_{mn}s_n & = & b_m. \end{array}$$

System of Linear Equations

Example 35 *Suppose we have the system of linear equations with variables x_1 and x_2 such that*

$$\begin{array}{rcrcl} x_1 & + & x_2 & = & 4 \\ 2x_1 & + & x_2 & = & 5. \end{array}$$

A vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ *is a solution of the system since*

$$\begin{array}{rcrcl} 1 & + & 3 & = & 4 \\ 2 \cdot 1 & + & 3 & = & 5. \end{array}$$

Geometrically, each linear equation in this system of linear equations defines a line (Figure 1.3). Since there are two linear equations in this system, there are two lines in this system. In this example, the two lines meet at the unique point, a vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. In the case of two variables, a solution is defined by where the linear equations in the system meet.

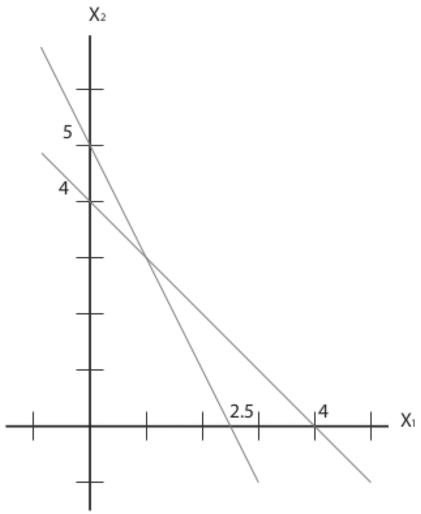


FIGURE 1.3
Geometrical view of the system of linear equations when there is a unique solution.

If these lines do not meet, then there is no solution to satisfy the system of linear equations. If they overlap, then there are multiple solutions, i.e., infinitely many solutions. For example, suppose we have a system of linear equations such that

$$\begin{array}{rcrcl} x_1 & + & x_2 & = & 4 \\ -2x_1 & - & 2x_2 & = & 5, \end{array}$$

then as Figure 1.4 shows, there is no crossing between two lines and therefore there is no solution.

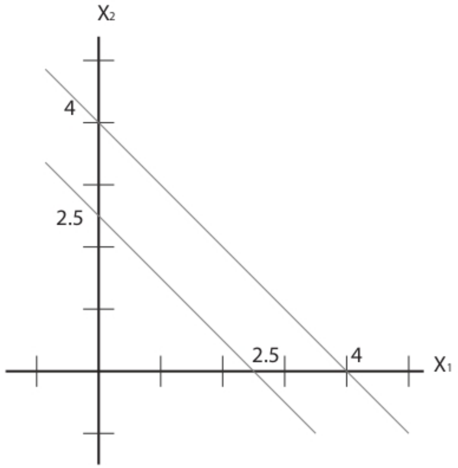


FIGURE 1.4
Geometrical view of the system of linear equations when there is no solution.

Suppose we have the following system of linear equations:

$$\begin{array}{rcrcl} -x_1 & - & x_2 & = & 4 \\ x_1 & + & x_2 & = & -4. \end{array}$$

Then since the lines overlap to each other, there are infinitely many solutions shown in Figure 1.5.

A system of linear equations has either:

1. no solution;
2. exactly one solution;
3. infinitely many solutions.

Matrix Notation

A system of linear equations can be written as a matrix. It is easy to visualize and using **matrix operations**, we can solve a system of linear equations. We will discuss how to solve a system of linear equations in the next section.

Suppose we have the system of linear equations with variables x_1 and x_2 such that

$$\begin{array}{rcrcr} x_1 & + & x_2 & = & 4 \\ 2x_1 & + & x_2 & = & 5. \end{array}$$

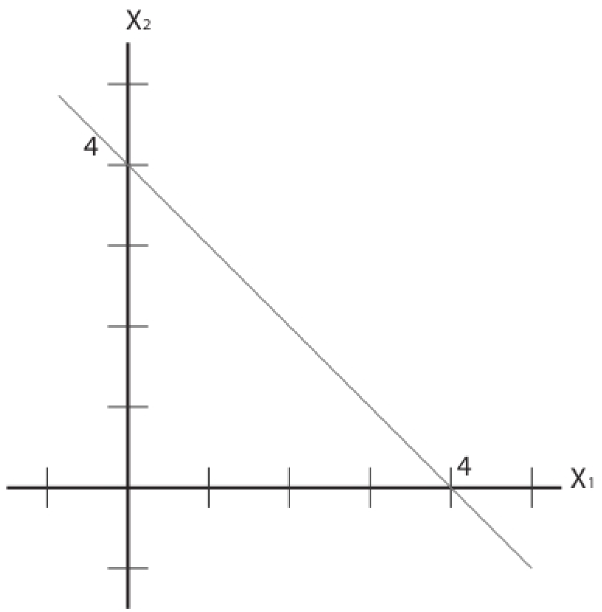


FIGURE 1.5
Geometrical view of the system of linear equations when there are infinitely many solutions.

Then we can write this system as a matrix:

$$\begin{bmatrix} 1 & 1 & 4 \\ 2 & 1 & 5 \end{bmatrix}.$$

This is called the **augmented matrix** of the system of linear equations. Also the matrix

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

without the last column of the augmented matrix is called the **coefficient matrix** of the system of linear equations.

Example 36 *The system of linear equations*

$$\begin{array}{rcrcr} x_1 & + & x_2 & = & 4 \\ -2x_1 & - & 2x_2 & = & 5 \end{array}$$

has the augmented matrix

$$\begin{bmatrix} 1 & 1 & 4 \\ -2 & -2 & 5 \end{bmatrix}.$$

Example 37 *The system of linear equations*

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 4x_3 & = & 1 \\ 2x_1 & & & - & x_3 & = & -1.5 \end{array}$$

Matrix Notation

has the augmented matrix

$$\begin{bmatrix} 1 & -1 & 4 & 1 \\ 2 & 0 & -1 & -1.5 \end{bmatrix}.$$

The element of the second row and the second column of the matrix is 0 *because we have a coefficient of* x_2 *equal to* 0 *in the second linear equation.*

Example 38 *Go back to the first example of this section. A beer company has two production sites* A *and* B *and they want to transport their beer to two distributors* C *and* D*. The demand from distributor* C *is 542 beers per week, and the demand from distributor* D *is 422 beers per week. The supply from production site* A *is 475 beers per week, and the supply from production site* B *is 489 beers per week. We want to know if these sites produce enough beers to satisfy the demands from the distributors. This problem can set up a system of linear equations such that*

$$\begin{array}{ccccccccc} x_{AC} & + & x_{AD} & & & & & = & 475 \\ & & & & x_{BC} & + & x_{BD} & = & 489 \\ x_{AC} & & & + & x_{BC} & & & = & 542 \\ & & x_{AD} & & & + & x_{BD} & = & 422 \end{array}$$

where the variables x_{ij} *are the amounts of beer shipped from production site* i *to distributor* j*. Then we have the augmented matrix:*

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 475 \\ 0 & 0 & 1 & 1 & 489 \\ 1 & 0 & 1 & 0 & 542 \\ 0 & 1 & 0 & 1 & 422 \end{bmatrix}.$$

In R*, you can type this as*

```
M <- matrix(c(1,0,1,0,1,0,0,1,0,1,1,0,0,1,0,1,475,489,542,422),
nrow=4,ncol=5)
```

Then, if you type M, you see the following:

```
> M
     [,1] [,2] [,3] [,4] [,5]
[1,]    1    1    0    0  475
[2,]    0    0    1    1  489
[3,]    1    0    1    0  542
[4,]    0    1    0    1  422
```

Solving a System of Linear Equations

via the elementary row reduction or Gaussian Elimination

Suppose we have the following system of 3 linear equations with the variables x_1, x_2, x_3:

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 4x_3 & = & 1 \\ 2x_1 & & & - & x_3 & = & -1.5 \\ -x_1 & + & x_2 & & & = & 2. \end{array}$$

With this example, we show how the elementary row reduction method works. First you compute the augmented matrix of this system:

$$\begin{bmatrix} 1 & -1 & 4 & 1 \\ 2 & 0 & -1 & -1.5 \\ -1 & 1 & 0 & 2 \end{bmatrix}.$$

Here we will go through, step by step, a notion of a system of linear equations and a notion of its augmented matrix:

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 4x_3 & = & 1 \\ 2x_1 & & & - & x_3 & = & -1.5 \\ -x_1 & + & x_2 & & & = & 2 \end{array} \quad \begin{bmatrix} 1 & -1 & 4 & 1 \\ 2 & 0 & -1 & -1.5 \\ -1 & 1 & 0 & 2 \end{bmatrix}.$$

1. We add the first equation to the third equation. This new equation becomes the new third equation:

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 4x_3 & = & 1 \\ -x_1 & + & x_2 & & & = & 2 \\ \hline & & & & 4x_3 & = & 3 \end{array}$$

Thus a new system of linear equations and its augmented matrix are:

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 4x_3 & = & 1 \\ 2x_1 & & & - & x_3 & = & -1.5 \\ & & & & 4x_3 & = & 3 \end{array} \quad \begin{bmatrix} 1 & -1 & 4 & 1 \\ 2 & 0 & -1 & -1.5 \\ 0 & 0 & 4 & 3 \end{bmatrix}.$$

2. Then we divide the third equation by 4. The new system of equations and its augmented matrix are:

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 4x_3 & = & 1 \\ 2x_1 & & & - & x_3 & = & -1.5 \\ & & & & x_3 & = & 0.75 \end{array} \quad \begin{bmatrix} 1 & -1 & 4 & 1 \\ 2 & 0 & -1 & -1.5 \\ 0 & 0 & 1 & 0.75 \end{bmatrix}.$$

3. Now we add the third equation to the second equation. This becomes the new second equation:

$$\begin{array}{rcrcr} 2x_1 & - & x_3 & = & -1.5 \\ & & x_3 & = & 0.75 \\ \hline 2x_1 & & & = & -0.75 \end{array}$$

Thus, the new system of linear equations and its augmented matrix are:

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 4x_3 & = & 1 \\ 2x_1 & & & & & = & -0.75 \\ & & & & x_3 & = & 0.75 \end{array} \quad \begin{bmatrix} 1 & -1 & 4 & 1 \\ 2 & 0 & 0 & -0.75 \\ 0 & 0 & 1 & 0.75 \end{bmatrix}.$$

4. Now we divide the second equation by 2. The new system of linear equations and its augmented matrix are:

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 4x_3 & = & 1 \\ x_1 & & & & & = & -0.375 \\ & & & & x_3 & = & 0.75 \end{array} \quad \begin{bmatrix} 1 & -1 & 4 & 1 \\ 1 & 0 & 0 & -0.375 \\ 0 & 0 & 1 & 0.75 \end{bmatrix}.$$

5. Exchange the first equation and the second equation: The new system of linear equations and its augmented matrix are:

$$\begin{array}{rcrcrcr} x_1 & & & & & = & -0.375 \\ x_1 & - & x_2 & + & 4x_3 & = & 1 \\ & & & & x_3 & = & 0.75 \end{array} \quad \begin{bmatrix} 1 & 0 & 0 & -0.375 \\ 1 & -1 & 4 & 1 \\ 0 & 0 & 1 & 0.75 \end{bmatrix}.$$

6. Now we take the second equation and subtract the first equation:

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 4x_3 & = & 1 \\ -(x_1 & & & & & = & -0.375) \\ \hline & - & x_2 & + & 4x_3 & = & 1.375 \end{array}$$

Solving a System of Linear Equations

via the elementary row reduction or Gaussian Elimination

This equation becomes the new second equation. The new system of linear equations and its augmented matrix are:

$$\begin{array}{rcrcrcr} x_1 & & & & & = & -0.375 \\ & - & x_2 & + & 4x_3 & = & 1.375 \\ & & & & x_3 & = & 0.75 \end{array} \quad \begin{bmatrix} 1 & 0 & 0 & -0.375 \\ 0 & -1 & 4 & 1.375 \\ 0 & 0 & 1 & 0.75 \end{bmatrix}.$$

7. Similarly we take the second equation and subtract 4 times the third equation:

$$\begin{array}{rcrcrcr} - & x_2 & + & 4x_3 & = & 1.375 \\ & & & -4(x_3 & = & 0.75) \\ \hline - & x_2 & & & = & -1.625 \end{array}$$

This becomes the new second equation. The new system of linear equations and its augmented matrix are:

$$\begin{array}{rcrcrcr} x_1 & & & & & = & -0.375 \\ & - & x_2 & & & = & -1.625 \\ & & & & x_3 & = & 0.75 \end{array} \quad \begin{bmatrix} 1 & 0 & 0 & -0.375 \\ 0 & -1 & 0 & -1.625 \\ 0 & 0 & 1 & 0.75 \end{bmatrix}.$$

8. The last step is to multiply the second equation by -1. Then we have the solution:

$$\begin{array}{rcrcr} x_1 & & & = & -0.375 \\ & x_2 & & = & 1.625 \\ & & x_3 & = & 0.75 \end{array} \quad \begin{bmatrix} 1 & 0 & 0 & -0.375 \\ 0 & 1 & 0 & 1.625 \\ 0 & 0 & 1 & 0.75 \end{bmatrix}.$$

What you have seen in the example above is an elementary row reduction applied to the augmented matrix. Elementary row operations are operations on an augmented matrix. These are equivalent to the operations on a system of linear equations such that adding or multiplying two equations in the system does not change a solution of the system, nor does multiplying the whole linear equation by a constant scalar.

Solving a System of Linear Equations

via the elementary row reduction or Gaussian Elimination

Definition 14 *The* **elementary row operations** *or* **Gaussian Eliminations** *are operations in a matrix defined as follows:*

1. *Replacement: Replace one row by the sum of the row itself and another row.*
2. *Interchange: Interchange two rows.*
3. *Scaling: Multiplication of a whole row by a constant.*

These operations in an augmented matrix can be translated to operations in its system of linear equations, which are shown in Table 1.5.

TABLE 1.5
Operations in a system of linear equations and elementary row operations on its augmented matrix.

Elementary row operations	System of linear equations
Replacement	Replace an equation by adding itself to another equation
Interchange	Interchanging two equations
Scaling	Multiplying a whole equation by a constant

Solving a System of Linear Equations in R

```
library(matlib)
```

Example 41 *We go back to Example 34. Recall we have the following system of linear equations such that*

$$\begin{array}{rcrcl} x_1 & + & x_2 & = & 4 \\ 2x_1 & + & x_2 & = & 5. \end{array}$$

Its augmented matrix is

$$\begin{bmatrix} 1 & 1 & 4 \\ 2 & 1 & 5 \end{bmatrix}.$$

In `R` *you type:*

```
A <- matrix(c(1,2,1,1), nrow = 2, ncol = 2)
```

If you type A, then `R` *will return*

```
> A
     [,1] [,2]
[1,]    1    1
[2,]    2    1
```

Now we create a vector for the right-hand side of the system of linear equations:

```
b <- c(4, 5)
```

If you type b, then `R` *will return*

```
> b
[1] 4 5
```

```
Solve(A, b)
```

Then, `R` *will return*

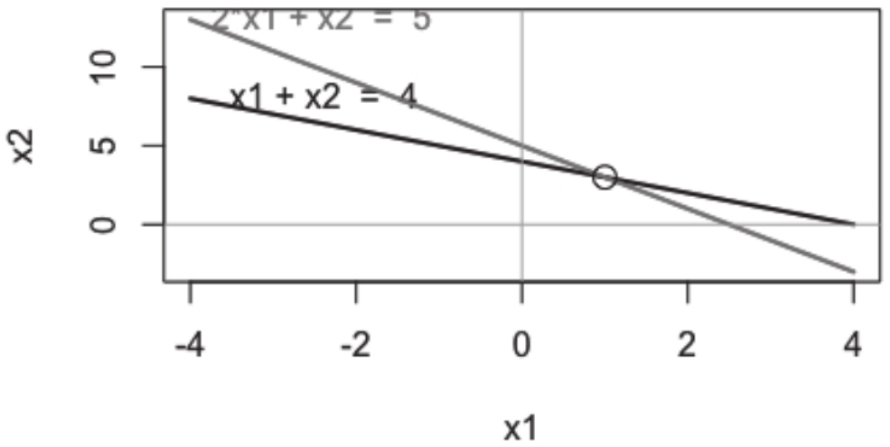
```
> Solve(A, b)
x1    =  1
  x2  =  3
```

Without this package, if you failed to install **`matlib`**, *just type*

```
solve(A, b)
```

Then, `R` *will return*

```
> solve(A, b)
[1] 1 3
```

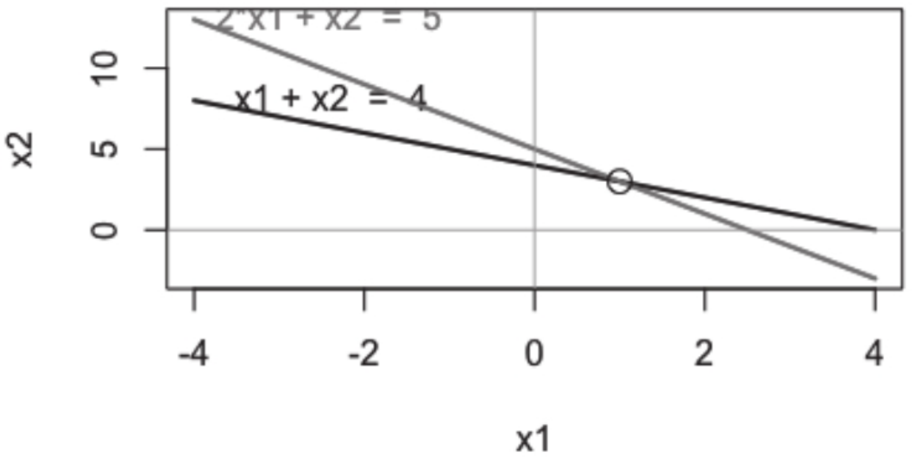


Visualization of the geometry of the system of linear equation

A nice thing about this package is that you can plot the geometry of the system of linear equations. If you type

```
plotEqn(A,b)
```

Then `R` *outputs the image shown in Figure 1.6. In this example the two lines defined by the two equations intersect at one point* $(1, 3)$*, which is the circle in the image.*



Visualization of the geometry of the system of linear equation

Example 42 *This is from Example 39. The system of linear equations is:*

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 6 \\ -2x_1 & + & 3x_2 & - & 2x_3 & = & -1 \\ -x_1 & + & 2x_2 & + & x_3 & = & 2. \end{array}$$

Its augmented matrix is

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ -2 & 3 & -2 & -1 \\ -1 & 2 & 1 & 2 \end{bmatrix}.$$

In R*, you type:*

```
A <- matrix(c(1,-2,-1,2,3,2,3,-2,1), nrow = 3, ncol = 3)
```

If you type A, then R *will return*

```
> A
     [,1] [,2] [,3]
[1,]    1    2    3
[2,]   -2    3   -2
[3,]   -1    2    1
```

Now we create a vector for the right-hand side of the system of linear equations:

```
b <- c(6, -1, 2)
```

If you type b, then R *will return*

```
> b
[1]  6 -1  2
```

If you type b, then R *will return*

```
> b
[1]  6 -1  2
```


Then, in order to solve the system of linear equations with the **matlib** *package, just type:*

```
Solve(A, b)
```

Then, R *will return*

```
> Solve(A, b)
x1       =  1
  x2     =  1
    x3   =  1
```

Without this package, if you failed to install **matlib***, just type*

Visualization of the geometry of the system of linear equation

```
solve(A, b)
```

Then, R *will return:*

```
> solve(A, b)
[1] 1 1 1
```

Again, this package can plot not only the dimensional geometry but also it can plot the three-dimensional geometry of the system of linear equations. If you type as follows:

```
plotEqn3d(A,b, xlim=c(0,4), ylim=c(0,4))
```

R *plots the figure shown in Figure 1.7.*

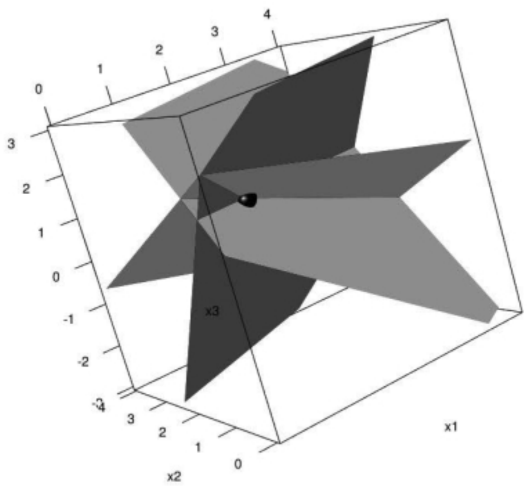


FIGURE 1.7
This is a visualization of the geometry of the system of linear equation in Example 42 from the `matlib` package in R.

In the argument of the function, "xlim=" is used to set up the range of the x-axis and "ylim" is used to set up the range of the y-axis. Then R *outputs the image shown in Figure 1.7. This is interactive. You hold the button of the mouse and rotate the image, so you can see how the linear equations intersect each other. In this example, the three linear planes defined by the three equations intersect at the solution* $(1, 1, 1)$*, which is the black dot in the image.*

Visualization of the geometry of the system of linear equation

Example 43 *A beer company has two production sites, A and B, and they want to transport their beers to two distributors, C and D. The demand from distributor C is 542 beers per week, and the demand from distributor D is 422 beers per week. The supply from production site A is 475 beers per week, and the supply from production site B is 489 beers per week. We want to know if these production sites produce enough beers to satisfy the demands from the distributors. This problem can be set up as a system of linear equations such that*

$$\begin{array}{ccccccccc} x_{AC} & + & x_{AD} & & & & & = & 475 \\ & & & & x_{BC} & + & x_{BD} & = & 489 \\ x_{AC} & & & + & x_{BC} & & & = & 542 \\ & & x_{AD} & & & + & x_{BD} & = & 422 \end{array}$$

where the variables x_{ij} are the number of beers shipped from a production site i to the distributor j. Then we have the augmented matrix:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 475 \\ 0 & 0 & 1 & 1 & 489 \\ 1 & 0 & 1 & 0 & 542 \\ 0 & 1 & 0 & 1 & 422 \end{bmatrix}.$$

In R*, first you create the coefficient matrix A by the following:*

```
A <- matrix(c(1,0,1,0,1,0,0,1,0,1,1,0,0,1,0,1),nrow=4,ncol=4)
```

Then if you type A, R *returns*

```
> A
     [,1] [,2] [,3] [,4]
[1,]    1    1    0    0
[2,]    0    0    1    1
[3,]    1    0    1    0
[4,]    0    1    0    1
```

Now we create a vector for the right-hand side of the system of linear equations such that

```
b <- c(475, 489, 542,422)
```

Then if you type b, R *returns*

```
> b
[1] 475 489 542 422
```

Now we will use the function Solve() to solve the system of linear equations:

```
Solve(A, b)
```

Then R *returns*

```
> Solve(A, b)
x1      - 1*x4  =   53
  x2      + x4  =  422
    x3    + x4  =  489
             0  =    0
```

This means that there are infinitely many solutions since x4 is an independent variable, i.e., you can plug in any number in the variable x4.

Visualize Gaussian Elimination

Example 44 *In* `R` *we can also see how Gaussian Elimination works using the* `matlib` *package [17]. We will use the example above. Suppose we have the following system of linear equations:*

$$\begin{array}{rcrcrcrcr} & & x_2 & + & 3x_3 & - & x_4 & = & 1 \\ -x_1 & + & x_2 & - & 4x_3 & & & = & 1 \\ x_1 & & & + & 2x_3 & + & 4x_4 & = & 5 \\ & & x_2 & & & - & 4x_4 & = & -2. \end{array}$$

Its augmented matrix is

$$\begin{bmatrix} 0 & 1 & 3 & -1 & 1 \\ -1 & 1 & -4 & 0 & 1 \\ 1 & 0 & 2 & 4 & 5 \\ 0 & 1 & 0 & -4 & -2 \end{bmatrix}.$$

First we load the package:

```
library(matlib)
```

Then we create the coefficient matrix and the vector for the right hand side of the system of linear equations.

```
A <- matrix(c(0, -1, 1, 0, 1, 1, 0, 1, 3, -4, 2, 0, -1, 0, 4, -4), 4, 4)
b <- c(1, 1, 5, -2)
```

Here, we did not type "nrow=" and "ncol=". If it is clear we can skip typying them.

Using the showEqn() function we can see the system of linear equations.

```
showEqn(A, b)
```

Then `R` *shows*

```
> showEqn(A, b)
 0*x1 + 1*x2 + 3*x3 - 1*x4  =   1
-1*x1 + 1*x2 - 4*x3 + 0*x4  =   1
 1*x1 + 0*x2 + 2*x3 + 4*x4  =   5
 0*x1 + 1*x2 + 0*x3 - 4*x4  =  -2
```

Now using the echelon() function we can see how Gaussian Elimination works. With this example, we can type in `R` *as:*

```
echelon(A, b, verbose=TRUE, fractions=TRUE)
```

If this option "fraction" is set as "TRUE", then it outputs in the form of rational numbers. `R` *outputs the following as how Gaussian Elimination works:*

```
> echelon(A, b, verbose=TRUE, fractions=TRUE)

Initial matrix:
     [,1] [,2] [,3] [,4] [,5]
[1,]  0    1    3   -1    1
[2,] -1    1   -4    0    1
[3,]  1    0    2    4    5
[4,]  0    1    0   -4   -2

row: 1

 exchange rows 1 and 2
     [,1] [,2] [,3] [,4] [,5]
[1,] -1    1   -4    0    1
[2,]  0    1    3   -1    1
[3,]  1    0    2    4    5
[4,]  0    1    0   -4   -2
```

```
 multiply row 4 by 1 and add to row 3
     [,1] [,2] [,3] [,4] [,5]
[1,] 1    0    0    0    1
[2,] 0    1    0    0    2
[3,] 0    0    1    0    0
[4,] 0    0    0    1    1
```

Therefore the solution for this system is $x_1 = 1$, $x_2 = 2$, $x_3 = 0$, $x_4 = 1$.

Visualize Gaussian Elimination

Example 45 *Now we have a system of linear equations such that*

$$\begin{array}{rcrcrcrcr} & & 5x_2 & + & 4x_3 & + & 7x_4 & = & -9 \\ -9x_1 & + & 6x_2 & + & 6x_3 & + & 7x_4 & = & -17 \\ 4x_1 & + & 3x_2 & & & + & 5x_4 & = & -3 \\ 4x_1 & + & 8x_2 & + & 4x_3 & + & 12x_4 & = & 0. \end{array}$$

Its augmented matrix is

$$\begin{bmatrix} 0 & 5 & 4 & 7 & -9 \\ -9 & 6 & 6 & 7 & -17 \\ 4 & 3 & 0 & 5 & -3 \\ 4 & 8 & 4 & 12 & 0 \end{bmatrix}.$$

We use the `matlib` *package for this example. We create the coefficient matrix and the vector for the right hand side of the equations for the system. Then we have*

```
A <- matrix(c(0, -9, 4, 4, 5, 6, 3, 8, 4, 6, 0, 4, 7, 7, 5, 12), 4, 4)
b <- c(-9, -17, -3, 0)
```

Using the showEqn() function we can see the system of linear equations.

```
showEqn(A, b)
```

Then `R` *shows*

```
> showEqn(A, b)
 0*x1 + 5*x2 + 4*x3  + 7*x4  =   -9
-9*x1 + 6*x2 + 6*x3  + 7*x4  =  -17
 4*x1 + 3*x2 + 0*x3  + 5*x4  =   -3
 4*x1 + 8*x2 + 4*x3 + 12*x4  =    0
```

Now using the echelon() function, we can obtain the reduced echelon form of the augmented matrix. From this we have an echelon form of the augmented matrix as follows:

```
     [,1]    [,2]    [,3]    [,4]    [,5]
[1,]       1       0       0    1/7    -1/7
[2,]       0       1       0  31/21   67/21
[3,]       0       0       1  -2/21 -131/21
[4,]       0       0       0      0     -12
```

Therefore, the reduced echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 1/7 & -1/7 \\ 0 & 1 & 0 & 31/21 & 67/21 \\ 0 & 0 & 1 & -2/21 & -131/21 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The last row has all zero elements except the last entry. This row represents the following equation

$$0 = 1,$$

which does not make sense. So if the last non-zero row has the form of

$$[0, 0, \ldots, 1]$$

in the reduced echelon form of the augmented matrix of the system, then there is no solution to the system of linear equations.

Visualize Gaussian Elimination

Example 46 *Now we have a system of linear equations such that*

$$\begin{array}{rcrcrcrcr} & & 5x_2 & + & 4x_3 & + & 7x_4 & = & -9 \\ -9x_1 & + & 6x_2 & + & 6x_3 & + & 7x_4 & = & -17 \\ 4x_1 & + & 3x_2 & & & + & 5x_4 & = & -3 \\ 4x_1 & + & 8x_2 & + & 4x_3 & + & 12x_4 & = & -12. \end{array}$$

Its augmented matrix is

$$\begin{bmatrix} 0 & 5 & 4 & 7 & -9 \\ -9 & 6 & 6 & 7 & -17 \\ 4 & 3 & 0 & 5 & -3 \\ 4 & 8 & 4 & 12 & -12 \end{bmatrix}.$$

We use the `matlib` *package for this example. We create the coefficient matrix and the vector for the right hand side of the equations for the system. Then we have*

```
A <- matrix(c(0, -9, 4, 4, 5, 6, 3, 8, 4, 6, 0, 4, 7, 7, 5, 12), 4, 4)
b <- c(-9, -17, -3, -12)
```

Using the showEqn() function we can see the system of linear equations.

```
showEqn(A, b)
```

Then `R` *shows*

```
> showEqn(A, b)
 0*x1 + 5*x2 + 4*x3  + 7*x4  =   -9
-9*x1 + 6*x2 + 6*x3  + 7*x4  =  -17
 4*x1 + 3*x2 + 0*x3  + 5*x4  =   -3
 4*x1 + 8*x2 + 4*x3 + 12*x4  =  -12
```

Now using the echelon() function we obtain the reduced echelon form of the augmented matrix. From this we have the reduced echelon form of the augmented matrix as follows:

```
 multiply row 3 by 7/8 and add to row 4
     [,1]   [,2]   [,3]   [,4]   [,5]
[1,]     1      0      0    1/7    5/7
[2,]     0      1      0  31/21 -41/21
[3,]     0      0      1  -2/21   4/21
[4,]     0      0      0      0      0
```

The reduced echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 1/7 & 5/7 \\ 0 & 1 & 0 & 31/21 & -41/21 \\ 0 & 0 & 1 & -2/21 & 4/21 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can write the corresponding system of linear equations

$$\begin{array}{ccccccc} x_1 & & & + & \frac{1}{7}x_4 & = & \frac{5}{7} \\ & x_2 & & + & \frac{31}{21}x_4 & = & -\frac{41}{21} \\ & & x_3 & - & \frac{2}{21}x_4 & = & \frac{4}{21} \\ & & & & 0 & = & 0. \end{array}$$

Since x_4 can be any real number then there are infinitely many solutions.

Matrix Operation: +

Example 47 *Suppose we have two vectors*

$$\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}.$$

Then the sum of these vectors is

$$\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}.$$

Example 48 *Suppose we have two 2×3 matrices*

$$\begin{bmatrix} 3 & 0 & -5 \\ -1 & -3 & 4 \end{bmatrix}, \begin{bmatrix} -5 & 5 & 2 \\ 1 & -2 & 0 \end{bmatrix}.$$

Then the sum of these matrices is

$$\begin{bmatrix} 3 & 0 & -5 \\ -1 & -3 & 4 \end{bmatrix} + \begin{bmatrix} -5 & 5 & 2 \\ 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 5 & -3 \\ 0 & -5 & 4 \end{bmatrix}.$$

Example 49 *From Example 47, suppose we have two vectors*

$$\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}.$$

In R, *we do the following: First we define two vectors:*

```
v1 <- c(2, -1, 3)
v2 <- c(-1, 0, 4)
```

Then the sum of these vectors is

```
v1 + v2
```

R *outputs the following*

```
> v1 + v2
[1]  1 -1  7
```

Example 50 *This is from Example 48. Suppose we have two 2×3 matrices*

$$\begin{bmatrix} 3 & 0 & -5 \\ -1 & -3 & 4 \end{bmatrix}, \begin{bmatrix} -5 & 5 & 2 \\ 1 & -2 & 0 \end{bmatrix}.$$

In R, *we do the following: First we define two matrices:*

```
A <- matrix(c(3, 0, -5, -1, -3, 4), nrow = 2, ncol = 3, byrow = TRUE)
B <- matrix(c(-5, 5, 2, 1, -2, 0), nrow = 2, ncol = 3, byrow = TRUE)
```

Then the sum of these matrices can be computed by

```
A + B
```

The output from R *is*

```
> A + B
     [,1] [,2] [,3]
[1,]   -2    5   -3
[2,]    0   -5    4
```

Matrix Operation: scalar multiplication

Definition 18 *Suppose we have a real number c and an $m \times n$ matrix*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

The **scalar multiplication** *of c to A is*

$$c \cdot A = \begin{bmatrix} c \cdot a_{11} & c \cdot a_{12} & \dots & c \cdot a_{1n} \\ c \cdot a_{21} & c \cdot a_{22} & \dots & c \cdot a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c \cdot a_{m1} & c \cdot a_{m2} & \dots & c \cdot a_{mn} \end{bmatrix}.$$

Example 51 *Suppose we have a vector*

$$v = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

and $c = -1$. Then we have the scalar multiplication of $c \cdot v$ is

$$c \cdot v = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}.$$

Example 52 *Suppose we have a 2×3 matrix*

$$A = \begin{bmatrix} 3 & 0 & -5 \\ -1 & -3 & 4 \end{bmatrix}$$

and $c = -3$. Then the scalar multiplication of $c \cdot A$ is

$$c \cdot A = \begin{bmatrix} -9 & 0 & 15 \\ 3 & 9 & -12 \end{bmatrix}.$$

Example 53 *In* R*, we can do the scalar multiplication from Example 52 as follows: First we define a matrix A:*

```
A <- matrix(c(3, 0, -5, -1, -3, 4), nrow = 2, ncol = 3, byrow = TRUE)
```

Then we can do the scalar multiplication in R *as*

```
-3 * A
```

Then R *outputs*

```
> -3 * A
     [,1] [,2] [,3]
[1,]   -9    0   15
[2,]    3    9  -12
```

Matrix Operation: multiplication of two matrices

Matrix multiplication is more complicated than matrix addition. Thus, we will show some examples first. First, we show a **dot product**, which is a special case of matrix multiplication.

Suppose we have two vectors, a 1×3 vector and a 3×1 vector, such that

$$v_1 = \begin{bmatrix} 1 & 2 & -3 \end{bmatrix}, \; v_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Note that the number of elements in both vectors are the same: more specifically, we have a 1×3 vector and a 3×1 vector, so that the number 3 is the same. In order to apply a dot product, they have to have the same number of entries in both vectors. In this example, the dot product $v_1 \cdot v_2$ is

$$v_1 \cdot v_2 = \begin{bmatrix} 1 & 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 1 \cdot 0 + 2 \cdot 1 + (-3) \cdot 2 = -4.$$

In general, the **dot product** of two vectors

$$v_1 = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}, \; v_2 = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

is

$$v_1 \cdot v_2 = x_1 \cdot y_1 + x_2 \cdot y_2 + \ldots + x_n \cdot y_n.$$

Matrix Operation: multiplication of two matrices

The multiplication of two matrices can be seen as a generalization of this dot product of two vectors. We will demonstrate matrix multiplication with an example. Suppose we have a 2×3 matrix and a 3×2 matrix such that

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -3 & 2 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}.$$

Let $a_1 = [1, -2, 3]$ and $a_2 = [-3, 2, -1]$, i.e., a_1 is the first row vector of matrix A and a_2 is the second row vector of matrix A. Also let

$$b_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, b_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

b_1 is the first column vector of matrix B and b_2 is the second column vector of matrix B.

Here, note the number of column vectors of A is the same as the number of row vectors of B. Also note that the dimension of the row vectors a_1, a_2 of A is the same as the dimension of column vectors b_1, b_2 of B. This is important for matrix multiplication. The matrix multiplication of A and B is a 2×2 matrix. Let this matrix be

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}.$$

For the entry in the first row and first column of C, we take the first row a_1 of A and the first column b_1 of B. Then we take the dot product of a_1 and b_1:

$$c_{11} = a_1 \cdot b_1 = 1 \cdot 0 + (-2) \cdot 1 + 3 \cdot 2 = 4.$$

So now we have

$$C = \begin{bmatrix} 4 & c_{12} \\ c_{21} & c_{22} \end{bmatrix}.$$

For the entry of the second row and first column of C, we take the dot product of the second row vector a_2 of A and the first column vector b_1 of B:

$$c_{21} = a_2 \cdot b_1 = (-3) \cdot 0 + 2 \cdot 1 + (-1) \cdot 2 = 0.$$

So now we have

$$C = \begin{bmatrix} 4 & c_{12} \\ 0 & c_{22} \end{bmatrix}.$$

For the entry of the first row and second column of C, we take the dot product of the first row vector a_1 of A and the second column vector b_2 of B:

$$c_{12} = a_1 \cdot b_2 = 1 \cdot 2 + (-2) \cdot 1 + 3 \cdot 0 = 0.$$

So now we have

$$C = \begin{bmatrix} 4 & 0 \\ 0 & c_{22} \end{bmatrix}.$$

Lastly, for the entry of the second row and second column of C, we take the dot product of the second row vector a_2 of A and the second column vector b_2 of B:

$$c_{22} = a_2 \cdot b_2 = (-3) \cdot 2 + 2 \cdot 1 + (-1) \cdot 0 = -4.$$

So now we have

$$C = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}.$$

In general, suppose we have an $m \times n$ matrix A and an $n \times k$ matrix B where the row vectors of A are a_1, a_2, a_m and the column vectors of B are b_1, b_2, $\ldots$, b_k. Then the matrix multiplication C of A and B is an $n \times k$ matrix such that

$$C = \begin{bmatrix} a_1 \cdot b_1 & a_1 \cdot b_2 & \ldots & a_1 \cdot b_k \\ a_2 \cdot b_1 & a_2 \cdot b_2 & \ldots & a_2 \cdot b_k \\ \vdots & \vdots & \vdots & \vdots \\ a_m \cdot b_1 & a_m \cdot b_2 & \ldots & a_m \cdot b_k \end{bmatrix}.$$

Matrix Operation: multiplication of two matrices

Definition 19 *Suppose we have $m \times n$ matrix A and $n \times k$ B such that*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \; B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1k} \\ b_{21} & b_{22} & \dots & b_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nk} \end{bmatrix}$$

The **multiplication** *of matrices A and B is an $m \times k$ matrix such that*

$$A \cdot B = \begin{bmatrix} \sum_{i=1}^{n} a_{1i} \cdot b_{i1} & \sum_{i=1}^{n} a_{1i} \cdot b_{i2} & \dots & \sum_{i=1}^{n} a_{1i} \cdot b_{ik} \\ \sum_{i=1}^{n} a_{2i} \cdot b_{i1} & \sum_{i=1}^{n} a_{2i} \cdot b_{i2} & \dots & \sum_{i=1}^{n} a_{2i} \cdot b_{ik} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{i=1}^{n} a_{mi} \cdot b_{i1} & \sum_{i=1}^{n} a_{mi} \cdot b_{i2} & \dots & \sum_{i=1}^{n} a_{mi} \cdot b_{ik} \end{bmatrix}.$$

Remark 2.1 *We call the multiplication of a $1 \times n$ vector and an $n \times 1$ vector as a* **dot product** *of two vectors.*

Matrix Operation: multiplication of two matrices

Example 54 *From Example 47, suppose we have two vectors*

$$\begin{bmatrix} 2 & -1 & 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}.$$

Then the dot product of these vectors is

$$\begin{bmatrix} 2 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} = 2 \cdot (-1) + (-1) \cdot 0 + 3 \cdot 4 = 10.$$

Example 55 *Suppose we have a 2×3 matrix and a 3×2 matrix*

$$\begin{bmatrix} 3 & 0 & -5 \\ -1 & -3 & 4 \end{bmatrix}, \begin{bmatrix} -5 & 5 \\ 2 & 1 \\ -2 & 0 \end{bmatrix}.$$

Then the multiplication of these matrices is

$$\begin{aligned} & \begin{bmatrix} 3 & 0 & -5 \\ -1 & -3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -5 & 5 \\ 2 & 1 \\ -2 & 0 \end{bmatrix} \\ = & \begin{bmatrix} 3 \cdot (-5) + 0 \cdot 2 + (-5) \cdot (-2) & 3 \cdot 5 + 0 \cdot 1 + (-5) \cdot 0 \\ (-1) \cdot (-5) + (-3) \cdot 2 + 4 \cdot (-2) & (-1) \cdot 5 + (-3) \cdot 1 + 4 \cdot 0 \end{bmatrix} \\ = & \begin{bmatrix} -5 & 15 \\ -9 & -8 \end{bmatrix} \end{aligned}$$

Example 56 *From Example 54, suppose we have two vectors*

$$\begin{bmatrix} 2 & -1 & 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}.$$

First we define these vectors

```
v1 <- c(2, -1, 3)
v2 <- c(-1, 0, 4)
```

Then the dot product of these vectors in `R` *is*

```
v1 %*% v2
```

The output from `R` *is*

```
> v1 %*% v2
     [,1]
[1,]   10
```

Matrix Operation: multiplication of two matrices

Example 57 *This is from Example 55. Suppose we have a 2×3 matrix and a 3×2 matrix*

$$\begin{bmatrix} 3 & 0 & -5 \\ -1 & -3 & 4 \end{bmatrix}, \begin{bmatrix} -5 & 5 \\ 2 & 1 \\ -2 & 0 \end{bmatrix}.$$

In R, *we do the following: First we define two matrices:*

```
A <- matrix(c(3, 0, -5, -1, -3, 4), nrow = 2, ncol = 3, byrow = TRUE)
B <- matrix(c(-5, 5, 2, 1, -2, 0), nrow = 3, ncol = 2, byrow = TRUE)
```

Then the dot product of these matrices can be computed by

```
A %*% B
```

The output from R *is*

```
> A %*% B
     [,1] [,2]
[1,]   -5   15
[2,]   -9   -8
```

Matrix Operation: transpose

One of the important operations in matrices is called the **transpose** of a matrix. With this operation, we can be more flexible with other matrix operations.

Definition 20 *Suppose we have an $m \times n$ matrix*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

The **transpose** *of the matrix A is an $n \times m$ matrix defined as*

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}.$$

Example 58 *Suppose we have a 2×3 matrix*

$$A = \begin{bmatrix} 4 & -1 & -5 \\ 0 & 1 & -2 \end{bmatrix}$$

The transpose of the matrix A is a 3×2 matrix

$$A^T = \begin{bmatrix} 4 & 0 \\ -1 & 1 \\ -5 & -2 \end{bmatrix}.$$

Example 59 *Suppose we have a 3×3 matrix*

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 5 & -4 & 4 \\ -5 & -3 & -1 \end{bmatrix}$$

The transpose of the matrix A is a 3×3 matrix

$$A^T = \begin{bmatrix} 2 & 5 & -5 \\ 1 & -4 & -3 \\ 3 & 4 & -1 \end{bmatrix}.$$

Example 60 *We can use* `R` *to compute the transpose of a matrix using the t() function. With the matrix A from Example 58, we first create a matrix using the matrix() function:*

```
A <- matrix(c(4, -1, -5, 0, 1, -2), 2, 3, byrow = TRUE)
```

Then we type in `R`

```
t(A)
```

Then `R` *outputs as follows:*

```
> t(A)
     [,1] [,2]
[1,]    4    0
[2,]   -1    1
[3,]   -5   -2
```

Properties of Matrix Operations

Theorem 2.2 *Suppose A, B, C are $m \times n$ matrices. Then we have the following rules:*

1. $A + B = B + A$ *(Commutative law for addition).*
2. $A + (B + C) = (A + B) + C$ *(Associative law for addition).*

Example 61 *Suppose we have*

$$A = \begin{bmatrix} -4 & -3 & 3 \\ -1 & -3 & 2 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 & 1 \\ 2 & -4 & 5 \end{bmatrix}, C = \begin{bmatrix} -5 & 2 & -4 \\ -5 & 5 & -1 \end{bmatrix}.$$

Then we have

$$A + B = \begin{bmatrix} 0 & -4 & 4 \\ 1 & -7 & 7 \end{bmatrix}$$

and

$$B + A = \begin{bmatrix} 0 & -4 & 4 \\ 1 & -7 & 7 \end{bmatrix}.$$

Also

$$(A + B) + C = \begin{bmatrix} -5 & -2 & 0 \\ -4 & -2 & 6 \end{bmatrix}$$

and

$$A + (B + C) = \begin{bmatrix} -5 & -2 & 0 \\ -4 & -2 & 6 \end{bmatrix}.$$

Theorem 2.3 *Suppose A, B, C are $m \times n$ matrices and suppose a, b, c are constants. Then we have the following rules:*

1. $c \cdot (A + B) = c \cdot A + c \cdot B$
2. $c \cdot (A - B) = c \cdot A - c \cdot B$
3. $(a + b) \cdot C = a \cdot C + b \cdot C$
4. $(a - b) \cdot C = a \cdot C - b \cdot C$

Example 62 *Suppose we have*

$$A = \begin{bmatrix} -4 & -3 & 3 \\ -1 & -3 & 2 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 & 1 \\ 2 & -4 & 5 \end{bmatrix}, C = \begin{bmatrix} -5 & 2 & -4 \\ -5 & 5 & -1 \end{bmatrix}$$

and $a = -1$, $b = 3$, $c = 5$. Then we have

$$c \cdot (A + B) = \begin{bmatrix} 0 & -20 & 20 \\ 5 & -35 & 35 \end{bmatrix}$$

and

$$c \cdot A + c \cdot B = \begin{bmatrix} 0 & -20 & 20 \\ 5 & -35 & 35 \end{bmatrix}.$$

$$(a + b) \cdot C = \begin{bmatrix} -10 & 4 & -8 \\ -10 & 10 & -2 \end{bmatrix}$$

and

$$a \cdot C + b \cdot C = \begin{bmatrix} -10 & 4 & -8 \\ -10 & 10 & -2 \end{bmatrix}.$$

Properties of Matrix Operations

Theorem 2.4 *Suppose A is an $m \times n$ matrix, B is an $n \times k$ matrix, and C is a $k \times s$ matrix. Then*

$$(A \cdot B) \cdot C = A \cdot (B \cdot C).$$

Example 63 *Suppose*

$$A = \begin{bmatrix} -4 & -3 & 3 \\ -1 & -3 & 2 \end{bmatrix}, \ B = \begin{bmatrix} -1 & -2 & 2 \\ -5 & -3 & -3 \\ 5 & 0 & 3 \end{bmatrix}, \ C = \begin{bmatrix} -4 & 5 \\ 4 & -5 \\ -3 & -4 \end{bmatrix}.$$

Then,

$$A \cdot B = \begin{bmatrix} 34 & 17 & 10 \\ 26 & 11 & 13 \end{bmatrix}$$

and

$$(A \cdot B) \cdot C = \begin{bmatrix} -98 & 45 \\ -99 & 23 \end{bmatrix}.$$

Also

$$B \cdot C = \begin{bmatrix} -10 & -3 \\ 17 & 2 \\ -29 & 13 \end{bmatrix}.$$

and

$$A \cdot (B \cdot C) = \begin{bmatrix} -98 & 45 \\ -99 & 23 \end{bmatrix}.$$

Properties of Matrix Operations

Theorem 2.5 *Suppose A is an $m \times n$ matrix, and B and C are $n \times k$ matrices. Then*

1. $A \cdot (B + C) = A \cdot B + A \cdot C$
2. $A \cdot (B - C) = A \cdot B - A \cdot C$.

Example 64 *Suppose*

$$A = \begin{bmatrix} -4 & -3 & 3 \\ -1 & -3 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & -2 \\ 0 & 2 \\ -5 & 1 \end{bmatrix}, C = \begin{bmatrix} -4 & 5 \\ 4 & -5 \\ -3 & -4 \end{bmatrix}.$$

$$B + C = \begin{bmatrix} -2 & 3 \\ 4 & -3 \\ -8 & -3 \end{bmatrix}$$

and

$$A \cdot (B + C) = \begin{bmatrix} -28 & -12 \\ -26 & 0 \end{bmatrix}.$$

Also

$$A \cdot B = \begin{bmatrix} -23 & 5 \\ -12 & -2 \end{bmatrix}, A \cdot C = \begin{bmatrix} -5 & -17 \\ -14 & 2 \end{bmatrix}$$

and

$$A \cdot B + A \cdot C = \begin{bmatrix} -28 & -12 \\ -26 & 0 \end{bmatrix}.$$

Theorem 2.6 *Suppose A is an $m \times n$ matrix, and B and C are $k \times m$ matrices. Then*

1. $(B + C) \cdot A = B \cdot A + C \cdot A$
2. $(B - C) \cdot A = B \cdot A - C \cdot A$.

Example 65 *Suppose*

$$A = \begin{bmatrix} -4 & -3 & 3 \\ -1 & -3 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & -2 \\ 0 & 2 \\ -5 & 1 \end{bmatrix}, C = \begin{bmatrix} -4 & 5 \\ 4 & -5 \\ -3 & -4 \end{bmatrix}.$$

Then,

$$B + C = \begin{bmatrix} -2 & 3 \\ 4 & -3 \\ -8 & -3 \end{bmatrix}$$

and

$$(B + C) \cdot A = \begin{bmatrix} 5 & -3 & 0 \\ -13 & -3 & 6 \\ 35 & 33 & -30 \end{bmatrix}.$$

Also

$$B \cdot A = \begin{bmatrix} -6 & 0 & 2 \\ -2 & -6 & 4 \\ 19 & 12 & -13 \end{bmatrix}, C \cdot A = \begin{bmatrix} 11 & -3 & -2 \\ -11 & 3 & 2 \\ 16 & 21 & -17 \end{bmatrix}$$

and

$$B \cdot A + C \cdot A = \begin{bmatrix} 5 & -3 & 0 \\ -13 & -3 & 6 \\ 35 & 33 & -30 \end{bmatrix}.$$

Inverse of Matrix

Definition 21 *Suppose A is a square matrix, i.e., an $m \times m$ matrix. Then the* **inverse** *of a matrix A is an $m \times m$ matrix A^{-1} such that*

$$A \cdot A^{-1} = A^{-1} \cdot A = I_m,$$

where I_m is the identity matrix of size m. If a matrix A has the inverse, then we say a matrix A is **invertible**.

Note that not all square matrices have their inverse.

Example 67 *Suppose*

$$A = \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}, A^{-1} = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}.$$

Then we have

$$A \cdot A^{-1} = A^{-1} \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Theorem 2.8 *Suppose A is an invertible matrix. Then*

$$(A^{-1})^{-1} = A.$$

Theorem 2.9 *Suppose we have a 2×2 matrix A such that*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where a, b, c, d are real numbers. Then A is invertible if and only if $a \cdot d - b \cdot c \neq 0$. If A is invertible, then the inverse of the matrix A is

$$A^{-1} = \frac{1}{a \cdot d - b \cdot c} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example 68 *Suppose*

$$A = \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}.$$

Then, we have

$$(-1) \cdot (-3) - 4 \cdot 1 = -1.$$

Therefore,

$$A^{-1} = \frac{1}{-1} \cdot \begin{bmatrix} -3 & -4 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}.$$

Inverse of Matrix

Example 69 *In* `R`*, if we want to compute the inverse of a matrix we can use the inv() function from the* `matlib` *package. For example, if we have a matrix*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 3 & -2 \\ -1 & 2 & 1 \end{bmatrix},$$

then we can type:

```
library(matlib)
A <- matrix(c(1,-2,-1,2,3,2,3,-2,1), nrow = 3, ncol = 3)
inv(A)
```

Then, `R` *returns*

```
> inv(A)
            [,1]       [,2]       [,3]
[1,]  0.58333333  0.3333333 -1.0833333
[2,]  0.33333333  0.3333333 -0.3333333
[3,] -0.08333333 -0.3333333  0.5833333
```

If we want to see the output in terms of rational numbers, then we can use the fractions() function from the `MASS` *package [42].*

```
library(MASS)
fractions(inv(A))
```

Then `R` *outputs*

```
> fractions(inv(A))
     [,1]   [,2]   [,3]
[1,]   7/12    1/3 -13/12
[2,]    1/3    1/3   -1/3
[3,]  -1/12   -1/3   7/12
```

Remark 2.10 *In* `R` *we can use the solve() function to find the inverse of a matrix instead of the inv() function. For example, with the matrix*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 3 & -2 \\ -1 & 2 & 1 \end{bmatrix},$$

we can do:

```
library(matlib)
A <- matrix(c(1,-2,-1,2,3,2,3,-2,1), nrow = 3, ncol = 3)
solve(A)
```

Then we obtained the same result as the inv() function:

```
> solve(A)
            [,1]       [,2]       [,3]
[1,]  0.58333333  0.3333333 -1.0833333
[2,]  0.33333333  0.3333333 -0.3333333
[3,] -0.08333333 -0.3333333  0.5833333
```

Inverse of Matrix

Remark 2.11 *There are some square matrices which do not have inverse matrices, i.e., they are not invertible. This relates to the existence of a unique solution to the system of linear equations.*

Theorem 2.12 *Suppose we have a system of n many linear equations in n variables such that*

$$A \cdot x = b$$

where A is an $n \times n$ matrix, x is an n-dimensional vector, and b is an n-dimensional vector. Then, if A is invertible, this system of linear equations has a unique solution.

Example 70 *This is from Example 39. The system of linear equations is:*

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 6 \\ -2x_1 & + & 3x_2 & - & 2x_3 & = & -1 \\ -x_1 & + & 2x_2 & + & x_3 & = & 2. \end{array}$$

Its coefficient matrix is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 3 & -2 \\ -1 & 2 & 1 \end{bmatrix}$$

and its inverse is

$$A^{-1} = \begin{bmatrix} 7/12 & 1/3 & -13/12 \\ 1/3 & 1/3 & -1/3 \\ -1/12 & -1/3 & 7/12 \end{bmatrix}.$$

Let

$$b = \begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix}.$$

Then the solution of this system of linear equations is

$$A^{-1} \cdot b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Now we discuss properties of matrix operations with the transpose of a matrix.

Transpose of tranpose of Matrix

Theorem 2.13 *Suppose we have a matrix A. Then*

$$(A^T)^T = A.$$

Example 71 *Suppose we have a matrix*

$$A = \begin{bmatrix} -4 & -3 & 3 \\ -1 & -3 & 2 \end{bmatrix}.$$

Then the transpose of A is

$$A^T = \begin{bmatrix} -4 & -1 \\ -3 & -3 \\ 3 & 2 \end{bmatrix}.$$

Then the transpose of A^T is

$$(A^T)^T = \begin{bmatrix} -4 & -3 & 3 \\ -1 & -3 & 2 \end{bmatrix}.$$

Theorem 2.14 *Suppose A and B are $m \times n$ matrices, Then*

1. $(A+B)^T = A^T + B^T$
2. $(A-B)^T = A^T - B^T$.

Example 72 *Suppose we have*

$$A = \begin{bmatrix} -4 & -3 & 3 \\ -1 & -3 & 2 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 & 1 \\ 2 & -4 & 5 \end{bmatrix}.$$

Then we have

$$A + B = \begin{bmatrix} 0 & -4 & 4 \\ 1 & -7 & 7 \end{bmatrix}$$

$$(A+B)^T = \begin{bmatrix} 0 & 1 \\ -4 & -7 \\ 4 & 7 \end{bmatrix}.$$

Also

$$A^T = \begin{bmatrix} -4 & -1 \\ -3 & -3 \\ 3 & 2 \end{bmatrix}$$

and

$$B^T = \begin{bmatrix} 4 & 2 \\ -1 & -4 \\ 1 & 5 \end{bmatrix}.$$

Then we have

$$A^T + B^T = \begin{bmatrix} 0 & 1 \\ -4 & -7 \\ 4 & 7 \end{bmatrix}.$$

Transpose of tranpose of Matrix

Theorem 2.15 *Suppose A is an $m \times n$ matrix and B is an $n \times k$ matrix, and c is a constant. Then*

1. $(c \cdot A)^T = c \cdot A^T$
2. $(A \cdot B)^T = B^T \cdot A^T$.

Example 73 *Suppose*

$$A = \begin{bmatrix} -4 & -3 & 3 \\ -1 & -3 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & -2 \\ 0 & 2 \\ -5 & 1 \end{bmatrix}.$$

Also let $c = -1$. Then

$$(-1) \cdot A = \begin{bmatrix} 4 & 3 & -3 \\ 1 & 3 & -2 \end{bmatrix}$$

and then

$$((-1) \cdot A)^T = \begin{bmatrix} 4 & 1 \\ 3 & 3 \\ -3 & -2 \end{bmatrix}.$$

Also we have

$$A^T = \begin{bmatrix} -4 & -1 \\ -3 & -3 \\ 3 & 2 \end{bmatrix}$$

and

$$(-1) \cdot A^T = \begin{bmatrix} 4 & 1 \\ 3 & 3 \\ -3 & -2 \end{bmatrix}.$$

Also we have

$$(A \cdot B) = \begin{bmatrix} -23 & 5 \\ -12 & -2 \end{bmatrix}$$

and

$$(A \cdot B)^T = \begin{bmatrix} -23 & -12 \\ 5 & -2 \end{bmatrix}.$$

$$A^T = \begin{bmatrix} -4 & -1 \\ -3 & -3 \\ 3 & 2 \end{bmatrix}$$

and

$$B^T = \begin{bmatrix} 2 & 0 & 5 \\ -2 & 2 & 1 \end{bmatrix}.$$

Thus

$$B^T \cdot A^T = \begin{bmatrix} -23 & -12 \\ 5 & -2 \end{bmatrix}.$$

Transpose of inverse of Matrix

Theorem 2.16 *Suppose A is a square invertible matrix. Then*

$$(A^T)^{-1} = (A^{-1})^T.$$

Example 74 *Suppose we have*

$$A = \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}.$$

Also we have

$$A^{-1} = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}$$

and

$$(A^{-1})^T = \begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}.$$

Also we have

$$A^T = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}$$

and

$$(A^T)^{-1} = \begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}.$$

Theorem 2.17 *If A is a square and symmetric matrix, then*

$$A^T = A.$$

Theorem 2.17 *If A is a square and symmetric matrix, then*

$$A^T = A.$$

Example 75 *Suppose we have*

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 0 \\ 4 & 0 & -2 \end{bmatrix}.$$

Then

$$A^T = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 0 \\ 4 & 0 & -2 \end{bmatrix}.$$

Thus $A^T = A$.

Practice: inverse of Matrix

1. https://cran.r-project.org/web/packages/matlib/vignettes/inv-ex1.html

2. https://cran.r-project.org/web/packages/matlib/vignettes/inv-ex2.html