Data Visualization

Lecture 5

Mathamatics Visualization

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Content

Plot a mathmatics graph

- *Calculus 2 and Gradient Descent*

Find the domain of a function of 2 varibles

EXAMPLE 4 Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

SOLUTION The domain of g is

$$D = \{(x, y) \mid 9 - x^2 - y^2 \geqslant 0\} = \{(x, y) \mid x^2 + y^2 \leqslant 9\}$$

which is the disk with center (0, 0) and radius 3. (See Figure 4.) The range of g is

$$\left\{z \mid z = \sqrt{9 - x^2 - y^2},\ (x, y) \in D\right\}$$

Since z is a positive square root, $z \geqslant 0$. Also, because $9 - x^2 - y^2 \leqslant 9$, we have

$$\sqrt{9 - x^2 - y^2} \leqslant 3$$

So the range is

$$\{z \mid 0 \leqslant z \leqslant 3\} = [0, 3]$$

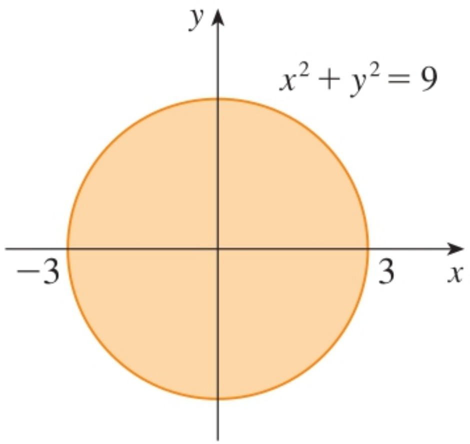


FIGURE 4
Domain of $g(x, y) = \sqrt{9 - x^2 - y^2}$

Graph a function of two variables

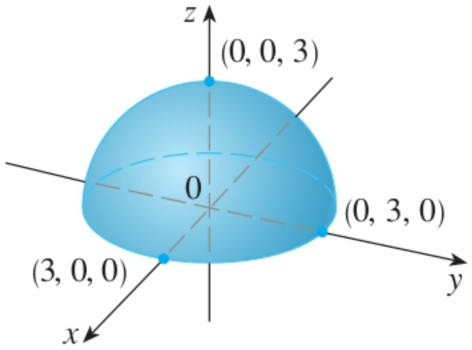


FIGURE 7
Graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$

V EXAMPLE 6 Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

SOLUTION The graph has equation $z = \sqrt{9 - x^2 - y^2}$. We square both sides of this equation to obtain $z^2 = 9 - x^2 - y^2$, or $x^2 + y^2 + z^2 = 9$, which we recognize as an equation of the sphere with center the origin and radius 3. But, since $z \geqslant 0$, the graph of g is just the top half of this sphere (see Figure 7).

NOTE An entire sphere can't be represented by a single function of x and y. As we saw in Example 6, the upper hemisphere of the sphere $x^2 + y^2 + z^2 = 9$ is represented by the function $g(x, y) = \sqrt{9 - x^2 - y^2}$. The lower hemisphere is represented by the function $h(x, y) = -\sqrt{9 - x^2 - y^2}$.

Plot a function of two variables

V EXAMPLE 8 Find the domain and range and sketch the graph of $h(x, y) = 4x^2 + y^2$.

SOLUTION Notice that $h(x, y)$ is defined for all possible ordered pairs of real numbers (x, y), so the domain is $\mathbb{R}^2$, the entire xy-plane. The range of h is the set $[0, \infty)$ of all nonnegative real numbers. [Notice that $x^2 \geqslant 0$ and $y^2 \geqslant 0$, so $h(x, y) \geqslant 0$ for all x and y.]

The graph of h has the equation $z = 4x^2 + y^2$, which is the elliptic paraboloid that we sketched in Example 4 in Section 12.6. Horizontal traces are ellipses and vertical traces are parabolas (see Figure 9).

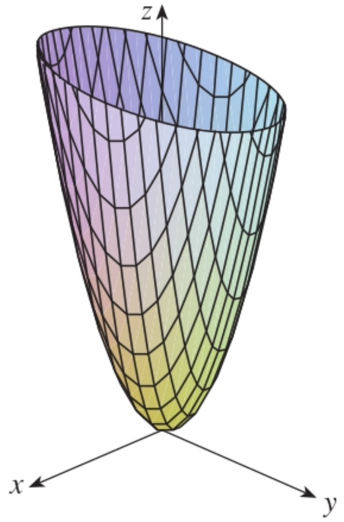


FIGURE 9
Graph of $h(x, y) = 4x^2 + y^2$

Plot a function of two variables: level curve or countour map

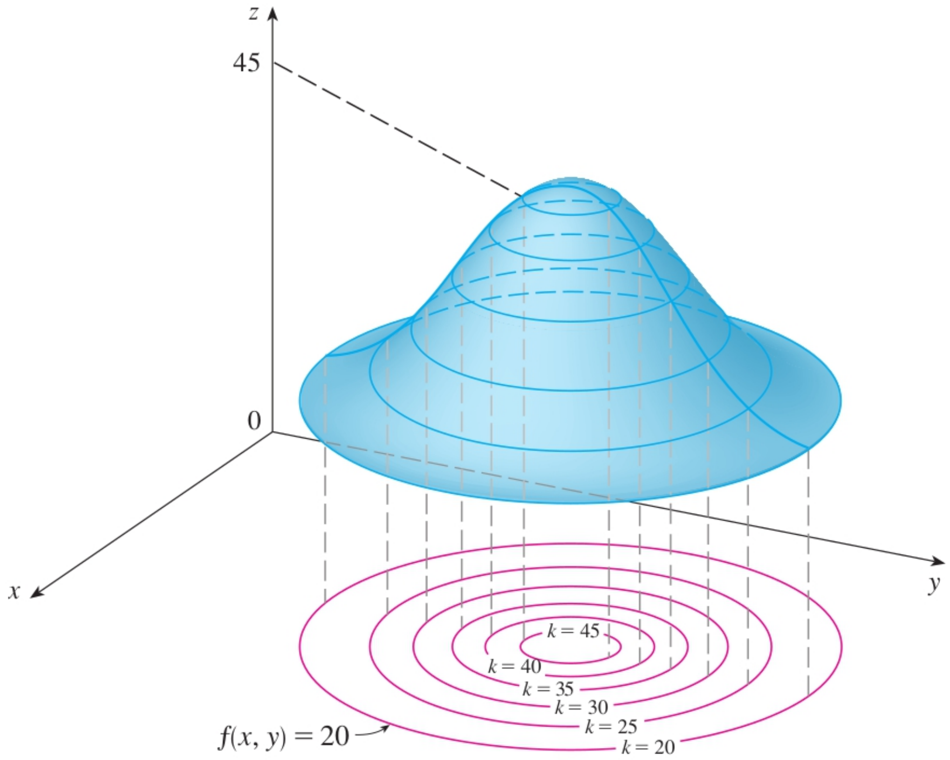
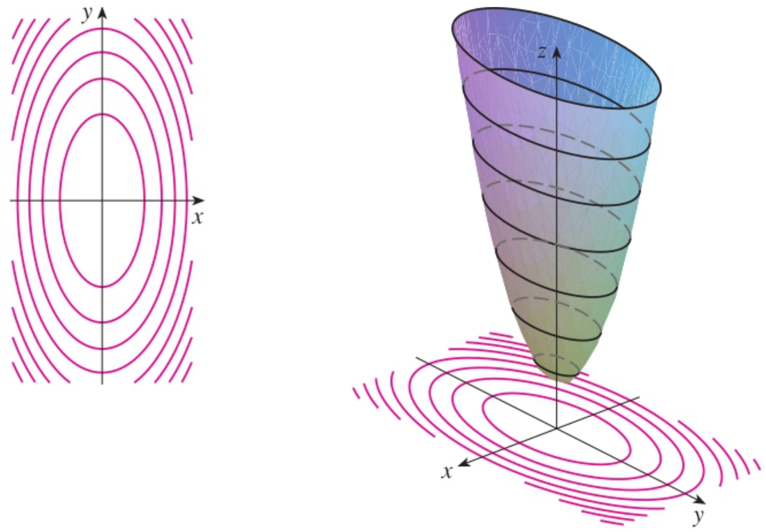


FIGURE 11

TEC Visual 14.1B demonstrates the connection between surfaces and their contour maps.



(a) Contour map

(b) Horizontal traces are raised level curves

FIGURE 17

The graph of $h(x, y) = 4x^2 + y^2 + 1$ is formed by lifting the level curves.

Understanding level curve

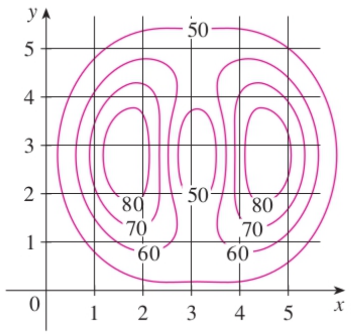


FIGURE 14

EXAMPLE 9 A contour map for a function f is shown in Figure 14. Use it to estimate the values of $f(1, 3)$ and $f(4, 5)$.

SOLUTION The point $(1, 3)$ lies partway between the level curves with z-values 70 and 80. We estimate that

$$f(1, 3) \approx 73$$

Similarly, we estimate that $\qquad f(4, 5) \approx 56$

EXAMPLE 10 Sketch the level curves of the function $f(x, y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6, 12$.

SOLUTION The level curves are

$$6 - 3x - 2y = k \qquad \text{or} \qquad 3x + 2y + (k - 6) = 0$$

This is a family of lines with slope $-\frac{3}{2}$. The four particular level curves with $k = -6, 0, 6$, and 12 are $3x + 2y - 12 = 0$, $3x + 2y - 6 = 0$, $3x + 2y = 0$, and $3x + 2y + 6 = 0$. They are sketched in Figure 15. The level curves are equally spaced parallel lines because the graph of f is a plane (see Figure 6).

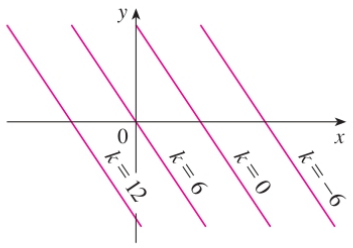


FIGURE 15
Contour map of
$f(x, y) = 6 - 3x - 2y$

Understanding level curve

V EXAMPLE 11 Sketch the level curves of the function

$$g(x, y) = \sqrt{9 - x^2 - y^2} \qquad \text{for} \quad k = 0, 1, 2, 3$$

SOLUTION The level curves are

$$\sqrt{9 - x^2 - y^2} = k \qquad \text{or} \qquad x^2 + y^2 = 9 - k^2$$

This is a family of concentric circles with center $(0, 0)$ and radius $\sqrt{9 - k^2}$. The cases $k = 0, 1, 2, 3$ are shown in Figure 16. Try to visualize these level curves lifted up to form a surface and compare with the graph of g (a hemisphere) in Figure 7. (See TEC Visual 14.1A.)

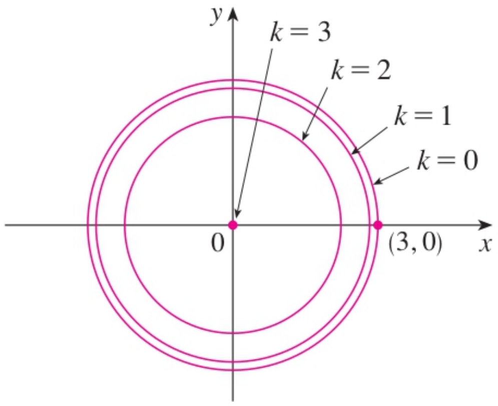
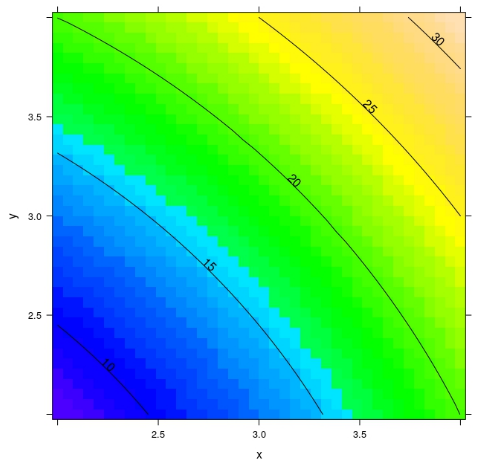


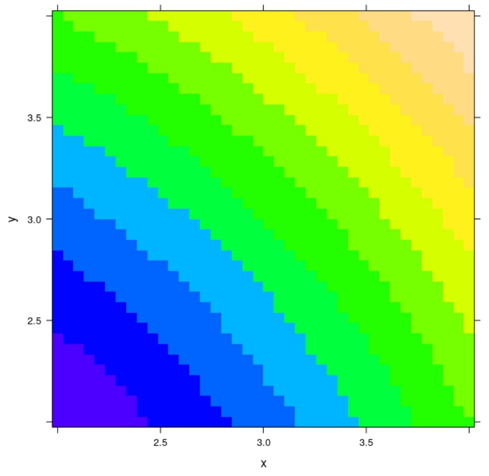
FIGURE 16
Contour map of $g(x, y) = \sqrt{9 - x^2 - y^2}$

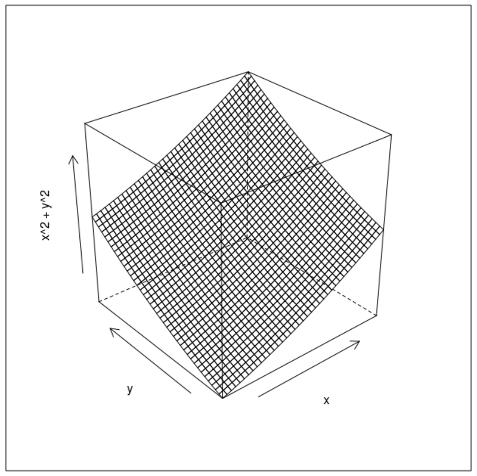
Plot function with two variables in R

$$f(x, y) = x^2 + y^2$$

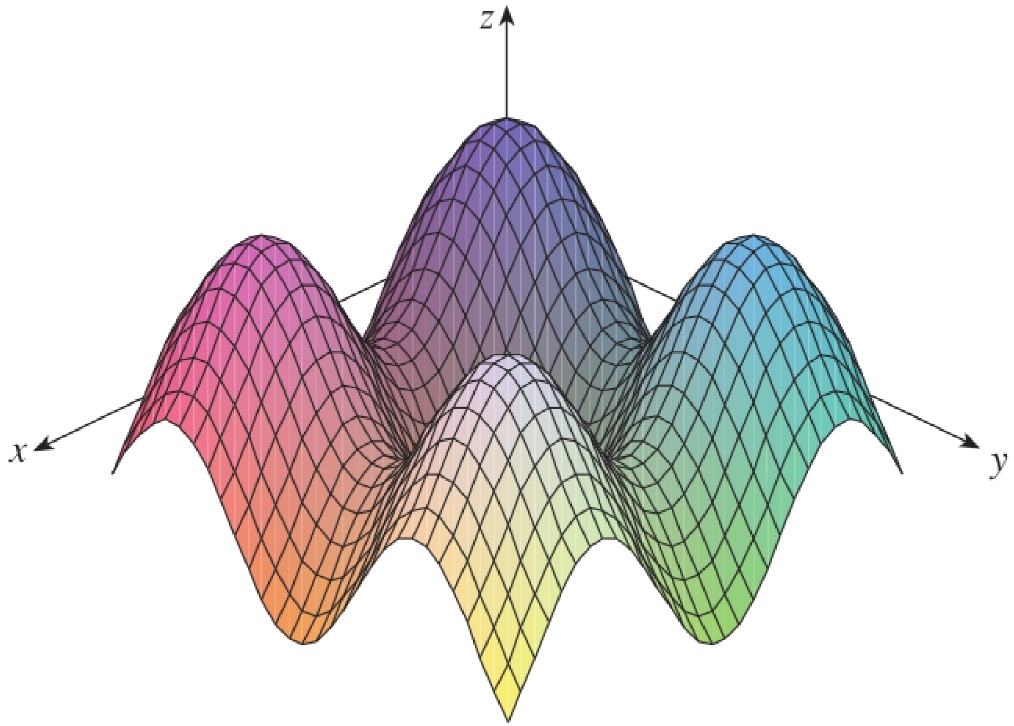
```
#https://dtkaplan.github.io/RforCalculus/repres
enting-mathematical-functions.html

library(mosaic)
library(mosaicCalc)

plotFun(x^2+y^2~x+y, x.lim=c(2,4),
y.lim=c(2,4), levels=2)

plotFun(x^2+y^2~x+y, x.lim=c(2,4),
y.lim=c(2,4), surface=T)
```





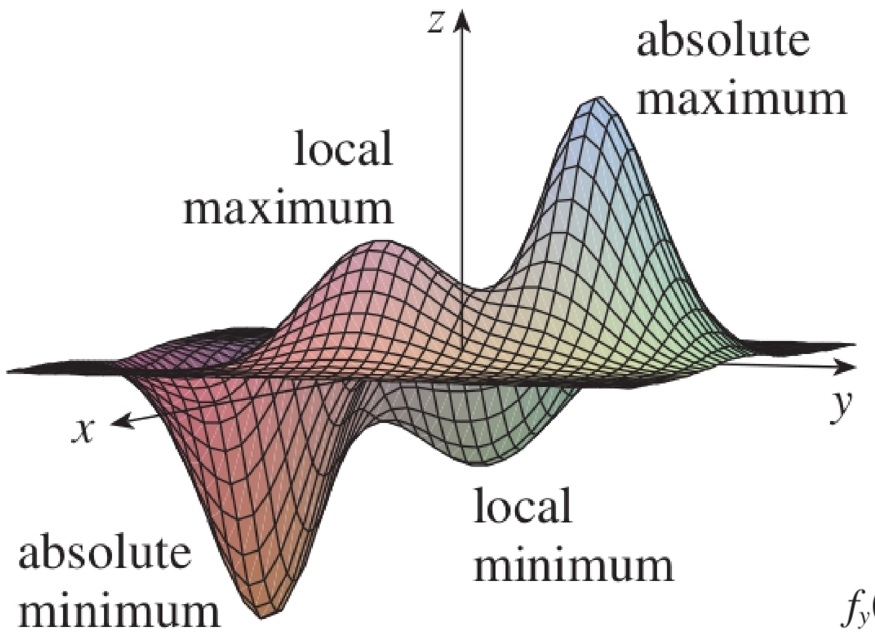


Plot function with two variables in R: exercise



(c) $f(x, y) = \sin x + \sin y$

Find Max and Min in a function of two variables



2 Theorem If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Notice that the conclusion of Theorem 2 can be stated in the notation of gradient vectors as $\nabla f(a, b) = \mathbf{0}$.

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist. Theorem 2 says that if f has a local maximum or minimum at (a, b), then (a, b) is a critical point of f. However, as in single-variable calculus, not all critical points give rise to maxima or minima. At a critical point, a function could have a local maximum or a local minimum or neither.

Find Max and Min in a function of two variables: Example 1

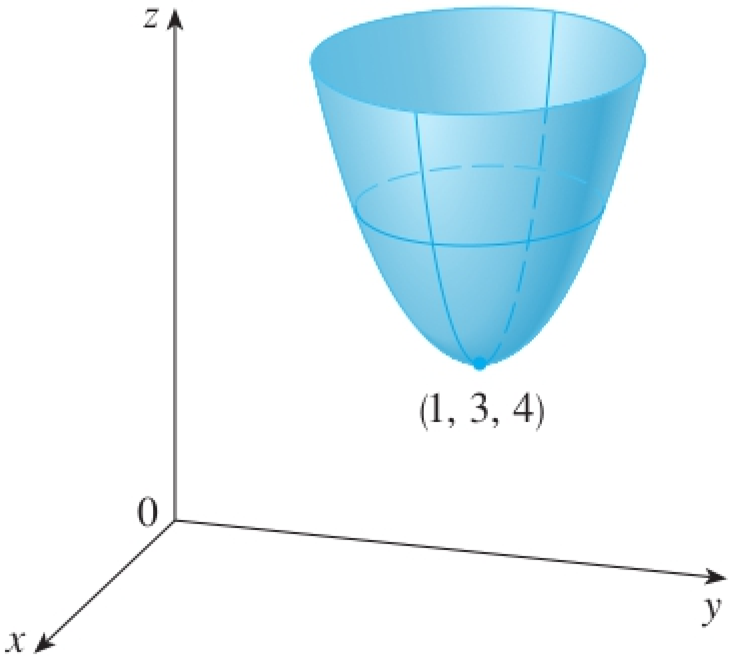


FIGURE 2
$z = x^2 + y^2 - 2x - 6y + 14$

EXAMPLE 1 Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Then

$$f_x(x, y) = 2x - 2 \qquad f_y(x, y) = 2y - 6$$

These partial derivatives are equal to 0 when $x = 1$ and $y = 3$, so the only critical point is $(1, 3)$. By completing the square, we find that

$$f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$$

Since $(x - 1)^2 \geqslant 0$ and $(y - 3)^2 \geqslant 0$, we have $f(x, y) \geqslant 4$ for all values of x and y. Therefore $f(1, 3) = 4$ is a local minimum, and in fact it is the absolute minimum of f.

This can be confirmed geometrically from the graph of f, which is the elliptic paraboloid with vertex $(1, 3, 4)$ shown in Figure 2.

Find Max and Min in a function of two variables: Example 2

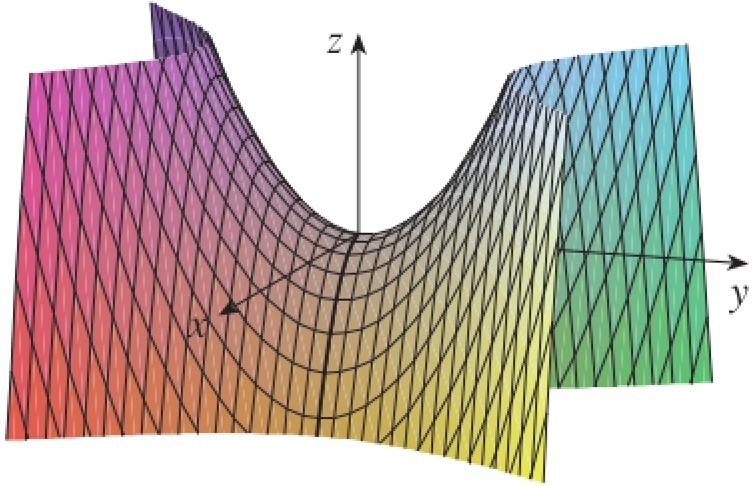


FIGURE 3
$z = y^2 - x^2$

EXAMPLE 2 Find the extreme values of $f(x, y) = y^2 - x^2$.

SOLUTION Since $f_x = -2x$ and $f_y = 2y$, the only critical point is $(0, 0)$. Notice that for points on the x-axis we have $y = 0$, so $f(x, y) = -x^2 < 0$ (if $x \neq 0$). However, for points on the y-axis we have $x = 0$, so $f(x, y) = y^2 > 0$ (if $y \neq 0$). Thus every disk with center $(0, 0)$ contains points where f takes positive values as well as points where f takes negative values. Therefore $f(0, 0) = 0$ can't be an extreme value for f, so f has no extreme value.

When it is Max or Min in a function of two variables?

$\boxed{3}$ **Second Derivatives Test** Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

Find Max and Min in a function of two variables: Example 3

V **EXAMPLE 3** Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

SOLUTION We first locate the critical points:

$$f_x = 4x^3 - 4y \qquad f_y = 4y^3 - 4x$$

Setting these partial derivatives equal to 0, we obtain the equations

$$x^3 - y = 0 \qquad \text{and} \qquad y^3 - x = 0$$

Find Max and Min in a function of two variables: Example 3

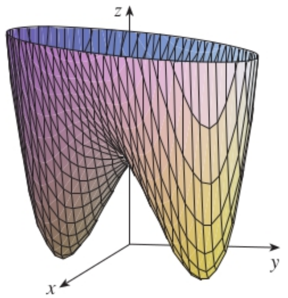


FIGURE 4
$z = x^4 + y^4 - 4xy + 1$

To solve these equations we substitute $y = x^3$ from the first equation into the second one. This gives

$$0 = x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1)$$

so there are three real roots: $x = 0, 1, -1$. The three critical points are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

Next we calculate the second partial derivatives and $D(x, y)$:

$$f_{xx} = 12x^2 \qquad f_{xy} = -4 \qquad f_{yy} = 12y^2$$

$$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$$

Since $D(0, 0) = -16 < 0$, it follows from case (c) of the Second Derivatives Test that the origin is a saddle point; that is, f has no local maximum or minimum at $(0, 0)$. Since $D(1, 1) = 128 > 0$ and $f_{xx}(1, 1) = 12 > 0$, we see from case (a) of the test that $f(1, 1) = -1$ is a local minimum. Similarly, we have $D(-1, -1) = 128 > 0$ and $f_{xx}(-1, -1) = 12 > 0$, so $f(-1, -1) = -1$ is also a local minimum.

The graph of f is shown in Figure 4.

A contour map of the function f in Example 3 is shown in Figure 5. The level curves near $(1, 1)$ and $(-1, -1)$ are oval in shape and indicate that as we move away from $(1, 1)$ or $(-1, -1)$ in any direction the values of f are increasing. The level curves near $(0, 0)$, on the other hand, resemble hyperbolas. They reveal that as we move away from the origin (where the value of f is 1), the values of f decrease in some directions but increase in other directions. Thus the contour map suggests the presence of the minima and saddle point that we found in Example 3.

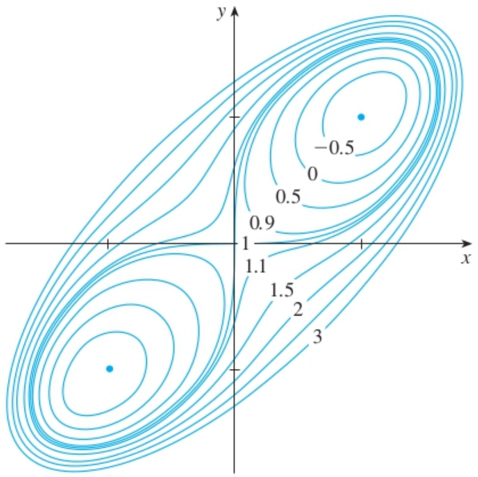


FIGURE 5

Find Max and Min in a function of two variables: Example 4

EXAMPLE 4 Find and classify the critical points of the function

$$f(x, y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$$

Also find the highest point on the graph of f.

V **EXAMPLE 5** Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

SOLUTION The distance from any point (x, y, z) to the point $(1, 0, -2)$ is

$$d = \sqrt{(x - 1)^2 + y^2 + (z + 2)^2}$$

Absolute Maximum and Minimum Values

$\boxed{9}$ To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

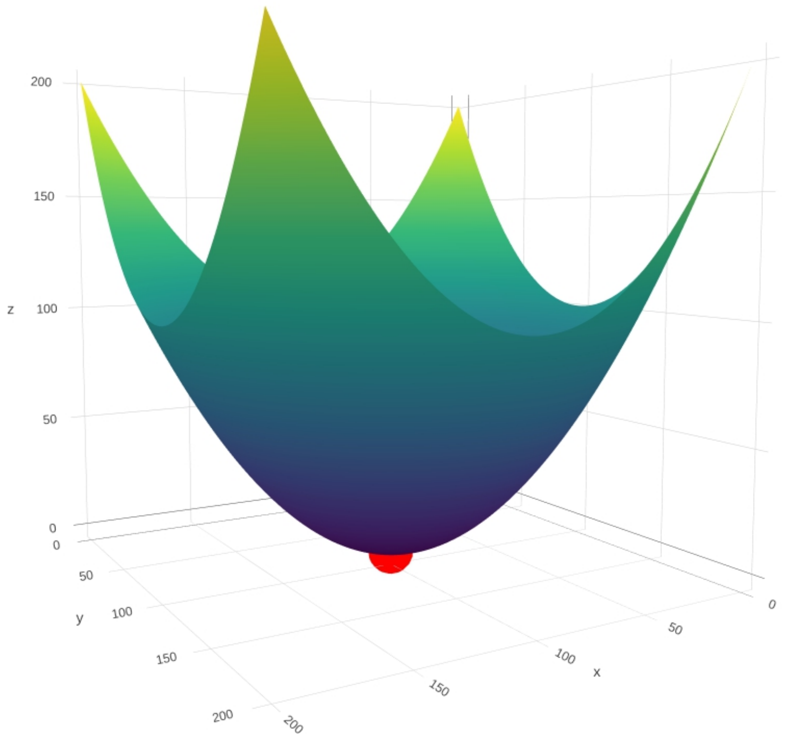
1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

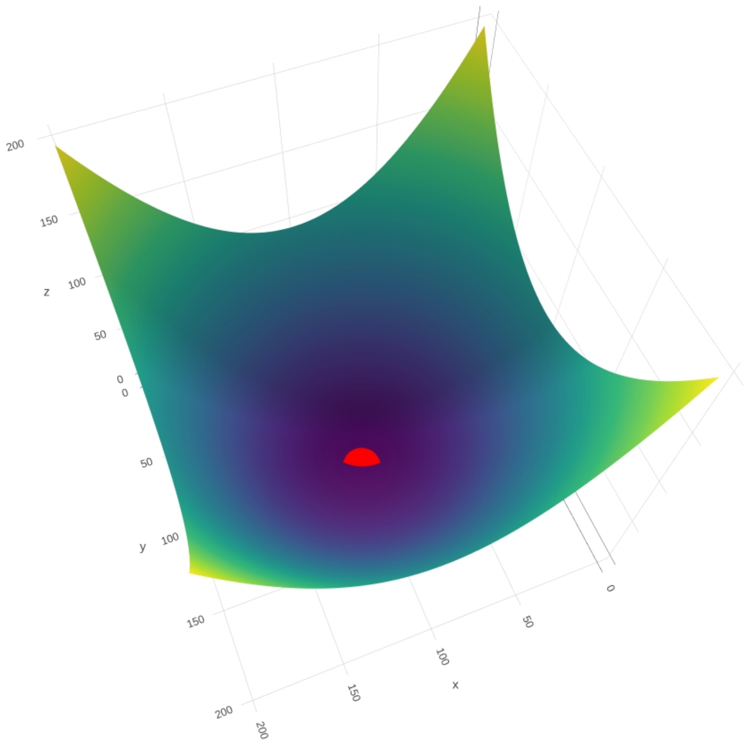
Absolute Maximum and Minimum Values using Gradient Descent of $f(x, y) = y^2 + x^2$

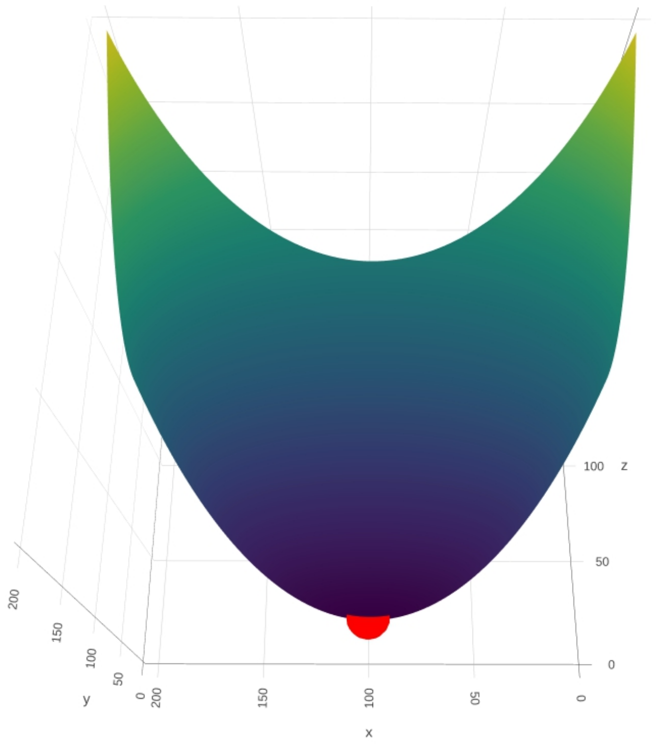
```
### Gradient descent of 2 variables function example 1
# function: x^2 + y^2
fxy <- function(x,y){x^2 + y^2}
# Gradient descent
gradient_descent <- function(x=3, y=3, rate=0.1, n=100){
   for(i in 1:n){
      xi <- x - rate*(2*x)
      x <- xi
      yi <- y - rate*(2*y)
      y <- yi
   }
   return(round(c(xi, yi, fxy(xi, yi)),5))
}
# find the min
dt <- as.data.frame(t(sapply(1:70, function(x) gradient_descent(n=x))))
colnames(dt) <- c("x", "y", "z")
```

```
# plot surface with contour
library(plotly)
# data for the function
x <- y <- seq(-10, 10, by=0.1)
names(x) <- names(y) <- x
fz <- outer(x, y, fxy)
# min of the function data
cfmin <- which(fz==min(fz), arr.ind=T)
fmin <- data.frame(x=cfmin[1], y=cfmin[2], z=fz[cfmin[1], cfmin[2]])
# plot
fig <- plot_ly(z=fz, type="surface") %>%
add_trace(data=fmin, x=fmin$x, y=fmin$y, z=fmin$z, mode="markers", type="scatter3d",
        marker = list(size=25, color="red", symbol=104)
  )
fig
```

Absolute Maximum and Minimum Values using Gradient Descent of $f(x, y) = y^2 + x^2$





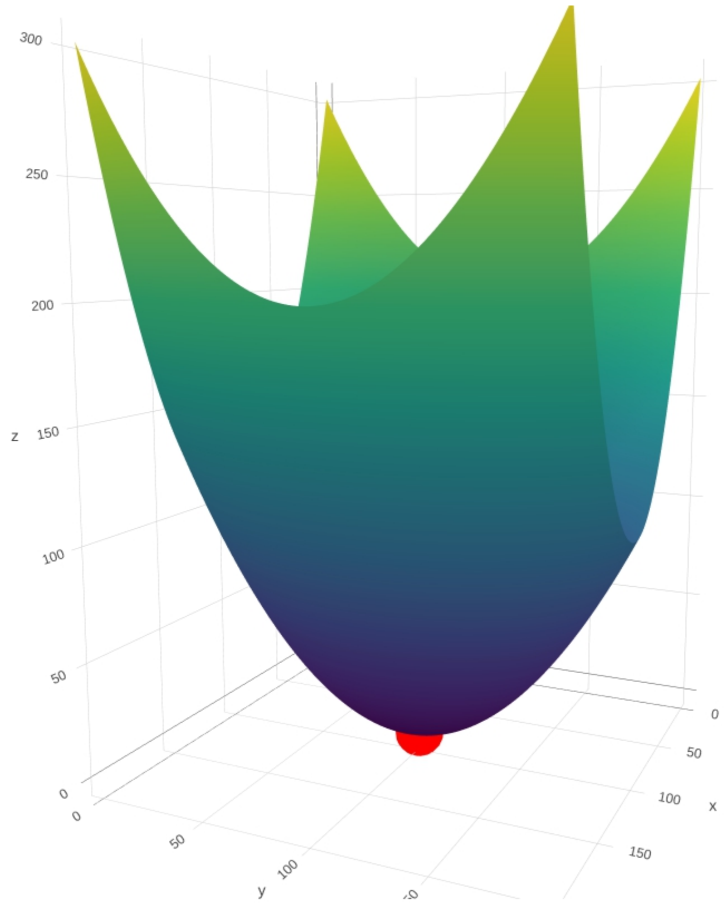


Absolute Maximum and Minimum Values using Gradient Descent of $f(x, y) = y^2 - x^2$

Absolute Maximum and Minimum Values using Gradient Descent of $f(x, y) = y^2 + 2x^2 + 1$

```
### Gradient descent of 2 variables
function example 2
# function: x^2 + 2*y^2 + 1
fxy <- function(x,y){x^2 + 2*y^2 + 1}
# Gradient descent
gradient_descent <- function(x=3, y=3,
rate=0.1, n=100){
    for(i in 1:n){
        xi <- x - rate*(2*x)
        x <- xi
        yi <- y - rate*(4*y)
        y <- yi
    }
    return(round(c(xi, yi, fxy(xi, yi)),5))
}
# find the min
dt <- as.data.frame(t(sapply(1:70,
function(x) gradient_descent(n=x))))
colnames(dt) <- c("x", "y", "z")
```

```
# plot surface with contour
library(plotly)
# data for the function
x <- y <- seq(-10, 10, by=0.1)
names(x) <- names(y) <- x
fz <- outer(x, y, fxy)
cfmin <- which(fz==min(fz), arr.ind=T)
# 101 101
fmin <- data.frame(x=cfmin[1],
y=cfmin[2], z=fz[cfmin[1], cfmin[2]])
# plot
fig <- plot_ly(z=fz, type="surface") %>%
add_trace(data=fmin, x=fmin$x,
y=fmin$y, z=fmin$z, mode="markers",
type="scatter3d",
            marker = list(size=25, color="red",
symbol=104)
  )
fig
```

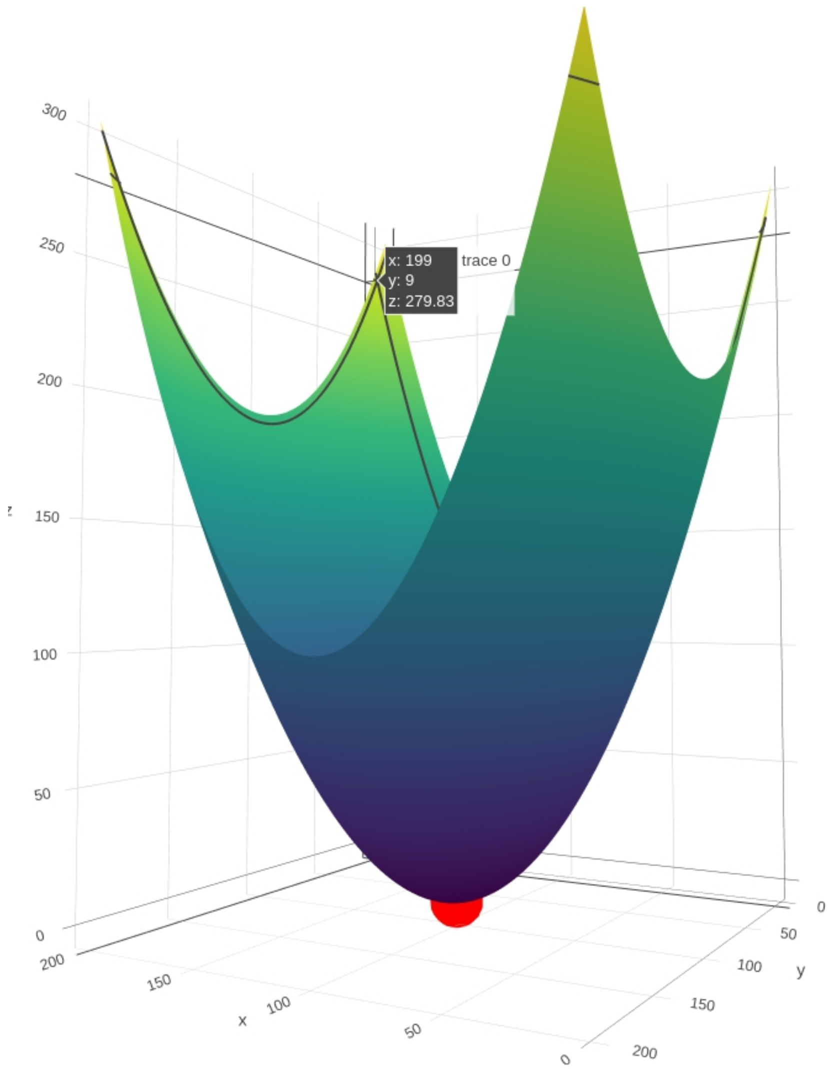


Absolute Maximum and Minimum Values using Gradient Descent of $f(x, y) = -y^2 - 2x^2 + 1$

Absolute Maximum and Minimum Values using Gradient Descent of $f(x,y) = x^2 - 2xy - 2x + 2y^2 - 2$

```
### Gradient descent of 2 variables
function example 3
# function: x^2 - 2*x*y - 2*x + 2*y^2 - 2
f <- function(x,y){x^2 - 2*x*y - 2*x +
2*y^2 - 2}
gradient_descent <- function(x=3, y=3,
rate=0.1, n=100){
    for(i in 1:n){
        xi <- x - rate*(2*x-2*y-2)
        yi <- y - rate*(4*y-2*x)
        x <- xi
        y <- yi
    }
    return(round(c(xi, yi, fxy(xi, yi)),5))
}
# find the min
dt <- as.data.frame(t(sapply(1:200,
function(x) gradient_descent(n=x))))
colnames(dt) <- c("x", "y", "z")
```

```
# plot surface with contour
library(plotly)
# data for the function
x <- y <- seq(-10, 10, by=0.1)
names(x) <- names(y) <- x
fz <- outer(x, y, fxy)
cfmin <- which(fz==min(fz), arr.ind=T)
fmin <- data.frame(x=cfmin[1], y=cfmin[2],
z=fz[cfmin[1], cfmin[2]])
# plot
fig <- plot_ly(z=fz, type="surface") %>%
add_trace(data=fmin, x=fmin$x, y=fmin$y,
z=fmin$z, mode="markers", type="scatter3d",
            marker = list(size=25, color="red",
symbol=104)
  )
fig
```



Lagrange Multipliers -> SVM

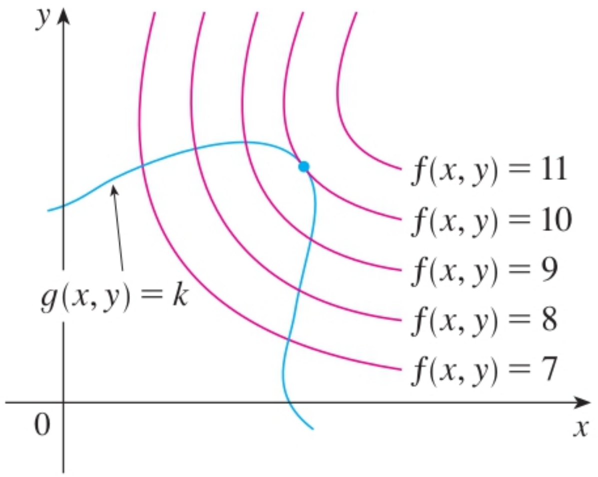


FIGURE 1

TEC Visual 14.8 animates Figure 1 for both level curves and level surfaces.

In Example 6 in Section 14.7 we maximized a volume function $V = xyz$ subject to the constraint $2xz + 2yz + xy = 12$, which expressed the side condition that the surface area was $12\ \text{m}^2$. In this section we present Lagrange's method for maximizing or minimizing a general function $f(x, y, z)$ subject to a constraint (or side condition) of the form $g(x, y, z) = k$.

It's easier to explain the geometric basis of Lagrange's method for functions of two variables. So we start by trying to find the extreme values of $f(x, y)$ subject to a constraint of the form $g(x, y) = k$. In other words, we seek the extreme values of $f(x, y)$ when the point (x, y) is restricted to lie on the level curve $g(x, y) = k$. Figure 1 shows this curve together with several level curves of f. These have the equations $f(x, y) = c$, where $c = 7, 8, 9, 10, 11$. To maximize $f(x, y)$ subject to $g(x, y) = k$ is to find the largest value of c such that the level curve $f(x, y) = c$ intersects $g(x, y) = k$. It appears from Figure 1 that this happens when these curves just touch each other, that is, when they have a common tangent line. (Otherwise, the value of c could be increased further.) This means that the normal lines at the point (x_0, y_0) where they touch are identical. So the gradient vectors are parallel; that is, $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some scalar λ.

This kind of argument also applies to the problem of finding the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$. Thus the point (x, y, z) is restricted to lie on the level surface S with equation $g(x, y, z) = k$. Instead of the level curves in Figure 1,

Lagrange Multipliers

we consider the level surfaces $f(x, y, z) = c$ and argue that if the maximum value of f is $f(x_0, y_0, z_0) = c$, then the level surface $f(x, y, z) = c$ is tangent to the level surface $g(x, y, z) = k$ and so the corresponding gradient vectors are parallel.

This intuitive argument can be made precise as follows. Suppose that a function f has an extreme value at a point $P(x_0, y_0, z_0)$ on the surface S and let C be a curve with vector equation $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ that lies on S and passes through P. If t_0 is the parameter value corresponding to the point P, then $\mathbf{r}(t_0) = \langle x_0, y_0, z_0\rangle$. The composite function $h(t) = f(x(t), y(t), z(t))$ represents the values that f takes on the curve C. Since f has an extreme value at (x_0, y_0, z_0), it follows that h has an extreme value at t_0, so $h'(t_0) = 0$. But if f is differentiable, we can use the Chain Rule to write

$$\begin{aligned} 0 &= h'(t_0) \\ &= f_x(x_0, y_0, z_0)x'(t_0) + f_y(x_0, y_0, z_0)y'(t_0) + f_z(x_0, y_0, z_0)z'(t_0) \\ &= \nabla f(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) \end{aligned}$$

This shows that the gradient vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the tangent vector $\mathbf{r}'(t_0)$ to every such curve C. But we already know from Section 14.6 that the gradient vector of g, $\nabla g(x_0, y_0, z_0)$, is also orthogonal to $\mathbf{r}'(t_0)$ for every such curve. (See Equation 14.6.18.) This means that the gradient vectors $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ must be parallel. Therefore, if $\nabla g(x_0, y_0, z_0) \neq \mathbf{0}$, there is a number λ such that

1

$$\boxed{\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)}$$

The number λ in Equation 1 is called a **Lagrange multiplier**. The procedure based on Equation 1 is as follows.

Method of Lagrange Multipliers To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Lagrange Multipliers: Example 1

V **EXAMPLE 1** A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

SOLUTION As in Example 6 in Section 14.7, we let x, y, and z be the length, width, and height, respectively, of the box in meters. Then we wish to maximize

$$V = xyz$$

subject to the constraint

$$g(x, y, z) = 2xz + 2yz + xy = 12$$

Using the method of Lagrange multipliers, we look for values of x, y, z, and λ such that $\nabla V = \lambda \nabla g$ and $g(x, y, z) = 12$. This gives the equations

$$V_x = \lambda g_x$$

$$V_y = \lambda g_y$$

$$V_z = \lambda g_z$$

$$2xz + 2yz + xy = 12$$

which become

$$\boxed{2} \qquad yz = \lambda(2z + y)$$

$$\boxed{3} \qquad xz = \lambda(2z + x)$$

$$\boxed{4} \qquad xy = \lambda(2x + 2y)$$

$$\boxed{5} \qquad 2xz + 2yz + xy = 12$$

There are no general rules for solving systems of equations. Sometimes some ingenuity is required. In the present example you might notice that if we multiply $\boxed{2}$ by x, $\boxed{3}$ by y, and $\boxed{4}$ by z, then the left sides of these equations will be identical. Doing this, we have

$$\boxed{6} \qquad xyz = \lambda(2xz + xy)$$

$$\boxed{7} \qquad xyz = \lambda(2yz + xy)$$

$$\boxed{8} \qquad xyz = \lambda(2xz + 2yz)$$

We observe that $\lambda \neq 0$ because $\lambda = 0$ would imply $yz = xz = xy = 0$ from $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ and this would contradict $\boxed{5}$. Therefore, from $\boxed{6}$ and $\boxed{7}$, we have

$$2xz + xy = 2yz + xy$$

which gives $xz = yz$. But $z \neq 0$ (since $z = 0$ would give $V = 0$), so $x = y$. From $\boxed{7}$ and $\boxed{8}$ we have

$$2yz + xy = 2xz + 2yz$$

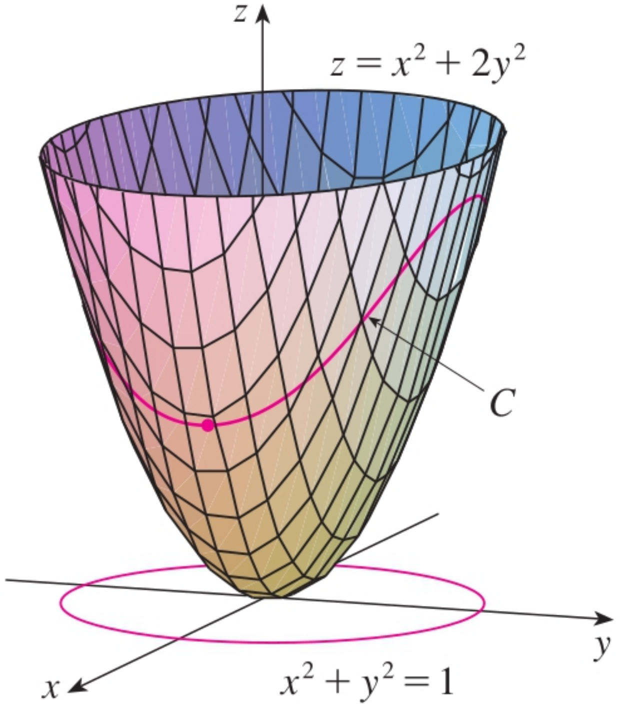
which gives $2xz = xy$ and so (since $x \neq 0$) $y = 2z$. If we now put $x = y = 2z$ in $\boxed{5}$, we get

$$4z^2 + 4z^2 + 4z^2 = 12$$

Since x, y, and z are all positive, we therefore have $z = 1$ and so $x = 2$ and $y = 2$. This agrees with our answer in Section 14.7.

Lagrange Multipliers: Example 2

In geometric terms, Example 2 asks for the highest and lowest points on the curve C in Figure 2 that lie on the paraboloid $z = x^2 + 2y^2$ and directly above the constraint circle $x^2 + y^2 = 1$.



V EXAMPLE 2 Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

SOLUTION We are asked for the extreme values of f subject to the constraint $g(x, y) = x^2 + y^2 = 1$. Using Lagrange multipliers, we solve the equations $\nabla f = \lambda \nabla g$ and $g(x, y) = 1$, which can be written as

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = 1$$

or as

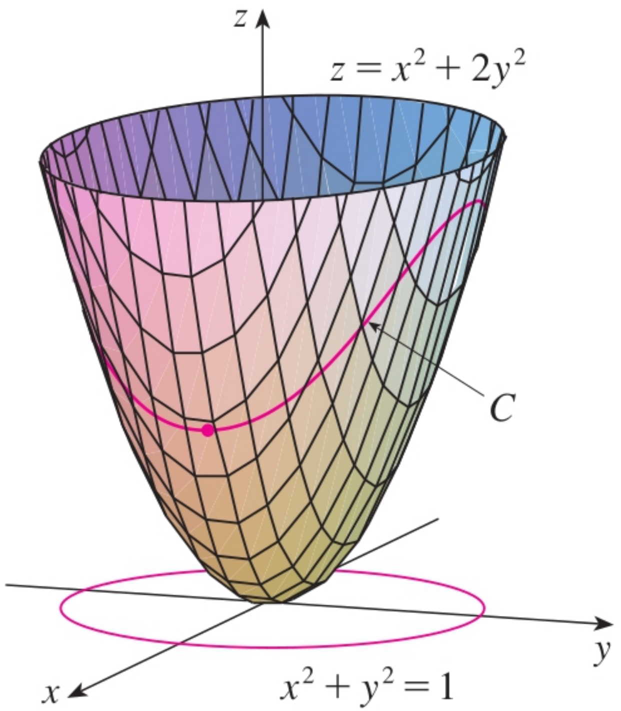
9 $$2x = 2x\lambda$$

10 $$4y = 2y\lambda$$

11 $$x^2 + y^2 = 1$$

Lagrange Multipliers: Example 2

In geometric terms, Example 2 asks for the highest and lowest points on the curve C in Figure 2 that lie on the paraboloid $z = x^2 + 2y^2$ and directly above the constraint circle $x^2 + y^2 = 1$.



From $\boxed{9}$ we have $x = 0$ or $\lambda = 1$. If $x = 0$, then $\boxed{11}$ gives $y = \pm 1$. If $\lambda = 1$, then $y = 0$ from $\boxed{10}$, so then $\boxed{11}$ gives $x = \pm 1$. Therefore f has possible extreme values at the points $(0, 1)$, $(0, -1)$, $(1, 0)$, and $(-1, 0)$. Evaluating f at these four points, we find that

$$f(0, 1) = 2 \qquad f(0, -1) = 2 \qquad f(1, 0) = 1 \qquad f(-1, 0) = 1$$

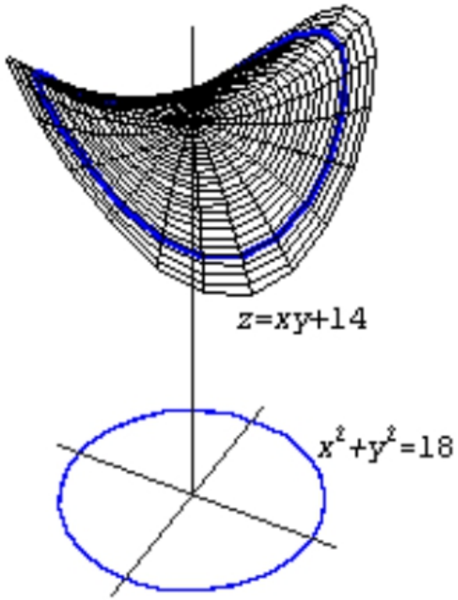
Therefore the maximum value of f on the circle $x^2 + y^2 = 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(\pm 1, 0) = 1$. Checking with Figure 2, we see that these values look reasonable.

Lagrange Multipliers: visually example 1

EXAMPLE 1 Find the extrema of $f(x, y) = xy + 14$ subject to

$$x^2 + y^2 = 18$$

Solution: That is, we want to find the highest and lowest points on the surface $z = xy + 14$ over the unit circle:



If we let $g(x, y) = x^2 + y^2$, then the constraint is $g(x, y) = 18$. The gradients of f and g are respectively

$$\nabla f = \langle y, x \rangle \qquad \text{and} \qquad \nabla g = \langle 2x, 2y \rangle$$

As a result, $\nabla f = \lambda \nabla g$ implies that

$$y = \lambda 2x \quad \textit{and} \quad x = \lambda 2y$$

. Clearly, $x = 0$ only if $y = 0$, but $(0, 0)$ is not on the unit circle. Thus, $x \neq 0$ and $y \neq 0$, so that solving for λ yields

$$\lambda = \frac{y}{2x} \quad \text{and} \quad \lambda = \frac{x}{2y} \qquad \Longrightarrow \qquad \frac{y}{2x} = \frac{x}{2y}$$

http://math.bu.edu/people/josborne/MA225and230/MA230/notes/ConstrainedOptimizationNotes.pdf

Lagrange Multipliers: visually example 1

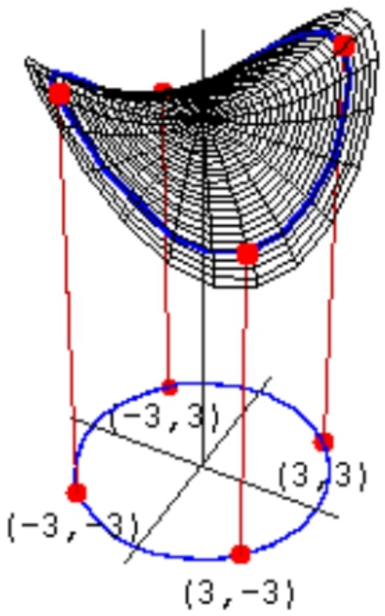
Cross-multiplying then yields $2y^2 = 2x^2$, which is the same as $y^2 = x^2$. Thus, the constraint $x^2 + y^2 = 18$ becomes

$$x^2 + x^2 = 18, \qquad x^2 = 9, \qquad x = \pm 3$$

Moreover, $y^2 = x^2$ implies that either $y = x$ or $y = -x$, so that the solutions to (2) are

$$(3,3), \ (-3,3), \ (3,-3), \ (-3,-3)$$

However, $f(3,3) = f(-3,-3) = 23$, while $f(-3,3) = f(3,-3) = 5$. Thus, the maxima of $f(x,y) = xy + 4$ over $x^2 + y^2 = 18$ occur at $(3,3)$ and $(-3,-3)$, while the minima of $f(x,y) = xy + 4$ occur at $(-3,3)$ and $(3,-3)$.



Solution: The Lagrangian for example 1 is

$$L(x,y,\lambda) = xy + 14 - \lambda\left(x^2 + y^2 - 18\right)$$

and correspondingly, $L_x = y - \lambda(2x)$, $L_y = y - \lambda(2y)$, and

$$L_\lambda = -\left(x^2 + y^2 - 18\right)$$

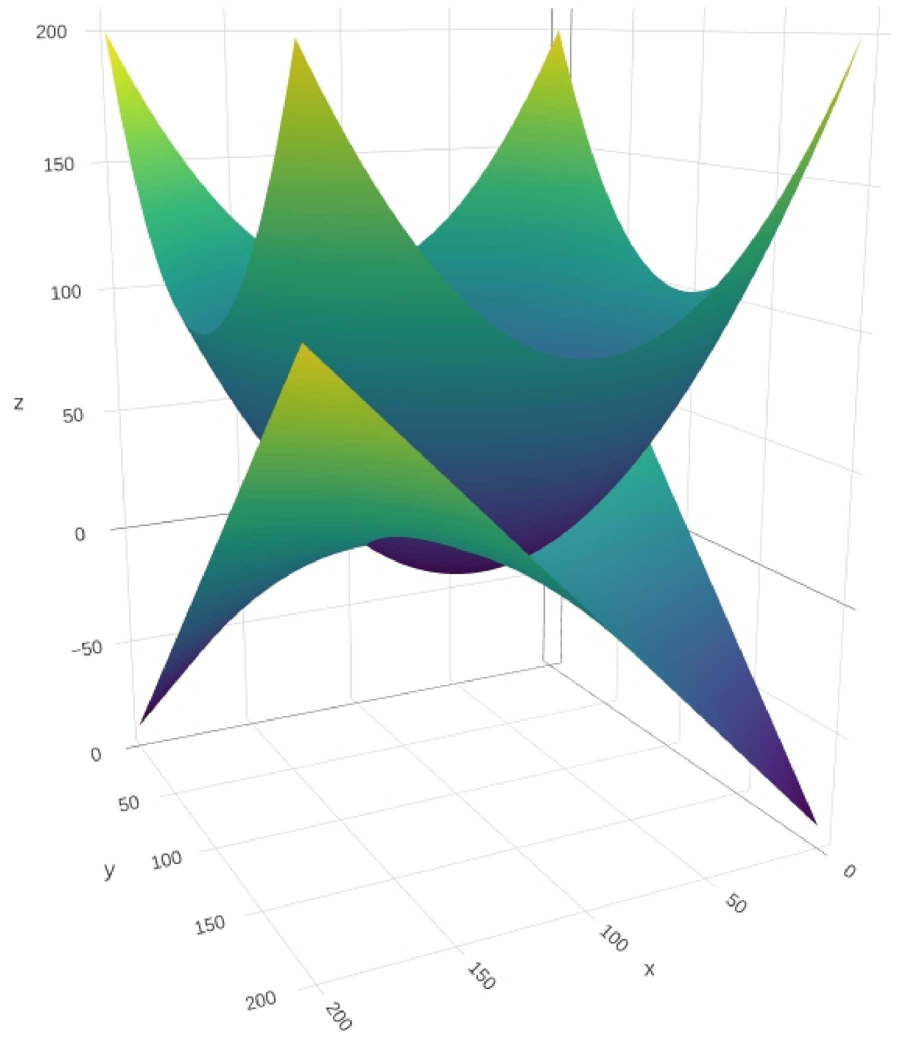
The critical points of L satisfy $L_x = 0$, $L_y = 0$, and $L_\lambda = 0$, which results in

$$y = \lambda 2x \quad and \quad x = \lambda 2y$$

along with $x^2 + y^2 = 18$. The remainder of the solution is the same as in example 1.

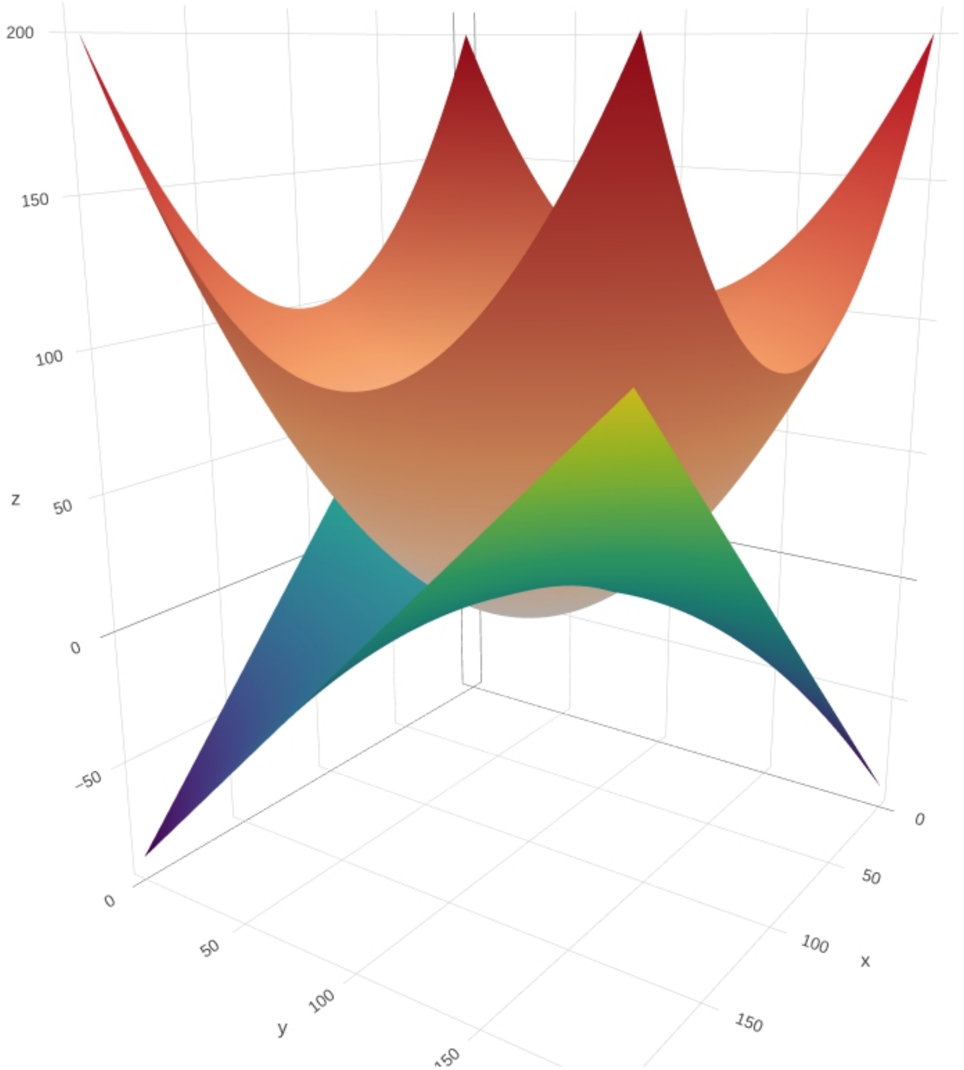
Lagrange Multipliers: visually

```
# Lagrange Multiplier example 1
# object function
f <- function(x,y){x*y+14}
g <- function(x,y){x^2+y^2}
# data for the function
x <- y <- seq(-10, 10, by=0.1)
names(x) <- names(y) <- x
fz <- outer(x, y, f)
gz <- outer(x, y, g)
# plot 1
fig <- plot_ly(z=fz, type="surface") %>%
add_trace(z=gz, type="surface")
fig
```



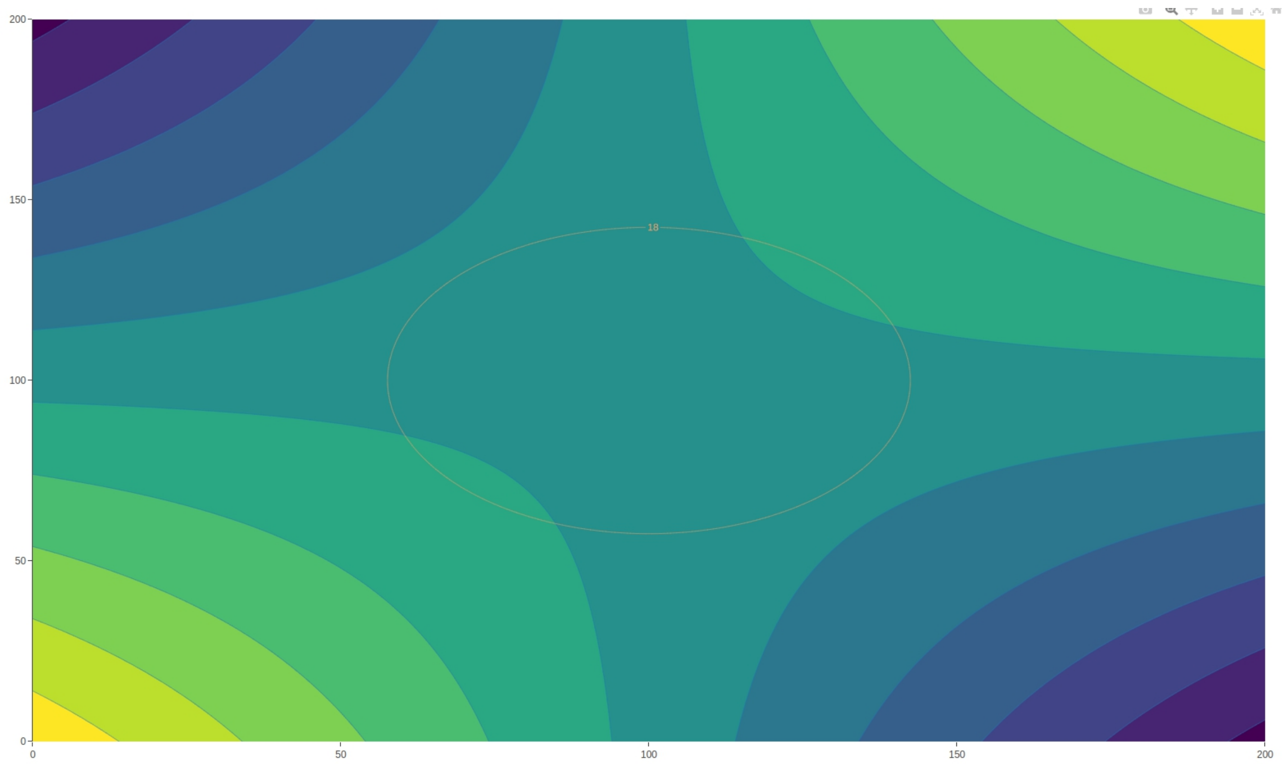
Lagrange Multipliers: visually

```
# plot 3
fig <- plot_ly(z=fz, type="surface") %>%
add_trace(z=gz, type="surface",
        colorscale = "Black",
        contours = list(
        start = 18,
        end = 18,
        showlabels = TRUE,
        showlines = TRUE,
        coloring = "lines"
      )
)
fig
```



Lagrange Multipliers: visually

```
# plot 4
fig <- plot_ly(z=fz, type="contour") %>%
add_trace(z=gz, type="contour",
        colorscale = "Black",
        contours = list(
        start = 18,
        end = 18,
        showlabels = TRUE,
        showlines = TRUE,
        coloring = "lines"
      )
)
```



Lagrange Multipliers: Exercise with solution

1. Find the maximum and minimum values of $f(x, y) = x^2 + x + 2y^2$ on the unit circle.

<u>**Answer:**</u> The objective function is $f(x, y)$. The constraint is $g(x, y) = x^2 + y^2 = 1$.

$$\begin{array}{llcl} \text{Lagrange equations:} & f_x = \lambda g_x & \Leftrightarrow & 2x + 1 = \lambda 2x \\ & f_y = \lambda g_y & \Leftrightarrow & 4y = \lambda 2y \\ \text{Constraint:} & & & x^2 + y^2 = 1 \end{array}$$

The second equation shows $y = 0$ or $\lambda = 2$.

$\lambda = 2 \Rightarrow x = 1/2,\ y = \pm\sqrt{3}/2.$

$y = 0 \Rightarrow x = \pm 1.$

Thus, the critical points are $(1/2, \sqrt{3}/2)$, $(1/2, -\sqrt{3}/2)$, $(1, 0)$, and $(-1, 0)$.

$f(1/2, \pm\sqrt{3/2}) = 9/4$ (maximum).

$f(1, 0) = 2$ (neither min. nor max).

$f(-1, 0) = 0$ (minimum).

Visually?

Lagrange Multipliers: Exercise with solution

2. Find the minimum and maximum values of $f(x,y) = x^2 - xy + y^2$ on the quarter circle $x^2 + y^2 = 1$, $x, y \geq 0$.

Answer: The constraint function here is $g(x,y) = x^2 + y^2 = 1$. The maximum and minimum values of $f(x,y)$ will occur where $\nabla f = \lambda \nabla g$ or at endpoints of the quarter circle.

$$\nabla f = \langle 2x - y, -x + 2y \rangle \quad \text{and} \quad \nabla g = \langle 2x, 2y \rangle.$$

Setting $\nabla f = \lambda \nabla g$, we get $2x - y = \lambda \cdot 2x$ and $-x + 2y = \lambda \cdot 2y$.

Solving for λ and setting the results equal to each other gives us:

$$\begin{aligned} \frac{2x - y}{2x} &= \frac{-x + 2y}{2y} \\ 2xy - y^2 &= -x^2 + 2xy \\ x^2 &= y^2. \end{aligned}$$

Because we're constrained to $x^2 + y^2 = 1$ with x and y non-negative, we conclude that $x = y = \frac{1}{\sqrt{2}}$.

Thus, the extreme points of $f(x,y)$ will be at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(1,0)$, or $(0,1)$.

$$f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{1}{2} \quad \text{is the minimum value of } f \text{ on this quarter circle.}$$

$$f(1,0) = f(0,1) = 1 \quad \text{are the maximal values of } f \text{ on this quarter circle.}$$

Visually?

Lagrange Multipliers: Exercise

1. *Find the extrema of* $f(x, y) = xy + 14$

   subject to $x^2 + y^2 = 18$

2. *Find the point(s) on the curve* $y = 1.5 - x^2$ *closet to the origin both visually and via the Lagrange Multiplier method.*

3. *Find the point(s) on the plane* $x + y - z = 3$ *that are closet to the origin both visually and via the Lagrange Multiplier method.*

4. *If x, y and z denote length, width, and height, respectively, find max of the volume*

   V (x,y,z) = f(x,y,z) = xyz subject to the contraints

   $$x + y + z = 45 \text{ and } y = 2x$$

   both visually and via the Lagrange Multiplier method.