Data Visualization

Lecture 5
Mathamatics Visualization

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Plot a mathmatics graph

- Introduction to Linear Algebra 3

Determinant of 2 by 2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where $a_{11}, a_{12}, a_{21}, a_{22}$ are real numbers, then A is invertible if and only if $a_{11} \cdot a_{22} - a_{12} \cdot a_{21} \neq 0$. For a 2×2 matrix A, the **determinant** of A is

$$a_{11} \cdot a_{22} - a_{12} \cdot a_{21}.$$

In this book, we notate the determinant of a square matrix A as $\det(A)$.

Determinant of 3 by 3 matrix

Suppose we have a 3×3 matrix A such that

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Now we will compute $\det(A)$. Now let A_{ij} be the square matrix computed from the matrix A by deleting the ith row and the jth column. For example, A_{12} is a 2×2 matrix computed from A by deleting the 1st row and the 2nd column. Specifically,

$$A_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}.$$

Then we have

$$\det(A) = a_{11} \cdot \det(A_{11}) - a_{12} \cdot \det(A_{12}) + a_{13} \det(A_{13}).$$

Example 80 *Suppose we have a 3×3 matrix A such that*

$$A = \begin{bmatrix} -2 & 4 & -5 \\ -1 & -1 & 1 \\ -5 & 0 & -3 \end{bmatrix}.$$

Then we have

$$A_{11} = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix}, A_{12} = \begin{bmatrix} -1 & 1 \\ -5 & -3 \end{bmatrix}, A_{13} = \begin{bmatrix} -1 & -1 \\ -5 & 0 \end{bmatrix}.$$

Also we have

$$\begin{aligned} \det(A_{11}) &= (-1) \cdot (-3) - 1 \cdot 0 = 3 \\ \det(A_{12}) &= (-1) \cdot (-3) - 1 \cdot (-5) = 8 \\ \det(A_{13}) &= (-1) \cdot 0 - (-1) \cdot (-5) = -5. \end{aligned}$$

Then we have

$$\begin{aligned} \det(A) &= a_{11} \cdot \det(A_{11}) - a_{12} \cdot \det(A_{12}) + a_{13} \det(A_{13}) \\ &= (-2) \cdot 3 - 4 \cdot 8 + (-5) \cdot (-5) = -13. \end{aligned}$$

Determinant in R

In `R` we can use the det() function to compute the determinant of a square matrix. We go back to Example 80. Suppose we have a 3×3 matrix such that

$$A = \begin{bmatrix} -2 & 4 & -5 \\ -1 & -1 & 1 \\ -5 & 0 & -3 \end{bmatrix}.$$

First we define the matrix in `R`:

```
A <- matrix(c(-2, 4, -5, -1, -1, 1, -5, 0, -3), nrow = 3, ncol = 3)
```

Then we use the det() function:

```
det(A)
```

Then `R` returns:

```
> det(A)
[1] -13
```

Example 81 *This is from Example 80. Suppose we have a 3×3 matrix A such that*

$$A = \begin{bmatrix} -2 & 4 & -5 \\ -1 & -1 & 1 \\ -5 & 0 & -3 \end{bmatrix}.$$

Then we have

$$A_{11} = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix}, A_{21} = \begin{bmatrix} 4 & -5 \\ 0 & -3 \end{bmatrix}, A_{31} = \begin{bmatrix} 4 & -5 \\ -1 & 1 \end{bmatrix}.$$

Also we have

$$\det(A_{11}) = (-1) \cdot (-3) - 1 \cdot 0 = 3$$
$$\det(A_{21}) = 4 \cdot (-3) - 0 \cdot (-5) = -12$$
$$\det(A_{31}) = 4 \cdot 1 - (-1) \cdot (-5) = -1.$$

Then we have

$$\begin{aligned} \det(A) &= a_{11} \cdot \det(A_{11}) - a_{21} \cdot \det(A_{21}) + a_{31} \det(A_{31}) \\ &= (-2) \cdot 3 - (-1) \cdot (-12) + (-5) \cdot (-1) = -13. \end{aligned}$$

Properties of Determinants

Theorem 3.4 *A square matrix A is invertible if and only if* $\det(A) \neq 0$.

This theorem implies the following corollary on the system of linear equations. This corollary was used in the practical applications in the previous section.

Corollary 3.5 *Suppose we have a system of n linear equations on n variables. Then the $n \times n$ coefficient matrix of the system of linear equations has a unique solution if and only if the determinant of the coefficient matrix of the system of linear equations is not equal to zero.*

Example 84 *This is from Example 44. Suppose we have the following system of linear equations:*

$$\begin{array}{ccccccccc} & & x_2 & + & 3x_3 & - & x_4 & = & 1 \\ -x_1 & + & x_2 & - & 4x_3 & & & = & 1 \\ x_1 & & & + & 2x_3 & + & 4x_4 & = & 5 \\ & & x_2 & & & - & 4x_4 & = & -2. \end{array}$$

Its coefficient matrix is

$$\begin{bmatrix} 0 & 1 & 3 & -1 \\ -1 & 1 & -4 & 0 \\ 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & -4 \end{bmatrix}.$$

We are going to use the det() function in R*. As we saw in the previous section, first we define the matrix in* R*:*

```
A <- matrix(c(0, 1, 3, -1, -1, 1, -4, 0, 1, 0, 2, 4, 0, 1, 0, -4),
nrow = 4, ncol = 4, byrow = TRUE)
```

Then we use the det() function:

```
det(A)
```

Then R *returns:*

```
> det(A)
[1] 30
```

Therefore, there exists a unique solution for the system of linear equations.

Properties of Determinants

Theorem 3.6 *This is again from Example 44. Suppose A and B are $n \times n$ matrices. Then*

$$\det(A \cdot B) = \det(A) \cdot \det(B).$$

Example 85 *This is from Example 44. Suppose we have the following system of linear equations:*

$$\begin{array}{rcrcrcrcr} & & x_2 & + & 3x_3 & - & x_4 & = & 1 \\ -x_1 & + & x_2 & - & 4x_3 & & & = & 1 \\ x_1 & & & + & 2x_3 & + & 4x_4 & = & 5 \\ & & x_2 & & & - & 4x_4 & = & -2. \end{array}$$

Its coefficient matrix is

$$A = \begin{bmatrix} 0 & 1 & 3 & -1 \\ -1 & 1 & -4 & 0 \\ 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & -4 \end{bmatrix}.$$

Then we exchange the first row and the third row of the system. This is equivalent to multiplying an elementary matrix

$$E = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

from the left side. The determinant of E is -1. Therefore

$$\det(E \cdot A) = \det(E) \cdot \det(A) = (-1) \cdot (30) = -30.$$

Properties of Determinants

Theorem 3.7 *Suppose A is a square matrix. Then*

$$\det(A^T) = \det(A).$$

Example 86 *This is from Example 44. Suppose we have the following system of linear equations:*

$$\begin{array}{rcrcrcrcr} & & x_2 & + & 3x_3 & - & x_4 & = & 1 \\ -x_1 & + & x_2 & - & 4x_3 & & & = & 1 \\ x_1 & & & + & 2x_3 & + & 4x_4 & = & 5 \\ & & x_2 & & & - & 4x_4 & = & -2. \end{array}$$

Its coefficient matrix is

$$A = \begin{bmatrix} 0 & 1 & 3 & -1 \\ -1 & 1 & -4 & 0 \\ 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & -4 \end{bmatrix}.$$

Then we consider the transpose of the matrix A. We are going to use the t() function to compute the transpose of A and the det() function to compute the determinant of A in R.

First we define the matrix in R*:*

```
A <- matrix(c(0, 1, 3, -1, -1, 1, -4, 0, 1, 0, 2, 4, 0, 1, 0, -4),
nrow = 4, ncol = 4, byrow = TRUE)
```

Then we use the t() function to compute the transpose of A and the det() function to compute the determinant of A:

```
det(t(A))
```

Then R *returns:*

```
> det(t(A))
[1] 30
```

Properties of Determinants

Theorem 3.8 *Suppose we have a square matrix A and suppose A is invertible. Then*

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Example 87 *This is again from Example 44. Suppose we have the following system of linear equations:*

$$\begin{array}{rcrcrcrcr} & & x_2 & + & 3x_3 & - & x_4 & = & 1 \\ -x_1 & + & x_2 & - & 4x_3 & & & = & 1 \\ x_1 & & & + & 2x_3 & + & 4x_4 & = & 5 \\ & & x_2 & & & - & 4x_4 & = & -2. \end{array}$$

Its coefficient matrix is

$$A = \begin{bmatrix} 0 & 1 & 3 & -1 \\ -1 & 1 & -4 & 0 \\ 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & -4 \end{bmatrix}.$$

Then we consider the transpose of the matrix A. We are going to use the inv() function from the `pracma` *package to compute the inverse of A and the det() function to compute the determinant of A in* `R`.

First we upload the `pracma` *package using the library() function and we define the matrix in* `R`*:*

```
library(pracma)
A <- matrix(c(0, 1, 3, -1, -1, 1, -4, 0, 1, 0, 2, 4, 0, 1, 0, -4),
nrow = 4, ncol = 4, byrow = TRUE)
```

Then we use the inv() function to compute the inverse of A and the det() function to compute the determinant of A:

```
det(inv(A))
```

Then `R` *returns:*

```
> det(inv(A))
[1] 0.03333333
```

Note that if we type "1/30" in `R` *it returns:*

```
> 1/30
[1] 0.03333333
```

Properties of Determinants

Theorem 3.9 *If an elementary matrix E represents an Exchange operation, then*

$$\det(E) = -1.$$

If an elementary matrix E represents a Scaling operation, i.e., multiplying a row by a constant c, then

$$\det(E) = c.$$

If an elementary matrix E represents a Replacement operation, i.e., multiplying a row by a constant c and adding the row to another row, then

$$\det(E) = 1.$$

Example 88 *Let us consider an example in Section 2.4. Suppose we have the following system of 3 linear equations with the variables x_1, x_2, x_3:*

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 4x_3 & = & 1 \\ 2x_1 & & & - & x_3 & = & -1.5 \\ -x_1 & + & x_2 & & & = & 2. \end{array}$$

Recall that the augmented matrix of this system is:

$$B = \begin{bmatrix} 1 & -1 & 4 & 1 \\ 2 & 0 & -1 & -1.5 \\ -1 & 1 & 0 & 2 \end{bmatrix}.$$

The reduced echelon form of the augmented matrix can be computed by the product of elementary matrices to the augmented matrix of the system of linear equations:

$$E_8 \cdot E_7 \cdot E_6 \cdot E_5 \cdot E_4 \cdot E_3 \cdot E_2 \cdot E_1 \cdot B$$

$$\begin{aligned}
E_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
E_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}, \\
E_3 &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
E_4 &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
E_5 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
E_6 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \\
E_7 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}, \\
E_8 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

Properties of Determinants

Let A be the coefficient matrix such that

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Then, we check the determinant of the matrix such that

$$E_8 \cdot E_7 \cdot E_6 \cdot E_5 \cdot E_4 \cdot E_3 \cdot E_2 \cdot E_1 \cdot A.$$

We have

$$\det(E_1) = -1,\ \det(E_2) = 1,\ \det(E_3) = 1,\ \det(E_4) = -1,\ \det(E_5) = 1/2,$$
$$\det(E_6) = 1,\ \det(E_7) = 1/4,\ \det(E_8) = 1.$$

Also we have the determinant of A

$$\det(A) = 8.$$

Thus, we have

$$\det(E_8 \cdot E_7 \cdot E_6 \cdot E_5 \cdot E_4 \cdot E_3 \cdot E_2 \cdot E_1 \cdot A) = \frac{1}{2} \cdot \frac{1}{4} \cdot 8 = 1.$$

In fact, we have

$$E_8 \cdot E_7 \cdot E_6 \cdot E_5 \cdot E_4 \cdot E_3 \cdot E_2 \cdot E_1 \cdot A = I_3$$

where I_3 is the identity matrix of size 3.

Cramer's rule

Theorem 3.10 (Cramer's rule) *Suppose that a system of n linear equations with n variables has a unique solution such that*

$$Ax = b$$

where A is the coefficient matrix, b is a vector for the right hand side, and x is a vector of variables such that

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \ b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \ x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Let $A_i(b)$ be an $n \times n$ matrix created from A and b by replacing the ith column of A with b such that

$$A_1(b) = \begin{bmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_n & a_{n2} & \dots & a_{nn} \end{bmatrix},$$

$$A_2(b) = \begin{bmatrix} a_{11} & b_1 & \dots & a_{1n} \\ a_{21} & b_2 & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & b_n & \dots & a_{nn} \end{bmatrix},$$

$$\vdots,$$

$$A_n(b) = \begin{bmatrix} a_{11} & a_{12} & \dots & b_1 \\ a_{21} & a_{22} & \dots & b_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & b_n \end{bmatrix}.$$

Then the unique solution for the system of linear equations is

$$x_1 = \frac{\det(A_1(b))}{\det(A)}, \ x_2 = \frac{\det(A_2(b))}{\det(A)}, \ \cdots, x_n = \frac{\det(A_n(b))}{\det(A)}.$$

Cramer's rule

Example 89 *We use the system of linear equations from Example 44. Suppose we have the following system of linear equations:*

$$\begin{array}{rcrcrcrcr} & & x_2 & + & 3x_3 & - & x_4 & = & 1 \\ -x_1 & + & x_2 & - & 4x_3 & & & = & 1 \\ x_1 & & & + & 2x_3 & + & 4x_4 & = & 5 \\ & & x_2 & & & - & 4x_4 & = & -2. \end{array}$$

Its coefficient matrix and the vector for the right hand side are

$$A = \begin{bmatrix} 0 & 1 & 3 & -1 \\ -1 & 1 & -4 & 0 \\ 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & -4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 5 \\ -2 \end{bmatrix}.$$

We showed in Example 84 that the determinant of the matrix A is 30. Therefore by Theorem 3.4, there exists a unique solution to the system.

Now we apply Cramer's rule to find the solution. Here we have

$$A_1(b) = \begin{bmatrix} 1 & 1 & 3 & -1 \\ 1 & 1 & -4 & 0 \\ 5 & 0 & 2 & 4 \\ -2 & 1 & 0 & -4 \end{bmatrix}, \quad A_2(b) = \begin{bmatrix} 0 & 1 & 3 & -1 \\ -1 & 1 & -4 & 0 \\ 1 & 5 & 2 & 4 \\ 0 & -2 & 0 & -4 \end{bmatrix},$$

$$A_3(b) = \begin{bmatrix} 0 & 1 & 1 & -1 \\ -1 & 1 & 1 & 0 \\ 1 & 0 & 5 & 4 \\ 0 & 1 & -2 & -4 \end{bmatrix}, \quad A_4(b) = \begin{bmatrix} 0 & 1 & 3 & 1 \\ -1 & 1 & -4 & 1 \\ 1 & 0 & 2 & 5 \\ 0 & 1 & 0 & -2 \end{bmatrix}.$$

We are going to use the det() function in R. *As usual, first we define the matrix A and a vector b:*

```
A <- matrix(c(0, 1, 3, -1, -1, 1, -4, 0, 1, 0, 2, 4, 0, 1, 0, -4),
nrow = 4, ncol = 4, byrow = TRUE)
b <- c(1, 1, 5, -2)
```

Then we define matrices $A_i(b)$ for $i = 1, 2, 3, 4$:

```
# Define A1(b)
A1 <- A
A1[, 1] <- b
# Define A2(b)
A2 <- A
A2[ ,2] <- b
# Define A3(b)
A3 <- A
A3[ ,3] <- b
# Define A4(b)
A4 <- A
A4[ ,4] <- b
```

Then we use the det() function to find the solution using Cramer's rule:

```
x1 <- det(A1)/det(A)
x2 <- det(A2)/det(A)
x3 <- det(A3)/det(A)
x4 <- det(A4)/det(A)
```

Cramer's rule

Then R *returns:*

```
> x1
[1] 1
> x2
[1] 2
> x3
[1] 7.401487e-17
> x4
[1] 1
```

Note that R *returns* $x_3 = 7.401487e - 17$*. This means* $x_3 = 7.401487 \cdot 10^{-17}$*, which is very small number and a very close to* 0*. This is caused by computational numerical errors.*

Now we check the solution by using the solve() function. Then R *returns:*

```
> solve(A,b)
[1] 1 2 0 1
```

Remark 3.11 *As you see from Example 89, Cramer's rule has larger numerical errors caused by divisions compared with the reduced echelon form used in the function solve(). Cramer's rule is nice in terms of mathematical theory but it is not stable in terms of numerical computations. Thus if we want to solve a system of linear equations we should use the reduced echelon form in stead of Cramer's rule.*

Pratice in class

1. Why is det(A) the area of its row vectors?
https://cran.r-project.org/web/packages/matlib/vignettes/det-ex1.html

2. Finding det() by Gaussian elimination
https://cran.r-project.org/web/packages/matlib/vignettes/det-ex2.html