Data Visualization

Lecture 5
Mathamatics Visualization

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Content

Plot a mathmatics graph

- Introduction to Linear Algebra 6: Inner Product Space

Inner Product Space

- An inner product is an operation on vectors in a vector space and it will define geometric properties of the vector space.
- A vector space paired with an inner product is called an inner product space.
- With an inner product we can define the length of a vector and also the distance between two vectors.

Definition 40 *An* **inner product** $\langle\,,\rangle$ *is a function sending a pair of vectors in a vector space to a real number such that: for all vectors u, v, w in a vector space V and a scale c,*

- $\langle u, v\rangle = \langle v, u\rangle$,
- $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$,
- $\langle c \cdot u, v\rangle = c \cdot \langle u, v\rangle$,
- $\langle u, u\rangle \geq 0$ *for all u in V, and*
- $\langle u, u\rangle = 0$ *if and only if* $u = \mathbf{0}$.

Example 123 *Suppose we have $V = \mathbb{R}^3$. Then we let*

$$\langle u, v\rangle = u_1 \cdot v_1 + u_2 \cdot v_2 + u_3 \cdot v_3,$$

where

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

are vectors in V. Now we want to show $\langle\,,\rangle$ is an inner product. We have to check all conditions in Definition 40.

- *Show* $\langle u, v\rangle = \langle v, u\rangle$:

$$\begin{aligned} \langle u, v\rangle &= u_1 \cdot v_1 + u_2 \cdot v_2 + u_3 \cdot v_3 \\ &= v_1 \cdot u_1 + v_2 \cdot u_2 + v_3 \cdot u_3 = \langle v, u\rangle \end{aligned}$$

- *Show* $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$: *Let*

$$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

Then,

$$\begin{aligned} \langle u + v, w\rangle &= (u_1 + v_1) \cdot w_1 + (u_2 + v_2) \cdot w_2 + (u_3 + v_3) \cdot w_3 \\ &= u_1 \cdot w_1 + v_1 \cdot w_1 + u_2 \cdot w_2 + v_2 \cdot w_2 + u_3 \cdot w_3 + v_3 \cdot w_3 \\ &= (u_1 \cdot w_1 + u_2 \cdot w_2 + u_3 \cdot w_3) + (v_1 \cdot w_1 + v_2 \cdot w_2 + v_3 \cdot w_3) \\ &= \langle u, w\rangle + \langle v, w\rangle. \end{aligned}$$

Inner Product Space

- *Show* $\langle c \cdot u, v\rangle = c \cdot \langle u, v\rangle$ *for a scalar* c.

$$\begin{aligned}\langle c \cdot u, v\rangle &= c \cdot u_1 \cdot v_1 + c \cdot u_2 \cdot v_2 + c \cdot u_3 \cdot v_3 \\ &= c(\cdot u_1 \cdot v_1 + \cdot u_2 \cdot v_2 + \cdot u_3 \cdot v_3) = c \cdot \langle u, v\rangle.\end{aligned}$$

- *Show* $\langle u, u\rangle \geq 0$.

$$\begin{aligned}\langle u, u\rangle &= u_1 \cdot u_1 + u_2 \cdot u_2 + u_3 \cdot u_3 \\ &= u_1^2 + u_2^2 + u_3^2 \geq 0\end{aligned}$$

since u_1, u_2, u_3 *are all real numbers.*

- *Now show* $\langle u, u\rangle = 0$ *if and only if* $u = \mathbf{0}$.

$$\begin{aligned}\langle u, u\rangle &= u_1 \cdot u_1 + u_2 \cdot u_2 + u_3 \cdot u_3 \\ &= u_1^2 + u_2^2 + u_3^2 = 0\end{aligned}$$

if and only if $u_1 = u_2 = u_3 = 0$.

In general, if we have $V = \mathbb{R}^d$, then we let

$$\langle u, v\rangle = u_1 \cdot v_1 + u_2 \cdot v_2 + \ldots + u_d \cdot v_d,$$

where

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_d \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{bmatrix}$$

are vectors in V. Then $\langle , \rangle$ is an inner product in V. This inner product is also called a **dot product**.

Example 124 *Suppose we have* $V = \mathbb{R}^2$. *Let*

$$u = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Then the inner product of u, v *is*

$$\langle u, v\rangle = 3 \cdot 1 + 2 \cdot 2 = 3 + 4 = 7.$$

Example 125 *Suppose we have* $V = \mathbb{R}^3$. *Let*

$$u = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}.$$

Then the inner product of u, v *is*

$$\langle u, v\rangle = 1 \cdot 1 + 5 \cdot 0 + 0 \cdot 3 = 1 + 0 + 0 = 1.$$

Example 126 *Suppose we have* $V = \mathbb{R}^3$. *Then we let*

$$\langle u, v\rangle = 3 \cdot u_1 \cdot v_1 + 2\dot{u}_2 \cdot v_2 + u_3 \cdot v_3,$$

where

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

are vectors in V. *Then* $\langle , \rangle$ *is an inner product in* V.

In R: compute Inner Product Space

To compute the inner product of two vectors in $\mathbb{R}^d$ in `R`, we can use the sum() function. From Example 125, suppose we have $V = \mathbb{R}^3$. Let

$$u = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}.$$

First we define two vectors:

```
u <- c(1, 5, 0)
v <- c(1, 0, 3)
```

Then we use the sum() function to find the product of two vectors:

```
sum(u * v)
```

This gives the inner product of these vectors:

```
> sum(u * v)
[1] 1
```

In R: compute Inner Product Space

To analyze data sets in $\mathbb{R}^d$ or to measure the strength of a force in mechanical physics, it is very useful to measure the length of a vector in $\mathbb{R}^d$. Here we can define the length of a vector in $\mathbb{R}^d$ using an inner product.

Definition 41 *The length of a vector v in a vector space $V = \mathbb{R}^d$ is defined as*

$$||v|| = \sqrt{\langle v, v \rangle}.$$

Example 127 *Suppose we have $V = \mathbb{R}^2$. Let*

$$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Then the length of v is

$$||v|| = \sqrt{1^2 + 2^2} = \sqrt{5}.$$

Example 128 *Suppose we have $V = \mathbb{R}^3$. Let*

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Then the length of v is

$$||v|| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}.$$

To compute the inner product of two vectors in $\mathbb{R}^d$ in `R`, we can use the sum() function. From Example 128, suppose we have $V = \mathbb{R}^3$. Let

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

First we define a vector:

```
v <- c(1, 2, 3)
```

Then we use the sum() function and the sqrt() to compute the inner product of two vectors:

```
sqrt(sum(v * v))
```

This gives the length of a vector:

```
> sqrt(sum(v * v))
[1] 3.741657
```

In R: Compute Inner Product Space

Measuring a distance between two vectors in $\mathbb{R}^d$ is used in many problems in many areas. For example, in statistics, to compute the sample standard deviation from a sample in $\mathbb{R}^d$ we need to measure distances between each data point and its sample mean. With the definition of the length of a vector in $\mathbb{R}^d$, we can define a distance between two vectors in $\mathbb{R}^d$.

Definition 42 *Suppose $V = \mathbb{R}^d$. Let v, u be vectors in $\mathbb{R}^d$. Then the distance between v and u is defined as*
$$||v - u||.$$

Example 129 *Suppose we have $V = \mathbb{R}^3$. Let*

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, u = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Then the length of v is

$$||v - u|| = \sqrt{(1-2)^2 + (2-2)^2 + (3-2)^2} - \sqrt{1+0+1} = \sqrt{2}.$$

To compute the inner product of two vectors in $\mathbb{R}^d$ in R, we can use the sum() function. From Example 129 suppose we have $V = \mathbb{R}^3$. Let

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, u = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

First we define vectors:

```
v <- c(1, 2, 3)
u <- c(2, 2, 2)
```

Then we use the sum() function and the sqrt() functions:

```
sqrt(sum((v - u)^2))
```

This gives the length of a vector:

```
> sqrt(sum((v - u)^2))
[1] 1.414214
```

The following theorem is useful when we simplify computations including inner products.

Theorem 5.1 *Suppose u, v, w are vectors in a vector space V with an inner product and c is a scalar. Then we have*

Inner Product Space

- $\langle u, \mathbf{0} \rangle = \langle \mathbf{0}, u \rangle = \mathbf{0}$,
- $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$,
- $\langle u, c \cdot v \rangle = c \cdot \langle u, v \rangle$,
- $\langle u - v, w \rangle = \langle u, w \rangle - \langle v, w \rangle$, *and*
- $\langle u, v - w \rangle = \langle u, v \rangle - \langle u, w \rangle$.

Example 130 *Suppose we have $V = \mathbb{R}^3$. Let*

$$u = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, v = \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}, w = \begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix}.$$

Then we want to compute $\langle u - 2 \cdot v, 3 \cdot u + w \rangle$.

$$\begin{aligned} \langle u - 2 \cdot v, 3 \cdot u + w \rangle &= \langle u, 3 \cdot u + w \rangle - \langle 2 \cdot v, 3 \cdot u + w \rangle \\ &= \langle u, 3 \cdot u \rangle + \langle u, w \rangle - (\langle 2 \cdot v, 3 \cdot u \rangle + \langle 2 \cdot v, w \rangle) \\ &= 3 \cdot \langle u, u \rangle + \langle u, w \rangle - 2 \cdot (3 \cdot \langle v, u \rangle + \langle v, w \rangle) \\ &= 3 \cdot ||u||^2 + \langle u, w \rangle - 2 \cdot (3 \cdot \langle u, v \rangle + \langle v, w \rangle). \end{aligned}$$

Thus we can compute only $||u||^2$, $\langle u, w \rangle$ $\langle u, v \rangle$, $\langle v, w \rangle$. We have

$$\begin{aligned} ||u||^2 &= 1^2 + 0^2 + 2^2 = 5, \\ \langle u, w \rangle &= 1 \cdot (-1) + 0 \cdot 5 + 2 \cdot 0 = -1, \\ \langle u, v \rangle &= 1 \cdot 1 + 0 \cdot (-5) + 2 \cdot 3 = 1 + 0 + 6 = 7, \\ \langle v, w \rangle &= 1 \cdot (-1) + 5 \cdot (-5) + 3 \cdot 0 = -1 - 25 + 0 = -26. \end{aligned}$$

Thus we have

$$\begin{aligned} &\langle u - 2 \cdot v, 3 \cdot u + w \rangle \\ &= 3 \cdot ||u||^2 + \langle u, w \rangle - 2 \cdot (3 \cdot \langle u, v \rangle + \langle v, w \rangle) \\ &= 3 \cdot 5 + (-1) - 2 \cdot (3 \cdot 7 + (-26)) \\ &= 24. \end{aligned}$$

Exercises

Exercise 5.1 *Let $V = \mathbb{R}^2$. Compute the lengths for the following vectors:*

1. $$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$
2. $$\begin{bmatrix} -4 \\ 0 \end{bmatrix}$$
3. $$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
4. $$\begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$$

Exercise 5.2 *Let $V = \mathbb{R}^3$. Compute the lengths for the following vectors:*

1. $$\begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$$
2. $$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$
3. $$\begin{bmatrix} -0.1 \\ 0.2 \\ -2 \end{bmatrix}$$
4. $$\begin{bmatrix} 2 \\ -1/2 \\ 2/3 \end{bmatrix}$$

Exercise 5.5 *Suppose we have the following vectors in $\mathbb{R}^3$:*

$$u = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, v = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}, w = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, z = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}.$$

Then compute the following:

1. $\langle 2 \cdot u, 3 \cdot v \rangle$,
2. $\langle u, v + w \rangle$,
3. $\langle 3 \cdot u - 2 \cdot w, v \rangle$,
4. $\langle z + y, v - w \rangle$,
5. $\langle 2 \cdot u + 5 \cdot z, v - z \rangle - \langle u, 2 \cdot v \rangle$, *and*
6. $\langle \frac{1}{2} \cdot u + \frac{5}{2} \cdot z, z \rangle - \langle \frac{1}{2} \cdot u - \frac{3}{2} \cdot 2 \cdot v, z \rangle$.

Homework

Lab Exercise 166 *Using the sum() function or/and the sqrt() function in* `R`*, repeat Exercise 5.2.*

Lab Exercise 167 *Using the sum() function or/and the sqrt() function in* `R`*, repeat Exercise 5.3.*

Lab Exercise 168 *Using the sum() function or/and the sqrt() function in* `R`*, repeat Exercise 5.5.*

Lab Exercise 169 *Suppose we have a matrix*

$$A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 2 & 0 & -2 & -1 \end{bmatrix}.$$

1. *Compute a basis for the null space of A.*
2. *Compute a basis for the row space of A.*
3. *For each vector in a basis for the null space of A and for each vector in a basis for the row space of A, take an inner product. What do you obtain?*
4. *Repeat (1), (2), (3) for the matrix*

$$A = \begin{bmatrix} -3 & -1 & -3 & -3 & 2 \\ -2 & 2 & 1 & -3 & 1 \\ 3 & 0 & -2 & 3 & -3 \end{bmatrix}.$$

Is there any pattern of inner products of a vector in a basis for the null space of A and a vector in a basis of row space of A? If so, what did you find?