

# Assignment 1

2022-10-05

## Question 1

a)

We have  $\hat{\alpha}_i = \frac{1}{n} \sum_{j=1}^n Y_{ij}$  so

$$E[\hat{\alpha}_i] = \frac{1}{n} \sum_{j=1}^n E[Y_{ij}] = \frac{1}{n} \sum_{j=1}^n \alpha_i = \alpha_i.$$

Thus  $\hat{\alpha}_i$  is an unbiased estimator.

b)

We have

$$\begin{aligned} \text{Var}(\hat{\alpha}_i) &= \text{Var} \left( \frac{1}{n} \sum_{j=1}^n Y_{ij} \right) \\ &= \frac{1}{n^2} \sum_{j=1}^n \text{Var}(Y_{ij}) && \text{since } Y_{ij} \text{ are i.i.d} \\ &= \frac{1}{n^2} \sum_{j=1}^n [\text{Var}(\alpha_i) + \sigma^2] \\ &= \frac{\text{Var}(\alpha_i) + \sigma^2}{n}. \end{aligned}$$

If  $i = j$ , we have

$$\text{Var}(\hat{\alpha}_i - 2\hat{\alpha}_i) = \text{Var}(-\hat{\alpha}_i) = \text{Var}(\hat{\alpha}_i) = \frac{\text{Var}(\alpha_i + \sigma^2)}{n}.$$

Otherwise if  $i \neq j$ , then

$$\begin{aligned} \text{Var}(\hat{\alpha}_i - 2\hat{\alpha}_j) &= \text{Var} \left( \frac{1}{n} \sum_{j=1}^n Y_{ij} - \frac{2}{n} \sum_{i=1}^n Y_{ij} \right) \\ &= \frac{1}{n^2} \sum_{j=1}^n \text{Var}(Y_{ij}) + \frac{4}{n^2} \sum_{j=1}^n \text{Var}(Y_{ij}) \\ &= \frac{5}{n^2} \sum_{j=1}^n \text{Var}(Y_{ij}) \\ &= \frac{5(\text{Var}(\hat{\alpha}_i) + \sigma^2)}{n}. \end{aligned}$$

c)

We have

$$\begin{aligned} E[S_{\Omega} - 3S_{\omega}] &= E[S_{\Omega} - S_{\omega} - 2S_{\omega}] \\ &= E[S_{\Omega} - S_{\omega}] - 2E[S_{\omega}] \\ &= I - 1 - 2(n - 1). \end{aligned}$$

Similarly,

$$\begin{aligned} E[3S_{\Omega} - 5S_{\omega}] &= E[3(S_{\Omega} - S_{\omega}) - 2S_{\omega}] \\ &= 3E[S_{\Omega} - S_{\omega}] - 2E[S_{\omega}] \\ &= 3(I - 1) - 2(n - 1). \end{aligned}$$

## Question 2

a)

Using the  $\mu^{st} = 0$  parametrization, we have

$$\hat{\alpha}_i^{st} = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij} = \bar{Y}_{i\cdot} = \hat{\alpha}_i^{tr} + \hat{\mu}^{tr}$$

and

$$\hat{\mu}^{st} = 0.$$

b)

We have

$$\hat{\mu}^{sum} = \frac{1}{I} \sum_{i=1}^I \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij} = \frac{1}{I} \sum_{i=1}^I \hat{\alpha}_i^{tr}$$

and

$$\hat{\alpha}_i^{sum} = \bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot} = \hat{\alpha}_i^{tr} - \hat{\mu}^{sum}.$$

c)

We have

$$\hat{\mu}^{sum} = \frac{1}{I} \sum_{i=1}^I \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij} = \frac{1}{I} \sum_{i=1}^I \hat{\alpha}_i^{tr}$$

and

$$\hat{\alpha}_i^{sum} = \bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot} = \hat{\alpha}_i^{tr} - \hat{\mu}^{sum}.$$

## Question 4

a) Using the parametrization  $\mu = 0$ :

```
data("iris");

Y <- iris[order(iris$Species), "Sepal.Width"];
```

```
X <- diag(3) %% rep(1, 50);
```

```
n = 150;
```

```
I = 3;
```

Then we calculate the estimated  $\hat{\beta} = (X^T X)^{-1} X^T Y$  as

```
beta = solve((t(X) %% X)) %% t(X) %% Y;
```

The residual sum of squares  $S_\Omega$  and  $S_\omega$  of the full and reduced models respectively are

```
s1 = norm(Y - X %% beta, type="2")^2;
```

```
s2 = norm(Y - matrix(rep(1, n), ncol=1) * mean(Y), type="2")^2;
```

The unbiased estimator of  $\sigma^2$  are  $\frac{S_\Omega}{n-1} = 16.962$  and  $\frac{S_\omega}{n-1} = 28.3069333$ .

```
unb_est = s1/(n - I);
```

```
bet_ss = s2 - s1;
```

```
bet_means = (s2 - s1)/(I);
```

```
f_val = ((s2 - s1)/(I - 1))/(s2/(n - I));
```

```
within_means = s1/(n - I);
```

The quantities needed to complete an ANOVA table are :

- Between groups sum of square:  $S_\omega - S_\Omega = 11.3449333$ .
- Between groups mean square:  $\frac{(S_\omega - S_\Omega)/(I-1)}{S_\Omega/(n-I)} = 3.7816444$ .
- Within groups sum of square:  $S_\Omega = 16.962$ .
- $F$  value = 29.4575393.

#### b) We first check for the model assumptions:

The normality of residuals with expectation zero are checked using QQ-plot, Shapiro-Wilk test, and one-sample t-test because the true standard deviation is not known.

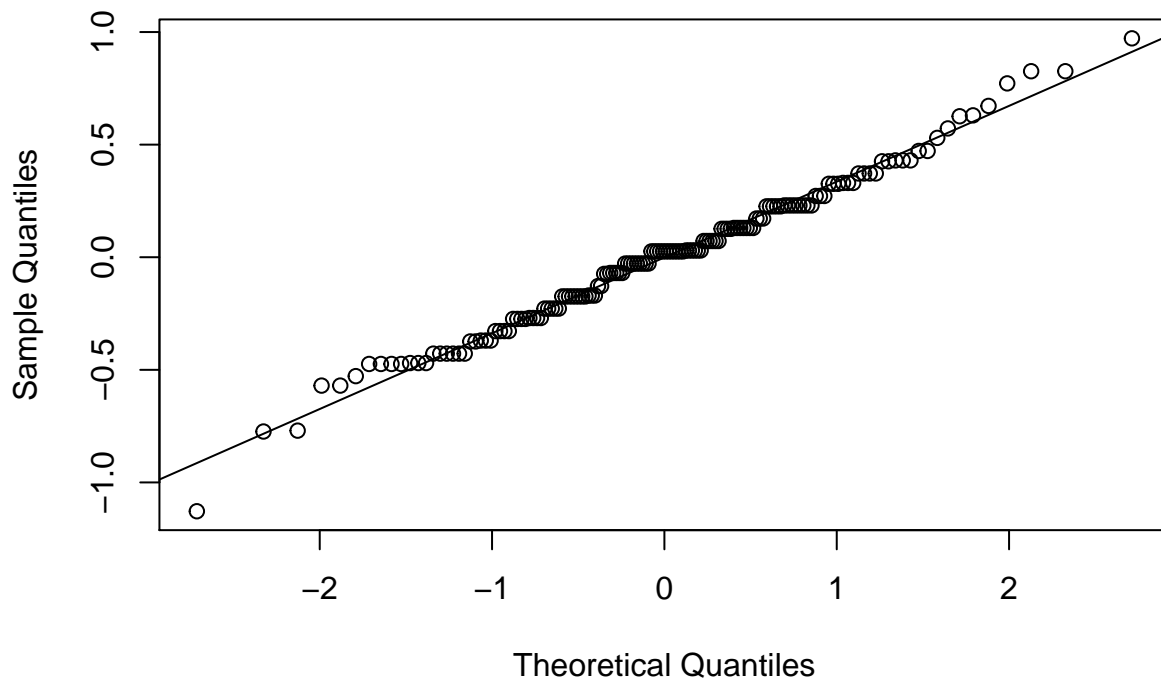
```
plot.new()
```

```
res <- Y - c(rep(beta, 1, each=50));
```

```
qqnorm(res)
```

```
qqline(res)
```

## Normal Q-Q Plot



```
shapiro.test(res)
```

```
## [...]  
##  Shapiro-Wilk normality test  
##  
## data:  res  
## W = 0.98948, p-value = 0.323
```

```
t.test(res, mu=0, alternative = "greater")
```

```
## [...]  
##  One Sample t-test  
##  
## data:  res  
## t = -1.2788e-14, df = 149, p-value = 0.5  
## alternative hypothesis: true mean is greater than 0  
## [...]
```

Since the  $p$ -values for both test are larger than 0.05, with the mean of the residuals being extremely close, we can say that the normality and zero mean assumptions hold. Next we check that  $Var(e_{ij}) = \sigma^2$  using Bartlett test.

```
species <- factor(c(rep(1, 50), rep(2, 50), rep(3, 50)),  
                  labels=c("setosa", "versicolor", "virginica"));  
data_iris <- data.frame(Y, species);  
  
bartlett.test(Y ~ species, data=data_iris)
```

```
##
## Bartlett test of homogeneity of variances
##
## data: Y by species
## Bartlett's K-squared = 2.0911, df = 2, p-value = 0.3515
```

Thus, the model assumptions hold. Now we test for the mean of iris sepal width of the three species using the produced  $F$ -statistic above.

```
pv <- pf(f_val, I - 1, n - I, lower.tail = FALSE);
```

The  $p$ -value is  $1.7447978 \times 10^{-11} < 0.05$  so we can reject the null hypothesis that the means are statistically the same.

c)

```
model <- aov(Y ~ species, data=data_iris);
summary(model)
```

```
##              Df Sum Sq Mean Sq F value Pr(>F)
## species         2  11.35    5.672   49.16 <2e-16 ***
## Residuals      147  16.96    0.115
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

It can be seen that the results from ANOVA agree with the final conclusion although some quantities are a bit off.

d)

```
kruskal.test(Y ~ species, data=data_iris);

##
## Kruskal-Wallis rank sum test
##
## data: Y by species
## Kruskal-Wallis chi-squared = 63.571, df = 2, p-value = 1.569e-14
```

Thus, the Kruskal-Wallis test agrees with our findings since its  $p$ -value is smaller than 0.05, and because we the normal distribution assumption holds, the location parameters are the means.

## Question 5

## Question 6

```
diet <- read.table("diet.txt", header = TRUE);
diet["weight.loss"] <- diet$preweight - diet$weight6weeks;
```

a) A short summary of the data is given:

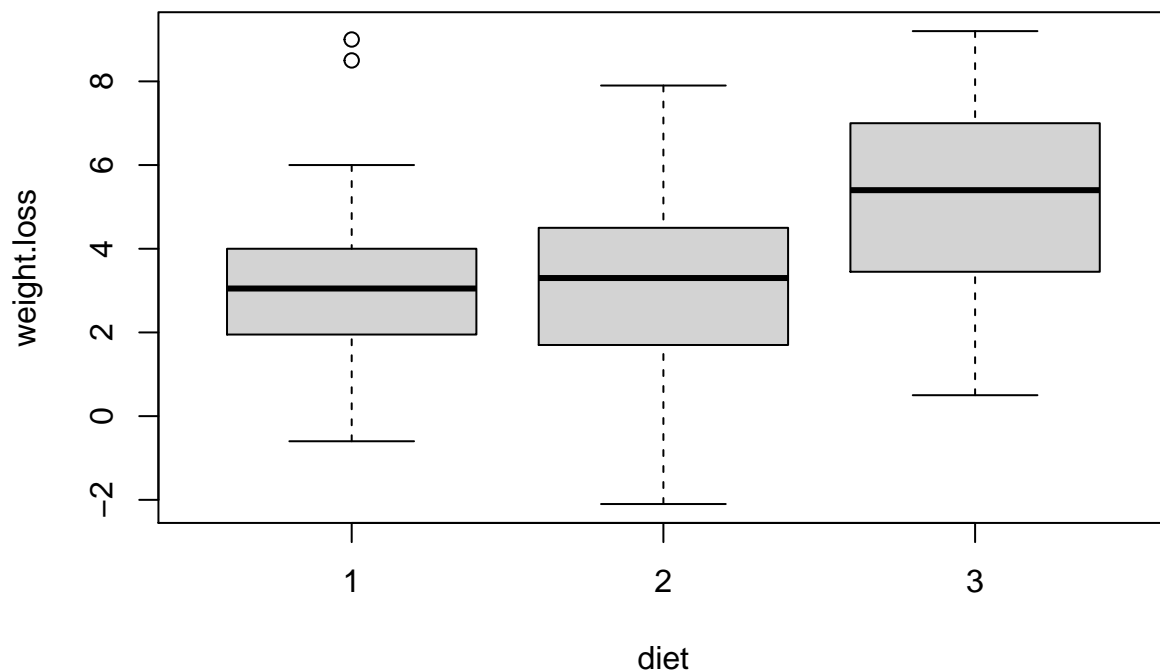
```
summary(diet);

##      person      gender      age      height
## Min.   : 1.00  Min.   :0.0000  Min.   :16.00  Min.   :141.0
## 1st Qu.:20.25  1st Qu.:0.0000  1st Qu.:32.25  1st Qu.:164.2
```

```
## Median :39.50    Median :0.0000    Median :39.00    Median :169.5
## Mean   :39.50    Mean   :0.4342    Mean   :39.15    Mean   :170.8
## 3rd Qu.:58.75    3rd Qu.:1.0000    3rd Qu.:46.75    3rd Qu.:174.8
## Max.   :78.00    Max.   :1.0000    Max.   :60.00    Max.   :201.0
##
##      preweight      diet      weight6weeks      weight.loss
## Min.   : 58.00    Min.   :1.000    Min.   : 53.00    Min.   : -2.100
## 1st Qu.: 66.00    1st Qu.:1.000    1st Qu.: 61.85    1st Qu.:  2.000
## Median : 72.00    Median :2.000    Median : 68.95    Median :  3.600
## Mean   : 72.53    Mean   :2.038    Mean   : 68.68    Mean   :  3.845
## 3rd Qu.: 78.00    3rd Qu.:3.000    3rd Qu.: 73.83    3rd Qu.:  5.550
## Max.   :103.00    Max.   :3.000    Max.   :103.00    Max.   :  9.200
##
```

To further see the effects of the diets on weight loss, we use boxplots.

```
boxplot(weight.loss ~ diet, data=diet)
```



It can be seen that there are a few outliers within the samples and it may affect our tests later thus we shall remove them.

```
Q1 <- quantile(diet$preweight, probs=c(.25, .75), na.rm = FALSE)
Q2 <- quantile(diet$weight6weeks, probs=c(.25, .75), na.rm = FALSE)

iqr_pre <- IQR(diet$preweight);
iqr_aft <- IQR(diet$weight6weeks);

diet_elim <- subset(diet, (preweight > (Q1[1] - 1.5*iqr_pre) &
```

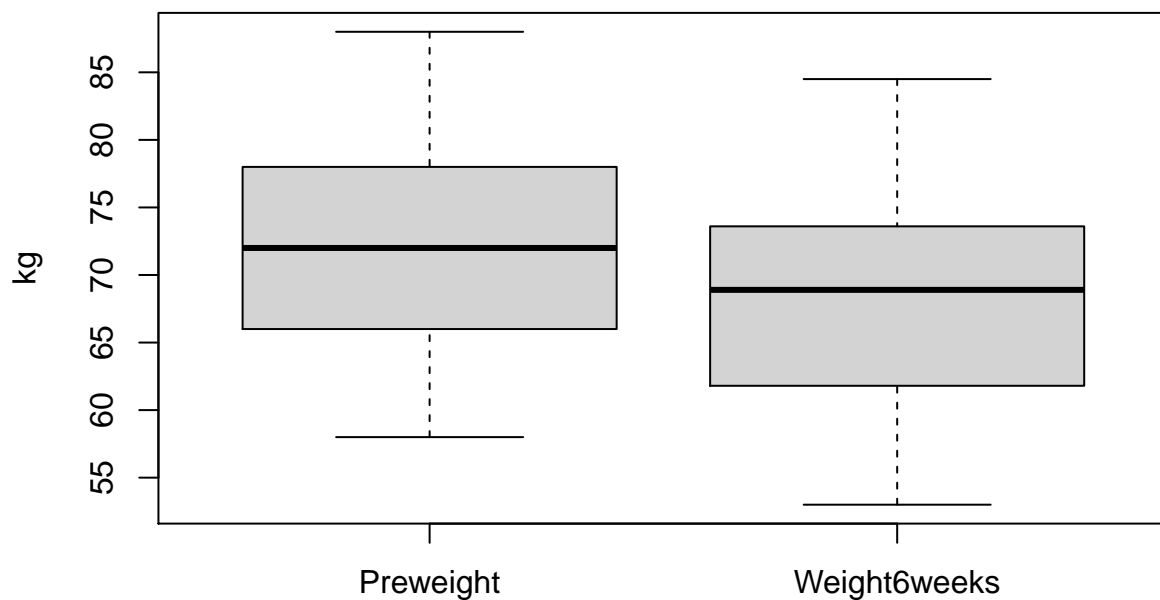
```

preweight < (Q1[2]+1.5*iqr_pre)) |
(weight6weeks < (Q2[1] - 1.5*iqr_aft) &
weight6weeks > (Q2[1] + 1.5*iqr_aft)));

boxplot(diet_elim$preweight, diet_elim$weight6weeks,
        names=c("Prewrite", "Weight6weeks"),
        ylab="kg",
        main="Boxplot of Prewrite and weight6weeks after removing outliers")

```

**Boxplot of Prewrite and weight6weeks after removing outliers**



To check whether the diets affect the weight loss, we can test for statistical difference between *preweight* and *weight6weeks*, if the diet does not affect then the mean is approximately the same and vice versa. We first use QQ-plot and Shapiro-Wilk test to check the normality of the samples.

```

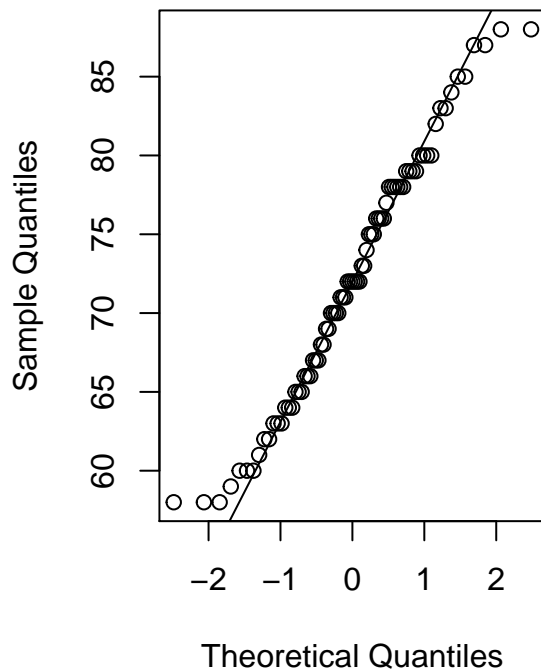
par(mfrow=(c(1,2)))

qqnorm(diet_elim$preweight, main="QQ-plot of preweight")
qqline(diet_elim$preweight)

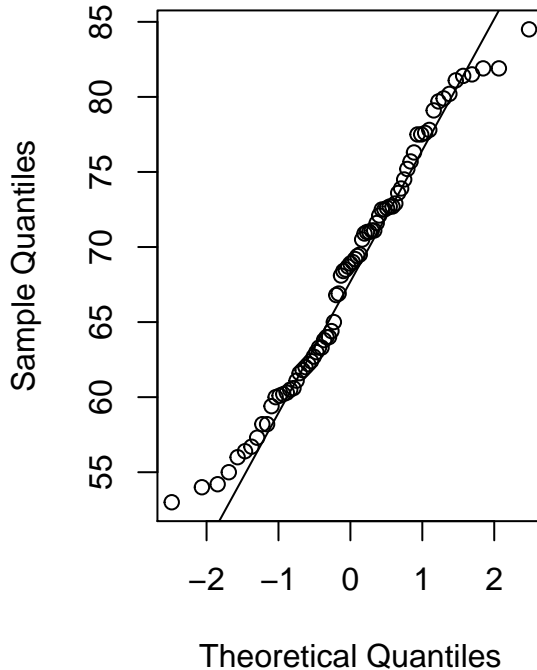
qqnorm(diet_elim$weight6weeks, main="QQ-plot of weight6weeks")
qqline(diet_elim$weight6weeks)

```

**QQ-plot of preweight**



**QQ-plot of weight6weeks**



```
shapiro.test(diet_elim$preweight)

##
##  Shapiro-Wilk normality test
##
## data:  diet_elim$preweight
## W = 0.97376, p-value = 0.1117
shapiro.test(diet_elim$weight6weeks)
```

```
##
##  Shapiro-Wilk normality test
##
## data:  diet_elim$weight6weeks
## W = 0.97222, p-value = 0.08964
```

The two  $p$ -values are higher than 0.05 thus we can safely assume that they do not significantly differ from normal distribution and the two samples  $t$ -test can be used.

```
t.test(diet_elim$preweight, diet_elim$weight6weeks)
```

```
## [...]
##  Welch Two Sample t-test
##
## data:  diet_elim$preweight and diet_elim$weight6weeks
## t = 3.0008, df = 152, p-value = 0.003148
## alternative hypothesis: true difference in means is not equal to 0
## [...]
```



Since the resulting  $p$ -value from the  $t$ -test is smaller than 0.05, we can reject the null hypothesis that their means are the same, i.e there is a statistical significant difference in the means and the diets do affect the weight loss.

b)

To check whether any type of diet has an effect on the lost weight, we use ANOVA to test the null hypothesis that across all three diets, the means of lost weights are the same.

```
an_mod <- aov(weight.loss ~ diet, data=diet_elim)
summary(an_mod)
```

```
##              Df Sum Sq Mean Sq F value    Pr(>F)
## diet           1   45.5   45.50    7.741 0.00682 **
## Residuals     75  440.8    5.88
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

The  $p$ -value is smaller than 0.05 so we can reject the null hypothesis and say that the diets have an effect on losing weight.

To check which diet is best for losing weight, we test for the their respective means to see which has the highest means i.e expected lost weight. By definition,

$$weight.loss = preweight - weight6weeks,$$

, but  $preweight$  and  $weight6weeks$  are normally distributed so we can assume that  $weight.loss$  is also normally distributed and the  $t$ -test can be used. We check if diet 3 is more effective than 1 and 2.

```
wl1 = subset(diet_elim, diet == "1");
wl2 = subset(diet_elim, diet == "2");
wl3 = subset(diet_elim, diet == "3");

t.test(wl3$weight.loss, wl1$weight.loss, alternative = "greater")

## [...]
## Welch Two Sample t-test
##
## data:  wl3$weight.loss and wl1$weight.loss
## t = 2.8462, df = 48.862, p-value = 0.003225
## alternative hypothesis: true difference in means is greater than 0

t.test(wl3$weight.loss, wl2$weight.loss, alternative = "greater")

## [...]
## Welch Two Sample t-test
##
## data:  wl3$weight.loss and wl2$weight.loss
## t = 2.9815, df = 50.672, p-value = 0.0022
## alternative hypothesis: true difference in means is greater than 0
```

Since the  $p$ -values are smaller than 0.05, we reject the null hypothesis that the means are the same so diet 3 is more effective than 1 and 2.

c)

We use two-way anova to investigate the effect of diet, gender, and their interaction on weight loss

```
tw_aov1 <- aov(weight.loss ~ diet * gender, data=diet_elim);
summary(tw_aov1)
```

```
##              Df Sum Sq Mean Sq F value    Pr(>F)
## diet          1   45.2    45.21   7.957 0.00619 **
## gender        1    0.1     0.14   0.025 0.87521
## diet:gender    1   16.5    16.47   2.898 0.09300 .
## Residuals    72  409.1     5.68
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
## 1 observation deleted due to missingness
```

The  $p$ -values for gender and interaction between diet and gender are larger than 0.05 so there are no statistical significance for their effects on weight loss as opposed to diet alone.

d)

We investigate the effect of diet and height using ANCOVA. We test the hypothesis  $H_A : \alpha_i = \dots = \alpha_I = 0$

```
anc1 <- lm(weight.loss ~ height + diet, data=diet_elim);
anova(anc1)
```

```
## Analysis of Variance Table
##
## Response: weight.loss
##              Df Sum Sq Mean Sq F value    Pr(>F)
## height        1   6.09   6.091  1.0292 0.313658
## diet          1  42.24  42.240  7.1370 0.009281 **
## Residuals    74 437.97   5.918
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

The  $p$ -value is smaller than 0.05 so we reject the null hypothesis that the diet does not affect the weight loss. Similarly, we test for  $H_\beta : \beta = 0$ .

```
anc2 <- lm(weight.loss ~ diet + height, data=diet_elim);
anova(anc2)
```

```
## Analysis of Variance Table
##
## Response: weight.loss
##              Df Sum Sq Mean Sq F value    Pr(>F)
## diet          1  45.50  45.499  7.6876 0.007031 **
## height        1   2.83   2.832  0.4786 0.491230
## Residuals    74 437.97   5.918
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

The  $p$ -value is larger than 0.49 so we can not reject the null hypothesis that the height does not have an effect. The interaction between diet and height is subsequently tested.

```
anc3 <- lm(weight.loss ~ height * diet, data=diet_elim);
anova(anc3)
```

```
## Analysis of Variance Table
##
## Response: weight.loss
##           Df Sum Sq Mean Sq F value    Pr(>F)
## height      1   6.09   6.091   1.0425 0.310602
## diet        1  42.24  42.240   7.2297 0.008879 **
## height:diet  1  11.46  11.458   1.9611 0.165629
## Residuals   73 426.51   5.843
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

The  $p$ -value for the interaction effect is larger than 0.05 so it does not bear any statistical significant effect towards weight loss. Furthermore, for the 3 types of diet, the effect of height is the same because of the hypothesis  $H_{A\beta} : \beta_i = \dots = \beta_I$  and we did not reject it.

e)

Out of two approaches, we prefer the b) one because in d), the height and the interaction effects statistically do not effect weight loss. Since diet is the only (tested) factor to have a significant effect on weight loss, we can do a simple linear regression model.

```
lm.model <- lm(weight.loss ~ diet, data=diet_elim)

summary(lm.model)
```

```
## [...]
## Coefficients:
##           Estimate Std. Error t value Pr(>|t|)
## (Intercept)   1.9667     0.7460   2.636  0.01018 *
## diet          0.9456     0.3399   2.782  0.00682 **
## [...]
```

So, based on the model, the lost weight of an average person only depend on their chosen diet and can be given as

$$lost\_weight = 1.9 + 0.9 \times diet.$$