ass3

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Question 1

a)

We have

$$
\begin{aligned}
\log f(y;r,\lambda_i) &= \log y^{r-1} + \log e^{-y/\lambda_i} - \log \lambda_i^r - \log \Gamma(x) \\
&= (r-1)\log y - \frac{y}{\lambda_i} - r\log\lambda_i - \log\Gamma(x) \\
&= (r-1)\log y - yr\theta_i - r\log\lambda_i - \log\Gamma(x) \\
&= (r-1)\log y - yr\theta_i - r\log\left((r\theta_i)^{-1}\right) - \log\Gamma(x) \\
&= (r-1)\log y - yr\theta_i + r\log(r\theta_i) - \log\Gamma(x) \\
&= (r-1)\log y - yr\theta_i + r\log(r) + r\log(\theta_i) - \log\Gamma(x) \\
&= -yr\theta_i + r\log\theta_i + r\log(r) + (r-1)\log y - \log\Gamma(x) \\
&= -r(y\theta_i - \log\theta_i) + r\log(r) + (r-1)\log y - \log\Gamma(x) \\
&= \frac{y\theta_i - \log\theta_i}{-1/r} + r\log(r) + (r-1)\log y - \log\Gamma(x) \\
&= \frac{y\theta_i - \log\theta_i}{-1/r} + \left(r\log r + (r-1)\log y - \log\Gamma(r)\right).
\end{aligned}
$$

Thus, it follows that $f(y;r,\lambda_i)$ belongs to the exponential family with $b(\theta_i) = \log\theta_i$, $\phi/A_i = -1/r$ with $\phi = 1/r$ and $A_i = -1$.

b)

We have

$$b(\theta_i) = \log\theta_i$$

so $\mathrm{E}\,Y_i = b'(\theta_i) = \frac{1}{\theta_i} = r\lambda_i = \mu_i$. Similarly,

$$\mathrm{Var}(Y_i) = b''(\theta_i)\phi/A_i = \frac{-1}{\theta_i^2}\frac{-1}{r} = \frac{1}{\theta_i^2 r} = \frac{r^2\lambda_i^2}{r} = r\lambda_i^2.$$

Furthermore, the CLF $g(\mu_i) = (b')^{-1}(\mu_i) = \frac{1}{\mu_i}$.

c)

We have the Pearson's statistic

$$
\begin{aligned}
P &= \phi \sum_{i=1}^{n} \frac{(Y_i - \mathrm{E}_{\hat{\beta}} Y_i)^2}{\mathrm{Var}_{\hat{\beta}}(Y_i)} \\
&= \frac{1}{r} \sum_{i=1}^{n} \frac{\left(Y_i - \frac{1}{\hat{\theta}_i}\right)^2}{1/(\hat{\theta}_i^2 r)} \\
&= \frac{1}{r} \sum_{i=1}^{n} \frac{\left(\frac{Y_i \hat{\theta}_i}{\hat{\theta}_i} - \frac{1}{\hat{\theta}_i}\right)^2}{1/(\hat{\theta}_i^2 r)} \\
&= \frac{1}{r} \sum_{i=1}^{n} \frac{\left(Y_i \hat{\theta}_i - 1\right)^2}{\hat{\theta}_i^2/(\hat{\theta}_i^2 r)} \\
&= \frac{1}{r} \sum_{i=1}^{n} \frac{(Y_i \hat{\theta}_i)^2 - 2Y_i \hat{\theta}_i + 1}{\hat{\theta}_i^2/(\hat{\theta}_i^2 r)} \\
&= \sum_{i=1}^{n} \left((Y_i \hat{\theta}_i)^2 - 2Y_i \hat{\theta}_i + 1\right).
\end{aligned}
$$

Furthermore, $\tilde{\theta}_i = (b')^{-1}(Y_i) = 1/Y_i$ then

$$
\begin{aligned}
D &= 2 \sum_{i=1}^{n} A_i (Y_i(\tilde{\theta}_i - \hat{\theta}_i) - b(\tilde{\theta}_i) + b(\hat{\theta}_i)) \\
&= -2 \sum_{i=1}^{n} \left(Y_i \left(\frac{1}{Y_i} - \hat{\theta}_i\right) - \log \frac{1}{Y_i} + \log \hat{\theta}_i\right) \\
&= -2 \sum_{i=1}^{n} \left(Y_i \left(\frac{1}{Y_i} - \hat{\theta}_i\right) + \log Y_i + \log \hat{\theta}_i\right) \\
&= -2 \sum_{i=1}^{n} (1 - Y_i \hat{\theta}_i + \log Y_i + \log \hat{\theta}_i).
\end{aligned}
$$

Finally, for the working matrix $\hat{W}$,

$$
\begin{aligned}
\hat{w}_{ii} &= \frac{A_i}{[g'(\mu_i)]^2 b''(\theta_i)} \\
&= \frac{1}{\frac{1}{\hat{\mu}_i^4} \frac{-1}{\hat{\theta}^2}} \\
&= \frac{1}{\hat{\theta}_i^4 \frac{-1}{\hat{\theta}^2}} \\
&= \frac{-1}{\hat{\theta}_i^2}.
\end{aligned}
$$

Question 2

a)

There are 18 observations. The estimated parameters $\hat{\beta} = (\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2) = (-0.021, 0.017, 0.01)$. Similarly, $\tilde{\phi} = \frac{D_\Omega}{n-p-1} = \frac{0.3}{18-2-1} = 0.02$. To test for $H_0 : \beta_1 = 0$ against $H_1 : \beta_1 \neq 0$, we compute $t_1 = \frac{\hat{\beta}_1}{\sqrt{(\hat{I}_F^{-1})_{11}}} =$

$\frac{0.017}{0.001} = 17$ where
$\left(\hat{I}_F^{-1}\right)_{11}$ is the square of the standard error of the first variable. Since the test statistic $t_1 = 17$ is larger than $t_{15;1-0.05/2} = 2.13$, we reject H_0. The 96% confidence interval for β_1 is

$$\hat{\beta}_1 \pm t_{15;1-0.04/2}\sqrt{\left(\hat{I}_F^{-1}\right)_{11}} = 0.017 \pm 2.25 \times 0.01 = [-0.0055, 0.0395].$$

b)

To investigate the relevance of the variables $X1$ and $X2$, it suffices to look at the corresponding p-values in the output table of the command *summary*(*model*). This shows that the p-value of $X2$ is smaller than 0.05 so we can not dismiss the relevance of $X2$. Similarly, we test for the relevance of $X1$ in a) and rejected the null hypothesis that $\beta_1 = 0$ so $X1$ is relevant.

c)

If not given, the standard errors in the first table could have been recovered as the square root of the variance of the sample. If the estimates are given, then the t-values could be (approximately) recovered by the formula

$$t_i = \frac{\hat{\beta}_i}{\sqrt{\left(\hat{I}_F^{-1}\right)_{ii}}}$$

where $\sqrt{\left(\hat{I}_F^{-1}\right)_{11}}$ is the standard error of the i predictor. It follows that the p-values could be recovered from the test statistic and the degree of freedom. Similarly, if the t-values are given, the estimates could be recovered.

For the second table, the degree of freedom can be recovered from the number of observations and the number of predictors. If the residuals deviance are given then we could calculate the deviances and the p-values since the test statistic will also be known. On the other hand, we are given the residual deviances for the variables but not the NULL one, then we could not calculate the deviance of the first variable since it depends on the residual deviance of the NULL one. Furthermore, if the deviances are known then we could not compute the residual one because the system of equation will be underdetermined. Also, by the result of Exercise 1, the residual deviance D of the models can be determined by the responses Y_i and the fitted canonical parameters $\hat{\theta}_i$ but since $\hat{\theta}_i = x_i^T\hat{\beta}$, we could recover $\hat{\theta}_i$ and therefore, the deviances of the models. And if the deviances are known, everything in the second table can be recovered.

d)

We use that option because the dispersion parameter ϕ is not known beforehand since the family is Poisson which does not have the dispersion parameter. It is not possible to derive what we would have obtained if we used the Chi-square test option because the Chisquare test assumes that ϕ is known and the test statistics use ϕ instead of the estimated one in the F test.

e)