Assignment 4

2022-12-09

Question 1

a) We have $\mathrm{E}(X_t) = \mathrm{E}(Y - 2Z_{t-1} + Z_t)$ and since Y is independent of $\{Z_t\}$, it follows that

$$\mathrm{E}(X_t) = \mathrm{E}(Y - 2Z_{t-1} + Z_t) = \mathrm{E}(Y) - 2\,\mathrm{E}(Z_{t-1} + \mathrm{E}(Z_t)) = 1.$$

Furthermore,

$$\mathrm{E}(X_t X_{t+h}) = \mathrm{E}((Y - 2Z_{t-1} + Z_t)(Y - 2Z_{t+h-1} + Z_{t+h})) = \mathrm{E}(Y^2) = \mathrm{Var}(Y) - \mathrm{E}(Y)^2 = 0$$

because in the expansion, any term with Z_{t+k} will have 0 expectation. Thus, $\{X_t\}$ is stationary. We can see that $\{Y_t\}$ is not stationary because X_t depends on t and thus, $\mathrm{E}(Y_t)$ depends on t.

b) We have $\nabla_d(W_t + 1) = W_t + 1 - W_{t-d} - 1 = W_t - W_{t-d}$ so it follows that

$$\begin{aligned}
\nabla_d^2(W_t + 1) &= \nabla_d(W_t - W_{t-d}) \\
&= W_t - W_{t-d} - W_{t-d} + W_{t-2d} \\
&= W_t - 2W_{t-d} + W_{t-2d} \\
&= at^2 + bts_t + X_t - 2(a(t-d)^2 + b(t-d)s_t + X_{t-d}) + a(t-2d)^2 + b(t-2d)s_t + X_{t-2d} \\
&= a(t^2 - 2t^2 + 4dt - 2d^2 + t^2 - 4dt + 4d^2) + bs_t(t - 2t + 2d + t - 2d) + X_t - 2X_{t-d} + X_{t-2d} \\
&= ad^2 + bs_t0 + X_t - 2X_{t-d} + X_{t-2d} = ad^2 + X_t - 2X_{t-d} + X_{t-2d}.
\end{aligned}$$

If $\{X_t\}$ is stationary then $Y_t = ad^2 + (X_t - X_{t-d}) - (X_{t-d} - X_{t-2d})$ is also stationary because $\{(X_t - X_{t-d})\}$ and $\{(X_{t-d} - X_{t-2d})\}$ are stationary.

Question 2

a) Note that $aY_{t-1} = 2aX_{t-1} - aX_{t-2}$ so

$$\begin{aligned}
aY_t &= 2aX_t - aX_{t-1} \\
&= 2a(0.2X_{t-1} + Z_t - Z_{t-1}) - a(0.2X_{t-2} + Z_{t-1} - Z_{t-2}) \\
&= 0.4aX_{t-1} - 0.2aX_{t-2} - aZ_{t-1} + aZ_{t-2} + 2aZ_t \\
&= \frac{1}{5}(2aX_{t-1} - aX_{t-2}) - aZ_{t-1} + aZ_{t-2} + 2aZ_t \\
&= \frac{1}{5}aY_{t-1} - aZ_{t-1} + aZ_{t-2} + 2aZ_t.
\end{aligned}$$

Let $a = 1/2$, we have

$$aY_t = \frac{1}{5}aY_{t-1} - aZ_{t-1} + aZ_{t-2} + Z_t$$

so $\{aY_t\}$ follows an ARMA(p, q) model for $a = 1/2$, $\alpha_1 = 1/5$, $\beta_1 = \beta_2 = a$, $p = 1$, and $q = 2$.

b) We have

$$\begin{aligned}\operatorname{Cov}(X_t, Z_t) &= \mathrm{E}[(X_t - \mathrm{E}(X_t))(Z_t - \mathrm{E}(Z_t))] \\ &= \mathrm{E}[(X_t - \mathrm{E}(X_t))Z_t] \\ &= \mathrm{E}(X_t Z_t) - \mathrm{E}[Z_t\,\mathrm{E}(X_t)].\end{aligned}$$

Assuming that $\{X_t\}$ is stationary,

$$\operatorname{Cov}(X_t, Z_t) = \mathrm{E}(X_t Z_t) - \mathrm{E}[Z_t\,\mathrm{E}(X_t)] = \mathrm{E}(X_t Z_t) - \mathrm{E}(Z_t)\,\mathrm{E}(X_t)$$

so

Question 3

a) The autocovariance function

$$\gamma_Z(h) = \begin{cases} \sigma^2 & \text{for } h = 0 \\ 0 & \text{for } h \neq 0 \end{cases}$$

then the spectral density

$$f_Z(\lambda) = \frac{1}{2\pi} \sum_{h \in \mathbb{Z}} \gamma_Z(h) e^{-ih\lambda} = \frac{1}{2\pi}\sigma^2.$$

b) $\{X_t\}$ is an MA(q) time series by introducing new WN $Z_t' = Z_{t+1}$ We have

$$\mathrm{E}(X_t) = \mathrm{E}\left(\frac{1}{2}Z_{t+1} + Z_t - \frac{1}{2}Z_{t-1}\right) = \frac{1}{2}\,\mathrm{E}(Z_{t+1}) + \mathrm{E}(Z_t) - \frac{1}{2}\,\mathrm{E}(Z_{t-1}) = \sigma^2$$

not dependent on t. Doing the same thing for $\mathrm{E}(X_t X_{t+h})$ to see that the expansion only contains Z_t terms so it is also independent of t. Thus, $\{X_t\}$ is stationary.

The autocovariance function

$$\gamma_X(h) = \begin{cases} \sigma^2 \sum_{i=0}^{2-h} \beta_i \beta_{i+h} & h = 0, 1, \dots, 2, \\ \gamma_X(-h) & h = -1, \dots, -2, \\ 0, & \text{otherwise} \end{cases}$$

and the spectral density

$$
\begin{aligned}
f_X(\lambda) &= \frac{1}{2\pi}\sum_{h\in\mathbb{Z}}\gamma_X(h)e^{-ih\lambda} \\
&= \frac{1}{2\pi}\sum_{h=-2}^{2}\gamma_X(h)e^{-ih\lambda} \\
&= \frac{1}{2\pi}\left(\sum_{h=-2}^{-1}\gamma_X(h)e^{-ih\lambda}+\sum_{h=0}^{2}\gamma_X(h)e^{-ih\lambda}\right) \\
&= \frac{1}{2\pi}\left(\gamma_X(0)+2\sum_{h=1}^{2}\gamma_X(h)e^{-ih\lambda}\right) \\
&= \frac{1}{2\pi}\left(\sigma^2\sum_{i=0}^{2}\beta_i^2+2\sum_{h=1}^{2}\gamma_X(h)e^{-ih\lambda}\right) \\
&= \frac{1}{2\pi}\left(\frac{\sigma^2}{2}+2\gamma_X(1)e^{-i\lambda}+2\gamma_X(2)e^{-i2\lambda}\right) \\
&= \frac{1}{2\pi}\left(\frac{\sigma^2}{2}+2\sigma^2(\beta_0\beta_1+\beta_1\beta_2)e^{-i\lambda}+2\sigma^2(\beta_0\beta_2)e^{-i2\lambda}\right) \\
&= \frac{1}{2\pi}\left(\frac{\sigma^2}{2}+2\sigma^2(0)e^{-i\lambda}+-\frac{\sigma^2}{2}e^{-i2\lambda}\right) \\
&= \frac{1}{2\pi}\left(\frac{\sigma^2}{2}+-\frac{\sigma^2}{2}e^{-i2\lambda}\right)
\end{aligned}
$$

c) $\{X_t\}$ is a linear transformation of $\{Z_t\}$ because $X_t = \frac{1}{2}Z_t^{'} + Z_{t-1}^{'} - \frac{1}{2}Z_{t-2}^{'}$ which is a linear combination of $\{Z_t^{'}\}$ by letting $Z_t^{'} = Z_{t+1}$. The corresponding filter coefficients $\psi_0 = 1/2$, $\psi_1 = 1$, $\psi_2 = -1/2$, and $\psi_k = 0$ for $k \notin \{0, 1, 2\}$. The transfer function

$$\psi(\lambda) = \sum_j \psi_j e^{-ij\lambda} = \frac{1}{2} + e^{-i\lambda} - \frac{1}{2}e^{-2\lambda i}.$$

Question 4

a)