Here is derivation of equations of motion for the roller coaster with hump. This derivation is using Lagrangian. The equation of the hump is
$y(x) = 3 - \frac{7}{6} x^2 + \frac{1}{6} x^4$

Kinetic energy = $T = \frac{1}{2} m v^2 = \frac{1}{2} m\left(\frac{ds}{dt}\right)^2$

Describe the curve parametrically as $x(\theta)$, $y(\theta)$.
In this case we have $x = \theta$.

Distance = $s(\tau) = \int_0^\tau \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

By Fundamental theorem of calculus, $\frac{ds}{d\theta}$ =the integrand above
(or does this only work because $\theta = x$?).

$\frac{ds}{dt} = \frac{ds}{d\theta}\frac{d\theta}{dt} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\ \frac{d\theta}{dt}$

This works out to

$\frac{ds}{dt} = \sqrt{1 + \frac{49\theta^2}{9} - \frac{28\theta^4}{9} + \frac{4\theta^6}{9}}\ \frac{d\theta}{dt}$

Let $\omega = \frac{d\theta}{dt}$

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T = (m / 2) (1 / 9) (9 + 49 θ^2 - 28 θ^4 + 4 θ^6) ω^2
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$\frac{1}{18} m \left(9 + 49 \theta^2 - 28 \theta^4 + 4 \theta^6\right) \omega^2$

Potential energy = $V = m g y$

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V = m g (1 / 6) (18 - 7 θ^2 + θ^4)
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$\frac{1}{6} g m \left(18 - 7 \theta^2 + \theta^4\right)$

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L = T - V
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$-\frac{1}{6} g m \left(18 - 7 \theta^2 + \theta^4\right) + \frac{1}{18} m \left(9 + 49 \theta^2 - 28 \theta^4 + 4 \theta^6\right) \omega^2$

```
D[L, θ]
```

$-\frac{1}{6} g m \left(-14 \theta + 4 \theta^3\right) + \frac{1}{18} m \left(98 \theta - 112 \theta^3 + 24 \theta^5\right) \omega^2$

```
D[L, ω]
```

$\frac{1}{9} m \left(9 + 49 \theta^2 - 28 \theta^4 + 4 \theta^6\right) \omega$

```
θ /: Dt[θ] = ω
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ω

```
SetAttributes[{m, g}, Constant]
```

```
Dt[D[L, ω]] // Simplify
```

$\frac{1}{9} m \left(2 \theta \left(49 - 56 \theta^2 + 12 \theta^4\right) \omega^2 + \left(9 + 49 \theta^2 - 28 \theta^4 + 4 \theta^6\right) \mathrm{Dt}[\omega]\right)$

```
eqn1 = Dt[D[L, ω]] - D[L, θ] /. Dt[ω] → W
```

$$\frac{1}{6} g m \left(-14 \theta + 4 \theta^3\right) + \frac{1}{9} m W \left(9 + 49 \theta^2 - 28 \theta^4 + 4 \theta^6\right) - \frac{1}{18} m \left(98 \theta - 112 \theta^3 + 24 \theta^5\right) \omega^2 + \frac{1}{9} m \omega \left(98 \theta \omega - 112 \theta^3 \omega + 24 \theta^5 \omega\right)$$

```
soln = Solve[eqn1 == 0, {W}] // Simplify
```

$$\left\{\left\{W \to -\frac{\theta \left(-7 + 2 \theta^2\right) \left(3 g + \left(-7 + 6 \theta^2\right) \omega^2\right)}{9 + 49 \theta^2 - 28 \theta^4 + 4 \theta^6}\right\}\right\}$$

```
soln[[1]][[1]] // CForm
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```
Rule(W,-((θ*(-7 + 2*Power(θ,2))*
        (3*g +
          (-7 + 6*Power(θ,2))*
           Power(ω,2)))/
      (9 + 49*Power(θ,2) -
        28*Power(θ,4) + 4*Power(θ,6))
      ))
```

Next I'm trying out a formula I developed for a general Lagrangian of a general roller coaster curve. I'll explain below....

```
y[θ_] := 3 - (7 / 6) θ^2 + (1 / 6) θ^4
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```
y'[θ]
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$$-\frac{7 \theta}{3} + \frac{2 \theta^3}{3}$$

```
(-g y'[θ] - ω^2 (y'[θ] y''[θ])) / (1 + (y'[θ])^2) // Simplify
```

$$-\frac{\theta \left(-7 + 2 \theta^2\right) \left(3 g + \left(-7 + 6 \theta^2\right) \omega^2\right)}{9 + 49 \theta^2 - 28 \theta^4 + 4 \theta^6}$$

Kinetic energy = $T = \frac{1}{2} m v^2 = \frac{1}{2} m\left(\frac{d s}{d t}\right)^2$

Describe the curve parametrically as $x(\theta)$, $y(\theta)$.

In this case we have $x = \theta$.

Distance = $s(\tau) = \int_0^\tau \sqrt{\left(\frac{d x}{d \theta}\right)^2 + \left(\frac{d y}{d \theta}\right)^2}\, d\theta$

By Fundamental theorem of calculus, $\frac{d s}{d \theta}$ =the integrand above

(or does this only work because $\theta = x$?).

$\frac{d s}{d t} = \frac{d s}{d \theta} \frac{d \theta}{d t} = \sqrt{\left(\frac{d x}{d \theta}\right)^2 + \left(\frac{d y}{d \theta}\right)^2}\ \dot{\theta}$

$V = m g y(\theta)$ potential energy

$L = T - V = \frac{m}{2}\left(\left(\frac{d x}{d \theta}\right)^2 + \left(\frac{d y}{d \theta}\right)^2\right)\dot{\theta}^2 - m g y(\theta)$

$\frac{\partial L}{\partial \theta} = \frac{m}{2}\left(2 \frac{d x}{d \theta} \frac{d^2 x}{d \theta^2} + 2 \frac{d y}{d \theta} \frac{d^2 y}{d \theta^2}\right)\dot{\theta}^2 - m g \frac{d y}{d \theta}$

$\frac{\partial L}{\partial \dot{\theta}} = \frac{m}{2}\left(\left(\frac{d x}{d \theta}\right)^2 + \left(\frac{d y}{d \theta}\right)^2\right) 2 \dot{\theta}$

note on next line: $\frac{d}{d t}\{X\} = \frac{d \theta}{d t} \frac{d}{d \theta}\{X\}$

$\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\left(\left(\frac{d x}{d \theta}\right)^2 + \left(\frac{d y}{d \theta}\right)^2\right)\ddot{\theta} + m \dot{\theta}\left(2 \frac{d x}{d \theta} \frac{d^2 x}{d \theta^2} \dot{\theta} + 2 \frac{d y}{d \theta} \frac{d^2 y}{d \theta^2} \dot{\theta}\right)$

Let W_θ be the work done by a tiny movement in θ.

$$W_\theta = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = m\left(\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right)\ddot{\theta} + m\,\dot{\theta}^2\left(\frac{dx}{d\theta}\,\frac{d^2x}{d\theta^2} + \frac{dy}{d\theta}\,\frac{d^2y}{d\theta^2}\right) + m\,g\,\frac{dy}{d\theta}$$

Figure work done by damping:

$$W_\theta = -b\,\frac{ds}{dt} = -b\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\;\dot{\theta}$$

Equate the above two expressions and solve for $\ddot{\theta}$

$$\ddot{\theta} = -\frac{m\,\dot{\theta}^2\left(\frac{dx}{d\theta}\,\frac{d^2x}{d\theta^2} + \frac{dy}{d\theta}\,\frac{d^2y}{d\theta^2}\right) + m\,g\,\frac{dy}{d\theta} + b\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\;\dot{\theta}}{m\left(\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right)}$$

That should work for any parmetrized curve!