Rigid Double Pendulum Algebra

This document calculates the equations of motion for the Rigid Double Pendulum as described in the accompanying paper *Double Pendulum as Rigid Bodies* by Erik Neumann, April 2, 2011.

We start with the equations given in that paper in sections 2 and 3. Substitute the four equations (1) thru (4) into the six equations (5) thru (10) to eliminate the unknowns x1'', y1'', x2'', y2''. This gives us modified versions of the six equations (5) thru (10) with six unknowns: T1x, T2x, T1y, T2y, θ1'', θ2''. We then solve for θ1'', θ2'' and eliminate the other four unknowns.

Naming conventions: here the letter d indicates derivative (with respect to time), dd indicates second derivative. For example:
x1' = dx1
x1'' = ddx1
θ1' = dθ1
θ1'' = ddθ1

```
Clear["Global`*"]
```

```
ddx1 = -dθ1^2 R1 Sin[θ1] + ddθ1 R1 Cos[θ1 ]
```

```
ddθ1 R1 Cos[θ1] - dθ1^2 R1 Sin[θ1]
```

```
ddy1 = dθ1^2 R1 Cos[θ1] + ddθ1 R1 Sin[θ1]
```

```
dθ1^2 R1 Cos[θ1] + ddθ1 R1 Sin[θ1]
```

```
ddx2 = -dθ1^2 L1 Sin[θ1 + ϕ] + ddθ1 L1 Cos[θ1 + ϕ] - dθ2^2 R2 Sin[θ2] + ddθ2 R2 Cos[θ2]
```

```
ddθ2 R2 Cos[θ2] + ddθ1 L1 Cos[θ1 + ϕ] - dθ2^2 R2 Sin[θ2] - dθ1^2 L1 Sin[θ1 + ϕ]
```

```
ddy2 = dθ1^2 L1 Cos[θ1 + ϕ] + ddθ1 L1 Sin[θ1 + ϕ] + dθ2^2 R2 Cos[θ2] + ddθ2 R2 Sin[θ2]
```

```
dθ2^2 R2 Cos[θ2] + dθ1^2 L1 Cos[θ1 + ϕ] + ddθ2 R2 Sin[θ2] + ddθ1 L1 Sin[θ1 + ϕ]
```

```
eq5 = m1 ddx1 == T1x + T2x
```

```
ddx1 m1 == T1x + T2x
```

```
eq6 = m1 ddy1 == T1y + T2y - m1 g
```

```
ddy1 m1 == -g m1 + T1y + T2y
```

```
eq7 = I1 ddθ1 == -(R1 Sin[θ1] T1y + R1 Cos[θ1] T1x) +
    (L1 Sin[θ1 + ϕ] - R1 Sin[θ1]) T2y + (L1 Cos[θ1 + ϕ] - R1 Cos[θ1]) T2x
```

```
ddθ1 I1 == -R1 T1x Cos[θ1] + T2x (-R1 Cos[θ1] + L1 Cos[θ1 + ϕ]) -
  R1 T1y Sin[θ1] + T2y (-R1 Sin[θ1] + L1 Sin[θ1 + ϕ])
```

```
eq8 = m2 ddx2 == -T2x
```

```
m2 (ddθ2 R2 Cos[θ2] + ddθ1 L1 Cos[θ1 + ϕ] - dθ2^2 R2 Sin[θ2] - dθ1^2 L1 Sin[θ1 + ϕ]) == -T2x
```

```
eq9 = m2 ddy2 == -T2y - m2 g
```

```
m2 (dθ2^2 R2 Cos[θ2] + dθ1^2 L1 Cos[θ1 + ϕ] + ddθ2 R2 Sin[θ2] + ddθ1 L1 Sin[θ1 + ϕ]) == -g m2 - T2y
```

```
eq10 = I2 ddθ2 == R2 Sin[θ2] T2y + R2 Cos[θ2] T2x
```

```
ddθ2 I2 == R2 T2x Cos[θ2] + R2 T2y Sin[θ2]
```

Solve for θ_1'' and θ_2'', eliminating the unknown forces T_{1x}, T_{1y}, T_{2x}, T_{2y}. (Note: //InputForm and //CForm is useful for copying out the result).

```
soln = Solve[{eq5, eq6, eq7, eq8, eq9, eq10}, {ddθ1, ddθ2}, {T1x, T1y, T2x, T2y}] // Simplify
```

$$\{\{dd\theta1 \to -\left(2\,g\,m1\,R1\,\left(I2+m2\,R2^2\right)\,Sin[\theta1]+L1\,m2\,\left(g\,\left(2\,I2+m2\,R2^2\right)\,Sin[\theta1+\phi]+R2\,\left(g\,m2\,R2\,Sin[\theta1-2\,\theta2+\phi]+2\,\left(d\theta2^2\,\left(I2+m2\,R2^2\right)+d\theta1^2\,L1\,m2\,R2\,Cos[\theta1-\theta2+\phi]\right)\,Sin[\theta1-\theta2+\phi]\right)\right)\right)\Big/\left(2\,I2\,L1^2\,m2+2\,I2\,m1\,R1^2+L1^2\,m2^2\,R2^2+2\,m1\,m2\,R1^2\,R2^2+2\,I1\,\left(I2+m2\,R2^2\right)-L1^2\,m2^2\,R2^2\,Cos[2\,(\theta1-\theta2+\phi)]\right),$$

$$dd\theta2 \to \left(m2\,R2\,\left(-g\,\left(2\,I1+L1^2\,m2+2\,m1\,R1^2\right)\,Sin[\theta2]+L1\,\left(g\,m1\,R1\,Sin[\theta2-\phi]+2\,d\theta1^2\,\left(I1+L1^2\,m2+m1\,R1^2\right)\,Sin[\theta1-\theta2+\phi]+d\theta2^2\,L1\,m2\,R2\,Sin[2\,(\theta1-\theta2+\phi)]+g\,m1\,R1\,Sin[2\,\theta1-\theta2+\phi]+g\,L1\,m2\,Sin[2\,\theta1-\theta2+2\,\phi]\right)\right)\right)\Big/\left(2\,I2\,L1^2\,m2+2\,I2\,m1\,R1^2+L1^2\,m2^2\,R2^2+2\,m1\,m2\,R1^2\,R2^2+2\,I1\,\left(I2+m2\,R2^2\right)-L1^2\,m2^2\,R2^2\,Cos[2\,(\theta1-\theta2+\phi)]\right)\}\}$$

To get a LaTeX version of this equation, I started with the InputForm, then did a bunch of regexp replacements.

```
soln // InputForm
```

```
{{ddθ1 -> -((2*g*m1*R1*(I2 + m2*R2^2)*Sin[θ1] + L1*m2*(g*(2*I2 + m2*R2^2)*Sin[θ1 + ϕ] + 
        R2*(g*m2*R2*Sin[θ1 - 2*θ2 + ϕ] + 2*(dθ2^2*(I2 + m2*R2^2) + dθ1^2*L1*m2*R2*
             Cos[θ1 - θ2 + ϕ])*Sin[θ1 - θ2 + ϕ])))/(2*I2*L1^2*m2 + 2*I2*m1*R1^2 + 
      L1^2*m2^2*R2^2 + 2*m1*m2*R1^2*R2^2 + 2*I1*(I2 + m2*R2^2) - 
      L1^2*m2^2*R2^2*Cos[2*(θ1 - θ2 + ϕ)])), 
  ddθ2 -> (m2*R2*(-(g*(2*I1 + L1^2*m2 + 2*m1*R1^2)*Sin[θ2]) + 
      L1*(g*m1*R1*Sin[θ2 - ϕ] + 2*dθ1^2*(I1 + L1^2*m2 + m1*R1^2)*Sin[θ1 - θ2 + ϕ] + 
        dθ2^2*L1*m2*R2*Sin[2*(θ1 - θ2 + ϕ)] + g*m1*R1*Sin[2*θ1 - θ2 + ϕ] + 
        g*L1*m2*Sin[2*θ1 - θ2 + 2*ϕ])))/(2*I2*L1^2*m2 + 2*I2*m1*R1^2 + L1^2*m2^2*R2^2 + 
     2*m1*m2*R1^2*R2^2 + 2*I1*(I2 + m2*R2^2) - L1^2*m2^2*R2^2*Cos[2*(θ1 - θ2 + ϕ)])}}
```

Change how CForm shows x^2 to be x*x instead of Power(x, 2). This is close to the form needed for Java.

```
Unprotect[Power];

Format[x_^2, CForm] := Format[StringForm["``*``", CForm[x], CForm[x]], OutputForm]

soln[[1]][[1]] // CForm
```

```
Rule(ddθ1,-((2*g*m1*R1*(I2 + m2*R2*R2)*Sin(θ1) + 
        L1*m2*(g*(2*I2 + m2*R2*R2)*Sin(θ1 + ϕ) + 
           R2*(g*m2*R2*Sin(θ1 - 2*θ2 + ϕ) + 
              2*(dθ2*dθ2*(I2 + m2*R2*R2) + dθ1*dθ1*L1*m2*R2*Cos(θ1 - θ2 + ϕ))*Sin(θ1 - θ2 + ϕ)))),
      (2*I2*L1*L1*m2 + 2*I2*m1*R1*R1 + L1*L1*m2*m2*R2*R2 + 2*m1*m2*R1*R1*R2*R2 + 
        2*I1*(I2 + m2*R2*R2) - L1*L1*m2*m2*R2*R2*Cos(2*(θ1 - θ2 + ϕ)))))
```

```
soln[[1]][[2]] // CForm
```

```
Rule(ddθ2,(m2*R2*(-(g*(2*I1 + L1*L1*m2 + 2*m1*R1*R1)*Sin(θ2)) + 
        L1*(g*m1*R1*Sin(θ2 - ϕ) + 2*dθ1*dθ1*(I1 + L1*L1*m2 + m1*R1*R1)*Sin(θ1 - θ2 + ϕ) + 
           dθ2*dθ2*L1*m2*R2*Sin(2*(θ1 - θ2 + ϕ)) + g*m1*R1*Sin(2*θ1 - θ2 + ϕ) + 
           g*L1*m2*Sin(2*θ1 - θ2 + 2*ϕ))))/
    (2*I2*L1*L1*m2 + 2*I2*m1*R1*R1 + L1*L1*m2*m2*R2*R2 + 2*m1*m2*R1*R1*R2*R2 + 2*I1*(I2 + m2*R2*R:
      L1*L1*m2*m2*R2*R2*Cos(2*(θ1 - θ2 + ϕ))))
```

In the ideal double pendulum there are two point masses at the end of each pendulum. This corresponds to setting L1=R1 and having rotational inertia be zero for both pendulums. Try substituting these values into the above equation.

```
soln /. {L1 → R1, ϕ → 0, I1 → 0, I2 → 0} // Simplify
```

$$\{\{dd\theta1 \to -\left(g\,(2\,m1+m2)\,Sin[\theta1]+m2\,\left(g\,Sin[\theta1-2\,\theta2]+2\,\left(d\theta2^2\,R2+d\theta1^2\,R1\,Cos[\theta1-\theta2]\right)\,Sin[\theta1-\theta2]\right)\right)\Big/\left(R1\,(2\,m1+m2-m2\,Cos[2\,(\theta1-\theta2)])\right),$$

$$dd\theta2 \to \frac{2\,\left(d\theta1^2\,(m1+m2)\,R1+g\,(m1+m2)\,Cos[\theta1]+d\theta2^2\,m2\,R2\,Cos[\theta1-\theta2]\right)\,Sin[\theta1-\theta2]}{R2\,(2\,m1+m2-m2\,Cos[2\,(\theta1-\theta2)])}\}\}$$

The above matches the equation for the ideal double pendulum.