Path Length of a Parabola

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We seek an expression for the path length of a parabola $y = x^2$ starting from $x = -1$.

The length of a plane curve is given by

$$L_a^b = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \tag{1}$$

For the parabola, we have:

$$\frac{dy}{dx} = 2x \tag{2}$$

and so we can write the path length starting at $x = -1$ as a function $p(x)$:

$$p(x) = \int_{-1}^{x} \sqrt{1 + (2t)^2} dt \tag{3}$$

Make a substitution $u = 2t$ and $du = 2dt$. The limits of integration also change accordingly.

$$p(x) = \int_{-2}^{2x} \sqrt{1 + u^2} \frac{du}{2} \tag{4}$$

$$p(x) = \frac{1}{2} \int_{-2}^{2x} \sqrt{1 + u^2} du \tag{5}$$

This is a standard integral for which we can find the solution for in a table of integrals.

$$p(x) = \frac{1}{2} \left(\frac{u}{2} \sqrt{1 + u^2} + \frac{1}{2} sinh^{-1} u \right) \Bigg]_{u=-2}^{u=2x} \tag{6}$$

$$p(x) = \frac{1}{4} \left(u \sqrt{1 + u^2} + sinh^{-1} u \right) \Bigg]_{u=-2}^{u=2x} \tag{7}$$

We can write this out as

$$p(x) = \frac{1}{4} \Big((2x \sqrt{1 + 4x^2} + sinh^{-1}(2x)) - (-2\sqrt{1 + (-2)^2} + sinh^{-1}(-2)) \Big) \tag{8}$$

$$p(x) = \frac{1}{4}\left(2x\sqrt{1+4x^2} + sinh^{-1}(2x) + 2\sqrt{5} - sinh^{-1}(-2)\right) \tag{9}$$

for $x >= -1$.

Note that $p(-1) = 0$ as required for the path length.

For reference, here is a formula for finding hyperbolic inverse sine using logarithms, which is valid for any real u:

$$sinh^{-1}(u) = ln(u + \sqrt{1+u^2}) \tag{10}$$

To find radius of curvature: it is inverse of κ given by:

$$\kappa = \frac{|\frac{d^2y}{dx^2}|}{(1 + (\frac{dy}{dx})^2)^{\frac{3}{2}}} \tag{11}$$

For the parabola:

$$\kappa = \frac{2}{(1 + (2x)^2)^{\frac{3}{2}}} \tag{12}$$