Double Pendulum

Equations of Motion

This notebook shows derivation of the equations of motion for the double pendulum simulation. See `http://www.myphysicslab.com/dbl_pendulum.html`.

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We regard *y* as increasing upwards. Origin is at the fixed base pivot.
θ1 is the angle at the (fixed) base pivot between the rod and the downward vertical position.
θ2 is the angle between the downward vertical from mass 1 and the second rod.
d is the derivative operator with respect to time.
L1, L2 are the lengths of the rods.
m1, m2 are masses.
g is gravitational constant.
x1,y1 is position of mass 1.
x2,y2 is position of mass2.
We start with the following:
x1 = L1 Sin[θ1]
y1 = −L1 Cos[θ1]
x2 = x1 + L2 Sin[θ2]
y2 = y1 − L2 Cos[θ2]
Then take two derivatives to get ddθ1 and ddθ2 as given below.

Naming convention: the letter d indicates derivative (with respect to time), dd indicates second derivative. For example:
x1' = dx1
x1'' = ddx1
θ1' = dθ1
θ1'' = ddθ1

```
ddx1 = -dθ1^2 L1 Sin[θ1] + ddθ1 L1 Cos[θ1]
```

```
General::spell1 :
 Possible spelling error: new symbol name "ddθ1" is similar to existing symbol "dθ1".
```

$$dd\theta1\, L1\, Cos[\theta1] - d\theta1^2\, L1\, Sin[\theta1]$$

```
ddy1 = dθ1^2 L1 Cos[θ1] + ddθ1 L1 Sin[θ1]
```

```
General::spell1 :
 Possible spelling error: new symbol name "ddy1" is similar to existing symbol "ddx1".
```

$$d\theta1^2\, L1\, Cos[\theta1] + dd\theta1\, L1\, Sin[\theta1]$$

```
ddx2 = ddx1 - dθ2^2 L2 Sin[θ2] + ddθ2 L2 Cos[θ2]
```

```
General::spell1 :
 Possible spelling error: new symbol name "ddθ2" is similar to existing symbol "dθ2".
```

$$dd\theta1\, L1\, Cos[\theta1] + dd\theta2\, L2\, Cos[\theta2] - d\theta1^2\, L1\, Sin[\theta1] - d\theta2^2\, L2\, Sin[\theta2]$$

```
ddy2 = ddy1 + dθ2^2 L2 Cos[θ2] + ddθ2 L2 Sin[θ2]
```

```
General::spell1 :
 Possible spelling error: new symbol name "ddy2" is similar to existing symbol "ddx2".
```

$$d\theta1^2\, L1\, Cos[\theta1] + d\theta2^2\, L2\, Cos[\theta2] + dd\theta1\, L1\, Sin[\theta1] + dd\theta2\, L2\, Sin[\theta2]$$

```
eq1 = -Cos[θ1] (m1 ddx1 + m2 ddx2) == Sin[θ1] (m1 ddy1 + m2 ddy2 + m2 g + m1 g)
```

```
-Cos[θ1] (m1 (ddθ1 L1 Cos[θ1] - dθ1^2 L1 Sin[θ1]) +
    m2 (ddθ1 L1 Cos[θ1] + ddθ2 L2 Cos[θ2] - dθ1^2 L1 Sin[θ1] - dθ2^2 L2 Sin[θ2])) ==
 Sin[θ1] (g m1 + g m2 + m1 (dθ1^2 L1 Cos[θ1] + ddθ1 L1 Sin[θ1]) +
    m2 (dθ1^2 L1 Cos[θ1] + dθ2^2 L2 Cos[θ2] + ddθ1 L1 Sin[θ1] + ddθ2 L2 Sin[θ2]))
```

```
eq2 = -Cos[θ2] m2 ddx2 == Sin[θ2] (m2 ddy2 + m2 g)
```

```
-m2 Cos[θ2] (ddθ1 L1 Cos[θ1] + ddθ2 L2 Cos[θ2] - dθ1^2 L1 Sin[θ1] - dθ2^2 L2 Sin[θ2]) ==
 Sin[θ2] (g m2 + m2 (dθ1^2 L1 Cos[θ1] + dθ2^2 L2 Cos[θ2] + ddθ1 L1 Sin[θ1] + ddθ2 L2 Sin[θ2]))
```

```
Solve[{eq1, eq2}, {ddθ1, ddθ2}] // Simplify
```

```
{{ddθ1 →
   -(g (2 m1 + m2) Sin[θ1] + m2 (g Sin[θ1 - 2 θ2] + 2 (dθ2^2 L2 + dθ1^2 L1 Cos[θ1 - θ2]) Sin[θ1 - θ2])) /
     (L1 (2 m1 + m2 - m2 Cos[2 (θ1 - θ2)])),
  ddθ2 → (2 (dθ1^2 L1 (m1 + m2) + g (m1 + m2) Cos[θ1] + dθ2^2 L2 m2 Cos[θ1 - θ2]) Sin[θ1 - θ2]) /
    (L2 (2 m1 + m2 - m2 Cos[2 (θ1 - θ2)]))}}
```