Contents

1 Lecture 01 - 方程组的几何解释

n linear equations, **n** unknowns

- row picture
- column picture ⋆
- matrix form

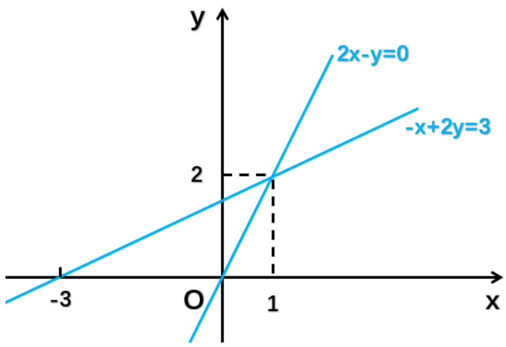
$$\begin{cases} 2x - y = 0 \\ -x + 2y = 3 \end{cases}$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \text{ i.e.}$$

$\mathbf{A}$ (matrix of coefficients) $= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $\mathbf{x}$ (vector of unknowns) $= \begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, such that

$$\mathbf{Ax} = \mathbf{b}$$

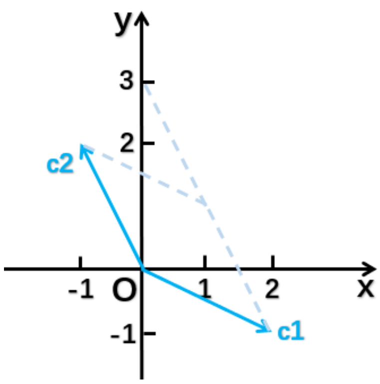
what's the **row** picture?



to find the point that lies on both two lines

what's the **column** picture?

$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$



$$1\overrightarrow{c_1} + 2\overrightarrow{c_2} = \overrightarrow{b}$$

to find the linear combination of columns of **A**, such that it equals **b**

what linear combination gives **b**?

what do all the linear combinations give?

what are all the possible, achievable right-hand sides be?

$$\begin{cases} 2x & - & y & & & = & 0 & \mathbf{1} \\ -x & + & 2y & - & z & = & -1 & \mathbf{2} \\ & - & 3y & + & 4z & = & 4 & \mathbf{3} \end{cases}$$

$$\left\{ \mathbf{1} \right. \quad : \text{ the plot of all the points that solve it are a plane}$$

$$\begin{cases} \mathbf{2} \\ \mathbf{3} \end{cases} \quad : \text{ two planes meet at a line}$$

$$\begin{cases} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{cases} \quad : \text{ meet at a point}$$

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

what's the **row** picture?

to find out all the points that satisfy all the equations

what's the **column** picture?

$$x \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

can I always solve $\mathbf{Ax} = \mathbf{b}$ for every right-hand side **b**?

do the linear combinations of the columns fill 3-dimensional space?

for this **A**, the answer is **YES** (non-singular, invertible)

but for some others **A**, the answer could be **NO** (singular, not-invertible)

if the 3 columns all lie in the same plane,

so I could solve it for some right-hand sides, when $\overrightarrow{b}$ is in the plane,

but most right-hand sides would be out of the plane and unreachable.

in some case, the combinations of **n** columns can only fill out **m**-D ($m < n$)

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

Ax means: **Ax** is a combination of columns of **A**

2 Lecture 02 - 矩阵消元

when solving equations-system,

Elimination, if it succeeds, it gets the answer.

It's always good to ask how could it fail.

$$\begin{cases} x + 2y + z = 2 \\ 3x + 8y + z = 12 \\ 4y + z = 2 \end{cases}$$

$$\begin{bmatrix} \underset{\text{first-pivot}}{\mathbf{1}} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow[row_3 - 0 \times row_1]{row_2 - 3 \times row_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & \underset{\text{second-pivot}}{\mathbf{2}} & -2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{row_3 - 2 \times row_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & \underset{\text{third-pivot}}{\mathbf{5}} \end{bmatrix}$$

pivots can **NOT** be 0 !

if there is a 0 in the pivot position, then try to switch lines

if 0 is in the pivot position and no place to exchange, then failure

let's bring the right-hand side in (Augmented Matrix)

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right] \Rightarrow \begin{cases} x + 2y + z = 2 \\ 2y - 2z = 6 \\ 5z = -10 \end{cases}$$

by back-substitution: $x = 2, y = 1, z = -2$

"elimination matrices"

$$\begin{bmatrix} \vdots & \vdots & \vdots \\ col_1 & col_2 & col_3 \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix} = 1 \times col_1 + 2 \times col_2 + 3 \times col_3$$

the result of multiplying a matrix by some vectors, is a combination of columns of the matrix

$$\begin{bmatrix} 1 & 2 & 7 \end{bmatrix} \begin{bmatrix} \cdots & row_1 & \cdots \\ \cdots & row_2 & \cdots \\ \cdots & row_3 & \cdots \end{bmatrix} = \begin{bmatrix} & & \end{bmatrix} = \begin{matrix} 1 \times row_1 \\ + \\ 2 \times row_2 \\ + \\ 7 \times row_3 \end{matrix}$$

the product of a row times a matrix, is a combination of rows of the matrix

when we do matrix multiplication, keep your eye on what it is doing with the whole vectors

what does the matrix, which can subtract $3 \times row_1$ from row_2 look like?

i.e. $\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$

$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$

$\xrightarrow{as\ R_1 = \mathbf{1} \times row_1 + \mathbf{0} \times row_2 + \mathbf{0} \times row_3} \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$

$\xrightarrow{as\ R_3 = \mathbf{0} \times row_1 + \mathbf{0} \times row_2 + \mathbf{1} \times row_3} \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{as\ R_2 = \mathbf{-3} \times row_1 + \mathbf{1} \times row_2 + \mathbf{0} \times row_3} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$: elementary matrix (初等矩阵)

$\mathbf{E_{i,j}}$ means it's the matrix that we use to fix the (i, j) position

e.g. $\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \xrightarrow{R_1 = row_1} \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \xrightarrow{R_2 = row_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ? & ? & ? \end{bmatrix} \xrightarrow{R_3 = row_3 - 2 \times row_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = E_{3,2}$

in elimination, we can use an elementary matrix to describe the change in each step

the next point in this lecture is to put these steps together, into a matrix that does these steps all in sequence, in another words, how could I create the matrix that does the whole job at once? i.e.

$$\mathbf{E_{3,2}}(\mathbf{E_{2,1}}\mathbf{A}) = \mathbf{U} \Longleftrightarrow \boxed{?}\mathbf{A} = \mathbf{U}$$

Associative Law

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$

permutation(置换):

- exchange rows, e.g.

  $\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is to exchange row_1 and row_2

- exchange columns, e.g.

  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$

  $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is to exchange col_1 and col_2

when I multiply a matrix on the left, I am doing row operations

if I want to do column operations, I should put a matrix on the right

if $\boxed{?}\mathbf{A} = \mathbf{U}$, then how can I "from $\mathbf{U}$ back to $\mathbf{A}$"?

this is about reversing steps, invertible, $\cdots$

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

"what steps can get me back?"

"what matrix can bring me back?"

3 Lecture 03 - 乘法和逆矩阵

key words:

- matrix multiplication (4 ways)
- inverse of $\mathbf{A}$, $\mathbf{AB}$, $\mathbf{A^T}$
- Gauss-Jordan, to find $\mathbf{A^{-1}}$

$$\underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} & & \\ & & \\ & & \\ & & \end{bmatrix}}_{\mathbf{B}} = \underbrace{\begin{bmatrix} & & \\ & c_{i,j} & \\ & & \end{bmatrix}}_{\mathbf{C=AB}}$$

$c_{i,j}$ comes from row_i of $\mathbf{A}$ and col_j of $\mathbf{B}$

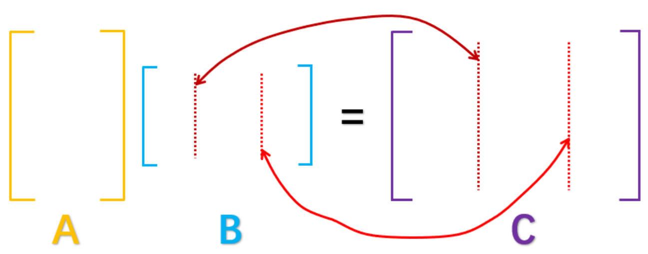
e.g.

$$\begin{aligned} c_{3,4} &= \begin{bmatrix} \\ row_3 \quad of \quad \mathbf{A} \\ \\ \end{bmatrix} \begin{bmatrix} col_4 \\ of \\ \mathbf{B} \end{bmatrix} \\ &= a_{3,1}b_{1,4} + a_{3,2}b_{2,4} + \cdots + a_{3,i}b_{i,4} + \cdots + a_{3,n}b_{n,4} \\ &= \sum_{k=1}^{n} a_{3,k}b_{k,4} \end{aligned}$$

the number of columns of $\mathbf{A}$ has to match the number of rows of $\mathbf{B}$

$$\mathbf{A}_{m\times n}\mathbf{B}_{n\times p} = \mathbf{C}_{m\times p}$$

the matrix times the n^{th} column is the n^{th} column of the answer



so I could think of multiplying a matrix by a vector, side by side

I can just think of having several columns, multiplying by $\mathbf{A}$, and getting the columns of answer

the columns of $\mathbf{C}$ are combinations of columns of $\mathbf{A}$

$\Longleftrightarrow$ every column of $\mathbf{C}$ is a combination of columns of $\mathbf{A}$, and numbers in $\mathbf{B}$ tell me what the combination is

in the same way, the rows of $\mathbf{C}$ are combinations of rows of $\mathbf{B}$

what about " $\underbrace{col\ \ of\ \ \mathbf{A}}_{m\times 1} \times \underbrace{row\ \ of\ \ \mathbf{B}}_{1\times p}$ "?

e.g.

$$\begin{bmatrix}2\\3\\4\end{bmatrix}\begin{bmatrix}1 & 6\end{bmatrix} = \begin{bmatrix}2 & 12\\3 & 18\\4 & 24\end{bmatrix}$$

$$\begin{bmatrix}2 & 7\\3 & 8\\4 & 9\end{bmatrix}\begin{bmatrix}1 & 6\\0 & 0\end{bmatrix} = \begin{bmatrix}2\\3\\4\end{bmatrix}\begin{bmatrix}1 & 6\end{bmatrix} + \begin{bmatrix}7\\8\\9\end{bmatrix}\begin{bmatrix}0 & 0\end{bmatrix}$$

$$\begin{aligned}\mathbf{AB} &= \text{sum of } \ (col_i\ of\ \mathbf{A}) \times (row_i\ of\ \mathbf{B})\\ &= \sum_{i=1}^{n}(col_i\ of\ \mathbf{A}) \times (row_i\ of\ \mathbf{B})\end{aligned}$$

the row space for $\begin{bmatrix}2 & 12\\3 & 18\\4 & 24\end{bmatrix}$, which is like all combinations of the rows, is the line through the row-vector $\begin{bmatrix}1 & 6\end{bmatrix}$, the same to the column space

you could also cut the matrix into blocks and do the multiplication by blocks, i.e.

$$\underbrace{\left[\begin{array}{c|c}\mathbf{A}_1 & \mathbf{A}_2\\ \hline \mathbf{A}_3 & \mathbf{A}_4\end{array}\right]}_{\mathbf{A}}\underbrace{\left[\begin{array}{c|c}\mathbf{B}_1 & \mathbf{B}_2\\ \hline \mathbf{B}_3 & \mathbf{B}_4\end{array}\right]}_{\mathbf{B}} = \underbrace{\left[\begin{array}{c|c}\mathbf{A}_1\mathbf{B}_1+\mathbf{A}_2\mathbf{B}_3 & \mathbf{A}_1\mathbf{B}_2+\mathbf{A}_2\mathbf{B}_4\\ \hline \mathbf{A}_3\mathbf{B}_1+\mathbf{A}_4\mathbf{B}_3 & \mathbf{A}_3\mathbf{B}_2+\mathbf{A}_4\mathbf{B}_4\end{array}\right]}_{\mathbf{AB}}$$

Inverses (square matrices)

not all matrices have inverses, if a matrix is square, is it invertible or not?

if $\mathbf{A}$ is invertible, non-singular, then

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$$

in singular case, no inverse!

e.g.

$$\mathbf{A} = \begin{bmatrix}1 & 3\\2 & 6\end{bmatrix}$$

thinking about columns here, if I multiply $\mathbf{A}$ by some other matrices, the columns of the results are all multiples of $\begin{bmatrix}1\\2\end{bmatrix}$, so no way to get the identity matrix $\mathbf{I}$

there is another more important reason

a square matrix has no inverse if I can find a vector $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$

but

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the matrix can't have an inverse if some columns give no contribution!

because

if $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ has an inverse, named $\mathbf{A^{-1}}$, then $\mathbf{A^{-1}A} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

and meanwhile, $\mathbf{A^{-1}A} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \mathbf{I} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$,

so that $\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which is not True

our conclusion is that for non-invertible/singular matrices, some combinations of their columns give the zero column

let's take a matrix that does have an inverse for example

e.g.

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\mathbf{A}^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{I}}$$

$$\text{then} \begin{cases} \mathbf{A} \begin{bmatrix} a \\ b \end{bmatrix} & = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \mathbf{A} \begin{bmatrix} c \\ d \end{bmatrix} & = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$

generally,

$$\mathbf{A} \cdot (col_j \ of \ \mathbf{A^{-1}}) = (col_j \ of \ \mathbf{I})$$

then how to solve the inverse for an invertible matrix?

here is the Gauss-Jordan idea, to solve two equations at once

$$\begin{cases} \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} & = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} & = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$

"solve them together!"

$$\underbrace{\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]}_{\left[\mathbf{A} \middle| \mathbf{I}\right]} \to \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \to \underbrace{\left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]}_{\left[\mathbf{I} \middle| \mathbf{A}^{-1}\right]}$$

把单位矩阵当成草稿纸，记录下对左侧矩阵的变换

相当于左右两边同时乘上逆矩阵，当左边变成单位矩阵时，右边即是该逆矩阵

i.e.

$$\begin{aligned} \mathbf{E}_{i_1,j_1}\mathbf{E}_{i_2,j_2}\cdots\mathbf{E}_{i_n,j_n}\left[\ \mathbf{A} \middle| \mathbf{I}\ \right] &= \left[\ \mathbf{E}_{i_1,j_1}\mathbf{E}_{i_2,j_2}\cdots\mathbf{E}_{i_n,j_n}\mathbf{A} \middle| \mathbf{E}_{i_1,j_1}\mathbf{E}_{i_2,j_2}\cdots\mathbf{E}_{i_n,j_n}\mathbf{I}\ \right] \\ &= \left[\ \mathbf{I} \middle| \mathbf{E}_{i_1,j_1}\mathbf{E}_{i_2,j_2}\cdots\mathbf{E}_{i_n,j_n}\ \right] \end{aligned}$$

then $\mathbf{A}^{-1} = \mathbf{E}_{i_1,j_1}\mathbf{E}_{i_2,j_2}\cdots\mathbf{E}_{i_n,j_n}$

注：

$\mathbf{E}_{i_t,j_t}\left[\ \mathbf{A} \middle| \mathbf{I}\ \right]$，即对 $\left[\ \mathbf{A} \middle| \mathbf{I}\ \right]$ 做行变换 $\Longleftrightarrow$ 对 $\mathbf{A}$ 与 $\mathbf{I}$ 同时、做同样的行变换

同理，可以对 $\left[\begin{array}{c} \mathbf{A} \\ \hline \mathbf{I} \end{array}\right]$ 做列变换，求得 $\mathbf{A}^{-1}$

4 Lecture 04 - 矩阵的 LU 分解

suppose $\mathbf{A}$ is invertible, and $\mathbf{B}$ is invertible, then what matrix gives me the inverse of $\mathbf{AB}$?

$$\mathbf{AB}(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{I}$$

$$(\mathbf{B}^{-1}\mathbf{A}^{-1})\mathbf{AB} = \mathbf{I}$$

if I transpose a matrix (square, invertible), what's its inverse?

$$\mathbf{AA}^{-1} = \mathbf{I}$$

$$(\mathbf{A}^{-1})^{\mathbf{T}}\mathbf{A}^{\mathbf{T}} = \mathbf{I}$$

$$\Updownarrow$$

$$(\mathbf{A}^{\mathbf{T}})^{-1} = (\mathbf{A}^{-1})^{\mathbf{T}}$$

the $\mathbf{A} = \mathbf{LU}$ is the most basic factorization of a matrix

think of the 2×2 case

$$\underbrace{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}}_{\mathbf{E}} \underbrace{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}}_{\mathbf{A}} = \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}_{\mathbf{U}}$$

if $\mathbf{A} = \mathbf{LU}$, then

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}}_{\mathbf{A}} = \underbrace{\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}_{\mathbf{U}}$$

so $\mathbf{L} = \mathbf{E}^{-1} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

$\mathbf{U}$ stands for upper triangular matrix, $\mathbf{L}$ stands for lower triangular matrix

what's more,

$$\begin{aligned} \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}}_{\mathbf{U}} \end{aligned}$$

$\mathbf{D}$ stands for diagonal matrix

if $\mathbf{A} = \begin{bmatrix} & \end{bmatrix}_{3\times 3}$,

suppose no row exchanges,

$\mathbf{E}_{3,2}\mathbf{E}_{3,1}\mathbf{E}_{2,1}\mathbf{A} = \mathbf{U}$

$$\mathbf{A} = \boxed{?}\mathbf{U}$$

$$\mathbf{A} = \underbrace{\mathbf{E}_{2,1}^{-1}\mathbf{E}_{3,1}^{-1}\mathbf{E}_{3,2}^{-1}}_{\mathbf{L}}\mathbf{U}$$

乘积的逆，只需要分别求逆

we know how to invert, we should take the separate inverses, but they go in the opposite order

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}}_{\mathbf{E_{3,2}}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{E_{2,1}}} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix}$$

I subtracted 2 of row_1 from row_2, and then I subtracted 5 of that new row_2 from row_3. So doing it in that order, how did row_1 affect row_3? Because 2 of row_1 got removed from row_2 and then 5 of those got removed from row_3, so altogether 10 of row_1 got thrown into row_3.

$$\mathbf{EA} = \mathbf{U} \quad \text{(elimination)}$$

$$\Downarrow$$

$$\mathbf{A} = \mathbf{E}^{-1}\mathbf{U} = \mathbf{LU} \quad (\mathbf{A} \text{ 的信息包含于 } \mathbf{LU})$$

if no row exchanges, the multipliers go directly into $\mathbf{L}$

how many operations on $n \times n$ matrix $\mathbf{A}$?

e.g.

$$\begin{bmatrix} \quad & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}_{100\times100} \xrightarrow{100\times99 \text{ numbers changed}} \begin{bmatrix} * & \cdots & \cdots & \cdots & \cdots \\ 0 & & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}_{100\times100}$$

$$\xrightarrow{99\times98 \text{ numbers changed}} \begin{bmatrix} * & \cdots & \cdots & \cdots & \cdots \\ 0 & * & \cdots & \cdots & \cdots \\ 0 & 0 & & & \\ \vdots & \vdots & & & \\ 0 & 0 & & & \end{bmatrix}_{100\times100}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

generally,

$$\sum_{i=1}^{n} i(i-1) = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = O(n^3)$$

I am ready to allow row exchanges.

There are some matrices that I will use to do row exchanges.

这些矩阵就是互换单位阵各行的所有的可能的情况。

e.g.

all 3×3 permutations

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{I} \qquad \text{no exhange}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{P_{1,2}} \qquad row_1 \leftrightarrow row_2$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{P_{1,3}} \qquad row_1 \leftrightarrow row_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{P_{2,3}} \qquad row_2 \leftrightarrow row_3$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

how about multiplying two of them together?

the answer is still in the list!

and if I invert, the inverses are all there too!

it's a little family of matrices there

$$\mathbf{P^{-1}} = \mathbf{P}$$

4×4 case $\longrightarrow$ 24 $\mathbf{P}'s$

5 Lecture 05 - 转置、置换、向量空间

书接上回 those are matrices $\mathbf{P}$ and they execute row exchanges

$\mathbf{A} = \mathbf{LU}$: assume no row exchanges

$\mathbf{PA} = \mathbf{LU}$: $\mathbf{P}$ gets the rows into the right order

permutations $\mathbf{P}$ is the identity matrix with reordered rows

$$\mathbf{P^{-1}} = \mathbf{P^T}$$

$$\mathbf{P^T P} = \mathbf{I}$$

we'll be interested in matrices that have $\mathbf{P^T P} = \mathbf{I}$, there are more of them than just permutations

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

Transpose: $\left(\mathbf{A^T}\right)_{i,j} = \mathbf{A}_{j,i}$
Symmetric Matrices: $\mathbf{A^T} = \mathbf{A}$

$\mathbf{R^T R}$ is always symmetric

e.g. $\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 7 \\ 11 & 13 & 11 \\ 7 & 11 & 17 \end{bmatrix}$

$\because \left(\mathbf{R^T R}\right)^{\mathbf{T}} = \mathbf{R^T} \left(\mathbf{R^T}\right)^{\mathbf{T}} = \mathbf{R^T R}$

what are vector spaces?

what are sub-spaces?

Example:

$\mathbb{R}^2 \to$ all 2-dim Real vectors, $\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\pi\\e\end{bmatrix}, \cdots$

the whole plane is $\mathbb{R}^2$, so $\mathbb{R}^2$ is the plane (xy plane)

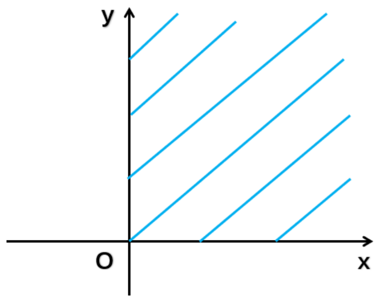
but the point is, it's a vector space

Every vector space has to ensure that zero vector in it.

$\mathbb{R}^3 \to$ all 3-dim Real vectors

$\mathbb{R}^n \to$ all vectors with n real components

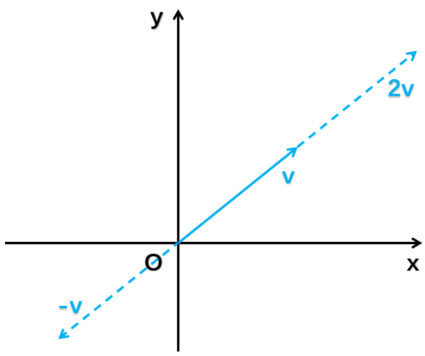
can we do additions and do we stay in the space?



in this figure, it's NOT a vector space, because it's not closed, for example, under multiplication by real numbers

a vector space has to be closed under multiplication and addition of vectors, in other words, linear combination

$\mathbb{R}^n$ is the most important, but we will be interested in vector spaces that are inside $\mathbb{R}^n$, vector spaces that follow the rules



this is a vector space inside $\mathbb{R}^2$ (sub-space of $\mathbb{R}^2$)

what are the possible sub-spaces of $\mathbb{R}^2$?

1. the whole space, $\mathbb{R}^2$ itself

2. lines through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (not the same as $\mathbb{R}^1$)

3. zero vector only

what are the possible sub-spaces of $\mathbb{R}^3$?

1. $\mathbb{R}^3$

2. plane through the origin

3. line through the origin

4. zero vector only

how do sub-spaces come from matrices?

I want to create some sub-spaces out of this matrix: $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$
all linear combinations of its columns (from $\mathbb{R}^3$) form a sub-space, called "column space", $C(\mathbf{A})$

the key idea is, we have to be able to take their combinations, still in the sub-space

if $col_1 // col_2$, then the column space is only a line through the origin

6 Lecture 06 - 列空间和零空间

vector space requirements

$\iff$ $\mathbf{v}+\mathbf{w}$ and $c\mathbf{v}$ are in the space

$\iff$ all combinations $c\mathbf{v}+d\mathbf{w}$ are in the space

notice that these two requirements mean
the sum and the scale of multiplication combine into linear combinations

Example: $\mathbb{R}^3$

2 subspaces: P - a plane, L - a line

❶ the union of those, $P \cup L$, has all vectors in P or L or both, is that a subspace?

NO!

❷ the intersection, $P \cap L$, has all vectors that are in both, is that a subspace?

YES!

the general question is, I have subspaces S and T, is their intersection $S \cap T$ a subspace?

YES!

proof:

if $\mathbf{v} \in S \cap T$, $\mathbf{w} \in S \cap T$

then $\mathbf{v}+\mathbf{w} \in S$ and $\mathbf{v}+\mathbf{w} \in T$

so $\mathbf{v}+\mathbf{w} \in S \cap T$

if $\mathbf{v} \in S \cap T$

then $c\mathbf{v} \in S$ and $c\mathbf{v} \in T$

so $c\mathbf{v} \in S \cap T$

in other words, when you take the intersection of two subspaces,
you get probably a smaller subspace, but it is still a subspace

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

the column space of $\mathbf{A}$, $C(\mathbf{A})$, is a subspace of $\mathbb{R}^4$
what's in that subspace?
not only the columns of $\mathbf{A}$, but also their linear combinations

so $C(\mathbf{A})$ is all linear combinations of $\mathbf{A}$'s columns

$$\text{so I would like to know} \begin{cases} \text{what's in that space?} \\ \text{how big is that space?} \\ \text{is that the whole of 4-dim space? or is it a subspace inside?} \end{cases}$$

取三个四维向量进行线性组合，怎么也得不到整个四维空间嘛！

let's make this question connected with linear equations,
does $\mathbf{Ax} = \mathbf{b}$ always have a solution for every $\mathbf{b}$?
NO, $\mathbf{Ax} = \mathbf{b}$ does not have a solution for every $\mathbf{b}$!

for example, 4 equations and 3 unknowns,
(the combinations of 3 columns cannot always fill the 4-dim space)
there's going to be some $\mathbf{b}$, are not linear combinations of the 3 columns, but sometimes can

what $\mathbf{b}$'s allow me to solve $\mathbf{Ax} = \mathbf{b}$?

I can solve $\mathbf{Ax} = \mathbf{b}$ exactly when 当且仅当 the right-hand side $\mathbf{b}$ is a vector in $C(\mathbf{A})$. (OR $\mathbf{b}$ is a linear combination of $\mathbf{A}$'s columns.)

so, $C(\mathbf{A})$ consists of all vectors $\mathbf{Ax}$ $(\forall \mathbf{x})$

if $\mathbf{b}$ is not a combination of $\mathbf{A}$'s columns, then there is no "$\mathbf{x}$", there is no way to solve $\mathbf{Ax} = \mathbf{b}$

Example: $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$

Question: Are those columns independent?

if I take the linear combinations of $\mathbf{A}$'s columns, does each column contributes something new or not? do I get a 3-D subspace?

NO!

can I throw away any column, and will get the same column space?

YES!

so for this $\mathbf{A}$, $C(\mathbf{A})$ is a 2-D subspace of $\mathbb{R}^4$

the null space 零空间, is going to be a totally different subspace

the null space of $\mathbf{A}$, what's in it?

- it contains not right-hand side $\mathbf{b}$
- it contains $\mathbf{x}$'s
- it contains all $\mathbf{x}$'s that solve "$\mathbf{Ax} = \mathbf{0}$"

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

the null space certainly contains zero ($\because$ the null space is a vector space as well)

for this $\mathbf{A}$,

$$N(\mathbf{A}) \text{ contains } \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}, \cdots, \begin{bmatrix} c \\ c \\ -c \end{bmatrix}$$

$$N(\mathbf{A}) = c \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

the null space is a line in $\mathbb{R}^3$

to check that the solutions to $\mathbf{Ax} = \mathbf{0}$ always give a subspace

proof:

if $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{Ax}^* = \mathbf{0}$

then $\mathbf{A}(\mathbf{x} + \mathbf{x}^*) = \mathbf{0}$

what's more, $\mathbf{A}(c\mathbf{x}) = c(\mathbf{Ax})$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

I would like to know all the solutions to this equation, and if these solutions form a subspace? NO! As zero vector is not a solution, and subspaces have to go through the origin.

the solutions is a plane/line that does not go through the origin