Contents

1 Lecture 01 - 方程组的几何解释

n linear equations, **n** unknowns

- row picture
- column picture ⋆
- matrix form

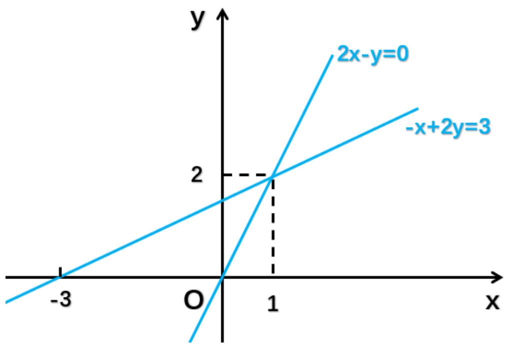
$$\begin{cases} 2x & - & y & = & 0 \\ -x & + & 2y & = & 3 \end{cases}$$

$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, i.e.

$\mathbf{A}$ (matrix of coefficients) $= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $\mathbf{x}$ (vector of unknowns) $= \begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, such that

$$\mathbf{Ax} = \mathbf{b}$$

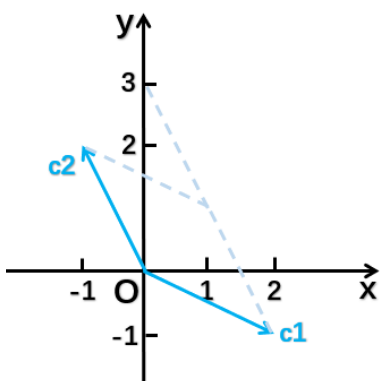
what's the **row** picture?



to find the point that lies on both two lines

what's the **column** picture?

$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$



$$1\overrightarrow{c_1} + 2\overrightarrow{c_2} = \overrightarrow{b}$$

to find the linear combination of columns of **A**, such that it equals **b**

what linear combination gives **b**?

what do all the linear combinations give?

what are all the possible, achievable right-hand sides be?

$$\begin{cases} 2x - y = 0 & \mathbf{1} \\ -x + 2y - z = -1 & \mathbf{2} \\ -3y + 4z = 4 & \mathbf{3} \end{cases}$$

$$\left\{ \mathbf{1} \right. \quad : \text{the plot of all the points that solve it are a plane}$$

$$\left\{ \begin{matrix} \mathbf{2} \\ \mathbf{3} \end{matrix} \right. \quad : \text{two planes meet at a line}$$

$$\left\{ \begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{matrix} \right. \quad : \text{meet at a point}$$

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

what's the **row** picture?

to find out all the points that satisfy all the equations

what's the **column** picture?

$$x \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

can I always solve $\mathbf{Ax} = \mathbf{b}$ for every right-hand side **b**?

do the linear combinations of the columns fill 3-dimensional space?

for this **A**, the answer is **YES** (non-singular, invertible)

but for some others **A**, the answer could be **NO** (singular, not-invertible)

if the 3 columns all lie in the same plane,

so I could solve it for some right-hand sides, when $\overrightarrow{b}$ is in the plane,

but most right-hand sides would be out of the plane and unreachable.

in some case, the combinations of **n** columns can only fill out **m**-D $(m < n)$

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

Ax means: **Ax** is a combination of columns of **A**

2 Lecture 02 - 矩阵消元

when solving equations-system,

Elimination, if it succeeds, it gets the answer.

It's always good to ask how could it fail.

$$\begin{cases} x + 2y + z = 2 \\ 3x + 8y + z = 12 \\ 4y + z = 2 \end{cases}$$

$$\begin{bmatrix} \underset{\text{first-pivot}}{\mathbf{1}} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow[row_3 - 0\times row_1]{row_2 - 3\times row_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & \underset{\text{second-pivot}}{\mathbf{2}} & -2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{row_3 - 2\times row_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & \underset{\text{third-pivot}}{\mathbf{5}} \end{bmatrix}$$

pivots can **NOT** be 0 !

if there is a 0 in the pivot position, then try to switch lines

if 0 is in the pivot position and no place to exchange, then failure

let's bring the right-hand side in (Augmented Matrix)

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right] \Rightarrow \begin{cases} x + 2y + z = 2 \\ 2y - 2z = 6 \\ 5z = -10 \end{cases}$$

by back-substitution: $x = 2, y = 1, z = -2$

"elimination matrices"

$$\begin{bmatrix} \vdots & \vdots & \vdots \\ col_1 & col_2 & col_3 \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix} = 1 \times col_1 + 2 \times col_2 + 3 \times col_3$$

the result of multiplying a matrix by some vectors, is a combination of columns of the matrix

$$\begin{bmatrix} 1 & 2 & 7 \end{bmatrix} \begin{bmatrix} \cdots & row_1 & \cdots \\ \cdots & row_2 & \cdots \\ \cdots & row_3 & \cdots \end{bmatrix} = \begin{bmatrix} & & \end{bmatrix} = \begin{matrix} 1 \times row_1 \\ + \\ 2 \times row_2 \\ + \\ 7 \times row_3 \end{matrix}$$

the product of a row times a matrix, is a combination of rows of the matrix

when we do matrix multiplication, keep your eye on what it is doing with the whole vectors

what does the matrix, which can subtract $3 \times row_1$ from row_2 look like?

i.e. $\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$

$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$

$\xrightarrow{as\ R_1=\mathbf{1}\times row_1+\mathbf{0}\times row_2+\mathbf{0}\times row_3} \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$

$\xrightarrow{as\ R_3=\mathbf{0}\times row_1+\mathbf{0}\times row_2+\mathbf{1}\times row_3} \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{as\ R_2=\mathbf{-3}\times row_1+\mathbf{1}\times row_2+\mathbf{0}\times row_3} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$: elementary matrix (初等矩阵)

$\mathbf{E_{i,j}}$ means it's the matrix that we use to fix the (i, j) position

e.g. $\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \xrightarrow{R_1=row_1} \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \xrightarrow{R_2=row_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ? & ? & ? \end{bmatrix} \xrightarrow{R_3=row_3-2\times row_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = E_{3,2}$

in elimination, we can use an elementary matrix to describe the change in each step

the next point in this lecture is to put these steps together, into a matrix that does these steps all in sequence, in another words, how could I create the matrix that does the whole job at once? i.e.

$$\mathbf{E_{3,2}}(\mathbf{E_{2,1}}\mathbf{A}) = \mathbf{U} \Longleftrightarrow \boxed{?}\mathbf{A} = \mathbf{U}$$

Associative Law

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$

permutation(置换):

- exchange rows, e.g.

  $\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is to exchange row_1 and row_2

- exchange columns, e.g.

  $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

  $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is to exchange col_1 and col_2

when I multiply a matrix on the left, I am doing row operations

if I want to do column operations, I should put a matrix on the right

if $\boxed{?}\mathbf{A} = \mathbf{U}$, then how can I "from $\mathbf{U}$ back to $\mathbf{A}$"?

this is about reversing steps, invertible, $\cdots$

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

"what steps can get me back?"

"what matrix can bring me back?"

3 Lecture 03 - 乘法和逆矩阵

key words:

- matrix multiplication (4 ways)
- inverse of $\mathbf{A}$, $\mathbf{AB}$, $\mathbf{A^T}$
- Gauss-Jordan, to find $\mathbf{A^{-1}}$

$$\underbrace{\begin{bmatrix} \quad & \quad \\ & \\ & \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \quad & \quad \\ & \\ & \\ & \end{bmatrix}}_{\mathbf{B}} = \underbrace{\begin{bmatrix} & \\ & c_{i,j} & \\ & \end{bmatrix}}_{\mathbf{C=AB}}$$

$c_{i,j}$ comes from row_i of $\mathbf{A}$ and col_j of $\mathbf{B}$

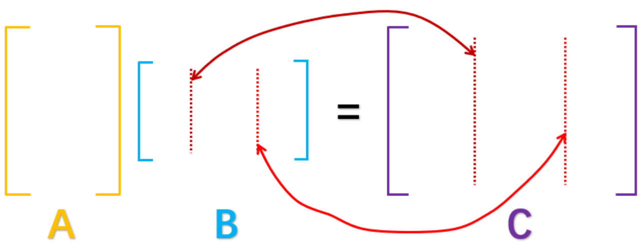
e.g.

$$\begin{aligned} c_{3,4} &= \begin{bmatrix} \\ row_3 \quad of \quad \mathbf{A} \\ \\ \end{bmatrix} \begin{bmatrix} col_4 \\ of \\ \mathbf{B} \end{bmatrix} \\ &= a_{3,1}b_{1,4} + a_{3,2}b_{2,4} + \cdots + a_{3,i}b_{i,4} + \cdots + a_{3,n}b_{n,4} \\ &= \sum_{k=1}^{n} a_{3,k}b_{k,4} \end{aligned}$$

the number of columns of $\mathbf{A}$ has to match the number of rows of $\mathbf{B}$

$$\mathbf{A}_{m\times n}\mathbf{B}_{n\times p} = \mathbf{C}_{m\times p}$$

the matrix times the n^{th} column is the n^{th} column of the answer



so I could think of multiplying a matrix by a vector, side by side
I can just think of having several columns, multiplying by $\mathbf{A}$, and getting the columns of answer

the columns of $\mathbf{C}$ are combinations of columns of $\mathbf{A}$

$\iff$ every column of $\mathbf{C}$ is a combination of columns of $\mathbf{A}$, and numbers in $\mathbf{B}$ tell me what the combination is

in the same way, the rows of $\mathbf{C}$ are combinations of rows of $\mathbf{B}$

what about " $\underbrace{col\ \ of\ \ \mathbf{A}}_{m\times 1} \quad \times \quad \underbrace{row\ \ of\ \ \mathbf{B}}_{1\times p}$ "?

e.g.

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \mathbf{AB} &= \text{sum of } \ (col_i\ of\ \ \mathbf{A}) \times (row_i\ of\ \ \mathbf{B}) \\ &= \sum_{i=1}^{n} (col_i\ of\ \ \mathbf{A}) \times (row_i\ of\ \ \mathbf{B}) \end{aligned}$$

the row space for $\begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$, which is like all combinations of the rows, is the line through the row-vector $\begin{bmatrix} 1 & 6 \end{bmatrix}$, the same to the column space

you could also cut the matrix into blocks and do the multiplication by blocks, i.e.

$$\underbrace{\left[\begin{array}{c|c} \mathbf{A}_1 & \mathbf{A}_2 \\ \hline \mathbf{A}_3 & \mathbf{A}_4 \end{array}\right]}_{\mathbf{A}} \underbrace{\left[\begin{array}{c|c} \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{B}_3 & \mathbf{B}_4 \end{array}\right]}_{\mathbf{B}} = \underbrace{\left[\begin{array}{c|c} \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_3 & \mathbf{A}_1\mathbf{B}_2 + \mathbf{A}_2\mathbf{B}_4 \\ \hline \mathbf{A}_3\mathbf{B}_1 + \mathbf{A}_4\mathbf{B}_3 & \mathbf{A}_3\mathbf{B}_2 + \mathbf{A}_4\mathbf{B}_4 \end{array}\right]}_{\mathbf{AB}}$$

Inverses (square matrices)

not all matrices have inverses, if a matrix is square, is it invertible or not?

if $\mathbf{A}$ is invertible, non-singular, then

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$$

in singular case, no inverse!

e.g.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

thinking about columns here, if I multiply $\mathbf{A}$ by some other matrices, the columns of the results are all multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so no way to get the identity matrix $\mathbf{I}$

there is another more important reason

a square matrix has no inverse if I can find a vector $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$

but

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the matrix can't have an inverse if some columns give no contribution!

because

if $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ has an inverse, named $\mathbf{A^{-1}}$, then $\mathbf{A^{-1}A} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \mathbf{A^{-1}} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

and meanwhile, $\mathbf{A^{-1}A} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \mathbf{I} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$,

so that $\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which is not True

our conclusion is that for non-invertible/singular matrices, some combinations of their columns give the zero column

let's take a matrix that does have an inverse for example

e.g.

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\mathbf{A^{-1}}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{I}}$$

then $\begin{cases} \mathbf{A} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \mathbf{A} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$

generally,

$$\mathbf{A} \cdot (col_j \ of \ \mathbf{A^{-1}}) = (col_j \ of \ \mathbf{I})$$

then how to solve the inverse for an invertible matrix?

here is the Gauss-Jordan idea, to solve two equations at once

$$\begin{cases} \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$

"solve them together!"

$$\underbrace{\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]}_{\left[\begin{array}{c|c}\mathbf{A} & \mathbf{I}\end{array}\right]} \to \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \to \underbrace{\left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]}_{\left[\begin{array}{c|c}\mathbf{I} & \mathbf{A}^{-1}\end{array}\right]}$$

把单位矩阵当成草稿纸，记录下对左侧矩阵的变换

相当于左右两边同时乘上逆矩阵，当左边变成单位矩阵时，右边即是该逆矩阵

i.e.

$$\begin{aligned} \mathbf{E}_{i_1,j_1}\mathbf{E}_{i_2,j_2}\cdots\mathbf{E}_{i_n,j_n}\left[\begin{array}{c|c}\mathbf{A} & \mathbf{I}\end{array}\right] &= \left[\begin{array}{c|c}\mathbf{E}_{i_1,j_1}\mathbf{E}_{i_2,j_2}\cdots\mathbf{E}_{i_n,j_n}\mathbf{A} & \mathbf{E}_{i_1,j_1}\mathbf{E}_{i_2,j_2}\cdots\mathbf{E}_{i_n,j_n}\mathbf{I}\end{array}\right] \\ &= \left[\begin{array}{c|c}\mathbf{I} & \mathbf{E}_{i_1,j_1}\mathbf{E}_{i_2,j_2}\cdots\mathbf{E}_{i_n,j_n}\end{array}\right] \end{aligned}$$

then $\mathbf{A}^{-1} = \mathbf{E}_{i_1,j_1}\mathbf{E}_{i_2,j_2}\cdots\mathbf{E}_{i_n,j_n}$

注：

$\mathbf{E}_{i_t,j_t}\left[\begin{array}{c|c}\mathbf{A} & \mathbf{I}\end{array}\right]$，即对 $\left[\begin{array}{c|c}\mathbf{A} & \mathbf{I}\end{array}\right]$ 做行变换 $\Longleftrightarrow$ 对 $\mathbf{A}$ 与 $\mathbf{I}$ 同时、做同样的行变换

同理，可以对 $\left[\begin{array}{c}\mathbf{A} \\ \hline \mathbf{I}\end{array}\right]$ 做列变换，求得 $\mathbf{A}^{-1}$

4 Lecture 04 - 矩阵的 LU 分解

suppose $\mathbf{A}$ is invertible, and $\mathbf{B}$ is invertible, then what matrix gives me the inverse of $\mathbf{AB}$?

$$\mathbf{AB}(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{I}$$

$$(\mathbf{B}^{-1}\mathbf{A}^{-1})\mathbf{AB} = \mathbf{I}$$

if I transpose a matrix (square, invertible), what's its inverse?

$$\mathbf{AA}^{-1} = \mathbf{I}$$

$$(\mathbf{A}^{-1})^{\mathbf{T}}\mathbf{A}^{\mathbf{T}} = \mathbf{I}$$

$$\Updownarrow$$

$$(\mathbf{A}^{\mathbf{T}})^{-1} = (\mathbf{A}^{-1})^{\mathbf{T}}$$

the $\mathbf{A} = \mathbf{LU}$ is the most basic factorization of a matrix

think of the 2×2 case

$$\underbrace{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}}_{\mathbf{E}} \underbrace{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}}_{\mathbf{A}} = \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}_{\mathbf{U}}$$

if $\mathbf{A} = \mathbf{LU}$, then

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}}_{\mathbf{A}} = \underbrace{\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}_{\mathbf{U}}$$

so $\mathbf{L} = \mathbf{E}^{-1} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

$\mathbf{U}$ stands for upper triangular matrix, $\mathbf{L}$ stands for lower triangular matrix

what's more,

$$\begin{aligned} \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}}_{\mathbf{U}} \end{aligned}$$

$\mathbf{D}$ stands for diagonal matrix

if $\mathbf{A} = \begin{bmatrix} & \end{bmatrix}_{3\times 3}$,

suppose no row exchanges,

$\mathbf{E}_{3,2}\mathbf{E}_{3,1}\mathbf{E}_{2,1}\mathbf{A} = \mathbf{U}$

$$\mathbf{A} = \boxed{?}\mathbf{U}$$

$$\mathbf{A} = \underbrace{\mathbf{E}_{2,1}^{-1}\mathbf{E}_{3,1}^{-1}\mathbf{E}_{3,2}^{-1}}_{\mathbf{L}}\mathbf{U}$$

乘积的逆，只需要分别求逆

we know how to invert, we should take the separate inverses, but they go in the opposite order

$$\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&-5&1\end{bmatrix}}_{\mathbf{E_{3,2}}}\underbrace{\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}}_{\mathbf{E_{2,1}}}=\begin{bmatrix}1&0&0\\-2&1&0\\10&-5&1\end{bmatrix}$$

I subtracted 2 of row_1 from row_2, and then I subtracted 5 of that new row_2 from row_3. So doing it in that order, how did row_1 affect row_3? Because 2 of row_1 got removed from row_2 and then 5 of those got removed from row_3, so altogether 10 of row_1 got thrown into row_3.

$$\mathbf{EA}=\mathbf{U}\quad\text{(elimination)}$$

$$\Downarrow$$

$$\mathbf{A}=\mathbf{E}^{-1}\mathbf{U}=\mathbf{LU}\quad(\mathbf{A}\text{ 的信息包含于 }\mathbf{LU})$$

if no row exchanges, the multipliers go directly into $\mathbf{L}$

how many operations on $n\times n$ matrix $\mathbf{A}$?

e.g.

$$\begin{bmatrix}&&&\\&&&\\&&&\\&&&\end{bmatrix}_{100\times100}\xrightarrow{100\times99\text{ numbers changed}}\begin{bmatrix}*&\cdots&\cdots&\cdots&\cdots\\0&&&&\\0&&&&\\\vdots&&&&\\0&&&&\end{bmatrix}_{100\times100}$$

$$\xrightarrow{99\times98\text{ numbers changed}}\begin{bmatrix}*&\cdots&\cdots&\cdots&\cdots\\0&*&\cdots&\cdots&\cdots\\0&0&&&\\\vdots&\vdots&&&\\0&0&&&\end{bmatrix}_{100\times100}$$

$$\cdots\quad\cdots\quad\cdots\quad\cdots$$

generally,

$$\sum_{i=1}^{n}i(i-1)=\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}=O(n^3)$$

I am ready to allow row exchanges.

There are some matrices that I will use to do row exchanges.

这些矩阵就是互换单位阵各行的所有的可能的情况。

e.g.

all 3×3 permutations

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{I} \qquad \text{no exhange}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{P_{1,2}} \qquad row_1 \leftrightarrow row_2$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{P_{1,3}} \qquad row_1 \leftrightarrow row_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{P_{2,3}} \qquad row_2 \leftrightarrow row_3$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

how about multiplying two of them together?

the answer is still in the list!

and if I invert, the inverses are all there too!

it's a little family of matrices there

$$\mathbf{P^{-1}} = \mathbf{P}$$

4×4 case $\longrightarrow$ 24 $\mathbf{P'}s$

5 Lecture 05 - 转置、置换、向量空间

书接上回 those are matrices $\mathbf{P}$ and they execute row exchanges

$\mathbf{A} = \mathbf{LU}$: assume no row exchanges

$\mathbf{PA} = \mathbf{LU}$: $\mathbf{P}$ gets the rows into the right order

permutations $\mathbf{P}$ is the identity matrix with reordered rows

$$\mathbf{P}^{-1} = \mathbf{P}^{\mathbf{T}}$$

$$\mathbf{P}^{\mathbf{T}}\mathbf{P} = \mathbf{I}$$

we'll be interested in matrices that have $\mathbf{P}^{\mathbf{T}}\mathbf{P} = \mathbf{I}$, there are more of them than just permutations

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

Transpose: $\left(\mathbf{A}^{\mathbf{T}}\right)_{i,j} = \mathbf{A}_{j,i}$

Symmetric Matrices: $\mathbf{A}^{\mathbf{T}} = \mathbf{A}$

$\mathbf{R}^{\mathbf{T}}\mathbf{R}$ is always symmetric

e.g. $\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 7 \\ 11 & 13 & 11 \\ 7 & 11 & 17 \end{bmatrix}$

$\because \left(\mathbf{R}^{\mathbf{T}}\mathbf{R}\right)^{\mathbf{T}} = \mathbf{R}^{\mathbf{T}}\left(\mathbf{R}^{\mathbf{T}}\right)^{\mathbf{T}} = \mathbf{R}^{\mathbf{T}}\mathbf{R}$

what are vector spaces?

what are sub-spaces?

Example:

$\mathbb{R}^2 \to$ all 2-dim Real vectors, $\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ e \end{bmatrix}, \cdots$

the whole plane is $\mathbb{R}^2$, so $\mathbb{R}^2$ is the plane (xy plane)

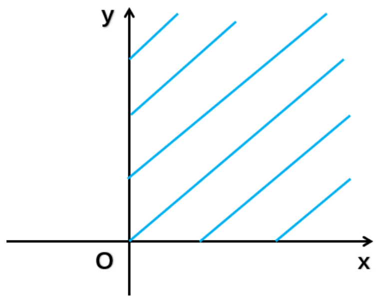
but the point is, it's a vector space

Every vector space has to ensure that zero vector in it.

$\mathbb{R}^3 \to$ all 3-dim Real vectors

$\mathbb{R}^n \to$ all vectors with n real components

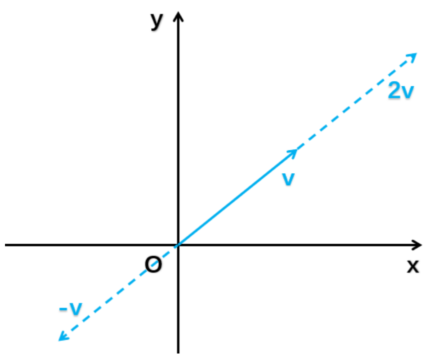
can we do additions and do we stay in the space?



in this figure, it's NOT a vector space, because it's not closed, for example, under multiplication by real numbers

a vector space has to be closed under multiplication and addition of vectors, in other words, linear combination

$\mathbb{R}^n$ is the most important, but we will be interested in vector spaces that are inside $\mathbb{R}^n$, vector spaces that follow the rules



this is a vector space inside $\mathbb{R}^2$ (sub-space of $\mathbb{R}^2$)

what are the possible sub-spaces of $\mathbb{R}^2$?

1. the whole space, $\mathbb{R}^2$ itself

2. lines through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (not the same as $\mathbb{R}^1$)

3. zero vector only

what are the possible sub-spaces of $\mathbb{R}^3$?

1. $\mathbb{R}^3$

2. plane through the origin

3. line through the origin

4. zero vector only

how do sub-spaces come from matrices?

I want to create some sub-spaces out of this matrix: $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$

all linear combinations of its columns (from $\mathbb{R}^3$) form a sub-space, called "column space", $C(\mathbf{A})$

the key idea is, we have to be able to take their combinations, still in the sub-space

if $col_1 // col_2$, then the column space is only a line through the origin

6 Lecture 06 - 列空间和零空间

vector space requirements

$\Longleftrightarrow$ $\mathbf{v}+\mathbf{w}$ and $c\mathbf{v}$ are in the space

$\Longleftrightarrow$ all combinations $c\mathbf{v}+d\mathbf{w}$ are in the space

notice that these two requirements mean
the sum and the scale of multiplication combine into linear combinations

Example: $\mathbb{R}^3$

2 subspaces: P - a plane, L - a line

❶ the union of those, $P \cup L$, has all vectors in P or L or both, is that a subspace?

NO!

❷ the intersection, $P \cap L$, has all vectors that are in both, is that a subspace?

YES!

the general question is, I have subspaces S and T, is their intersection $S \cap T$ a subspace?

YES!

proof:

if $\mathbf{v} \in S \cap T$, $\mathbf{w} \in S \cap T$

then $\mathbf{v}+\mathbf{w} \in S$ and $\mathbf{v}+\mathbf{w} \in T$

so $\mathbf{v}+\mathbf{w} \in S \cap T$

if $\mathbf{v} \in S \cap T$

then $c\mathbf{v} \in S$ and $c\mathbf{v} \in T$

so $c\mathbf{v} \in S \cap T$

in other words, when you take the intersection of two subspaces,
you get probably a smaller subspace, but it is still a subspace

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

the column space of $\mathbf{A}$, $C(\mathbf{A})$, is a subspace of $\mathbb{R}^4$

what's in that subspace?

not only the columns of $\mathbf{A}$, but also their linear combinations

so $C(\mathbf{A})$ is all linear combinations of $\mathbf{A}$'s columns

$$\text{so I would like to know} \begin{cases} \text{what's in that space?} \\ \text{how big is that space?} \\ \text{is that the whole of 4-dim space? or is it a subspace inside?} \end{cases}$$

取三个四维向量进行线性组合，怎么也得不到整个四维空间嘛！

let's make this question connected with linear equations,
does $\mathbf{Ax} = \mathbf{b}$ always have a solution for every $\mathbf{b}$?
NO, $\mathbf{Ax} = \mathbf{b}$ does not have a solution for every $\mathbf{b}$!

for example, 4 equations and 3 unknowns,
(the combinations of 3 columns cannot always fill the 4-dim space)
there's going to be some $\mathbf{b}$, are not linear combinations of the 3 columns, but sometimes can

what $\mathbf{b}$'s allow me to solve $\mathbf{Ax} = \mathbf{b}$?

I can solve $\mathbf{Ax} = \mathbf{b}$ exactly when 当且仅当 the right-hand side $\mathbf{b}$ is a vector in $C(\mathbf{A})$. (OR $\mathbf{b}$ is a linear combination of $\mathbf{A}$'s columns.)

so, $C(\mathbf{A})$ consists of all vectors $\mathbf{Ax}$ $(\forall \mathbf{x})$

if $\mathbf{b}$ is not a combination of $\mathbf{A}$'s columns, then there is no "$\mathbf{x}$", there is no way to solve $\mathbf{Ax} = \mathbf{b}$

Example: $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$

Question: Are those columns independent?

if I take the linear combinations of $\mathbf{A}$'s columns, does each column contributes something new or not? do I get a 3-D subspace?
NO!

can I throw away any column, and will get the same column space?
YES!

so for this $\mathbf{A}$, $C(\mathbf{A})$ is a 2-D subspace of $\mathbb{R}^4$

the null space 零空间, is going to be a totally different subspace

the null space of $\mathbf{A}$, what's in it?

- it contains not right-hand side $\mathbf{b}$
- it contains $\mathbf{x}$'s
- it contains all $\mathbf{x}$'s that solve "$\mathbf{Ax} = \mathbf{0}$"

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

the null space certainly contains zero ($\because$ the null space is a vector space as well)

for this $\mathbf{A}$,

$$N(\mathbf{A}) \text{ contains } \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}, \cdots, \begin{bmatrix} c \\ c \\ -c \end{bmatrix}$$

$$N(\mathbf{A}) = c \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

the null space is a line in $\mathbb{R}^3$

to check that the solutions to $\mathbf{Ax} = \mathbf{0}$ always give a subspace

proof:

if $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{Ax}^* = \mathbf{0}$

then $\mathbf{A}(\mathbf{x} + \mathbf{x}^*) = \mathbf{0}$

what's more, $\mathbf{A}(c\mathbf{x}) = c(\mathbf{Ax})$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

I would like to know all the solutions to this equation, and if these solutions form a subspace? NO! As zero vector is not a solution, and subspaces have to go through the origin.

the solutions is a plane/line that does not go through the origin

7 Lecture 07 - 求解 Ax=0 ： 主变量与特解

what's the algorithm for solving $\mathbf{Ax} = \mathbf{0}$?

that's the null space that I'm interested in

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{U}$$

while I am doing elimination,

I am not changing the solutions $\Longrightarrow$ I am not changing the null space

$$\left[\begin{array}{ccccccc} 1 & & 2 & & 2 & & 2 \\ = & = & = & = & & & \\ 0 & & 0 & \| & 2 & & 4 \\ & & & = & = & = & = \\ 0 & & 0 & & 0 & & 0 \end{array}\right] : \text{ echelon form, staircase form}$$

there are two pivots only

the number of pivots = the rank of the matrix = the number of pivot variables

$\mathbf{Ax} = \mathbf{0} \Longrightarrow \mathbf{Ux} = \mathbf{0}$ same solutions, same null space

how do I describe the solutions?
四个三维向量一定线性相关

$$\begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad 2 \text{ free columns, } 2 \text{ pivot columns}$$

free means that we can assign values freely, and we can find the other values accordingly

for convenient purpose, we choose 1 and 0 to those free variables

$$\mathbf{x_1} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x_2} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \qquad \text{2 special solutions (I gave special numbers to free variables)}$$

what are all the solutions to $\mathbf{Ax} = \mathbf{0}$ or $\mathbf{Ux} = \mathbf{0}$?

$$\mathbf{x} = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} = c\mathbf{x_1} + d\mathbf{x_2}$$

I am taking all the linear combinations of my 2 special solutions, and they are null space.

how many special solution are there? 每个自由变量对应一个特解

$$\begin{bmatrix} & & & \\ & & & \end{bmatrix}_{m\times n} \quad \text{with rank } r,\ (n-r) \text{ free variables}$$

we get r pivot variables, so there are really r equations there, only r independent equations, and there are $(n-r)$ variables that we can choose freely

Algorithms to Solve $\mathbf{Ax} = \mathbf{0}$

1. do elimination
2. decide which are pivot columns and which are free columns
3. give values to free variables
4. complete pivot values accordingly
5. do linear combinations

reduced row echelon form ($\mathbf{U}$ 还可以简化)

$$\mathbf{U} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{R}$$

in rref, it has zeros above and below the pivots

$$\begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

notice that there is an identity sitting in the pivot rows and pivot columns!

$\mathbf{Ax} = \mathbf{0} \Longrightarrow \mathbf{Ux} = \mathbf{0} \Longrightarrow \mathbf{Rx} = \mathbf{0}$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

$$\mathbf{A^T} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so the rank is 2 again!

the number of special solutions is 1

the fact: the number of pivot columns of $\mathbf{A}$ and $\mathbf{A^T}$ is the same

8 Lecture 08 - 可解性与解的结构

$$\begin{cases} 1x_1 & + & 2x_2 & + & 2x_3 & + & 2x_4 & = & b_1 \\ 2x_1 & + & 4x_2 & + & 6x_3 & + & 8x_4 & = & b_2 \\ 3x_1 & + & 6x_2 & + & 8x_3 & + & 10x_4 & = & b_3 \end{cases}$$

there is a condition on b_1, b_2, b_3 for this system to have a solution

$$\underbrace{\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right]}_{\text{augmented matrix}=\left[\begin{array}{c|c} \mathbf{A} & \mathbf{b} \end{array}\right]} \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right]$$

$b_3 - b_2 - b_1 = 0$, this is the condition for solvability

suppose that $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$,

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{b} \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

what are the conditions on $\mathbf{b}$ that make the equation system solvable?

Solvability Condition on $\mathbf{b}$

- 表述一

  当且仅当 $\mathbf{b}$ 属于 $\mathbf{A}$ 的列空间时成立

  或者 $\mathbf{b}$ 必须是 $\mathbf{A}$ 各列的线性组合

- 表述二

  if a combination of the rows of $\mathbf{A}$ gives the zero row,

  the same combination of the components of $\mathbf{b}$ has to give zero

what's the algorithm to find the solutions?

1. a particular solution, $\mathbf{x}_{particular}$

   to set all free variables to zero,

   since those free variables can be anything,

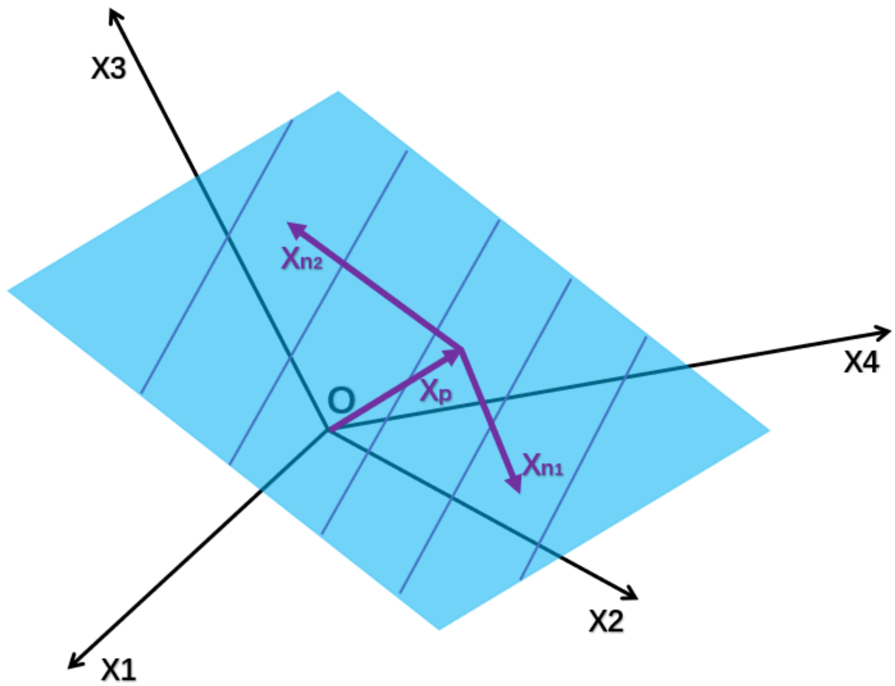
   then solve $\mathbf{Ax} = \mathbf{b}$ to get the pivot variables in this case, $\mathbf{x}_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$

2. $\mathbf{x}$ from null space, $\mathbf{x}_{nullspace}$

   in this case, $\mathbf{x}_n = c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

3. $\mathbf{x}_c = \mathbf{x}_p + \mathbf{x}_n$, $\mathbf{x}_{complete}$

   in this case, $\mathbf{x}_c = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$



“由零空间这个子空间从原点平移过来得到的平面”

think of a $m \times n$ matrix $\mathbf{A}$ of rank r, $\begin{cases} r \le m \\ r \le n \end{cases}$

- full column rank ($r = n$)

  意味着全部向量撑开全部的维数

there's a pivot in every column, no free variables

there are no free variables to give values, so the null space is only the zero vector

i.e. $N(\mathbf{A}) = \{\mathbf{0}\}$

the complete solution to $\mathbf{Ax} = \mathbf{b}$ is just $\mathbf{x}_p$, it's the unique solution if $\mathbf{x}_p$ exists

“列满秩时，如果解存在（$\mathbf{b} \in C(\mathbf{A})$），那么解唯一”“此时只有零个解或一个解”

- full row rank ($r = m$)

  every row has a pivot

  I can solve $\mathbf{Ax} = \mathbf{b}$ for any $\mathbf{b}$

  “在消元时没有得到零行!”

  number(free-variables) $= n - m$

- $r = m = n$

  invertible!

  此时 $\mathbf{Ax} = \mathbf{b}$ 必定有解且解唯一

- $r < m$ and $r < n$

  0 solution or ∞ solutions

9 Lecture 09 - 线性相关性、基、维数

key words:

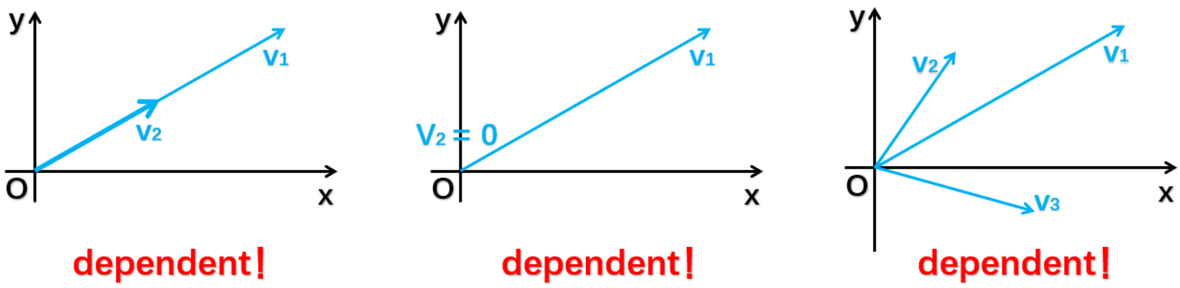
- linear independence
- spanning a space
- basis for a subspace / basis for a vector space
- the dimension of a subspace

$$\text{we talk about a bunch of vectors} \begin{cases} \text{being independent} \\ \text{spanning a space} \\ \text{being a basis} \end{cases}$$

suppose $\mathbf{A}$ is a $m \times n$ $(m < n)$ matrix, $(\#unknowns > \#equations)$
then there are non-zero solutions to $\mathbf{Ax} = \mathbf{0}$
$\Longrightarrow$ 在 $\mathbf{A}$ 的零空间中，除了零向量之外，还包含了其他一些向量

the reason is, there will be free variables, at least one, which I can assign non-zero values to

vectors $\mathbf{x_1}, \mathbf{x_2}, \cdots, \mathbf{x_n}$ are independent if no combination, except the "zero combination", gives the zero vector



corollary

- if the zero vector is in there, "independent is dead"
- 如果零空间 $N(\mathbf{A})$ 里存在非零向量，那么各列相关

when $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}$ are columns of $\mathbf{A}$,
they are independent if null space of $\mathbf{A}$ is only the zero vector, $rank = n$, NO free variables,
they are dependent if there exists non-zero $\mathbf{c}$, such that $\mathbf{Ac} = \mathbf{0}$, $rank < n$, YES free variables

vectors $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_l}$ span a space means,
the space consists of all combinations of those vectors,

the space will be the smallest space with those combinations in it

the column space of a matrix, is the space spanned by its columns

a basis for a vector space is a sequence of vectors, $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_d}$, with 2 properties:
（向量的个数刚刚好）

- they are independent
- they can span the space

Example: space is $\mathbb{R}^3$

$$\text{one basis is } \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\text{another basis is } \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\4\\5\end{bmatrix}$$

for $\mathbb{R}^n$, n vectors give a basis if the $n \times n$ matrix with those columns is invertible

the basis is not unique, for $\mathbb{R}^3$, any invertible 3×3 matrix, its columns are a basis for $\mathbb{R}^3$

but there is a common character of those bases, that's the number of vectors!

Given a space, every basis for the space has the same number[1] of vectors.

Example: $\mathbf{A} = \begin{bmatrix}1 & 2 & 3 & 1\\1 & 1 & 2 & 1\\1 & 2 & 3 & 1\end{bmatrix}$

$$\text{one of the bases for } C(\mathbf{A}) \text{ is } \left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}\right\}$$

$$\text{one of the bases for } N(\mathbf{A}) \text{ is } \left\{\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}\right\}$$

the rank of $\mathbf{A}$ is 2 = the number of pivot columns = the dimension of $C(\mathbf{A})$

$dim\ C(\mathbf{A}) = rank(\mathbf{A}) = r$
$dim\ N(\mathbf{A}) = the\ number\ of\ free\ variables = n - r$

[1] Def. dimension of the space

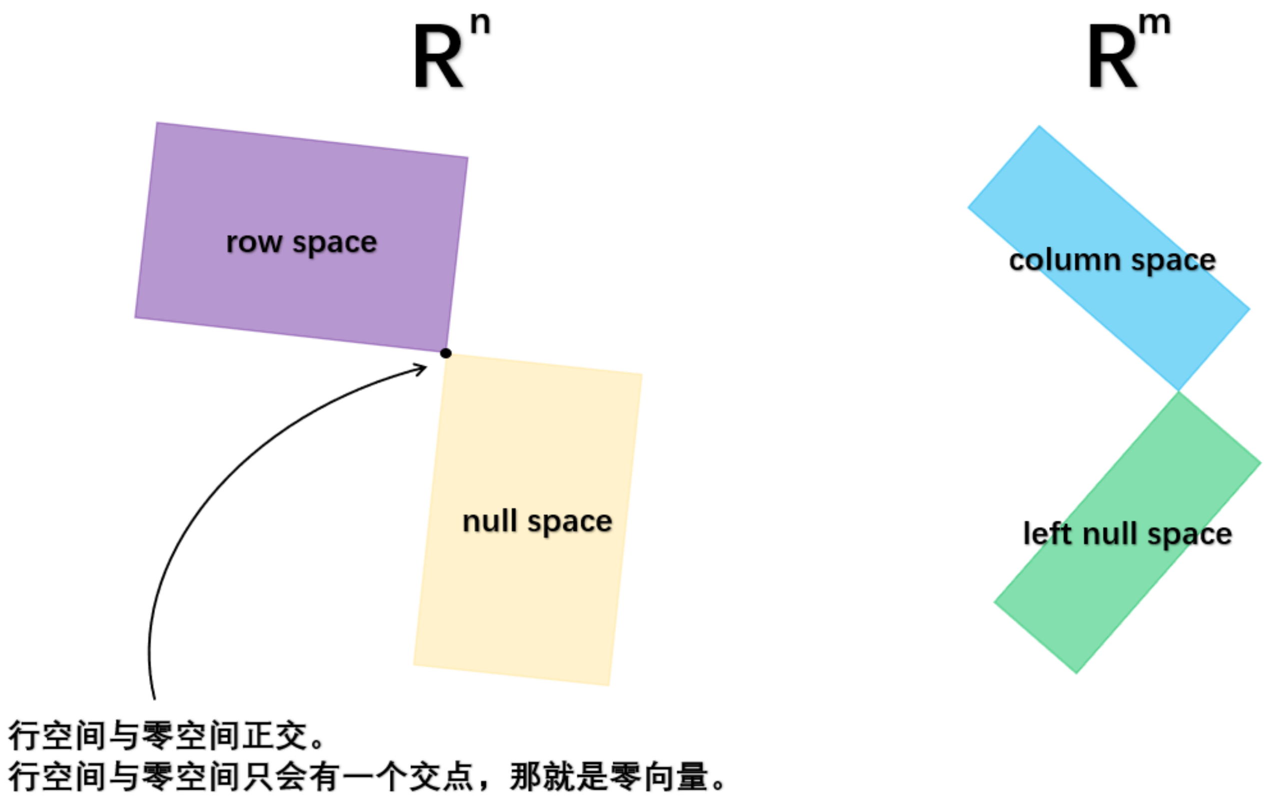
行秩等于列秩！

standard basis for $\mathbb{R}^3$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

10 Lecture 10 - 四个基本子空间

4 fundamental subspaces of $\mathbf{A}_{m\times n}$

- the column space, $C(\mathbf{A})$, $C(\mathbf{A}) \subset \mathbb{R}^m$
- the null space, $N(\mathbf{A})$, $N(\mathbf{A}) \subset \mathbb{R}^n$
- the row space, it's all the combinations of the rows, $C(\mathbf{A}^\mathbf{T})$, $C(\mathbf{A}^\mathbf{T}) \subset \mathbb{R}^n$
- the null space of $\mathbf{A}^\mathbf{T}$, $N(\mathbf{A}^\mathbf{T})$, the left null space of $\mathbf{A}$, $N(\mathbf{A}^\mathbf{T}) \subset \mathbb{R}^m$



	$C(\mathbf{A})$	$C(\mathbf{A}^\mathbf{T})$	$N(\mathbf{A})$	$N(\mathbf{A}^\mathbf{T})$
basis	the pivot columns	the first r rows of $\mathbf{R}$	the special solutions	
dimension	rank r	rank r	$n-r$	$m-r$

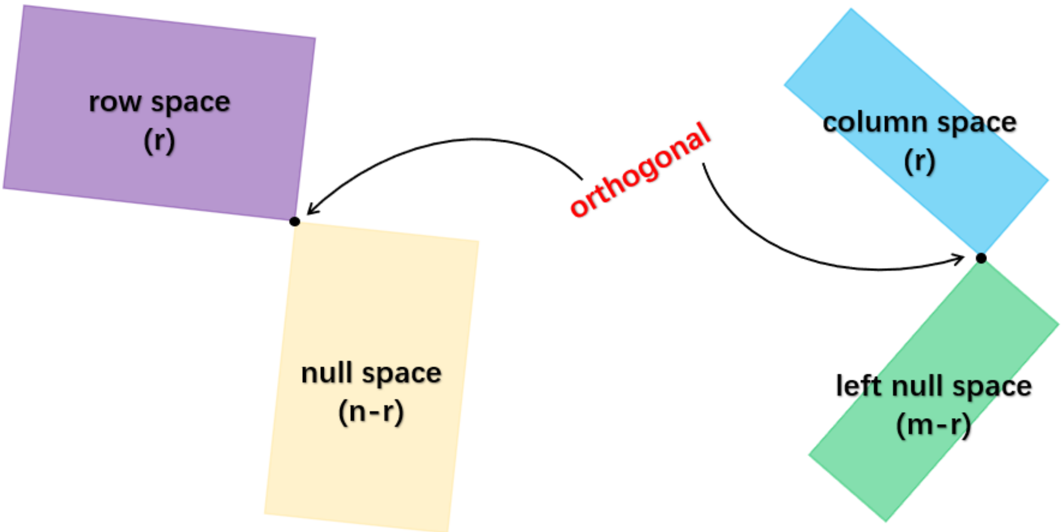
the row space and the column space have the same dimension!

$\mathbf{A} \xrightarrow{rref()} \mathbf{R}$ different column space, same row space
因为 $\mathbf{R}$ 是由 $\mathbf{A}$ 经过行变化而来，所以它们共享同一个行空间。

11 Lecture 14 - 正交向量与子空间

key words:

- orthogonal vectors
- orthogonal subspaces
- orthogonal bases



orthogonal(perpendicular) vectors: $\mathbf{x}^{\mathbf{T}}\mathbf{y} = \mathbf{0}$

proof:

$\because \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = \|\mathbf{x}+\mathbf{y}\|^2 \quad$ (Pythagoras)

即 $\mathbf{x}^{\mathbf{T}}\mathbf{x} + \mathbf{y}^{\mathbf{T}}\mathbf{y} = (\mathbf{x}+\mathbf{y})^{\mathbf{T}}(\mathbf{x}+\mathbf{y})$

$\Leftrightarrow \mathbf{x}^{\mathbf{T}}\mathbf{x} + \mathbf{y}^{\mathbf{T}}\mathbf{y} = (\mathbf{x}^{\mathbf{T}}+\mathbf{y}^{\mathbf{T}})(\mathbf{x}+\mathbf{y})$

$\therefore \mathbf{x}^{\mathbf{T}}\mathbf{y} + \mathbf{x}\mathbf{y}^{\mathbf{T}} = \mathbf{0}$

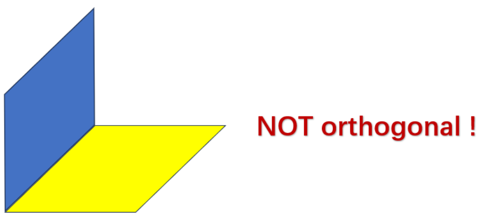
$\because \mathbf{x}^{\mathbf{T}}\mathbf{y} = \mathbf{x}\mathbf{y}^{\mathbf{T}}$

$\therefore \mathbf{x}^{\mathbf{T}}\mathbf{y} = \mathbf{0}$

Q.E.D.

subspace S is orthogonal to subspace T,

means that every vector in S is orthogonal to every vector in T



"non-zero vectors are not orthogonal to themselves"
"if two subspaces meet at some non-zero vectors, they are not orthogonal"

row space is orthogonal to null space, why?

$\mathbf{Ax} = \mathbf{0}$

$$\begin{bmatrix} row_1 \\ row_2 \\ \cdots \\ row_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

therefore,

$c_1(\mathbf{row_1})^\mathbf{T}\mathbf{x} = 0$

$c_2(\mathbf{row_2})^\mathbf{T}\mathbf{x} = 0$

$\cdots \quad \cdots \quad \cdots$

$c_m(\mathbf{row_m})^\mathbf{T}\mathbf{x} = 0$

$\therefore \left[c_1(\mathbf{row_1})^\mathbf{T} + c_2(\mathbf{row_2})^\mathbf{T} + \cdots + c_m(\mathbf{row_m})^\mathbf{T}\right] \mathbf{x} = \mathbf{0}$

$\therefore \left[c_1\mathbf{row_1} + c_2\mathbf{row_2} + \cdots + c_m\mathbf{row_m}\right]^\mathbf{T} \mathbf{x} = \mathbf{0}$

Q.E.D.

正交子空间可以不同维！

null space and row space are orthogonal complements (补集) in $\mathbb{R}^n$

注：空间的正交补，包含了所有与之正交的向量，而不只是部分。

null space contains all, not just some, vectors that are perpendicular to row space

12 Lecture 15 - 子空间投影

“如何求一个无解的方程组的解？”（回归问题、拟合、坏数据）

"solve" $\mathbf{Ax} = \mathbf{b}$ when there is no solution, what's the best solution?

$\mathbf{A^T A}$ is symmetric

proof:

$$(\mathbf{A^T A})^\mathbf{T} = \mathbf{A^T A^{TT}} = \mathbf{A^T A}$$

$rank(\mathbf{A^T A}) = rank(\mathbf{A})$

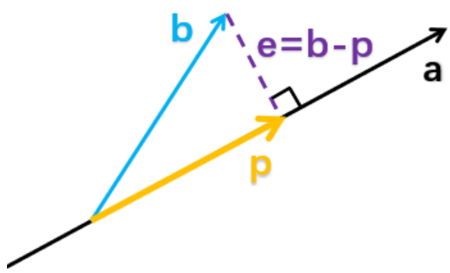
$rank(\mathbf{AB}) \leq min\{rank(\mathbf{A}), rank(\mathbf{B})\}$

$\mathbf{Ax} = \mathbf{b} \longrightarrow \mathbf{A^T A \hat{x}} = \mathbf{A^T b}$

$\mathbf{A^T A}$ 不一定是可逆的 ，例如零矩阵，或者 $\begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix}$

$\mathbf{A^T A}$ is invertible exactly if $\mathbf{A}$ has independent columns

考虑二维的情况



$\mathbf{a^T e} = \mathbf{a^T(b - p)} = \mathbf{a^T}(\mathbf{b} - \lambda\mathbf{a}) = 0$

$\Longrightarrow \lambda\mathbf{a^T a} = \mathbf{a^T b}$

$$\Longrightarrow \lambda = \frac{\mathbf{a^T b}}{\mathbf{a^T a}} \in \mathbb{R}$$

$$\Longrightarrow \mathbf{p} = \lambda \cdot \mathbf{a} = \mathbf{a} \cdot \lambda = \mathbf{a} \cdot \frac{\mathbf{a^T b}}{\mathbf{a^T a}} = \frac{\mathbf{aa^T}}{\mathbf{a^T a}} \cdot \mathbf{b} = \mathbf{Pb}$$

$\mathbf{P}$ is the projection matrix acting on the input, $\mathbf{P}^2 = \mathbf{P}$, $\mathbf{P^T} = \mathbf{P}$

if $\mathbf{b}$ is doubled, $\mathbf{p}$ will be doubled

if $\mathbf{a}$ is doubled, $\mathbf{p}$ will not change at all

为什么这里要引入投影矩阵？

$\because$ $\mathbf{Ax} = \mathbf{b}$ may have no solutions

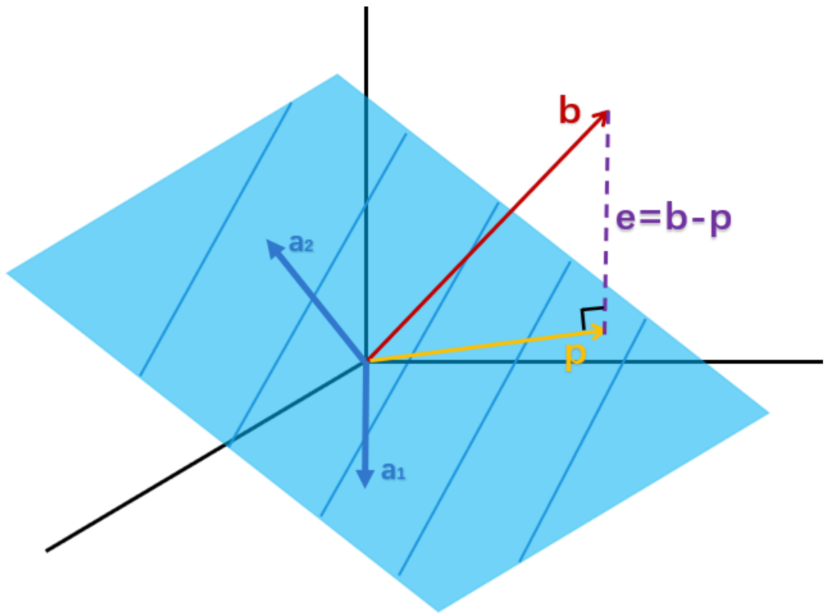
$\therefore$ I turn to solve the closest problem that I can solve

$\because$ $\mathbf{Ax}$ will always be in the column space of $\mathbf{A}$, but $\mathbf{b}$ is probably not

（所以我要怎么微调 $\mathbf{b}$?）

$\therefore$ solve $\mathbf{A}\widehat{\mathbf{x}} = \mathbf{p}$ instead, where $\mathbf{p}$ is the projection of $\mathbf{b}$ onto the column space

考虑三维的情况



$\{\mathbf{a_1}, \mathbf{a_2}\}$ is a basis for the plane, plane is the column space of $\mathbf{A}$

若 $\mathbf{b}$ 不在列空间中，则投影到列空间里，若 $\mathbf{b}$ 在列空间中，则投影结果就是 $\mathbf{b}$ 自己

$\mathbf{e}$ is perpendicular to the plane

投影 $\mathbf{p}$ 是基向量 $\{\mathbf{a_1}, \mathbf{a_2}\}$ 的线性组合，$\mathbf{p} = \hat{x}_1\mathbf{a_1} + \hat{x}_2\mathbf{a_2}$（在 $\mathbf{a_1}$ 方向和 $\mathbf{a_2}$ 方向的投影之和）

如前文所说，我们改为求解 $\mathbf{A}\widehat{\mathbf{x}} = \mathbf{p}$ (to find $\widehat{\mathbf{x}}$)

key: $\mathbf{e} = \mathbf{b} - \mathbf{p} = \mathbf{b} - \mathbf{A}\widehat{\mathbf{x}}$ is perpendicular to the plane

$\because$ $\mathbf{A^T e} = \mathbf{0}$

$\therefore$ $\mathbf{A^T}(\mathbf{b} - \mathbf{A}\widehat{\mathbf{x}}) = \mathbf{0}$

$\therefore$ $\widehat{\mathbf{x}} = (\mathbf{A^T A})^{-1}\mathbf{A^T b}$ $\Longleftarrow$ 这就是方程的近似解

$\mathbf{p} = \mathbf{A}\widehat{\mathbf{x}} = \mathbf{A}(\mathbf{A^T A})^{-1}\mathbf{A^T b}$

$\mathbf{P} = \mathbf{A}(\mathbf{A^T A})^{-1}\mathbf{A^T}$

if $\mathbf{A}$ is invertible, then $\mathbf{A}(\mathbf{A^T A})^{-1}\mathbf{A^T} = \mathbf{I}$

注意，以上对一般的情况不成立，毕竟 $\mathbf{A}$ 甚至不是方阵！

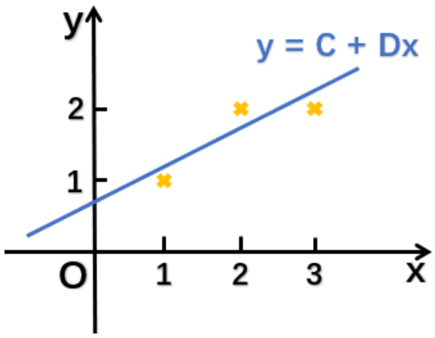
if $\mathbf{A}$ is square and invertible,

此时 $C(\mathbf{A})$ 为整个 $\mathbb{R}^n$ 空间，则 $\mathbf{P} = \mathbf{I}$ ！

“汝即此间人，不借此间物。”

13 Lecture 16 - 投影矩阵、最小二乘

Least Squares: fitting $(1,1)$, $(2,2)$, $(3,2)$ by a line $y = C + Dx$



$$\begin{cases} C + D = 1 \\ C + 2D = 2 \\ C + 3D = 2 \end{cases} \quad \text{or} \quad \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} C \\ D \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}_{\mathbf{b}}$$

$\mathbf{A}\mathbf{x} = \mathbf{b}$ NO solution!

$\mathbf{A}^{\mathrm{T}}\mathbf{A}\widehat{\mathbf{x}} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$ $\exists solution!$

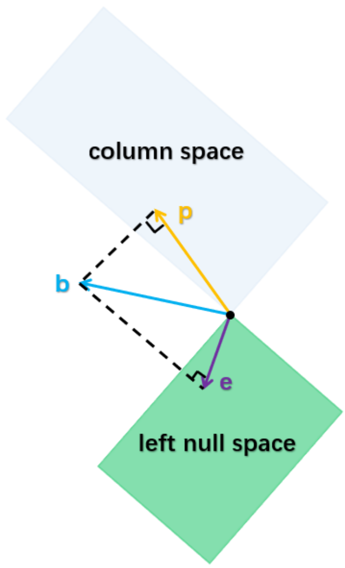
复习一下 projection matrix 投影矩阵

$\mathbf{P} = \mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}$

two extreme cases:

- if $\mathbf{b}$ is in column space of $\mathbf{A}$, then $\mathbf{P}\mathbf{b} = \mathbf{b}$
- if $\mathbf{b}$ is perpendicular to column space of $\mathbf{A}$ ($\mathbf{b} \in N(\mathbf{A}^{\mathrm{T}})$), then $\mathbf{P}\mathbf{b} = \mathbf{0}$

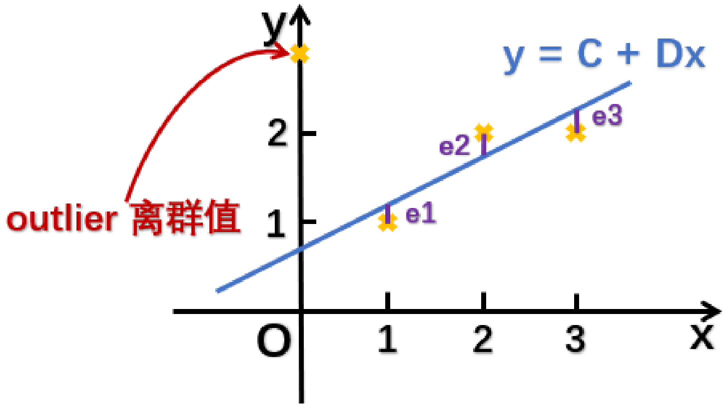
usual cases:



$\mathbf{p}+\mathbf{e}=\mathbf{b}$, where $\mathbf{p}$ is projection to $C(\mathbf{A})$, $\mathbf{e}$ is projection to $N(\mathbf{A^T})$

$\iff \mathbf{Pb}+(\mathbf{I}-\mathbf{P})\mathbf{b}=\mathbf{b}$, where $(\mathbf{I}-\mathbf{P})$ is a projection matrix too, onto the perpendicular space

最小二乘，即最小平方和　　min　error^2



“最小二乘法很容易被离群值影响” so suppose no outliers!

$$\min \quad e_1^2+e_2^2+e_3^2=(C+D-1)^2+(C+2D-2)^2+(C+3D-2)^2$$

to find $\begin{bmatrix}\widehat{C}\\ \widehat{D}\end{bmatrix}=\widehat{\mathbf{x}}$, to solve $\mathbf{A^T A}\widehat{\mathbf{x}}=\mathbf{A^T A}\begin{bmatrix}\widehat{C}\\ \widehat{D}\end{bmatrix}=\mathbf{A^T b}$

then $\begin{cases}3\widehat{C} + 6\widehat{D} = 5\\ 6\widehat{C} + 14\widehat{D} = 11\end{cases} \implies \begin{cases}\widehat{C} = \frac{2}{3}\\ \widehat{D} = \frac{1}{2}\end{cases} \implies y=\frac{1}{2}x+\frac{2}{3}$

suppose p_1, p_2, p_3 are the three values to substitute b_1, b_2, b_3

$\mathbf{b}=\mathbf{p}+\mathbf{e}$ 　 i.e. $\begin{bmatrix}1\\2\\2\end{bmatrix}=\begin{bmatrix}7/6\\5/3\\13/6\end{bmatrix}+\begin{bmatrix}-1/6\\1/3\\-1/6\end{bmatrix}$

$\mathbf{p}\perp\mathbf{e}$ $(\because \mathbf{p}\in C(\mathbf{A}),\ \mathbf{e}\in N(\mathbf{A^T}))$

fact: if $\mathbf{A}$ has independent columns, then $\mathbf{A^T A}$ is invertible

proof:

suppose $\mathbf{A^T A x}=\mathbf{0}$, to prove $\mathbf{x}$ must be $\mathbf{0}$

(trick) $\mathbf{x^T A^T A x}=\mathbf{0}$

$\implies (\mathbf{Ax})^\mathbf{T}\mathbf{Ax}=\mathbf{0}$

$\xrightarrow{\mathbf{y}=\mathbf{Ax},\text{ is a col vector}} \mathbf{y^T y}=\mathbf{0}$

$\iff \mathbf{y}=\mathbf{0}$

$\therefore$ $\mathbf{Ax}$ has to be $\mathbf{0}$

to use the hypothesis: $\mathbf{A}$ has independent columns

$\mathbf{x}$ has to be $\mathbf{0}$

14 Lecture 17 - 正交矩阵、格拉姆-施密特正交化

有一种特别的线性无关：columns are definitely independent if they are perpendicular unit vectors, i.e. orthogonal-normal vectors, or orthonormal vectors in short

orthonormal basis $\mathbf{q_1}, \mathbf{q_2}, \cdots, \mathbf{q_n}$ such that $\mathbf{q_i^T q_j} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$

How does having an orthonormal basis make things better? "they never overflow or underflow"[2]

orthonormal matrix $\mathbf{Q}$ (square)

$$\mathbf{Q} = \begin{bmatrix} | & & | \\ \mathbf{q_1} & \cdots & \mathbf{q_n} \\ | & & | \end{bmatrix}$$

$$\mathbf{Q^TQ} = \begin{bmatrix} -- & \mathbf{q_1^T} & -- \\ & \vdots & \\ -- & \mathbf{q_n^T} & -- \end{bmatrix} \begin{bmatrix} | & & | \\ \mathbf{q_1} & \cdots & \mathbf{q_n} \\ | & & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \mathbf{I}$$

if $\mathbf{Q}$ is square, then $\mathbf{Q^TQ} = \mathbf{I}$ tells that $\mathbf{Q^T} = \mathbf{Q}^{-1}$

注：若 $\mathbf{Q}$ 不是方阵，则未必有 $\mathbf{QQ^T} = \mathbf{I}$ ，但仍然有 $\mathbf{Q^TQ} = \mathbf{I}$ ！

in case that the columns of $\mathbf{A}$ are not orthonormal, how can I make them so?

Graham-Schmidt

to start with independent vectors, and we want to make "them"[3] orthonormal

思想：原来的 $\mathbf{A}$ 的各列张成（span）一个列空间 $C(\mathbf{A})$ ，$\mathbf{A}$ 的列是 $C(\mathbf{A})$ 的一组基，但这组基还不够好（不正交）。现在通过 Graham-Schmidt 方法去找 $C(\mathbf{A})$ 的一组标准正交基。

what's the good of having a $\mathbf{Q}$? what formulas become easier?

suppose $\mathbf{Q}$ has orthonormal columns, to project onto its column space,

$\mathbf{P} = \mathbf{Q}(\mathbf{Q^TQ})^{-1}\mathbf{Q^T} = \mathbf{Q} \cdot \mathbf{I} \cdot \mathbf{Q^T} = \mathbf{QQ^T}$ $\quad (= \mathbf{I}$ if $\mathbf{Q}$ is square)

$(\mathbf{QQ^T})(\mathbf{QQ^T}) = \mathbf{QQ^T}$

to solve the normal equations,

$\mathbf{A^TA\widehat{x}} = \mathbf{A^Tb}$

[2]☆意☆義☆不☆明☆

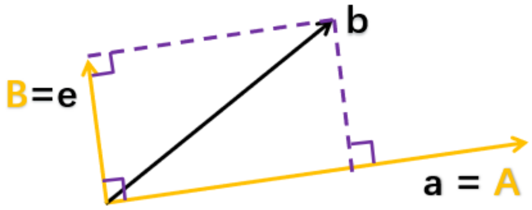
[3]not original ones

now $\mathbf{A}$ is $\mathbf{Q}$,

$\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\widehat{\mathbf{x}} = \mathbf{Q}^{\mathrm{T}}\mathbf{b} \Longleftrightarrow \widehat{\mathbf{x}} = \mathbf{Q}^{\mathrm{T}}\mathbf{b}$

二维的情形：

independent vectors $\mathbf{a}$, $\mathbf{b}$ $\longrightarrow$ orthogonal vectors $\mathbf{A}$, $\mathbf{B}$ $\longrightarrow$ orthonormal vectors $\dfrac{\mathbf{A}}{\|\mathbf{A}\|}, \dfrac{\mathbf{B}}{\|\mathbf{B}\|}$



$$\mathbf{B} = \mathbf{b} - \frac{\mathbf{A}\mathbf{A}^{\mathrm{T}}}{\mathbf{A}^{\mathrm{T}}\mathbf{A}}\mathbf{b} = \mathbf{e} \qquad \text{then } \mathbf{B} \perp \mathbf{A}$$

三维的情形：

independent vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$

$\longrightarrow$ orthogonal vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$

$\longrightarrow$ orthonormal vectors $\dfrac{\mathbf{A}}{\|\mathbf{A}\|}, \dfrac{\mathbf{B}}{\|\mathbf{B}\|}, \dfrac{\mathbf{C}}{\|\mathbf{C}\|}$

idea: $\mathbf{C}$ 向 $\mathbf{A}$、$\mathbf{B}$ 所张成的平面投影

$$\mathbf{C} = \mathbf{c} - \frac{\mathbf{A}\mathbf{A}^{\mathrm{T}}}{\mathbf{A}^{\mathrm{T}}\mathbf{A}}\mathbf{c} - \frac{\mathbf{B}\mathbf{B}^{\mathrm{T}}}{\mathbf{B}^{\mathrm{T}}\mathbf{B}}\mathbf{c} \qquad \text{then } \mathbf{C} \perp \mathbf{A} \text{ and } \mathbf{C} \perp \mathbf{B}$$

example: $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \left(\mathbf{T} = \begin{bmatrix} | & | \\ \mathbf{a} & \mathbf{b} \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \right)$

solution:

$$\mathbf{A} = \mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

check: $\mathbf{A} \perp \mathbf{B}$

$$\mathbf{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \mathbf{q}_2 = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} | & | \\ \mathbf{q}_1 & \mathbf{q}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

the fact: $C(\mathbf{Q}) = C(\mathbf{T})$!

$$\begin{bmatrix} | & | \\ \mathbf{a} & \mathbf{b} \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \mathbf{q}_1 & \mathbf{q}_2 \\ | & | \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^\mathbf{T}\mathbf{a} & \mathbf{q}_1^\mathbf{T}\mathbf{b} \\ \mathbf{q}_2^\mathbf{T}\mathbf{a} & \mathbf{q}_2^\mathbf{T}\mathbf{b} \end{bmatrix} = \begin{bmatrix} | & | \\ \mathbf{q}_1 & \mathbf{q}_2 \\ | & | \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^\mathbf{T}\mathbf{a} & \mathbf{q}_1^\mathbf{T}\mathbf{b} \\ 0 & \mathbf{q}_2^\mathbf{T}\mathbf{b} \end{bmatrix}$$

可以直接看出 $\mathbf{a}$ 与 $\mathbf{b}$ 分别在 $\{\mathbf{q}_1, \mathbf{q}_2\}$ 坐标系中的坐标为 $(\mathbf{q}_1^\mathbf{T}\mathbf{a}, 0)$ 与 $(\mathbf{q}_1^\mathbf{T}\mathbf{b}, \mathbf{q}_2^\mathbf{T}\mathbf{b})$ 。

15 Lecture 18 - 行列式及其性质

Determinants $\quad \det \mathbf{A} = |\mathbf{A}|$

the big reason we need the determinants is for the Eigen Values

3 basic properties

① $\det \mathbf{I} = 1$

② if to exchange 2 rows, then to reverse the sign of the determinant

③ each row 的线性性质

a. $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

b. $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

注：$\det(\mathbf{A}+\mathbf{B}) \neq \det \mathbf{A} + \det \mathbf{B}$

7 corollaries

④ 2 equal rows lead to determinant equals 0

proof:

exchange those 2 rows $\Longrightarrow$ same matrix $\Longrightarrow \det \mathbf{A} = -\det \mathbf{A} \Longrightarrow \det \mathbf{A} = 0$

⑤ to subtract $l \times row_i$ from row_j $(i \neq j)$, the determinant doesn't change

proof:

$$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \begin{vmatrix} a & b \\ a & b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

⑥ row of zeros ($rank(\mathbf{A}) < n$) leads to determinant equals 0

⑦ $\mathbf{U} = \begin{bmatrix} d_1 & * & \cdots & * \\ 0 & d_2 & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & d_n \end{bmatrix}$, $\det \mathbf{U} = d_1 d_2 \cdots d_n =$ product of pivots

“MATLAB 计算行列式的原理”

⑧ $\begin{cases} \det \mathbf{A} = 0 & \text{exactly when } \mathbf{A} \text{ is singular/non-invertible} \\ \det \mathbf{A} \neq 0 & \text{exactly when } \mathbf{A} \text{ is non-singular/invertible} \end{cases}$

⑨ $\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B})$

$\det(\mathbf{A}^{-1}) = (\det \mathbf{A})^{-1} = \dfrac{1}{\det \mathbf{A}}$

$\det(\mathbf{A}^2) = (\det \mathbf{A})^2$

注：$\det(2\mathbf{A}) = 2^n \cdot (\det \mathbf{A})$

⑩ $\det(\mathbf{A}^{\mathrm{T}}) = \det \mathbf{A}$ “行有的性质，列也有”

proof:

$\left|\mathbf{A}^{\mathrm{T}}\right| = |\mathbf{A}| \xleftarrow{\text{只需证明}} \left|\mathbf{U}^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}\right| = |\mathbf{LU}| \xleftarrow{\text{只需证明}} \left|\mathbf{U}^{\mathrm{T}}\right| \left|\mathbf{L}^{\mathrm{T}}\right| = |\mathbf{L}|\,|\mathbf{U}|$

而 $\mathbf{L}$ 、$\mathbf{U}$ 都是三角阵，其行列式只与对角线元素有关。

16 Lecture 19 - 行列式公式、代数余子式

$$
\begin{aligned}
\begin{vmatrix} a & b \\ c & d \end{vmatrix} &= \begin{vmatrix} a+0 & 0+b \\ c & d \end{vmatrix} \\
&= \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} \\
&= \begin{vmatrix} a & 0 \\ c+0 & 0+d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c+0 & 0+d \end{vmatrix} \\
&= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} \\
&= 0 + ad + (-bc) + 0 \\
&= ad - bc
\end{aligned}
$$

$$
\begin{aligned}
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} \\
&\quad + \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix} \\
&= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}
\end{aligned}
$$

BIG FORMULA “逆序数”

$\det \mathbf{A} = \sum_{n!\ terms} \pm a_{1\alpha}a_{2\beta}a_{3\gamma}\cdots a_{n\omega}$, where $(\alpha, \beta, \gamma, \cdots, \omega)$ is a permutation of $(1, 2, 3, \cdots, n)$

onward to cofactors 代数余子式的作用是把 n 阶行列式化简为 $n-1$ 阶行列式

$$
\text{cofactor of } a_{ij} = C_{ij} = \begin{cases} +\det\left((n-1)\times(n-1) \text{ matrix with } row_i, col_j \text{ erased }\right) &, \text{ if } i+j \text{ is even} \\ -\det\left((n-1)\times(n-1) \text{ matrix with } row_i, col_j \text{ erased }\right) &, \text{ if } i+j \text{ is odd} \end{cases}
$$

去掉符号的部分称为余子式 minor

cofactor formula (along row_1): $\det \mathbf{A} = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$

$$
\begin{aligned}
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{12}\left(a_{21}a_{33} - a_{23}a_{31}\right) + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right) \\
&= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix}
\end{aligned}
$$

Example:

$$\det \mathbf{A_1} = \begin{vmatrix} 1 \end{vmatrix} = 1$$

$$\det \mathbf{A_2} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

$$\det \mathbf{A_3} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -1$$

$$\det \mathbf{A_4} = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = \det \mathbf{A_3} - \det \mathbf{A_2} = -1$$

$\cdots \quad \cdots \quad \cdots$

$\det \mathbf{A_n} = \det \mathbf{A_{n-1}} - \det \mathbf{A_{n-2}}$ （递推公式）

so,

$\det \mathbf{A_5} = 0$

$\det \mathbf{A_6} = 1$

$\det \mathbf{A_7} = 1$

$\det \mathbf{A_8} = 0$

1, 0, -1, -1, 0, 1 如此循环，周期为 6

17 Lecture 20 - 克拉默法则、逆矩阵、体积

formula for $\mathbf{A}^{-1}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix}$$

after observing, $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} = \frac{1}{\det \mathbf{A}}\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^{\mathbf{T}} = \frac{1}{\det \mathbf{A}}\mathbf{C}^{\mathbf{T}}$

"algebra instead of algorithm"

to prove that $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\mathbf{C}^{\mathbf{T}}$

proof:

GOAL is that $\mathbf{A}\mathbf{C}^{\mathbf{T}} = (\det \mathbf{A})\,\mathbf{I}$

$\because a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij} = \det \mathbf{A}$

$\because$ when $i \neq j$, $a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} = \begin{vmatrix} \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0$

$$\therefore \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} = \begin{bmatrix} |\mathbf{A}| & & & \\ & |\mathbf{A}| & & \\ & & \ddots & \\ & & & |\mathbf{A}| \end{bmatrix}$$

Q.E.D.

as long as the determinant is not zero, that's exactly when there exists an inverse

$$\mathbf{A}\mathbf{x} = \mathbf{b} \implies \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{1}{\det \mathbf{A}}\mathbf{C}^{\mathbf{T}}\mathbf{b} \implies \text{CRAMER's RULE}$$

$$x_1 = \frac{1}{\det \mathbf{A}}(C_{11}b_1 + C_{21}b_2 + \cdots + C_{n1}b_n) = \frac{1}{\det \mathbf{A}}\begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$x_2 = \frac{1}{\det \mathbf{A}}(C_{12}b_1 + C_{22}b_2 + \cdots + C_{n2}b_n) = \frac{1}{\det \mathbf{A}}\begin{vmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{vmatrix}$$

$\cdots \quad \cdots \quad \cdots \quad \cdots$

$$x_i = \frac{1}{\det \mathbf{A}}(C_{1i}b_1 + C_{2i}b_2 + \cdots + C_{ni}b_n) = \frac{1}{\det \mathbf{A}}\underbrace{\begin{vmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix}}_{\mathbf{A} \text{ with } col_i \text{ replaced by the right-hand side } \mathbf{b}} \triangleq \frac{1}{\det \mathbf{A}}\det(\mathbf{B_i})$$

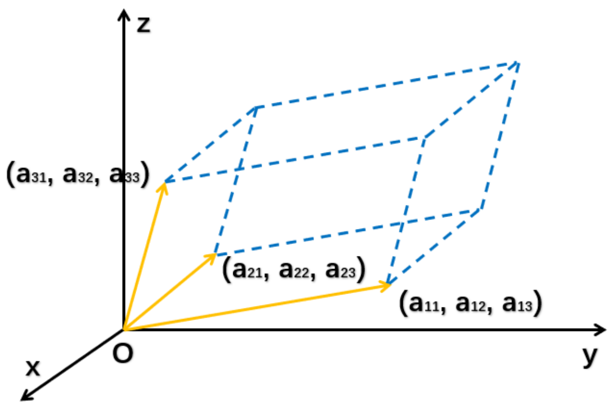
$\cdots \quad \cdots \quad \cdots \quad \cdots$

$$x_n = \frac{1}{\det \mathbf{A}}(C_{1n}b_1 + C_{2n}b_2 + \cdots + C_{nn}b_n) = \frac{1}{\det \mathbf{A}}\begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{vmatrix}$$

the fact

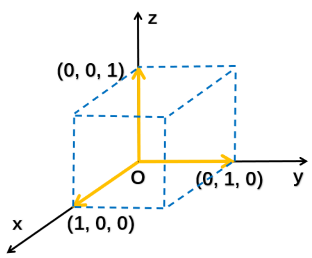
the determinant gives a volume

the determinant is actually equal to the volume of something



$$V = |\det \mathbf{A}| = \left\|\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}\right\| > 0$$ 符号代表了是左手系还是右手系

special case: $\mathbf{A} = \mathbf{I}$



special case: $\mathbf{A} = \mathbf{Q}$ (orthonormal matrix), the cube is a rotated identity cube

proof:

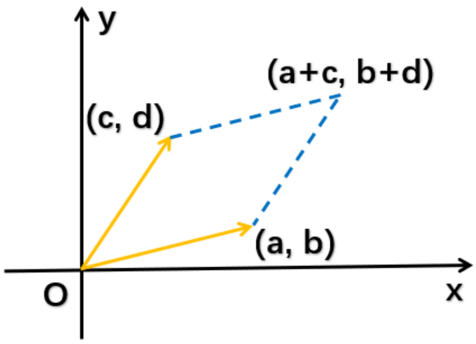
$\because \mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}$

$\therefore \left|\mathbf{Q}^{\mathrm{T}}\right| |\mathbf{Q}| = |\mathbf{I}| = 1$

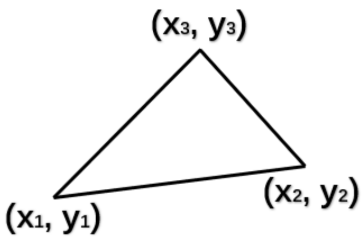
$\because \left|\mathbf{Q}^{\mathrm{T}}\right| = |\mathbf{Q}|$

$\therefore |\mathbf{Q}|^2 = 1$

$\therefore |\mathbf{Q}| = \pm 1$



$$S = \left|\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right| = |ad - bc|$$



$$S = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

18 Lecture 21 - 特征值与特征向量