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1 Lecture 01 - 方程组的几何解释

n linear equations, **n** unknowns

- row picture
- column picture ⋆
- matrix form

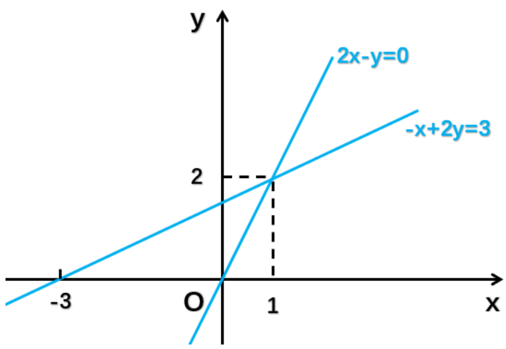
$$\begin{cases} 2x & - & y & = & 0 \\ -x & + & 2y & = & 3 \end{cases}$$

$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, i.e.

$\mathbf{A}$ (matrix of coefficients) $= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $\mathbf{x}$ (vector of unknowns) $= \begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, such that

$$\mathbf{Ax} = \mathbf{b}$$

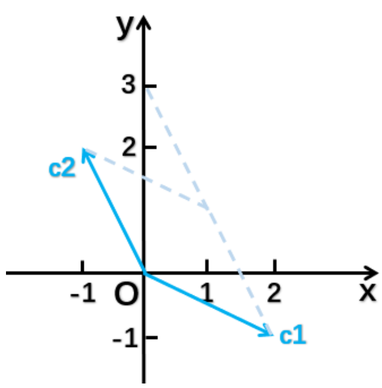
what's the **row** picture?



to find the point that lies on both two lines

what's the **column** picture?

$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$



$$1\overrightarrow{c_1} + 2\overrightarrow{c_2} = \overrightarrow{b}$$

to find the linear combination of columns of **A**, such that it equals **b**

what linear combination gives **b**?

what do all the linear combinations give?

what are all the possible, achievable right-hand sides be?

$$\begin{cases} 2x & - & y & & & = & 0 & \mathbf{1} \\ -x & + & 2y & - & z & = & -1 & \mathbf{2} \\ & - & 3y & + & 4z & = & 4 & \mathbf{3} \end{cases}$$

$$\left\{ \mathbf{1} \right. \quad : \text{the plot of all the points that solve it are a plane}$$

$$\begin{cases} \mathbf{2} \\ \mathbf{3} \end{cases} \quad : \text{two planes meet at a line}$$

$$\begin{cases} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{cases} \quad : \text{meet at a point}$$

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

what's the **row** picture?

to find out all the points that satisfy all the equations

what's the **column** picture?

$$x \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

can I always solve $\mathbf{Ax} = \mathbf{b}$ for every right-hand side **b**?

do the linear combinations of the columns fill 3-dimensional space?

for this **A**, the answer is **YES** (non-singular, invertible)

but for some others **A**, the answer could be **NO** (singular, not-invertible)

if the 3 columns all lie in the same plane,

so I could solve it for some right-hand sides, when $\overrightarrow{b}$ is in the plane,

but most right-hand sides would be out of the plane and unreachable.

in some case, the combinations of **n** columns can only fill out **m**-D $(m < n)$

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

Ax means: **Ax** is a combination of columns of **A**

2 Lecture 02 - 矩阵消元

when solving equations-system,

Elimination, if it succeeds, it gets the answer.

It's always good to ask how could it fail.

$$\begin{cases} x + 2y + z = 2 \\ 3x + 8y + z = 12 \\ 4y + z = 2 \end{cases}$$

$$\begin{bmatrix} \underset{\text{first}-\text{pivot}}{\mathbf{1}} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow[row_3 - 0\times row_1]{row_2 - 3\times row_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & \underset{\text{second}-\text{pivot}}{\mathbf{2}} & -2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{row_3 - 2\times row_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & \underset{\text{third}-\text{pivot}}{\mathbf{5}} \end{bmatrix}$$

pivots can **NOT** be 0 !

if there is a 0 in the pivot position, then try to switch lines

if 0 is in the pivot position and no place to exchange, then failure

let's bring the right-hand side in (Augmented Matrix)

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right] \Rightarrow \begin{cases} x + 2y + z = 2 \\ 2y - 2z = 6 \\ 5z = -10 \end{cases}$$

by back-substitution: $x = 2, y = 1, z = -2$

"elimination matrices"

$$\begin{bmatrix} \vdots & \vdots & \vdots \\ col_1 & col_2 & col_3 \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix} = 1 \times col_1 + 2 \times col_2 + 3 \times col_3$$

the result of multiplying a matrix by some vectors, is a combination of columns of the matrix

$$\begin{bmatrix} 1 & 2 & 7 \end{bmatrix} \begin{bmatrix} \cdots & row_1 & \cdots \\ \cdots & row_2 & \cdots \\ \cdots & row_3 & \cdots \end{bmatrix} = \begin{bmatrix} & & \end{bmatrix} = \begin{matrix} 1 \times row_1 \\ + \\ 2 \times row_2 \\ + \\ 7 \times row_3 \end{matrix}$$

the product of a row times a matrix, is a combination of rows of the matrix

when we do matrix multiplication, keep your eye on what it is doing with the whole vectors

what does the matrix, which can subtract $3 \times row_1$ from row_2 look like?

i.e. $\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$

$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$

$\xrightarrow{as\ R_1=1\times row_1+0\times row_2+0\times row_3} \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$

$\xrightarrow{as\ R_3=0\times row_1+0\times row_2+1\times row_3} \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{as\ R_2=-3\times row_1+1\times row_2+0\times row_3} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$: elementary matrix (初等矩阵)

$\mathbf{E_{i,j}}$ means it's the matrix that we use to fix the (i, j) position

e.g. $\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \xrightarrow{R_1=row_1} \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \xrightarrow{R_2=row_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ? & ? & ? \end{bmatrix} \xrightarrow{R_3=row_3-2\times row_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = E_{3,2}$

in elimination, we can use an elementary matrix to describe the change in each step

the next point in this lecture is to put these steps together, into a matrix that does these steps all in sequence, in another words, how could I create the matrix that does the whole job at once? i.e.

$$\mathbf{E_{3,2}}(\mathbf{E_{2,1}A}) = \mathbf{U} \iff \boxed{?}\mathbf{A} = \mathbf{U}$$

Associative Law

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$

permutation(置换):

- exchange rows, e.g.

  $\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is to exchange row_1 and row_2

- exchange columns, e.g.

  $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

  $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is to exchange col_1 and col_2

when I multiply a matrix on the left, I am doing row operations

if I want to do column operations, I should put a matrix on the right

if $\boxed{?}\mathbf{A} = \mathbf{U}$, then how can I "from $\mathbf{U}$ back to $\mathbf{A}$"?

this is about reversing steps, invertible, $\cdots$

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

"what steps can get me back?"

"what matrix can bring me back?"

3 Lecture 03 - 乘法和逆矩阵

key words:

- matrix multiplication (4 ways)
- inverse of **A**, **AB**, $\mathbf{A}^{\mathbf{T}}$
- Gauss-Jordan, to find $\mathbf{A}^{-1}$

$$\underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} & \\ & \\ & \end{bmatrix}}_{\mathbf{B}} = \underbrace{\begin{bmatrix} & & \\ & c_{i,j} & \\ & & \end{bmatrix}}_{\mathbf{C=AB}}$$

$c_{i,j}$ comes from row_i of **A** and col_j of **B**

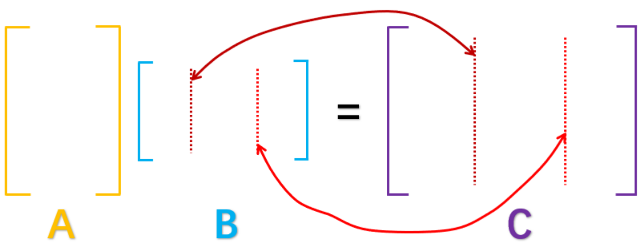
e.g.

$$\begin{aligned} c_{3,4} &= \begin{bmatrix} \\ row_3 \quad of \quad \mathbf{A} \\ \\ \end{bmatrix} \begin{bmatrix} col_4 \\ of \\ \mathbf{B} \end{bmatrix} \\ &= a_{3,1}b_{1,4} + a_{3,2}b_{2,4} + \cdots + a_{3,i}b_{i,4} + \cdots + a_{3,n}b_{n,4} \\ &= \sum_{k=1}^{n} a_{3,k}b_{k,4} \end{aligned}$$

the number of columns of **A** has to match the number of rows of **B**

$$\mathbf{A}_{m\times n}\mathbf{B}_{n\times p} = \mathbf{C}_{m\times p}$$

the matrix times the n^{th} column is the n^{th} column of the answer



so I could think of multiplying a matrix by a vector, side by side

I can just think of having several columns, multiplying by **A**, and getting the columns of answer

the columns of **C** are combinations of columns of **A**

$\Longleftrightarrow$ every column of $\mathbf{C}$ is a combination of columns of $\mathbf{A}$, and numbers in $\mathbf{B}$ tell me what the combination is

in the same way, the rows of $\mathbf{C}$ are combinations of rows of $\mathbf{B}$

what about " $\underbrace{col\ \ of\ \ \mathbf{A}}_{m\times 1} \times \underbrace{row\ \ of\ \ \mathbf{B}}_{1\times p}$ "?

e.g.

$$\begin{bmatrix}2\\3\\4\end{bmatrix}\begin{bmatrix}1 & 6\end{bmatrix}=\begin{bmatrix}2 & 12\\3 & 18\\4 & 24\end{bmatrix}$$

$$\begin{bmatrix}2 & 7\\3 & 8\\4 & 9\end{bmatrix}\begin{bmatrix}1 & 6\\0 & 0\end{bmatrix}=\begin{bmatrix}2\\3\\4\end{bmatrix}\begin{bmatrix}1 & 6\end{bmatrix}+\begin{bmatrix}7\\8\\9\end{bmatrix}\begin{bmatrix}0 & 0\end{bmatrix}$$

$$\begin{aligned}\mathbf{AB} &= \text{sum of }\ (col_i\ of\ \mathbf{A})\times(row_i\ of\ \mathbf{B})\\ &= \sum_{i=1}^{n}(col_i\ of\ \mathbf{A})\times(row_i\ of\ \mathbf{B})\end{aligned}$$

the row space for $\begin{bmatrix}2 & 12\\3 & 18\\4 & 24\end{bmatrix}$, which is like all combinations of the rows, is the line through the row-vector $\begin{bmatrix}1 & 6\end{bmatrix}$, the same to the column space

you could also cut the matrix into blocks and do the multiplication by blocks, i.e.

$$\underbrace{\left[\begin{array}{c|c}\mathbf{A}_1 & \mathbf{A}_2\\ \hline \mathbf{A}_3 & \mathbf{A}_4\end{array}\right]}_{\mathbf{A}}\underbrace{\left[\begin{array}{c|c}\mathbf{B}_1 & \mathbf{B}_2\\ \hline \mathbf{B}_3 & \mathbf{B}_4\end{array}\right]}_{\mathbf{B}}=\underbrace{\left[\begin{array}{c|c}\mathbf{A}_1\mathbf{B}_1+\mathbf{A}_2\mathbf{B}_3 & \mathbf{A}_1\mathbf{B}_2+\mathbf{A}_2\mathbf{B}_4\\ \hline \mathbf{A}_3\mathbf{B}_1+\mathbf{A}_4\mathbf{B}_3 & \mathbf{A}_3\mathbf{B}_2+\mathbf{A}_4\mathbf{B}_4\end{array}\right]}_{\mathbf{AB}}$$

Inverses (square matrices)

not all matrices have inverses, if a matrix is square, is it invertible or not?

if $\mathbf{A}$ is invertible, non-singular, then

$$\mathbf{A}^{-1}\mathbf{A}=\mathbf{I}=\mathbf{A}\mathbf{A}^{-1}$$

in singular case, no inverse!

e.g.

$$\mathbf{A}=\begin{bmatrix}1 & 3\\2 & 6\end{bmatrix}$$

thinking about columns here, if I multiply $\mathbf{A}$ by some other matrices, the columns of the results are all multiples of $\begin{bmatrix}1\\2\end{bmatrix}$, so no way to get the identity matrix $\mathbf{I}$

there is another more important reason

a square matrix has no inverse if I can find a vector $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$

but

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the matrix can't have an inverse if some columns give no contribution!

because

if $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ has an inverse, named $\mathbf{A^{-1}}$, then $\mathbf{A^{-1}A} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \mathbf{A^{-1}} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

and meanwhile, $\mathbf{A^{-1}A} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \mathbf{I} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$,

so that $\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which is not True

our conclusion is that for non-invertible/singular matrices, some combinations of their columns give the zero column

let's take a matrix that does have an inverse for example

e.g.

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\mathbf{A^{-1}}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{I}}$$

then $\begin{cases} \mathbf{A} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \mathbf{A} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$

generally,

$$\mathbf{A} \cdot (col_j \ of \ \mathbf{A^{-1}}) = (col_j \ of \ \mathbf{I})$$

then how to solve the inverse for an invertible matrix?

here is the Gauss-Jordan idea, to solve two equations at once

$$\begin{cases} \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$

"solve them together!"

$$\underbrace{\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]}_{\left[\,\mathbf{A}\,\middle|\,\mathbf{I}\,\right]} \to \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \to \underbrace{\left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]}_{\left[\,\mathbf{I}\,\middle|\,\mathbf{A}^{-1}\,\right]}$$

把单位矩阵当成草稿纸，记录下对左侧矩阵的变换

相当于左右两边同时乘上逆矩阵，当左边变成单位矩阵时，右边即是该逆矩阵

i.e.

$$\begin{aligned} \mathbf{E}_{i_1,j_1}\mathbf{E}_{i_2,j_2}\cdots\mathbf{E}_{i_n,j_n}\left[\,\mathbf{A}\,\middle|\,\mathbf{I}\,\right] &= \left[\,\mathbf{E}_{i_1,j_1}\mathbf{E}_{i_2,j_2}\cdots\mathbf{E}_{i_n,j_n}\mathbf{A}\,\middle|\,\mathbf{E}_{i_1,j_1}\mathbf{E}_{i_2,j_2}\cdots\mathbf{E}_{i_n,j_n}\mathbf{I}\,\right] \\ &= \left[\,\mathbf{I}\,\middle|\,\mathbf{E}_{i_1,j_1}\mathbf{E}_{i_2,j_2}\cdots\mathbf{E}_{i_n,j_n}\,\right] \end{aligned}$$

then $\mathbf{A}^{-1} = \mathbf{E}_{i_1,j_1}\mathbf{E}_{i_2,j_2}\cdots\mathbf{E}_{i_n,j_n}$

注：

$\mathbf{E}_{i_t,j_t}\left[\,\mathbf{A}\,\middle|\,\mathbf{I}\,\right]$，即对 $\left[\,\mathbf{A}\,\middle|\,\mathbf{I}\,\right]$ 做行变换 $\Longleftrightarrow$ 对 $\mathbf{A}$ 与 $\mathbf{I}$ 同时、做同样的行变换

同理，可以对 $\left[\begin{array}{c} \mathbf{A} \\ \hline \mathbf{I} \end{array}\right]$ 做列变换，求得 $\mathbf{A}^{-1}$

4 Lecture 04 - 矩阵的 LU 分解

suppose $\mathbf{A}$ is invertible, and $\mathbf{B}$ is invertible, then what matrix gives me the inverse of $\mathbf{AB}$?

$$\mathbf{AB}(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{I}$$

$$(\mathbf{B}^{-1}\mathbf{A}^{-1})\mathbf{AB} = \mathbf{I}$$

if I transpose a matrix (square, invertible), what's its inverse?

$$\mathbf{AA}^{-1} = \mathbf{I}$$

$$(\mathbf{A}^{-1})^{\mathbf{T}}\mathbf{A}^{\mathbf{T}} = \mathbf{I}$$

$$\Updownarrow$$

$$(\mathbf{A}^{\mathbf{T}})^{-1} = (\mathbf{A}^{-1})^{\mathbf{T}}$$

the $\mathbf{A} = \mathbf{LU}$ is the most basic factorization of a matrix

think of the 2×2 case

$\underbrace{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}}_{\mathbf{E}} \underbrace{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}}_{\mathbf{A}} = \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}_{\mathbf{U}}$

if $\mathbf{A} = \mathbf{LU}$, then

$\underbrace{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}}_{\mathbf{A}} = \underbrace{\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}_{\mathbf{U}}$

so $\mathbf{L} = \mathbf{E}^{-1} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

$\mathbf{U}$ stands for upper triangular matrix, $\mathbf{L}$ stands for lower triangular matrix

what's more,

$$\begin{aligned} \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}}_{\mathbf{U}} \end{aligned}$$

$\mathbf{D}$ stands for diagonal matrix

if $\mathbf{A} = \begin{bmatrix} & \end{bmatrix}_{3\times 3}$,

suppose no row exchanges,

$\mathbf{E}_{3,2}\mathbf{E}_{3,1}\mathbf{E}_{2,1}\mathbf{A} = \mathbf{U}$

$$\mathbf{A} = \boxed{?}\,\mathbf{U}$$

$$\mathbf{A} = \underbrace{\mathbf{E}_{2,1}^{-1}\mathbf{E}_{3,1}^{-1}\mathbf{E}_{3,2}^{-1}}_{\mathbf{L}}\mathbf{U}$$

乘积的逆，只需要分别求逆

we know how to invert, we should take the separate inverses, but they go in the opposite order

$$\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&-5&1\end{bmatrix}}_{\mathbf{E_{3,2}}}\underbrace{\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}}_{\mathbf{E_{2,1}}}=\begin{bmatrix}1&0&0\\-2&1&0\\10&-5&1\end{bmatrix}$$

I subtracted 2 of row_1 from row_2, and then I subtracted 5 of that new row_2 from row_3. So doing it in that order, how did row_1 affect row_3? Because 2 of row_1 got removed from row_2 and then 5 of those got removed from row_3, so altogether 10 of row_1 got thrown into row_3.

$$\mathbf{EA}=\mathbf{U}\quad\text{(elimination)}$$

$$\Downarrow$$

$$\mathbf{A}=\mathbf{E}^{-1}\mathbf{U}=\mathbf{LU}\quad(\mathbf{A}\text{ 的信息包含于 }\mathbf{LU})$$

if no row exchanges, the multipliers go directly into $\mathbf{L}$

how many operations on $n\times n$ matrix $\mathbf{A}$?

e.g.

$$\begin{bmatrix}\quad&&&&\\&&&&\\&&&&\\&&&&\\&&&&\end{bmatrix}_{100\times100}\xrightarrow{100\times99\text{ numbers changed}}\begin{bmatrix}*&\cdots&\cdots&\cdots&\cdots\\0&&&&\\0&&&&\\\vdots&&&&\\0&&&&\end{bmatrix}_{100\times100}$$

$$\xrightarrow{99\times98\text{ numbers changed}}\begin{bmatrix}*&\cdots&\cdots&\cdots&\cdots\\0&*&\cdots&\cdots&\cdots\\0&0&&&\\\vdots&\vdots&&&\\0&0&&&\end{bmatrix}_{100\times100}$$

$$\cdots\quad\cdots\quad\cdots\quad\cdots$$

generally,

$$\sum_{i=1}^{n}i(i-1)=\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}=O(n^3)$$

I am ready to allow row exchanges.

There are some matrices that I will use to do row exchanges.

这些矩阵就是互换单位阵各行的所有的可能的情况。

e.g.

all 3×3 permutations

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{I} \qquad \text{no exhange}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{P_{1,2}} \qquad row_1 \leftrightarrow row_2$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{P_{1,3}} \qquad row_1 \leftrightarrow row_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{P_{2,3}} \qquad row_2 \leftrightarrow row_3$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

how about multiplying two of them together?

the answer is still in the list!

and if I invert, the inverses are all there too!

it's a little family of matrices there

$$\mathbf{P^{-1}} = \mathbf{P}$$

4×4 case $\longrightarrow$ 24 $\mathbf{P'}s$

5 Lecture 05 - 转置、置换、向量空间

书接上回 those are matrices $\mathbf{P}$ and they execute row exchanges

$\mathbf{A} = \mathbf{LU}$: assume no row exchanges

$\mathbf{PA} = \mathbf{LU}$: $\mathbf{P}$ gets the rows into the right order

permutations $\mathbf{P}$ is the identity matrix with reordered rows

$$\mathbf{P^{-1}} = \mathbf{P^T}$$

$$\mathbf{P^T P} = \mathbf{I}$$

we'll be interested in matrices that have $\mathbf{P^T P} = \mathbf{I}$, there are more of them than just permutations

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

Transpose: $\left(\mathbf{A^T}\right)_{i,j} = \mathbf{A}_{j,i}$

Symmetric Matrices: $\mathbf{A^T} = \mathbf{A}$

$\mathbf{R^T R}$ is always symmetric

e.g. $\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 7 \\ 11 & 13 & 11 \\ 7 & 11 & 17 \end{bmatrix}$

$\because \left(\mathbf{R^T R}\right)^\mathbf{T} = \mathbf{R^T} \left(\mathbf{R^T}\right)^\mathbf{T} = \mathbf{R^T R}$

what are vector spaces?

what are sub-spaces?

Example:

$\mathbb{R}^2 \rightarrow$ all 2-dim Real vectors, $\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ e \end{bmatrix}, \cdots$

the whole plane is $\mathbb{R}^2$, so $\mathbb{R}^2$ is the plane (xy plane)

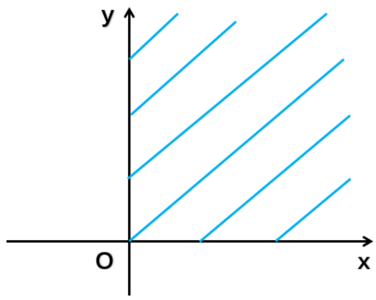
but the point is, it's a vector space

Every vector space has to ensure that zero vector in it.

$\mathbb{R}^3 \rightarrow$ all 3-dim Real vectors

$\mathbb{R}^n \rightarrow$ all vectors with n real components

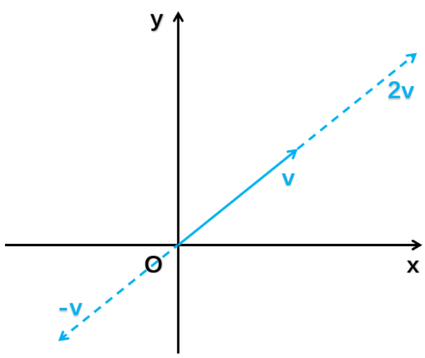
can we do additions and do we stay in the space?



in this figure, it's NOT a vector space, because it's not closed, for example, under multiplication by real numbers

a vector space has to be closed under multiplication and addition of vectors, in other words, linear combination

$\mathbb{R}^n$ is the most important, but we will be interested in vector spaces that are inside $\mathbb{R}^n$, vector spaces that follow the rules



this is a vector space inside $\mathbb{R}^2$ (sub-space of $\mathbb{R}^2$)

what are the possible sub-spaces of $\mathbb{R}^2$?

1. the whole space, $\mathbb{R}^2$ itself

2. lines through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (not the same as $\mathbb{R}^1$)

3. zero vector only

what are the possible sub-spaces of $\mathbb{R}^3$?

1. $\mathbb{R}^3$

2. plane through the origin

3. line through the origin

4. zero vector only

how do sub-spaces come from matrices?

I want to create some sub-spaces out of this matrix: $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$

all linear combinations of its columns (from $\mathbb{R}^3$) form a sub-space, called "column space", $C(\mathbf{A})$

the key idea is, we have to be able to take their combinations, still in the sub-space

if $col_1 // col_2$, then the column space is only a line through the origin

6 Lecture 06 - 列空间和零空间

vector space requirements

$\iff$ $\mathbf{v}+\mathbf{w}$ and $c\mathbf{v}$ are in the space

$\iff$ all combinations $c\mathbf{v}+d\mathbf{w}$ are in the space

notice that these two requirements mean
the sum and the scale of multiplication combine into linear combinations

Example: $\mathbb{R}^3$

2 subspaces: P - a plane, L - a line

❶ the union of those, $P \cup L$, has all vectors in P or L or both, is that a subspace?

NO!

❷ the intersection, $P \cap L$, has all vectors that are in both, is that a subspace?

YES!

the general question is, I have subspaces S and T, is their intersection $S \cap T$ a subspace?

YES!

proof:

if $\mathbf{v} \in S \cap T$, $\mathbf{w} \in S \cap T$

then $\mathbf{v}+\mathbf{w} \in S$ and $\mathbf{v}+\mathbf{w} \in T$

so $\mathbf{v}+\mathbf{w} \in S \cap T$

if $\mathbf{v} \in S \cap T$

then $c\mathbf{v} \in S$ and $c\mathbf{v} \in T$

so $c\mathbf{v} \in S \cap T$

in other words, when you take the intersection of two subspaces,
you get probably a smaller subspace, but it is still a subspace

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

the column space of $\mathbf{A}$, $C(\mathbf{A})$, is a subspace of $\mathbb{R}^4$

what's in that subspace?

not only the columns of $\mathbf{A}$, but also their linear combinations

so $C(\mathbf{A})$ is all linear combinations of $\mathbf{A}$'s columns

$$\text{so I would like to know} \begin{cases} \text{what's in that space?} \\ \text{how big is that space?} \\ \text{is that the whole of 4-dim space? or is it a subspace inside?} \end{cases}$$

取三个四维向量进行线性组合，怎么也得不到整个四维空间嘛！

let's make this question connected with linear equations,
does $\mathbf{Ax} = \mathbf{b}$ always have a solution for every $\mathbf{b}$?
NO, $\mathbf{Ax} = \mathbf{b}$ does not have a solution for every $\mathbf{b}$!

for example, 4 equations and 3 unknowns,
(the combinations of 3 columns cannot always fill the 4-dim space)
there's going to be some $\mathbf{b}$, are not linear combinations of the 3 columns, but sometimes can

what $\mathbf{b}$'s allow me to solve $\mathbf{Ax} = \mathbf{b}$?

I can solve $\mathbf{Ax} = \mathbf{b}$ exactly when 当且仅当 the right-hand side $\mathbf{b}$ is a vector in $C(\mathbf{A})$. (OR $\mathbf{b}$ is a linear combination of $\mathbf{A}$'s columns.)

so, $C(\mathbf{A})$ consists of all vectors $\mathbf{Ax}$ $(\forall \mathbf{x})$

if $\mathbf{b}$ is not a combination of $\mathbf{A}$'s columns, then there is no "$\mathbf{x}$", there is no way to solve $\mathbf{Ax} = \mathbf{b}$

Example: $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$

Question: Are those columns independent?

if I take the linear combinations of $\mathbf{A}$'s columns, does each column contributes something new or not? do I get a 3-D subspace?
NO!

can I throw away any column, and will get the same column space?
YES!

so for this $\mathbf{A}$, $C(\mathbf{A})$ is a 2-D subspace of $\mathbb{R}^4$

the null space 零空间, is going to be a totally different subspace

the null space of $\mathbf{A}$, what's in it?

- it contains not right-hand side $\mathbf{b}$
- it contains $\mathbf{x}$'s
- it contains all $\mathbf{x}$'s that solve "$\mathbf{Ax} = \mathbf{0}$"

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

the null space certainly contains zero ($\because$ the null space is a vector space as well)

for this $\mathbf{A}$,

$$N(\mathbf{A}) \text{ contains } \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}, \cdots, \begin{bmatrix} c \\ c \\ -c \end{bmatrix}$$

$$N(\mathbf{A}) = c \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

the null space is a line in $\mathbb{R}^3$

to check that the solutions to $\mathbf{Ax} = \mathbf{0}$ always give a subspace

proof:

if $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{Ax}^* = \mathbf{0}$

then $\mathbf{A}(\mathbf{x} + \mathbf{x}^*) = \mathbf{0}$

what's more, $\mathbf{A}(c\mathbf{x}) = c(\mathbf{Ax})$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

I would like to know all the solutions to this equation, and if these solutions form a subspace?

NO! As zero vector is not a solution, and subspaces have to go through the origin.

the solutions is a plane/line that does not go through the origin

7 Lecture 07 - 求解 Ax=0 ： 主变量与特解

what's the algorithm for solving $\mathbf{Ax} = \mathbf{0}$?

that's the null space that I'm interested in

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{U}$$

while I am doing elimination,

I am not changing the solutions $\Longrightarrow$ I am not changing the null space

$$\begin{bmatrix} 1 & & 2 & & 2 & & 2 \\ = & = & = & = & & & \\ 0 & & 0 & \| & 2 & & 4 \\ & & & = & = & = & = \\ 0 & & 0 & & 0 & & 0 \end{bmatrix} : \text{echelon form, staircase form}$$

there are two pivots only

the number of pivots = the rank of the matrix = the number of pivot variables

$\mathbf{Ax} = \mathbf{0} \Longrightarrow \mathbf{Ux} = \mathbf{0}$ same solutions, same null space

how do I describe the solutions?

四个三维向量一定线性相关

$$\begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad 2 \text{ free columns, } 2 \text{ pivot columns}$$

free means that we can assign values freely, and we can find the other values accordingly

for convenient purpose, we choose 1 and 0 to those free variables

$$\mathbf{x_1} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x_2} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \qquad \text{2 special solutions (I gave special numbers to free variables)}$$

what are all the solutions to $\mathbf{Ax} = \mathbf{0}$ or $\mathbf{Ux} = \mathbf{0}$?

$$\mathbf{x} = c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} = c\mathbf{x_1} + d\mathbf{x_2}$$

I am taking all the linear combinations of my 2 special solutions, and they are null space.

how many special solution are there? 每个自由变量对应一个特解

$$\begin{bmatrix} & & & \\ & & & \end{bmatrix}_{m\times n} \quad \text{with rank } r, (n-r) \text{ free variables}$$

we get r pivot variables, so there are really r equations there, only r independent equations, and there are $(n-r)$ variables that we can choose freely

Algorithms to Solve $\mathbf{Ax} = \mathbf{0}$

1. do elimination
2. decide which are pivot columns and which are free columns
3. give values to free variables
4. complete pivot values accordingly
5. do linear combinations

reduced row echelon form (**U** 还可以简化)

$$\mathbf{U} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{R}$$

in rref, it has zeros above and below the pivots

$$\begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

notice that there is an identity sitting in the pivot rows and pivot columns!

$\mathbf{Ax} = \mathbf{0} \Longrightarrow \mathbf{Ux} = \mathbf{0} \Longrightarrow \mathbf{Rx} = \mathbf{0}$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

$$\mathbf{A^T} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so the rank is 2 again!

the number of special solutions is 1

the fact: the number of pivot columns of $\mathbf{A}$ and $\mathbf{A^T}$ is the same