Contents

1 Lecture 01 - 方程组的几何解释

n linear equations, **n** unknowns

- row picture
- column picture ⋆
- matrix form

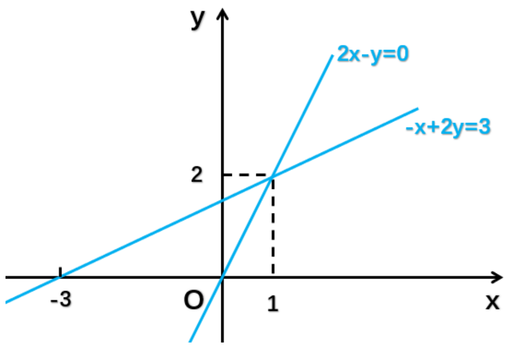
$$\begin{cases} 2x & - & y & = & 0 \\ -x & + & 2y & = & 3 \end{cases}$$

$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, i.e.

$\mathbf{A}$ (matrix of coefficients) $= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $\mathbf{x}$ (vector of unknowns) $= \begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, such that

$$\mathbf{Ax} = \mathbf{b}$$

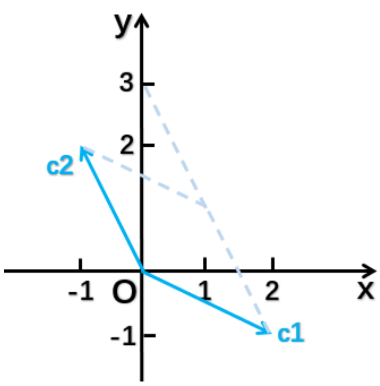
what's the **row** picture?



to find the point that lies on both two lines

what's the **column** picture?

$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$



$$1\overrightarrow{c_1} + 2\overrightarrow{c_2} = \overrightarrow{b}$$

to find the linear combination of columns of **A**, such that it equals **b**

what linear combination gives **b**?

what do all the linear combinations give?

what are all the possible, achievable right-hand sides be?

$$\begin{cases} 2x - y = 0 & \mathbf{1} \\ -x + 2y - z = -1 & \mathbf{2} \\ -3y + 4z = 4 & \mathbf{3} \end{cases}$$

$$\begin{cases} \mathbf{1} \end{cases} : \text{the plot of all the points that solve it are a plane}$$

$$\begin{cases} \mathbf{2} \\ \mathbf{3} \end{cases} : \text{two planes meet at a line}$$

$$\begin{cases} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{cases} : \text{meet at a point}$$

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

what's the **row** picture?

to find out all the points that satisfy all the equations

what's the **column** picture?

$$x \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

can I always solve $\mathbf{Ax} = \mathbf{b}$ for every right-hand side **b**?

do the linear combinations of the columns fill 3-dimensional space?

for this **A**, the answer is **YES** (non-singular, invertible)

but for some others **A**, the answer could be **NO** (singular, not-invertible)

if the 3 columns all lie in the same plane,

so I could solve it for some right-hand sides, when $\overrightarrow{b}$ is in the plane,

but most right-hand sides would be out of the plane and unreachable.

in some case, the combinations of **n** columns can only fill out **m**-D (m < n)

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

Ax means: **Ax** is a combination of columns of **A**

2 Lecture 02 - 矩阵消元

when solving equations-system,

Elimination, if it succeeds, it gets the answer.

It's always good to ask how could it fail.

$$\begin{cases} x + 2y + z = 2 \\ 3x + 8y + z = 12 \\ \quad\;\; 4y + z = 2 \end{cases}$$

$$\begin{bmatrix} \underset{\text{first}-\text{pivot}}{\mathbf{1}} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow[row_3 - 0\times row_1]{row_2 - 3\times row_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & \underset{\text{second}-\text{pivot}}{\mathbf{2}} & -2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{row_3 - 2\times row_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & \underset{\text{third}-\text{pivot}}{\mathbf{5}} \end{bmatrix}$$

pivots can **NOT** be 0 !

if there is a 0 in the pivot position, then try to switch lines

if 0 is in the pivot position and no place to exchange, then failure

let's bring the right-hand side in (Augmented Matrix)

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right] \Rightarrow \begin{cases} x + 2y + z = 2 \\ \quad\;\; 2y - 2z = 6 \\ \qquad\quad\;\; 5z = -10 \end{cases}$$

by back-substitution: $x = 2, y = 1, z = -2$

"elimination matrices"

$$\begin{bmatrix} \vdots & \vdots & \vdots \\ col_1 & col_2 & col_3 \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix} = 1 \times col_1 + 2 \times col_2 + 3 \times col_3$$

the result of multiplying a matrix by some vectors, is a combination of columns of the matrix

$$\begin{bmatrix} 1 & 2 & 7 \end{bmatrix} \begin{bmatrix} \cdots & row_1 & \cdots \\ \cdots & row_2 & \cdots \\ \cdots & row_3 & \cdots \end{bmatrix} = \begin{bmatrix} & & \end{bmatrix} = \begin{array}{ccc} 1 & \times & row_1 \\ & + & \\ 2 & \times & row_2 \\ & + & \\ 7 & \times & row_3 \end{array}$$

the product of a row times a matrix, is a combination of rows of the matrix

when we do matrix multiplication, keep your eye on what it is doing with the whole vectors

what does the matrix, which can subtract $3 \times row_1$ from row_2 look like?

i.e. $\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$

$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$

$\xrightarrow{as\ R_1=1\times row_1+0\times row_2+0\times row_3} \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$

$\xrightarrow{as\ R_3=0\times row_1+0\times row_2+1\times row_3} \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{as\ R_2=-3\times row_1+1\times row_2+0\times row_3} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$: elementary matrix (初等矩阵)

$\mathbf{E_{i,j}}$ means it's the matrix that we use to fix the (i, j) position

e.g. $\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \xrightarrow{R_1=row_1} \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \xrightarrow{R_2=row_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ? & ? & ? \end{bmatrix} \xrightarrow{R_3=row_3-2\times row_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = E_{3,2}$

in elimination, we can use an elementary matrix to describe the change in each step

the next point in this lecture is to put these steps together, into a matrix that does these steps all in sequence, in another words, how could I create the matrix that does the whole job at once? i.e.

$$\mathbf{E_{3,2}}(\mathbf{E_{2,1}A}) = \mathbf{U} \Longleftrightarrow \boxed{?}\mathbf{A} = \mathbf{U}$$

Associative Law

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$

permutation(置换):

- exchange rows, e.g.

  $\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ is to exchange } row_1 \text{ and } row_2$$

- exchange columns, e.g.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ is to exchange } col_1 \text{ and } col_2$$

when I multiply a matrix on the left, I am doing row operations
if I want to do column operations, I should put a matrix on the right

if $\boxed{?}\mathbf{A} = \mathbf{U}$, then how can I "from $\mathbf{U}$ back to $\mathbf{A}$"?
this is about reversing steps, invertible, $\cdots$

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

"what steps can get me back?"
"what matrix can bring me back?"

3 Lecture 03 - 乘法和逆矩阵