Contents

1 Lecture 01 - 方程组的几何解释

n linear equations, **n** unknowns

- row picture
- column picture ⋆
- matrix form

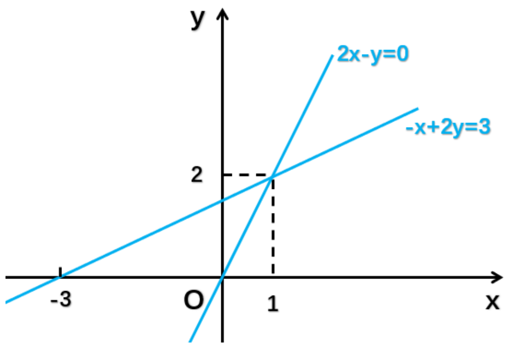
$$\begin{cases} 2x & - & y & = & 0 \\ -x & + & 2y & = & 3 \end{cases}$$

$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, i.e.

$\mathbf{A}$ (matrix of coefficients) $= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $\mathbf{x}$ (vector of unknowns) $= \begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, such that

$$\mathbf{Ax} = \mathbf{b}$$

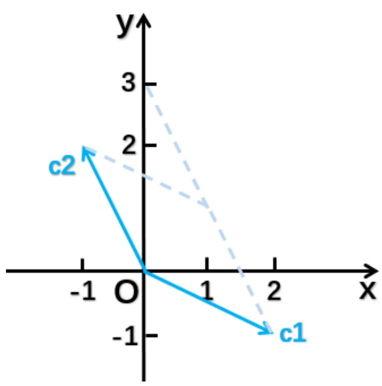
what's the **row** picture?



to find the point that lies on both two lines

what's the **column** picture?

$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$



$$1\overrightarrow{c_1} + 2\overrightarrow{c_2} = \overrightarrow{b}$$

to find the linear combination of columns of **A**, such that it equals **b**

what linear combination gives **b**?

what do all the linear combinations give?

what are all the possible, achievable right-hand sides be?

$$\begin{cases} 2x & - & y & & & = & 0 & \mathbf{1} \\ -x & + & 2y & - & z & = & -1 & \mathbf{2} \\ & - & 3y & + & 4z & = & 4 & \mathbf{3} \end{cases}$$

$$\left\{ \mathbf{1} \right. \quad : \text{the plot of all the points that solve it are a plane}$$

$$\begin{cases} \mathbf{2} \\ \mathbf{3} \end{cases} \quad : \text{two planes meet at a line}$$

$$\begin{cases} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{cases} \quad : \text{meet at a point}$$

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

what's the **row** picture?

to find out all the points that satisfy all the equations

what's the **column** picture?

$$x \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

can I always solve $\mathbf{Ax} = \mathbf{b}$ for every right-hand side **b**?

do the linear combinations of the columns fill 3-dimensional space?

for this **A**, the answer is **YES** (non-singular, invertible)

but for some others **A**, the answer could be **NO** (singular, not-invertible)

if the 3 columns all lie in the same plane,

so I could solve it for some right-hand sides, when $\overrightarrow{b}$ is in the plane,

but most right-hand sides would be out of the plane and unreachable.

in some case, the combinations of **n** columns can only fill out **m**-D $(m < n)$

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

Ax means: **Ax** is a combination of columns of **A**

2 Lecture 02 - 矩阵消元

when solving equations-system,

Elimination, if it succeeds, it gets the answer.

It's always good to ask how could it fail.

$$\begin{cases} x + 2y + z = 2 \\ 3x + 8y + z = 12 \\ 4y + z = 2 \end{cases}$$

$$\begin{bmatrix} \underset{\text{first-pivot}}{\mathbf{1}} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow[row_3 - 0\times row_1]{row_2 - 3\times row_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & \underset{\text{second-pivot}}{\mathbf{2}} & -2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{row_3 - 2\times row_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & \underset{\text{third-pivot}}{\mathbf{5}} \end{bmatrix}$$

pivots can **NOT** be 0 !

if there is a 0 in the pivot position, then try to switch lines

if 0 is in the pivot position and no place to exchange, then failure

let's bring the right-hand side in (Augmented Matrix)

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right] \Rightarrow \begin{cases} x + 2y + z = 2 \\ 2y - 2z = 6 \\ 5z = -10 \end{cases}$$

by back-substitution: $x = 2, y = 1, z = -2$

"elimination matrices"

$$\begin{bmatrix} \vdots & \vdots & \vdots \\ col_1 & col_2 & col_3 \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix} = 1 \times col_1 + 2 \times col_2 + 3 \times col_3$$

the result of multiplying a matrix by some vectors, is a combination of columns of the matrix

$$\begin{bmatrix} 1 & 2 & 7 \end{bmatrix} \begin{bmatrix} \cdots & row_1 & \cdots \\ \cdots & row_2 & \cdots \\ \cdots & row_3 & \cdots \end{bmatrix} = \begin{bmatrix} & & \end{bmatrix} = \begin{array}{c} 1 \times row_1 \\ + \\ 2 \times row_2 \\ + \\ 7 \times row_3 \end{array}$$

the product of a row times a matrix, is a combination of rows of the matrix

when we do matrix multiplication, keep your eye on what it is doing with the whole vectors

what does the matrix, which can subtract $3 \times row_1$ from row_2 look like?

i.e. $\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$

$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$

$\xrightarrow{as\ R_1=\mathbf{1}\times row_1+\mathbf{0}\times row_2+\mathbf{0}\times row_3} \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$

$\xrightarrow{as\ R_3=\mathbf{0}\times row_1+\mathbf{0}\times row_2+\mathbf{1}\times row_3} \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{as\ R_2=\mathbf{-3}\times row_1+\mathbf{1}\times row_2+\mathbf{0}\times row_3} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$: elementary matrix (初等矩阵)

$\mathbf{E_{i,j}}$ means it's the matrix that we use to fix the (i, j) position

e.g. $\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \xrightarrow{R_1=row_1} \begin{bmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \xrightarrow{R_2=row_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ? & ? & ? \end{bmatrix} \xrightarrow{R_3=row_3-2\times row_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = E_{3,2}$

in elimination, we can use an elementary matrix to describe the change in each step

the next point in this lecture is to put these steps together, into a matrix that does these steps all in sequence, in another words, how could I create the matrix that does the whole job at once? i.e.

$$\mathbf{E_{3,2}}(\mathbf{E_{2,1}A}) = \mathbf{U} \Longleftrightarrow \boxed{?}\mathbf{A} = \mathbf{U}$$

Associative Law

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$

permutation(置换):

- exchange rows, e.g.

  $\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is to exchange row_1 and row_2

- exchange columns, e.g.

  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$

  $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is to exchange col_1 and col_2

when I multiply a matrix on the left, I am doing row operations

if I want to do column operations, I should put a matrix on the right

if $\boxed{?}\mathbf{A} = \mathbf{U}$, then how can I "from $\mathbf{U}$ back to $\mathbf{A}$"?

this is about reversing steps, invertible, $\cdots$

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

"what steps can get me back?"

"what matrix can bring me back?"

3 Lecture 03 - 乘法和逆矩阵

key words:

- matrix multiplication (4 ways)
- inverse of $\mathbf{A}$, $\mathbf{AB}$, $\mathbf{A^T}$
- Gauss-Jordan, to find $\mathbf{A}^{-1}$

$$\underbrace{\begin{bmatrix} \quad & \quad \\ & \\ & \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \quad & \quad \\ & \\ & \\ & \end{bmatrix}}_{\mathbf{B}} = \underbrace{\begin{bmatrix} & & \\ & c_{i,j} & \\ & & \end{bmatrix}}_{\mathbf{C=AB}}$$

$c_{i,j}$ comes from row_i of $\mathbf{A}$ and col_j of $\mathbf{B}$

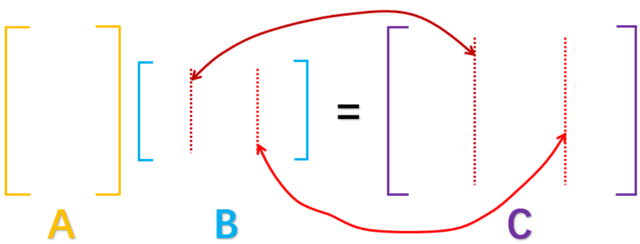
e.g.

$$\begin{aligned} c_{3,4} &= \begin{bmatrix} \\ row_3 \quad of \quad \mathbf{A} \\ \\ \end{bmatrix} \begin{bmatrix} col_4 \\ of \\ \mathbf{B} \end{bmatrix} \\ &= a_{3,1}b_{1,4} + a_{3,2}b_{2,4} + \cdots + a_{3,i}b_{i,4} + \cdots + a_{3,n}b_{n,4} \\ &= \sum_{k=1}^{n} a_{3,k}b_{k,4} \end{aligned}$$

the number of columns of $\mathbf{A}$ has to match the number of rows of $\mathbf{B}$

$$\mathbf{A}_{m\times n}\mathbf{B}_{n\times p} = \mathbf{C}_{m\times p}$$

the matrix times the n^{th} column is the n^{th} column of the answer



so I could think of multiplying a matrix by a vector, side by side
I can just think of having several columns, multiplying by $\mathbf{A}$, and getting the columns of answer

the columns of $\mathbf{C}$ are combinations of columns of $\mathbf{A}$

$\Longleftrightarrow$ every column of $\mathbf{C}$ is a combination of columns of $\mathbf{A}$, and numbers in $\mathbf{B}$ tell me what the combination is

in the same way, the rows of $\mathbf{C}$ are combinations of rows of $\mathbf{B}$

what about " $\underbrace{col\ \ of\ \ \mathbf{A}}_{m\times 1} \quad \times \quad \underbrace{row\ \ of\ \ \mathbf{B}}_{1\times p}$ "?

e.g.

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \mathbf{AB} &= \text{sum of } \ (col_i\ of\ \ \mathbf{A}) \times (row_i\ of\ \ \mathbf{B}) \\ &= \sum_{i=1}^{n} (col_i\ of\ \ \mathbf{A}) \times (row_i\ of\ \ \mathbf{B}) \end{aligned}$$

the row space for $\begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$, which is like all combinations of the rows, is the line through the row-vector $\begin{bmatrix} 1 & 6 \end{bmatrix}$, the same to the column space

you could also cut the matrix into blocks and do the multiplication by blocks, i.e.

$$\underbrace{\left[\begin{array}{c|c} \mathbf{A}_1 & \mathbf{A}_2 \\ \hline \mathbf{A}_3 & \mathbf{A}_4 \end{array}\right]}_{\mathbf{A}} \underbrace{\left[\begin{array}{c|c} \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathbf{B}_3 & \mathbf{B}_4 \end{array}\right]}_{\mathbf{B}} = \underbrace{\left[\begin{array}{c|c} \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_3 & \mathbf{A}_1\mathbf{B}_2 + \mathbf{A}_2\mathbf{B}_4 \\ \hline \mathbf{A}_3\mathbf{B}_1 + \mathbf{A}_4\mathbf{B}_3 & \mathbf{A}_3\mathbf{B}_2 + \mathbf{A}_4\mathbf{B}_4 \end{array}\right]}_{\mathbf{AB}}$$

Inverses (square matrices)

not all matrices have inverses, if a matrix is square, is it invertible or not?

if $\mathbf{A}$ is invertible, non-singular, then

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$$

in singular case, no inverse!

e.g.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

thinking about columns here, if I multiply $\mathbf{A}$ by some other matrices, the columns of the results are all multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so no way to get the identity matrix $\mathbf{I}$

there is another more important reason

a square matrix has no inverse if I can find a vector $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$

but
$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the matrix can't have an inverse if some columns give no contribution!

because

if $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ has an inverse, named $\mathbf{A^{-1}}$, then $\mathbf{A^{-1}A} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

and meanwhile, $\mathbf{A^{-1}A} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \mathbf{I} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$,

so that $\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which is not True

our conclusion is that for non-invertible/singular matrices, some combinations of their columns give the zero column

let's take a matrix that does have an inverse for example

e.g.
$$\underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\mathbf{A}^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{I}}$$

$$\text{then} \begin{cases} \mathbf{A} \begin{bmatrix} a \\ b \end{bmatrix} & = & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \mathbf{A} \begin{bmatrix} c \\ d \end{bmatrix} & = & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$

generally,
$$\mathbf{A} \cdot (col_j \ of \ \mathbf{A^{-1}}) = (col_j \ of \ \mathbf{I})$$

then how to solve the inverse for an invertible matrix?

here is the Gauss-Jordan idea, to solve two equations at once
$$\begin{cases} \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} & = & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} & = & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$
"solve them together!"

$$\underbrace{\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]}_{\left[\begin{array}{c|c} \mathbf{A} & \mathbf{I} \end{array}\right]} \rightarrow \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \rightarrow \underbrace{\left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]}_{\left[\begin{array}{c|c} \mathbf{I} & \mathbf{A}^{-1} \end{array}\right]}$$

把单位矩阵当成草稿纸，记录下对左侧矩阵的变换

相当于左右两边同时乘上逆矩阵，当左边变成单位矩阵时，右边即是该逆矩阵

i.e.

$$\begin{aligned} \mathbf{E}_{i_1,j_1}\mathbf{E}_{i_2,j_2}\cdots\mathbf{E}_{i_n,j_n}\left[\begin{array}{c|c} \mathbf{A} & \mathbf{I} \end{array}\right] &= \left[\begin{array}{c|c} \mathbf{E}_{i_1,j_1}\mathbf{E}_{i_2,j_2}\cdots\mathbf{E}_{i_n,j_n}\mathbf{A} & \mathbf{E}_{i_1,j_1}\mathbf{E}_{i_2,j_2}\cdots\mathbf{E}_{i_n,j_n}\mathbf{I} \end{array}\right] \\ &= \left[\begin{array}{c|c} \mathbf{I} & \mathbf{E}_{i_1,j_1}\mathbf{E}_{i_2,j_2}\cdots\mathbf{E}_{i_n,j_n} \end{array}\right] \end{aligned}$$

then $\mathbf{A}^{-1} = \mathbf{E}_{i_1,j_1}\mathbf{E}_{i_2,j_2}\cdots\mathbf{E}_{i_n,j_n}$

注：

$\mathbf{E}_{i_t,j_t}\left[\begin{array}{c|c} \mathbf{A} & \mathbf{I} \end{array}\right]$，即对 $\left[\begin{array}{c|c} \mathbf{A} & \mathbf{I} \end{array}\right]$ 做行变换 $\Longleftrightarrow$ 对 $\mathbf{A}$ 与 $\mathbf{I}$ 同时、做同样的行变换

同理，可以对 $\left[\begin{array}{c} \mathbf{A} \\ \hline \mathbf{I} \end{array}\right]$ 做列变换，求得 $\mathbf{A}^{-1}$