Chapter 1: PROBABILITY

NGUYỄN THỊ THU THỦY(1)

SCHOOL OF APPLIED MATHEMATICS AND INFORMATICS
HANOI UNIVERSITY OF SCIENCE AND TECHNOLOGY

HANOI – 9/2020

(1)Email: thuy.nguyenthithu2@hust.edu.vn

Content

1.1.1. Experiment

Definition 1.1 (Experiment)

An **experiment** is the process by which an observation (or measurement) is obtained.

Example 1.1 (Some examples of experiment)

1. Flip a coin. Did it land with heads or tails facing up?
2. Walk to a bus stop. How long do you wait for the arrival of a bus?
3. Give a lecture. How many students are seated in the fourth row?

Example 1.2 An experiment consists of the following procedure, observation, and model

1. **Procedure**: Flip a coin and let it land on a table.
2. **Observation**: Observe which side (head or tail) faces you after the coin lands.
3. **Model**: Heads and tails are equally likely. The result of each flip is unrelated to the results of previous flips.

1.1.1. Experiment

Definition 1.1 (Experiment)

An **experiment** is the process by which an observation (or measurement) is obtained.

Example 1.1 (Some examples of experiment)

1. Flip a coin. Did it land with heads or tails facing up?
2. Walk to a bus stop. How long do you wait for the arrival of a bus?
3. Give a lecture. How many students are seated in the fourth row?

Example 1.2 An experiment consists of the following procedure, observation, and model

1. **Procedure**: Flip a coin and let it land on a table.
2. **Observation**: Observe which side (head or tail) faces you after the coin lands.
3. **Model**: Heads and tails are equally likely. The result of each flip is unrelated to the results of previous flips.

1.1.1. Experiment

Definition 1.1 (Experiment)

An **experiment** is the process by which an observation (or measurement) is obtained.

Example 1.1 (Some examples of experiment)

1. Flip a coin. Did it land with heads or tails facing up?
2. Walk to a bus stop. How long do you wait for the arrival of a bus?
3. Give a lecture. How many students are seated in the fourth row?

Example 1.2 An experiment consists of the following procedure, observation, and model

1. **Procedure**: Flip a coin and let it land on a table.
2. **Observation**: Observe which side (head or tail) faces you after the coin lands.
3. **Model**: Heads and tails are equally likely. The result of each flip is unrelated to the results of previous flips.

1.1.1. Experiment

Definition 1.1 (Experiment)

An **experiment** is the process by which an observation (or measurement) is obtained.

Example 1.1 (Some examples of experiment)

1. Flip a coin. Did it land with heads or tails facing up?
2. Walk to a bus stop. How long do you wait for the arrival of a bus?
3. Give a lecture. How many students are seated in the fourth row?

Example 1.2 An experiment consists of the following procedure, observation, and model

1. **Procedure**: Flip a coin and let it land on a table.
2. **Observation**: Observe which side (head or tail) faces you after the coin lands.
3. **Model**: Heads and tails are equally likely. The result of each flip is unrelated to the results of previous flips.

1.1.1. Experiment

Definition 1.1 (Experiment)

An **experiment** is the process by which an observation (or measurement) is obtained.

Example 1.1 (Some examples of experiment)

1. Flip a coin. Did it land with heads or tails facing up?
2. Walk to a bus stop. How long do you wait for the arrival of a bus?
3. Give a lecture. How many students are seated in the fourth row?

Example 1.2 An experiment consists of the following procedure, observation, and model

1. **Procedure**: Flip a coin and let it land on a table.
2. **Observation**: Observe which side (head or tail) faces you after the coin lands.
3. **Model**: Heads and tails are equally likely. The result of each flip is unrelated to the results of previous flips.

1.1.1. Experiment

Definition 1.1 (Experiment)

An **experiment** is the process by which an observation (or measurement) is obtained.

Example 1.1 (Some examples of experiment)

1. Flip a coin. Did it land with heads or tails facing up?
2. Walk to a bus stop. How long do you wait for the arrival of a bus?
3. Give a lecture. How many students are seated in the fourth row?

Example 1.2 An experiment consists of the following procedure, observation, and model

1. **Procedure**: Flip a coin and let it land on a table.
2. **Observation**: Observe which side (head or tail) faces you after the coin lands.
3. **Model**: Heads and tails are equally likely. The result of each flip is unrelated to the results of previous flips.

1.1.1. Experiment

Definition 1.1 (Experiment)

An **experiment** is the process by which an observation (or measurement) is obtained.

Example 1.1 (Some examples of experiment)

1. Flip a coin. Did it land with heads or tails facing up?
2. Walk to a bus stop. How long do you wait for the arrival of a bus?
3. Give a lecture. How many students are seated in the fourth row?

Example 1.2 An experiment consists of the following procedure, observation, and model

1. **Procedure**: Flip a coin and let it land on a table.
2. **Observation**: Observe which side (head or tail) faces you after the coin lands.
3. **Model**: Heads and tails are equally likely. The result of each flip is unrelated to the results of previous flips.

1.1.1. Experiment

Definition 1.1 (Experiment)

An **experiment** is the process by which an observation (or measurement) is obtained.

Example 1.1 (Some examples of experiment)

1. Flip a coin. Did it land with heads or tails facing up?
2. Walk to a bus stop. How long do you wait for the arrival of a bus?
3. Give a lecture. How many students are seated in the fourth row?

Example 1.2 An experiment consists of the following procedure, observation, and model

1. **Procedure**: Flip a coin and let it land on a table.
2. **Observation**: Observe which side (head or tail) faces you after the coin lands.
3. **Model**: Heads and tails are equally likely. The result of each flip is unrelated to the results of previous flips.

1.1.1. Experiment

Note

- An experiment consists of both a procedure and observations.
- It is important to understand that two experiments with the same procedure but with different observations are different experiments.

Example 1.3 Flip a coin three times

(a) Observe the sequence of heads and tails.

(b) Observe the number of heads.

Note

- These two experiments have the same procedure: flip a coin three times.
- They are different experiments because they require different observations.

1.1.1. Experiment

Note

- An experiment consists of both a procedure and observations.
- It is important to understand that two experiments with the same procedure but with different observations are different experiments.

Example 1.3 Flip a coin three times

(a) Observe the sequence of heads and tails.

(b) Observe the number of heads.

Note

- These two experiments have the same procedure: flip a coin three times.
- They are different experiments because they require different observations.

1.1.1. Experiment

Note

- An experiment consists of both a procedure and observations.
- It is important to understand that two experiments with the same procedure but with different observations are different experiments.

Example 1.3 Flip a coin three times

(a) Observe the sequence of heads and tails.

(b) Observe the number of heads.

Note

- These two experiments have the same procedure: flip a coin three times.
- They are different experiments because they require different observations.

1.1.1. Experiment

Note

- An experiment consists of both a procedure and observations.
- It is important to understand that two experiments with the same procedure but with different observations are different experiments.

Example 1.3 Flip a coin three times

(a) Observe the sequence of heads and tails.

(b) Observe the number of heads.

Note

- These two experiments have the same procedure: flip a coin three times.
- They are different experiments because they require different observations.

1.1.1. Experiment

Note

- An experiment consists of both a procedure and observations.
- It is important to understand that two experiments with the same procedure but with different observations are different experiments.

Example 1.3 Flip a coin three times

(a) Observe the sequence of heads and tails.

(b) Observe the number of heads.

Note

- These two experiments have the same procedure: flip a coin three times.
- They are different experiments because they require different observations.

1.1.1. Experiment

Note

- An experiment consists of both a procedure and observations.
- It is important to understand that two experiments with the same procedure but with different observations are different experiments.

Example 1.3 Flip a coin three times

(a) Observe the sequence of heads and tails.

(b) Observe the number of heads.

Note

- These two experiments have the same procedure: flip a coin three times.
- They are different experiments because they require different observations.

1.1.2. Sample Space

Definition 1.2 (Outcome)

An **outcome** of an experiment is any possible observation of that experiment.

Definition 1.3 (Sample Space)

The **sample space** of an experiment is the set of all possible outcomes for an experiment.

Denoted by S.

Sample point

Each outcome in a sample space is called an element or a member of the sample space, or simply a **sample point**.

1.1.2. Sample Space

Definition 1.2 (Outcome)

An **outcome** of an experiment is any possible observation of that experiment.

Definition 1.3 (Sample Space)

The **sample space** of an experiment is the set of all possible outcomes for an experiment.

Denoted by S.

Sample point

Each outcome in a sample space is called an element or a member of the sample space, or simply a **sample point**.

1.1.2. Sample Space

Example 1.4

(a) The sample space in Example 1.2 is $S = \{H, T\}$ where H is the outcome "observe head," and T is the outcome "observe tail."

(b) The sample space in Example 1.3(a) is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

(c) The sample space in Example 1.3(b) is $S = \{0, 1, 2, 3\}$.

1.1.2. Sample Space

Example 1.4

(a) The sample space in Example 1.2 is $S = \{H, T\}$ where H is the outcome "observe head," and T is the outcome "observe tail."

(b) The sample space in Example 1.3(a) is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

(c) The sample space in Example 1.3(b) is $S = \{0, 1, 2, 3\}$.

1.1.2. Sample Space

Example 1.4

(a) The sample space in Example 1.2 is $S = \{H, T\}$ where H is the outcome "observe head," and T is the outcome "observe tail."

(b) The sample space in Example 1.3(a) is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

(c) The sample space in Example 1.3(b) is $S = \{0, 1, 2, 3\}$.

1.1.2. Sample Space

Example 1.5

An experiment consists of flipping a coin and then flipping it a second time if a head occurs. If a tail occurs on the first flip, then a die is tossed once.

Example 1.5 Solution

To list the elements of the sample space providing the most information, we construct the tree diagram of Figure 1.1. The sample space is

$$S = \{HH, HT, T1, T2, T3, T4, T5, T6\}.$$

1.1.2. Sample Space

Figure 1.1

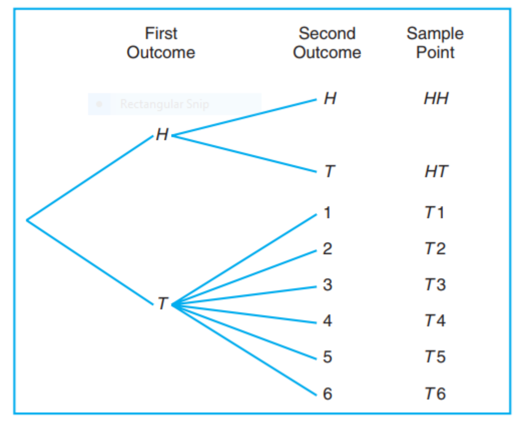


Figure: Tree diagram for Example 1.5

1.1.2. Sample Space

Note

Sample spaces with a large or infinite number of sample points are best described by a **statement** or **rule method**.

Examples

- If the possible outcomes of an experiment are the set of cities in the world with a population over 1 million, our sample space is written

$$S = \{x \mid x \text{ is a city with a population over 1 million}\}.$$

- If S is the set of all points (x, y) on the boundary or the interior of a circle of radius 2 with center at the origin, we write the rule

$$S = \{(x, y) \mid x^2 + y^2 \leq 4\}.$$

1.1.2. Sample Space

Note

Sample spaces with a large or infinite number of sample points are best described by a **statement** or **rule method**.

Examples

- If the possible outcomes of an experiment are the set of cities in the world with a population over 1 million, our sample space is written

$$S = \{x \mid x \text{ is a city with a population over 1 million}\}.$$

- If S is the set of all points (x, y) on the boundary or the interior of a circle of radius 2 with center at the origin, we write the rule

$$S = \{(x, y) \mid x^2 + y^2 \leq 4\}.$$

1.1.2. Sample Space

Note

Sample spaces with a large or infinite number of sample points are best described by a **statement** or **rule method**.

Examples

- If the possible outcomes of an experiment are the set of cities in the world with a population over 1 million, our sample space is written

$$S = \{x \mid x \ \text{ is a city with a population over 1 million}\}.$$

- If S is the set of all points (x, y) on the boundary or the interior of a circle of radius 2 with center at the origin, we write the rule

$$S = \{(x, y) \mid x^2 + y^2 \leq 4\}.$$

1.1.3. Event

Definition 1.4 (Event)

An **event** is a set of outcomes of an experiment (or a subset of a sample space).

Example 1.7

Suppose we roll a six-sided die and observe the number of dots on the side facing upwards. We can label these outcomes E_i, $i = 1, \dots, 6$ where E_i denotes the outcome that i dots appear on the up face.

1. The sample space is $S = \{E_1, E_2, \dots, E_6\}$.
2. Each subset of S is an event. Examples of events are
   - The event $A = \{$Roll 4 or higher $\} = \{E_4, E_5, E_6\}$.
   - The event $B = \{$The roll is even$\} = \{E_2, E_4, E_6\}$.
   - The event $C = \{$The roll is the square of an integer$\} = \{E_1, E_4\}$.

1.1.3. Event

Definition 1.4 (Event)

An **event** is a set of outcomes of an experiment (or a subset of a sample space).

Example 1.7

Suppose we roll a six-sided die and observe the number of dots on the side facing upwards. We can label these outcomes E_i, $i = 1, \dots, 6$ where E_i denotes the outcome that i dots appear on the up face.

1. The sample space is $S = \{E_1, E_2, \dots, E_6\}$.
2. Each subset of S is an event. Examples of events are
   - The event $A = \{\text{Roll } 4 \text{ or higher }\} = \{E_4, E_5, E_6\}$.
   - The event $B = \{\text{The roll is even}\} = \{E_2, E_4, E_6\}$.
   - The event $C = \{\text{The roll is the square of an integer}\} = \{E_1, E_4\}$.

1.1.3. Event

Definition 1.4 (Event)

An **event** is a set of outcomes of an experiment (or a subset of a sample space).

Example 1.7

Suppose we roll a six-sided die and observe the number of dots on the side facing upwards. We can label these outcomes E_i, $i = 1, \ldots, 6$ where E_i denotes the outcome that i dots appear on the up face.

1. The sample space is $S = \{E_1, E_2, \ldots, E_6\}$.
2. Each subset of S is an event. Examples of events are
   - The event $A = \{\text{Roll } 4 \text{ or higher }\} = \{E_4, E_5, E_6\}$.
   - The event $B = \{\text{The roll is even}\} = \{E_2, E_4, E_6\}$.
   - The event $C = \{\text{The roll is the square of an integer}\} = \{E_1, E_4\}$.

1.1.3. Event

Definition 1.4 (Event)

An **event** is a set of outcomes of an experiment (or a subset of a sample space).

Example 1.7

Suppose we roll a six-sided die and observe the number of dots on the side facing upwards. We can label these outcomes E_i, $i = 1, \ldots, 6$ where E_i denotes the outcome that i dots appear on the up face.

1. The sample space is $S = \{E_1, E_2, \ldots, E_6\}$.
2. Each subset of S is an event. Examples of events are
   - The event $A = \{$Roll 4 or higher $\} = \{E_4, E_5, E_6\}$.
   - The event $B = \{$The roll is even$\} = \{E_2, E_4, E_6\}$.
   - The event $C = \{$The roll is the square of an integer$\} = \{E_1, E_4\}$.

1.1.3. Event

Definition 1.4 (Event)

An **event** is a set of outcomes of an experiment (or a subset of a sample space).

Example 1.7

Suppose we roll a six-sided die and observe the number of dots on the side facing upwards. We can label these outcomes E_i, $i = 1, \ldots, 6$ where E_i denotes the outcome that i dots appear on the up face.

1. The sample space is $S = \{E_1, E_2, \ldots, E_6\}$.
2. Each subset of S is an event. Examples of events are
   - The event $A = \{$Roll 4 or higher $\} = \{E_4, E_5, E_6\}$.
   - The event $B = \{$The roll is even$\} = \{E_2, E_4, E_6\}$.
   - The event $C = \{$The roll is the square of an integer$\} = \{E_1, E_4\}$.

1.1.3. Event

Definition 1.4 (Event)

An **event** is a set of outcomes of an experiment (or a subset of a sample space).

Example 1.7

Suppose we roll a six-sided die and observe the number of dots on the side facing upwards. We can label these outcomes E_i, $i = 1, \dots, 6$ where E_i denotes the outcome that i dots appear on the up face.

1. The sample space is $S = \{E_1, E_2, \dots, E_6\}$.
2. Each subset of S is an event. Examples of events are
   - The event $A = \{\text{Roll } 4 \text{ or higher }\} = \{E_4, E_5, E_6\}$.
   - The event $B = \{\text{The roll is even}\} = \{E_2, E_4, E_6\}$.
   - The event $C = \{\text{The roll is the square of an integer}\} = \{E_1, E_4\}$.

1.1.3. Event

Definition 1.4 (Event)

An **event** is a set of outcomes of an experiment (or a subset of a sample space).

Example 1.7

Suppose we roll a six-sided die and observe the number of dots on the side facing upwards. We can label these outcomes E_i, $i = 1, \ldots, 6$ where E_i denotes the outcome that i dots appear on the up face.

1. The sample space is $S = \{E_1, E_2, \ldots, E_6\}$.
2. Each subset of S is an event. Examples of events are
   - The event $A = \{\text{Roll } 4 \text{ or higher }\} = \{E_4, E_5, E_6\}$.
   - The event $B = \{\text{The roll is even}\} = \{E_2, E_4, E_6\}$.
   - The event $C = \{\text{The roll is the square of an integer}\} = \{E_1, E_4\}$.

1.1.3. Event

Example 1.8

Wait for someone to make a phone call and observe the duration of the call in minutes.

1. An outcome x is a nonnegative real number.
2. The sample space is $S = \{x \mid x \geq 0\}$.
3. The event A "the phone call lasts longer than five minutes" is $A = \{x \mid x > 5\}$.

1.1.3. Event

Example 1.8

Wait for someone to make a phone call and observe the duration of the call in minutes.

1. An outcome x is a nonnegative real number.
2. The sample space is $S = \{x \mid x \geq 0\}$.
3. The event A "the phone call lasts longer than five minutes" is $A = \{x \mid x > 5\}$.

1.1.3. Event

Example 1.8

Wait for someone to make a phone call and observe the duration of the call in minutes.

1. An outcome x is a nonnegative real number.
2. The sample space is $S = \{x \mid x \geq 0\}$.
3. The event A "the phone call lasts longer than five minutes" is $A = \{x \mid x > 5\}$.

1.1.3. Event

Example 1.8

Wait for someone to make a phone call and observe the duration of the call in minutes.

1. An outcome x is a nonnegative real number.
2. The sample space is $S = \{x \mid x \geq 0\}$.
3. The event A "the phone call lasts longer than five minutes" is $A = \{x \mid x > 5\}$.

1.1.3. Event

Definition 1.5 (Simple Event)

A **simple event** is an event that consists of exactly one outcome.

Example 1.9

See Example 1.7. List the simple events in the experiment.

- Event E_1: Observe a 1
- Event E_2: Observe a 2
- Event E_3: Observe a 3
- Event E_4: Observe a 4
- Event E_5: Observe a 5
- Event E_6: Observe a 6

1.1.3. Event

Definition 1.5 (Simple Event)

A **simple event** is an event that consists of exactly one outcome.

Example 1.9

See Example 1.7. List the simple events in the experiment.

- Event E_1: Observe a 1
- Event E_2: Observe a 2
- Event E_3: Observe a 3
- Event E_4: Observe a 4
- Event E_5: Observe a 5
- Event E_6: Observe a 6

1.1.3. Event

Definition 1.5 (Simple Event)

A **simple event** is an event that consists of exactly one outcome.

Example 1.9

See Example 1.7. List the simple events in the experiment.

- Event E_1: Observe a 1
- Event E_2: Observe a 2
- Event E_3: Observe a 3
- Event E_4: Observe a 4
- Event E_5: Observe a 5
- Event E_6: Observe a 6

Union. Intersection

Definition 1.6 (Union)

The **union** of events A and B, denoted by $A \cup B$ (or $A + B$) is the event that either A or B or both occur.

Definition 1.7 (Intersection)

The **intersection** of events A and B, denoted by $A \cap B$ (or AB), is the event that both A and B occur.

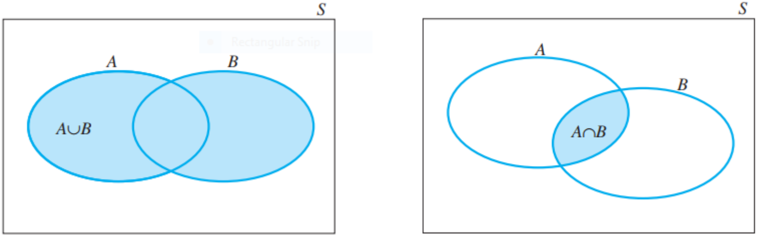


Figure: Venn diagram of $A \cup B$ and $A \cap B$

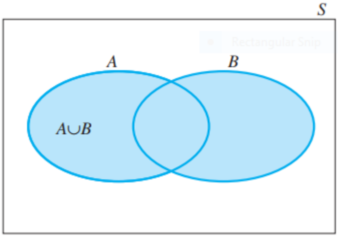
Union. Intersection

Definition 1.6 (Union)

The **union** of events A and B, denoted by $A \cup B$ (or $A + B$) is the event that either A or B or both occur.

Definition 1.7 (Intersection)

The **intersection** of events A and B, denoted by $A \cap B$ (or AB), is the event that both A and B occur.



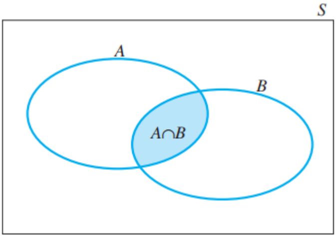


Figure: Venn diagram of $A \cup B$ and $A \cap B$

Union. Intersection

Remark 1.1

We will use a shorthand for unions and intersections of n events:

$$\cup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n = A_1 + A_2 + \cdots + A_n, \tag{1.1}$$

$$\cap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n = A_1 A_2 \ldots A_n. \tag{1.2}$$

Mutually Exclusive

Definition 1.8 (Mutually Exclusive)

(a) Two events, A and B, are **mutually exclusive/disjoint** if, when one event occurs, the others cannot, and vice versa.
That is, $A \cap B = AB = \emptyset$.

(b) A collection of events $A_1, A_2, \dots, A_n$ is mutually exclusive if and only if

$$A_i \cap A_j = \emptyset, \quad i \neq j. \tag{1.3}$$

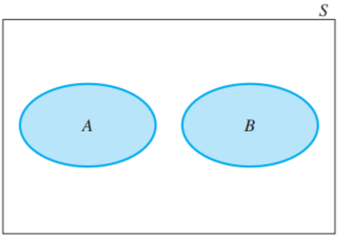


Figure: Two disjoint events

Mutually Exclusive

Definition 1.8 (Mutually Exclusive)

(a) Two events, A and B, are **mutually exclusive/disjoint** if, when one event occurs, the others cannot, and vice versa.
That is, $A \cap B = AB = \emptyset$.

(b) A collection of events $A_1, A_2, \ldots, A_n$ is mutually exclusive if and only if

$$A_i \cap A_j = \emptyset, \quad i \neq j. \tag{1.3}$$

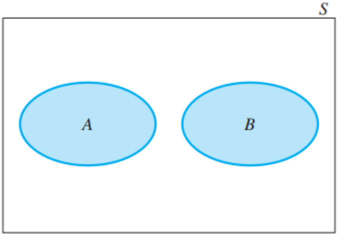


Figure: Two disjoint events

Mutually Exclusive

Example 1.10

(a) In Example 1.7, events A and B are not mutually exclusive, because they have one outcome in common.

(b) In Example 1.9 the six simple events $E_1, E_2, \ldots, E_6$ form a set of all mutually exclusive outcomes of the experiment.

Mutually Exclusive

Example 1.10

(a) In Example 1.7, events A and B are not mutually exclusive, because they have one outcome in common.

(b) In Example 1.9 the six simple events $E_1, E_2, \ldots, E_6$ form a set of all mutually exclusive outcomes of the experiment.

Mutually Exclusive

Figure 1.5

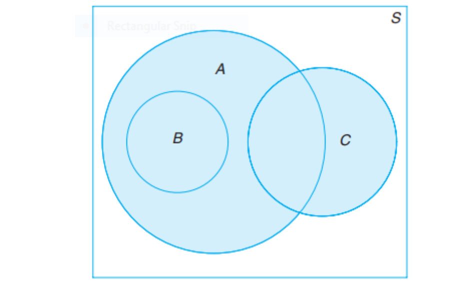


Figure: Events of the sample space S

Events B and C are mutually exclusive; event $A \cap C$ has at least one element; event $A \cap B = B$; and event $A \cup B = A$

Collectively Exhaustive. Complementary

Definition 1.9 (Collectively Exhaustive)

A collection of events $A_1, A_2, \ldots, A_n$ is **collectively exhaustive** if and only if

$$A_1 \cup A_2 \cup \cdots \cup A_n = S. \tag{1.4}$$

Definition 1.10 (Complementary Events)

The **complement** of event A is the set of all outcomes in the sample space that are not included in event A. Denoted by $\overline{A}$ or A^c.

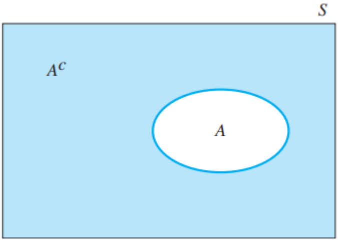


Figure: The complement of an event

Collectively Exhaustive. Complementary

Definition 1.9 (Collectively Exhaustive)

A collection of events $A_1, A_2, \ldots, A_n$ is **collectively exhaustive** if and only if

$$A_1 \cup A_2 \cup \cdots \cup A_n = S. \tag{1.4}$$

Definition 1.10 (Complementary Events)

The **complement** of event A is the set of all outcomes in the sample space that are not included in event A. Denoted by $\overline{A}$ or A^c.

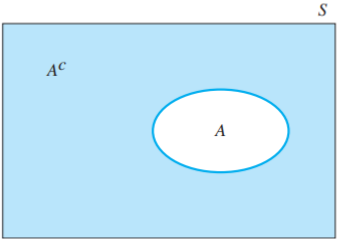


Figure: The complement of an event

1.1.4 Event Relations

Properties

1. $A \cap \emptyset = \emptyset$.
2. $A \cup \emptyset = A$.
3. $A \cap A^c = \emptyset$.
4. $A \cup A^c = S$.
5. $S^c = \emptyset$.
6. $\emptyset^c = S$.
7. $(A^c)^c = A$.
8. $(A \cap B)^c = A^c \cup B^c$.
9. $(A \cup B)^c = A^c \cap B^c$.

1.1.5 Event Space

Definition 1.11 (Event Space)

An **event space** is a collectively exhaustive, mutually exclusive set of events.

Note

The set $A = \{A_1, A_2, \ldots, A_n\}$ is an event space if and only if

$$A_1 \cup A_2 \cup \cdots \cup A_n = S, \quad (1.5)$$

$$A_i \cap A_j = \emptyset, \quad i \neq j,\ i, j = 1, 2, \ldots, n. \quad (1.6)$$

Remark 1.2

An event space and a sample space have a lot in common.

- The members of both are mutually exclusive and collectively exhaustive.
- They differ in the finest-grain property that applies to a sample space but not to an event space.

1.1.5 Event Space

Definition 1.11 (Event Space)

An **event space** is a collectively exhaustive, mutually exclusive set of events.

Note

The set $A = \{A_1, A_2, \ldots, A_n\}$ is an event space if and only if

$$A_1 \cup A_2 \cup \cdots \cup A_n = S, \tag{1.5}$$

$$A_i \cap A_j = \emptyset, \quad i \neq j,\ i, j = 1, 2, \ldots, n. \tag{1.6}$$

Remark 1.2

An event space and a sample space have a lot in common.

- The members of both are mutually exclusive and collectively exhaustive.
- They differ in the finest-grain property that applies to a sample space but not to an event space.

1.1.5 Event Space

Definition 1.11 (Event Space)

An event space is a collectively exhaustive, mutually exclusive set of events.

Note

The set $A = \{A_1, A_2, \dots, A_n\}$ is an event space if and only if

$$A_1 \cup A_2 \cup \cdots \cup A_n = S, \tag{1.5}$$

$$A_i \cap A_j = \emptyset, \quad i \neq j,\ i, j = 1, 2, \dots, n. \tag{1.6}$$

Remark 1.2

An event space and a sample space have a lot in common.

- The members of both are mutually exclusive and collectively exhaustive.
- They differ in the finest-grain property that applies to a sample space but not to an event space.

1.1.5 Event Space

Definition 1.11 (Event Space)

An **event space** is a collectively exhaustive, mutually exclusive set of events.

Note

The set $A = \{A_1, A_2, \ldots, A_n\}$ is an event space if and only if

$$A_1 \cup A_2 \cup \cdots \cup A_n = S, \tag{1.5}$$

$$A_i \cap A_j = \emptyset, \quad i \neq j,\ i, j = 1, 2, \ldots, n. \tag{1.6}$$

Remark 1.2

An event space and a sample space have a lot in common.

- The members of both are mutually exclusive and collectively exhaustive.
- They differ in the finest-grain property that applies to a sample space but not to an event space.

1.1.5 Event Space

Example 1.11

Flip four coins, a penny, a nickel, a dime, and a quarter. Examine the coins in order (penny, then nickel, then dime, then quarter) and observe whether each coin shows a head (H) or a tail (T). What is the sample space? How many elements are in the sample space?

Solution Example 1.11

- The sample space consists of 16 four-letter words, with each letter either H or T. For example, the outcome $TTHH$ refers to the penny and the nickel showing tails and the dime and quarter showing heads.
- There are 16 members of the sample space.

1.1.5 Event Space

Example 1.11

Flip four coins, a penny, a nickel, a dime, and a quarter. Examine the coins in order (penny, then nickel, then dime, then quarter) and observe whether each coin shows a head (H) or a tail (T). What is the sample space? How many elements are in the sample space?

Solution Example 1.11

- The sample space consists of 16 four-letter words, with each letter either H or T. For example, the outcome $TTHH$ refers to the penny and the nickel showing tails and the dime and quarter showing heads.
- There are 16 members of the sample space.

1.1.5 Event Space

Example 1.11

Flip four coins, a penny, a nickel, a dime, and a quarter. Examine the coins in order (penny, then nickel, then dime, then quarter) and observe whether each coin shows a head (H) or a tail (T). What is the sample space? How many elements are in the sample space?

Solution Example 1.11

- The sample space consists of 16 four-letter words, with each letter either H or T. For example, the outcome $TTHH$ refers to the penny and the nickel showing tails and the dime and quarter showing heads.
- There are 16 members of the sample space.

1.1.5 Event Space

Example 1.12

Continuing Example 1.11, let $A_i = \{\text{outcomes with } i \text{ heads}\}$.

- Each A_i is an event containing one or more outcomes. For example, $A_1 = \{TTTH, TTHT, THTT, HTTT\}$ contains four outcomes.
- The set $B = \{A_0, A_1, A_2, A_3, A_4\}$ is an event space.

1.1.5 Event Space

Theorem 1.1

For an event space $B = \{B_1, B_2, \ldots\}$ and any event A in the sample space, let $C_i = A \cap B_i$. For $i \neq j$, the events C_i and C_j are mutually exclusive and

$$A = C_1 \cup C_2 \cup \cdots$$

Figure 1.7

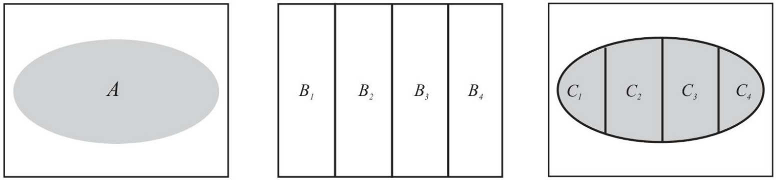


Figure: Figure for Theorem 1.1

1.1.5 Event Space

Theorem 1.1

For an event space $B = \{B_1, B_2, \dots\}$ and any event A in the sample space, let $C_i = A \cap B_i$. For $i \neq j$, the events C_i and C_j are mutually exclusive and

$$A = C_1 \cup C_2 \cup \cdots$$

Figure 1.7

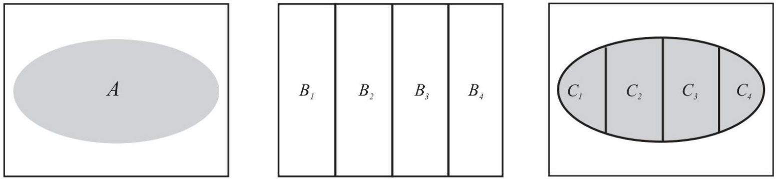


Figure: Figure for Theorem 1.1

1.1.5 Event Space

Example 1.13

In the coin-tossing experiment of Example 1.10, let A equal the set of outcomes with less than three heads:

$$A = \{TTTT, HTTT, THTT, TTHT, TTTH, HHTT, HTHT, HTTH, TTHH, THTH, T$$

From Example 1.12, let $B_i = \{\text{outcomes with } i \text{ heads}\}$. Since $\{B_0, B_1, B_2, B_3, B_4\}$ is an event space, Theorem 1.1 states that

$$A = (A \cap B_0) \cup (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cup (A \cap B_4).$$

(a) Multiplication Rule

Rule 1.1

If an operation can be performed in n_1 ways, and if for each of these ways a second operation can be performed in n_2 ways, then the two operations can be performed together in $n_1 n_2$ ways.

Example 1.14

How many sample points are there in the sample space when a pair of dice is thrown once?

Example 1.14 Solution

The first die can land face-up in any one of $n_1 = 6$ ways. For each of these 6 ways, the second die can also land face-up in $n_2 = 6$ ways. Therefore, the pair of dice can land in $n_1 n_2 = (6)(6) = 36$ possible ways.

(a) Multiplication Rule

Rule 1.1

If an operation can be performed in n_1 ways, and if for each of these ways a second operation can be performed in n_2 ways, then the two operations can be performed together in $n_1 n_2$ ways.

Example 1.14

How many sample points are there in the sample space when a pair of dice is thrown once?

Example 1.14 Solution

The first die can land face-up in any one of $n_1 = 6$ ways. For each of these 6 ways, the second die can also land face-up in $n_2 = 6$ ways. Therefore, the pair of dice can land in $n_1 n_2 = (6)(6) = 36$ possible ways.

(a) Multiplication Rule

Rule 1.1

If an operation can be performed in n_1 ways, and if for each of these ways a second operation can be performed in n_2 ways, then the two operations can be performed together in $n_1 n_2$ ways.

Example 1.14

How many sample points are there in the sample space when a pair of dice is thrown once?

Example 1.14 Solution

The first die can land face-up in any one of $n_1 = 6$ ways. For each of these 6 ways, the second die can also land face-up in $n_2 = 6$ ways. Therefore, the pair of dice can land in $n_1 n_2 = (6)(6) = 36$ possible ways.

(a) Multiplication Rule

Rule 1.2

If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1 n_2 \dots n_k$ ways.

Example 1.16

Shuffle the deck and choose three cards in order. How many outcomes are there?

Example 1.16 Solution

In this experiment, there are 52 possible outcomes for the first card, 51 for the second card, and 50 for the third card. The total number of outcomes is $52 \times 51 \times 50$.

(a) Multiplication Rule

Rule 1.2

If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1 n_2 \dots n_k$ ways.

Example 1.16

Shuffle the deck and choose three cards in order. How many outcomes are there?

Example 1.16 Solution

In this experiment, there are 52 possible outcomes for the first card, 51 for the second card, and 50 for the third card. The total number of outcomes is $52 \times 51 \times 50$.

(a) Multiplication Rule

Rule 1.2

If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1 n_2 \dots n_k$ ways.

Example 1.16

Shuffle the deck and choose three cards in order. How many outcomes are there?

Example 1.16 Solution

In this experiment, there are 52 possible outcomes for the first card, 51 for the second card, and 50 for the third card. The total number of outcomes is $52 \times 51 \times 50$.

(b) Permutation

Theorem 1.2

The number of permutations of n distinct objects taken k at a time is

$$A_n^k = n(n-1)(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!}. \tag{1.8}$$

Note

Permutation of n objects taken k at a time: ORDER, NO REPEAT.

Example 1.17

Three lottery tickets are drawn from a total of 50. If the tickets will be distributed to each of three employees in the order in which they are drawn, the order will be important. How many simple events are associated with the experiment?

Example 1.17 Solution

The total number of simple events is $A_{50}^3 = \frac{50!}{47!} = 50 \times 49 \times 48 = 117600.$

(b) Permutation

Theorem 1.2

The number of permutations of n distinct objects taken k at a time is

$$A_n^k = n(n-1)(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!}. \quad (1.8)$$

Note

Permutation of n objects taken k at a time: ORDER, NO REPEAT.

Example 1.17

Three lottery tickets are drawn from a total of 50. If the tickets will be distributed to each of three employees in the order in which they are drawn, the order will be important. How many simple events are associated with the experiment?

Example 1.17 Solution

The total number of simple events is $A_{50}^3 = \frac{50!}{47!} = 50 \times 49 \times 48 = 117600.$

(b) Permutation

Theorem 1.2

The number of permutations of n distinct objects taken k at a time is

$$A_n^k = n(n-1)(n-2)\dots(n-k+1) = \frac{n!}{(n-k)!}. \tag{1.8}$$

Note

Permutation of n objects taken k at a time: ORDER, NO REPEAT.

Example 1.17

Three lottery tickets are drawn from a total of 50. If the tickets will be distributed to each of three employees in the order in which they are drawn, the order will be important. How many simple events are associated with the experiment?

Example 1.17 Solution

The total number of simple events is $A_{50}^3 = \frac{50!}{47!} = 50 \times 49 \times 48 = 117600.$

(b) Permutation

Theorem 1.2

The number of permutations of n distinct objects taken k at a time is

$$A_n^k = n(n-1)(n-2)\dots(n-k+1) = \frac{n!}{(n-k)!}. \quad (1.8)$$

Note

Permutation of n objects taken k at a time: ORDER, NO REPEAT.

Example 1.17

Three lottery tickets are drawn from a total of 50. If the tickets will be distributed to each of three employees in the order in which they are drawn, the order will be important. How many simple events are associated with the experiment?

Example 1.17 Solution

The total number of simple events is $A_{50}^3 = \frac{50!}{47!} = 50 \times 49 \times 48 = 117600$.

(b) Permutation

Theorem 1.3 (a spectial case)

The number of n-permutations of n distinguishable objects is

$$P_n = A_n^n = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 = n!$$

Note

Arranging n items: ORDER, NO REPEAT from n objects taken n at a time.

Example 1.18

A piece of equipment is composed of five parts that can be assembled in any order. A test is to be conducted to determine the time necessary for each order of assembly. If each order is to be tested once, how many tests must be conducted?

Example 1.18 Solution

The total number of tests is $P_5 = 5! = 120.$

(b) Permutation

Theorem 1.3 (a spectial case)

The number of n-permutations of n distinguishable objects is

$$P_n = A_n^n = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 = n!$$

Note

Arranging n items: ORDER, NO REPEAT from n objects taken n at a time.

Example 1.18

A piece of equipment is composed of five parts that can be assembled in any order. A test is to be conducted to determine the time necessary for each order of assembly. If each order is to be tested once, how many tests must be conducted?

Example 1.18 Solution

The total number of tests is $P_5 = 5! = 120$.

(b) Permutation

Theorem 1.3 (a spectial case)

The number of n-permutations of n distinguishable objects is

$$P_n = A_n^n = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 = n!$$

Note

Arranging n items: ORDER, NO REPEAT from n objects taken n at a time.

Example 1.18

A piece of equipment is composed of five parts that can be assembled in any order. A test is to be conducted to determine the time necessary for each order of assembly. If each order is to be tested once, how many tests must be conducted?

Example 1.18 Solution

The total number of tests is $P_5 = 5! = 120$.

(b) Permutation

Theorem 1.3 (a spectial case)

The number of n-permutations of n distinguishable objects is

$$P_n = A_n^n = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 = n!$$

Note

Arranging n items: ORDER, NO REPEAT from n objects taken n at a time.

Example 1.18

A piece of equipment is composed of five parts that can be assembled in any order. A test is to be conducted to determine the time necessary for each order of assembly. If each order is to be tested once, how many tests must be conducted?

Example 1.18 Solution

The total number of tests is $P_5 = 5! = 120$.

(c) Combination

Theorem 1.6 (combination)

The number of distinct combinations of n distinct objects that can be formed, taking them k at a time, is

$$C_n^k = \frac{n!}{k!(n-k)!}. \tag{1.9}$$

Note

Combination of n objects taken k at a time: NO REPEAT, NO ORDER

Example 1.21

A printed circuit board may be purchased from five suppliers. In how many ways can three suppliers be chosen from the five?

Example 1.21 Solution

The number of ways is $C_5^3 = \dfrac{5!}{3!2!} = 10.$

(c) Combination

Theorem 1.6 (combination)

The number of distinct combinations of n distinct objects that can be formed, taking them k at a time, is

$$C_n^k = \frac{n!}{k!(n-k)!}. \tag{1.9}$$

Note

Combination of n objects taken k at a time: NO REPEAT, NO ORDER

Example 1.21

A printed circuit board may be purchased from five suppliers. In how many ways can three suppliers be chosen from the five?

Example 1.21 Solution

The number of ways is $C_5^3 = \dfrac{5!}{3!2!} = 10.$

(c) Combination

Theorem 1.6 (combination)

The number of distinct combinations of n distinct objects that can be formed, taking them k at a time, is

$$C_n^k = \frac{n!}{k!(n-k)!}. \tag{1.9}$$

Note

Combination of n objects taken k at a time: NO REPEAT, NO ORDER

Example 1.21

A printed circuit board may be purchased from five suppliers. In how many ways can three suppliers be chosen from the five?

Example 1.21 Solution

The number of ways is $C_5^3 = \frac{5!}{3!2!} = 10.$

(c) Combination

Theorem 1.6 (combination)

The number of distinct combinations of n distinct objects that can be formed, taking them k at a time, is

$$C_n^k = \frac{n!}{k!(n-k)!}. \tag{1.9}$$

Note

Combination of n objects taken k at a time: NO REPEAT, NO ORDER

Example 1.21

A printed circuit board may be purchased from five suppliers. In how many ways can three suppliers be chosen from the five?

Example 1.21 Solution

The number of ways is $C_5^3 = \dfrac{5!}{3!2!} = 10.$

(d) Sampling with Replacement

Theorem 1.7

Given n distinguishable objects, there are

$$\overline{A}_n^k = n^k \tag{1.11}$$

ways to choose with replacement an ordered sample of k objects.

Note

ORDER, with REPEAT, from n objects taken k at a time

Example 1.22

The access code to a house's security system consists of 3 digits. Each digit can be 0 through 9. How many different codes are available if each digit can be repeated?

Example 1.22 Solution

Because each digit can be repeated, there are 10 choices for each of the 3 digits $10 \times 10 \times 10 = 10^3 = 1000$ codes.

(d) Sampling with Replacement

Theorem 1.7

Given n distinguishable objects, there are

$$\overline{A}_n^k = n^k \tag{1.11}$$

ways to choose with replacement an ordered sample of k objects.

Note

ORDER, with REPEAT, from n objects taken k at a time

Example 1.22

The access code to a house's security system consists of 3 digits. Each digit can be 0 through 9. How many different codes are available if each digit can be repeated?

Example 1.22 Solution

Because each digit can be repeated, there are 10 choices for each of the 3 digits $10 \times 10 \times 10 = 10^3 = 1000$ codes.

(d) Sampling with Replacement

Theorem 1.7

Given n distinguishable objects, there are

$$\overline{A}_n^k = n^k \tag{1.11}$$

ways to choose with replacement an ordered sample of k objects.

Note

ORDER, with REPEAT, from n objects taken k at a time

Example 1.22

The access code to a house's security system consists of 3 digits. Each digit can be 0 through 9. How many different codes are available if each digit can be repeated?

Example 1.22 Solution

Because each digit can be repeated, there are 10 choices for each of the 3 digits $10 \times 10 \times 10 = 10^3 = 1000$ codes.

(d) Sampling with Replacement

Theorem 1.7

Given n distinguishable objects, there are

$$\overline{A}_n^k = n^k \tag{1.11}$$

ways to choose with replacement an ordered sample of k objects.

Note

ORDER, with REPEAT, from n objects taken k at a time

Example 1.22

The access code to a house's security system consists of 3 digits. Each digit can be 0 through 9. How many different codes are available if each digit can be repeated?

Example 1.22 Solution

Because each digit can be repeated, there are 10 choices for each of the 3 digits $10 \times 10 \times 10 = 10^3 = 1000$ codes.

Content

1.2.1 Definition

Definition 1.14 (Probability)

The **probability** $P[A]$ of an event A is a measure of our belief that the event A will occur.

Property 1.1

The probability of an event A is the sum of the weights of all sample points in A. Therefore,

1. $0 \leq P[A] \leq 1.$
2. $P[\emptyset] = 0.$
3. $P[S] = 1.$
4. If $A \subset B$, $P[A] \leq P[B].$

1.2.1 Definition

Definition 1.14 (Probability)

The **probability** $P[A]$ of an event A is a measure of our belief that the event A will occur.

Property 1.1

The probability of an event A is the sum of the weights of all sample points in A. Therefore,

1. $0 \leq P[A] \leq 1$.
2. $P[\emptyset] = 0$.
3. $P[S] = 1$.
4. If $A \subset B$, $P[A] \leq P[B]$.

1.2.1 Definition

Definition 1.14 (Probability)

The **probability** $P[A]$ of an event A is a measure of our belief that the event A will occur.

Property 1.1

The probability of an event A is the sum of the weights of all sample points in A. Therefore,

1. $0 \leq P[A] \leq 1$.
2. $P[\emptyset] = 0$.
3. $P[S] = 1$.
4. If $A \subset B$, $P[A] \leq P[B]$.

1.2.1 Definition

Definition 1.14 (Probability)

The **probability** $P[A]$ of an event A is a measure of our belief that the event A will occur.

Property 1.1

The probability of an event A is the sum of the weights of all sample points in A. Therefore,

1. $0 \leq P[A] \leq 1$.
2. $P[\emptyset] = 0$.
3. $P[S] = 1$.
4. If $A \subset B$, $P[A] \leq P[B]$.

1.2.1 Definition

Definition 1.14 (Probability)

The **probability** $P[A]$ of an event A is a measure of our belief that the event A will occur.

Property 1.1

The probability of an event A is the sum of the weights of all sample points in A. Therefore,

1. $0 \leq P[A] \leq 1$.
2. $P[\emptyset] = 0$.
3. $P[S] = 1$.
4. If $A \subset B$, $P[A] \leq P[B]$.

1.2.2 Equally Likely Outcomes

Definition 1.15 (Equally Likely Outcomes)

A large number of experiments have a sample space $S = \{E_1, \dots, E_n\}$ in which our knowledge of the practical situation leads us to believe that no one outcome is any more likely than any other. In these experiments we say that the n outcomes are equally likely.

Theorem 1.8

For an experiment with sample space $S = \{E_1, \dots, E_n\}$ in which each outcome E_i is equally likely,

$$P[E_i] = \frac{1}{n}, \quad i = 1, \dots, n. \tag{1.12}$$

1.2.2 Equally Likely Outcomes

Definition 1.15 (Equally Likely Outcomes)

A large number of experiments have a sample space $S = \{E_1, \ldots, E_n\}$ in which our knowledge of the practical situation leads us to believe that no one outcome is any more likely than any other. In these experiments we say that the n outcomes are equally likely.

Theorem 1.8

For an experiment with sample space $S = \{E_1, \ldots, E_n\}$ in which each outcome E_i is equally likely,

$$P[E_i] = \frac{1}{n}, \quad i = 1, \ldots, n. \tag{1.12}$$

1.2.2 Equally Likely Outcomes

Example 1.23

A coin is tossed twice. What is the probability that at least 1 head occurs?

Example 1.23 Solution

The sample space for this experiment is $S = \{HH, HT, TH, TT\}$. If the coin is balanced, each of these outcomes is equally likely to occur. Therefore, we assign a probability of ω to each sample point. Then $4\omega = 1$, or $\omega = 1/4$. If A represents the event of at least 1 head occurring, then

$$A = \{HH, HT, TH\} \quad \text{and} \quad P[A] = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

1.2.2 Equally Likely Outcomes

Example 1.23

A coin is tossed twice. What is the probability that at least 1 head occurs?

Example 1.23 Solution

The sample space for this experiment is $S = \{HH, HT, TH, TT\}$. If the coin is balanced, each of these outcomes is equally likely to occur. Therefore, we assign a probability of ω to each sample point. Then $4\omega = 1$, or $\omega = 1/4$. If A represents the event of at least 1 head occurring, then

$$A = \{HH, HT, TH\} \quad \text{and} \quad P[A] = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

1.2.3 Theoretical Probability. Empirical Probability

Rule 1.3 (Theoretical Probability)

If an experiment can result in any one of n different equally likely outcomes, and if exactly m of these outcomes correspond to event A, then the probability of event A is

$$P[A] = \frac{m}{n}. \tag{1.13}$$

Example 1.25

A statistics class for engineers consists of 25 industrial, 10 mechanical, 10 electrical, and 8 civil engineering students. If a person is randomly selected by the instructor to answer a question, find the probability that the student chosen is (a) an industrial engineering major and (b) a civil engineering or an electrical engineering major.

Example 1.25 Solution

Denote by I, M, E, and C the students majoring in industrial, mechanical, electrical, and civil engineering, respectively. $P[I] = \frac{25}{53}$. $P[C \cup E] = \frac{18}{53}$.

1.2.3 Theoretical Probability. Empirical Probability

Rule 1.3 (Theoretical Probability)

If an experiment can result in any one of n different equally likely outcomes, and if exactly m of these outcomes correspond to event A, then the probability of event A is

$$P[A] = \frac{m}{n}. \tag{1.13}$$

Example 1.25

A statistics class for engineers consists of 25 industrial, 10 mechanical, 10 electrical, and 8 civil engineering students. If a person is randomly selected by the instructor to answer a question, find the probability that the student chosen is (a) an industrial engineering major and (b) a civil engineering or an electrical engineering major.

Example 1.25 Solution

Denote by I, M, E, and C the students majoring in industrial, mechanical, electrical, and civil engineering, respectively. $P[I] = \frac{25}{53}$. $P[C \cup E] = \frac{18}{53}$.

1.2.3 Theoretical Probability. Empirical Probability

Rule 1.3 (Theoretical Probability)

If an experiment can result in any one of n different equally likely outcomes, and if exactly m of these outcomes correspond to event A, then the probability of event A is

$$P[A] = \frac{m}{n}. \tag{1.13}$$

Example 1.25

A statistics class for engineers consists of 25 industrial, 10 mechanical, 10 electrical, and 8 civil engineering students. If a person is randomly selected by the instructor to answer a question, find the probability that the student chosen is (a) an industrial engineering major and (b) a civil engineering or an electrical engineering major.

Example 1.25 Solution

Denote by I, M, E, and C the students majoring in industrial, mechanical, electrical, and civil engineering, respectively. $P[I] = \dfrac{25}{53}$. $P[C \cup E] = \dfrac{18}{53}$.

1.2.3 Theoretical Probability. Empirical Probability

Definition 1.15 (Relative frequency. Probability)

If an experiment is performed n times, then the relative frequency of a particular occurrence say, A is

$$\text{Relative frequency} = \frac{\text{Frequency}}{n}$$

where the frequency is the number of times the event A occurred. If you let n, the number of repetitions of the experiment, become larger and larger ($n \to \infty$), you will eventually generate the entire population. In this population, the relative frequency of the event A is defined as the probability of event A; that is

$$P[A] = \lim_{n\to\infty} \frac{\text{Frequency}}{n}. \tag{1.14}$$

1.2.3 Theoretical Probability. Empirical Probability

Theorem 1.3 Since $P[A]$ behaves like a relative frequency,

(a) $P[A]$ must be a proportion lying between 0 and 1.

(b) $P[A] = 0$ if the event A never occurs.

(c) $P[A] = 1$ if the event A always occurs.

(d) If $A \subset B$, then $P[A] \leq P[B]$.

Note: The closer $P[A]$ is to 1, the more likely it is that A will occur.

Remark 1.3 (Requirements for simple event probabilities)

(a) Each probability must lie between 0 and 1.

(b) The sum of the probabilities for all simple events in S equals 1.

1.2.3 Theoretical Probability. Empirical Probability

Theorem 1.3 Since $P[A]$ behaves like a relative frequency,

(a) $P[A]$ must be a proportion lying between 0 and 1.

(b) $P[A] = 0$ if the event A never occurs.

(c) $P[A] = 1$ if the event A always occurs.

(d) If $A \subset B$, then $P[A] \leq P[B]$.

Note: The closer $P[A]$ is to 1, the more likely it is that A will occur.

Remark 1.3 (Requirements for simple event probabilities)

(a) Each probability must lie between 0 and 1.

(b) The sum of the probabilities for all simple events in S equals 1.

1.2.3 Theoretical Probability. Empirical Probability

Theorem 1.3 Since $P[A]$ behaves like a relative frequency,

(a) $P[A]$ must be a proportion lying between 0 and 1.

(b) $P[A] = 0$ if the event A never occurs.

(c) $P[A] = 1$ if the event A always occurs.

(d) If $A \subset B$, then $P[A] \leq P[B]$.

Note: The closer $P[A]$ is to 1, the more likely it is that A will occur.

Remark 1.3 (Requirements for simple event probabilities)

(a) Each probability must lie between 0 and 1.

(b) The sum of the probabilities for all simple events in S equals 1.

1.2.3 Theoretical Probability. Empirical Probability

Theorem 1.3 Since $P[A]$ behaves like a relative frequency,

(a) $P[A]$ must be a proportion lying between 0 and 1.

(b) $P[A] = 0$ if the event A never occurs.

(c) $P[A] = 1$ if the event A always occurs.

(d) If $A \subset B$, then $P[A] \leq P[B]$.

Note: The closer $P[A]$ is to 1, the more likely it is that A will occur.

Remark 1.3 (Requirements for simple event probabilities)

(a) Each probability must lie between 0 and 1.

(b) The sum of the probabilities for all simple events in S equals 1.

Content

1.3. Additive Rules

Theorem 1.10 (Additive Rules)

If A and B are two events, then

$$P[A \cup B] = P[A] + P[B] - P[AB]. \tag{1.15}$$

Figure 1.8

Notice in the Venn diagram in Figure 1.8 that the sum $P[A] + P[B]$ double counts the simple events that are common to both A and B.

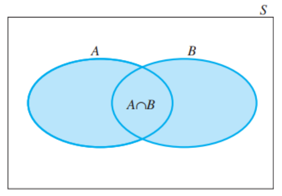


Figure: The Addition Rule

1.3. Additive Rules

Theorem 1.10 (Additive Rules)

If A and B are two events, then

$$P[A \cup B] = P[A] + P[B] - P[AB]. \tag{1.15}$$

Figure 1.8

Notice in the Venn diagram in Figure 1.8 that the sum $P[A] + P[B]$ double counts the simple events that are common to both A and B.

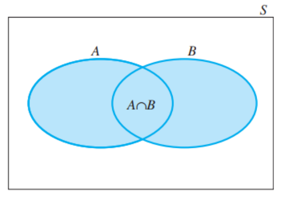


Figure: The Addition Rule

1.3. Additive Rules

Corollary 1.1

If A and B are mutually exclusive, then

$$P[A \cup B] = P[A] + P[B]. \tag{1.16}$$

Figure 1.9

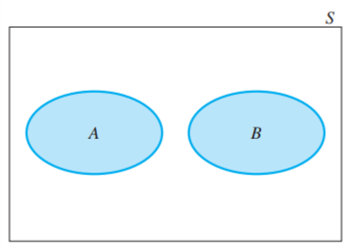


Figure: Two disjoint events

1.3. Additive Rules

Corollary 1.1

If A and B are mutually exclusive, then

$$P[A \cup B] = P[A] + P[B]. \tag{1.16}$$

Figure 1.9

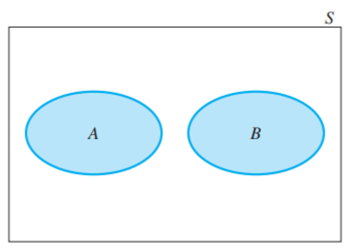


Figure: Two disjoint events

1.3. Additive Rules

Corollary 1.2

If $A_1, A_2, \ldots, A_n$ are mutually exclusive, then

$$P[A_1 \cup A_2 \cup \cdots \cup A_n] = P[A_1] + P[A_2] + \cdots + P[A_n]. \tag{1.17}$$

Corollary 1.3

If $\{A_1, A_2, \ldots, A_n\}$ is a event space, then

$$P[A_1 \cup A_2 \cup \cdots \cup A_n] = P[A_1] + P[A_2] + \cdots + P[A_n] = P[S] = 1. \tag{1.18}$$

Theorem 1.11

For three events A, B, and C,

$$P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[AB] - P[AC] - P[BC] + P[ABC]. \tag{1.19}$$

1.3. Additive Rules

Corollary 1.2

If $A_1, A_2, \ldots, A_n$ are mutually exclusive, then

$$P[A_1 \cup A_2 \cup \cdots \cup A_n] = P[A_1] + P[A_2] + \cdots + P[A_n]. \tag{1.17}$$

Corollary 1.3

If $\{A_1, A_2, \ldots, A_n\}$ is a event space, then

$$P[A_1 \cup A_2 \cup \cdots \cup A_n] = P[A_1] + P[A_2] + \cdots + P[A_n] = P[S] = 1. \tag{1.18}$$

Theorem 1.11

For three events A, B, and C,

$$P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[AB] - P[AC] - P[BC] + P[ABC]. \tag{1.19}$$

1.3. Additive Rules

Corollary 1.2

If $A_1, A_2, \ldots, A_n$ are mutually exclusive, then

$$P[A_1 \cup A_2 \cup \cdots \cup A_n] = P[A_1] + P[A_2] + \cdots + P[A_n]. \tag{1.17}$$

Corollary 1.3

If $\{A_1, A_2, \ldots, A_n\}$ is a event space, then

$$P[A_1 \cup A_2 \cup \cdots \cup A_n] = P[A_1] + P[A_2] + \cdots + P[A_n] = P[S] = 1. \tag{1.18}$$

Theorem 1.11

For three events A, B, and C,

$$P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[AB] - P[AC] - P[BC] + P[ABC]. \tag{1.19}$$

1.3.3 The Addition Rule

Example 1.26

John is going to graduate from an industrial engineering department in a university by the end of the semester. After being interviewed at two companies he likes, he assesses that his probability of getting an offer from company A is 0.8, and his probability of getting an offer from company B is 0.6. If he believes that the probability that he will get offers from both companies is 0.5, what is the probability that he will get at least one offer from these two companies?

Example 1.26 Solution

Using the additive rule, we have

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] = 0.8 + 0.6 - 0.5 = 0.9.$$

1.3.3 The Addition Rule

Example 1.26

John is going to graduate from an industrial engineering department in a university by the end of the semester. After being interviewed at two companies he likes, he assesses that his probability of getting an offer from company A is 0.8, and his probability of getting an offer from company B is 0.6. If he believes that the probability that he will get offers from both companies is 0.5, what is the probability that he will get at least one offer from these two companies?

Example 1.26 Solution

Using the additive rule, we have

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] = 0.8 + 0.6 - 0.5 = 0.9.$$

1.3.3 The Addition Rule

Theorem 1.12

If A and A^c are complementary events, then

$$P[A] + P[A^c] = 1. \tag{1.20}$$

Proof

Since $A \cup A^c = S$ and the sets A and A^c are disjoint,

$$1 = P[S] = P[A \cup A^c] = P[A] + P[A^c].$$

1.3.3 The Addition Rule

Theorem 1.12

If A and A^c are complementary events, then

$$P[A] + P[A^c] = 1. \tag{1.20}$$

Proof

Since $A \cup A^c = S$ and the sets A and A^c are disjoint,

$$1 = P[S] = P[A \cup A^c] = P[A] + P[A^c].$$

1.3.3 The Addition Rule

Example 1.27

If the probabilities that an automobile mechanic will service 3, 4, 5, 6, 7, or 8 or more cars on any given workday are, respectively, 0.12, 0.19, 0.28, 0.24, 0.10, and 0.07, what is the probability that he will service at least 5 cars on his next day at work?

Example 1.27 Solution

Let E be the event that at least 5 cars are serviced. Now, $P[E] = 1 - P[E^c]$, where E^c is the event that fewer than 5 cars are serviced. Since

$$P[E^c] = 0.12 + 0.19 = 0.31,$$

it follows from Theorem 1.12 that

$$P[E] = 1 - 0.31 = 0.69.$$

1.3.3 The Addition Rule

Example 1.27

If the probabilities that an automobile mechanic will service 3, 4, 5, 6, 7, or 8 or more cars on any given workday are, respectively, 0.12, 0.19, 0.28, 0.24, 0.10, and 0.07, what is the probability that he will service at least 5 cars on his next day at work?

Example 1.27 Solution

Let E be the event that at least 5 cars are serviced. Now, $P[E] = 1 - P[E^c]$, where E^c is the event that fewer than 5 cars are serviced. Since

$$P[E^c] = 0.12 + 0.19 = 0.31,$$

it follows from Theorem 1.12 that

$$P[E] = 1 - 0.31 = 0.69.$$

1.3.3 The Addition Rule

Theorem 1.13

For m events $A_1, A_2, \cdots, A_m$, then

$$P\left[\sum_{i=1}^{m} A_i\right] = \sum_{i=1}^{m} P[A_i] - \sum_{i<j} P[A_i A_j] + \sum_{i<j<k} P[A_i A_j A_k] - \dots + (-1)^{m-1} P[A_1 A_2 \dots A_m]. \quad (1.21)$$

Theorem 1.14

For any event A, and event space $\{B_1, B_2, \dots, B_m\}$,

$$P[A] = \sum_{i=1}^{m} P[A \cap B_i]. \quad (1.22)$$

1.3.3 The Addition Rule

Theorem 1.13

For m events $A_1, A_2, \cdots, A_m$, then

$$P\Big[\sum_{i=1}^{m} A_i\Big] = \sum_{i=1}^{m} P[A_i] - \sum_{i<j} P[A_iA_j] + \sum_{i<j<k} P[A_iA_jA_k] - \ldots + (-1)^{m-1} P[A_1A_2\ldots A_m]. \tag{1.21}$$

Theorem 1.14

For any event A, and event space $\{B_1, B_2, \ldots, B_m\}$,

$$P[A] = \sum_{i=1}^{m} P[A \cap B_i]. \tag{1.22}$$

Content

1.4.1 Conditional Probability

Definition 1.17 (Conditional Probability)

The **conditional probability** of event A, given that event B has occurred is

$$P[A|B] = \frac{P[AB]}{P[B]} \quad \text{if} \quad P(B) \neq 0. \tag{1.24}$$

The **conditional probability** of event B, given that event A has occurred is

$$P[B|A] = \frac{P[AB]}{P[A]} \quad \text{if} \quad P(A) \neq 0. \tag{1.25}$$

Remark 1.5

A conditional probability measure $P[A|B]$ has the following **properties**:

(a) $P[A|B] \geq 0$.

(b) $P[B|B] = 1$.

1.4.1 Conditional Probability

Definition 1.17 (Conditional Probability)

The **conditional probability** of event A, given that event B has occurred is

$$P[A|B] = \frac{P[AB]}{P[B]} \quad \text{if} \quad P(B) \neq 0. \tag{1.24}$$

The **conditional probability** of event B, given that event A has occurred is

$$P[B|A] = \frac{P[AB]}{P[A]} \quad \text{if} \quad P(A) \neq 0. \tag{1.25}$$

Remark 1.5

A conditional probability measure $P[A|B]$ has the following **properties**:

(a) $P[A|B] \geq 0$.

(b) $P[B|B] = 1$.

1.4.1 Conditional Probability

Definition 1.17 (Conditional Probability)

The **conditional probability** of event A, given that event B has occurred is

$$P[A|B] = \frac{P[AB]}{P[B]} \quad \text{if} \quad P(B) \neq 0. \tag{1.24}$$

The **conditional probability** of event B, given that event A has occurred is

$$P[B|A] = \frac{P[AB]}{P[A]} \quad \text{if} \quad P(A) \neq 0. \tag{1.25}$$

Remark 1.5

A conditional probability measure $P[A|B]$ has the following **properties**:

(a) $P[A|B] \geq 0$.

(b) $P[B|B] = 1$.

1.4.1 Conditional Probability

Example 1.28

The probability that a regularly scheduled flight departs on time is $P[D] = 0.83$; the probability that it arrives on time is $P[A] = 0.82$; and the probability that it departs and arrives on time is $P[D \cap A] = 0.78$. Find the probability that a plane (a) arrives on time, given that it departed on time, and (b) departed on time, given that it has arrived on time.

Example 1.28 Solution

(a) The probability that a plane arrives on time, given that it departed on time, is

$$P[A|D] = \frac{P[A \cap D]}{P[D]} = \frac{0.78}{0.83} = 0.94.$$

(b) The probability that a plane departed on time, given that it has arrived on time, is

$$P[D|A] = \frac{P[A \cap D]}{P[A]} = \frac{0.78}{0.82} = 0.95.$$

1.4.1 Conditional Probability

Example 1.28

The probability that a regularly scheduled flight departs on time is $P[D] = 0.83$; the probability that it arrives on time is $P[A] = 0.82$; and the probability that it departs and arrives on time is $P[D \cap A] = 0.78$. Find the probability that a plane (a) arrives on time, given that it departed on time, and (b) departed on time, given that it has arrived on time.

Example 1.28 Solution

(a) The probability that a plane arrives on time, given that it departed on time, is

$$P[A|D] = \frac{P[A \cap D]}{P[D]} = \frac{0.78}{0.83} = 0.94.$$

(b) The probability that a plane departed on time, given that it has arrived on time, is

$$P[D|A] = \frac{P[A \cap D]}{P[A]} = \frac{0.78}{0.82} = 0.95.$$

1.4.1 Conditional Probability

Example 1.29

Consider an experiment that consists of testing two integrated circuits that come from the same silicon wafer, and observing in each case whether a circuit is accepted (A) or rejected (R). Consider the a priori probability model

$$P[RR] = 0.01, \quad P[RA] = 0.01, \quad P[AR] = 0.01, \quad P[AA] = 0.97.$$

Find the probability of $B =$"second chip rejected" and $C =$"first chip rejected." Also find the conditional probability that the second chip is a reject given that the first chip is a reject.

1.4.1 Conditional Probability

Example 1.29 Solution

- The sample space of the experiment is $S = \{RR, RA, AR, AA\}$.
- $B = RR + AR$ and $P[B] = P[RR] + P[AR] = 0.02$.
- $C = RR + RA$ and $P[C] = P[RR] + P[RA] = 0.02$.
- The conditional probability of the second chip being rejected given that the first chip is rejected is,

$$P[B|C] = \frac{P[BC]}{P[C]} = \frac{0.01}{0.02} = 0.5,$$

where $P[BC] = P[\text{both rejected}] = P[RR] = 0.01$.

1.4.1 Conditional Probability

Example 1.30

Roll two fair four-sided dice. Let X_1 and X_2 denote the number of dots that appear on die 1 and die 2, respectively. Let A be the event $X_1 \geq 2$. What is $P[A]$? Let B denote the event $X_2 > X_1$. What is $P[B]$? What is $P[A|B]$?

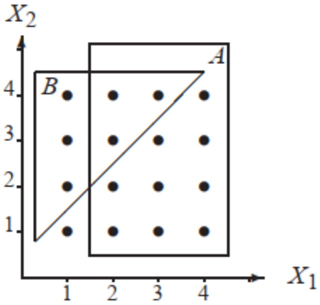


Figure: Figure for Example 1.30

1.4.1 Conditional Probability

Example 1.30

Roll two fair four-sided dice. Let X_1 and X_2 denote the number of dots that appear on die 1 and die 2, respectively. Let A be the event $X_1 \geq 2$. What is $P[A]$? Let B denote the event $X_2 > X_1$. What is $P[B]$? What is $P[A|B]$?

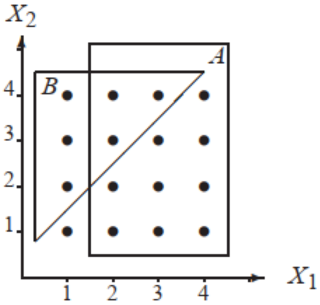


Figure: Figure for Example 1.30

1.4.1 Conditional Probability

Example 1.30 Solution

- The sample space has 16 elements corresponding to the four possible values of X_1 and the same four values of X_2.
- Since the dice are fair, the outcomes are equally likely, each with probability 1/16.
- The rectangle represents A. It contains 12 outcomes, each with probability 1/16. So $P[A] = 12/16 = 3/4$.
- The triangle represents B. It contains six outcomes. Therefore $P[B] = 6/16 = 3/8$.
- The event AB has three outcomes, $(2,3)$, $(2,4)$, $(3,4)$, so $P[AB] = 3/16$.
- From the definition of conditional probability, we write

$$P[A|B] = \frac{P[AB]}{P[B]} = \frac{1}{2}.$$

1.4.2 Independent Events

Definition 1.18 (Independent)

Two events, A and B, are said to be independent if and only if the probability of event B is not influenced or changed by the occurrence of event A, or vice versa.

Note

Two events, A and B, are independent if and only if

$$P[B|A] = P[B] \quad \text{or} \quad P[A|B] = P[A],$$

assuming the existences of the conditional probabilities. Otherwise, A and B are dependent.

1.4.2 Independent Events

Definition 1.18 (Independent)

Two events, A and B, are said to be independent if and only if the probability of event B is not influenced or changed by the occurrence of event A, or vice versa.

Note

Two events, A and B, are independent if and only if

$$P[B|A] = P[B] \quad \text{or} \quad P[A|B] = P[A],$$

assuming the existences of the conditional probabilities. Otherwise, A and B are dependent.

1.4.2 Independent Events

Example 1.31

On the other hand, consider tossing a single die two times, and define two events: A "Observe a 2 on the first toss". B "Observe a 2 on the second toss". If the die is fair, the probability of event A is $P[A] = \dfrac{1}{6}$. Consider the probability of event B. Regardless of whether event A has or has not occurred, the probability of observing a 2 on the second toss is still $\dfrac{1}{6}$. We could write:

$$P(B \text{ given that } A \text{ occurred}) = \frac{1}{6}.$$

$$P(B \text{ given that } A \text{ did not occur}) = \frac{1}{6}.$$

Since the probability of event B is not changed by the occurrence of event A, we say that A and B are independent events.

1.4.3 The Product Rule, or the Multiplicative Rule

Theorem 1.15

If in an experiment the events A and B can both occur, then

$$P[A \cap B] = P[A]P[B|A], \quad \text{provided} \quad P[A] > 0. \tag{1.26}$$

and

$$P[A \cap B] = P[B \cap A] = P[B]P[A|B], \quad \text{provided} \quad P[B] > 0. \tag{1.27}$$

1.4.3 The Product Rule, or the Multiplicative Rule

Theorem 1.15

If in an experiment the events A and B can both occur, then

$$P[A \cap B] = P[A]P[B|A], \quad \text{provided} \quad P[A] > 0. \tag{1.26}$$

and

$$P[A \cap B] = P[B \cap A] = P[B]P[A|B], \quad \text{provided} \quad P[B] > 0. \tag{1.27}$$

1.4.3 The Product Rule, or the Multiplicative Rule

Theorem 1.16

Two events A and B are independent if and only if

$$P[A \cap B] = P[A]P[B]. \quad (1.28)$$

Example 1.33

Toss two coins and observe the outcome. Define these events: A "Head (H) on the first coin", B "Tail (T) on the second coin". Are events A and B independent?

Example 1.33 Solution

From previous examples, you know that $S = \{HH, HT, TH, TT\}$. Use these four simple events to find

$$P[A] = \frac{1}{2}, \quad P[B] = \frac{1}{2}, \quad P[AB] = \frac{1}{4}.$$

Since $P[A]P[B] = P[AB] = \frac{1}{4}$, and the two events must be independent.

1.4.3 The Product Rule, or the Multiplicative Rule

Theorem 1.16

Two events A and B are independent if and only if

$$P[A \cap B] = P[A]P[B]. \tag{1.28}$$

Example 1.33

Toss two coins and observe the outcome. Define these events: A "Head (H) on the first coin", B "Tail (T) on the second coin". Are events A and B independent?

Example 1.33 Solution

From previous examples, you know that $S = \{HH, HT, TH, TT\}$. Use these four simple events to find

$$P[A] = \frac{1}{2}, \quad P[B] = \frac{1}{2}, \quad P[AB] = \frac{1}{4}.$$

Since $P[A]P[B] = P[AB] = \frac{1}{4}$, and the two events must be independent.

1.4.3 The Product Rule, or the Multiplicative Rule

Theorem 1.16

Two events A and B are independent if and only if

$$P[A \cap B] = P[A]P[B]. \tag{1.28}$$

Example 1.33

Toss two coins and observe the outcome. Define these events: A "Head (H) on the first coin", B "Tail (T) on the second coin". Are events A and B independent?

Example 1.33 Solution

From previous examples, you know that $S = \{HH, HT, TH, TT\}$. Use these four simple events to find

$$P[A] = \frac{1}{2}, \quad P[B] = \frac{1}{2}, \quad P[AB] = \frac{1}{4}.$$

Since $P[A]P[B] = P[AB] = \dfrac{1}{4}$, and the two events must be independent.

1.4.3 The Product Rule, or the Multiplicative Rule

Definition 1.19 (Three Independent Events)

A_1, A_2, and A_3 are **independent** if and only if

(a) A_1 and A_2 are independent,

(b) A_2 and A_3 are independent,

(c) A_1 and A_3 are independent, and

(d) $P[A_1 \cap A_2 \cap A_3] = P[A_1]P[A_2]P[A_3]$.

1.4.3 The Product Rule, or the Multiplicative Rule

Definition 1.19 (Three Independent Events)

A_1, A_2, and A_3 are **independent** if and only if

(a) A_1 and A_2 are independent,

(b) A_2 and A_3 are independent,

(c) A_1 and A_3 are independent, and

(d) $P[A_1 \cap A_2 \cap A_3] = P[A_1]P[A_2]P[A_3]$.

1.4.3 The Product Rule, or the Multiplicative Rule

Definition 1.19 (Three Independent Events)

A_1, A_2, and A_3 are **independent** if and only if

(a) A_1 and A_2 are independent,

(b) A_2 and A_3 are independent,

(c) A_1 and A_3 are independent, and

(d) $P[A_1 \cap A_2 \cap A_3] = P[A_1]P[A_2]P[A_3]$.

1.4.3 The Product Rule, or the Multiplicative Rule

Definition 1.19 (Three Independent Events)

A_1, A_2, and A_3 are **independent** if and only if

(a) A_1 and A_2 are independent,

(b) A_2 and A_3 are independent,

(c) A_1 and A_3 are independent, and

(d) $P[A_1 \cap A_2 \cap A_3] = P[A_1]P[A_2]P[A_3]$.

1.4.3 The Product Rule, or the Multiplicative Rule

Example 1.35

In an experiment with equiprobable outcomes, the event space is $S = \{1, 2, 3, 4\}$. $P[s] = 1/4$ for all $s \in S$. Are the events $A_1 = \{1, 3, 4\}$, $A_2 = \{2, 3, 4\}$, and $A_3 = \emptyset$ independent?

Example 1.35 Solution

These three sets satisfy the final condition of Definition 1.19 because $A_1 \cap A_2 \cap A_3 = \emptyset$, and

$$P[A_1 \cap A_2 \cap A_3] = P[A_1]P[A_2]P[A_3] = 0.$$

However, A_1 and A_2 are not independent because, with all outcomes equiprobable,

$$P[A_1 \cap A_2] = P[\{3, 4\}] = \frac{1}{2} \neq P[A_1]P[A_2] = \frac{3}{4} \times \frac{3}{4}.$$

Hence the three events are not independent.

1.4.3 The Product Rule, or the Multiplicative Rule

Example 1.35

In an experiment with equiprobable outcomes, the event space is $S = \{1, 2, 3, 4\}$. $P[s] = 1/4$ for all $s \in S$. Are the events $A_1 = \{1, 3, 4\}$, $A_2 = \{2, 3, 4\}$, and $A_3 = \emptyset$ independent?

Example 1.35 Solution

These three sets satisfy the final condition of Definition 1.19 because $A_1 \cap A_2 \cap A_3 = \emptyset$, and

$$P[A_1 \cap A_2 \cap A_3] = P[A_1]P[A_2]P[A_3] = 0.$$

However, A_1 and A_2 are not independent because, with all outcomes equiprobable,

$$P[A_1 \cap A_2] = P[\{3, 4\}] = \frac{1}{2} \neq P[A_1]P[A_2] = \frac{3}{4} \times \frac{3}{4}.$$

Hence the three events are not independent.

1.4.3 The Product Rule, or the Multiplicative Rule

Example 1.35

In an experiment with equiprobable outcomes, the event space is $S = \{1, 2, 3, 4\}$. $P[s] = 1/4$ for all $s \in S$. Are the events $A_1 = \{1, 3, 4\}$, $A_2 = \{2, 3, 4\}$, and $A_3 = \emptyset$ independent?

Example 1.35 Solution

These three sets satisfy the final condition of Definition 1.19 because $A_1 \cap A_2 \cap A_3 = \emptyset$, and

$$P[A_1 \cap A_2 \cap A_3] = P[A_1]P[A_2]P[A_3] = 0.$$

However, A_1 and A_2 are not independent because, with all outcomes equiprobable,

$$P[A_1 \cap A_2] = P[\{3, 4\}] = \frac{1}{2} \neq P[A_1]P[A_2] = \frac{3}{4} \times \frac{3}{4}.$$

Hence the three events are not independent.

1.4.3 The Product Rule, or the Multiplicative Rule

Example 1.35

In an experiment with equiprobable outcomes, the event space is $S = \{1, 2, 3, 4\}$. $P[s] = 1/4$ for all $s \in S$. Are the events $A_1 = \{1, 3, 4\}$, $A_2 = \{2, 3, 4\}$, and $A_3 = \emptyset$ independent?

Example 1.35 Solution

These three sets satisfy the final condition of Definition 1.19 because $A_1 \cap A_2 \cap A_3 = \emptyset$, and

$$P[A_1 \cap A_2 \cap A_3] = P[A_1]P[A_2]P[A_3] = 0.$$

However, A_1 and A_2 are not independent because, with all outcomes equiprobable,

$$P[A_1 \cap A_2] = P[\{3, 4\}] = \frac{1}{2} \neq P[A_1]P[A_2] = \frac{3}{4} \times \frac{3}{4}.$$

Hence the three events are not independent.

1.4.3 The Product Rule, or the Multiplicative Rule

Definition 1.20 (More than Three Independent Events)

If $n > 3$, the sets $A_1, A_2, \ldots, A_n$ are **independent** if and only if

(a) every set of $n-1$ sets taken from $A_1, A_2, \ldots, A_n$ is independent,

(b) $P[A_1 \cap A_2 \cap \cdots A_n] = P[A_1]P[A_2] \cdots P[A_n]$.

Remark 1.6

1. This definition and Example 1.35 show us that when $n > 3$ it is a complex matter to determine whether or not a set of n events is independent.
2. On the other hand, if we know that a set is independent, it is a simple matter to determine the probability of the intersection of any subset of the events. Just multiply the probabilities of the events in the subset.

1.4.3 The Product Rule, or the Multiplicative Rule

Definition 1.20 (More than Three Independent Events)

If $n > 3$, the sets $A_1, A_2, \ldots, A_n$ are **independent** if and only if

(a) every set of $n-1$ sets taken from $A_1, A_2, \ldots, A_n$ is independent,

(b) $P[A_1 \cap A_2 \cap \cdots A_n] = P[A_1]P[A_2]\cdots P[A_n]$.

Remark 1.6

1. This definition and Example 1.35 show us that when $n > 3$ it is a complex matter to determine whether or not a set of n events is independent.
2. On the other hand, if we know that a set is independent, it is a simple matter to determine the probability of the intersection of any subset of the events. Just multiply the probabilities of the events in the subset.

1.4.3 The Product Rule, or the Multiplicative Rule

Theorem 1.17

If, in an experiment, the events $A_1, A_2, \ldots, A_n$ can occur, then

$$P[A_1 \cap A_2 \cap \cdots \cap A_n] = P[A_1]P[A_2|A_1]P[A_3|A_1 \cap A_2] \ldots P[A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1}]. \quad (1.29)$$

If the events $A_1, A_2, \ldots, A_n$ are independent, then

$$P[A_1 \cap A_2 \cap \cdots \cap A_k] = P[A_1]P[A_2] \ldots P[A_n]. \quad (1.30)$$

1.4.3 The Product Rule, or the Multiplicative Rule

Example 1.36

Three cards are drawn in succession, without replacement, from an ordinary deck of playing cards. Find the probability that the event $A_1 \cap A_2 \cap A_3$ occurs, where A_1 is the event that the first card is a red ace, A_2 is the event that the second card is a 10 or a jack, and A_3 is the event that the third card is greater than 3 but less than 7.

1.4.3 The Product Rule, or the Multiplicative Rule

Example 1.36 Solution

First we define the events A_1: the first card is a red ace, A_2: the second card is a 10 or a jack, A_3: the third card is greater than 3 but less than 7. Now

$$P[A_1] = \frac{2}{52}, \quad P[A_2|A_1] = \frac{8}{51}, \quad P[A_3|A_1 \cap A_2] = \frac{12}{50}$$

and hence, by Theorem 1.17,

$$\begin{aligned} P[A_1 \cap A_2 \cap A_3] &= P[A_1]P[A_2|A_1]P[A_3|A_1 \cap A_2] \\ &= \left(\frac{2}{52}\right)\left(\frac{8}{51}\right)\left(\frac{12}{50}\right) = \frac{8}{5525}. \end{aligned}$$

1.4.3 The Product Rule, or the Multiplicative Rule

Remark 1.7

What's the difference between mutually exclusive and independent events?

(a) When two events are mutually exclusive or disjoint, they cannot both happen when the experiment is performed. Once the event B has occurred, event A cannot occur, so that $P[A|B] = 0$, or vice versa. The occurrence of event B certainly affects the probability that event A can occur. Therefore, mutually exclusive events must be dependent.

(b) When two events are mutually exclusive or disjoint,
$P[AB] = 0$ and $P[A \cup B] = P[A] + P[B]$.

(c) When two events are independent,
$P[AB] = P[A]P[B]$ and $P[A \cup B] = P[A] + P[B] - P[A]P[B]$.

Content

1.5.1 Total Probability

Theorem 1.18 (Total Probability)

For an **event space** $\{B_1, B_2, \ldots, B_m\}$ with $P[B_i] > 0$ for all i and an event A, the probability of the event A can be expressed as

$$P[A] = P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + \cdots + P[B_m]P[A|B_m] \quad (1.31)$$

or

$$P[A] = \sum_{i=1}^{m} P[B_i]P[A|B_i]. \quad (1.311)$$

1.5.1 Total Probability

Theorem 1.18 (Total Probability)

For an **event space** $\{B_1, B_2, \ldots, B_m\}$ with $P[B_i] > 0$ for all i and an event A, the probability of the event A can be expressed as

$$P[A] = P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + \cdots + P[B_m]P[A|B_m] \tag{1.31}$$

or

$$P[A] = \sum_{i=1}^{m} P[B_i]P[A|B_i]. \tag{1.311}$$

1.5.1 Total Probability

Theorem 1.18 (Total Probability)

For an **event space** $\{B_1, B_2, \ldots, B_m\}$ with $P[B_i] > 0$ for all i and an event A, the probability of the event A can be expressed as

$$P[A] = P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + \cdots + P[B_m]P[A|B_m] \tag{1.31}$$

or

$$P[A] = \sum_{i=1}^{m} P[B_i]P[A|B_i]. \tag{1.311}$$

1.5.1 Total Probability

Proof

Consider the Venn diagram of Figure 1.11.

Figure 1.11

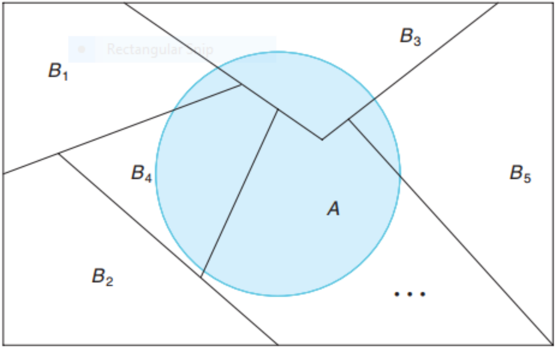


Figure: Partitioning the sample space S

1.5.1 Total Probability

Proof

Consider the Venn diagram of Figure 1.11.

Figure 1.11

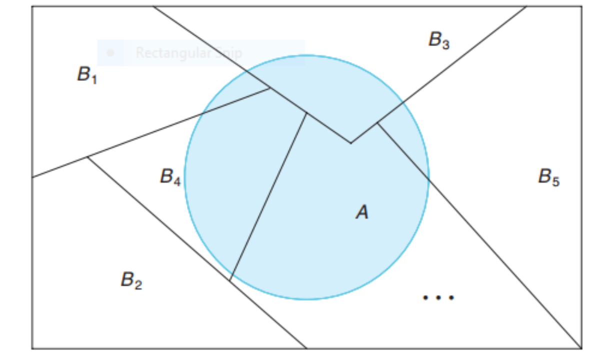


Figure: Partitioning the sample space S

1.5.1 Total Probability

Proof

- $A = AS = A(B_1 + B_2 + \cdots + B_m) = AB_1 + AB_2 + \cdots + AB_m.$
- $P[A] = P[AB_1 + AB_2 + \cdots + AB_m] = P[AB_1] + P[AB_2] + \cdots + P[AB_m]$
  $= P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + \cdots + P[B_m]P[A|B_m].$

1.5.1 Total Probability

Proof

- $A = AS = A(B_1 + B_2 + \cdots + B_m) = AB_1 + AB_2 + \cdots + AB_m.$
- $P[A] = P[AB_1 + AB_2 + \cdots + AB_m] = P[AB_1] + P[AB_2] + \cdots + P[AB_m]$
  $= P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + \cdots + P[B_m]P[A|B_m].$

1.5.1 Total Probability

Proof

- $A = AS = A(B_1 + B_2 + \cdots + B_m) = AB_1 + AB_2 + \cdots + AB_m.$
- $P[A] = P[AB_1 + AB_2 + \cdots + AB_m] = P[AB_1] + P[AB_2] + \cdots + P[AB_m]$
  $= P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + \cdots + P[B_m]P[A|B_m].$

1.5.1 Total Probability

Proof

- $A = AS = A(B_1 + B_2 + \cdots + B_m) = AB_1 + AB_2 + \cdots + AB_m.$
- $P[A] = P[AB_1 + AB_2 + \cdots + AB_m] = P[AB_1] + P[AB_2] + \cdots + P[AB_m]$
  $= P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + \cdots + P[B_m]P[A|B_m].$

1.5.1 Total Probability

Proof

- $A = AS = A(B_1 + B_2 + \cdots + B_m) = AB_1 + AB_2 + \cdots + AB_m.$
- $P[A] = P[AB_1 + AB_2 + \cdots + AB_m] = P[AB_1] + P[AB_2] + \cdots + P[AB_m]$
  $= P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + \cdots + P[B_m]P[A|B_m].$

1.5.1 Total Probability

Proof

- $A = AS = A(B_1 + B_2 + \cdots + B_m) = AB_1 + AB_2 + \cdots + AB_m.$
- $P[A] = P[AB_1 + AB_2 + \cdots + AB_m] = P[AB_1] + P[AB_2] + \cdots + P[AB_m]$
  $= P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + \cdots + P[B_m]P[A|B_m].$

1.5.1 Total Probability

Proof

- $A = AS = A(B_1 + B_2 + \cdots + B_m) = AB_1 + AB_2 + \cdots + AB_m.$
- $P[A] = P[AB_1 + AB_2 + \cdots + AB_m] = P[AB_1] + P[AB_2] + \cdots + P[AB_m]$
  $= P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + \cdots + P[B_m]P[A|B_m].$

1.5.1 Total Probability

Example 1.37

A company has three machines B1, B2, and B3 for making 1 $k\Omega$ resistors. It has been observed that 80% of resistors produced by B_1 are within 50 of the nominal value. Machine B_2 produces 90% of resistors within 50 of the nominal value. The percentage for machine B_3 is 60%. Each hour, machine B_1 produces 3000 resistors, B_2 produces 4000 resistors, and B_3 produces 3000 resistors. All of the resistors are mixed together at random in one bin and packed for shipment. What is the probability that the company ships a resistor that is within 50 of the nominal value?

1.4.1 Law of Total Probability

Example 1.37 Solution

- Let $A =$ "resistor is within 50 of the nominal value". $P[A] =?$.
- B_1: the product is made by machine B1; B_2: the product is made by machine B2; B_3: the product is made by machine B3.
- $\{B_1, B_2, B_3\}$ is an event space.
- Hence, $P[A] = P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + P[B_3]P[A|B_3]$,
- where $P[B_1] = \dfrac{3000}{10000} = 0.3; \quad P[B_2] = \dfrac{4000}{10000} = 0.4; \; P[B_3] = \dfrac{3000}{10000} = 0.3$,
- and $P[A|B_1] = 0.8; \quad P[A|B_2] = 0.9; \quad P[A|B_3] = 0.6$.
- Hence,

$$P[A] = 0.3 \times 0.8 + 0.4 \times 0.9 + 0.3 \times 0.6 = 0.78.$$

For the whole factory, 78% of resistors are within 50 of the nominal value.

1.4.1 Law of Total Probability

Example 1.37 Solution

- Let $A =$ "resistor is within 50 of the nominal value". $P[A] =?$.
- B_1: the product is made by machine B1; B_2: the product is made by machine B2; B_3: the product is made by machine B3.
- $\{B_1, B_2, B_3\}$ is an event space.
- Hence, $P[A] = P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + P[B_3]P[A|B_3]$,
- where $P[B_1] = \dfrac{3000}{10000} = 0.3; \quad P[B_2] = \dfrac{4000}{10000} = 0.4; \; P[B_3] = \dfrac{3000}{10000} = 0.3$,
- and $P[A|B_1] = 0.8; \quad P[A|B_2] = 0.9; \quad P[A|B_3] = 0.6.$
- Hence,

$$P[A] = 0.3 \times 0.8 + 0.4 \times 0.9 + 0.3 \times 0.6 = 0.78.$$

For the whole factory, 78% of resistors are within 50 of the nominal value.

1.4.1 Law of Total Probability

Example 1.37 Solution

- Let $A =$ "resistor is within 50 of the nominal value". $P[A] =?$.
- B_1: the product is made by machine B1; B_2: the product is made by machine B2; B_3: the product is made by machine B3.
- $\{B_1, B_2, B_3\}$ is an event space.
- Hence, $P[A] = P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + P[B_3]P[A|B_3]$,
- where $P[B_1] = \dfrac{3000}{10000} = 0.3; \quad P[B_2] = \dfrac{4000}{10000} = 0.4; \; P[B_3] = \dfrac{3000}{10000} = 0.3,$
- and $P[A|B_1] = 0.8; \quad P[A|B_2] = 0.9; \quad P[A|B_3] = 0.6.$
- Hence,

$$P[A] = 0.3 \times 0.8 + 0.4 \times 0.9 + 0.3 \times 0.6 = 0.78.$$

For the whole factory, 78% of resistors are within 50 of the nominal value.

1.4.1 Law of Total Probability

Example 1.37 Solution

- Let $A =$ "resistor is within 50 of the nominal value". $P[A] =?$.
- B_1: the product is made by machine B1; B_2: the product is made by machine B2; B_3: the product is made by machine B3.
- $\{B_1, B_2, B_3\}$ is an event space.
- Hence, $P[A] = P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + P[B_3]P[A|B_3]$,
- where $P[B_1] = \dfrac{3000}{10000} = 0.3; \quad P[B_2] = \dfrac{4000}{10000} = 0.4; \; P[B_3] = \dfrac{3000}{10000} = 0.3$,
- and $P[A|B_1] = 0.8; \quad P[A|B_2] = 0.9; \quad P[A|B_3] = 0.6$.
- Hence,

$$P[A] = 0.3 \times 0.8 + 0.4 \times 0.9 + 0.3 \times 0.6 = 0.78.$$

For the whole factory, 78% of resistors are within 50 of the nominal value.

1.4.1 Law of Total Probability

Example 1.37 Solution

- Let $A =$ "resistor is within 50 of the nominal value". $P[A] =?$.
- B_1: the product is made by machine B1; B_2: the product is made by machine B2; B_3: the product is made by machine B3.
- $\{B_1, B_2, B_3\}$ is an event space.
- Hence, $P[A] = P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + P[B_3]P[A|B_3]$,
- where $P[B_1] = \dfrac{3000}{10000} = 0.3; \quad P[B_2] = \dfrac{4000}{10000} = 0.4; \; P[B_3] = \dfrac{3000}{10000} = 0.3$,
- and $P[A|B_1] = 0.8; \quad P[A|B_2] = 0.9; \quad P[A|B_3] = 0.6$.
- Hence,

$$P[A] = 0.3 \times 0.8 + 0.4 \times 0.9 + 0.3 \times 0.6 = 0.78.$$

For the whole factory, 78% of resistors are within 50 of the nominal value.

1.4.1 Law of Total Probability

Example 1.37 Solution

- Let $A =$ "resistor is within 50 of the nominal value". $P[A] =?$.
- B_1: the product is made by machine B1; B_2: the product is made by machine B2; B_3: the product is made by machine B3.
- $\{B_1, B_2, B_3\}$ is an event space.
- Hence, $P[A] = P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + P[B_3]P[A|B_3]$,
- where $P[B_1] = \dfrac{3000}{10000} = 0.3$; $P[B_2] = \dfrac{4000}{10000} = 0.4$; $P[B_3] = \dfrac{3000}{10000} = 0.3$,
- and $P[A|B_1] = 0.8$; $P[A|B_2] = 0.9$; $P[A|B_3] = 0.6$.
- Hence,

$$P[A] = 0.3 \times 0.8 + 0.4 \times 0.9 + 0.3 \times 0.6 = 0.78.$$

For the whole factory, 78% of resistors are within 50 of the nominal value.

1.4.1 Law of Total Probability

Example 1.37 Solution

- Let $A =$ "resistor is within 50 of the nominal value". $P[A] =?$.
- B_1: the product is made by machine B1; B_2: the product is made by machine B2; B_3: the product is made by machine B3.
- $\{B_1, B_2, B_3\}$ is an event space.
- Hence, $P[A] = P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + P[B_3]P[A|B_3]$,
- where $P[B_1] = \dfrac{3000}{10000} = 0.3$; $P[B_2] = \dfrac{4000}{10000} = 0.4$; $P[B_3] = \dfrac{3000}{10000} = 0.3$,
- and $P[A|B_1] = 0.8$; $P[A|B_2] = 0.9$; $P[A|B_3] = 0.6$.
- Hence,

$$P[A] = 0.3 \times 0.8 + 0.4 \times 0.9 + 0.3 \times 0.6 = 0.78.$$

For the whole factory, 78% of resistors are within 50 of the nominal value.

1.4.1 Law of Total Probability

Example 1.37 Solution

- Let $A =$ "resistor is within 50 of the nominal value". $P[A] =?$.
- B_1: the product is made by machine B1; B_2: the product is made by machine B2; B_3: the product is made by machine B3.
- $\{B_1, B_2, B_3\}$ is an event space.
- Hence, $P[A] = P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + P[B_3]P[A|B_3]$,
- where $P[B_1] = \dfrac{3000}{10000} = 0.3$; $\quad P[B_2] = \dfrac{4000}{10000} = 0.4$; $P[B_3] = \dfrac{3000}{10000} = 0.3$,
- and $P[A|B_1] = 0.8$; $\quad P[A|B_2] = 0.9$; $\quad P[A|B_3] = 0.6$.
- Hence,

$$P[A] = 0.3 \times 0.8 + 0.4 \times 0.9 + 0.3 \times 0.6 = 0.78.$$

For the whole factory, 78% of resistors are within 50 of the nominal value.

1.4.1 Law of Total Probability

Example 1.37 Solution

- Let $A =$ "resistor is within 50 of the nominal value". $P[A] =?$.
- B_1: the product is made by machine B1; B_2: the product is made by machine B2; B_3: the product is made by machine B3.
- $\{B_1, B_2, B_3\}$ is an event space.
- Hence, $P[A] = P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + P[B_3]P[A|B_3]$,
- where $P[B_1] = \frac{3000}{10000} = 0.3;\quad P[B_2] = \frac{4000}{10000} = 0.4;\ P[B_3] = \frac{3000}{10000} = 0.3,$
- and $P[A|B_1] = 0.8;\quad P[A|B_2] = 0.9;\quad P[A|B_3] = 0.6.$
- Hence,

$$P[A] = 0.3 \times 0.8 + 0.4 \times 0.9 + 0.3 \times 0.6 = 0.78.$$

For the whole factory, 78% of resistors are within 50 of the nominal value.

1.4.1 Law of Total Probability

Example 1.37 Solution

- Let $A =$ "resistor is within 50 of the nominal value". $P[A] = ?$.
- B_1: the product is made by machine B1; B_2: the product is made by machine B2; B_3: the product is made by machine B3.
- $\{B_1, B_2, B_3\}$ is an event space.
- Hence, $P[A] = P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + P[B_3]P[A|B_3]$,
- where $P[B_1] = \dfrac{3000}{10000} = 0.3;\quad P[B_2] = \dfrac{4000}{10000} = 0.4;\ P[B_3] = \dfrac{3000}{10000} = 0.3,$
- and $P[A|B_1] = 0.8;\quad P[A|B_2] = 0.9;\quad P[A|B_3] = 0.6.$
- Hence,

$$P[A] = 0.3 \times 0.8 + 0.4 \times 0.9 + 0.3 \times 0.6 = 0.78.$$

For the whole factory, 78% of resistors are within 50 of the nominal value.

1.4.1 Law of Total Probability

Example 1.37 Solution

- Let $A =$ "resistor is within 50 of the nominal value". $P[A] =?$.
- B_1: the product is made by machine B1; B_2: the product is made by machine B2; B_3: the product is made by machine B3.
- $\{B_1, B_2, B_3\}$ is an event space.
- Hence, $P[A] = P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + P[B_3]P[A|B_3]$,
- where $P[B_1] = \dfrac{3000}{10000} = 0.3; \quad P[B_2] = \dfrac{4000}{10000} = 0.4; \; P[B_3] = \dfrac{3000}{10000} = 0.3$,
- and $P[A|B_1] = 0.8; \quad P[A|B_2] = 0.9; \quad P[A|B_3] = 0.6.$
- Hence,

$$P[A] = 0.3 \times 0.8 + 0.4 \times 0.9 + 0.3 \times 0.6 = 0.78.$$

For the whole factory, 78% of resistors are within 50 of the nominal value.

1.4.1 Law of Total Probability

Example 1.37 Solution

- Let $A =$ "resistor is within 50 of the nominal value". $P[A] =?$.
- B_1: the product is made by machine B1; B_2: the product is made by machine B2; B_3: the product is made by machine B3.
- $\{B_1, B_2, B_3\}$ is an event space.
- Hence, $P[A] = P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + P[B_3]P[A|B_3]$,
- where $P[B_1] = \dfrac{3000}{10000} = 0.3$; $P[B_2] = \dfrac{4000}{10000} = 0.4$; $P[B_3] = \dfrac{3000}{10000} = 0.3$,
- and $P[A|B_1] = 0.8$; $P[A|B_2] = 0.9$; $P[A|B_3] = 0.6$.
- Hence,

$$P[A] = 0.3 \times 0.8 + 0.4 \times 0.9 + 0.3 \times 0.6 = 0.78.$$

For the whole factory, 78% of resistors are within 50 of the nominal value.

1.4.1 Law of Total Probability

Example 1.37 Solution

- Let $A =$ "resistor is within 50 of the nominal value". $P[A] =?$.
- B_1: the product is made by machine B1; B_2: the product is made by machine B2; B_3: the product is made by machine B3.
- $\{B_1, B_2, B_3\}$ is an event space.
- Hence, $P[A] = P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + P[B_3]P[A|B_3]$,
- where $P[B_1] = \dfrac{3000}{10000} = 0.3; \quad P[B_2] = \dfrac{4000}{10000} = 0.4; \; P[B_3] = \dfrac{3000}{10000} = 0.3$,
- and $P[A|B_1] = 0.8; \quad P[A|B_2] = 0.9; \quad P[A|B_3] = 0.6$.
- Hence,

$$P[A] = 0.3 \times 0.8 + 0.4 \times 0.9 + 0.3 \times 0.6 = 0.78.$$

For the whole factory, 78% of resistors are within 50 of the nominal value.

1.4.1 Law of Total Probability

Example 1.37 Solution

- Let $A =$ "resistor is within 50 of the nominal value". $P[A] =?$.
- B_1: the product is made by machine B1; B_2: the product is made by machine B2; B_3: the product is made by machine B3.
- $\{B_1, B_2, B_3\}$ is an event space.
- Hence, $P[A] = P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + P[B_3]P[A|B_3]$,
- where $P[B_1] = \dfrac{3000}{10000} = 0.3; \quad P[B_2] = \dfrac{4000}{10000} = 0.4; \; P[B_3] = \dfrac{3000}{10000} = 0.3$,
- and $P[A|B_1] = 0.8; \quad P[A|B_2] = 0.9; \quad P[A|B_3] = 0.6$.
- Hence,

$$P[A] = 0.3 \times 0.8 + 0.4 \times 0.9 + 0.3 \times 0.6 = 0.78.$$

For the whole factory, 78% of resistors are within 50 of the nominal value.

1.4.1 Law of Total Probability

Example 1.37 Solution

- Let $A =$ "resistor is within 50 of the nominal value". $P[A] =?$.
- B_1: the product is made by machine B1; B_2: the product is made by machine B2; B_3: the product is made by machine B3.
- $\{B_1, B_2, B_3\}$ is an event space.
- Hence, $P[A] = P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + P[B_3]P[A|B_3]$,
- where $P[B_1] = \dfrac{3000}{10000} = 0.3; \quad P[B_2] = \dfrac{4000}{10000} = 0.4; \; P[B_3] = \dfrac{3000}{10000} = 0.3$,
- and $P[A|B_1] = 0.8; \quad P[A|B_2] = 0.9; \quad P[A|B_3] = 0.6$.
- Hence,

$$P[A] = 0.3 \times 0.8 + 0.4 \times 0.9 + 0.3 \times 0.6 = 0.78.$$

For the whole factory, 78% of resistors are within 50 of the nominal value.

1.4.1 Law of Total Probability

Example 1.37 Solution

- Let $A =$ "resistor is within 50 of the nominal value". $P[A] = ?$.
- B_1: the product is made by machine B1; B_2: the product is made by machine B2; B_3: the product is made by machine B3.
- $\{B_1, B_2, B_3\}$ is an event space.
- Hence, $P[A] = P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + P[B_3]P[A|B_3]$,
- where $P[B_1] = \dfrac{3000}{10000} = 0.3; \quad P[B_2] = \dfrac{4000}{10000} = 0.4; \ P[B_3] = \dfrac{3000}{10000} = 0.3$,
- and $P[A|B_1] = 0.8; \quad P[A|B_2] = 0.9; \quad P[A|B_3] = 0.6$.
- Hence,

$$P[A] = 0.3 \times 0.8 + 0.4 \times 0.9 + 0.3 \times 0.6 = 0.78.$$

For the whole factory, 78% of resistors are within 50 of the nominal value.

1.5.1 Total Probability

Practice Test 1

In a certain assembly plant, three machines, B1, B2, and B3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is non-defective?

1.5.2 Bayes' Rule

Theorem 1.19 (Bayes' Theorem)

$$P[B|A] = \frac{P[B]P[A|B]}{P[A]}. \tag{1.32}$$

Proof

- $P[B|A] = \frac{P[AB]}{P[A]}$ (by the definition of conditional probability).
- $P[AB] = P[B]P[A|B]$ (by the multiplicative rule).

1.5.2 Bayes' Rule

Theorem 1.19 (Bayes' Theorem)

$$P[B|A] = \frac{P[B]P[A|B]}{P[A]}. \tag{1.32}$$

Proof

- $P[B|A] = \dfrac{P[AB]}{P[A]}$ (by the definition of conditional probability).
- $P[AB] = P[B]P[A|B]$ (by the multiplicative rule).

1.5.2 Bayes' Rule

Theorem 1.19 (Bayes' Theorem)

$$P[B|A] = \frac{P[B]P[A|B]}{P[A]}. \tag{1.32}$$

Proof

- $P[B|A] = \dfrac{P[AB]}{P[A]}$ (by the definition of conditional probability).
- $P[AB] = P[B]P[A|B]$ (by the multiplicative rule).

1.5.2 Bayes' Rule

Theorem 1.20 (Bayes' Rule)

Let $\{B_1, B_2, \ldots, B_m\}$ be an **event space** with prior probabilities $P[B_1]$, $P[B_2]$, ..., $P[B_m]$. If an event A occurs, the posterior probability of B_i given A is the conditional probability

$$P[B_i|A] = \frac{P[B_i]P[A|B_i]}{\sum_{i=1}^{m} P[B_i]P[A|B_i]}. \quad (1.33)$$

1.5.2 Bayes' Rule

Theorem 1.20 (Bayes' Rule)

Let $\{B_1, B_2, \ldots, B_m\}$ be an **event space** with prior probabilities $P[B_1]$, $P[B_2]$, ..., $P[B_m]$. If an event A occurs, the posterior probability of B_i given A is the conditional probability

$$P[B_i|A] = \frac{P[B_i]P[A|B_i]}{\sum_{i=1}^{m} P[B_i]P[A|B_i]}. \tag{1.33}$$

1.5.2 Bayes' Rule

Proof

- $P[B_i|A] = \dfrac{P[AB_i]}{P[A]}$ (by the definition of conditional probability).
- $P[AB_i] = P[B_i]P[A|B_i]$ (by the multiplicative rule).
- $P[A] = P[B_1]P[A|B_1] + \cdots + P[B_m]P[A|B_m]$ (by total probability).

1.5.2 Bayes' Rule

Proof

- $P[B_i|A] = \dfrac{P[AB_i]}{P[A]}$ (by the definition of conditional probability).
- $P[AB_i] = P[B_i]P[A|B_i]$ (by the multiplicative rule).
- $P[A] = P[B_1]P[A|B_1] + \cdots + P[B_m]P[A|B_m]$ (by total probability).

1.5.2 Bayes' Rule

Proof

- $P[B_i|A] = \dfrac{P[AB_i]}{P[A]}$ (by the definition of conditional probability).
- $P[AB_i] = P[B_i]P[A|B_i]$ (by the multiplicative rule).
- $P[A] = P[B_1]P[A|B_1] + \cdots + P[B_m]P[A|B_m]$ (by total probability).

1.5.2 Bayes' Rule

Proof

- $P[B_i|A] = \dfrac{P[AB_i]}{P[A]}$ (by the definition of conditional probability).
- $P[AB_i] = P[B_i]P[A|B_i]$ (by the multiplicative rule).
- $P[A] = P[B_1]P[A|B_1] + \cdots + P[B_m]P[A|B_m]$ (by total probability).

1.5.2 Bayes' Rule

Proof

- $P[B_i|A] = \dfrac{P[AB_i]}{P[A]}$ (by the definition of conditional probability).
- $P[AB_i] = P[B_i]P[A|B_i]$ (by the multiplicative rule).
- $P[A] = P[B_1]P[A|B_1] + \cdots + P[B_m]P[A|B_m]$ (by total probability).

1.5.2 Bayes' Rule

Proof

- $P[B_i|A] = \dfrac{P[AB_i]}{P[A]}$ (by the definition of conditional probability).
- $P[AB_i] = P[B_i]P[A|B_i]$ (by the multiplicative rule).
- $P[A] = P[B_1]P[A|B_1] + \cdots + P[B_m]P[A|B_m]$ (by total probability).

1.5.2 Bayes' Rule

Example 1.39

In Example 1.37 about a shipment of resistors from the factory, we learned that:

(a) The probability that a resistor is from machine B_3 is $P[B_3] = 0.3$.

(b) The probability that a resistor is acceptable, i.e., within 50 of the nominal value, is $P[A] = 0.78$.

(c) Given that a resistor is from machine B_3, the conditional probability that it is acceptable is $P[A|B_3] = 0.6$.

What is the probability that an acceptable resistor comes from machine B_3?

1.5.2 Bayes' Rule

Example 1.39 Solution

Now we are given the event A that a resistor is within 50 of the nominal value, and we need to find $P[B_3|A]$.

- Using Bayes' theorem, we have

$$P[B_3|A] = \frac{P[B_3]P[A|B_3]}{P[A]}.$$

- Since all of the quantities we need are given in the problem description, our answer is

$$P[B_3|A] = \frac{0.3 \times 0.6}{0.78} = 0.23.$$

- Similarly we obtain $P[B_1|A] = 0.31$ and $P[B_2|A] = 0.46$.
- Of all resistors within 50 of the nominal value, only 23% come from machine B_3 (even though this machine produces 30% of all resistors). Machine B_1 produces 31% of the resistors that meet the 50 criterion and machine B_2 produces 46% of them.

1.5.2 Bayes' Rule

Example 1.39 Solution

Now we are given the event A that a resistor is within 50 of the nominal value, and we need to find $P[B_3|A]$.

- Using Bayes' theorem, we have

$$P[B_3|A] = \frac{P[B_3]P[A|B_3]}{P[A]}.$$

- Since all of the quantities we need are given in the problem description, our answer is

$$P[B_3|A] = \frac{0.3 \times 0.6}{0.78} = 0.23.$$

- Similarly we obtain $P[B_1|A] = 0.31$ and $P[B_2|A] = 0.46$.
- Of all resistors within 50 of the nominal value, only 23% come from machine B_3 (even though this machine produces 30% of all resistors). Machine B_1 produces 31% of the resistors that meet the 50 criterion and machine B_2 produces 46% of them.

1.5.2 Bayes' Rule

Example 1.39 Solution

Now we are given the event A that a resistor is within 50 of the nominal value, and we need to find $P[B_3|A]$.

- Using Bayes' theorem, we have

$$P[B_3|A] = \frac{P[B_3]P[A|B_3]}{P[A]}.$$

- Since all of the quantities we need are given in the problem description, our answer is

$$P[B_3|A] = \frac{0.3 \times 0.6}{0.78} = 0.23.$$

- Similarly we obtain $P[B_1|A] = 0.31$ and $P[B_2|A] = 0.46$.
- Of all resistors within 50 of the nominal value, only 23% come from machine B_3 (even though this machine produces 30% of all resistors). Machine B_1 produces 31% of the resistors that meet the 50 criterion and machine B_2 produces 46% of them.

1.5.2 Bayes' Rule

Example 1.39 Solution

Now we are given the event A that a resistor is within 50 of the nominal value, and we need to find $P[B_3|A]$.

- Using Bayes' theorem, we have

$$P[B_3|A] = \frac{P[B_3]P[A|B_3]}{P[A]}.$$

- Since all of the quantities we need are given in the problem description, our answer is

$$P[B_3|A] = \frac{0.3 \times 0.6}{0.78} = 0.23.$$

- Similarly we obtain $P[B_1|A] = 0.31$ and $P[B_2|A] = 0.46$.
- Of all resistors within 50 of the nominal value, only 23% come from machine B_3 (even though this machine produces 30% of all resistors). Machine B_1 produces 31% of the resistors that meet the 50 criterion and machine B_2 produces 46% of them.

1.5.2 Bayes' Rule

Example 1.39 Solution

Now we are given the event A that a resistor is within 50 of the nominal value, and we need to find $P[B_3|A]$.

- Using Bayes' theorem, we have

$$P[B_3|A] = \frac{P[B_3]P[A|B_3]}{P[A]}.$$

- Since all of the quantities we need are given in the problem description, our answer is

$$P[B_3|A] = \frac{0.3 \times 0.6}{0.78} = 0.23.$$

- Similarly we obtain $P[B_1|A] = 0.31$ and $P[B_2|A] = 0.46$.
- Of all resistors within 50 of the nominal value, only 23% come from machine B_3 (even though this machine produces 30% of all resistors). Machine B_1 produces 31% of the resistors that meet the 50 criterion and machine B_2 produces 46% of them.

1.5.2 Bayes' Rule

Example 1.39 Solution

Now we are given the event A that a resistor is within 50 of the nominal value, and we need to find $P[B_3|A]$.

- Using Bayes' theorem, we have

$$P[B_3|A] = \frac{P[B_3]P[A|B_3]}{P[A]}.$$

- Since all of the quantities we need are given in the problem description, our answer is

$$P[B_3|A] = \frac{0.3 \times 0.6}{0.78} = 0.23.$$

- Similarly we obtain $P[B_1|A] = 0.31$ and $P[B_2|A] = 0.46$.
- Of all resistors within 50 of the nominal value, only 23% come from machine B_3 (even though this machine produces 30% of all resistors). Machine B_1 produces 31% of the resistors that meet the 50 criterion and machine B_2 produces 46% of them.

1.5.2 Bayes' Rule

Practice Test 2

A manufacturing firm employs three analytical plans for the design and development of a particular product. For cost reasons, all three are used at varying times. In fact, plans 1, 2, and 3 are used for 30%, 20%, and 50% of the products, respectively. The defect rate is different for the three procedures as follows:

$$P(D|B_1) = 0.01, \quad P(D|B_2) = 0.03, \quad P(D|B_3) = 0.02,$$

where $P(D|B_i)$ is the probability of a defective product, given plan B_i, $i = 1, 2, 3$. If a random product was observed and found to be defective, which plan was most likely used and thus responsible?

1.5.2 Bayes' Rule

Practice Test 3

Police plan to enforce speed limits by using radar traps at four different locations within the city limits. The radar traps at each of the locations L_1, L_2, L_3, and L_4 will be operated 20%, 30%, 20%, and 30% of the time. If a person who is speeding on her way to work has probabilities of 0.2, 0.1, 0.3, and 0.2, respectively, of passing through these locations, (a) what is the probability that she will receive a speeding ticket? (b) If the person received a speeding ticket on her way to work, what is the probability that she passed through the radar trap located at L_2?

Content

1.6.1 Bernoulli trial/Independent Trials

Definition 1.21 (Bernoulli trial)

In the theory of probability and statistics, a **Bernoulli trial (or binomial trial)** is a random experiment with exactly two possible outcomes, "success" and "failure", in which the probability of success is the same every time the experiment is conducted.

Examples 1.41 (Examples of Bernoulli trials)

(a) **Flipping a coin.**

- In this context, obverse "heads" conventionally denotes success and reverse "tails" denotes failure.
- A fair coin has the probability of success 0.5 by definition.
- In this case there are exactly two possible outcomes.

1.6.1 Bernoulli trial/Independent Trials

Definition 1.21 (Bernoulli trial)

In the theory of probability and statistics, a **Bernoulli trial (or binomial trial)** is a random experiment with exactly two possible outcomes, "success" and "failure", in which the probability of success is the same every time the experiment is conducted.

Examples 1.41 (Examples of Bernoulli trials)

(a) **Flipping a coin.**

- In this context, obverse "heads" conventionally denotes success and reverse "tails" denotes failure.
- A fair coin has the probability of success 0.5 by definition.
- In this case there are exactly two possible outcomes.

1.6.1 Bernoulli trial/Independent Trials

Definition 1.21 (Bernoulli trial)

In the theory of probability and statistics, a **Bernoulli trial (or binomial trial)** is a random experiment with exactly two possible outcomes, "success" and "failure", in which the probability of success is the same every time the experiment is conducted.

Examples 1.41 (Examples of Bernoulli trials)

(a) Flipping a coin.

- In this context, obverse "heads" conventionally denotes success and reverse "tails" denotes failure.
- A fair coin has the probability of success 0.5 by definition.
- In this case there are exactly two possible outcomes.

1.6.1 Bernoulli trial/Independent Trials

Definition 1.21 (Bernoulli trial)

In the theory of probability and statistics, a **Bernoulli trial (or binomial trial)** is a random experiment with exactly two possible outcomes, "success" and "failure", in which the probability of success is the same every time the experiment is conducted.

Examples 1.41 (Examples of Bernoulli trials)

(a) **Flipping a coin.**

- In this context, obverse "heads" conventionally denotes success and reverse "tails" denotes failure.
- A fair coin has the probability of success 0.5 by definition.
- In this case there are exactly two possible outcomes.

1.6.1 Bernoulli trial/Independent Trials

Definition 1.21 (Bernoulli trial)

In the theory of probability and statistics, a **Bernoulli trial (or binomial trial)** is a random experiment with exactly two possible outcomes, "success" and "failure", in which the probability of success is the same every time the experiment is conducted.

Examples 1.41 (Examples of Bernoulli trials)

(a) **Flipping a coin.**

- In this context, obverse "heads" conventionally denotes success and reverse "tails" denotes failure.
- A fair coin has the probability of success 0.5 by definition.
- In this case there are exactly two possible outcomes.

1.6.1 Bernoulli trial/Independent Trials

Definition 1.21 (Bernoulli trial)

In the theory of probability and statistics, a **Bernoulli trial (or binomial trial)** is a random experiment with exactly two possible outcomes, "success" and "failure", in which the probability of success is the same every time the experiment is conducted.

Examples 1.41 (Examples of Bernoulli trials)

(a) Flipping a coin.

- In this context, obverse "heads" conventionally denotes success and reverse "tails" denotes failure.
- A fair coin has the probability of success 0.5 by definition.
- In this case there are exactly two possible outcomes.

1.6.1 Bernoulli trial/Independent Trials

Examples 1.41 (Examples of Bernoulli trials)

(b) Rolling a die.

- A six is "success" and everything else a "failure".
- In this case there are six possible outcomes, and the event is a six;
- The complementary event "not a six" corresponds to the other five possible outcomes.

1.6.1 Bernoulli trial/Independent Trials

Examples 1.41 (Examples of Bernoulli trials)

(b) Rolling a die.

- A six is "success" and everything else a "failure".
- In this case there are six possible outcomes, and the event is a six;
- The complementary event "not a six" corresponds to the other five possible outcomes.

1.6.1 Bernoulli trial/Independent Trials

Examples 1.41 (Examples of Bernoulli trials)

(b) **Rolling a die**.

- A six is "success" and everything else a "failure".
- In this case there are six possible outcomes, and the event is a six;
- The complementary event "not a six" corresponds to the other five possible outcomes.

1.6.1 Bernoulli trial/Independent Trials

Examples 1.41 (Examples of Bernoulli trials)

(b) Rolling a die.

- A six is "success" and everything else a "failure".
- In this case there are six possible outcomes, and the event is a six;
- The complementary event "not a six" corresponds to the other five possible outcomes.

1.6.1 Bernoulli trial/Independent Trials

Examples 1.41 (Examples of Bernoulli trials)

(b) Rolling a die.

- A six is "success" and everything else a "failure".
- In this case there are six possible outcomes, and the event is a six;
- The complementary event "not a six" corresponds to the other five possible outcomes.

1.6.1 Bernoulli trial/Independent Trials

Examples 1.41 (Examples of Bernoulli trials)

(b) **Rolling a die**.

- A six is "success" and everything else a "failure".
- In this case there are six possible outcomes, and the event is a six;
- The complementary event "not a six" corresponds to the other five possible outcomes.

1.6.2 Bernoulli Trial Calculator

Introduction (Bernoulli Trial Calculator)

- We start with a simple subexperiment in which there are two outcomes: a success occurs with probability p; otherwise, a failure occurs with probability $1-p$.
- The results of all trials of the subexperiment are mutually independent.
- An outcome of the complete experiment is a sequence of successes and failures denoted by a sequence of ones and zeroes. For example, 10101... is an alternating sequence of successes and failures.
- Let A denote the event k successes and $n-k$ failures in n trials. Find $P[A] = P_n(k)$.

1.6.2 Bernoulli Trial Calculator

Introduction (Bernoulli Trial Calculator)

- We start with a simple subexperiment in which there are two outcomes: a success occurs with probability p; otherwise, a failure occurs with probability $1 - p$.
- The results of all trials of the subexperiment are mutually independent.
- An outcome of the complete experiment is a sequence of successes and failures denoted by a sequence of ones and zeroes. For example, $10101\ldots$ is an alternating sequence of successes and failures.
- Let A denote the event k successes and $n - k$ failures in n trials. Find $P[A] = P_n(k)$.

1.6.2 Bernoulli Trial Calculator

Introduction (Bernoulli Trial Calculator)

- We start with a simple subexperiment in which there are two outcomes: a success occurs with probability p; otherwise, a failure occurs with probability $1 - p$.
- The results of all trials of the subexperiment are mutually independent.
- An outcome of the complete experiment is a sequence of successes and failures denoted by a sequence of ones and zeroes. For example, 10101... is an alternating sequence of successes and failures.
- Let A denote the event k successes and $n - k$ failures in n trials. Find $P[A] = P_n(k)$.

1.6.2 Bernoulli Trial Calculator

Introduction (Bernoulli Trial Calculator)

- We start with a simple subexperiment in which there are two outcomes: a success occurs with probability p; otherwise, a failure occurs with probability $1 - p$.
- The results of all trials of the subexperiment are mutually independent.
- An outcome of the complete experiment is a sequence of successes and failures denoted by a sequence of ones and zeroes. For example, 10101... is an alternating sequence of successes and failures.
- Let A denote the event k successes and $n - k$ failures in n trials. Find $P[A] = P_n(k)$.

1.6.2 Bernoulli Trial Calculator

Example 1.42

What is the probability $P_5(3)$ of three successes and two failures in five Bernoulli trials with success probability p.

Example 1.42 Solution

- We observe that the outcomes with three successes in five trials are 11100, 11010, 11001, 10110, 10101, 10011, 01110, 01101, 01011, and 00111.
- We note that the probability of each outcome is a product of five probabilities, each related to one subexperiment. In outcomes with three successes, three of the probabilities are p and the other two are $1-p$. Therefore each outcome with three successes has probability $p^3(1-p)^2$.
- The number of such sequences is C_5^3. So

$$P_5(3) = C_5^3 \times p^3 \times (1-p)^{5-3}.$$

1.6.2 Bernoulli Trial Calculator

Example 1.42

What is the probability $P_5(3)$ of three successes and two failures in five Bernoulli trials with success probability p.

Example 1.42 Solution

- We observe that the outcomes with three successes in five trials are 11100, 11010, 11001, 10110, 10101, 10011, 01110, 01101, 01011, and 00111.
- We note that the probability of each outcome is a product of five probabilities, each related to one subexperiment. In outcomes with three successes, three of the probabilities are p and the other two are $1-p$. Therefore each outcome with three successes has probability $p^3(1-p)^2$.
- The number of such sequences is C_5^3. So

$$P_5(3) = C_5^3 \times p^3 \times (1-p)^{5-3}.$$

1.6.2 Bernoulli Trial Calculator

Example 1.42

What is the probability $P_5(3)$ of three successes and two failures in five Bernoulli trials with success probability p.

Example 1.42 Solution

- We observe that the outcomes with three successes in five trials are 11100, 11010, 11001, 10110, 10101, 10011, 01110, 01101, 01011, and 00111.
- We note that the probability of each outcome is a product of five probabilities, each related to one subexperiment. In outcomes with three successes, three of the probabilities are p and the other two are $1-p$. Therefore each outcome with three successes has probability $p^3(1-p)^2$.
- The number of such sequences is C_5^3. So

$$P_5(3) = C_5^3 \times p^3 \times (1-p)^{5-3}.$$

1.6.2 Bernoulli Trial Calculator

Example 1.42

What is the probability $P_5(3)$ of three successes and two failures in five Bernoulli trials with success probability p.

Example 1.42 Solution

- We observe that the outcomes with three successes in five trials are 11100, 11010, 11001, 10110, 10101, 10011, 01110, 01101, 01011, and 00111.
- We note that the probability of each outcome is a product of five probabilities, each related to one subexperiment. In outcomes with three successes, three of the probabilities are p and the other two are $1-p$. Therefore each outcome with three successes has probability $p^3(1-p)^2$.
- The number of such sequences is C_5^3. So

$$P_5(3) = C_5^3 \times p^3 \times (1-p)^{5-3}.$$

1.6.2 Bernoulli Trial Calculator

Theorem 1.21

The probability of k successes and $n-k$ failures in n Becnoulli trials is

$$P_n(k) = C_n^k p^k (1-p)^{n-k} = C_n^{n-k} (1-p)^{n-k} p^k, \tag{1.34}$$

where a success occurs with probability p and a failure occurs with probability $1-p$.

Example 1.43

In Example 1.25, we found that a randomly tested resistor was acceptable with probability $P[A] = 0.78$. If we randomly test 100 resistors, what is the probability of T_k, the event that k resistors test acceptable?

Example 1.43 Solution

Testing each resistor is a Bernoulli trial with a success occurring when a resistor is acceptable. Thus for $0 \le k \le 100$,

$$P[T_k] = P_{100}(k) = C_{100}^k (0.78)^k (1-0.78)^{100-k}.$$

1.6.2 Bernoulli Trial Calculator

Theorem 1.21

The probability of k successes and $n-k$ failures in n Becnoulli trials is

$$P_n(k) = C_n^k p^k (1-p)^{n-k} = C_n^{n-k}(1-p)^{n-k} p^k, \tag{1.34}$$

where a success occurs with probability p and a failure occurs with probability $1-p$.

Example 1.43

In Example 1.25, we found that a randomly tested resistor was acceptable with probability $P[A] = 0.78$. If we randomly test 100 resistors, what is the probability of T_k, the event that k resistors test acceptable?

Example 1.43 Solution

Testing each resistor is a Bernoulli trial with a success occurring when a resistor is acceptable. Thus for $0 \le k \le 100$,

$$P[T_k] = P_{100}(k) = C_{100}^k (0.78)^k (1-0.78)^{100-k}.$$

1.6.2 Bernoulli Trial Calculator

Theorem 1.21

The probability of k successes and $n-k$ failures in n Becnoulli trials is

$$P_n(k) = C_n^k p^k (1-p)^{n-k} = C_n^{n-k}(1-p)^{n-k} p^k, \quad (1.34)$$

where a success occurs with probability p and a failure occurs with probability $1-p$.

Example 1.43

In Example 1.25, we found that a randomly tested resistor was acceptable with probability $P[A] = 0.78$. If we randomly test 100 resistors, what is the probability of T_k, the event that k resistors test acceptable?

Example 1.43 Solution

Testing each resistor is a Bernoulli trial with a success occurring when a resistor is acceptable. Thus for $0 \le k \le 100$,

$$P[T_k] = P_{100}(k) = C_{100}^k (0.78)^k (1-0.78)^{100-k}.$$

1.6.2 Bernoulli Trial Calculator

Remark 1.9

Bernoulli trial computation can only done under the following circumstances.

1. **2 outcomes only**

   When there only only 2 possibile outcomes, most of the time expressed as success or failure. This can represent many different results such as heads or tails, win or lose, go or don't go. It can only be done when there are exactly 2 outcomes. Most of the times it will be expressed as success or failure.

2. **Each trial must be independent**

   Each trial (each time the event occurs) must be independent of each other. This means that the events are completely independent; they do not depend on the previous trial or the trial after. A classic example is heads or tails. Each flip is independent of all others.

3. **Probability of success is the same for each trial**

   The probability of the event occurring or there being success in the desired outcome must be the same. For example, for each flip of a coin, there is always a 50% success rate of getting a heads. In other words, the probability of an event occurring must be the same, not different, for each trial.

1.6.2 Bernoulli Trial Calculator

Remark 1.9

Bernoulli trial computation can only done under the following circumstances.

1. **2 outcomes only**

   When there only only 2 possibile outcomes, most of the time expressed as success or failure. This can represent many different results such as heads or tails, win or lose, go or don't go. It can only be done when there are exactly 2 outcomes. Most of the times it will be expressed as success or failure.

2. **Each trial must be independent**

   Each trial (each time the event occurs) must be independent of each other. This means that the events are completely independent; they do not depend on the previous trial or the trial after. A classic example is heads or tails. Each flip is independent of all others.

3. **Probability of success is the same for each trial**

   The probability of the event occurring or there being success in the desired outcome must be the same. For example, for each flip of a coin, there is always a 50% success rate of getting a heads. In other words, the probability of an event occurring must be the same, not different, for each trial.

1.6.2 Bernoulli Trial Calculator

Remark 1.9

Bernoulli trial computation can only done under the following circumstances.

1. **2 outcomes only**

   When there only only 2 possibile outcomes, most of the time expressed as success or failure. This can represent many different results such as heads or tails, win or lose, go or don't go. It can only be done when there are exactly 2 outcomes. Most of the times it will be expressed as success or failure.

2. **Each trial must be independent**

   Each trial (each time the event occurs) must be independent of each other. This means that the events are completely independent; they do not depend on the previous trial or the trial after. A classic example is heads or tails. Each flip is independent of all others.

3. **Probability of success is the same for each trial**

   The probability of the event occurring or there being success in the desired outcome must be the same. For example, for each flip of a coin, there is always a 50% success rate of getting a heads. In other words, the probability of an event occurring must be the same, not different, for each trial.

1.6.2 Bernoulli Trial Calculator

Remark 1.9

Bernoulli trial computation can only done under the following circumstances.

1. **2 outcomes only**

   When there only only 2 possibile outcomes, most of the time expressed as success or failure. This can represent many different results such as heads or tails, win or lose, go or don't go. It can only be done when there are exactly 2 outcomes. Most of the times it will be expressed as success or failure.

2. **Each trial must be independent**

   Each trial (each time the event occurs) must be independent of each other. This means that the events are completely independent; they do not depend on the previous trial or the trial after. A classic example is heads or tails. Each flip is independent of all others.

3. **Probability of success is the same for each trial**

   The probability of the event occurring or there being success in the desired outcome must be the same. For example, for each flip of a coin, there is always a 50% success rate of getting a heads. In other words, the probability of an event occurring must be the same, not different, for each trial.

1.6.2 Bernoulli Trial Calculator

Remark 1.9

Bernoulli trial computation can only done under the following circumstances.

1. **2 outcomes only**

   When there only only 2 possibile outcomes, most of the time expressed as success or failure. This can represent many different results such as heads or tails, win or lose, go or don't go. It can only be done when there are exactly 2 outcomes. Most of the times it will be expressed as success or failure.

2. **Each trial must be independent**

   Each trial (each time the event occurs) must be independent of each other. This means that the events are completely independent; they do not depend on the previous trial or the trial after. A classic example is heads or tails. Each flip is independent of all others.

3. **Probability of success is the same for each trial**

   The probability of the event occurring or there being success in the desired outcome must be the same. For example, for each flip of a coin, there is always a 50% success rate of getting a heads. In other words, the probability of an event occurring must be the same, not different, for each trial.

1.6.2 Bernoulli Trial Calculator

Remark 1.9

Bernoulli trial computation can only done under the following circumstances.

1. **2 outcomes only**

   When there only only 2 possibile outcomes, most of the time expressed as success or failure. This can represent many different results such as heads or tails, win or lose, go or don't go. It can only be done when there are exactly 2 outcomes. Most of the times it will be expressed as success or failure.

2. **Each trial must be independent**

   Each trial (each time the event occurs) must be independent of each other. This means that the events are completely independent; they do not depend on the previous trial or the trial after. A classic example is heads or tails. Each flip is independent of all others.

3. **Probability of success is the same for each trial**

   The probability of the event occurring or there being success in the desired outcome must be the same. For example, for each flip of a coin, there is always a 50% success rate of getting a heads. In other words, the probability of an event occurring must be the same, not different, for each trial.

1.6.2 Bernoulli Trial Calculator

Remark 1.9

Bernoulli trial computation can only done under the following circumstances.

1. **2 outcomes only**

   When there only only 2 possibile outcomes, most of the time expressed as success or failure. This can represent many different results such as heads or tails, win or lose, go or don't go. It can only be done when there are exactly 2 outcomes. Most of the times it will be expressed as success or failure.

2. **Each trial must be independent**

   Each trial (each time the event occurs) must be independent of each other. This means that the events are completely independent; they do not depend on the previous trial or the trial after. A classic example is heads or tails. Each flip is independent of all others.

3. **Probability of success is the same for each trial**

   The probability of the event occurring or there being success in the desired outcome must be the same. For example, for each flip of a coin, there is always a 50% success rate of getting a heads. In other words, the probability of an event occurring must be the same, not different, for each trial.

1.6.2 Bernoulli Trial Calculator

Practice Test 4

If 10 percent of the balls in a certain box are red, and if 20 balls are selected from the box at random, with replacement, what is the probability that more than three red balls will be obtained? (b) Also find the probability that exactly two red balls will be obtained.