Fundamentals of Optimization

Constraint Programming

Pham Quang Dung
dungpq@soict.hust.edu.vn
Department of Computer Science

Content

- Constraint Satisfaction Optimization Problems
- Constraint Propagation
- Branching and Backtrack Search
- Examples

Constraint Satisfaction Problems

- Variables
  - $X = \{X_0, X_1, X_2, X_3, X_4\}$
- Domain
  - $X_0, X_1, X_2, X_3, X_4 \in \{1,2,3,4,5\}$
- Constraints
  - C_1: $X_2 + 3 \neq X_1$
  - C_2: $X_3 \leq X_4$
  - C_3: $X_2 + X_3 = X_0 + 1$
  - C_4: $X_4 \leq 3$
  - C_5: $X_1 + X_4 = 7$
  - C_6: $X_2 = 1 \Rightarrow X_4 \neq 2$

Constraint Satisfaction Problems

- CSP = (X,D,C), in which:
  - $X = \{X_1,...,X_N\}$ – set of variables
  - $D = \{D(X_1),\ldots, D(X_N)\}$ – domains of variables
  - $C = \{C_1,..,C_K\}$ – set of constraints over variables
  - Denote $X(c)$ – set of variables appearing in the constraint c

Constraint Satisfaction Optimization Problems

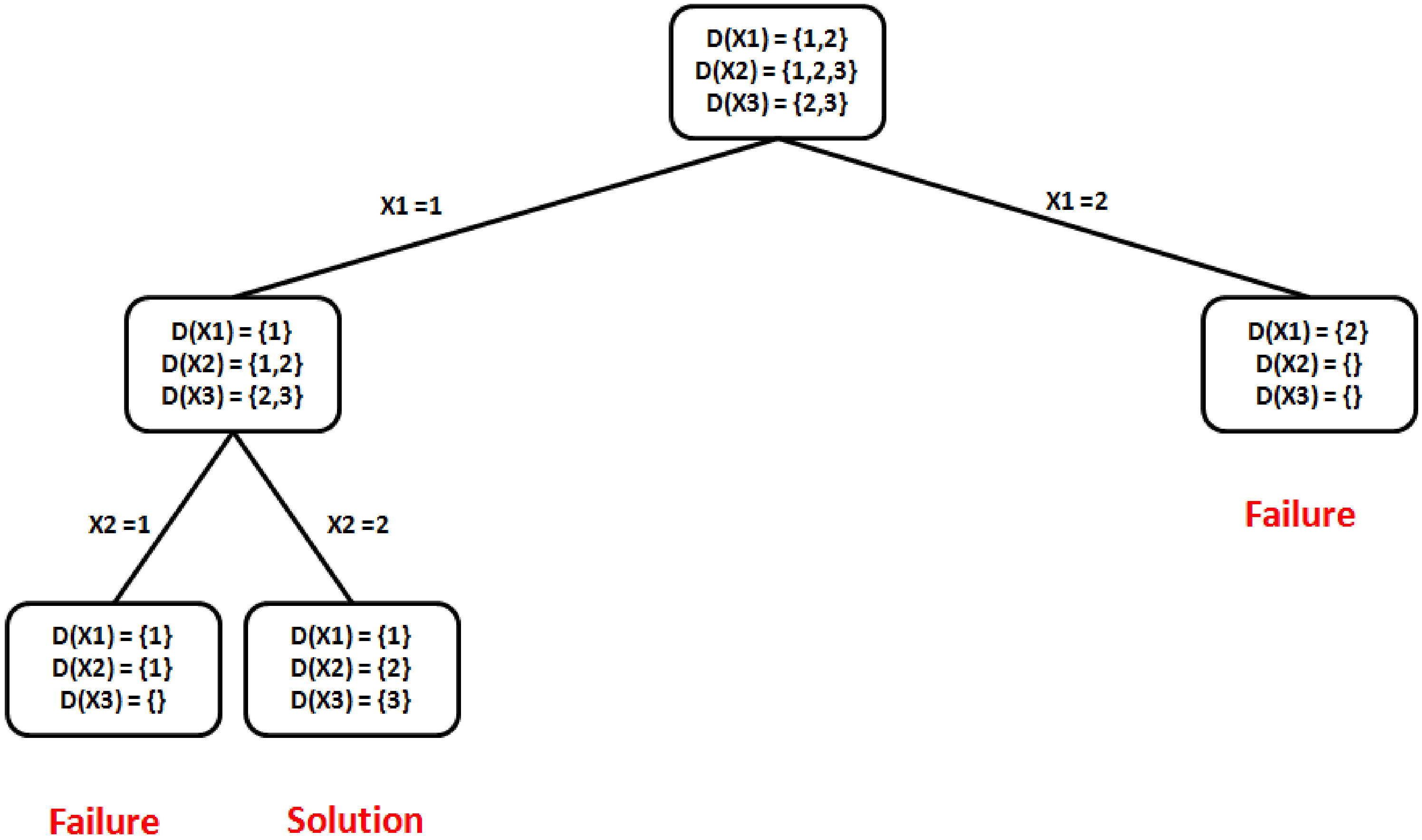
- COP = (X,D,C,f), in which:
  - $X = \{X_1,...,X_N\}$ – set of variables
  - $D = \{D(X_1),\ldots, D(X_N)\}$ – domains of variables
  - $C = \{C_1,..,C_K\}$ – set of constraints over variables
  - Denote $X(c)$ – set of variables appearing in the constraint c
  - f: objective function to be optimized

Constraint Programming

- A computation paradigm for solving CSP, COP combining
  - Constraint Propagation: narrow the search space by pruning redundant values from the domains of variables
  - Branching (backtracking search): split the problem into equivalent sub-problems by
    - Instantiating some variables with values of its domain
    - Split the domain of a selected variable into sub-domains

Constraint Programming

X = {X1,X2,X3}
D(X1) = {1,2,3,4}, D(X2) = {1,2,3}, D(X3) = {1,2,3}
C1: X1 ≤ X2
C2: X3 = X1+X2
C3: X2+X3 = 5



Constraint Propagation

- Domain consistency (DC)
  - Given a CSP = (X,D,C), a constraint $c \in C$ is called domain consistent if for each variable $X_i \in X(c)$ and each value $v \in D(X_i)$, there exists values for variables of $X(c) \setminus \{X_i\}$ such that c is satisfied
  - A CSP is called domain consistent if c is domain consistent for all c $\in C$

Constraint Propagation

- DC algorithms aim at pruning redundant values from the domains of variables so that the obtained equivalent CSP is domain consistent

Constraint Propagation

- Example: CSP = (X,D,C) in which:
  - $X = \{X_1, X_2, X_3, X_4\}$
  - $D(X_1) = \{1,2,3,4\}$, $D(X_2) =\{1,2,3,4,5,6,7\}$, $D(X_3) = \{2,3,4,5\}$, $D(X_4) = \{1,2,3,4,5,6\}$
  - $C = \{c_1,c_2,c_3\}$ với
    - $c_1 \equiv X_1 + X_2 \geq 5$
    - $c_2 \equiv X_1 + X_3 \geq X_4$
    - $c_3 \equiv X_1 + 3 \geq X_3$
  - CSP is domain consistent
  - When branching, consider $X_1 = 1$, a DC algorithm will transform the given CSP to an equivalent domain consistent CSP^1 having : $D^1(X_1) = \{1\}$, $D^1(X_2) = \{4,5,6,7\}$, $D^1(X_3) = \{2,3,4\}$, $D^1(X_4) = \{1,2,3,4,5\}$

Constraint Propagation

- A domain consistent CSP does not ensure to have feasible solutions
- Example:
  - $X = \{X_1, X_2, X_3\}$
  - $D(X_1) = D(X_2) = D(X_3) = \{0,1\}$
  - $c_1 \equiv X_1 \neq X_2$, $c_2 \equiv X_1 \neq X_3$, $c_3 \equiv X_2 \neq X_3$

 The CSP is domain consistent but does not have any feasible solution

Constraint Propagation

```
Algorithm AC3(X,D,C){
  Q = {(x,c) | c ∈ C ∧ x ∈X(c)};
  while(Q not empty){
    select and remove (x,c) from Q;
    if ReviseAC3(x,c) then{
      if D(x) = {} then
        return false;
      else
        Q = Q ∪{(x',c') | c'∈C\{c} ∧ x,x' ∈X(c') ∧ x≠ x'}
    }
  }
  return true;
}
```

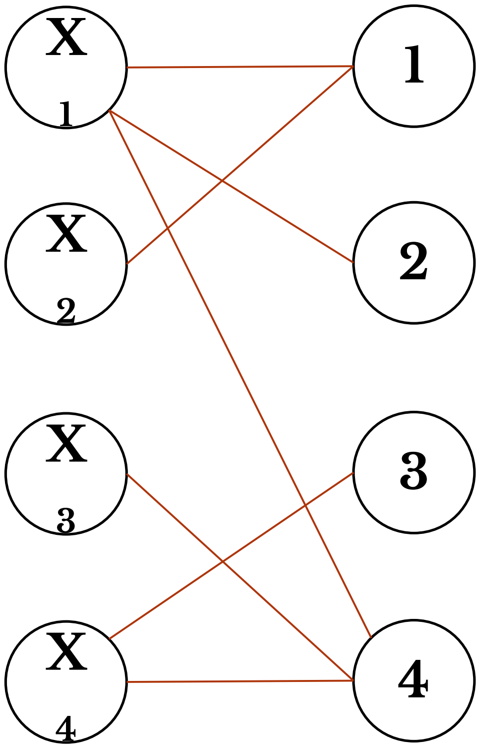
```
Algorithm ReviseAC3(x,c){
  CHANGE = false;
  for v ∈ D(x) do{
    if there does not exists other values
      of X(c) \ {x} such that c
        is satisfied then{
      remove v from D(x);
      CHANGE = true;
    }
  }
  return CHANGE;
}
```

Constraint Propagation

- Some constraints, e.g., binary constraints (related 2 variables) have efficient DC algorithm
- Constraint AllDifferent($X_1, X_2, \ldots, X_N$), the DC algorithm is efficient based on the matching (Max-Matching) algorithm on bipartite graphs
  - Nodes on the right-hand side are variables and nodes on the left-hand side are values
  - For each edge (X_i, v), (với $v \in D(X_i)$), if there does not exist a matching of size N containing (X_i, v), then v is removed from $D(X_i)$

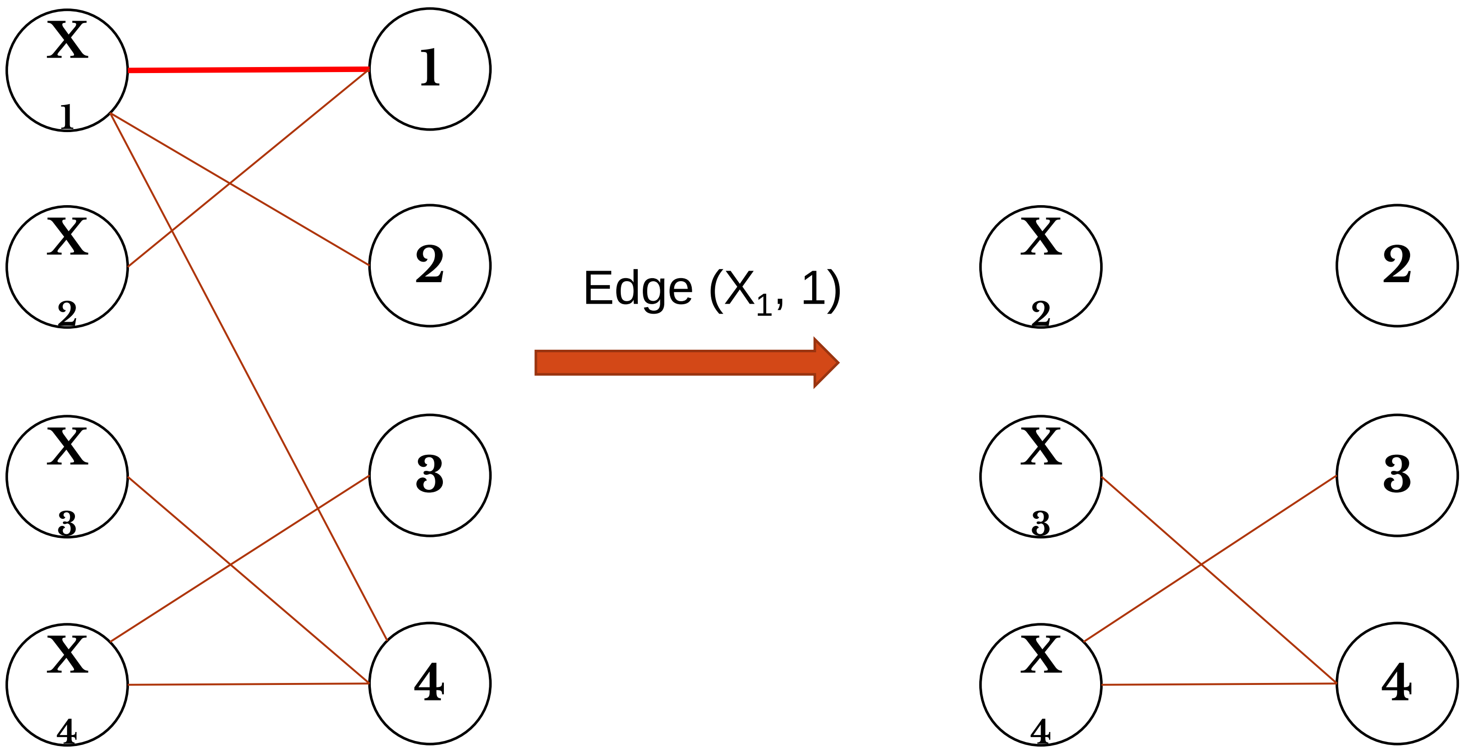
AllDifferent

- $X = \{X_1, X_2, X_3, X_4\}$
- $D(X_1) = \{1,2,4\}$, $D(X_2) = \{1\}$, $D(X_3) = \{4\}$, $D(X_4) = \{3,4\}$



AllDifferent

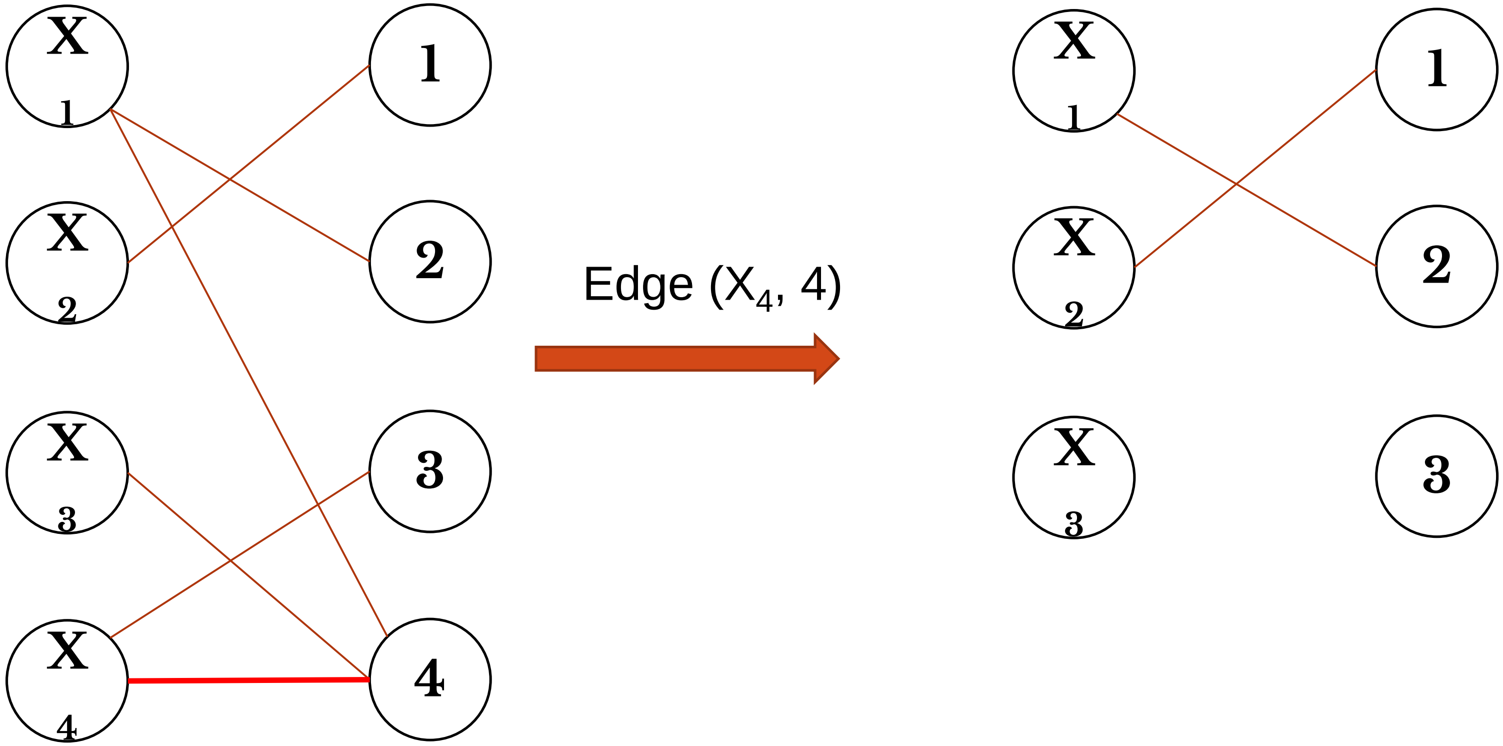
- $X = \{X_1, X_2, X_3, X_4\}$
- $D(X_1) = \{1,2,4\}$, $D(X_2) = \{1\}$, $D(X_3) = \{4\}$, $D(X_4) = \{3,4\}$



No matching of size 3 □ remove 1 from $D(X_1)$

AllDifferent

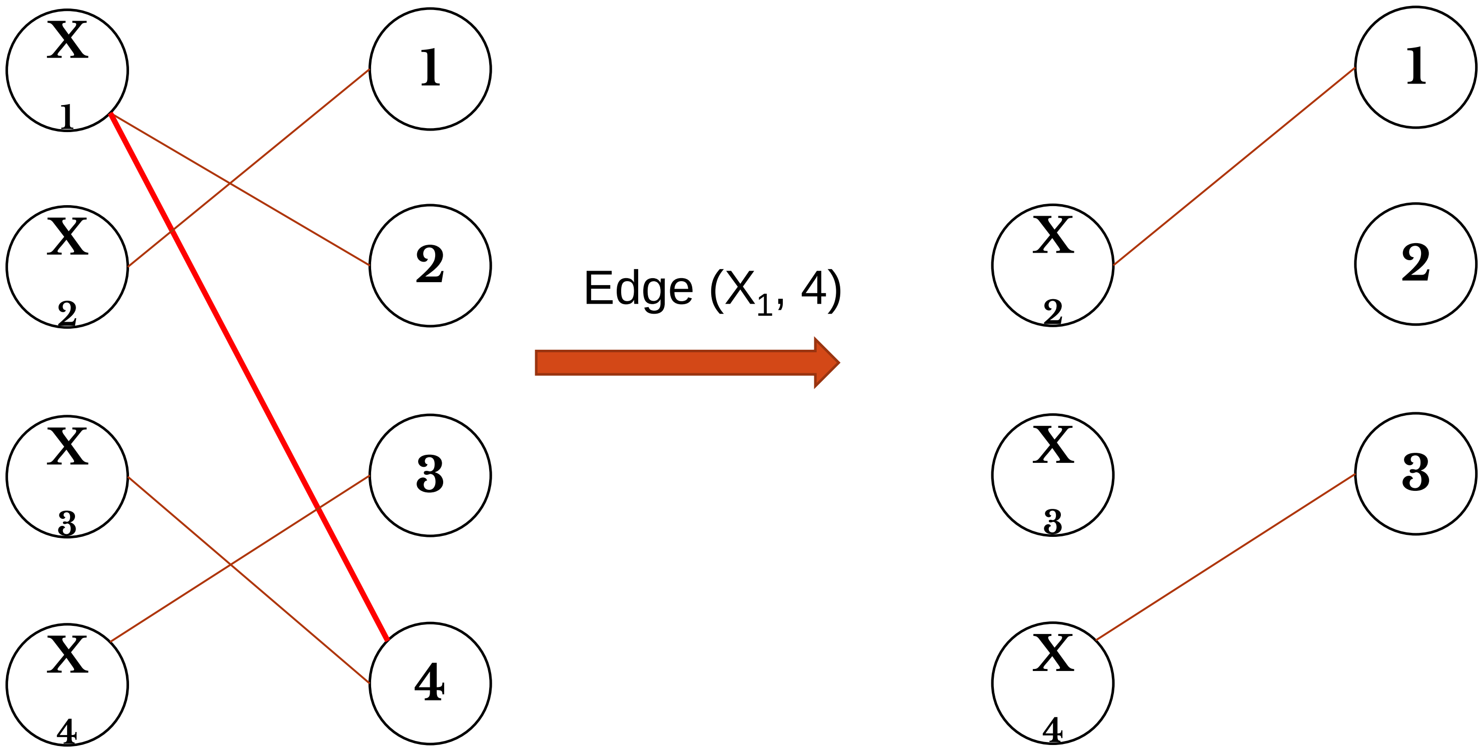
- $X = \{X_1, X_2, X_3, X_4\}$
- $D(X_1) = \{2,4\}$, $D(X_2) = \{1\}$, $D(X_3) = \{4\}$, $D(X_4) = \{3,4\}$



No matching of size 3 □ removed 4 from $D(X_4)$

AllDifferent

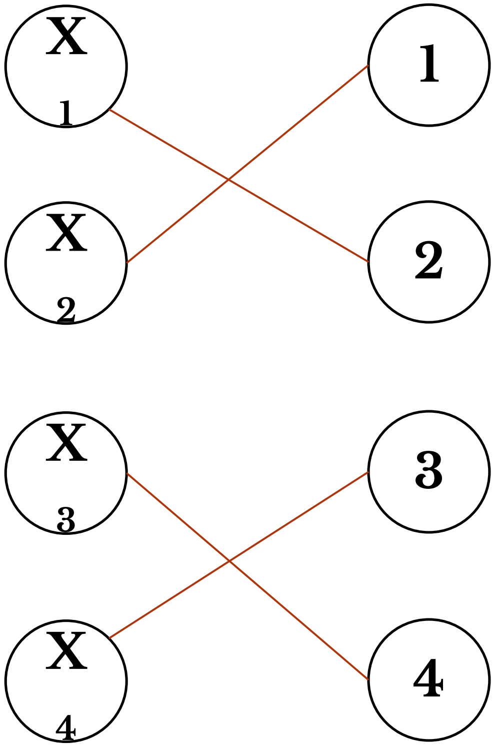
- $X = \{X_1, X_2, X_3, X_4\}$
- $D(X_1) = \{2,4\}$, $D(X_2) = \{1\}$, $D(X_3) = \{4\}$, $D(X_4) = \{3\}$



No matching of size 3 □ removed 4 from $D(X_1)$

AllDifferent

- $X = \{X_1, X_2, X_3, X_4\}$
- $D(X_1) = \{2\}$, $D(X_2) = \{1\}$, $D(X_3) = \{4\}$, $D(X_4) = \{3\}$

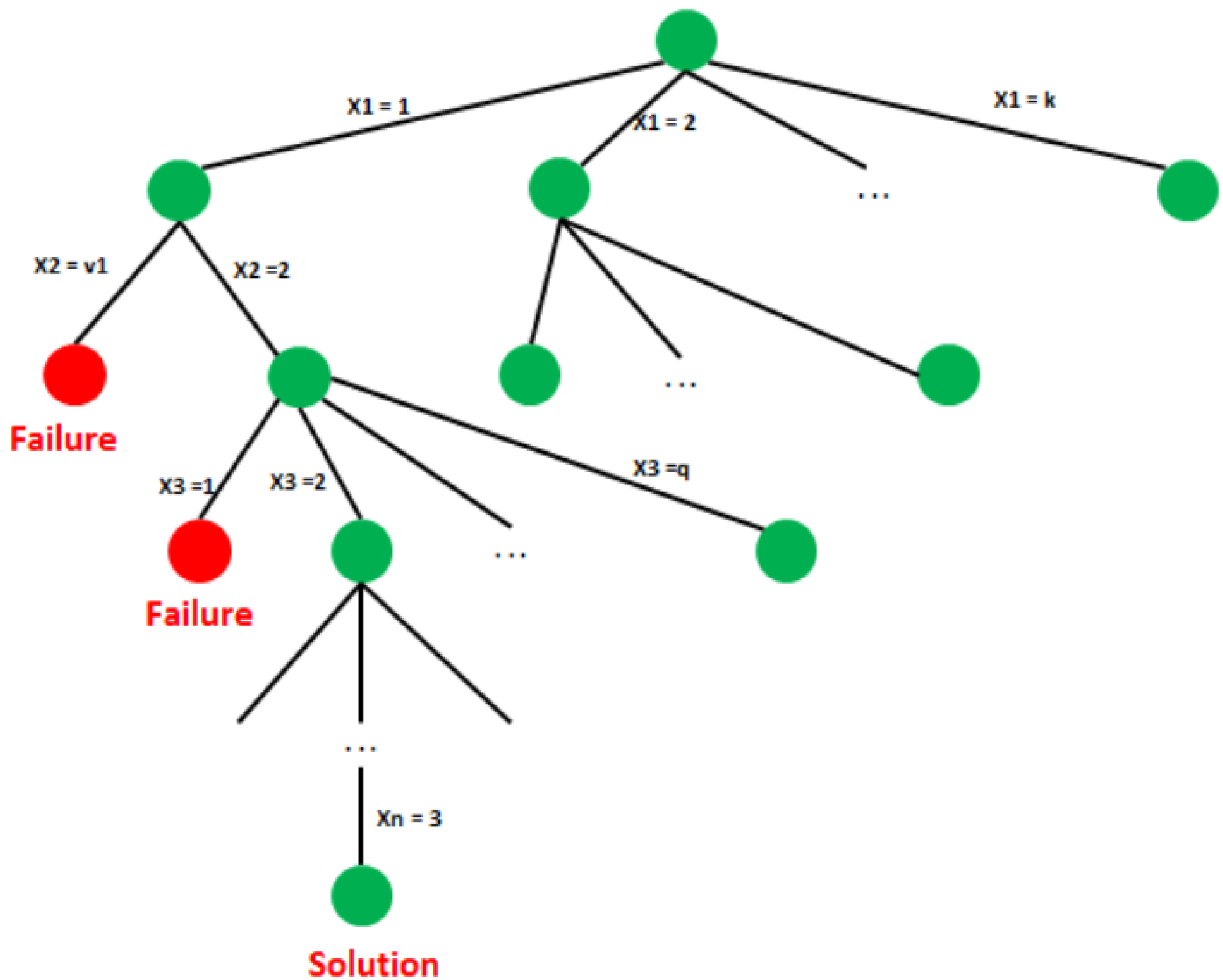


Feasible solution!

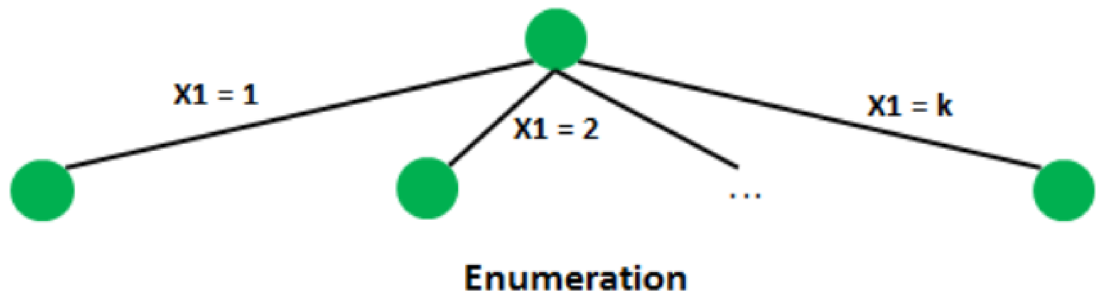
Branching and Backtracking Search

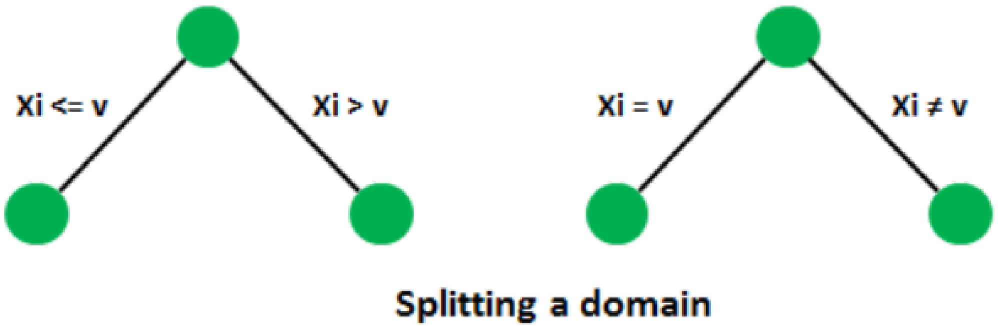
- Constraint propagation is not enough for finding feasible solutions
- Combine constraint propagation with branching and backtracking search
  - Split the original CSP P_0 into sub-problems CSP $P_1,\ldots,P_M$
    - Set of solutions of P_0 is equivalent to the union of sets of solutions to $P_1,\ldots,P_M$
    - Domain of each variable in $P_1,\ldots,P_M$ is not greater than the domain of that variable in P_0
  - Search Tree
    - Root is the original CSP P_0
    - Each node of the tree is a CSP
    - If $P_1,\ldots,P_M$ are children of P_0 then the set of solutions of P_0 is equivalent to the union of sets of solutions to $P_1,\ldots,P_M$
    - Leaves
      - A feasible solution
      - Failure (a variable has an empty domain)

Branching and Backtracking Search



Branching and Backtracking Search





Branching and Backtracking Search

- Search strategies
  - Variable selection
    - **dom** heuristic: select a variable having the smallest domain
    - **deg** heuristic: select a variable participating in most of the constraints
    - **dom+deg** heuristic: first apply **dom**, then use **deg** when tie break (when there are more than one variable with the same smallest domain size)
    - **dom/deg**: select a variable having the smallest dom/deg
  - Value selection
    - Increasing order
    - Decreasing order

Example

- Variables
  - $X = \{X_0, X_1, X_2, X_3, X_4\}$
- Domain
  - $X_0, X_1, X_2, X_3, X_4 \in \{1,2,3,4,5\}$
- Constraints
  - C_1: $X_2 + 3 \neq X_1$
  - C_2: $X_3 \leq X_4$
  - C_3: $X_2 + X_3 = X_0 + 1$
  - C_4: $X_4 \leq 3$
  - C_5: $X_1 + X_4 = 7$
  - C_6: $X_2 = 1 \Rightarrow X_4 \neq 2$

Example

```
'''
If-Then-Else expression
if x[2] = 1 then x[4] != 2
'''
from ortools.sat.python import cp_model
class VarArraySolutionPrinter(cp_model.CpSolverSolutionCallback):
        #print intermediate solution
        def __init__(self,variables):
                cp_model.CpSolverSolutionCallback.__init__(self)
                self.__variables = variables
                self.__solution_count = 0

        def on_solution_callback(self):
                self.__solution_count += 1
                for v in self.__variables:
                        print('%s = %i'% (v,self.Value(v)), end = ' ')
                print()
        def solution_count():
                return self.__solution_count
```

Example

```
model = cp_model.CpModel()

x = {}
for i in range(5):
         x[i] = model.NewIntVar(1,5,'x[' + str(i) + ']')

c1 = model.Add(x[2] + 3 != x[1])
c2 = model.Add(x[3] <= x[4])
c3 = model.Add(x[2] + x[3] == x[0] + 1)
c4 = model.Add(x[4] <= 3)
c5 = model.Add(x[1] + x[4] == 7)


b = model.NewBoolVar('b')

#constraints
model.Add(x[2] == 1).OnlyEnforceIf(b)
model.Add(x[2] != 1).OnlyEnforceIf(b.Not())

model.Add(x[4] != 2).OnlyEnforceIf(b)
```

Example

```
solver = cp_model.CpSolver()

#Force the solver to follow the decision strategy exactly
solver.parameters.search_branching = cp_model.FIXED_SEARCH

vars = [x[i] for i in range(5)]


solution_printer = VarArraySolutionPrinter(vars)
solver.SearchForAllSolutions(model,solution_printer)
```