Problem 2:

2.1:

$$P_N(n) = \begin{cases} c(1/2)^n & n = 0,1,2 \\ 0 & \text{other wise} \end{cases}$$

a, $P_N(n) \geq 0 \quad \forall n$

$\Rightarrow c\left(\frac{1}{2}\right)^n \geq 0 \Rightarrow c \geq 0$

$\sum_{n=0}^{2} P_N(n) = 1 \Leftrightarrow c.\left(\frac{1}{2}\right)^0 + c.\left(\frac{1}{2}\right) + c\left(\frac{1}{2}\right)^2 = 1$

$\Rightarrow c = \frac{4}{7}$

b, $P[N \leq 1] = P[N=0] + P[N=1]$

$= \frac{4}{7} + \frac{2}{7} = \frac{6}{7}$.

2.2:

$$P_V(v) = \begin{cases} cv^2 & v = 1,2,3,4 \\ 0 & \text{other wise} \end{cases}$$

a, $P_V(v) \geq 0 \quad \forall v \Rightarrow c \geq 0$

$\sum_{v \in S_v} P_V(v) = 1 \Rightarrow c.1^2 + c.2^2 + c3^2 + c.4^2 = 1$

$\Leftrightarrow 30c = 1 \Leftrightarrow c = \frac{1}{30}$

b, $P[V \in u^2 / u = 1,2,3\ldots]$

$= P[V=1] + P[V=4]$

$= \frac{1}{30} + \frac{16}{30} = \frac{17}{30} = \cancel{\frac{1}{6}}$.

c, V is an even number.

$$P_{[V=2]} + P_{[V=4]} = \frac{16}{30} + \frac{4}{30} = \frac{20}{30} = \frac{2}{3}$$

d, $P_{[V>2]} = P[V=3] + P[V=4]$

$$= \frac{9}{30} + \frac{16}{30} = \frac{25}{30} = \frac{1}{6}.$$

2.3

Probability og single $P_B(1)$

Probability og doable $P_B(2)$

Probability og tripble $P_B(3)$

Probability og home run $P_B(4)$

we have: $\begin{cases} P_B(1) + P_B(2) + P_B(3) + P_B(4) = 0,3 \\ P_B(1) = 2P_B(2) = 4P_B(3) = 8P_B(4) \end{cases}$

$$\Rightarrow P_B(n) = \begin{cases} 0,7 & n = 0 \\ 0,16 & n = 1 \\ 0,08 & n = 2 \\ 0,04 & n = 3 \\ 0,02 & n = 4 \\ 0 & \text{other wise} \end{cases}$$

2.4

a, $P_Y(n) = \begin{cases} 1/11 & n = 5, 6, 7 \ldots 15 \\ 0 & \text{other wise} \end{cases}$

b, $P[Y < 10] = P_{[Y=5]} + P_{[Y=6]} + P_{[Y=7]} + P_{[Y=8]}$
$+ P_{[Y=9]}$
$= \frac{5}{11}$

c, $P[Y > 12] = P_{[Y=13]} + P_{[Y=14]} + P_{[Y=15]}$
$= \frac{3}{11}$

d, $P[8 \leq Y < 12] = P_{[Y=8]} + P_{[Y=9]} + P_{[Y=10]}$
$+ P[Y=11] + P[Y=12]$
$= \frac{5}{11}$

2.5

a, Bernoulli Theorem:

$P_K(k) = \begin{cases} C_n^k \cdot p^k \cdot (1-p)^{n-k} & , k = 0, 1, \ldots, n. \\ 0 & \text{else} \end{cases}$

b, Messege was received at least once : R

$P(R) \geq 0,95$

$P_{(R^c)} = (1-p)^n$

$\Rightarrow 1 - P_{(R^c)} \geq 0,95 \Leftrightarrow 1-(1-p)^n \geq 0,95$

$\Rightarrow n \geq 1,9.$

$\Rightarrow n = 2.$

2.6:

a,

$$P_N(n) = \begin{cases} (1-p)^{n-1}\cdot p & n = 1,2\ldots \\ 0 & \text{otherwise} \end{cases}$$

b,

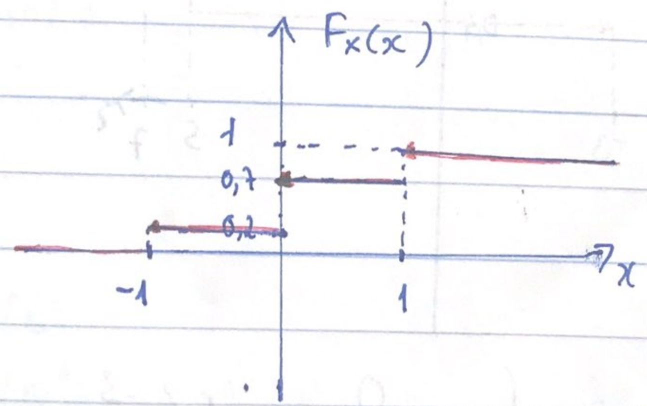
$$P[N \leq 3] = 1 - P[N > 3] = 1 - \sum_{n=4}^{\infty} P_N(n) = 1 - (1-p)^3$$

$$1 - (1-p)^3 \geq 0{,}95 \Rightarrow p \geq 1 - (0{,}05)^{1/3} \approx 0{,}6316$$

2.7:

$$F_X(x) = \begin{cases} 0,\cancel{1} & x \leq -1 \\ 0{,}2 & -1 < x \leq 0 \\ 0{,}7 & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$$

a,



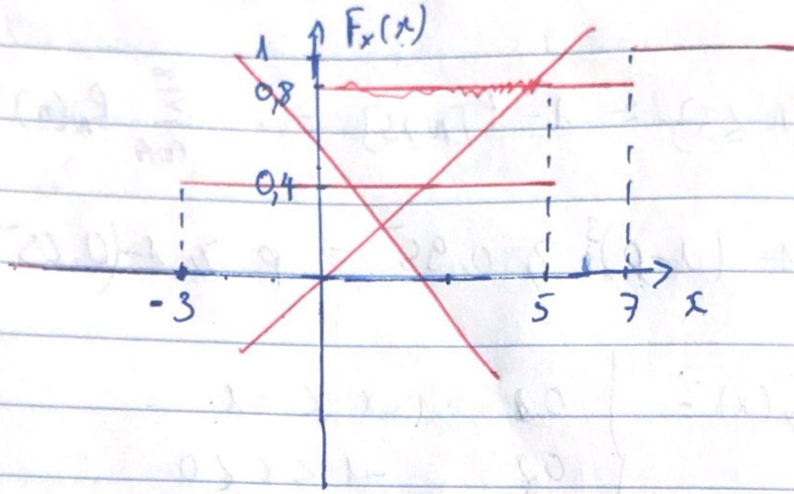
b,

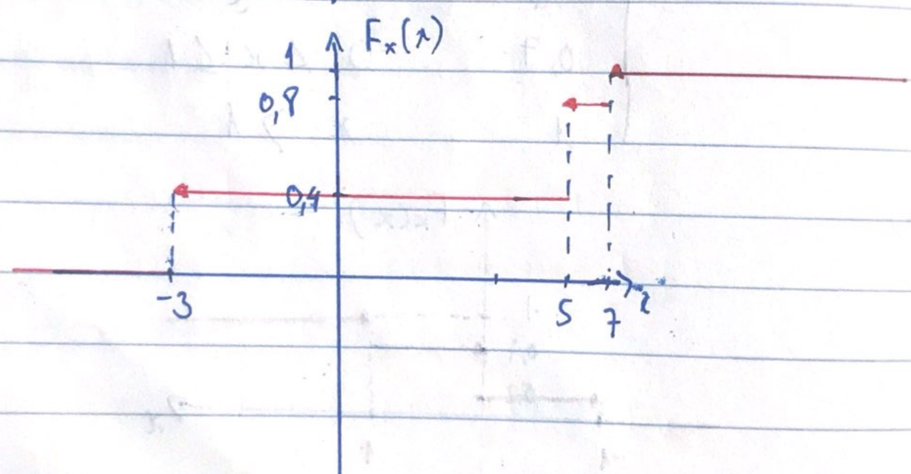
$$P_X(x) = \begin{cases} 0 & x \leq -1 \\ 0{,}2 & -1 < x \leq 0 \\ 0{,}5 & 0 < x \leq 1 \\ 0{,}3 & x > 1 \end{cases}$$

2.8

$$F_x(x) = \begin{cases} 0 & x \leq -3 \\ 0,4 & -3 < x \leq 5 \\ 0,8 & 5 < x \leq 7 \\ 1 & x \geq 7. \end{cases}$$

a,





b, $$P_x(x) = \begin{cases} 0 & x \leq -3 \\ 0,4 & -3 < x \leq 5 \\ 0,4 & 5 < x \leq 7 \\ 0,2 & x \geq 7 \end{cases}$$

2.9.

$$P_X(x) = \begin{cases} 0{,}01 & x = 1,2 \ldots 100, \\ 0 & \text{other wise} \end{cases}$$

a, $X_{mod} = \{ \cancel{0}, 1, 2 \ldots 100 \}$

b, $x_{med} = \cancel{50}.\ 5{,}1$

2.10.

$$E[X] = \sum_{x \in S_X} x . P_X(x) = 0{,}1.$$

2.11:

$$E[X] = \sum_{x \in S_X} x . P_X(x) = -3 . 0{,}4 + 5 . 0{,}4 + 7 . 0{,}2$$

$$= -1{,}2 + 2 + 1{,}4$$

$$= 2{,}2.$$

2.12

$$P_N(n) = \begin{cases} 0{,}2 & n = 0 \\ 0{,}7 & n = 1 \\ 0{,}1 & n = 2 \\ 0 & \text{other wise} \end{cases}$$

a, $E(N) = \sum_{n=0}^{2} n . P_N(n) = 0{,}7 . 1 + 0{,}1 . 2 = 0{,}9$

b, $E(N^2) = \sum_{n=0}^{2} n^2 . P_N(n) = 0{,}7 . 1 + 0{,}1 . 2^2 = 1{,}1$

c,

$$Var[N] = E[N^2] - E^2[N] = 1{,}1 - 0{,}9^2 = 1{,}1 - 0{,}81$$

$$= 0{,}29.$$

d, $\sigma_N = \sqrt{Var[N]} = 0{,}538$

2.13:

$$\operatorname{Var} X = E[X^2] - E^2[X] =$$
$$= (-1)^2 \cdot 0{,}2 + 1 \cdot 0{,}3 - 0{,}1^2$$
$$= 0{,}49.$$

2.14. $Y = aX + b$

$$\operatorname{Var}[Y] = \operatorname{Var}[aX+b] = E[(aX+b)^2] - E^2(aX+b)$$
$$= E[a^2X^2 + 2abX + b^2] - (aE[X] + b)^2$$
$$= a^2 E[X^2] + \cancel{2ab E[X]} + \cancel{b^2} - a^2 E^2[X] - \cancel{2ab E[X]} - \cancel{b^2}$$
$$= a^2 E[X^2] - a^2 E^2[X]$$
$$= a^2 \cdot \operatorname{Var}[X]$$

2.15. Let $f_X(x)$ be the PDF of X.

$$E[Y] = E\left[\frac{1}{\sigma_X}(x - \mu_X)\right] = \int \frac{1}{\sigma_X}(x-\mu_X) f_X(x)\,dx.$$
$$= \frac{1}{\sigma_X}\left[\int (x-\mu_X) \cdot f_X(x)\,dx\right]$$
$$= \frac{1}{\sigma_X}\int x f_X(x)\,dx - \frac{1}{\sigma}\int \mu_X f_X(x)\,dx = \frac{\mu_X}{\sigma_X} - \frac{\mu_X}{\sigma_X} = 0$$

$$\operatorname{Var}[Y] = E[Y^2] - (E[Y])^2 = E[Y^2]$$
$$= \int \frac{1}{\sigma_X^2} \cdot (x-\mu_X)^2 f_X(x)\,dx$$
$$= \frac{1}{\sigma_X^2}\int (x-\mu_X)^2 \cdot f_X(x)\,dx = \frac{\sigma_X^2}{\sigma_X^2} = 1.$$

2.16.

$$F_X(x) = \begin{cases} 0 & x \leq -1 \\ (x+1)/2 & -1 < x \leq 1 \\ 1 & x > 1 \end{cases}$$

a, $P[X > 1/2] = \cancel{\int_{1/2}^{\infty} F_{X}(x)\, dx} = \cancel{\int_{1/2}^{1} \frac{x+1}{2}\, dx} +$

$$= F_X(\infty) - F_X(1/2) = 1 - \frac{1/2+1}{2}$$

$$= \frac{1}{4}.$$

b, $P[-1/2 < x \leq 3/4] = F_X[\tfrac{3}{4}] - F_X[-\tfrac{1}{2}]$

$$= \frac{7}{8} - \frac{1}{4} = \frac{5}{8}$$

c, $P[|X| \leq \frac{1}{2}] = P[X \leq \frac{1}{2}] + P[X \geq -\frac{1}{2}]$

$$= F_X[\tfrac{1}{2}] - F_X[-\tfrac{1}{2}] = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}.$$

d, $P[X \leq a] = 0,8 \Rightarrow -1 < a < 1.$

$$\frac{a+1}{2} = 0,8 \Rightarrow a = 0,6.$$

2.17:

$$F_V(v) = \begin{cases} 0 & v \le -5 \\ c(v+5)^2 & -5 < v \le 7. \\ 1 & v > 7. \end{cases}$$

a, $0 \le F_V(v) \le 1$

$\Leftrightarrow \quad 0 \le c(v+5)^2 \le 1$

$\Leftrightarrow \quad 0 \le c \le \dfrac{1}{(v+5)^2}.$

$$\begin{cases} F_V(-5) = \lim\limits_{v \to -5^-} F_V(v) = \lim\limits_{v \to -5^+} F_V(v) = 0 \\ \Rightarrow \cancel{\lim\limits_{v \to} c(v+5)^2} \\ F_V(7) = \lim\limits_{v \to 7^+} F_V(v) = \lim\limits_{v \to 7^-} F_V(v) \end{cases}$$

$\Rightarrow \lim\limits_{v \to 7} c(v+5)^2 = 1$

$\Rightarrow c \cdot 12^2 = 1$

$\Rightarrow c = \dfrac{1}{144}.$

b, $P[V > 4] = F_V(\infty) - F_V(4)$

$= 1 - \dfrac{81}{144} = \dfrac{63}{144}.$

c, $P[-3 \le V \le 0] = F_V[0] - F_V[-3]$

$= \dfrac{25}{144} - \dfrac{4}{144} = \dfrac{21}{144}.$

d, $P[V > a] = 2/3 \Rightarrow -5 < a \leq 7.$

$$\frac{1}{144}\cdot(a+5)^2 = \frac{2}{3} \Rightarrow (a+5)^2 = 96$$

$$\Rightarrow a = \pm 9{,}8 - 5.$$

$$\left[\begin{array}{l} a = 4{,}8 \\ a = -14{,}8. \end{array}\right.$$

2.18

$$f_X(x) = \begin{cases} cx & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

a, $f_X(x) \geqslant 0 \quad \forall x \Rightarrow c \geqslant 0$

$$\int_0^2 cx\,dx = 1 \Rightarrow \frac{cx^2}{2}\Big|_0^2 = 1$$

$$\Leftrightarrow 2c = 1 \Rightarrow c = \frac{1}{2}.$$

b, $P[0 \leq X \leq 1] = \int_0^1 f_X(x)dx = \int_0^1 \frac{x}{2}\cdot dx$

$$= \frac{x^2}{4}\Big|_0^1 = \frac{1}{4}.$$

c, $P[-1/2 \leq X \leq \frac{1}{2}]$

$$= \int_{-1/2}^0 f_X(x)dx + \int_0^{1/2} f_X(x)\,dx$$

$$= 0 + \int_0^{1/2} \frac{x}{2}\,dx$$

$$= \frac{x^2}{4}\Big|_0^{1/2} = \frac{1}{16}.$$

d. $F_X(x) = \int f_X(x) = \int^0 \frac{x}{2} = \frac{x^2}{4}$

2.19.

$$F_x(x) = \begin{cases} 0 & x \leq -1 \\ (x+1)/2 & -1 < x \leq 1 \\ 1 & x > 1. \end{cases}$$

$$f_X(x) = F_x'(x) = \begin{cases} 0 & x \leq -1 \\ \frac{1}{2} & -1 < x \leq 1 \\ 0 & x > 1 \end{cases}$$

2.20

$$f_x(x) = \begin{cases} 1/4 & -1 \leq x \leq 3 \\ 0 & \text{other wise} \end{cases}$$

$Y = h(x) = X^2.$

$$\cancel{F_x(x)} = \begin{cases} \cancel{1/4}x & -1 \leq x \cancel{\leq} 3 \end{cases}$$

$$E[\cancel{X}] = \sum_{x \in S_x} \cancel{x}.$$

a, $E[X] = \int_{-1}^{3} x . f_x(x)\, dx$

$= \int_{-1}^{3} \frac{x}{4}\, dx = \frac{x^2}{8}\Big|_{-1}^{3} = \frac{9}{8} - \frac{1}{8} = 1$

$Var[\cancel{X}] =$

$E[X^2] = \int_{-1}^{3} x^2 . f_x(x)\, dx = \int_{-1}^{3} \frac{x^2}{4}\, dx$

$= \frac{x^3}{12}\Big|_{-1}^{3} = \frac{27}{12} + \frac{1}{12} = \frac{28}{12} = \frac{13}{6}$

$Var[X] = E[X^2] - E^2[X] = \frac{7}{3} - 1^2 = \frac{4}{3}$

b, $h(E[X]) = h(1) = 1.$

$$E[h(X)] = E[X^2] = \cancel{1/4}\ \cancel{6} \cdot \frac{7}{3}.$$

c,

$$E[Y] = E[X^2] = \frac{7}{3}.$$

$$E[Y^2] = E[X^4] = \int_{-1}^{3} x^4 f_X(x)\,dx = \frac{x^5}{20}\Big|_{-1}^{3} = \frac{244}{20}$$

$$Var[Y] = E[Y^2] - E^2[Y]$$

$$= \frac{244}{20} - \left(\frac{7}{3}\right)^2 =$$

2.21.

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ x/2 & 0 < x \leq 2 \\ 1 & x > 2. \end{cases}$$

a, $E[\cancel{X}] = \int_0^2 x.$,

$$f_X(x) = \begin{cases} \frac{1}{2} & 0 < x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

$$E[X] = \int_0^2 x \cdot f_X(x)\,dx = \int_0^2 \frac{x}{2}\,dx$$

$$= \frac{x^2}{4}\Big|_0^2 = 1$$

b, $Var[X] = E[X^2] - E^2[X]$

$$= \int_0^2 x^2 f_X(x)\,dx - 1$$

$$= \frac{x^3}{6}\Big|_0^2 - 1 = \frac{1}{3}.$$

2.22.

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{2} - k\cos x & 0 < x \leq \pi \\ 1 & x > \pi \end{cases}$$

a, $F_X(x) \geq 0 \quad \forall x.$

$\Leftrightarrow \frac{1}{2} - k\cos x \geq 0 \Leftrightarrow k \leq \frac{1}{2\cos x} \Leftrightarrow -\frac{1}{2} \leq k \leq \frac{1}{2}$

$$\begin{cases} F_X(0) = \lim_{x\to 0^+} F_X(x) = \lim_{x\to 0^-} F_X(x) \\ F_X(\pi) = \lim_{x\to \pi^+} F_X(x) = \lim_{x\to \pi^-} F_X(x) \end{cases}$$

$$\Leftrightarrow \begin{cases} 0 = \frac{1}{2} - k.\cos 0 \\ 1 = \frac{1}{2} - k.\cos\pi \end{cases} \Leftrightarrow k = \frac{1}{2}.$$

b,

$$P[0 < X < \tfrac{\pi}{2}] = F_X[\pi/2] - F_X[0]$$

$$= \frac{1}{2} - \frac{1}{2}\cos\frac{\pi}{2} = \frac{1}{2}.$$

c, $$f_X(x) = \begin{cases} \frac{\sin x}{2} & 0 \leq x \leq \pi. \\ 0 & \text{other wise} \end{cases}$$

$$E[X] = \int_0^\pi \frac{\sin x}{2}.x \; dx.$$

$$= \frac{1}{2}\left[x.(-\cos x)\Big|_0^\pi - \int_0^\pi -\cos x \; dx \right]$$

$$= \frac{1}{2}\pi + \int_0^\pi \cos x \; dx$$

$$= \frac{1}{2}\pi + \sin x\Big|_0^\pi = \frac{\pi}{2}.$$

223:

$$F_X(x) = \begin{cases} 0 & x \le -a \\ A + B\arcsin\frac{x}{a} & x \in (-a, a) \\ 1 & x \geq a \end{cases}$$

a, $F_X(x) \geq 0 \Leftrightarrow A + B\arc\sin\frac{x}{a} \geq 0$

$$\begin{cases} F_X(-a) = \lim\limits_{x\to -a^-} F_X(x) = \lim\limits_{x\to a^+} F_X(x) \\ F_X(a) = \lim\limits_{x\to a^-} F_X(x) = \lim\limits_{x\to a^+} F_X(x) \end{cases}$$

$$\Leftrightarrow \begin{cases} 0 = A + B\arcsin\frac{-a}{a} \\ 1 = A + B\arcsin\frac{a}{a} \end{cases} \Leftrightarrow \begin{cases} A + B\left(-\frac{\pi}{2}\right) = 0 \\ A + B\left(\frac{\pi}{2}\right) = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} A = \frac{1}{2} \\ B = \frac{1}{\pi} \end{cases}$$

$$f_X(x) = \begin{cases} \frac{1}{\pi} \cdot \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} & x \in (-a, a) \\ 0 & \text{otherwise.} \end{cases}$$

224. $F_X(x) = a + b\arctan x \quad x \in \mathbb{R}$.

a, $\begin{cases} F_X(-\infty) = 0 \\ F_X(+\infty) = 1 \end{cases} \Leftrightarrow \begin{cases} a - \frac{\pi}{2}.b = 0 \\ a + \frac{\pi}{2}b = 1 \end{cases} \Leftrightarrow \begin{cases} a = \frac{1}{2} \\ b = \frac{1}{\pi} \end{cases}$

b, $f_X(x) = F_X'(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$

c, $P[-1 < X < 1] = F_X[1] - F_X[-1] =$

2.25.

$$F_X(x) = \frac{1}{2} + \frac{1}{\pi} \arctan \frac{x}{2}.$$

$$P(X > x_1) = \frac{1}{4}.$$

$$\Leftrightarrow \quad F_X(\infty) - F_X(x_1) = \frac{1}{4}.$$

$$\Leftrightarrow \quad \frac{1}{2} + \frac{1}{\pi} \cdot \frac{\pi}{2} - \frac{1}{2} - \frac{1}{\pi} \arctan \frac{x_1}{2} = \frac{1}{4}.$$

$$\Leftrightarrow \quad \cancel{\frac{1}{\pi}} \arctan \frac{x_1}{2} = \frac{\pi}{4}.$$

$$\Leftrightarrow \quad \frac{x_1}{2} = 1 \Rightarrow x_1 = 2.$$

2.26. $f(x) = a \cdot e^{-|x|} \quad (-\infty < x < +\infty)$

a) $\begin{cases} f(x) \geq 0 \Rightarrow a \geq 0 \\ \int_{-\infty}^{\infty} a e^{-|x|} dx = 1. \end{cases}$

$$\Leftrightarrow \int_{-\infty}^{0} a e^{x} dx + \int_{0}^{\infty} a \cdot e^{-x} dx = 1$$

$$\Leftrightarrow \cancel{a} \; a + (+1) a = 1 \Rightarrow a = \frac{1}{2}.$$

b)

+) $x < 0$: $F_Y(x) = \int_{-\infty}^{x^2} f(u) \cdot du = \int_{-\infty}^{x^2} \frac{e^{u}}{2} \cdot du$

$$= \frac{e^{x^2}}{2}.$$

+) $x > 0$ $F_Y(x) = \int_{-\infty}^{x^2} f(u) du = \int_{-\infty}^{x^2} \frac{e^{-u}}{2} du$

$$= \frac{e^{-x^2}}{2}.$$

(a) $F_Y(x) = P(Y < x)$

$= P(X^2 < x)$

$= P(-\sqrt{x} < X < \sqrt{x})$

$= \int_{-\sqrt{x}}^{0} f(u)\,du + \int_{0}^{\sqrt{x}} f(u)\,du$

$= \int_{-\sqrt{x}}^{0} \frac{e^{u}}{2}\,du + \int_{0}^{\sqrt{x}} \frac{e^{-u}}{2}\,du.$

$= \frac{e^{u}}{2}\Big|_{-\sqrt{x}}^{0} + \frac{e^{-u}}{2}\Big|_{0}^{\sqrt{x}}$

$= \frac{1}{2} - \frac{e^{-\sqrt{x}}}{2} + \frac{e^{-\sqrt{x}}}{2} - \frac{1}{2}.$

$= 0$

c)

$E[X] = \int_{-\infty}^{\infty} x.f(x)\,dx. = \int_{-\infty}^{0} x.\frac{e^{x}}{2}\,dx + \int_{0}^{+\infty} x.\frac{e^{-x}}{2}\,dx.$

$= \frac{1}{2}\Big[\, x.e^{x}\Big|_{-\infty}^{0} - \int_{-\infty}^{0} e^{x}\,dx + x.(-e^{-x})\Big|_{0}^{+\infty} - \int_{0}^{+\infty} -e^{-x}.\,dx.$

$= \frac{1}{2}\Big[-e^{x}\Big|_{-\infty}^{0} + (-e^{-x})\Big|_{0}^{+\infty} \Big]$

$= \frac{1}{2}\big[-1 + (-1)\big] = \cancel{0} \; -1.$

$Var[X] = \cancel{0} \; E[X^2] - (E[X])^2.$

$$E[X^2] = \int_{-\infty}^{0} x^2 \cdot \frac{e^x}{2} dx + \int_{0}^{+\infty} x^2 \cdot \frac{e^{-x}}{2} dx$$

$$= \frac{1}{2}\left[x^2 e^x \Big|_{-\infty}^{0} - 2x \cdot e^x \Big|_{-\infty}^{0} + 2 e^x \Big|_{-\infty}^{0} \right]$$

$$+ \frac{1}{2}\left[x^2(-e^{-x}) \Big|_{0}^{+\infty} - 2x \cdot e^{-x} \Big|_{0}^{+\infty} + 2 e^{-x} \Big|_{0}^{+\infty} \right].$$

$$= \frac{1}{2}[2] + \frac{1}{2} \cdot [-2] = 0.$$

$$\Rightarrow \quad Var[X] = E[X^2] - E^2[X]$$

$$= -1.$$