

Fundamentals of Optimization

Lagrangian relaxation

Pham Quang Dung

dungpq@soict.hust.edu.vn

Department of Computer Science

Content

- Lagrange dual function
- Lagrange dual problem
- KKT condition

Lagrange dual function

- Optimization problem in the standard form

$$\begin{array}{ll} (P) & \text{minimize } f(x) \\ \text{s.t.} & g_i(x) \leq 0, i = 1, 2, \dots, m \\ & h_i(x) = 0, i = 1, 2, \dots, p \\ & x \in X \subseteq R^n \end{array}$$

with $x \in R^n$, and assume $D = (\cap_{i=1}^m \text{dom } g_i) \cap (\cap_{i=1}^p \text{dom } h_i)$ is not empty.

- Denote f^* the optimal value of $f(x)$
- If $f, g_i (i = 1, 2, \dots, m)$ are convex functions, $h_i (i = 1, \dots, p)$ are linear \rightarrow **convex program**

Lagrange dual function

- Define Lagrangian function $L: R^n \times R^m \times R^p \rightarrow R$

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^m \lambda_i g_i(x) + \sum_{i=1}^p \mu_i h_i(x)$$

- Lagrange dual function (or dual function)

$$q(\lambda, \mu) = \inf_{x \in D} L(x, \lambda, \mu)$$

Lagrange dual problem

- For each pair (λ, μ) , the Lagrange dual function provides a lower bound on the optimal value of the primal problem (original optimization problem)

$$q(\lambda, \mu) \leq f^*$$

- Question: what is the *best* lower bound \rightarrow Lagrange dual problem

$$\begin{aligned} \max q(\lambda, \mu) \\ \lambda \geq 0 \end{aligned}$$

Lagrange Dual problem

- **Weak duality theorem** if x^* is an optimal solution to the primal problem and (λ^*, μ^*) is an optimal solution to the dual problem, then $f(x^*) \geq q(\lambda^*, \mu^*)$
- **Corollary** If there exist x^* and (λ^*, μ^*) such that $f(x^*) = q(\lambda^*, \mu^*)$, then x^* and (λ^*, μ^*) are respectively optimal solutions to the primal and dual problems

KKT Conditions

- **Theorem** (Fritz John necessary conditions) Let x^* be a feasible solution of (P) . If x^* is a local minimum of (P) , then there exists (u, λ, μ) such that:
 - $u \nabla f(x^*) + \sum_{i=1}^m \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) = 0$
 - $u, \lambda \geq 0, (u, \lambda, \mu) \neq 0$
 - $\lambda_i g_i(x^*) = 0, i = 1, \dots, m$

KKT Conditions

- **Theorem** (Karush-Kuhn-Tucker (KKT) necessary conditions) Let x^* be a feasible solution of (P) and $I = \{i: g_i(x^*) = 0\}$. Further, suppose that $\nabla h_i(x^*)$ for $i = 1, \dots, p$ and $\nabla g_i(x^*)$ for $i \in I$ are linearly independent. If x^* is a local minimum of (P) , then there exists (λ, μ) such that:
 - $\nabla f(x^*) + \sum_{i=1}^m \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) = 0$
 - $\lambda_i \geq 0$,
 - $\lambda_i g_i(x^*) = 0, i = 1, \dots, m$

KKT Conditions

- **Theorem** (KKT sufficient conditions) Let x^* be a feasible solution of (P) which is convex program (f, g_i are convex functions, h_i are linear functions). If there exists (λ, μ) such that:
 - $\nabla f(x^*) + \sum_{i=1}^m \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) = 0$
 - $\lambda \geq 0,$
 - $\lambda_i g_i(x^*) = 0, i = 1, \dots, m$then x^* is a global optimal solution of (P)

Examples

$$\begin{array}{ll}\text{minimize} & f(x,y) = 2x - y \\ \text{s.t.} & g_1(x,y) = x^2 + y^2 - 2 \leq 0 \\ & g_2(x,y) = x - y - 1 \leq 0\end{array}$$

- $\nabla f(x,y) = [2 \ 1]^T$, $\nabla g_1(x,y) = [2x \ 2y]^T$, $\nabla g_2(x,y) = [1 \ -1]^T$
- f and g_2 are linear, so they are convex
- $\nabla^2 g_1(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ which is positive definite, so g_1 is convex

Examples

$$\begin{array}{ll}\text{minimize} & f(x,y) = 2x - y \\ \text{s.t.} & g_1(x,y) = x^2 + y^2 - 2 \leq 0 \\ & g_2(x,y) = x - y - 1 \leq 0\end{array}$$

- Apply the KKT necessary conditions
 - $[2 \ -1]^T + \lambda_1[2x \ 2y]^T + \lambda_2[1 \ -1]^T = 0$
 - $x^2 + y^2 - 2 \leq 0$
 - $x - y - 1 \leq 0$
 - $\lambda_1(x^2 + y^2 - 2) = 0$
 - $\lambda_2(x - y - 1) = 0$
 - $\lambda_1, \lambda_2 \geq 0$

Examples

$$\begin{array}{ll}\text{minimize} & f(x,y) = 2x - y \\ \text{s.t.} & g_1(x,y) = x^2 + y^2 - 2 \leq 0 \\ & g_2(x,y) = x - y - 1 \leq 0\end{array}$$

- Apply the KKT necessary conditions (rewrite)
 - $2 + 2\lambda_1 x + \lambda_2 = 0$ (1)
 - $-1 + 2\lambda_1 y - \lambda_2 = 0$ (2)
 - $x^2 + y^2 - 2 \leq 0$ (3)
 - $x - y - 1 \leq 0$ (4)
 - $\lambda_1(x^2 + y^2 - 2) = 0$ (5)
 - $\lambda_2(x - y - 1) = 0$ (6)
 - $\lambda_1, \lambda_2 \geq 0$ (7)

Examples

$$\begin{array}{ll} \text{minimize} & f(x,y) = 2x - y \\ \text{s.t.} & g_1(x,y) = x^2 + y^2 - 2 \leq 0 \\ & g_2(x,y) = x - y - 1 \leq 0 \end{array}$$

- There are 4 cases
 - (I) $\lambda_1 = \lambda_2 = 0$
 - (II) $\lambda_1 = 0, \lambda_2 > 0$
 - (III) $\lambda_1 > 0, \lambda_2 = 0$
 - (IV) $\lambda_1 > 0, \lambda_2 > 0$

Examples

$$\begin{array}{ll}\text{minimize} & f(x,y) = 2x - y \\ \text{s.t.} & g_1(x,y) = x^2 + y^2 - 2 \leq 0 \\ & g_2(x,y) = x - y - 1 \leq 0\end{array}$$

- Case (I) $\lambda_1 = \lambda_2 = 0$, from (1) and (2) \rightarrow NOT POSSIBLE

Examples

$$\begin{array}{ll}\text{minimize} & f(x,y) = 2x - y \\ \text{s.t.} & g_1(x,y) = x^2 + y^2 - 2 \leq 0 \\ & g_2(x,y) = x - y - 1 \leq 0\end{array}$$

- Case (II) $\lambda_1 = 0, \lambda_2 > 0$, from (1) and (2) we have
 - $2 + \lambda_2 = 0$
 - $-1 - \lambda_2 = 0$ (NOT POSSIBLE as $\lambda_2 > 0$)

Examples

$$\begin{array}{ll}\text{minimize} & f(x,y) = 2x - y \\ \text{s.t.} & g_1(x,y) = x^2 + y^2 - 2 \leq 0 \\ & g_2(x,y) = x - y - 1 \leq 0\end{array}$$

- Case (III) $\lambda_1 > 0, \lambda_2 = 0$, we have
 - $2 + 2\lambda_1 x = 0$ (8)
 - $-1 + 2\lambda_1 y = 0$ (9)
 - $x^2 + y^2 - 2 \leq 0$ (10)
 - $x - y - 1 \leq 0$ (11)
- Solve this equation, we get $x = -1$ and $y = 1$, $\lambda_1 = 1$, $\lambda_2 = 0$. This solution satisfies (1) - (7) \rightarrow $x = -1, y = 1$ is a global optimal solution to the problem (based on the KKT sufficient conditions)

Examples

$$\begin{array}{ll}\text{minimize} & f(x,y) = 2x - y \\ \text{s.t.} & g_1(x,y) = x^2 + y^2 - 2 \leq 0 \\ & g_2(x,y) = x - y - 1 \leq 0\end{array}$$

- Case (IV) $\lambda_1 > 0, \lambda_2 > 0$, we have
 - $2 + 2\lambda_1 x + \lambda_2 = 0$ (12)
 - $-1 + 2\lambda_1 y - \lambda_2 = 0$ (13)
 - $x^2 + y^2 - 2 = 0$ (14)
 - $x - y - 1 = 0$ (15)
 - $\lambda_1, \lambda_2 \geq 0$ (16)

Examples

$$\begin{array}{ll}\text{minimize} & f(x,y) = 2x - y \\ \text{s.t.} & g_1(x,y) = x^2 + y^2 - 2 \leq 0 \\ & g_2(x,y) = x - y - 1 \leq 0\end{array}$$

- Case (IV) $\lambda_1 > 0, \lambda_2 > 0$, we have
 - From (15), we have $x = y + 1$, replace in (12) and (13), we have
 - $2 + 2\lambda_1(y + 1) + \lambda_2 = 0$ (17)
 - $-1 + 2\lambda_1 y - \lambda_2 = 0$ (18)
 - Subtract (17) and (18), we have $3 + 2\lambda_1 + 2\lambda_2 = 0$ (NOT POSSIBLE as $\lambda_1 > 0, \lambda_2 > 0$)

Lagrange relaxation for Integer Programming

$$\begin{array}{ll} Z_{IP} = \text{minimize } f(x) = & c^T x \\ \text{s.t.} & Ax \geq b \\ & Dx \geq d \\ & x \text{ integer} \end{array}$$

- Let $X = \{x \text{ integer} \mid Dx \geq d\}$
- Assume optimizing over X can be solved easily, but adding constraint $Ax \geq b$ makes the problem too difficult

Lagrange relaxation for Integer Programming

$$\begin{array}{ll} Z(\lambda) = \min_x & c^T x + \lambda^T (b - Ax) \\ \text{s.t.} & Dx \geq d \\ & x \text{ integer} \end{array}$$

- For a fixed λ , $Z(\lambda)$ is assumed to be computed easily
- Important: compute the best lower bound

$$Z_D = \max_{\lambda \geq 0} Z(\lambda)$$

Lagrange relaxation for Integer Programming

- Subgradient method for computing Z_D

```
Choose starting point  $\lambda^{(0)}$  (e.g.,  $\lambda^{(0)} = 0$ );  $k = 0$   
while (STOP condition not reach) {  
     $x^{(k)}$  is the solution of  $Z(\lambda^{(0)})$   
    Compute subgradient  $s^{(k)} = b - Ax^{(k)}$  of function  $Z$  at  $\lambda^{(k)}$   
    if  $s^{(k)} = 0$  then BREAK  
     $\lambda^{(k+1)} = \max\{0, \lambda^{(k)} + \alpha^{(k)}s^{(k)}\}$  /*  $\alpha^{(k)}$  denote the step size */  
     $k = k + 1$   
}
```