

EXERCISES - CALCULUS 3 - MI1131E

Chapter 1

Series

1.1 Number series

Exercise 1.1. Test for convergence and find the sum (if exists):

a) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

e) $\sum_{n=1}^{\infty} \left(\frac{9}{10^n} - \frac{2}{5^n} \right)$

b) $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots$

f) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot 3^n}{10^{n+2}}$

c) $\sum_{n=1}^{\infty} (\sin n + 1 - \sin n)$

d) $\sum_{n=1}^{\infty} \ln \left(1 + \frac{1}{n} \right)$

g) $\sum_{n=1}^{\infty} \arctan \frac{1}{n^2 + n + 1}$

Exercise 1.2. Test for convergence:

1. Divergence test

a) $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{n+1}$

c) $\sum_{n=1}^{\infty} \cos \left(\frac{1}{n^2} \right)$

e) $\sum_{n=1}^{\infty} (-1)^n \cos \left(\frac{1}{n} \right)$

b) $\sum_{n=1}^{\infty} \frac{2n+3}{6n-1}$

d) $\sum_{n=1}^{\infty} \left(\frac{n+1}{n+2} \right)^n$

2. Comparison tests

a) $\sum_{n=1}^{\infty} \frac{\sqrt{n^2 + n + 1}}{n^2 \sqrt{n} + 2}$

d) $\sum_{n=1}^{\infty} \frac{\sqrt[n]{e} - 1}{n}$

g) $\sum_{n=1}^{\infty} \frac{2}{\ln(2n+1)}$

b) $\sum_{n=1}^{\infty} \frac{\sqrt{n+2} - \sqrt{n}}{2n+1}$

e) $\sum_{n=2}^{\infty} \arctan(2^{-n})$

h) $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \sin \frac{1}{\sqrt{n}} \right)$

c) $\sum_{n=1}^{\infty} \ln \left(1 + \frac{1}{n} \right)$

f) $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

i) $\sum_{n=1}^{\infty} \frac{4 + \cos n}{n^2(1 + e^{-n})}$

3. Ratio test

$$\begin{array}{lll} \text{a)} \sum_{n=1}^{\infty} \frac{2019^n}{n!} & \text{c)} \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n+1)!} & \text{e)} \sum_{n=1}^{\infty} \frac{n^n}{4^n \cdot n!} \\ \text{b)} \sum_{n=2}^{\infty} \frac{1}{3^n} \frac{(2n+1)!}{n^2-1} & \text{d)} \sum_{n=1}^{\infty} \frac{n!}{3n^2} & \text{f)} \sum_{n=2}^{\infty} \frac{e^n n!}{n^n} \end{array}$$

4. Root test

$$\begin{array}{lll} \text{a)} \sum_{n=1}^{\infty} \left(\frac{3n+1}{3n+2} \right)^{n^2} & \text{c)} \sum_{n=2}^{\infty} \left(\frac{n}{n+2} \right)^{n^2-1} & \text{e)} \sum_{n=2}^{\infty} \left(\cos \frac{1}{n} \right)^{n^3} \\ \text{b)} \sum_{n=2}^{\infty} \frac{1}{4^n} \left(1 - \frac{1}{n} \right)^{n^2} & \text{d)} \sum_{n=1}^{\infty} \frac{1}{3^n} \left(\frac{n-2}{n} \right)^{n^2+1} & \end{array}$$

5. Integral test

$$\begin{array}{lll} \text{a)} \sum_{n=2}^{\infty} \frac{\ln n}{n^2} & \text{c)} \sum_{n=4}^{\infty} \frac{1}{n \ln n \ln(\ln n)} & \text{e)} \sum_{n=2}^{\infty} \frac{1}{\ln(n!)} \\ \text{b)} \sum_{n=2}^{\infty} \frac{\ln n}{n} & \text{d)} \sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}} & \end{array}$$

6. Series with sign-changing terms

$$\begin{array}{lll} \text{a)} \sum_{n=1}^{\infty} \frac{\cos n}{\sqrt{n^3+1}} & \text{d)} \sum_{n=1}^{\infty} \left(\frac{3-2n}{2^n+5} \right)^{n^2} & \text{g)} \sum_{n=1}^{\infty} \frac{(-1)^n}{3^n + \cos n} \\ \text{b)} \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1} & \text{e)} \sum_{n=1}^{\infty} \frac{n \cos(n\pi)}{2n^2+1} & \text{h)} \sum_{n=2}^{\infty} \frac{(-1)^n + \cos n}{n \ln^2 n} \\ \text{c)} \sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^3}{2^n - 1} & \text{f)} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot n^3}{(n^2+1)^{\frac{4}{3}}} & \text{i)} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \sin \frac{1}{\sqrt{n}} \end{array}$$

Exercise 1.3. Test for absolute and conditional convergence:

$$\begin{array}{lll} \text{a)} \sum_{n=2}^{\infty} \frac{(-1)^n n}{n^2+1} & \text{c)} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} & \text{e)} \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n} + (-1)^n} \\ \text{b)} \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+100} & \text{d)} \sum_{n=1}^{\infty} (-1)^n \left(\frac{2n+100}{3n+1} \right)^n & \end{array}$$

Exercise 1.4. Test for convergence

$$\begin{array}{lll} \text{a)} \sum_{n=1}^{\infty} \frac{n+1}{(n^2+2) \ln(n+3)} & \text{c)} \sum_{n=1}^{\infty} \frac{n^5}{3^n+2^n} & \text{e)} \sum_{n=2}^{\infty} \left(e^{\frac{(-1)^n}{\sqrt{n}}} - 1 \right) \\ \text{b)} \sum_{n=1}^{\infty} \frac{2 - n^2 \cdot 3^{-n^2}}{n^2} & \text{d)} \sum_{n=1}^{\infty} \left(\cos \frac{1}{n+1} - \cos \frac{1}{n} \right) & \text{f)} \sum_{n=2}^{\infty} \frac{(-1)^n (n-1)}{n^2+1} \end{array}$$

1.2 Function series

Exercise 1.5. Determine the domain of convergence of the following function series:

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|---|---|--|
| a) $\sum_{n=1}^{\infty} \frac{1}{1+n^{-x}}$ | d) $\sum_{n=1}^{\infty} \frac{x^n}{x^{2n}+1}$ | g) $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{(x^2+1)n^x}$ |
| b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^x}$ | e) $\sum_{n=1}^{\infty} \frac{n^x + (-1)^n}{n}$ | h) $\sum_{n=1}^{\infty} n \cdot e^{-nx}$ |
| c) $\sum_{n=1}^{\infty} \frac{1}{x^n+1}$ | f) $\sum_{n=1}^{\infty} \left(x + \frac{1}{n}\right)^n$ | i) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$ |

Exercise 1.6. Determine the domain of convergence of the following power series:

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|--|---|---|
| a) $\sum_{n=1}^{\infty} \frac{x^{2n}}{n}$ | d) $\sum_{n=1}^{\infty} \frac{e^{nx}}{n^2+n+1}$ | g) $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{\sqrt{n^3+1}} (1-3x)^n$ |
| b) $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n\sqrt{n}}$ | e) $\sum_{n=1}^{\infty} \frac{n^2}{1+n^3} (2x+1)^n$ | h) $\sum_{n=1}^{\infty} \left(\frac{1-2n}{2n+3}\right)^n x^{2n+1}$ |
| c) $\sum_{n=1}^{\infty} \frac{n}{2n+1} (x-2)^n$ | f) $\sum_{n=1}^{\infty} \frac{x^n}{2^n+3^n}$ | |

Exercise 1.7. Test for uniform convergence on the given set of the following series:

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|---|--|
| a) $\sum_{n=1}^{\infty} \frac{\sin nx}{2x^2+n^2}, \text{ on } \mathbb{R}$ | e) $\sum_{n=1}^{\infty} \frac{x}{1+n^4x^2}, \text{ on } [0, \infty)$ |
| b) $\sum_{n=1}^{\infty} \frac{e^{-nx}+1}{n^2}, \text{ on } [0, \infty)$ | f) $\sum_{n=1}^{\infty} \frac{x}{(1+(n-1)x)(1+nx)}, \text{ on } (0, 1]$ |
| c) $\sum_{n=1}^{\infty} \frac{x^n}{(4x^2+9)^n}, x \in \mathbb{R}$ | g) $\sum_{n=1}^{\infty} (1-x)x^n, \text{ on } [0, 1]$ |
| d) $\sum_{n=1}^{\infty} \frac{1}{2^n} \left(\frac{2x+1}{x+2}\right)^n, x \in [-1; 1]$ | h) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{x^2+n+2}, \text{ on } \mathbb{R}.$ |

Exercise 1.8. 1. Let $F(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^3}$. Prove that

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|--------------------------------------|---------------------------------------|---|
| (a) $F(x)$ is continuous $\forall x$ | (b) $\lim_{x \rightarrow 0} F(x) = 0$ | (c) $F'(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ |
|--------------------------------------|---------------------------------------|---|

2. Prove that $\int_0^{\pi} \left(\frac{\cos 2x}{1.3} + \frac{\cos 4x}{3.5} + \frac{\cos 6x}{5.7} + \dots \right) dx = 0$.

Exercise 1.9. Find the sum of the following function series or number series:

- | | |
|---|--|
| a) $\sum_{n=1}^{\infty} nx^n, x \in (-1; 1)$ | d) $\sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)}, x \in (-1; 1)$ |
| b) $\sum_{n=1}^{\infty} \frac{x^n}{n+1}, x \in (-1, 1)$ | e) $\sum_{n=1}^{\infty} \frac{x^{4n-3}}{4n-3}, x \in (-1; 1)$ |
| c) $\sum_{n=1}^{\infty} (n^2+n)x^{n+1}, x \in (-1, 1)$ | f) $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot \pi^{2n+1}}{(2n+1)!}$ |

g) $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1) \cdot 2^n}$

i) $\sum_{n=1}^{\infty} \frac{3n+1}{8^n}$

h) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)3^n}$

j) $\sum_{n=1}^{\infty} \frac{1}{(2n)!!}$

Exercise 1.10. Find the Maclaurin series of the following functions:

a) $y = \sin^2 x \cos^2 x$

e) $y = \frac{2x-1}{x^2+2x-3}$

h) $y = \ln(1+2x)$

b) $y = \sin x \sin 3x$

f) $y = \frac{1}{x^2+x+1}$

i) $y = x \ln(x+2)$

c) $y = e^{2x} + 3x \cos x$

g) $y = \frac{1}{\sqrt{4-x^2}}$

j) $y = \ln(1+x-2x^2)$

d) $y = \frac{2x+1}{x^2-3x+2}$

k) $y = \arcsin x$

Exercise 1.11. Find the Taylor series of y at the given point:

a) $y = \frac{1}{2x+3}, x_0 = 4$

b) $y = \sin \frac{\pi x}{3}, x_0 = 1$

c) $y = \sqrt{x}, x_0 = 4$

Exercise 1.12. Graph each of the following periodic functions and find compute corresponding Fourier series

a) $y = x, x \in (-\pi, \pi), T = 2\pi$

d) $y = \begin{cases} 2x, & 0 \leq x < 3, \\ 0, & -3 < x < 0, \end{cases} T = 6$

b) $y = |x|, x \in (-\pi, \pi), T = 2\pi$

e) $y = 2x, 0 < x < 10, T = 10$

c) $y = \begin{cases} 4, & 0 < x < 2, \\ -4, & 2 < x < 4, \end{cases} T = 4$

f) $y = \begin{cases} 2-x, & 0 < x < 4, \\ x-6, & 4 < x < 8, \end{cases} T = 8$

In each part, find the points of discontinuity of the function. To what value does the series converge at those points?

Exercise 1.13. Expand the function into a Fourier series

a) $f(x) = x, x \in [0, \pi], f(x)$ is an odd and periodic function of $T = 2\pi$.

b) $f(x) = 2-x, x \in (0, 2), f(x)$ is an even and periodic of $T = 4$.

c) $f(x) = x+1, x \in [0, \pi)$.

d) $f(x) = x-1, x \in (0, \pi)$ into a Fourier sine series.

e) $f(x) = x(\pi-x), x \in [0, \pi]$ into a Fourier cosine series. Then prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Chapter 2

Ordinary differential equations

2.1 First order ODEs

Exercise 2.1. 1) Separable equations

a) $2y(x^2 + 4)dy = (y^2 + 1)dx$

b) $y' + e^{y+x} = 0$

c) $1 + x + xy'y = 0$

d) $y' = \cos^2 x \cos^2(2y)$

e) $y' = x^2y, y(1) = 1$

f) $x dx + ye^{-x}dy = 0, y(0) = 1.$

g) $y^2\sqrt{1-x^2}dy = \arcsin x dx, y(0) = 0$

h) $y' = \frac{2x}{y + x^2y}, y(0) = -2.$

2) Homogeneous equations:

a) $y' = \frac{y}{x} + \frac{x}{y} + 1$

b) $xy' = x \sin \frac{y}{x} + y$

c) $2y' + \left(\frac{y}{x}\right)^2 = -1$

d) $(x + 2y)dx - xdy = 0$

e) $xy' = y + e^{\frac{y}{x}}, y(1) = 0$

f) $xy' = y + 2x^3 \sin^2 \frac{y}{x}, y(1) = \frac{\pi}{2}$

g) $y' = \frac{y^2}{x^2} - \frac{y}{x} + 1, y(1) = 2$

h) $(2x - y + 4)dx + (x + 2y - 3)dy = 0.$

3) Linear equations:

a) $xy' - 4y = 4x^8$

b) $(x^2 + 1)y' + 2xy = e^x$

c) $xy' - y = x^2 \cos x, y(\pi) = \pi$

d) $y' + y \sin x = \sin x, y(0) = 0$

e) $y' - \frac{3}{x} = 2x^2, y(1) = 2$

f) $(2xy + 3)dy - y^2dx = 0.$

4) Bernoulli equations:

a) $y' + \frac{2}{x}y = \frac{y^3}{x^2}$

b) $xy' + y = -x^3y^2, y(1) = 1$

c) $y' + xy = \frac{xe^{-2x^2}}{y}$

d) $xy' = \frac{x^3}{y^2} - 2y, y(1) = 2.$

5) Exact equations:

- a) $(x^2 + y)dx = (2y - x)dy$ d) $\frac{y}{x}dx + (e^y + 1 + \ln x)dy = 0$
 b) $e^y dx = (2y - xe^y)dy$
 c) $(3x^2y^2 + 2y + 1)dx + 2(x + x^3y)dy = 0$ e) $(e^x \sin y + y^2)dx + (e^x \cos y + 2xy)dy = 0$

Exercise 2.2. In each of the following parts, find an integrating factor and solve the given equation

1. $\left(\frac{y}{x} - 1\right)dx + \left(\frac{y}{x} + 1\right)dy = 0$
2. $(3x^2y + 2xy + y^3)dx + (x^2 + y^2)dy = 0$
3. $ydx + (2xy - e^{-2y})dy = 0$

Exercise 2.3. Solve the ODE $y' = y^2 - \frac{2}{x^2}$ by change of function $y = \frac{z}{x}$.

Exercise 2.4. Solve the ODE $xy' - (2x + 1)y + y^2 + x^2 = 0$ by change of function $y = z + x$.

Exercise 2.5. Solve the following ODEs:

- a) $y' = (x + y)^2$ e) $(x^2y^2 - x)dy = ydx$
 b) $y' = 1 + x + y + xy$ f) $3xy^2y' - y^3 = x, y(1) = 3$
 c) $(2xy^2 - 3y^3)dx = (3xy^2 - y)dy$ g) $(8xy^2 - y)dx + xdy = 0, y(1) = 1$
 d) $xy' = y + x^3 \sin x, y(\pi) = 0$ h) $x = (y')^2 - y' + 2$

2.2 Second order ODEs

Exercise 2.6. Solve the following ODEs

- a) $xy'' + 2y' = 12x^2$ c) $2yy'' = (y')^2 + 1$
 b) $\begin{cases} (1 - x^2)y'' - xy' = 2, \\ y(0) = 0, y'(0) = 0 \end{cases}$ d) $\begin{cases} (1 + x)y'' + x(y')^2 = y', \\ y(0) = 1, y'(0) = 2 \end{cases}$

Exercise 2.7. Solve the following equations

1. $(x - 1)^2y'' + 4(x - 1)y' + 2y = 0$, given a particular solution $y_1 = \frac{1}{1 - x}$.
2. $xy'' + 2y' + xy = 0$, given a particular solution $y_1 = \frac{\sin x}{x}$.
3. $y'' - \frac{2xy'}{x^2 + 1} + \frac{2y}{x^2 + 1} = 0$ given a particular solution $y_1 = x$.
4. $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0, x > 0$, given a particular solution $y_1 = \frac{\cos x}{\sqrt{x}}$.

Exercise 2.8. Solve the ODEs with constant coefficients:

- | | |
|---|---|
| a) $y'' - 4y' + 3y = (15x + 37)e^{-2x}$ | f) $y'' + y = 2 \cos x \cos 2x$ |
| b) $y'' - y = 4(x + 1)e^x$ | g) $y'' + 2y' + 2y = 8 \cos x - \sin x$ |
| c) $y'' - 2y' + y = (12x + 4)e^x$ | h) $y'' + y' - 2y = x + \sin 2x$ |
| d) $y'' - y' - 2y = xe^x \cos x$ | i) $y'' + 3y' - 4y = 3 \sin^2 x$ |
| e) $y'' + 2y' + 10y = e^{-x} \cos 3x$ | j) $y'' + 4y = e^{3x} + x \sin 2x$ |

Exercise 2.9. Solve the ODEs using the method of variation of parameters:

- | | |
|------------------------------------|--|
| a) $y'' - 2y' + y = \frac{e^x}{x}$ | b) $y'' - 3y' + 2y = \frac{1}{1 + e^{-x}}$ |
|------------------------------------|--|

Exercise 2.10. Solve the ODE $(2x - x^2)y'' + 2(x - 1)y' - 2y = -2$, given two particular solutions $y_1 = 1, y_2 = x$.

Exercise 2.11. Solve the following Euler equations

- $x^2 y'' - 3xy' + 4y = x^3, y(1) = 1, y'(1) = 2$
- $y'' - \frac{y'}{x+1} + \frac{y}{(x+1)^2} = \frac{2}{x+1}, x > -1.$

2.3 Systems of first order ODEs

Exercise 2.12. Solve the following systems of ODEs

- | | |
|---|---|
| a) $\begin{cases} \frac{dy}{dx} = 5y + 4z \\ \frac{dz}{dx} = 4y + 5z \end{cases}$ | c) $\begin{cases} \frac{dx}{dt} = \frac{y}{x-y} \\ \frac{dy}{dt} = \frac{x}{x-y} \end{cases}$ |
| b) $\begin{cases} \frac{dy}{dx} = y + 5z \\ \frac{dz}{dx} = -y - 3z \end{cases}$ | d) $\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -x + \frac{1}{\cos t} \end{cases}$ |

2.4 Power series method

Exercise 2.13. Solve the following series by the power series method

- | | |
|-----------------------------------|--|
| a) $(x^2 + 1)y'' - 4xy' + 6y = 0$ | b) $y'' + xy' + y = 0, y(0) = 0, y'(0) = 1.$ |
|-----------------------------------|--|

Chapter 3

Laplace transform

3.1 Laplace and inverse Laplace transforms

Exercise 3.1. Using the definition, find the Laplace transforms of the following functions:

$$\text{a) } f(t) = t \qquad \text{b) } f(t) = e^{2t+3} \qquad \text{c) } f(t) = \sin(2t).$$

Exercise 3.2. Find the Laplace transforms of the following functions:

$$\begin{aligned} \text{a) } f(t) &= \sqrt{t} + 3t - 2t^2\sqrt{t} & \text{c) } f(t) &= (e^t + e^{-2t})^2 & \text{e) } f(t) &= 2\sin\left(t + \frac{\pi}{3}\right) \\ \text{b) } f(t) &= (t+2)^2 - 2e^{3t} & \text{d) } f(t) &= 2\sin 3t \cdot \cos 5t & \text{f) } f(t) &= e^{-2t} - 3u(t-2) \end{aligned}$$

Exercise 3.3. Find the inverse Laplace transforms of the following functions:

$$\begin{aligned} \text{a) } F(s) &= \frac{3}{s^4} - \frac{2}{s^{\frac{5}{2}}} + \frac{4}{s} & \text{c) } F(s) &= \frac{5-3s}{s^2+9} & \text{e) } F(s) &= \frac{e^{-2s}+5}{s} \\ \text{b) } F(s) &= \frac{3}{s-4} + \frac{10}{s+2} & \text{d) } F(s) &= \frac{10s-3}{s^2+25} & \text{f) } F(s) &= \frac{e^{-\pi s}}{s} - \frac{2s+3}{s^2+4} \end{aligned}$$

3.2 Transformation of initial value problems

Exercise 3.4. Solve the following IVPs:

$$\begin{aligned} \text{a) } \begin{cases} x^{(3)} - x'' - x' + x = e^{2t} \\ x(0) = x'(0) = x''(0) = 0 \end{cases} & \text{c) } \begin{cases} x^{(4)} - 16x = 240 \cos t \\ x(0) = x'(0) = x''(0) = x^{(3)}(0) = 0 \end{cases} \\ \text{b) } \begin{cases} x^{(3)} - 6x'' + 11x' - 6x = 0 \\ x(0) = x'(0) = 0, x''(0) = 2 \end{cases} & \text{d) } \begin{cases} x^{(4)} + 8x'' + 16x = 0 \\ x(0) = x'(0) = x''(0) = 0, x^{(3)}(0) = 1 \end{cases} \end{aligned}$$

Exercise 3.5. Solve the following IVPs

$$\begin{aligned} \text{a) } \begin{cases} y' = 2y + z \\ z' = y + 2z \\ y(0) = 1, z(0) = 3 \end{cases} & \text{b) } \begin{cases} z' + 2y = e^x \\ y' - 2z = 1 + x \\ y(0) = 1, z(0) = 2(HW - submit) \end{cases} \end{aligned}$$

3.3 Translation. Rational functions

Exercise 3.6. Find the Laplace transforms of the following functions:

$$\text{a) } f(t) = t^4 e^{\pi t} \qquad \text{b) } f(t) = e^{-2t} \sin 3t \qquad \text{c) } f(t) = e^t \sin \left(t + \frac{\pi}{3} \right)$$

Exercise 3.7. Find the inverse Laplace transforms of the following functions:

$$\begin{array}{lll} \text{a) } F(s) = \frac{2+s}{s^2-3s+2} & \text{e) } F(s) = \frac{2s-1}{s^2-4} & \text{i) } F(s) = \frac{s^2-2s}{s^4+5s^2+4} \\ \text{b) } F(s) = \frac{3s-2}{s(s^2+4)} & \text{f) } F(s) = \frac{5-2s}{s^2+7s+10} & \text{j) } F(s) = \frac{3s+1}{s^2+4s+4} \\ \text{c) } F(s) = \frac{2s+1}{s^2(s^2+1)} & \text{g) } F(s) = \frac{1}{s^3-5s^2} & \text{k) } F(s) = \frac{3s+5}{s^2-6s+25} \\ \text{d) } F(s) = \frac{1}{s(s+1)(s+2)} & \text{h) } F(s) = \frac{1}{s^4-16} & \text{l) } F(s) = \frac{s^2+3}{(s^2+2s+2)^2} \end{array}$$

3.4 Derivatives, integrals and products of Laplace transforms

Exercise 3.8. Find the Laplace transforms of the following functions:

$$\begin{array}{lll} \text{a) } f(t) = t \cos^2 t & \text{c) } f(t) = t e^{2t} \sin 3t & \text{e) } f(t) = \frac{e^{2t}-1}{t} \\ \text{b) } f(t) = (t - e^{2t})^2 & \text{d) } f(t) = (2t - \sin 3t)^2 & \text{f) } f(t) = \frac{1 - \cos 2t}{t} \end{array}$$

Exercise 3.9. Find the inverse Laplace transforms of the following functions:

$$\text{a) } F(s) = \arctan \frac{1}{s} \qquad \text{b) } F(s) = \ln \frac{s-2}{s+2} \qquad \text{c) } F(s) = \ln \frac{s^2+1}{(s+2)(s-3)}$$

Exercise 3.10. Solve the following IVPs:

$$\begin{array}{ll} \text{a) } \begin{cases} tx'' + (t-2)x' + x = 0 \\ x(0) = 0 \end{cases} & \text{c) } \begin{cases} tx'' + (4t-2)x' + (13t-4)x = 0 \\ x(0) = 0 \end{cases} \\ \text{b) } \begin{cases} tx'' - (4t+1)x' + 2(2t+1)x = 0 \\ x(0) = 0 \end{cases} & \text{d) } \begin{cases} ty'' - ty' + y = 2 \\ y(0) = 2, y'(0) = -4 \end{cases} \end{array}$$

Exercise 3.11. Solve the following IVPs:

$$\begin{array}{l} \text{a) } \begin{cases} x'' - 3x' + 2x = u(t-2) \\ x(0) = 0, x'(0) = 1 \end{cases} \\ \text{b) } \begin{cases} x'' + 4x = \sin t - u(t-2\pi) \sin(t-2\pi) \\ x(0) = 0, x'(0) = 0 \end{cases} \\ \text{c) } \begin{cases} y'' + 2y' + 2y = e^{-(t-1)} u(t-1), \\ y(0) = y'(0) = 0. \end{cases} \end{array}$$

$$\text{d) } \begin{cases} x'' + 4x' + 4x = f(t) \\ x(0) = x'(0) = 0 \end{cases} \quad \text{where } f(t) = \begin{cases} t, & 0 \leq t < 2 \\ 0, & t \geq 2 \end{cases}$$

$$\text{e) } \begin{cases} x'' + x = f(t) \\ x(0) = 0, x'(0) = 1 \end{cases} \quad \text{where } f(t) = \begin{cases} \frac{t}{2}, & 0 \leq t < 6 \\ 3, & t \geq 6 \end{cases}$$

$$\text{f) } \begin{cases} x'' + x = f(t) \\ x(0) = x'(0) = 0 \end{cases} \quad \text{where } f(t) = \begin{cases} \cos t, & 0 \leq t < 2\pi \\ 0, & t \geq 2\pi. \end{cases}$$