Fundamentals of optimization

Linear Programming

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Content

- Linear Programs
- Geometric approach
- Simplex method

Linear programs

Standard form

$$f(x) = c_1x_1 + c_2x_2 + \ldots + c_nx_n \rightarrow \max$$

$$a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n \leq b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n \leq b_2$$

$$\ldots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n \leq b_m$$

$$x_1, x_2, \ \ldots \ x_n \in R, \ x_1, x_2, \ \ldots \ X_n \geq 0$$

Convert general linear program forms into standard form

- $f(x) \rightarrow \min \Leftrightarrow -f(x) \rightarrow \max$
- $g(x) \geq b \Leftrightarrow -g(x) \leq -b$
- $A = B \Leftrightarrow (A \leq B)$ and $(A \geq B)$
- A variable $x_j \in R$ can be represented by $x_j = x_j^+ - x_j^-$ where $x_j^+, x_j^- \geq 0$

Convert general linear program forms into standard form

- Example

$$f(x_1, x_2) = 3x_1 + 2x_2 \rightarrow \min$$

$$2x_1 + x_2 \leq 7$$

$$x_1 + 2x_2 = 8$$

$$x_1 - x_2 \geq 2$$

$$x_1, x_2 \in R, x_2 \geq 0$$

Convert general linear program forms into standard form

- Represent: $x_1 = x_1^+ - x_1^-$

$$
\begin{aligned}
f(x_1^+, x_1^-, x_2) = -3x_1^+ + 3x_1^- - 2x_2 &\rightarrow \max \\
2x_1^+ - 2x_1^- + x_2 &\leq 7 \\
x_1^+ - x_1^- + 2x_2 &\leq 8 \\
-x_1^+ + x_1^- - 2x_2 &\leq -8 \\
-x_1^+ + x_1^- + x_2 &\leq 2 \\
x_1^+, x_1^-, x_2 \in R,\ x_1^+, x_1^-, x_2 &\geq 0
\end{aligned}
$$

Geometric approach

- Constraints (inequalities) form a feasible region
- Optimal points will be one of the corners of the feasible region

Geometric approach

$$
\begin{aligned}
f(x_1, x_2) \quad = \quad & 3x_1 + 2x_2 \rightarrow \max \\
& 2x_1 + x_2 \leq 7 \\
& x_1 + 2x_2 \leq 8 \\
& x_1 - x_2 \leq 2 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

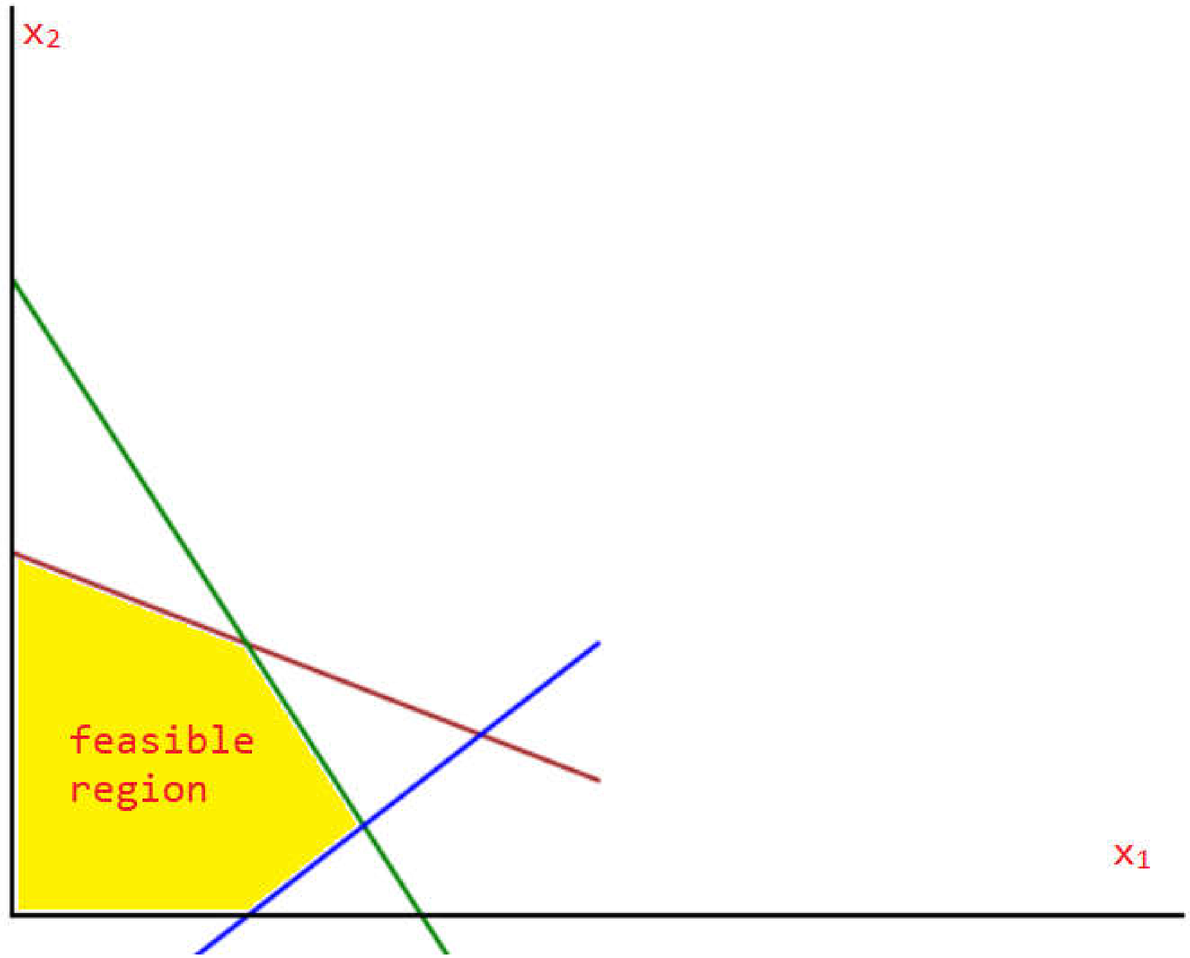
Geometric approach

$2x_1 + x_2 \leq 7$

$x_1 + 2x_2 \leq 8$

$x_1 - x_2 \leq 2$

$x_1, x_2 \geq 0$



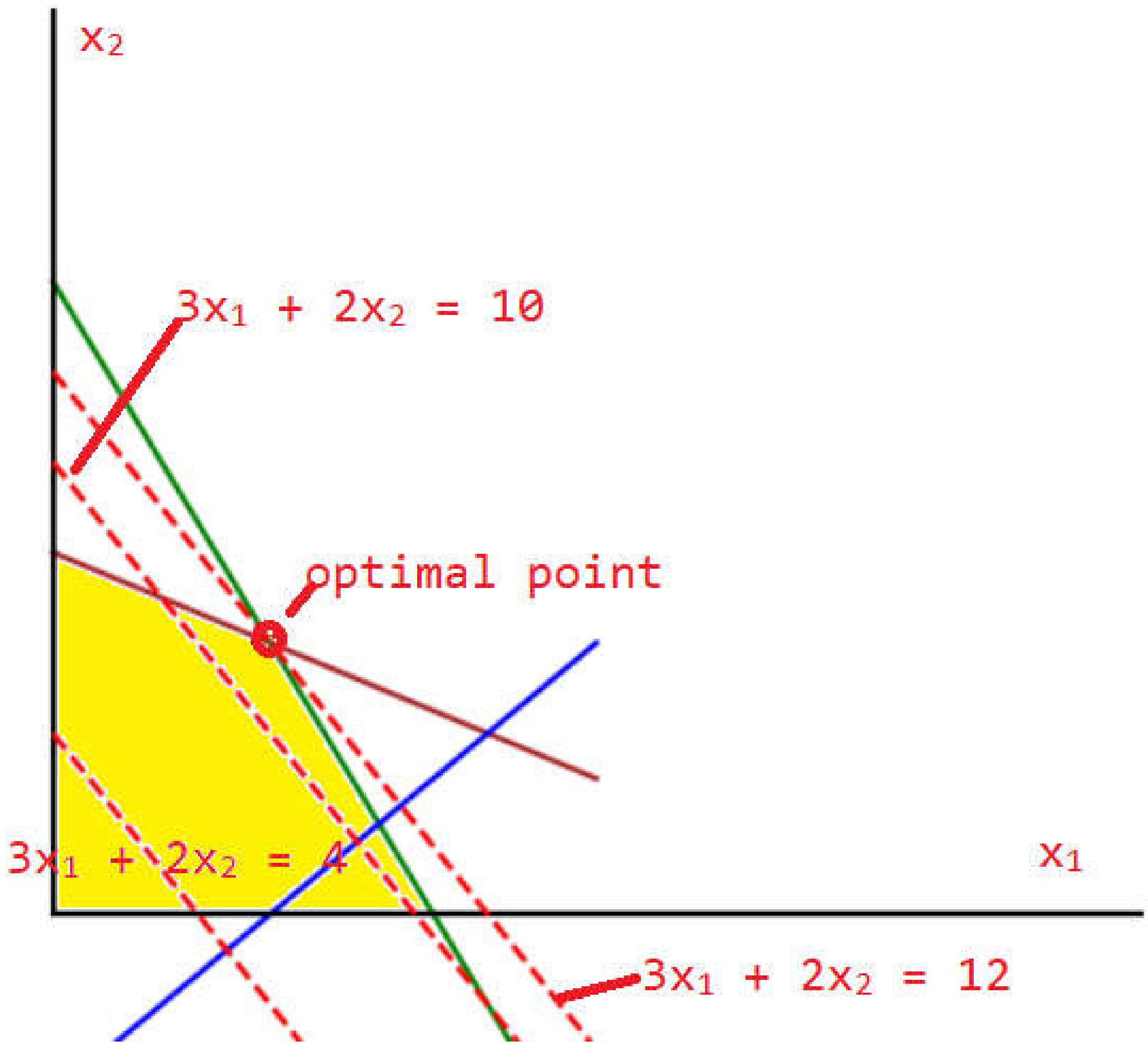
Geometric approach

$2x_1 + x_2 \leq 7$

$x_1 + 2x_2 \leq 8$

$x_1 - x_2 \leq 2$

$x_1, x_2 \geq 0$



Geometric approach

- Special cases
  - Problem does not have optimal solutions
  - Problem does not have feasible solutions

$$f(x_1, x_2) = 3x_1 + 2x_2 \rightarrow \max$$
$$-2x_1 - x_2 \le -7$$
$$x_1 - x_2 \le 2$$
$$x_1, x_2 \in R,\ x_1, x_2 \ge 0$$

$$f(x_1, x_2) = 3x_1 + 2x_2 \rightarrow \max$$
$$2x_1 + x_2 \le 7$$
$$-4x_1 - 2x_2 \le -16$$
$$x_1, x_2 \in R,\ x_1, x_2 \ge 0$$

Example

- A company must decide to make a plan to produce 2 products P1, P2.
  - The revenue received when selling 1 unit of P1 and P2 are respectively 5$ and 7$
  - The manufacturing cost when producing P1 and P2 are respectively 5$ and 3$
  - The storage cost in warehouses for 1 unit of P1 and P2 are respectively 2$ and 3$
- Compute the production plan so that
  - Total manufacturing cost is less than or equal to 200$
  - Total storage cost is less than or equal to 150$
  - Total revenue is maximal

Simplex method

$$f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \ldots + c_nx_n \rightarrow \max$$

$$a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n \le b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n \le b_2$$

$$\ldots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n \le b_m$$

$$x_1, x_2, \ldots, x_n \ge 0$$

Simplex method

Add slack variables $s_1, s_2, \ldots, s_m \geq 0$

$$f(x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m) = c_1x_1 + c_2x_2 + \ldots + c_nx_n \to \max$$

$$\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n + s_1 \qquad\qquad &= b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n + 0 + s_2 + \ldots &= b_2 \\
\ldots \\
a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n + 0 + \ldots + s_m &= b_m \\
x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m \geq 0
\end{aligned}$$

Simplex method

Consider the linear program under augmented form

$f(x) = cx \rightarrow \max$

$Ax = b$

$x \geq 0$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix} \qquad c = (c_1, c_2, \ldots, c_n)$$

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ & \ldots & & \\ a_{m,1} & a_{m,2} & \ldots & a_{m,n} \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{pmatrix}$$

Denote $J = (1,2,\ldots, n)$, $I = (1,2,\ldots, m)$ - set of indices

Basic feasible solutions

- Denote
  - $A(j)$ the j^{th} column of matrix A
  - If $J = (j_1, j_2, \ldots, j_k)$ – vector of indices (we also use J as a set of indices), then $x_J = (x_{j_1}, \ldots, x_{j_k})$, $A(J) = (A(j_1), \ldots, A(j_k))$ submatrix of A composed by selecting columns $A(j_1), \ldots, A(j_k)$
- Suppose rank(A) = m
- $J_B = (j_1, j_2, \ldots, j_m)$ such that columns $A(j_1), \ldots, A(j_m)$ are linear independent, J_N – vector of remaining indices of J (out of J_B)
- $B = A(J_B)$ - basic matrix
- $N = A(J_N)$
- $x = (x_B, x_N)$, $x_B = x(J_B) = (x_{j_1}, \ldots, x_{j_m})$ such that $Bx = b$, $x_B = B^{-1} b$, $x_N = 0$

Basic feasible solutions

Example

$f(x) = cx \to \max$

$Ax = b$

$x \geq 0$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \qquad c = (3, 2, 0, 0, 0)$$

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 7 \\ 8 \\ 2 \end{pmatrix}$$

$J = (1,2,3,4,5)$ – set of variable indices, $I = (1,2,3)$ - set of constraint indices

$J_B = (3,4,5)$, $J_N = (1,2)$,

$$x_B = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 2 \end{pmatrix} \qquad x_N = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad N = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & -1 \end{pmatrix}$$

Simplex method

- Consider another feasible solution $x' = x + \Delta x$

$\rightarrow$ $Ax' = Ax \rightarrow A\Delta x = 0 \rightarrow B\Delta x_B + N\Delta x_N = 0 \rightarrow \Delta x_B = -B^{-1}N\Delta x_N$

$\rightarrow \Delta f = c\Delta x = c_B\Delta x_B + c_N\Delta x_N = -(c_B B^{-1}N - c_N)\Delta x = -\Delta_N \Delta x_N =$

$-\sum_{j\in J_N} \Delta_j \Delta x_j$, where $u = c_B B^{-1}$, $\Delta_N = uN - c_N$

- $\Delta x_j = x_j \geq 0, \forall j \in J_N$
- If $\Delta_j \geq 0, \forall j \in J_N$ then $\Delta f \leq 0$: x is a maximizer

Simplex method

- If there exists index $p \in J_N$ such that $\Delta_p < 0$, build another feasible solution $x' = x + \Delta x$ as follows
  - $\Delta x_p = \theta \geq 0$
  - $\Delta x_j = 0, \forall j \in J_N \setminus \{p\}$
  - $\Delta x_B = -B^{-1}N\Delta x_N = -\theta B^{-1}A(p)$, $x'_B = x_B - \theta B^{-1}A(p)$,
  - $\Delta f = -\theta \Delta_p$

→ with $\theta > 0$ and small, we have $\Delta f > 0$: x is not an optimal solution because x' is better than x.

- If $B^{-1}A(p) \leq 0$, then $x'_B = x_B - \theta B^{-1}A(p) \geq 0, \forall \theta > 0$. It means that $f \rightarrow +\infty$ when $\theta \rightarrow +\infty$ (cannot find maximizer)

Simplex method

- Denote $B^{-1}A(p) = (x_{j_1,p}, x_{j_2,p}, \ldots, x_{j_m,p})^T$ and

$$\theta_i = \begin{cases} \dfrac{x_i}{x_{i,p}} \text{ if } x_{i,p} > 0 \\ +\infty, \text{ otherwise} \end{cases} \qquad \forall i \in J_B$$

→ select $\theta = \theta_q = \min\{\theta_i \mid i \in J_B\}$ (we have $\theta < +\infty$, otherwise the objective function is unbounded)

Simplex method

$J_B = (j_1, j_2, \ldots, j_m)$ s.t. $B = A(J_B)$ is a basic

J_N – vector of remaining indices of J (out of J_B)

While stop condition not reach{

    u = $c_B B^{-1}$, $\Delta_N = uN - c_N$, $x_N = 0$, $x_B = B^{-1}b$

    if $\Delta_N \geq 0$ then {

        print('found optimal solution!'), break

    } else {

        p = argMin$\{\Delta_j \mid j \in J_N\}$

        $Y = B^{-1}A(p)$

        if Y ≤ 0 then {

            print('objective is unbounded'), Return null

        } else {

            q = argMin $\{x_i/Y_i \mid i \in J_B$ s.t. $Y_i > 0\}$

            remove q from J_B and add p to J_B

            remove p from J_N and add q to J_N

            $B = A(J_B)$, $N = A(J_N)$

        }

    }

}

Return (x_B, x_N) where $x_B = B^{-1}b$, $x_N = 0$

Simplex method

Example

$$f(x_1, x_2) = 3x_1 + 2x_2 \rightarrow \max$$

$$\begin{aligned} 2x_1 + x_2 &\leq 7 \\ x_1 + 2x_2 &\leq 8 \\ x_1 - x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Add 3 slack variables x_3, x_4, x_5

$$f(x_1, x_2) = 3x_1 + 2x_2 \rightarrow \max$$

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 7 \\ x_1 + 2x_2 + x_4 &= 8 \\ x_1 - x_2 + x_5 &= 2 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

Simplex method

Init: $J_B = (3,4,5)$, $J_N = (1,2)$

$$x_B = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 2 \end{pmatrix} \quad x_N = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad N = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & -1 \end{pmatrix}$$

$f(x) = 0$

Simplex method

Step 1: $\Delta_N = \begin{pmatrix} -3 & -2 \end{pmatrix}$ → select $p = 1$

$Y = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ → select $q = 5$, $J_B = (1,3,4)$, $J_N = (2,5)$

$x_B = \begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$ $x_N = \begin{bmatrix} x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ $N = \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ -1 & 1 \end{pmatrix}$

$B^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{pmatrix}$

$f(x) = 6$

Simplex method

Step 2: $\Delta_N = \begin{pmatrix} -5 & 3 \end{pmatrix} \rightarrow$ select $p = 2$

$$Y = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix} \quad \rightarrow \text{select } q = 3,\ J_B = (1,2,4),\ J_N = (3,5)$$

$$x_B = \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} \quad x_N = \begin{pmatrix} x_3 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & -1 & 0 \end{pmatrix} \quad N = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} 1/3 & 0 & 1/3 \\ 1/3 & 0 & -2/3 \\ -1 & 1 & 1 \end{pmatrix}$$

$f(x) = 11$

Simplex method

Step 3: $\Delta_N = \begin{pmatrix} 5/3 & -1/3 \end{pmatrix}$ → select $p = 5$

$Y = \begin{pmatrix} 1/3 \\ -2/3 \\ 1 \end{pmatrix}$ → select $q = 4$, $J_B = (1,2,5)$, $J_N = (3,4)$

$x_B = \begin{pmatrix} x_1 \\ x_2 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$ $\quad x_N = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix}$ $\quad N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

$B^{-1} = \begin{pmatrix} 2/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ -1 & 1 & 1 \end{pmatrix}$

f(x) = 12

Simplex method

Step 3: $\Delta_N =$ $(4/3 \quad 1/3) > 0$ → STOP, found optimal solution

$$x_B = \begin{pmatrix} x_1 \\ x_2 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} \qquad x_N = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$f(x) = 12$

Simplex method
Tabular form

c_B	B	x_B	c_1	…	c_p	…	c_n	θ
			A_1	…	A_p	…	A_n	
c_{j_1}	A_{j_1}	x_{j_1}	$x_{j_1,1}$		$x_{j_1,p}$		$x_{j_1,n}$	θ_{j_1}
…	…	…			…			…
c_q	A_{i_0}	x_q	$x_{q,1}$		$x_{q,p}$		$x_{q,n}$	θ_q
…	…	…			…			…
c_{j_m}	A_{j_m}	x_{j_m}	$x_{j_m,1}$		$x_{j_m,p}$		$x_{j_m,n}$	θ_{j_m}
Δ			Δ_1	…	Δ_p	…	Δ_n	

Simplex method
Tabular form

- Update schema
  - Update $x_{i,j}$: $x'_{i,j} = x_{i,j} - (x_{q,j} * x_{i,p}) / x_{q,p}$, $\forall\, i \in J_B \setminus \{q\}$, $\forall\, j \in J \setminus \{p\}$
  - $\Delta'_j = \Delta_j - (x_{q,j} * \Delta_p) / x_{q,p}$, $\forall\, j \in J \setminus \{p\}$
  - $J_B = J_B \setminus \{q\} \cup \{p\}$
  - On row q: $x'_{q,j} = x_{q,j} / x_{q,p}$, $\forall\, j \in J$
  - On column p: $x'_{q,p} = 1$, $x'_{i,p} = 0$, $\forall\, i \in J_B \setminus \{q\}$
  - $x'_q = x_q / x_{q,p}$
  - $\Delta_p = 0$

Simplex method
Tabular form

Example

$$f(x_1, x_2, x_3, x_4, x_5) = 3x_1 + 2x_2 \rightarrow \max$$

$$2x_1 + x_2 + x_3 = 7$$

$$x_1 + 2x_2 + x_4 = 8$$

$$x_1 - x_2 + x_5 = 2$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Simplex method
Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
0	A_3	7	2	1	1	0	0	
0	A_4	8	1	2	0	1	0	
0	A_5	2	1	-1	0	0	1	
Δ			-3	-2	0	0	0	

Simplex method
Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
0	A_3	7	2	1	1	0	0	
0	A_4	8	1	2	0	1	0	
0	A_5	2	1	-1	0	0	1	
Δ			-3	-2	0	0	0	

Select p = 1

Simplex method
Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
0	A_3	7	2	1	1	0	0	7/2
0	A_4	8	1	2	0	1	0	8
0	A_5	2	1	-1	0	0	1	2
Δ			-3	-2	0	0	0	

Select p = 1

Simplex method
Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
0	A_3	7	2	1	1	0	0	7/2
0	A_4	8	1	2	0	1	0	8
0	A_5	2	1	-1	0	0	1	2
Δ			-3	-2	0	0	0	

Select p = 1

Select q = 5

Simplex method
Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
0	A_3	3	2	3	1	0	-2	
0	A_4	6	1	3	0	1	-1	
0	A_5	2	1	-1	0	0	1	
Δ			-3	-5	0	0	3	

Update Δ, x_B, $x_{i,j}$ except row A5, and except columns 1

Simplex method
Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
0	A_3	3	0	3	1	0	-2	
0	A_4	6	0	3	0	1	-1	
3	A_1	2	1	-1	0	0	1	
Δ			0	-5	0	0	3	

Replace A5 by A1 in the B, Update row corresponding to A1, and column 1

Simplex method
Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
0	A_3	3	0	3	1	0	-2	
0	A_4	6	0	3	0	1	-1	
3	A_1	2	1	-1	0	0	1	
Δ			0	-5	0	0	3	

Select p = 2

Simplex method
Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
0	A_3	3	0	3	1	0	-2	1
0	A_4	6	0	3	0	1	-1	2
3	A_1	2	1	-1	0	0	1	$+\infty$
Δ			0	-5	0	0	3	

Select p = 2

Select q = 1

Simplex method
Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
0	A_3	3	0	3	1	0	-2	
0	A_4	3	0	3	-1	1	1	
3	A_1	3	1	-1	1/3	0	1/3	
Δ			0	-5	5/3	0	-1/3	

Update Δ, x_B, $x_{i,j}$ except row A3, and except columns 2

Simplex method
Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
2	A_2	1	0	1	1/3	0	-2/3	
0	A_4	3	0	0	-1	1	1	
3	A_1	3	1	0	1/3	0	1/3	
Δ			0	0	5/3	0	-1/3	

Replace A_3 by A_2 in B, update x_B, $x_{i,j}$ on row A_3, and columns 2

Simplex method
Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
2	A_2	1	0	1	1/3	0	-2/3	$+\infty$
0	A_4	3	0	0	-1	1	1	3
3	A_1	3	1	0	1/3	0	1/3	$+\infty$
Δ			0	0	5/3	0	-1/3	

Select p = 5

Simplex method
Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
2	A_2	1	0	1	1/3	0	-2/3	$+\infty$
0	A_4	3	0	0	-1	1	1	3
3	A_1	3	1	0	1/3	0	1/3	$+\infty$
Δ			0	0	5/3	0	-1/3	

Select p = 5

Select q = 4

Simplex method
Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
2	A_2	3	0	1	-1	2/3	-2/3	
0	A_4	3	0	0	-1	1	1	
3	A_1	2	1	0	2/3	-1/3	1/3	
	Δ		0	0	4/3	1/3	-1/3	

Simplex method
Tabular form

c_B	B	x_B	3	2	0	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
2	A_2	3	0	1	-1/3	2/3	0	
0	A_5	3	0	0	-1	1	1	
3	A_1	2	1	0	2/3	-1/3	0	
Δ			0	0	4/3	1/3	0	

STOP, found optimal solution!!!!