Preferential Attachment

Introduction to Network Science
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Topic 05

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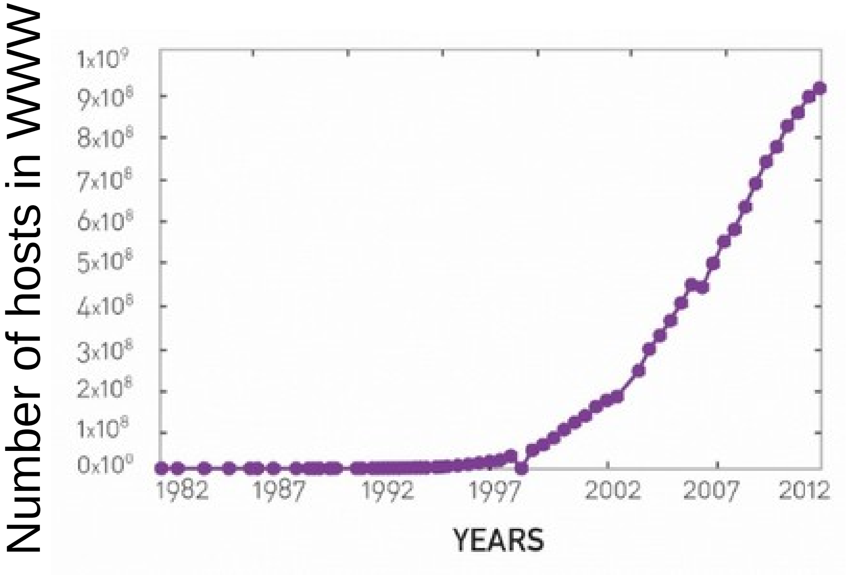
Contents

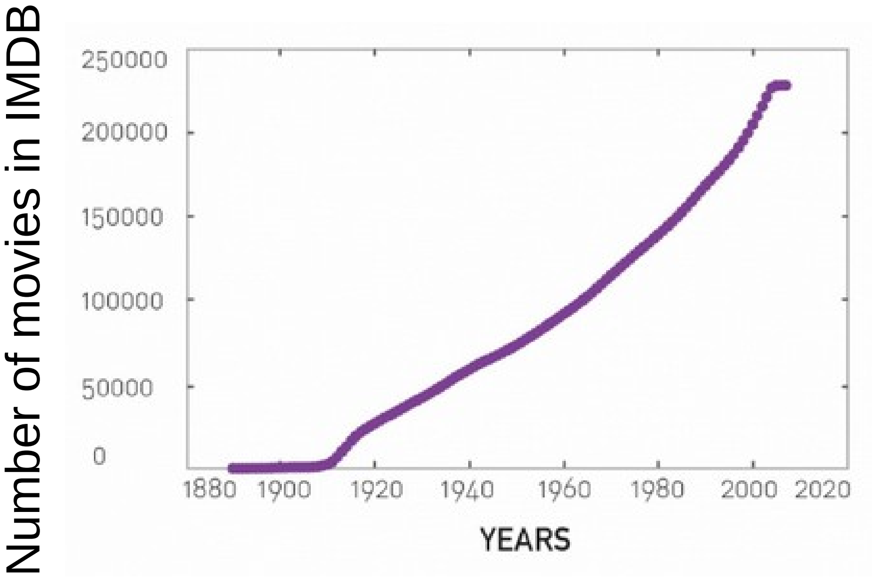
- The uniform random attachment model
- The BA or preferential attachment model
- Degree distribution under the BA model
- Distance distribution under the BA model
- Clustering coefficient under the BA model

Sources

- Albert László Barabási (2016) Network Science
  - Preferential attachment follows chapter 05
- Ravi Srinivasan 2013 Complex Networks Ch 12
- Networks, Crowds, and Markets Ch 18
- Data-Driven Social Analytics course by Vicenç Gómez and Andreas Kaltenbrunner

The number of nodes N increases: we need models of network growth





Preliminary:
Uniform Random Attachment

Growth in an ER network

- Two assumptions in ER networks:
  - There are N nodes that **pre-exist**
  - Nodes connect **at random**
- Let’s challenge the first assumption

Uniform Attachment

- Network starts with m fully-connected nodes
- Time starts at $t_0=m$
- At every time step we add 1 node
- This node will have m outlinks

Expected degree over time

- Probability of obtaining one link: m/t
  - Decreases over time
- Expected degree of node born at $m < i < t$

$$m + \frac{m}{i} + \frac{m}{i+1} + \frac{m}{i+2} + \cdots + \frac{m}{t} \approx m\left(1 + \log\left(\frac{t}{i}\right)\right)$$

Tail of degree distribution

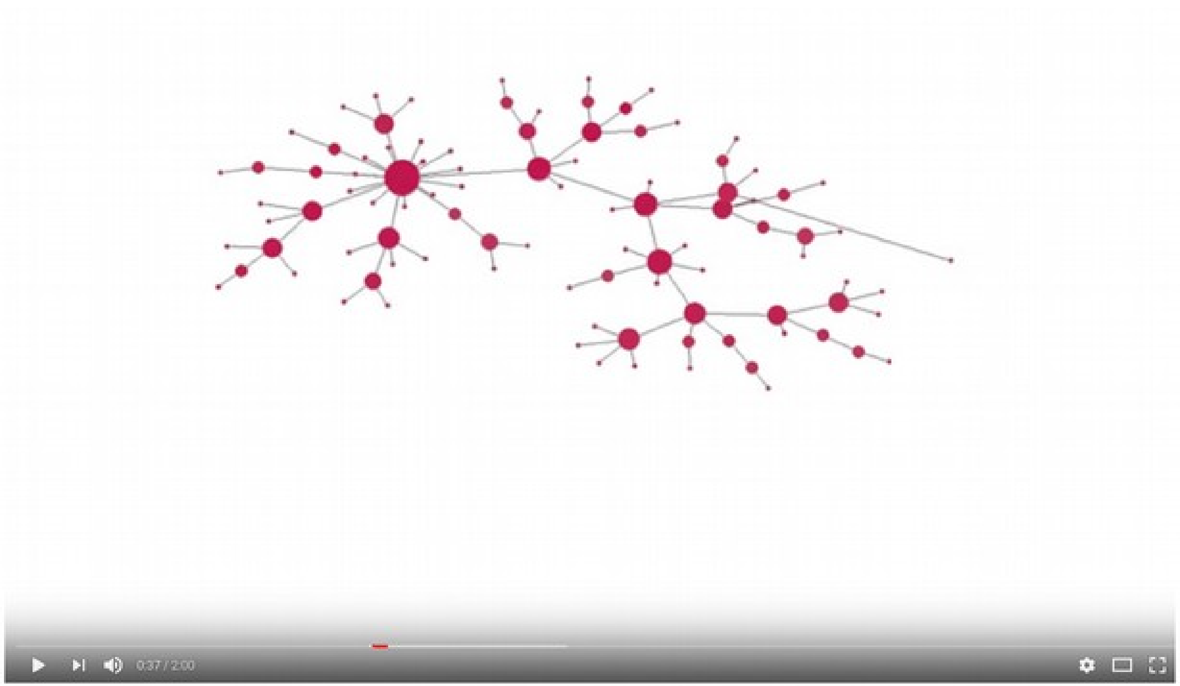
- How many nodes of degree larger than K are there at time t? (Computation in "Advanced materials" at the end of these slides)

$$e^{-\frac{K-m}{m}}$$

- Decreases exponentially with *K*: it's vanishingly rare to find high-degree nodes

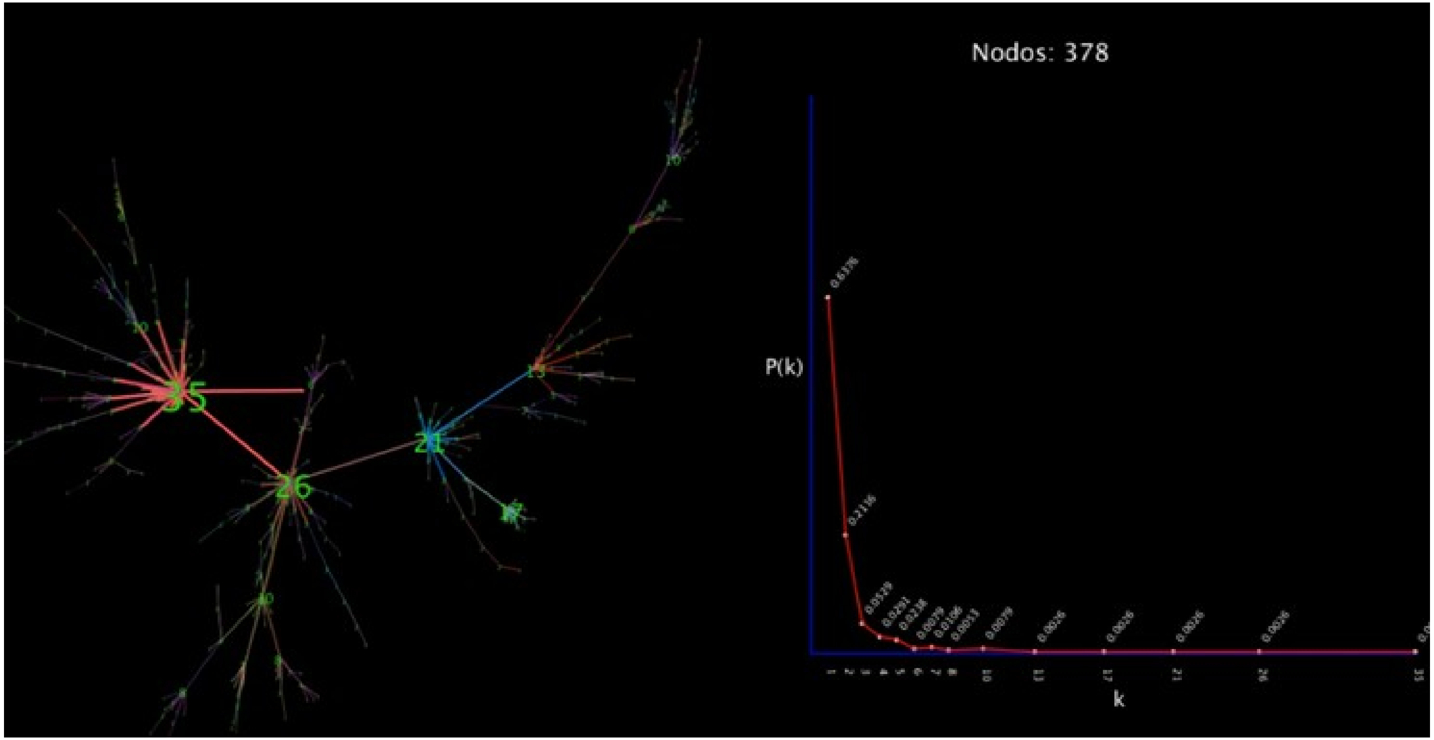
Preferential Attachment

Preferential attachment simulation



https://www.youtube.com/watch?v=4GDqJVtPEGg

Degree distribution in simulation



https://www.youtube.com/watch?v=5RIQweqPT6A

We have seen what but not why

- Power-law degree distributions are prevalent
  - Why?
- Two assumptions in ER networks:
  - There are N nodes that **pre-exist**
  - Nodes connect **at random**
- Let’s challenge both assumptions

Growth

- Suppose there are two web pages on a topic, one with many inlinks the other with few, which one am I most likely to link to?
- Which scientific papers are read? Which are cited?
- Which actors are more sought after for new movies?


BURNS
UNIVERSITY
Ut
Uberiores
Divites
Our motto: *Ut uberiores divites.*
FOX 32
CHICAGO

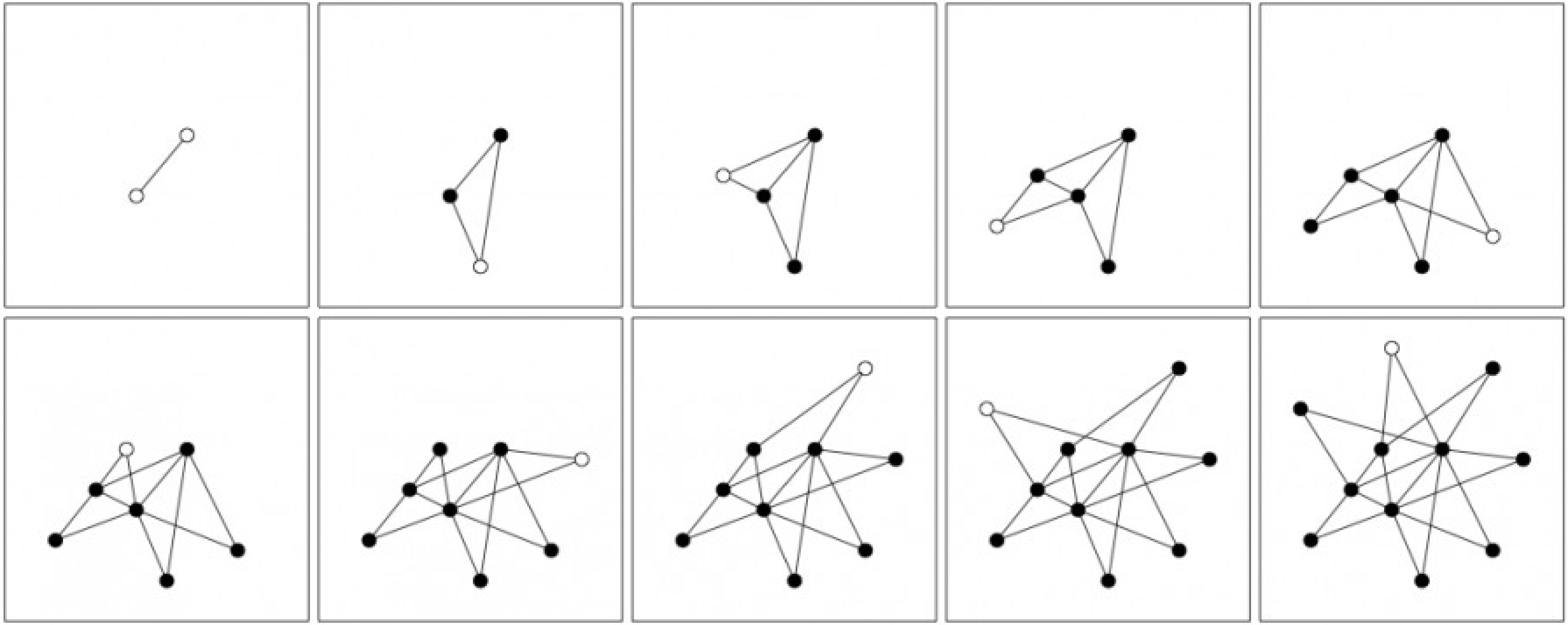
The Barabási-Albert (BA) model

- Network starts with m_0 nodes connected arbitrarily as long as their degree is ≥ 1
- At every time step we add 1 node
- This node will have $m \leq m_0$ outlinks
- The probability of an existing node of degree k_i to gain one such link is

$$\Pi(k_i) = \frac{k_i}{\sum_{j=1}^{N-1} k_j}$$

In an ER network, $\Pi(k_i) = \frac{1}{N-1}$

Example ($m_0 = 2; m=2$)



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$$\Pi(k_i) = \frac{k_i}{\sum_{j=1}^{N-1} k_j}$$

Write the formula for $N(t)$ and $L(t)$: at $t=0$ the network has m_0 nodes and $L(0)$ links

Degree $k_i(t)$ as a function of time

$$\frac{d}{dt}k_i = m\Pi(k_i) = m\frac{k_i}{\sum_{j=1}^{N-1} k_j}$$

$$\sum_{j=1}^{N-1} k_j = 2m(t-1)$$ (All nodes minus the current)

$$\frac{d}{dt}k_i = \frac{k_i}{2t-2} \approx \frac{k_i}{2t}$$ (For large t)

Degree $k_i(t)$ ... continued

$$\frac{d}{dt}k_i(t) = \frac{k_i(t)}{2t}$$

$$\frac{1}{k_i(t)}\frac{d}{dt}k_i(t) = \frac{1}{2t}$$

$$\int_{t=t_i}^{t}\frac{1}{k_i(t)}\frac{d}{dt}k_i(t)dt = \int_{t=t_i}^{t}\frac{1}{2t}dt$$

$$\log k_i(t) - \log k_i(t_i) = \frac{1}{2}\log t - \frac{1}{2}\log t_i$$

$$\log k_i(t) = \frac{1}{2}\log t - \frac{1}{2}\log t_i + \log m$$

Note: in exams for this course, you will **not** be asked to solve differential equations on your own

(t_i is the creation time of node i)

Degree $k_i(t)$ … continued

$$\log k_i(t) = \frac{1}{2}\log t - \frac{1}{2}\log t_i + \log m$$

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{\frac{1}{2}}$$

Is the degree growth linear, super-linear, or sub-linear? Intuitively, why?

Degree $k_i(t)$ … consequences

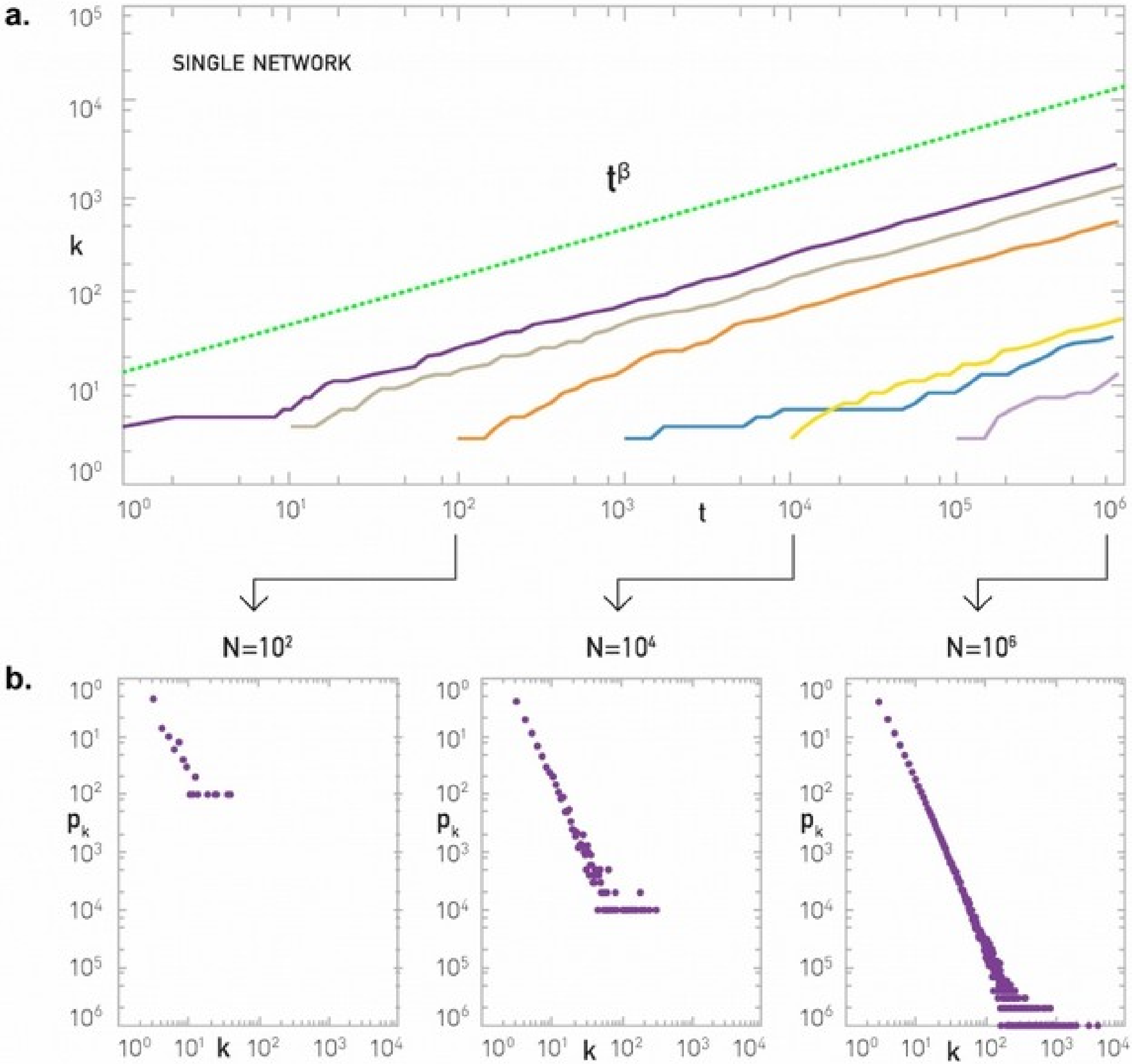
$$\log k_i(t) = \frac{1}{2}\log t - \frac{1}{2}\log t_i + \log m$$

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{\frac{1}{2}}$$

$$\frac{dk_i(t)}{dt} = \frac{k_i(t)}{2t} = \frac{m\left(\frac{t}{t_i}\right)^{\frac{1}{2}}}{2t} = \frac{m}{2\left(t \cdot t_i\right)^{\frac{1}{2}}}$$

If $t_i < t_j$ (node i is older than node j), what do we expect of k_i and k_j?

Simulation results



Model

Nodes with t_i = 1, 10, 100, 1000, 10000, ...

Degree distribution

- Let's calculate the CDF of the degree distribution

$$
\begin{aligned}
Pr(k_i \leq k) &= 1 - Pr(k_i > k) \\
&= 1 - Pr\left(m\left(\frac{t}{t_i}\right)^{\beta} > k\right) \\
&= 1 - Pr\left(\left(\frac{m}{k}\right)^{1/\beta} > \frac{t_i}{t}\right) \\
&= 1 - \left(\frac{m}{k}\right)^{1/\beta}
\end{aligned}
$$

$$\frac{ti}{t} \sim \mathrm{Uniform}(0,1)$$

Degree distribution

Now let's take the derivative of the CDF to obtain the PDF

$$p_k = \frac{d}{dk}Pr(k_i \leq k) = \frac{d}{dk}\left(1 - \left(\frac{m}{k}\right)^{1/\beta}\right)$$

$$= -\frac{d}{dk}\left(\left(\frac{m}{k}\right)^{1/\beta}\right) = -m^{1/\beta}\frac{d}{dk}\left(\frac{1}{k^{1/\beta}}\right)$$

$$= \frac{1}{\beta}\frac{m^{1/\beta}}{k^{1/\beta+1}} \qquad (\beta = 1/2)$$

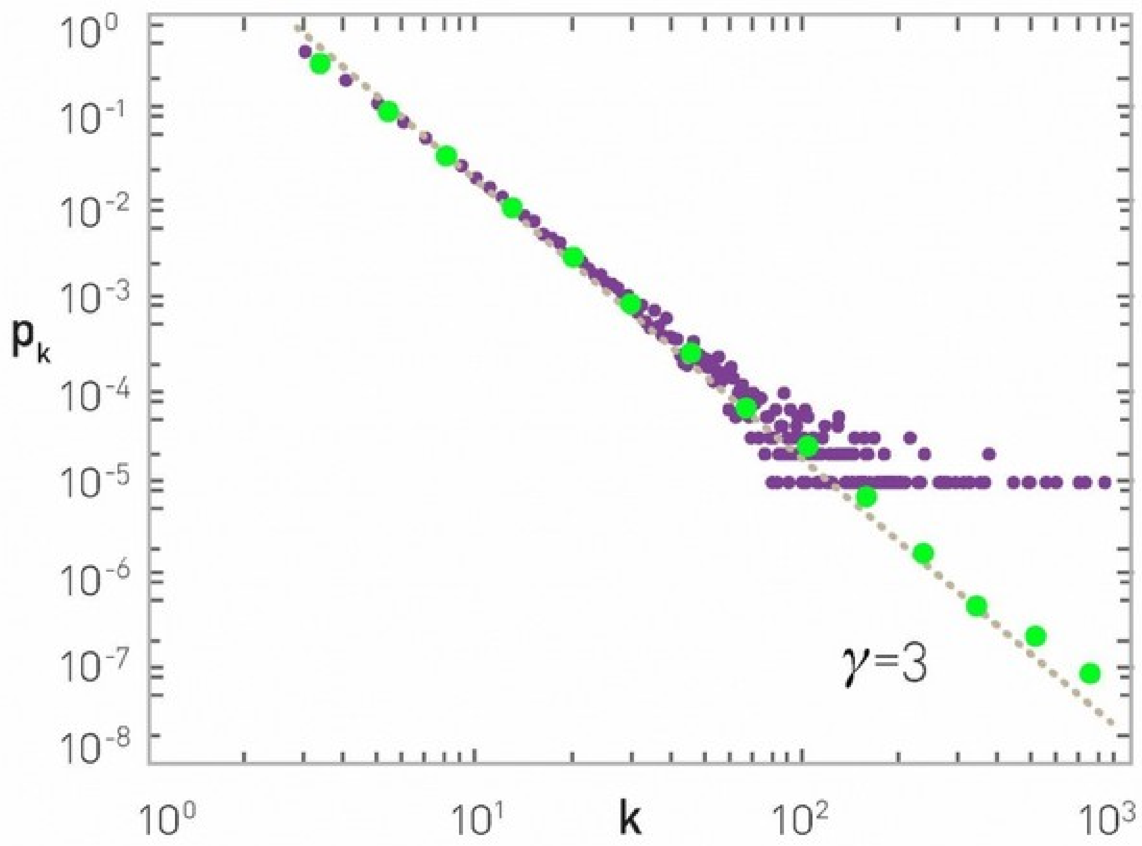
$$= 2\frac{m^2}{k^3} \longrightarrow p(k) \propto k^{-3}$$

Degree distribution

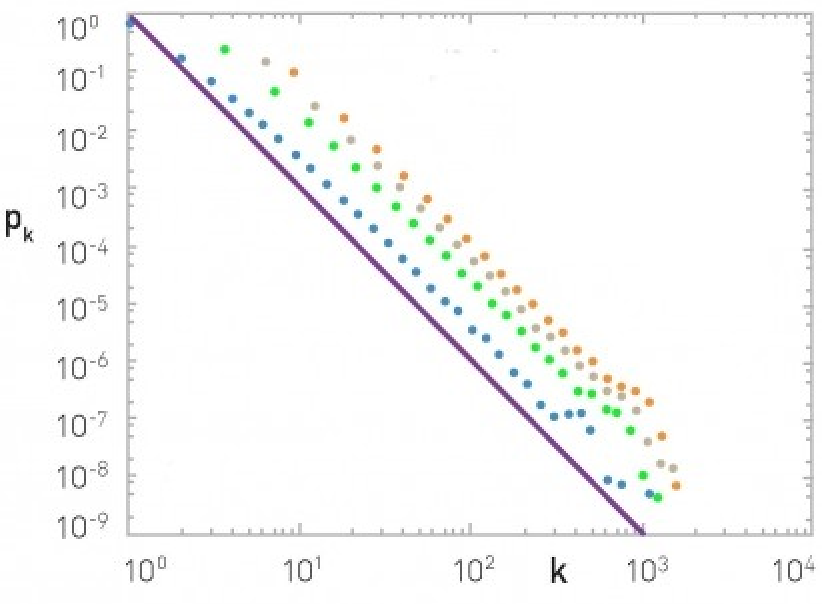
- $\beta = 1/2$ is called the dynamical exponent
- $\gamma = \frac{1}{\beta} + 1 = 3$ is the power-law exponent

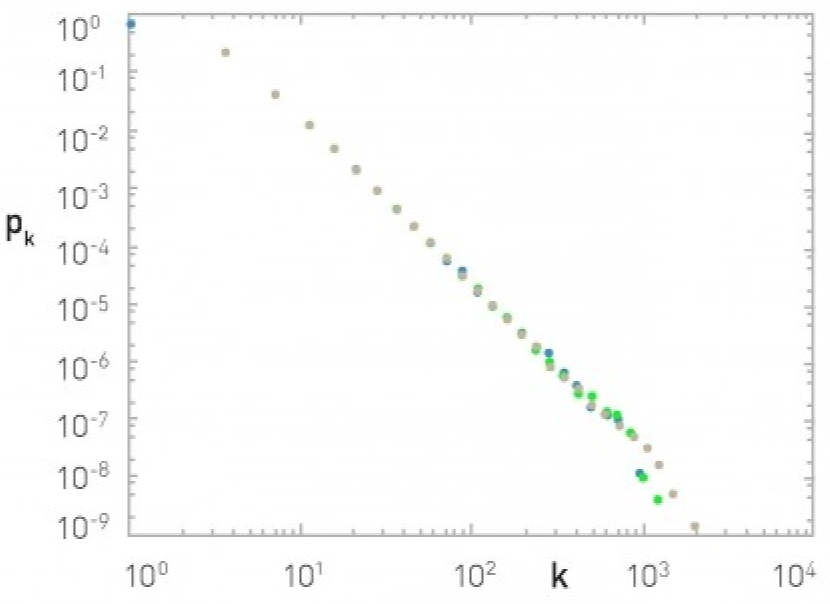
- Note that $p(k) \approx 2m^2/k^3$
  does not depend on t
  hence, it describes a stationary network

Degree distribution, simulation results
N=100,000   m=3



More simulations





N = 100,000; m_0 = m =

1 (blue), 3 (green), 5 (gray), 7 (orange)

Observe γ is independent of m (and m_0)

The slope of the purple line is -3

m_0 = m = 3; N =

50K (blue), 100K (green), 200K (gray)

Observe p_k is independent of N

Processes that generate scale-free networks

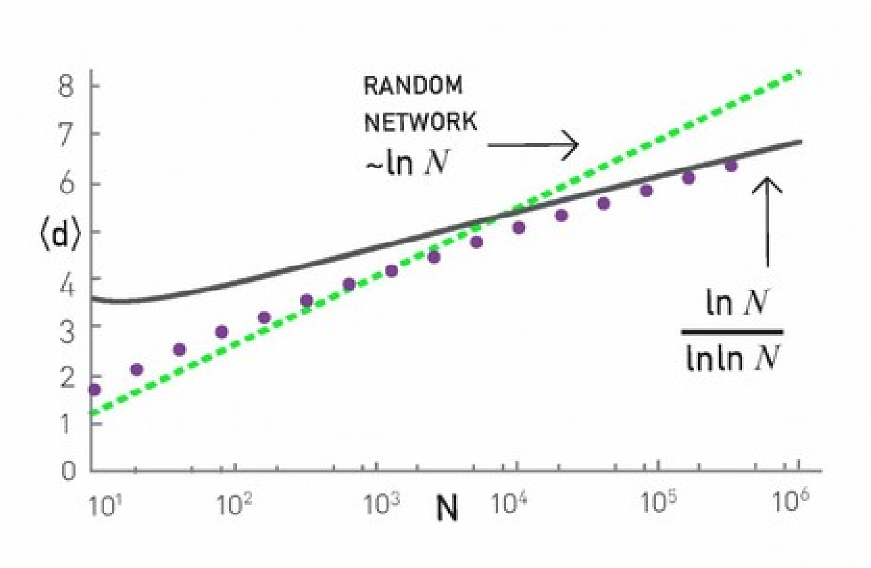
- Link-selection model (step)
  - Add one new node v to the network
  - Select an existing link at random and connect v to one of the edges of that existing link
- Copy model (step)
  - Add one new node v to the network
  - Pick a random existing node u
  - With probability p link to u
  - With probability 1-p link to a neighbor of u

Average distance

- Distances grow slower than log N

$$\langle d \rangle \approx \frac{\log N}{\log \log N}$$

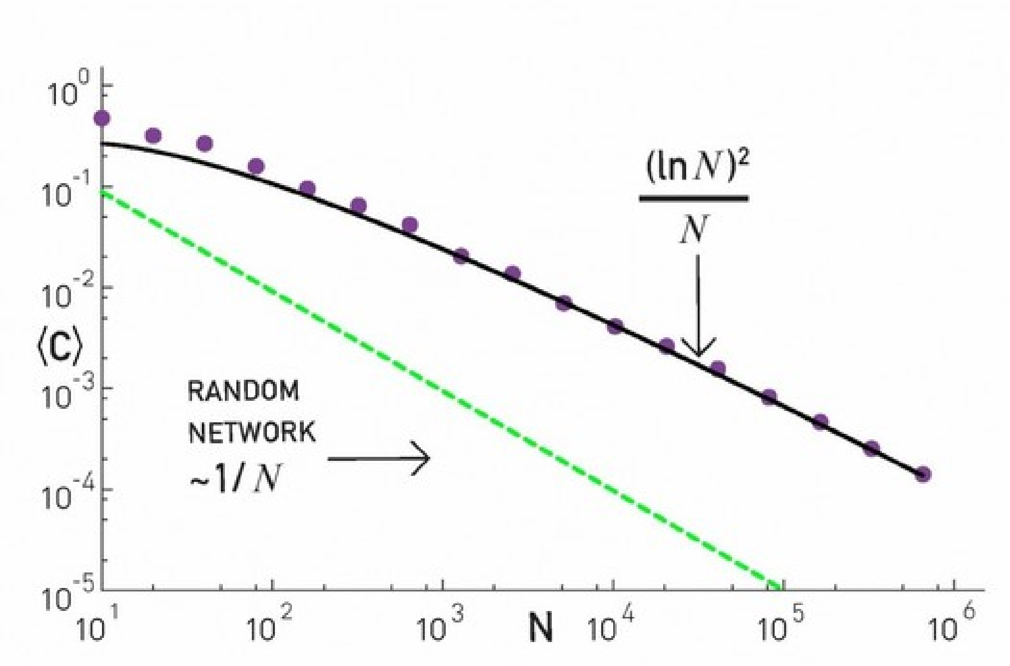
(Scale free network with $\gamma = 3$)



Clustering coefficient

- BA networks are locally more clustered than ER networks

$$\langle C \rangle \approx \frac{(\log N)^2}{N}$$



Limitations of the BA model

- Predicts a fixed exponent of -3
- Assumes an undirected network, while many real complex networks are directed
- Does not consider node deletions or edge deletions which are common in practice
- Considers that all nodes are equal except for their arrival times

Exercise: the copy model

In the copy model, start at $t=1$ with one node, and at every step t :

- Add one new node v to the network
- Pick a random existing node u
- If u has no out-links, link to u
- If u has out-links:
  - With probability p link to u
  - With probability $1-p$ link to one of the out-links of u chosen at random

- Simulate it on paper for 7 nodes with $p=0.5$
  - Make sure you understand the model fully!
- What is $N(t)$ and $L(t)$?

In the copy model, at every step t:

1) Add one new node v to the network
2) Pick a random existing node u
3) With probability p link to u
4) With probability 1-p link to a neighbor of u

- What is k_i^{out} ?
- We will compute k_i^{in}
- How many links on average gets node i at time t? In other words, what is …

$$\frac{d}{dt} k_i^{\mathrm{in}}(t)$$

- Hint: it has a term with p and a term with $1\text{-}p$

Practice on your own

- Try to reconstruct the derivations we have done in class, including the exercise
  - Try to understand every step
- Insert a small change in the model and try to recalculate what we have done

Advanced materials #1: Copy model cont. (not included in the exam)

- Integrate between t_i and t to obtain an expression for $k_i(t_i)$
(***we drop the "in" superscript just for simplicity during this exercise***)
- Note that now $k_i(t_i) = 0$

- Once you have a expression for $k_i(t_i)$
- Compute $Pr(k_i(t_i) > k)$
- Now write the cumulative distribution function of $k_i(t_i)$
- And compute its derivative to obtain

$$p_k = Pr(k_i(t) = k) = \frac{d}{dk} Pr(k_i(t) \leq k)$$

- It should show exponent $\gamma = \dfrac{2-p}{1-p}$

Advanced materials #2: Expected degree under uniform random attachment (not included in the exam)

Expected degree in uniform random attachment using a differential equation

$$\frac{d}{dt}k_i(t) = \frac{m}{t}$$

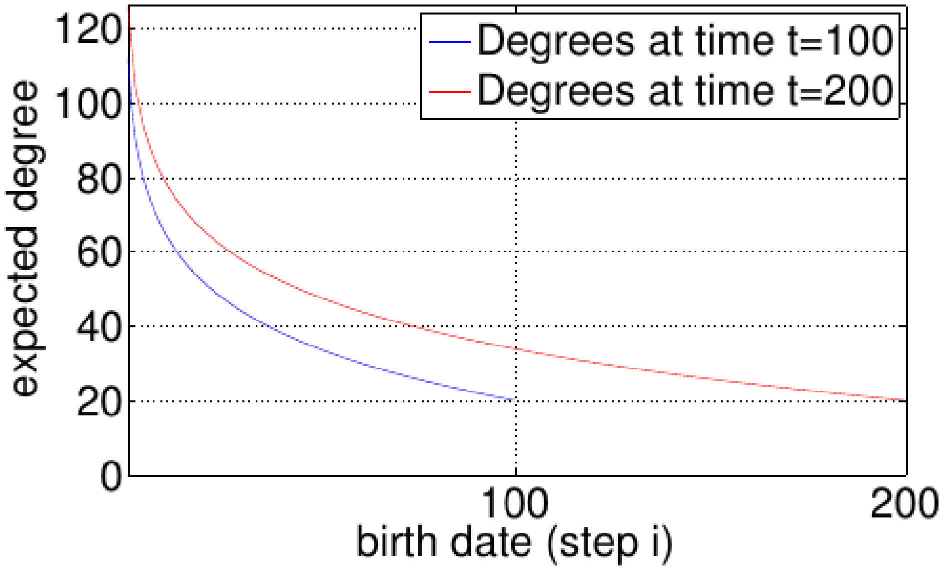
(1) Integrate between time i and time t

(2) Use initial condition $k_i(i) = m$

$$\int \frac{1}{t} = \log t + C$$

Degree distribution over time is not static

Degree of node born at time $m < i < t = m\left(1 + \log\left(\frac{t}{i}\right)\right)$



Tail of degree distribution

$$m\left(1+\log\left(\frac{t}{i}\right)\right) > K$$

$$1+\log\left(\frac{t}{i}\right) > \frac{K}{m}$$

$$\log\left(\frac{t}{i}\right) > \frac{K-m}{m}$$

$$\frac{t}{i} > e^{\frac{K-m}{m}}$$

$$i < te^{-\frac{K-m}{m}}$$

How many nodes of degree larger than K are there at time t?

The fraction is $$\frac{te^{-\frac{K-m}{m}}}{t} = e^{-\frac{K-m}{m}}$$

Decreases exponentially with K: it's vanishingly rare to find high-degree nodes