

Preferential Attachment

Introduction to Network Science

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Topic 05

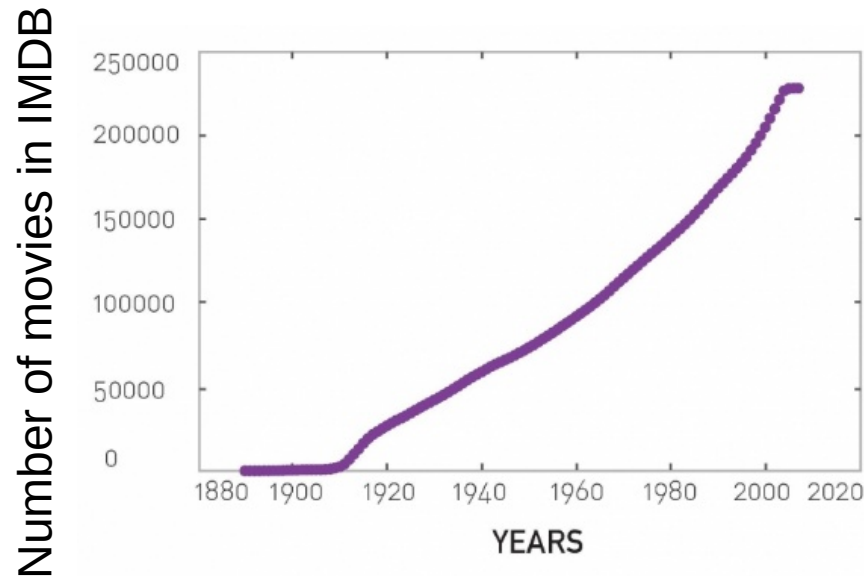
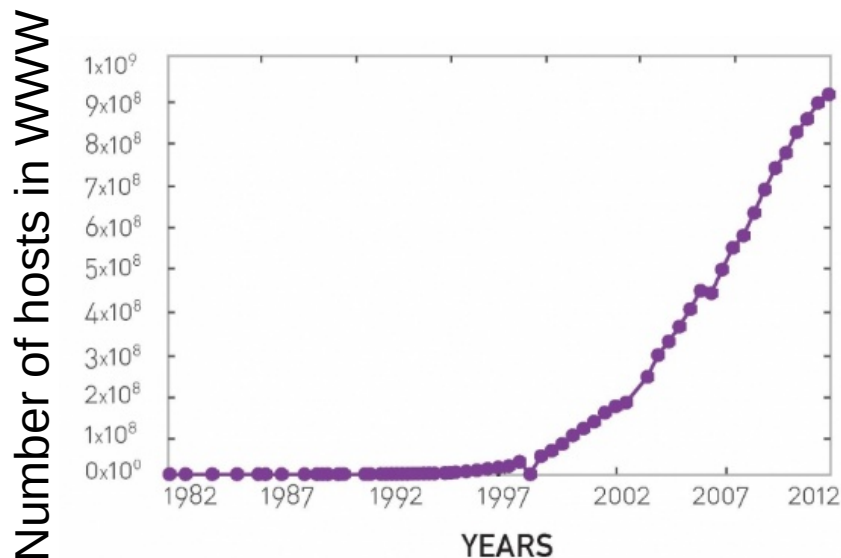
Contents

- The uniform random attachment model
- The BA or preferential attachment model
- Degree distribution under the BA model
- Distance distribution under the BA model
- Clustering coefficient under the BA model

Sources

- Albert-László Barabási (2016) Network Science
 - Preferential attachment follows [chapter 05](#)
- [Ravi Srinivasan 2013 Complex Networks Ch 12](#)
- [Networks, Crowds, and Markets Ch 18](#)
- [Data-Driven Social Analytics](#) course by Vicenç Gómez and Andreas Kaltenbrunner

The number of nodes N increases: we need models of network growth



Preliminary: Uniform Random Attachment

Growth in an ER network

- Two assumptions in ER networks:
 - There are N nodes that **pre-exist**
 - Nodes connect **at random**
- Let's challenge the first assumption

Uniform Attachment

- Network starts with m fully-connected nodes
- Time starts at $t_0=m$
- At every time step we add 1 node
- This node will have m outlinks

Expected degree over time

- Probability of obtaining one link: m/t
 - Decreases over time
- Expected degree of node born at $m < i < t$

$$m + \frac{m}{i} + \frac{m}{i+1} + \frac{m}{i+2} + \dots + \frac{m}{t} \approx m \left(1 + \log \left(\frac{t}{i} \right) \right)$$

Tail of degree distribution

- How many nodes of degree larger than K are there at time t ? (Computation in “Advanced materials” at the end of these slides)

$$e^{-\frac{K-m}{m}}$$

- Decreases exponentially with K : it's vanishingly rare to find high-degree nodes

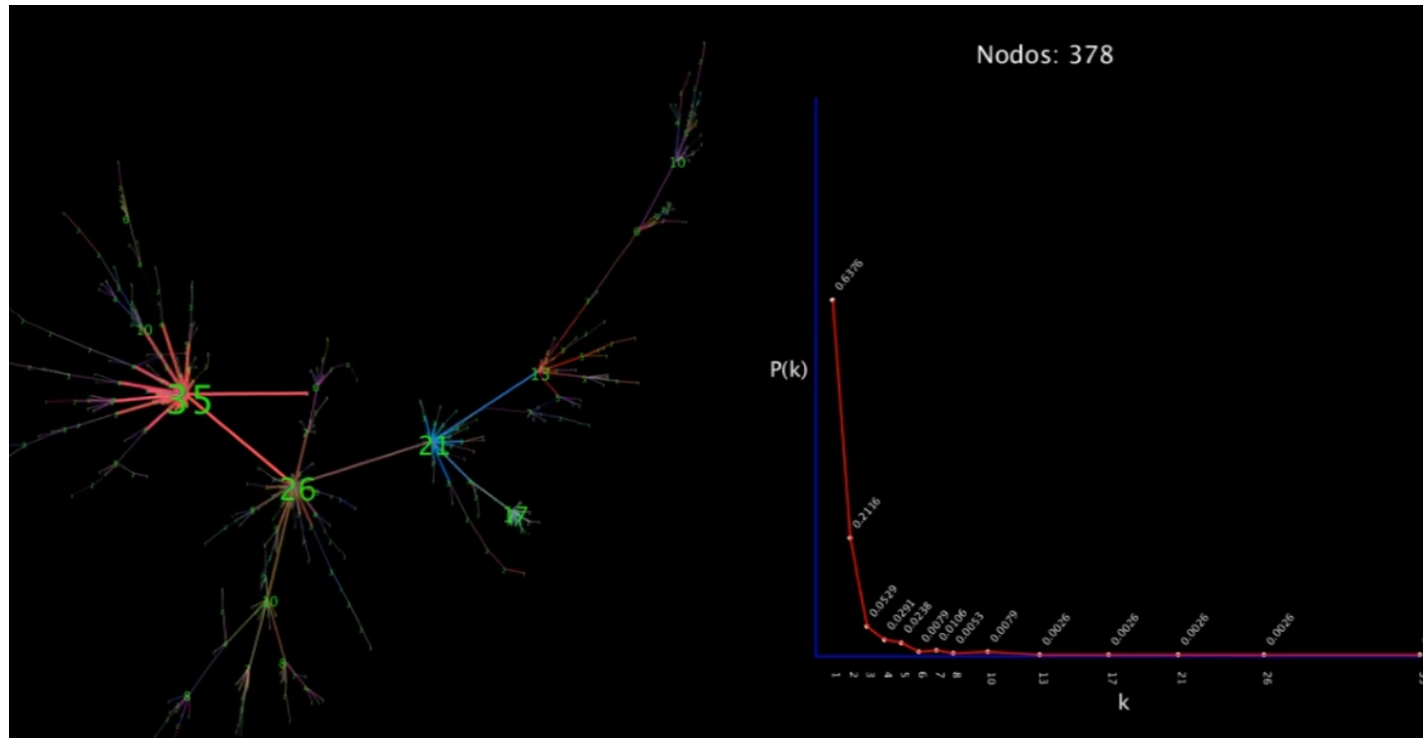
Preferential Attachment

Preferential attachment simulation



<https://www.youtube.com/watch?v=4GDqJVtPEGg>

Degree distribution in simulation



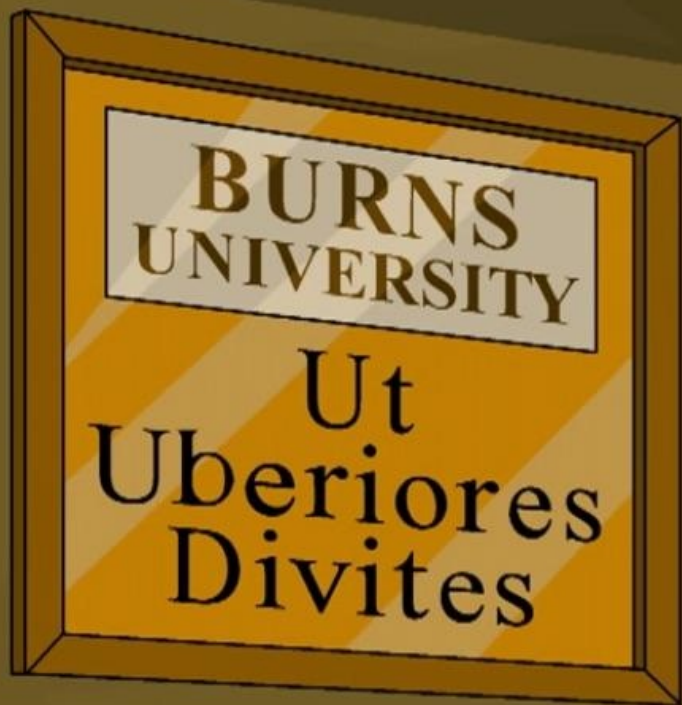
<https://www.youtube.com/watch?v=5RIQweqPT6A>

We have seen what but not why

- Power-law degree distributions are prevalent
 - Why?
- Two assumptions in ER networks:
 - There are N nodes that **pre-exist**
 - Nodes connect **at random**
- Let's challenge both assumptions

Growth

- Suppose there are two web pages on a topic, one with many inlinks the other with few, which one am I most likely to link to?
- Which scientific papers are read?
- Which book authors sell more?
- Which actors are more sought after?



Our motto: *Ut uberiores divites.*

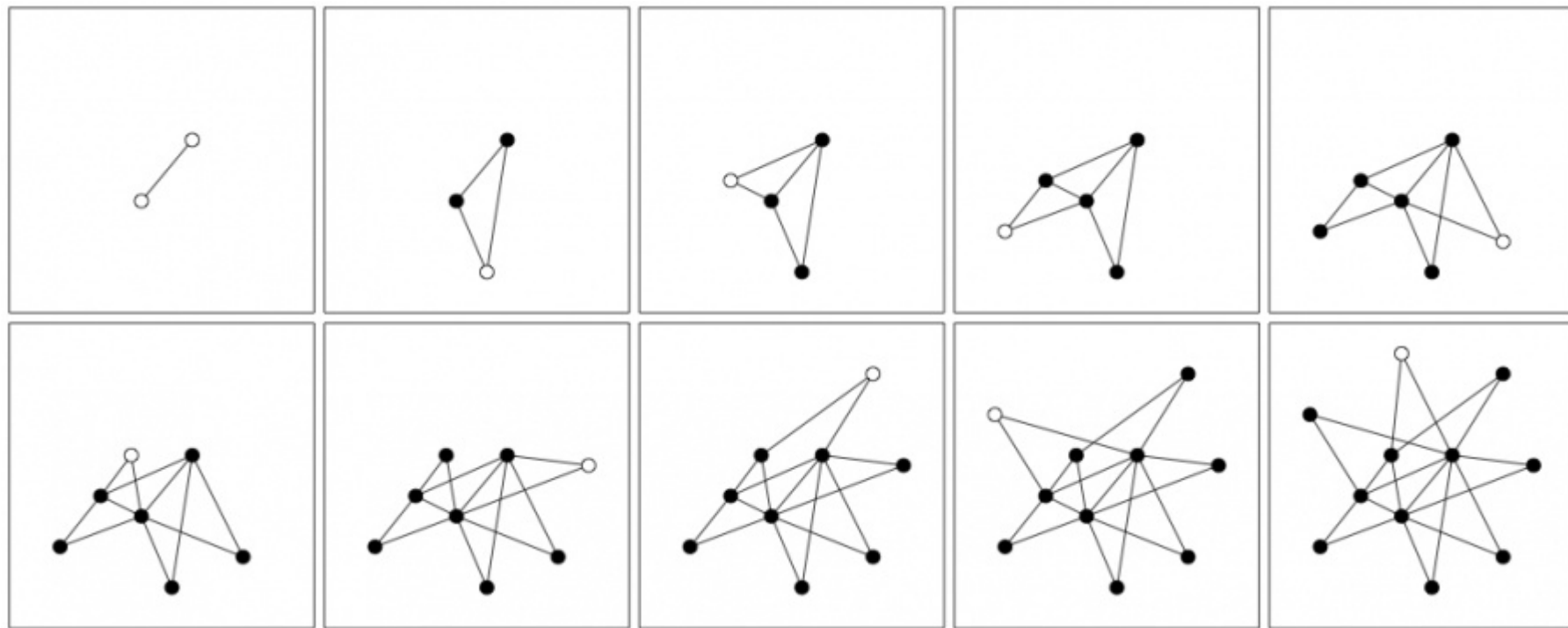
The Barabási-Albert (BA) model

- Network starts with m_0 nodes connected arbitrarily as long as their degree is ≥ 1
- At every time step we add 1 node
- This node will have $m \leq m_0$ outlinks
- The probability of an existing node of degree k_i to gain one such link is

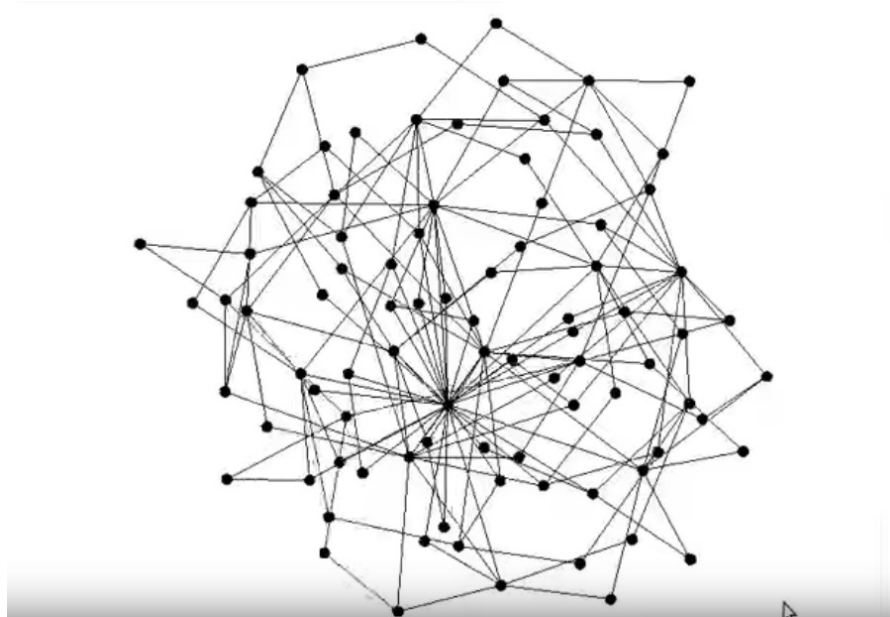
$$\Pi(k_i) = \frac{k_i}{\sum_{j=1}^{N-1} k_j}$$

In an ER network, $\Pi(k_i) = \frac{1}{N-1}$

Example ($m_0 = 2; m=2$)



Network growth with $m=2$



<https://www.youtube.com/watch?v=wocaGeNKn7Y>

The Barabási-Albert (BA) model

- Network starts with m_0 nodes connected arbitrarily as long as their degree is ≥ 1
- At every time step we add 1 node
- This node will have m outlinks ($m \leq m_0$)
- The probability of an existing node of degree k_i to gain one such link is

$$\Pi(k_i) = \frac{k_i}{\sum_{j=1}^{N-1} k_j}$$

Write the formula for $N(t)$ and $L(t)$: at $t=0$ the network has m_0 nodes and $L(0)$ links

Degree $k_i(t)$ as a function of time

$$\frac{d}{dt}k_i = m\Pi(k_i) = m \frac{k_i}{\sum_{j=1}^{N-1} k_j}$$

$$\sum_{j=1}^{N-1} k_j = L(0) + 2m(t-1) \approx 2m(t-1)$$

(For large t)

$$\frac{d}{dt}k_i = \frac{mk_i}{2m(t-1)} = \frac{k_i}{2t-2} \approx \frac{k_i}{2t}$$

Degree $k_i(t)$... continued

$$\frac{d}{dt} k_i(t) = \frac{k_i(t)}{2t}$$

$$\frac{1}{k_i(t)} \frac{d}{dt} k_i(t) = \frac{1}{2t}$$

$$\int_{t=t_i}^t \frac{1}{k_i(t)} \frac{d}{dt} k_i(t) dt = \int_{t=t_i}^t \frac{1}{2t} dt$$

Note: in exams for this course, you will **not** be asked to solve differential equations on your own

(t_i is the creation time of node i)

$$\log k_i(t) - \log k_i(t_i) = \frac{1}{2} \log t - \frac{1}{2} \log t_i$$

$$\log k_i(t) = \frac{1}{2} \log t - \frac{1}{2} \log t_i + \log m$$

Degree $k_i(t)$... continued

$$\log k_i(t) = \frac{1}{2} \log t - \frac{1}{2} \log t_i + \log m$$

$$k_i(t) = m \left(\frac{t}{t_i} \right)^{\frac{1}{2}}$$

Is the degree growth linear, super-linear, or sub-linear? Intuitively, why?

Degree $k_i(t)$... consequences

$$\log k_i(t) = \frac{1}{2} \log t - \frac{1}{2} \log t_i + \log m$$

$$k_i(t) = m \left(\frac{t}{t_i} \right)^{\frac{1}{2}}$$

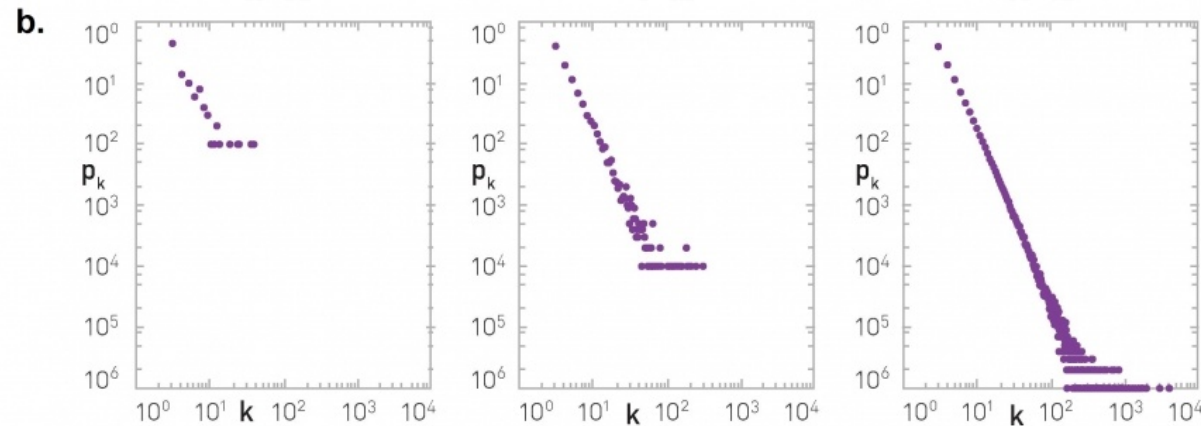
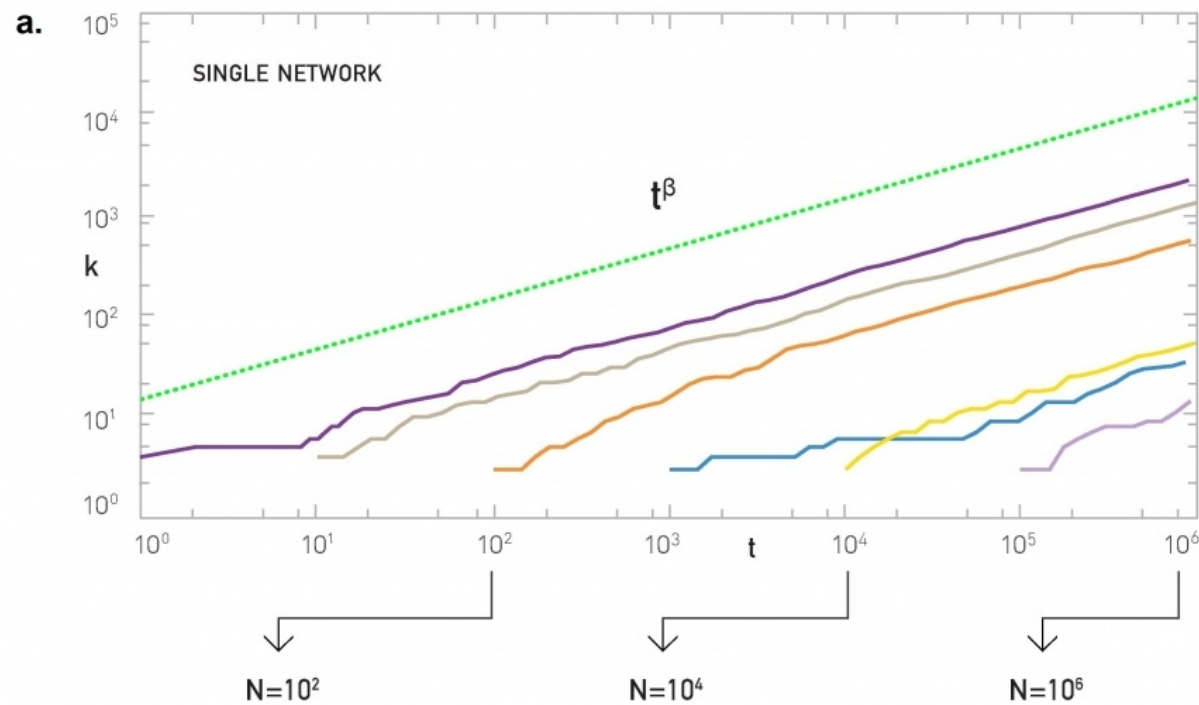
$$\frac{dk_i(t)}{dt} = \frac{k_i(t)}{2t} = \frac{m \left(\frac{t}{t_i} \right)^{\frac{1}{2}}}{2t} = \frac{m}{2(t \cdot t_i)^{\frac{1}{2}}}$$

If $t_i < t_j$ (node i is older than node j), what do we expect of k_i and k_j ?

Simulation results

Model

Nodes with $t_i = 1, 10, 100, 1000, 10000, \dots$



Degree distribution

- Let's calculate the CDF of the degree distribution

$$Pr(k_i \leq k) = 1 - Pr(k_i > k)$$

$$= 1 - Pr\left(m \left(\frac{t}{t_i}\right)^\beta > k\right)$$

$$= 1 - Pr\left(\left(\frac{m}{k}\right)^{1/\beta} > \frac{t_i}{t}\right)$$

$$\frac{t_i}{t} \sim \text{Uniform}(0, 1)$$

$$= 1 - \left(\frac{m}{k}\right)^{1/\beta}$$

Degree distribution

Now let's take the derivative of the CDF to obtain the PDF

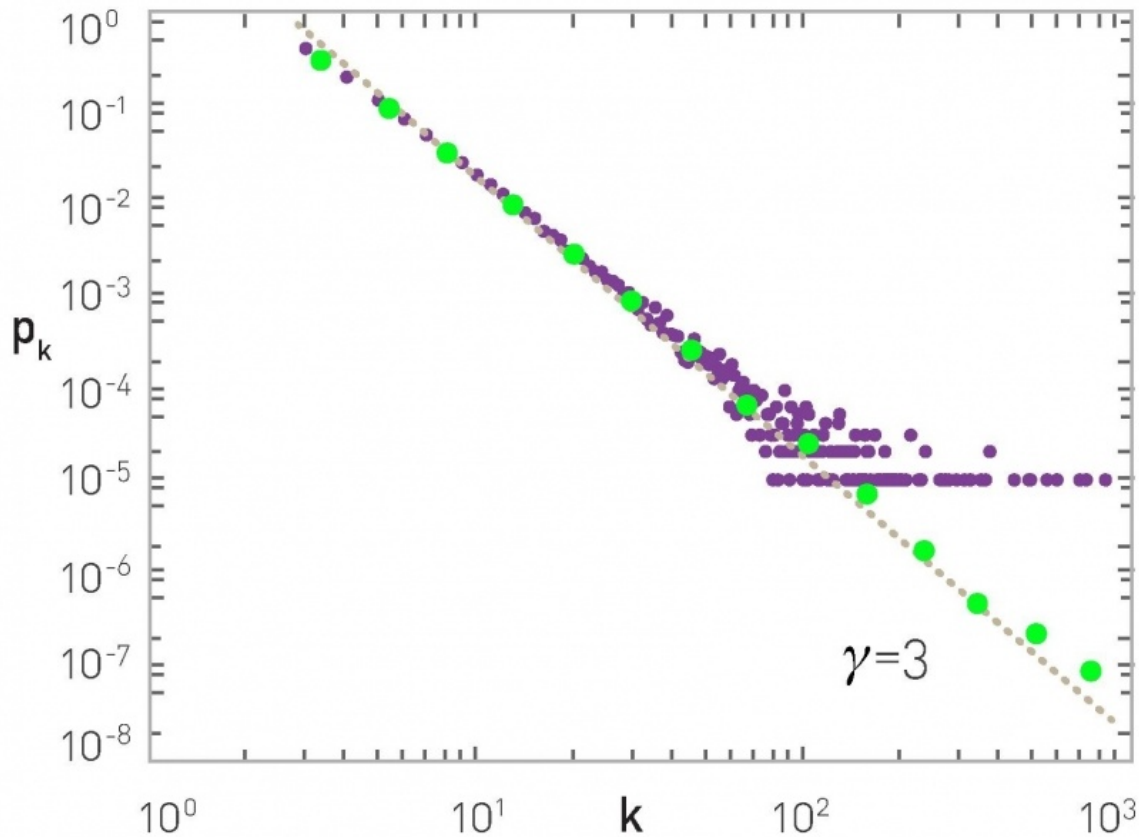
$$\begin{aligned} p_k &= \frac{d}{dk} \Pr(k_i \leq k) = \frac{d}{dk} \left(1 - \left(\frac{m}{k} \right)^{1/\beta} \right) \\ &= -\frac{d}{dk} \left(\left(\frac{m}{k} \right)^{1/\beta} \right) = -m^{1/\beta} \frac{d}{dk} \left(\frac{1}{k^{1/\beta}} \right) \\ &= \frac{1}{\beta} \frac{m^{1/\beta}}{k^{1/\beta+1}} \quad (\beta = 1/2) \\ &= 2 \frac{m^2}{k^3} \longrightarrow p(k) \propto k^{-3} \end{aligned}$$

Degree distribution

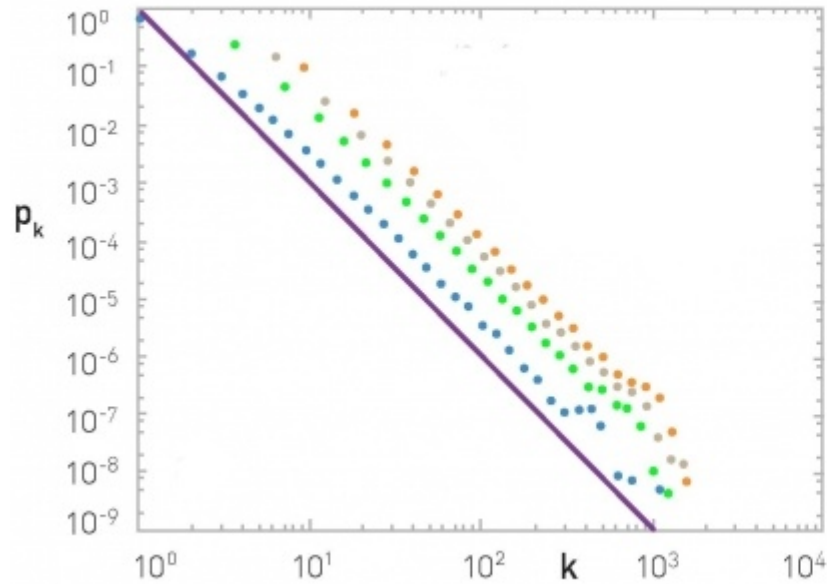
- $\beta = 1/2$ is called the dynamical exponent
- $\gamma = \frac{1}{\beta} + 1 = 3$ is the power-law exponent
- Note that $p(k) \approx 2m^2/k^3$
does not depend on t
hence, it describes a stationary network

Degree distribution, simulation results

$N=100,000$ $m=3$



More simulations

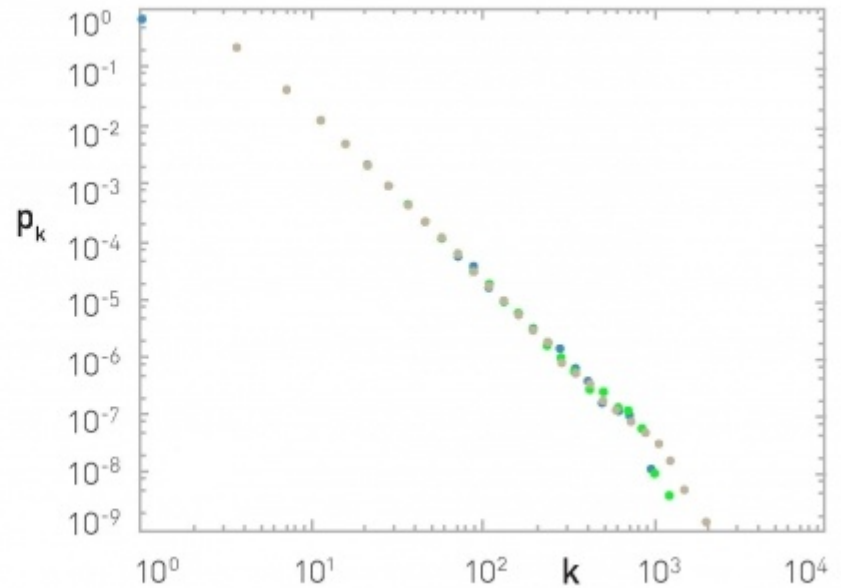


$N = 100,000$; $m_0 = m =$

1 (blue), 3 (green), 5 (gray), 7 (orange)

Observe γ is independent of m (and m_0)

The slope of the purple line is -3



$m_0 = m = 3$; $N =$

50K (blue), 100K (green), 200K (gray)

Observe p_k is independent of N

Processes that generate scale-free networks

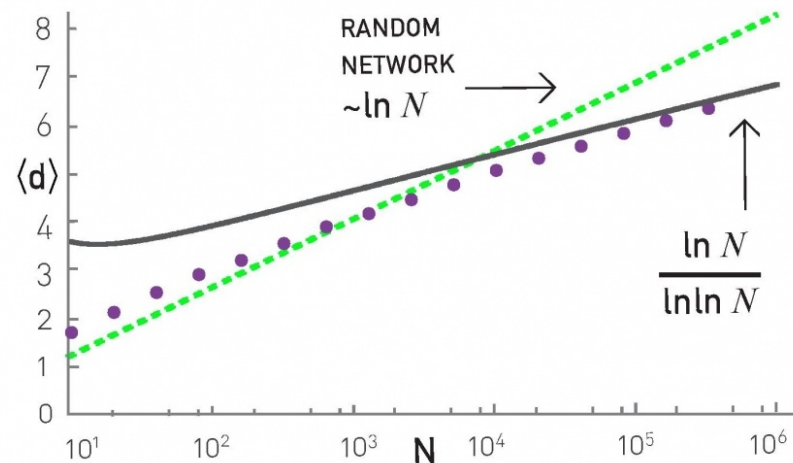
- **Link-selection model** — step:
 - Add one new node v to the network
 - Select an existing link at random and connect v to one of the edges of that existing link
- **Copy model** — step:
 - Add one new node v to the network
 - Pick a random existing node u
 - With probability p link to u
 - With probability $1-p$ link to a neighbor of u

Average distance

- Distances grow slower than $\log N$

$$\langle d \rangle \approx \frac{\log N}{\log \log N}$$

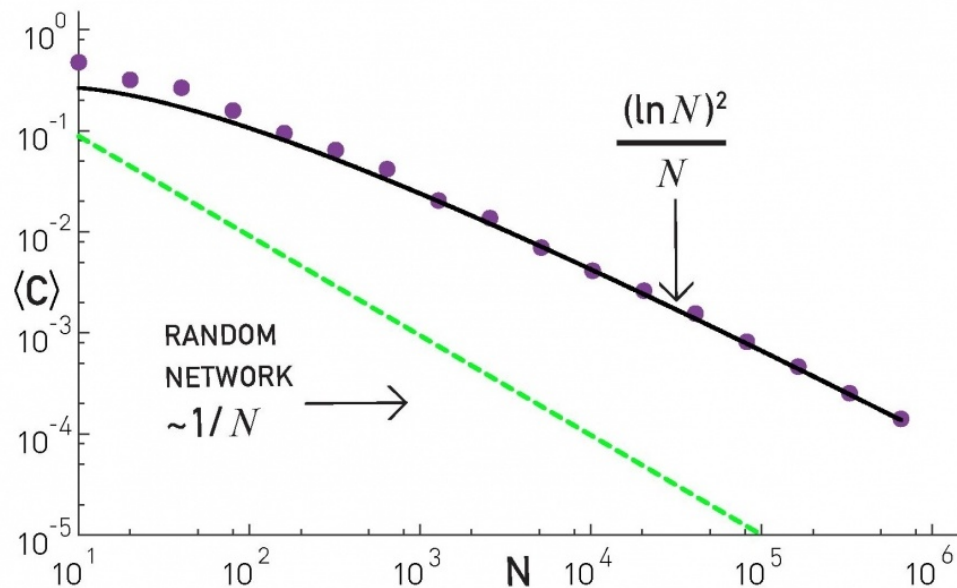
(Scale free network with $\gamma = 3$)



Clustering coefficient

- BA networks are locally more clustered than ER networks

$$\langle C \rangle \approx \frac{(\log N)^2}{N}$$



Limitations of the BA model

- Predicts a fixed exponent of -3
- Assumes an undirected network, while many real complex networks are directed
- Does not consider node deletions or edge deletions which are common in practice
- Considers that all nodes are equal except for their arrival times

Exercise: the copy model

In the copy model, start at $t=1$ with one node, and at every step t :

- Add one new node v to the network
- Pick a random existing node u
- If u has no out-links, link to u
- If u has out-links:
 - With probability p link to u
 - With probability $1-p$ link to one of the out-links of u chosen at random
- Simulate it on paper for 7 nodes with $p=0.5$
 - Make sure you understand the model fully!
- What is $N(t)$ and $L(t)$?

In the copy model, at every step t :

1) Add one new node v to the network

2) Pick a random existing node u

3) With probability p link to u

4) With probability $1-p$ link to a neighbor of u

- What is k_i^{out} ?
- We will compute k_i^{in}
- How many links on average gets node i at time t ?
In other words, what is ...

$$\frac{d}{dt} k_i^{\text{in}}(t)$$

- Hint: it has a term with p and a term with $1-p$

Summary

Things to remember

- Preferential attachment
- How to create a BA network step by step
- Degree distribution in the BA model
- Distances and clustering coefficient in BA
- The copy model

Practice on your own

- Try to reconstruct the derivations we have done in class, including the exercise
 - Try to understand every step
- Insert a small change in the model and try to recalculate what we have done

Advanced materials #1:
Copy model cont.
(not included in the exam)

- Integrate between t_i and t to obtain an expression for $k_i(t_i)$
(we drop the “in” superscript just for simplicity during this exercise)
- Note that now $k_i(t_i) = 0$

- Once you have an expression for $k_i(t_i)$
- Compute $Pr(k_i(t_i) > k)$
- Now write the cumulative distribution function of $k_i(t_i)$
- And compute its derivative to obtain
$$p_k = Pr(k_i(t) = k) = \frac{d}{dk} Pr(k_i(t) \leq k)$$
- It should show exponent $\gamma = \frac{2 - p}{1 - p}$

Advanced materials #2:
Expected degree under
uniform random attachment
(not included in the exam)

Expected degree in uniform random attachment using a differential equation

$$\frac{d}{dt}k_i(t) = \frac{m}{t}$$

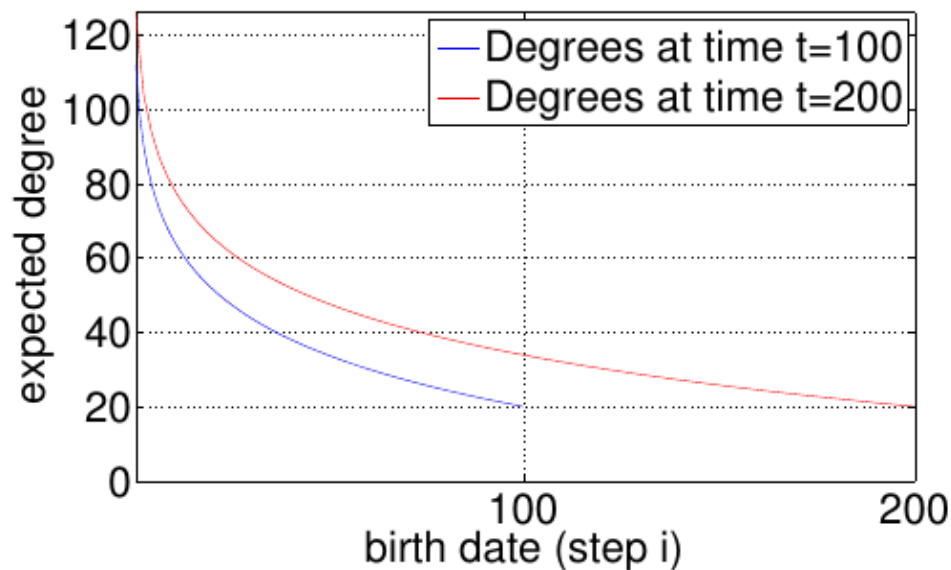
(1) Integrate between time i and time t

(2) Use initial condition $k_i(i) = m$

$$\int \frac{1}{t} = \log t + C$$

Degree distribution over time is not static

Degree of node born at time $m < i < t = m \left(1 + \log \left(\frac{t}{i} \right) \right)$



Tail of degree distribution

How many nodes of degree larger than K are there at time t ?

The fraction is $\frac{te^{-\frac{K-m}{m}}}{t} = e^{-\frac{K-m}{m}}$

Decreases exponentially with K : it's vanishingly rare to find high-degree nodes

$$m \left(1 + \log \left(\frac{t}{i} \right) \right) > K$$

$$1 + \log \left(\frac{t}{i} \right) > \frac{K}{m}$$

$$\log \left(\frac{t}{i} \right) > \frac{K - m}{m}$$

$$\frac{t}{i} > e^{\frac{K-m}{m}}$$

$$i < te^{-\frac{K-m}{m}}$$