Network flows

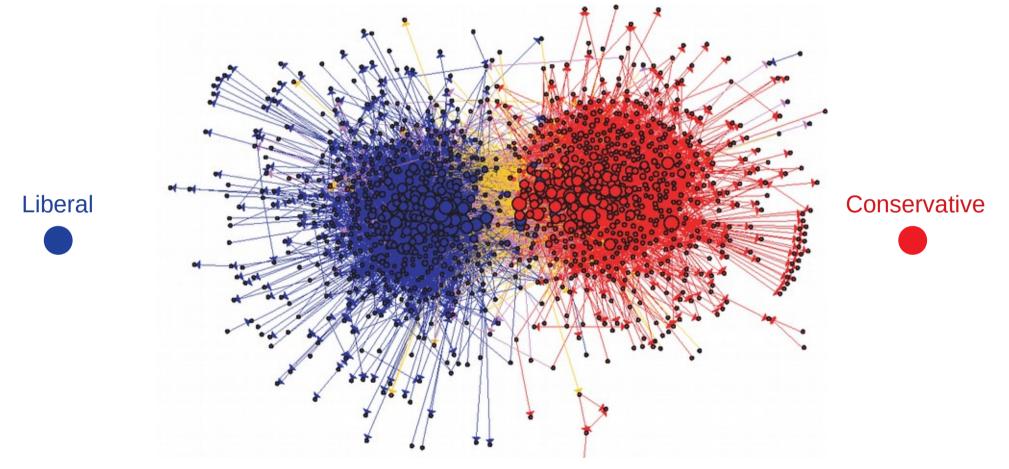
Introduction to Network Science Carlos Castillo Topic 12

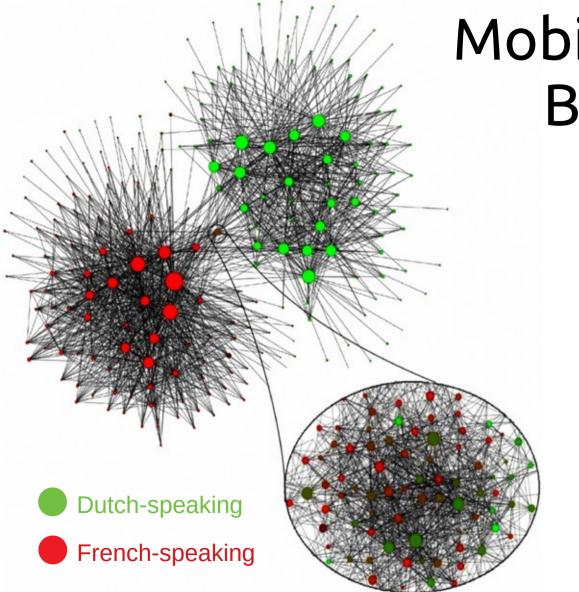


Sources

- Barabási 2016 Chapter 9
- Networks, Crowds, and Markets Ch 3
- C. Castillo: Graph partitioning 2017

US Political Blogs (2004)



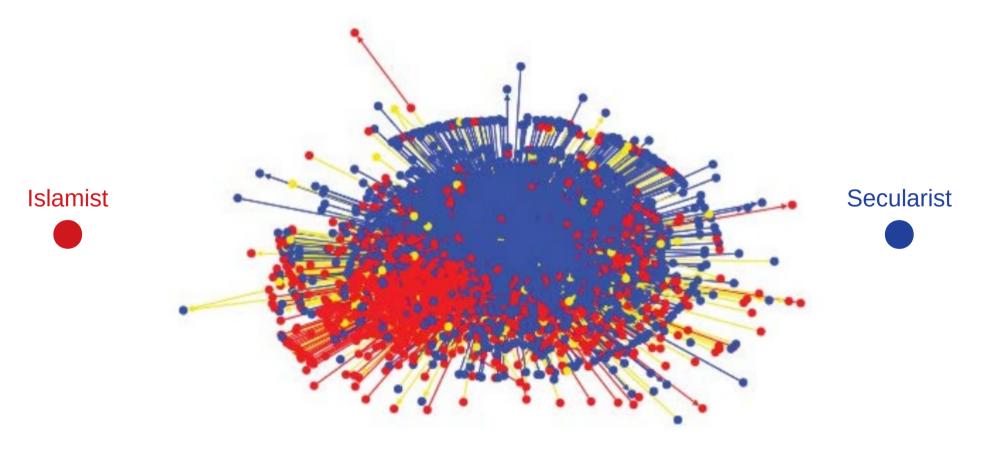


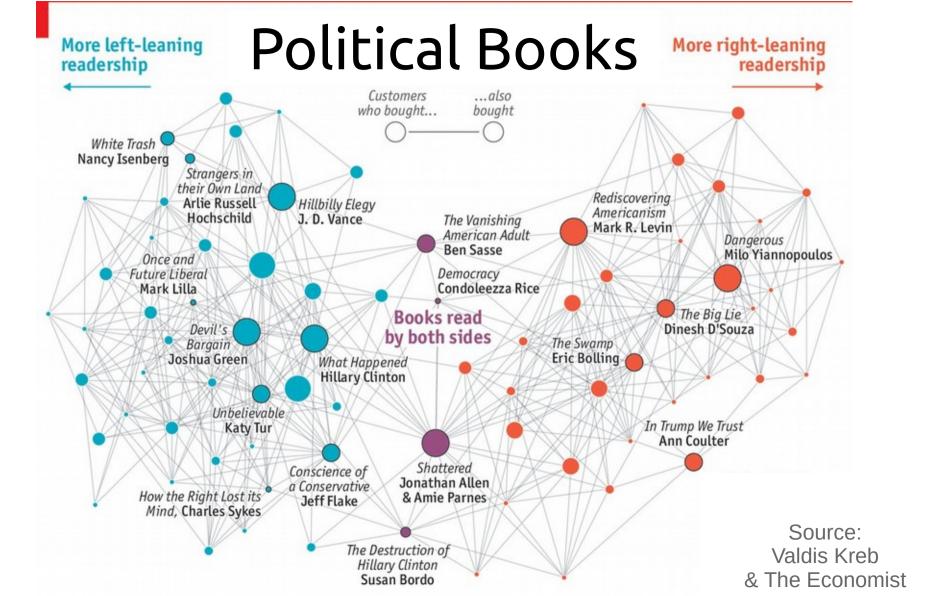
Mobile phone users in Belgium (2008)

Each node is a community of 100 mobile users or more that tend to call each other

V. D. Blondel, J.-L. Guillaume, R. Lambiotte, and E. Lefebvre. Fast unfolding of communities in large networks. J. Stat. Mech., 2008.

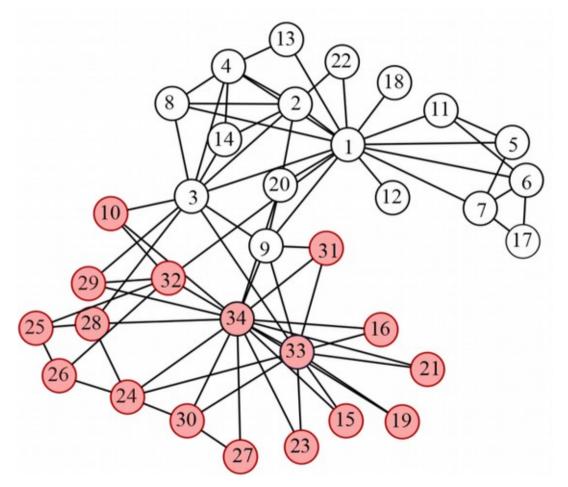
Egyptian Twitter Users (2013)





Wayne Zachary's PhD Thesis (1972)

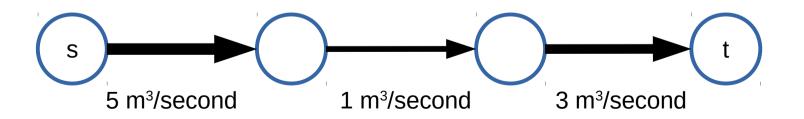
- Studied 34 members of a karate club
- Found 78 links between members who regularly interacted outside the club
- The club splitted in two during the study
- 1=sensei, 34=president



Splitting into two communities: Max-flow and Min-cut

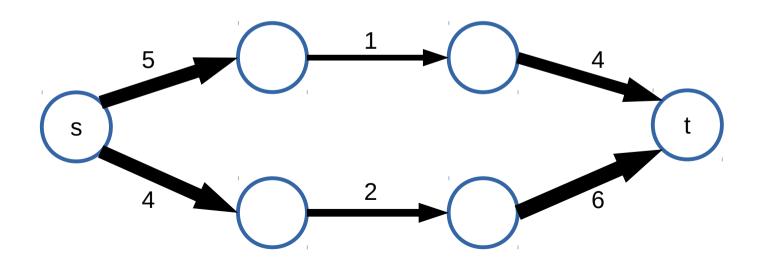
Maximum flow: example 1

 If edge weights were capacities, what is the maximum flow that can be sent from s to t?



Maximum flow: example 2

 If edge weights were capacities, what is the maximum flow that can be sent from s to t?



Maximum flow problem

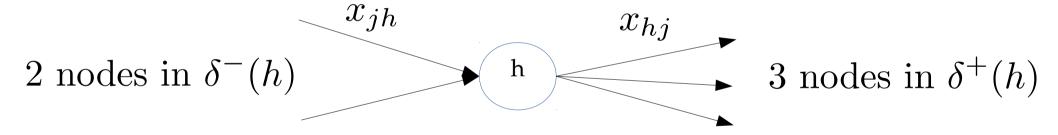
- What is the maximum "flow" that can be carried from s to t?
 - Think of edge weights as capacities (e.g. m³/s of water)
- What is the flow of an edge?
 - The amount sent through that edge (an assignment)
- What is the net flow of a node?
 - The amount exiting the node minus the amount entering the node

Formulating the max flow problem

- The flow through each edge should be $\leq k_{ij}$
- Node s has only out_flow, should have positive flow v
- Node t has only in_flow, should have negative flow -v
- What should be the flow of the other nodes?

Formulating the max flow problem

- Let v be a feasible flow
- Node s should have positive flow v
- Node t should have negative flow -v



• What should be the flow of an arbitrary node h?

$$\sum_{(h,j)\in\delta^{+}(h)} x_{hj} - \sum_{(i,h)\in\delta^{-}(h)} x_{ij} = ?$$

Max flow as a linear program

$$\max v \qquad (1)$$

$$\sum_{(s,j)\in\delta^{+}(s)} x_{sj} = v \qquad (2)$$

$$-\sum_{(i,t)\in\delta^{-}(t)} x_{it} = -v \qquad (3)$$

$$\sum_{(h,j)\in\delta^{+}(h)} x_{hj} - \sum_{(i,h)\in\delta^{-}(h)} x_{ih} = 0, h \in N - \{s,t\}$$

$$x_{ij} \leq k_{ij} \quad (i,j) \in A \qquad (5)$$

$$x_{ij} \geq 0 \quad (i,j) \in A \qquad (6)$$

Primal-Dual in Linear Programming

PRIMAL

DUAL

$$\min \sum_{j} c_j x_j$$
 subject to

$$\max \sum_{i} y_i b_i$$
 subject to

$$\sum_{i} a_{ij} x_j \ge b_i \quad \forall i \in [m]$$

$$\sum_{i} y_i a_{ij} \le c_j \quad \forall j \in [n]$$

$$x_j \ge 0 \ \forall j \in [n]$$

$$y_i \ge 0 \ \forall i \in [m]$$

Writing the dual: each constraint will become a variable

$$\max_{(s,j)\in\delta^{+}(s)} v \qquad (1)$$

$$\sum_{(s,j)\in\delta^{+}(s)} x_{sj} = v \qquad \text{variable } u_{s} \qquad (2)$$

$$-\sum_{(i,t)\in\delta^{-}(t)} x_{it} = -v \qquad \text{variable } u_{t} \qquad (3)$$

$$\sum_{(h,j)\in\delta^{+}(h)} x_{hj} - \sum_{(i,h)\in\delta^{-}(h)} x_{ih} = 0, h \in N - \{s,t\} \qquad \text{variables } u_{j} \qquad (4)$$

$$x_{ij} \leq k_{ij} \quad (i,j) \in A \qquad \text{variables } y_{ij} \qquad (5)$$

$$x_{ij} \geq 0 \quad (i,j) \in A \qquad (6)$$

Writing the dual

 Remember: the infimum of the solutions of the dual is the supremum of the solutions of primal

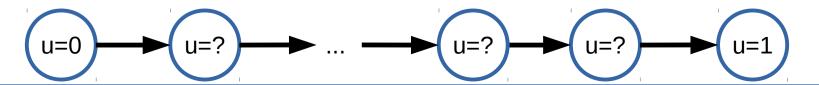
$$\begin{aligned} &\min \sum_{(i,j) \in A} k_{ij} y_{ij} \\ &u_i - u_j + y_{ij} \geq 0, (i,j) \in A \\ &-u_s + u_t = 1 \\ &y_{ij} \geq 0 \end{aligned} \qquad \qquad \text{ Think of } y_{ij} \text{ as } \\ &0 \text{ or } 1 \end{aligned}$$

- Variables u_i don't enter the objective, only their difference is in the constraints
- We can set them arbitrarily, in particular $u_s = 0$, $u_t = 1$

Dual (after simplification)

$$min \sum_{(i,j)\in A} k_{ij}y_{ij}$$
 $u_i - u_j + y_{ij} \ge 0, (i,j) \in A$
 $y_{ij} \ge 0$
 $u_s = 0, u_t = 1$

 What happens with the values of u in every simple path going from s to t?



Dual (after simplification)

$$min \sum_{(i,j)\in A} k_{ij}y_{ij}$$

$$u_i - u_j + y_{ij} \ge 0, (i,j) \in A$$

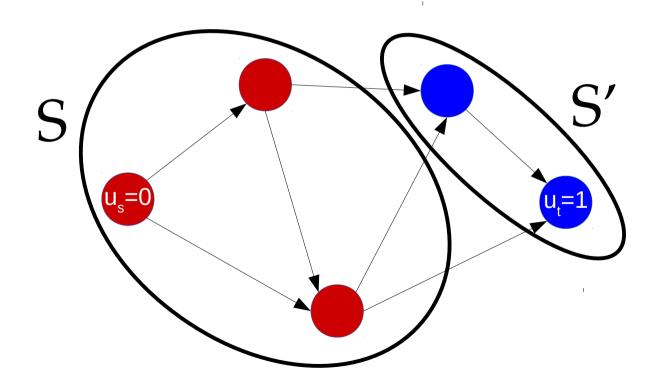
$$y_{ij} \ge 0$$

$$u_s = 0, u_t = 1$$

Every feasible solution represents a cut (S, S')

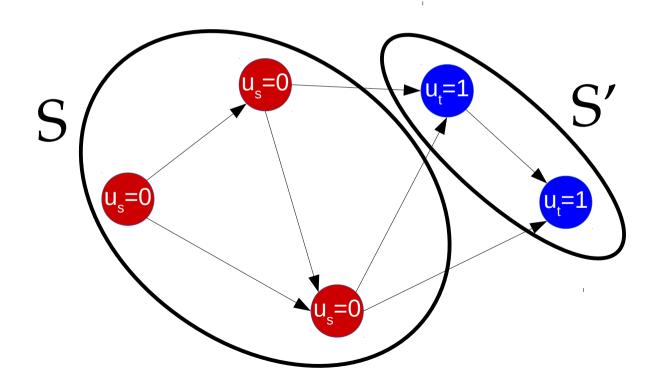
Dual solutions are cuts

• Every feasible solution of the dual has the form of a cut (S, S')



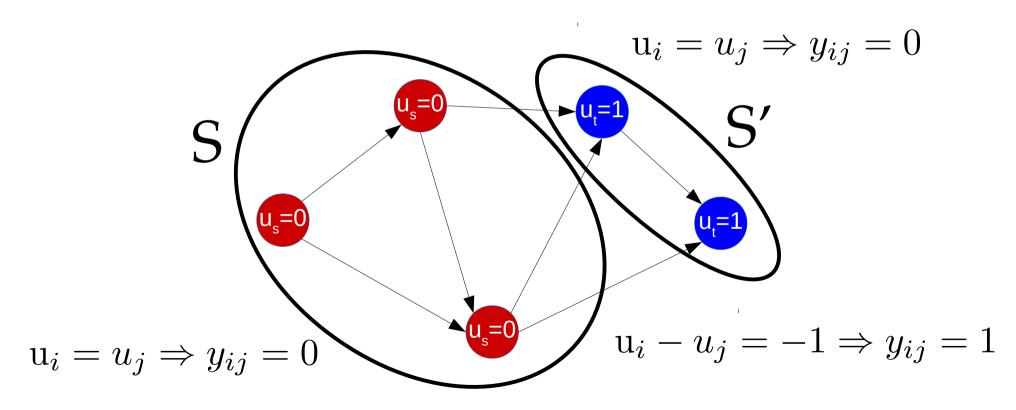
Dual solutions are cuts

 Every feasible solution of the dual has the form of a cut (S, S')



Dual solutions are (s-t)-cuts

$$\mathbf{u}_i - u_j + y_{ij} \geq 0$$
 and remember we're trying to minimize $\sum k_{ij}y_{ij}$



One more thing about the solution

$$min \sum_{(i,j) \in A} k_{ij} y_{ij}$$
 $u_i - u_j + y_{ij} \ge 0, (i,j) \in A$
 $y_{ij} \ge 0$
 $u_s = 1, u_t = 0$

 y_{ij} is a dual variable corresponding to primal constraint $x_{ij} \leq k_{ij}$ If y_{ij} is non-zero, then the corresponding constraint is tight What does it mean for the edges in the cut?

This is an efficient method

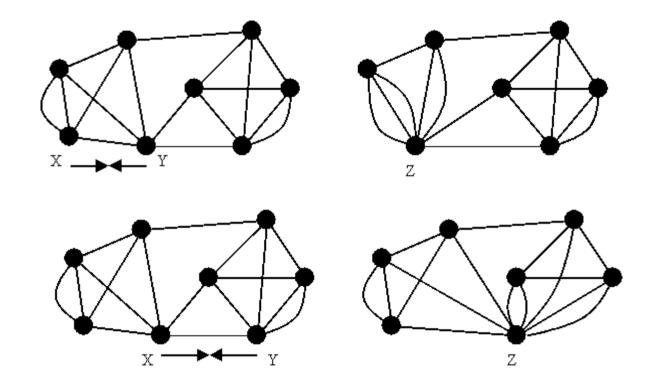
- Min-cut and Max-flow are equivalent problems
 - Their solutions are also equal: the value of the maximum flow is equivalent to the minimum cut
- Think of a chain that breaks at the weakest link
- Both can be solved exactly in polynomial time

Randomized algorithm for (s-t)-cuts

Randomized algorithm for (s-t)-cuts

- Pick an edge at random (u,v)
- Merge u and v in new vertex uv
- Edges between u and v are removed
- Edges pointing to u or v are added as multi-edges to vertex uv
- When only s and t remain, the multi-edges are a cut, probably the minimum one

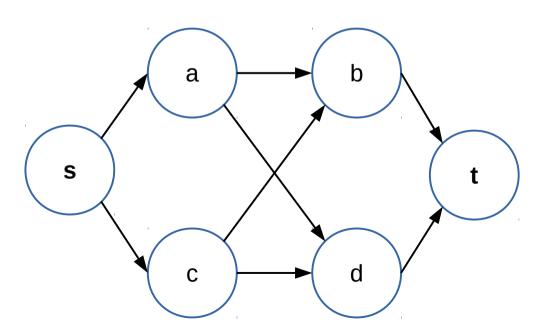
Example merges ("contractions")



Try it!

Run the randomized algorithm on this graph

- Pick an edge at random (*u*, *v*)
- Merge u and v in new vertex uv
- Edges between *u* and *v* are removed
- Edges pointing to u or v are added as multiedges to vertex uv
- When only *s* and *t* remain, the multi-edges are a cut, probably the minimum one



The randomized algorithm might miss the min cut

- Multiple runs are required
- The probability that this finds the min cut in one run is about 1/log(n), so O(log n) iterations are required to find min cut
- Each iteration costs O(n² log n)
- O(n² log² n) operations needed to find min cut
- Exact algorithm: O(n³ + n² log n); the n³ is because of |V||E| operations required

To remember

- Minimum s-t cut in a graph = set of edges
- The sum of the capacities of those edges is the maximum s-t flow the graph can carry
- Solvable in polynomial time
- Approximate randomized algorithm exists