

Distances in Scale-Free Networks

Introduction to Network Science

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Topic 10



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Contents

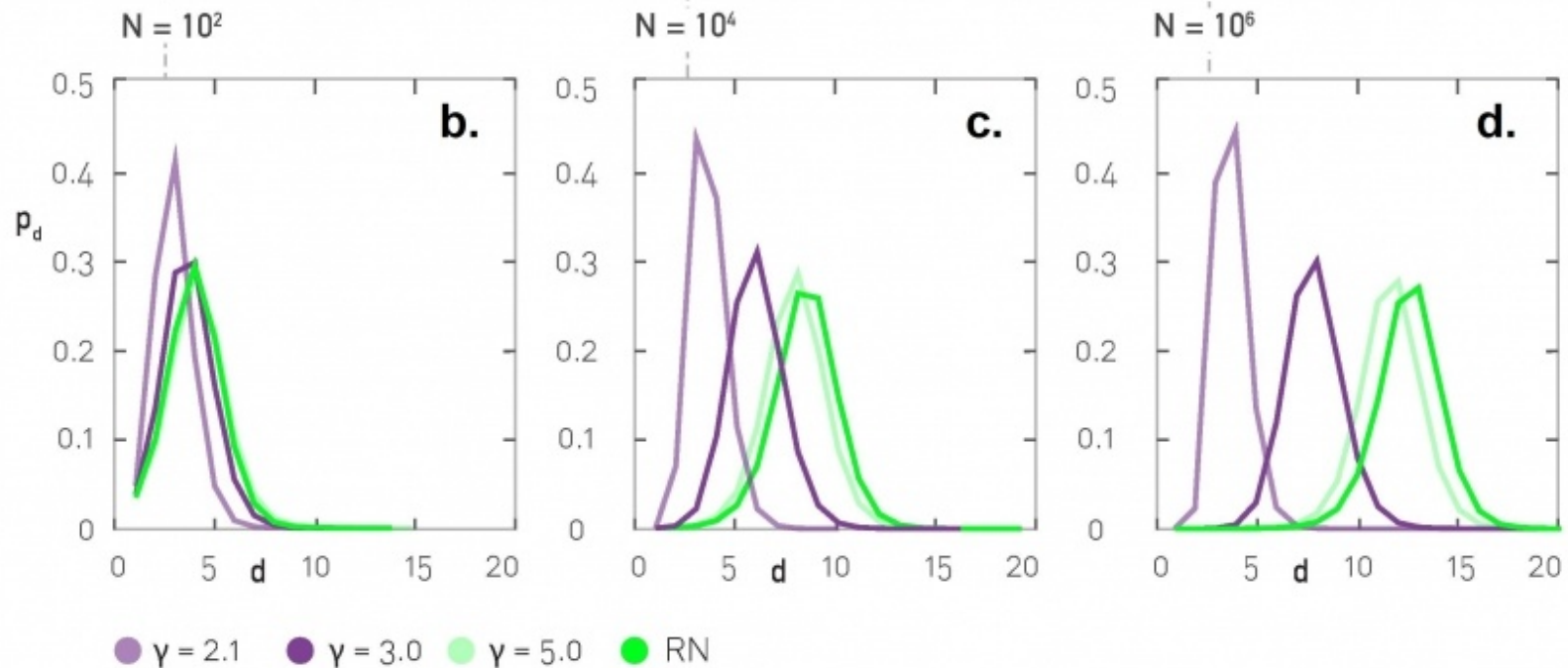
- Distance distribution of scale-free networks

Sources

- Albert-László Barabási: Network Science. Cambridge University Press, 2016.
 - Follows almost section-by-section chapter 04
- URLs cited in the footer of specific slides

Distance distributions: simulation results

Scale-free networks of increasing size, $\langle k \rangle = 3$



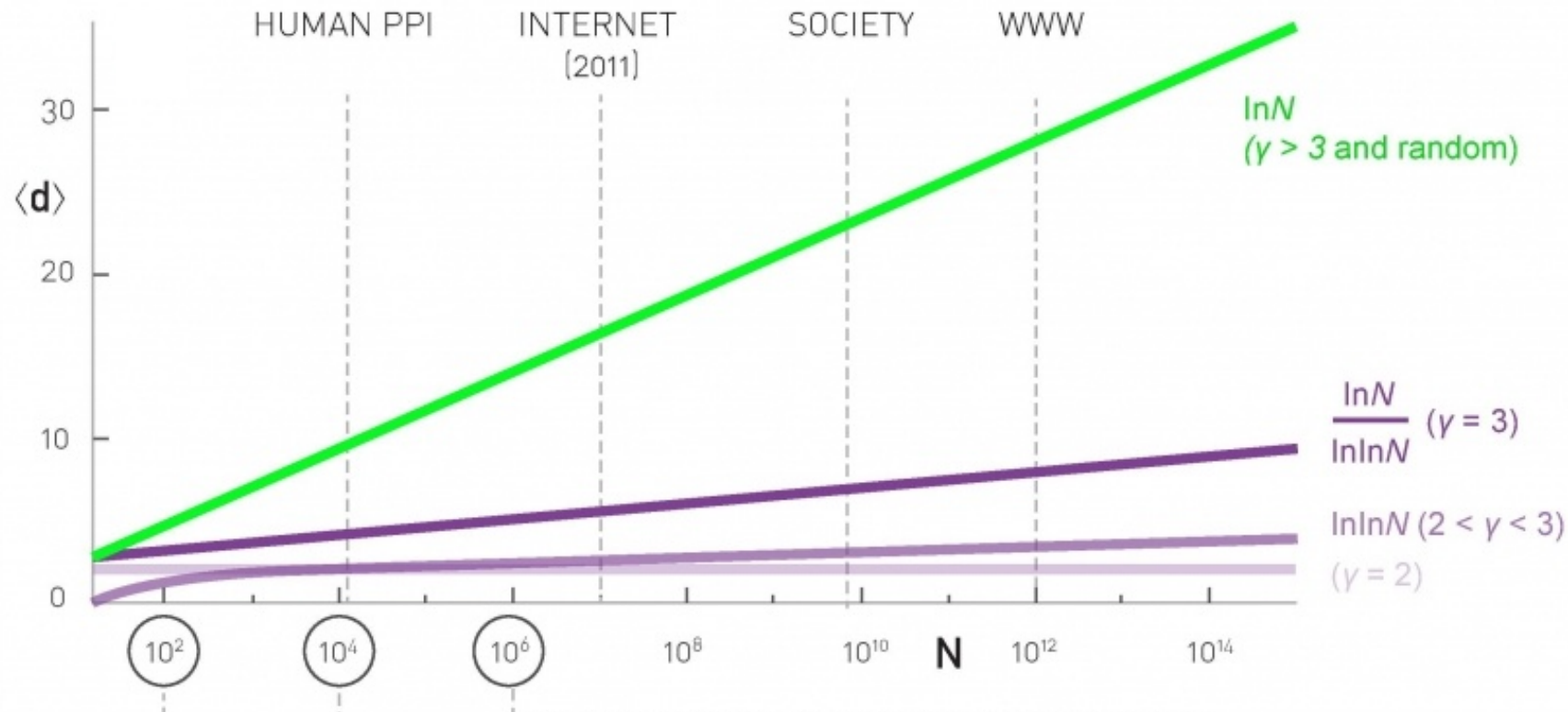
Average distance

- Depends on γ and N

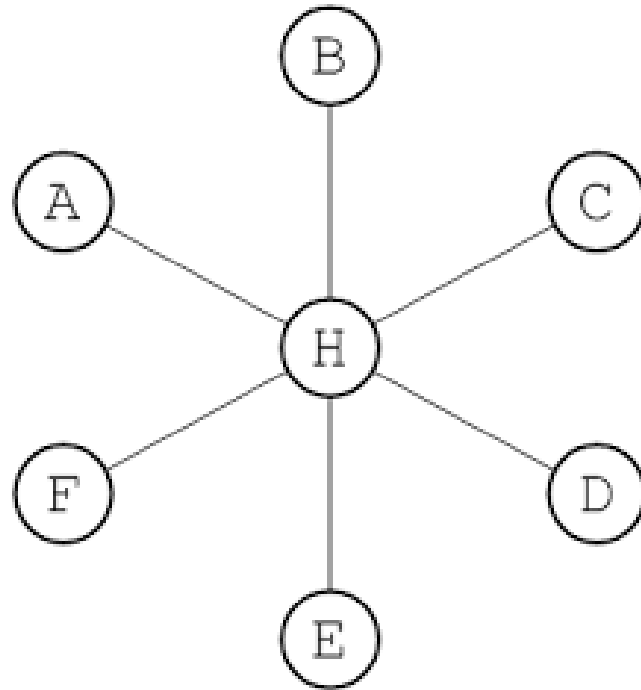
$$\langle d \rangle = \begin{cases} \text{const.} & \text{if } \gamma = 2 \\ \log \log N & \text{if } 2 < \gamma < 3 \\ \log N / \log \log N & \text{if } \gamma = 3 \\ \log N & \text{if } \gamma > 3 \end{cases}$$

← Same as in
ER graphs

Average distance and N



Anomalous regime $\gamma = 2$



Ultra-small world $2 < \gamma < 3$

- Average distance follows $\log(\log(N))$
- Example (humans):

$$N \approx 7 \times 10^9$$

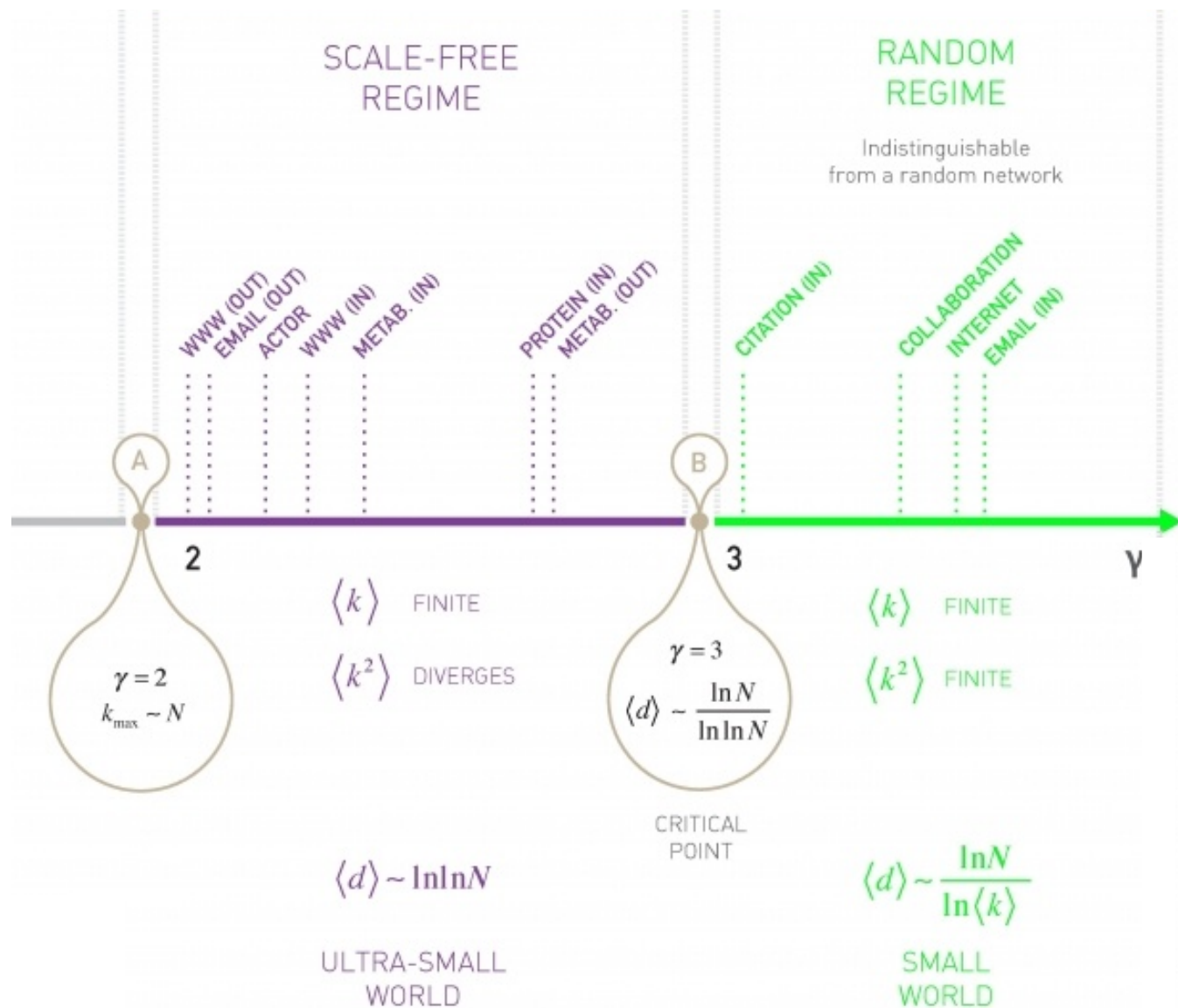
$$\log N \approx 22.66$$

$$\log \log N \approx 3.12$$

Small world $\gamma > 3$

- Average distance follows $\log(N)$
- Similar to ER graphs where it followed $\log(N)/\log(\langle k \rangle)$

The degree distribution exponent plays an important role



When $\gamma > 3$

- In this case it is hard to distinguish this case from an ER graph
- In most real complex networks (but not all)

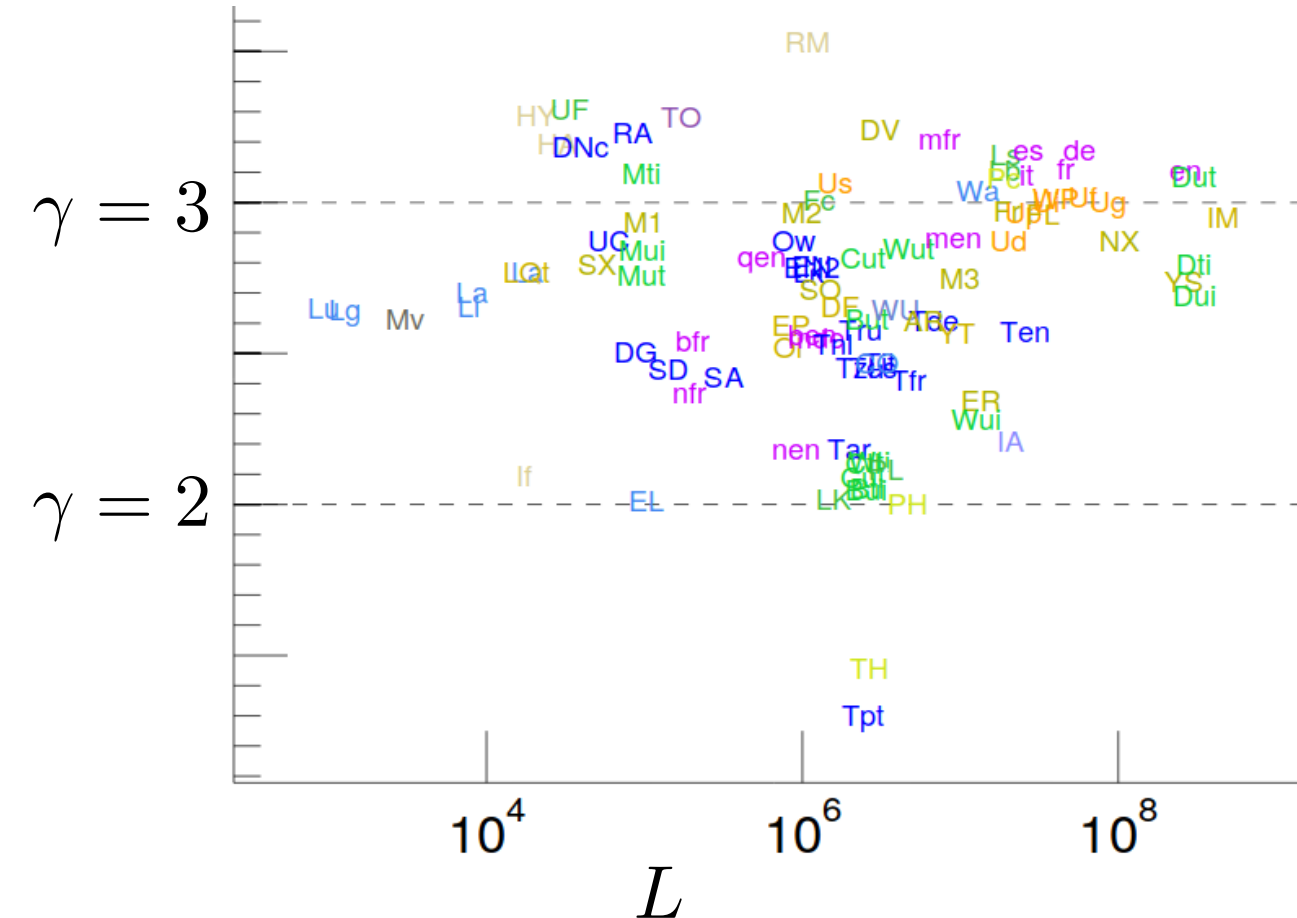
$$2 < \gamma < 3$$

When $\gamma > 3$

- Remember $k_{\max} = k_{\min} N^{\frac{1}{\gamma-1}}$ $N = \left(\frac{k_{\max}}{k_{\min}} \right)^{\gamma-1}$
- Observing the scale-free properties requires that $k_{\max} \gg k_{\min}$, e.g. $k_{\max} = 10 k_{\min}$
- Then if $\gamma = 5$, $N > 10^8$
- Hence we won't find many such networks

Examples

<http://konect.uni-koblenz.de/statistics/prefatt>



EL	Wikipedia elections
LK	Linux kernel mailing list threads
Bul	BibSonomy u-i
Bti	BibSonomy t-i
Cul	CiteULike u-i
If	Infectious
PL	Prosper loans
Cti	CiteULike t-i
Wti	Twitter t-i
nen	Wikinews (en)
Tar	Wikipedia talk, Arabic
Wul	Twitter u-i
ER	Epinions
nfr	Wikinews (fr)
Tfr	Wikipedia talk, French
SD	Slashdot
Tzh	Wikipedia talk, Chinese
Tes	Wikipedia talk, Spanish

Etc.

The friendship paradox

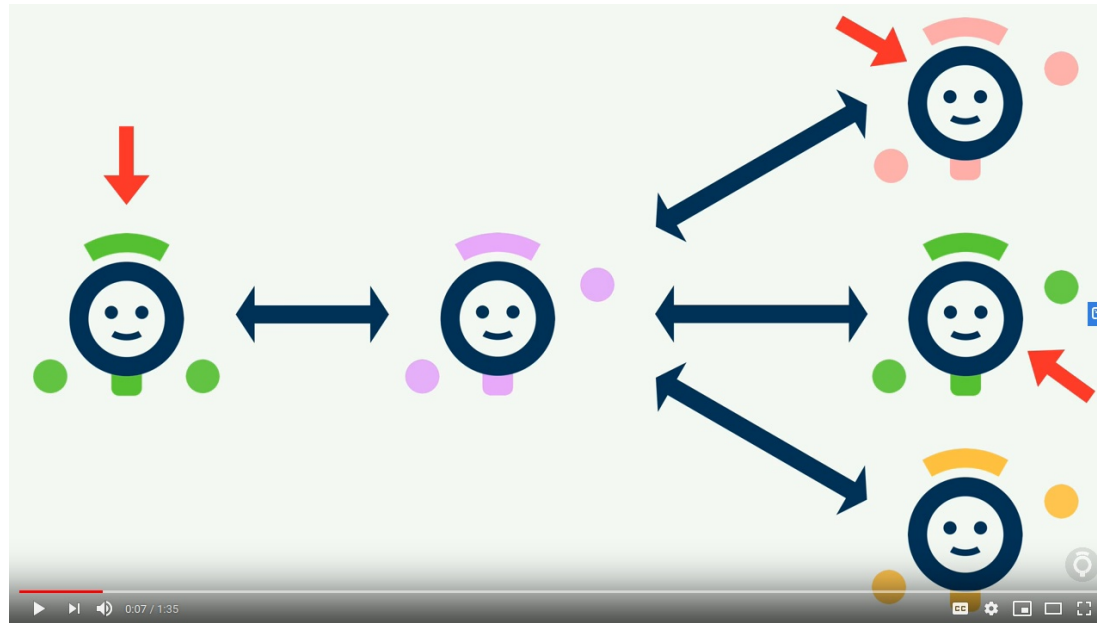
The friendship paradox

- Take a random person x , what is the expected degree of this person? $\langle k \rangle$
- Take a random person x , now pick one of x 's neighbors, let's say y

What is the expected degree of y ?

It is not $\langle k \rangle$

Sampling bias and the friendship paradox (1'35'')



<https://www.youtube.com/watch?v=httLvVufAYs>

Imagine you're at a random airport on earth

- Is it more likely to be ...
a large airport or a small airport?
- If you take a random flight out of it ...
will it go to a large airport or a small airport?

An example of friendship paradox

- Pick a random airport on Earth
 - Most likely it will be a small airport
- However, no matter how small it is, it **will** have flights to big airports
- On average those airports will have much larger degree



Time	Flight	Airline	Destination	Gate	Exp.	Remarks
11:00	KA 376	DRAGONAIR	Hong Kong	4		Chk-in closed
12:25	DG 7792	tigerair	Singapore	1		On Time
12:25	QR 931	QATAR	Doha, Qatar	5		On Time
17:40	EK 339	Emirates	Dubai	5		On Time
00:50	OZ 708	ASIANA AIRLINES	Seoul Incheon	5		On Time
07:05	5J 150		Hong Kong	1		On Time
07:20	DG 7924	tigerair	Hong Kong	1		On Time
08:00	DG 7792	tigerair	Singapore	1		On Time
12:10	5J 537		Singapore	1		On Time
12:25	QR 931	QATAR	Doha, Qatar	5		On Time

Exercise [B. 2016, Ex. 4.10.2]

"Friendship Paradox"

- Remember p_k is the probability that a node has k "friends"
- If we randomly select a link, the probability that a node at any end of the link has k friends is $q_k = C k p_k$ where C is a normalization factor
 - (a) Find C (the sum of q_k must be 1)

Answer in Nearpod Collaborate
<https://nearpod.com/student/>
Access to be provided during class

Exercise [B. 2016, Ex. 4.10.2]

"Friendship Paradox"

- If we randomly select a link, the probability that a node at any end of the link has k friends is $q_k = C k p_k$ where C is a normalization factor
- q_k is also the prob. that a randomly chosen node has a neighbor of degree k

(b) Find its expectation $E[q_k]$ which we will call $\langle k_F \rangle$

Remember

$$E[X] = \sum_{X_{\min}}^{X_{\max}} x \cdot P(X = x)$$

Exercise [B. 2016, Ex. 4.10.2]

"Friendship Paradox"

(c) Compute the expected number of friends of a neighbor of a randomly chosen node in the case below

(d) compare with the expected number of friends of a randomly chosen node

$$N = 10000$$

$$\gamma = 2.3$$

$$k_{\min} = 1$$

$$k_{\max} = 1000$$

$$\langle k^n \rangle = C \frac{k_{\max}^{n-\gamma+1} - k_{\min}^{n-\gamma+1}}{n - \gamma + 1}$$

$$C = (\gamma - 1) k_{\min}^{\gamma-1}$$

Code

```
def degree_moment(kmin, kmax, moment, gamma):  
    C = (gamma-1.0)*(kmin**(gamma-1.0))  
    numerator = (kmax**(moment-gamma+1.0) - kmin**(moment-gamma+1.0))  
    denominator = (moment-gamma+1.0)  
    return C * numerator / denominator
```

```
kavg = degree_moment(kmin=1, kmax=1000, moment=1, gamma=2.3)  
print(kavg)
```

3.787798988222529

```
ksqavg = degree_moment(kmin=1, kmax=1000, moment=2, gamma=2.3)  
print(ksqavg)
```

231.94329076177414

```
print(ksqavg / kavg)
```

61.23431879119234

Summary

Things to remember

- Regimes of distance and connectivity
- The friendship paradox

Practice on your own

- Remember the regimes of a graph given $\langle k \rangle$
(It's useful to know this by heart)
- Estimate degree distributions and distance distributions for some graphs
- Draw a small graph, and sample from that graph until you're convinced $\langle k_F \rangle > \langle k \rangle$