

Other growth models

Introduction to Network Science

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Topic 13

Other graph evolution models

- Good-get-richer
- Sub-linear and super-linear preferential attachment
- Aging effects
- No preference and no growth



Sources

- Albert-László Barabási: Network Science. Cambridge University Press, 2016.
 - Chapters 05 and 06

Actual network growth is complex

A snapshot of the Autodesk organizational hierarchy was taken each day between May 2007 and June 2011, a span of 1498 days.

Each day the entire hierarchy of the company is constructed as a tree with each employee represented by a circle, and a line connecting each employee with his or her manager.

Larger circles represent managers with more employees working under them. The tree is then laid out using a force-directed layout algorithm.

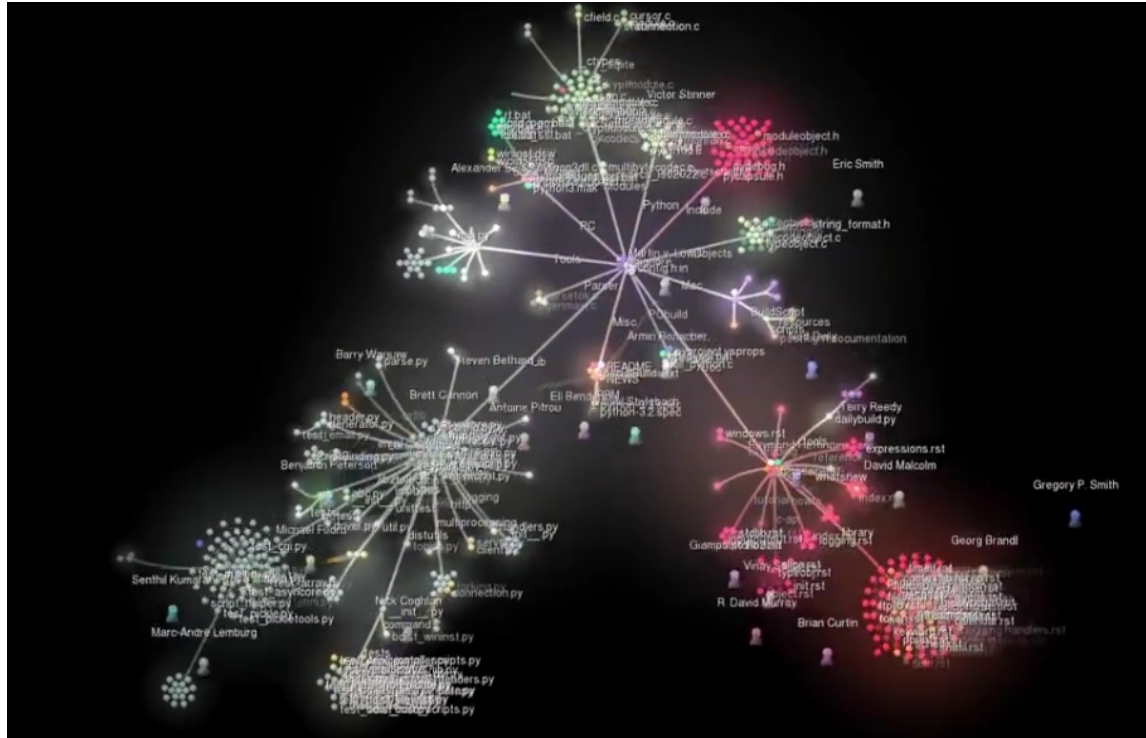
From day to day, there are three types of changes that are possible:

- Employees join the company
- Employees leave the company
- Employees change managers



<https://www.youtube.com/watch?v=mkJ-Uy5dt5g>

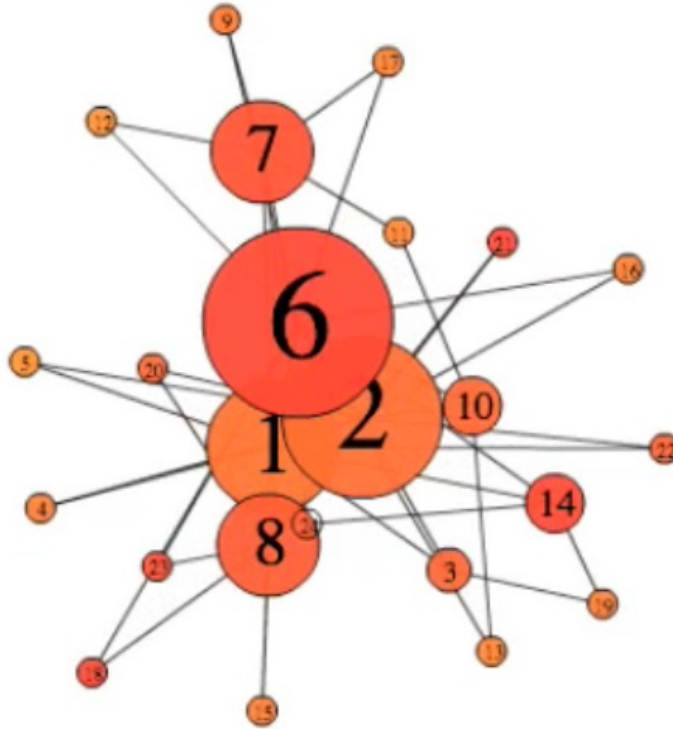
Growth of Python (Gource visualization)



<https://www.youtube.com/watch?v=cNBtDstOTmA>

“Good get richer”
(incl. Bianconi-Barabási model)

“Good get richer” simulation (number is attractiveness)



“Good get richer”

- A “good get richer” model is one where
 - Each node has an “attractiveness” (called “fitness”)
 - Preferential attachment is guided by this fitness η_i
- The probability of connecting to node i is:

$$\Pi_i = \frac{\eta_i k_i}{\sum_j \eta_j k_j}$$

Degree dynamics

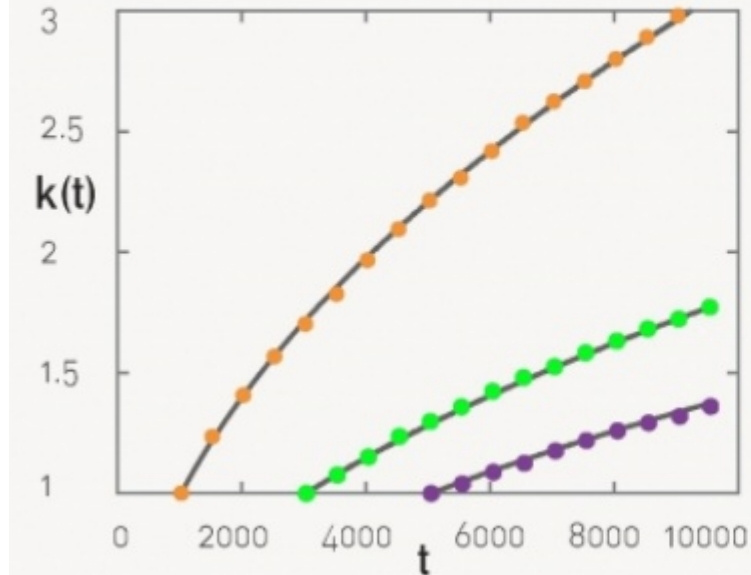
$$\frac{d}{dt}k_i = m \frac{\eta_i k_i}{\sum_j \eta_j k_j}$$

$$k_i(t; t_i, \eta_i) = m \left(\frac{t}{t_i} \right)^{\beta(\eta_i)}$$

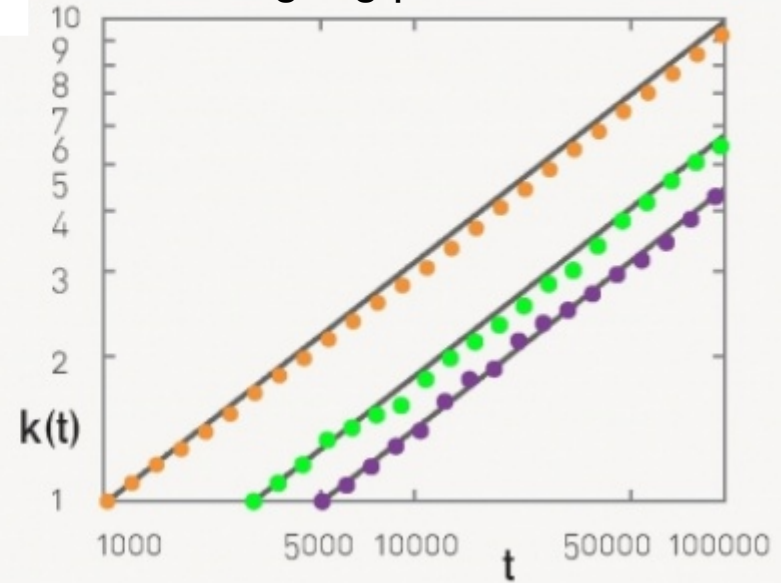
- With the dynamic exponent $\beta(\eta_i) \propto \eta_i$
- Remember that in linear preferential attachment $\beta = 1/2$ (for all nodes)

In preferential attachment (BA)
a “younger” node cannot overtake
an “older” node

Linear plot

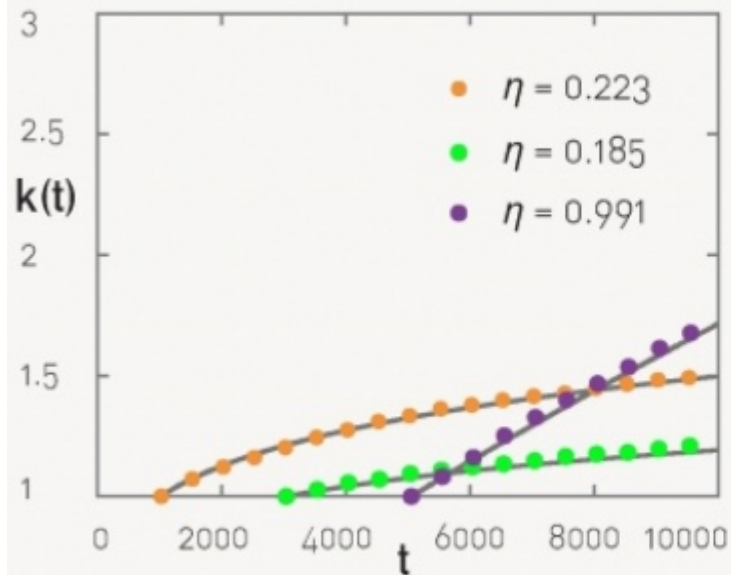


Log-log plot

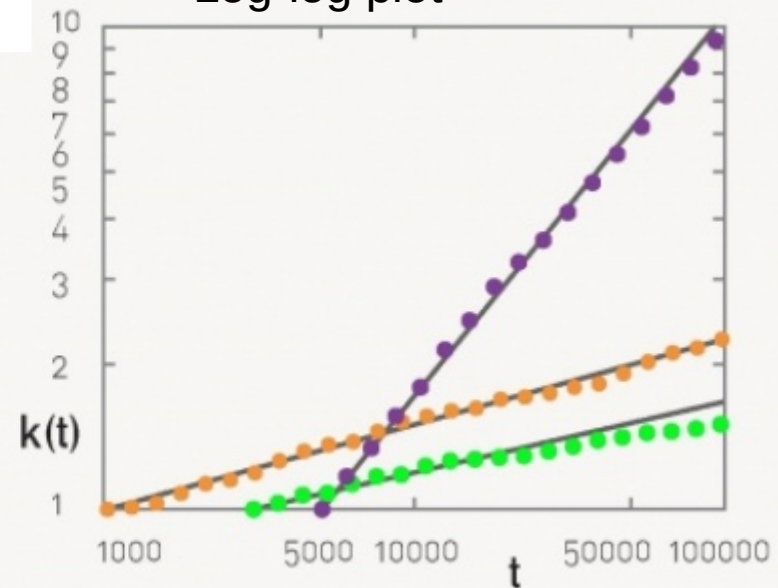


In good-get-richer (Bianconi-Barabási) this depends on node fitness

Linear plot



Log-log plot



Degree distribution

$$p_k \propto \int \frac{\rho(\eta)}{\eta} \left(\frac{m}{k}\right)^{\frac{c}{\eta}+1} d\eta \quad \eta \sim \rho(\eta)$$

- When η is constant this reduces to BA
- When η is uniformly distributed in $[0, 1]$ this also yields a power law but instead of $\gamma = 3$ we get $\gamma \approx 2.3$

Which distribution is more heterogeneous?

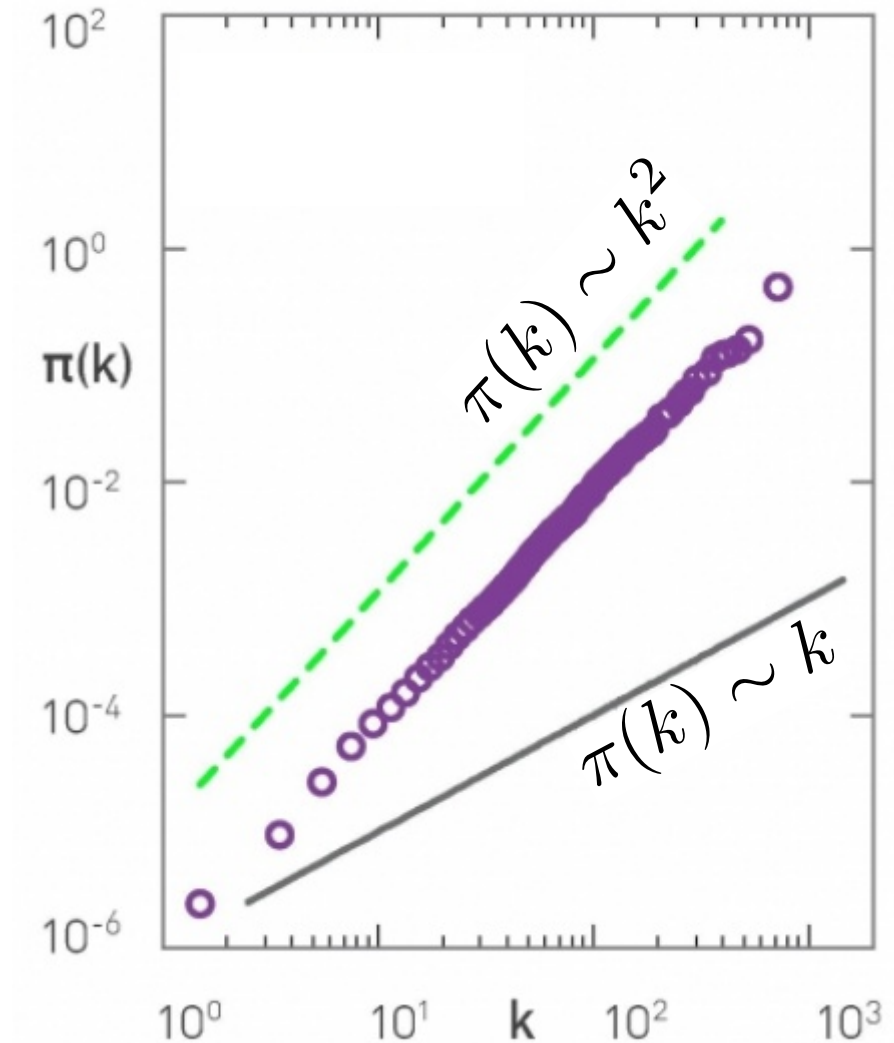
Measuring preferential attachment

Measuring preferential attachment

- We should try to measure $\Pi(k_i) \approx \frac{\Delta k_i}{\Delta t}$
- This can be too noisy
 - Why?
- Instead we will measure $\pi(k) = \sum_{k_i=0}^k \Pi(k_i)$
- If $\Pi(k_i)$ is constant $\pi(k) \propto k$
- If $\Pi(k_i) \propto k$ then $\pi(k) \propto k^2$

Preferential attachment in a citation network

- We observe it follows preferential attachment (with $\alpha = 1$) in this case



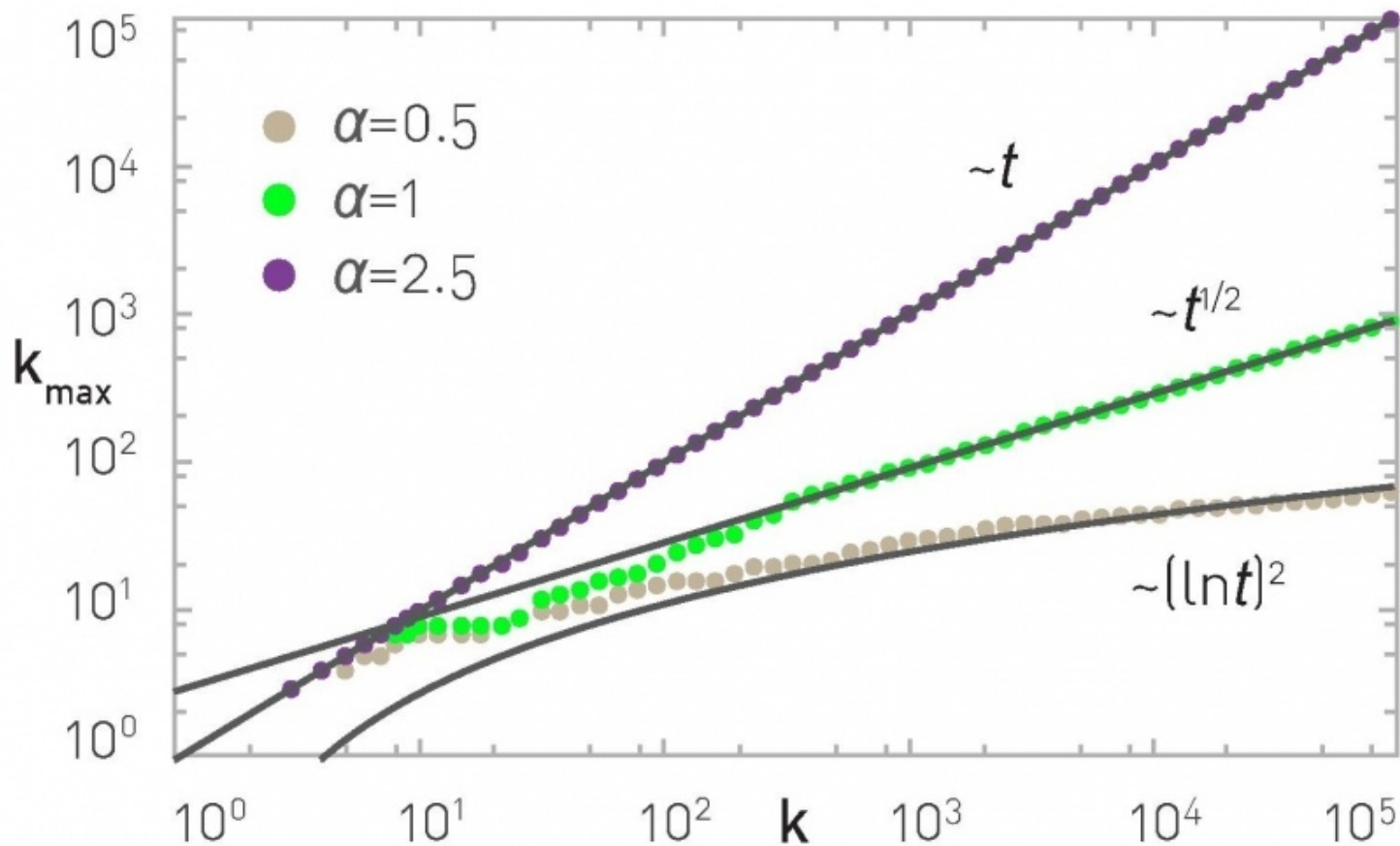
Sub-linear and super-linear preferential attachment

Sub-linear and super-linear preferential attachment

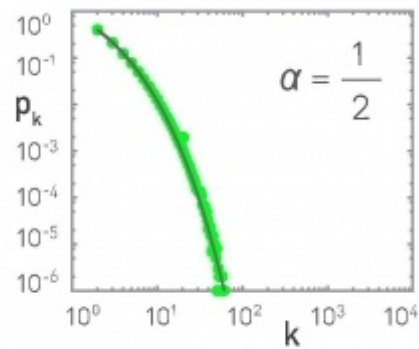
- The model we have studied so far has **linear preferential attachment** because $\frac{d}{dt}k_i \propto k_i$
- We could imagine cases where $\frac{d}{dt}k_i \propto k_i^\alpha$ for $\alpha > 1$ or $\alpha < 1$

What do you think should happen in each case?

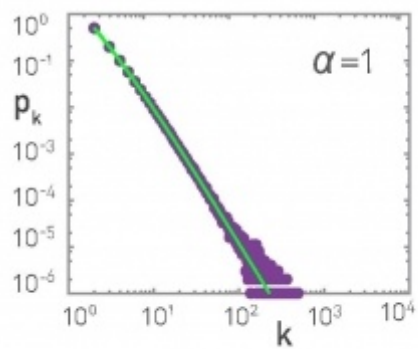
The degree of the largest hub k_{\max}



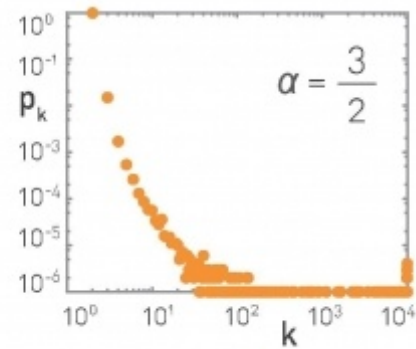
SUBLINEAR

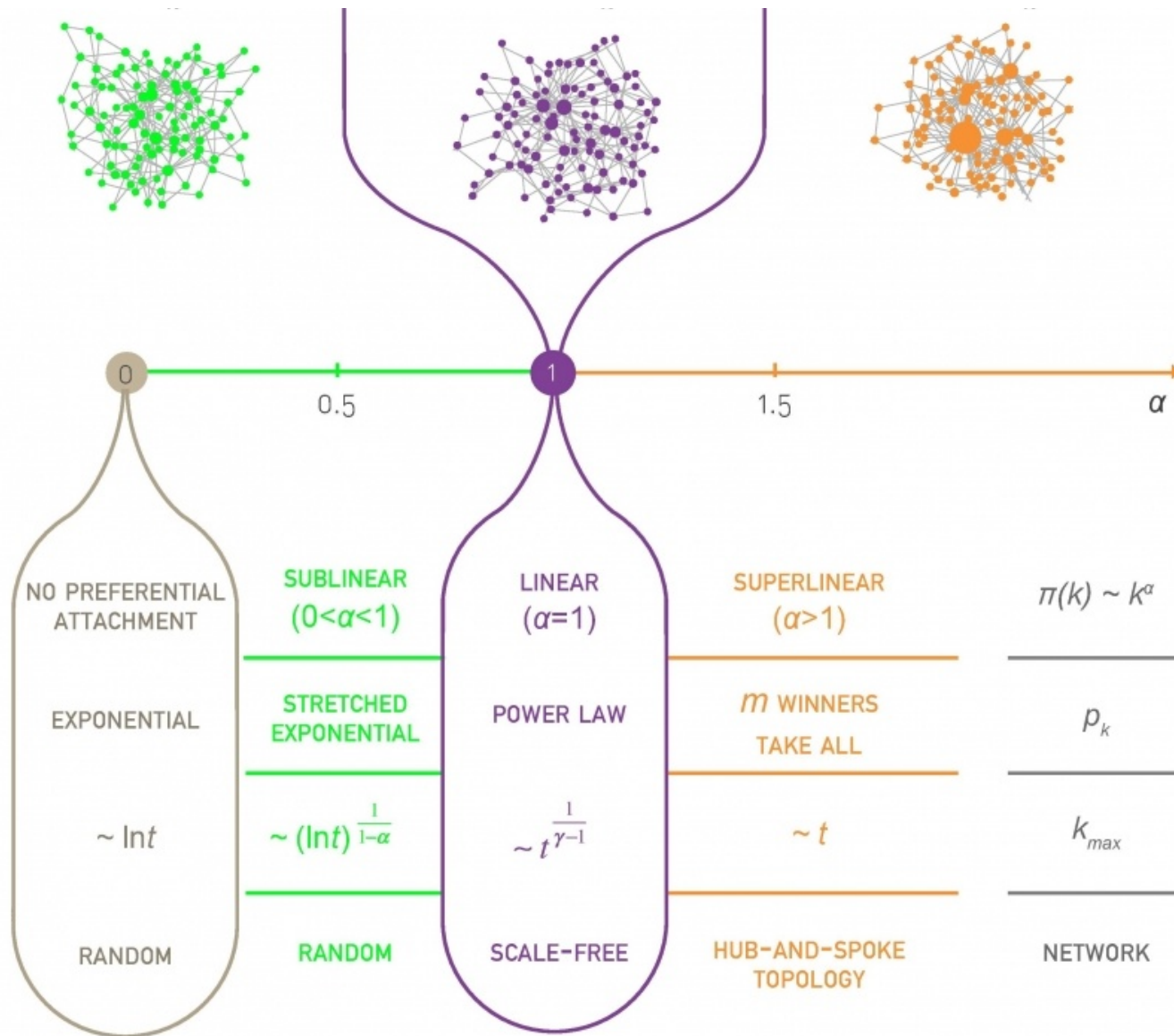


LINEAR



SUPERLINEAR





Aging effects

Sick Boy's unified theory of life from *Trainspotting* (1996)



In English: <https://www.youtube.com/watch?v=pQD-dXfHrvk>

In Spanish: https://www.youtube.com/watch?v=cN_WbiuqyQU

English (bad audio) subs in Spanish: <https://www.youtube.com/watch?v=4xTWD9GNRFA>

Aging effects

- Models without fitness but with a negative effect of age

$$\Pi(k_i, t - t_i) \approx k_i(t - t_i)^{-v}$$

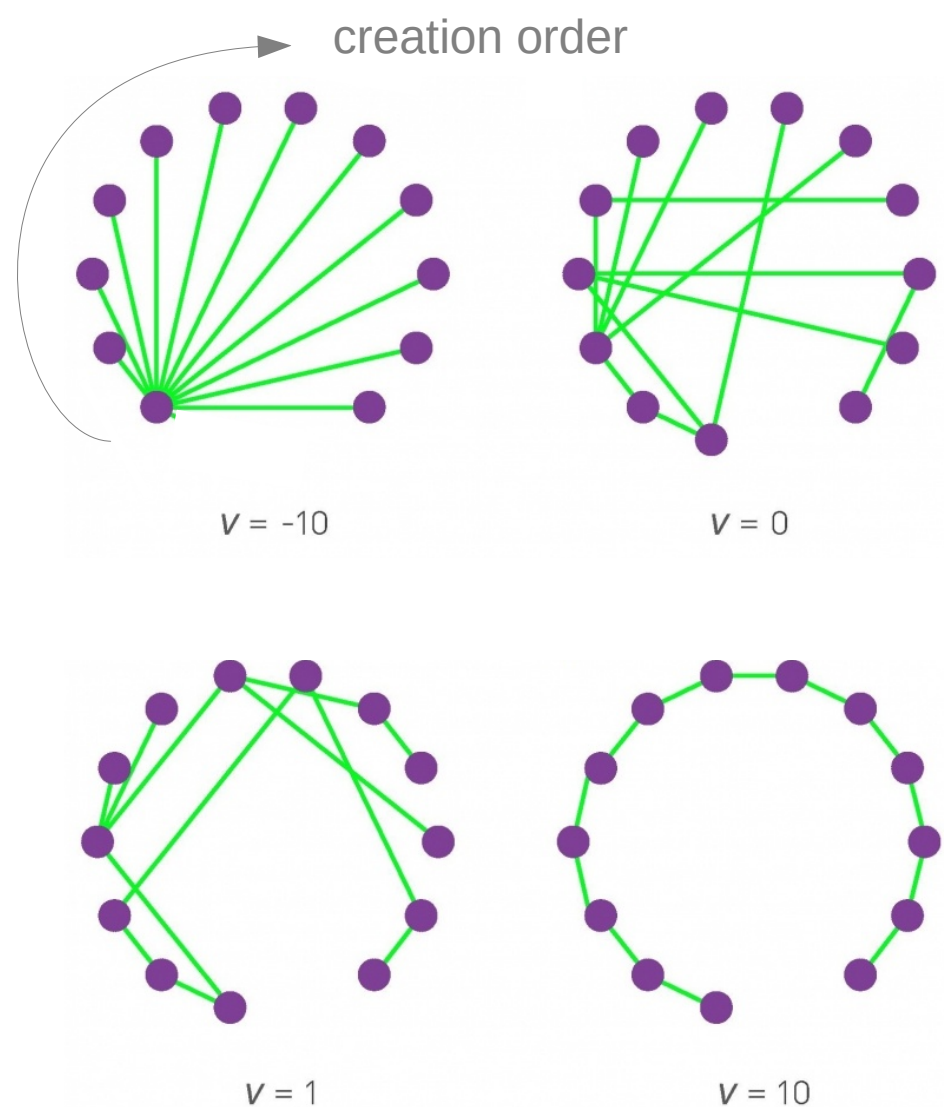
- Older nodes accumulate links more slowly
- Parameter v is the decay factor

Qualitatively, what would you expect if:

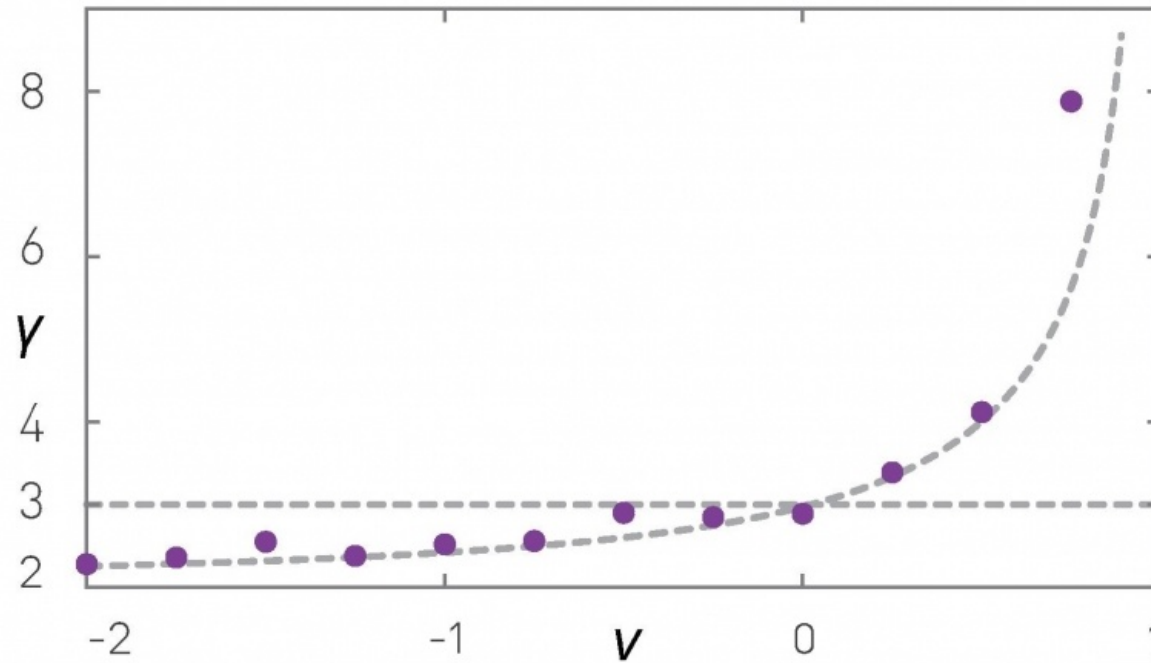
$$v < 0 \quad v = 0 \quad v \approx 1 \quad v \gg 1$$

Aging effects

- $v < 0$ favors older nodes
- $v = 0$ is simply preferential attachment
- $v \gg 1$ means only youngest are linked



Power-law exponent in models with aging ($N=10K$, $m=1$)



More
heterogeneous



More
homogeneous

Summary

Things to remember

- “Good-get-richer” model
- Sub-linear and super-linear preferential attachment
- Model with aging effects

Practice on your own

- Practice creating graphs using the different models
 - By hand
 - Or write your own code (it's not a lot of code)

Advanced materials
(not included in the exam):

(1) No preference (2) No growth

Remember preferential attachment

- Start with m_0 nodes
- At every time step
 - Add one new node u
 - Repeat m times
 - Pick a node v with probability $\Pi(k_v) = \frac{k_v}{\sum_j k_j}$
 - Connect u to v

Two simple variants

- No preference
 - Nodes receiving inlinks are picked uniformly at random
- No growth
 - The network starts with N nodes
 - No new nodes are created

No preference model

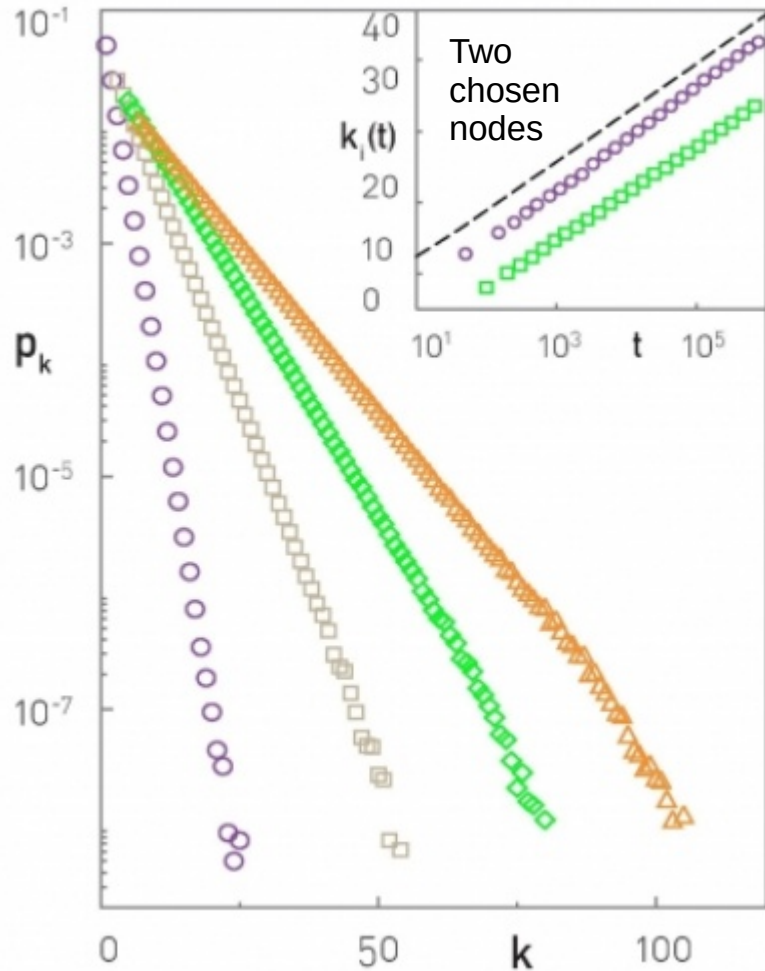
- Write the process on paper
- Write $\Pi(k_i)$
- Noting that $\frac{d}{dt}k_i = m\Pi(k_i)$ obtain $k_i(t)$

$$\int \frac{a}{b+x} = a \log(b+x) + C$$

No preference model (cont.)

- Compute $Pr(k_i(t) > k)$ assuming large t , t_i
- Use it to compute
$$Pr(k_i(t) \leq k) = 1 - Pr(k_i(t) > k)$$
- Derive to obtain $p_k = Pr(k_i(t) = k)$

Consequences of the “no preference” model



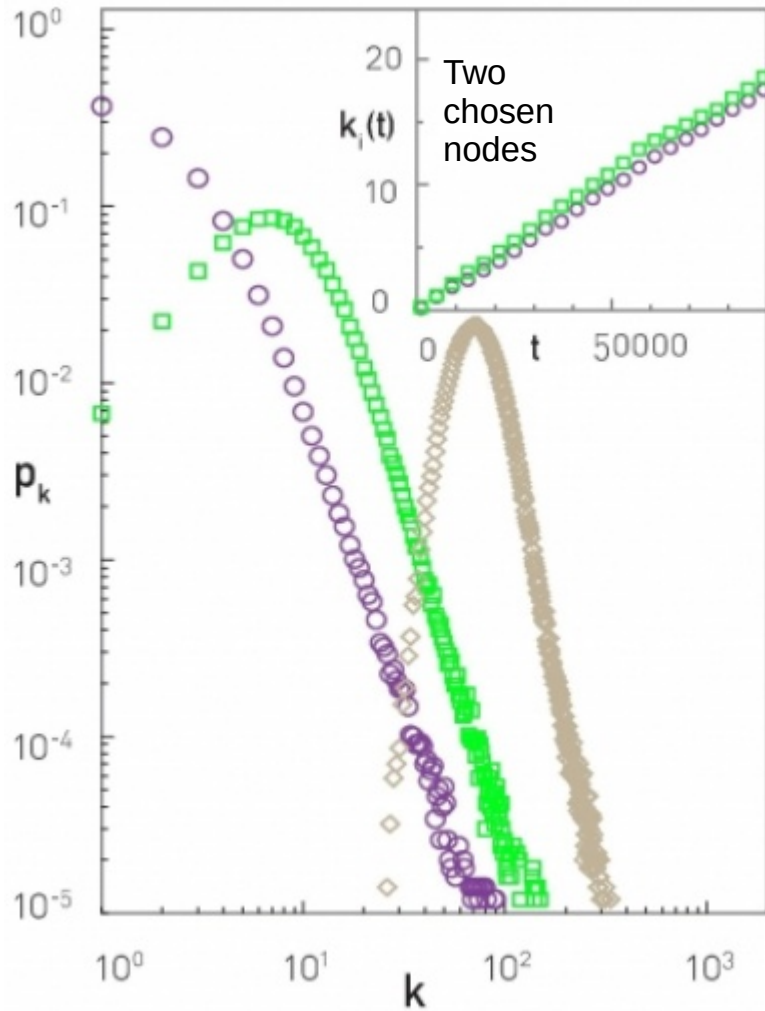
$m=1$, $m=3$, $m=5$, $m=7$

- Degree decays exponentially $p_k \propto e^{-k/m}$
- No power-law
- No large hubs

No growth model

- Write the process on paper
- You will need to impose $k_i(t_i) \neq 0$ why?
- Write $\Pi(k_i)$
- Noting that $\frac{d}{dt}k_i = \Pi(k_i)$ obtain $k_i(t)$

Consequences of the “no growth” model



$N=100K$

$t=N$, $t=5N$, $t=40N$

- Degree grows linearly $k_i(t) \propto t$
- Degree distribution is not stationary