

Epidemics

Introduction to Network Science

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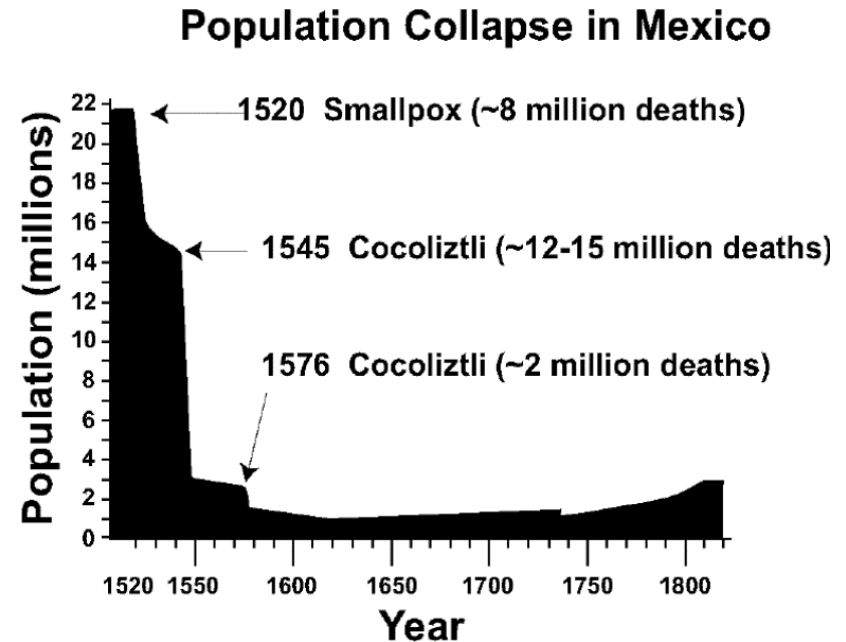
Topic 18

Sources

- Barabási (2016): [Network Science Ch. 10](#)
- Easley and Kleinberg (2010): Networks, Crowds, and Markets [Ch 21](#).

Examples: human epidemics

- Influenza, measles, STDs
- The “Black Death”
[next slide]
- Smallpox and other diseases brought by Europeans to America since early 1500s



1300s

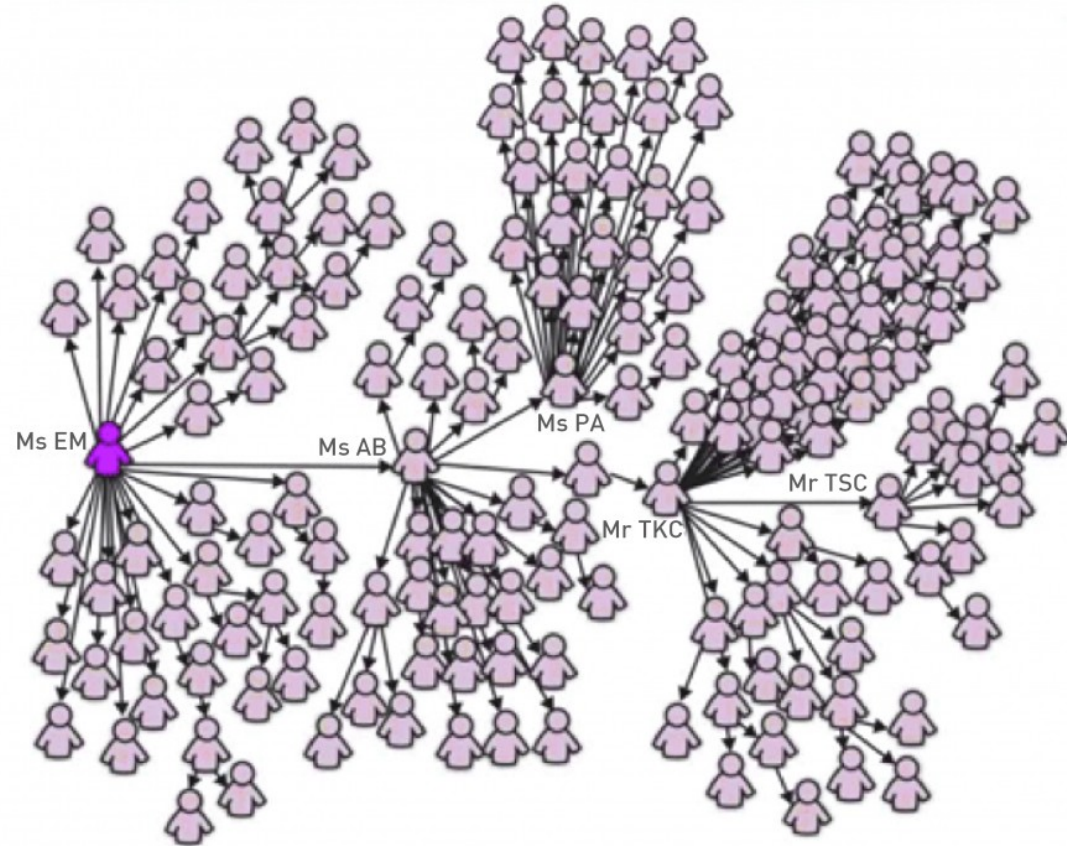
Killed 30%-60% of
the total population
of Europe



1346 1347 1348 1349 1350 1351 1352 1353

SARS Outbreak (2003)

- February 21st: Chinese doctor who have been several treating “atypical pneumonia” cases check-ins into hotel in Hong Kong
 - Hospitalized on Feb 22nd
 - Died on March 4th
- March 1st: “Ms. E. M.” returns to Singapore after visiting Hong Kong
 - Graph depicts 144 out of the first 206 SARS patients in Singapore
 - Ms. E. M. lived, various of her family members died



Diffusion of ideas vs diseases

- Adopting a new idea, behavior, fashion, product, taste, may also spread from person to person: “social contagion”
- There is a certain agency of the receiver
- In diffusion of diseases, we assume **there is no agency: each contagion is random**

Simple model: branching process

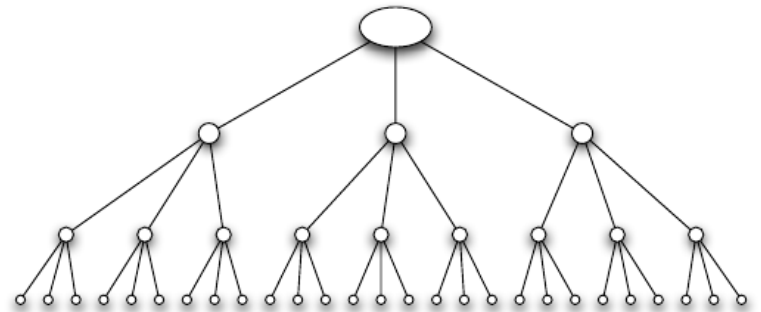
Modeling epidemics

- There are many factors:
 - Contagiousness
 - Length of infectious period,
 - Severity
 - ...
- Structure of contacts in a population

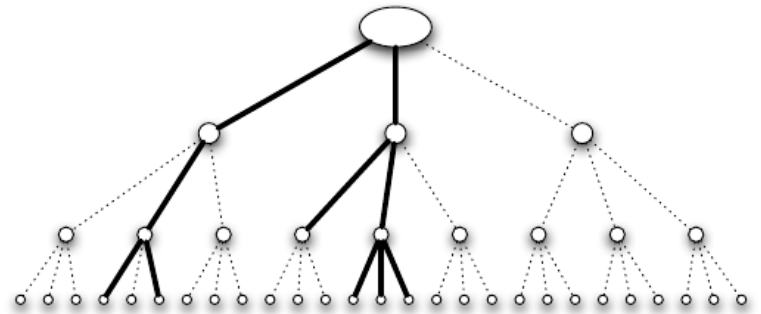
Simple model: branching process

- Each person interacts with other k people
- Each interaction ends in infection with probability β

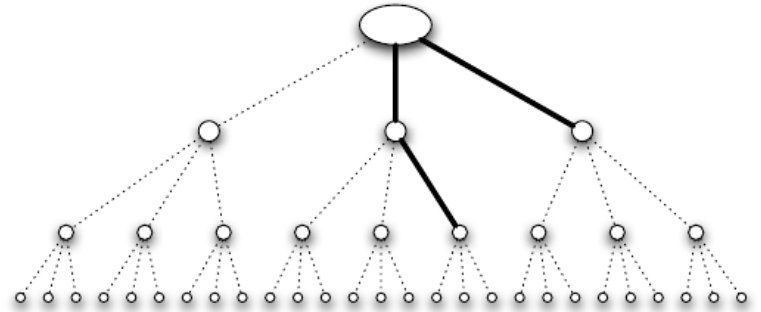
Example: $k=3$



(a) The contact network for a branching process



(b) With high contagion probability, the infection spreads widely



(c) With low contagion probability, the infection is likely to die out quickly

Transmission rate or “Basic reproductive number” R_0

- Each person interacts with other k people
- Each interaction ends in infection with probability β

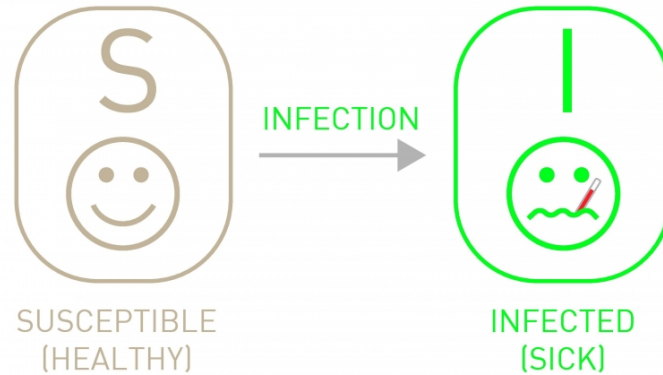
- What is the expected number of cases caused by a single individual, R_0 ?
- What do you think happens if $R_0 < 1$?
- What do you think happens if $R_0 > 1$?

Disease	Transmission	R_0
Measles	Airborne	12-18
Pertussis	Airborne droplet	12-17
Diphtheria	Saliva	6-7
Smallpox	Social contact	5-7
Polio	Fecal-oral route	5-7
Rubella	Airborne droplet	5-7
Mumps	Airborne droplet	4-7
HIV/AIDS	Sexual contact	2-5
SARS	Airborne droplet	2-5
Influenza (1918 strain)	Airborne droplet	2-3

Changing $R_0 = \beta k$

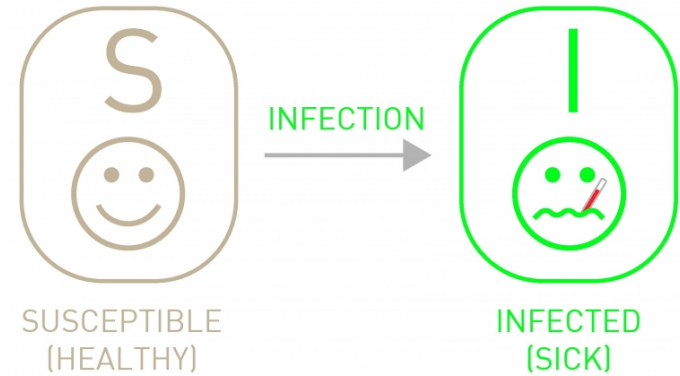
- Sanitary practices
(reduce β)
- Quarantine
(reduces k)

The SI model



The SI model

- Susceptible:
 - The node can catch the disease
- Infected:
 - The node has the disease and can spread it
 - It will stay sick forever



Notation

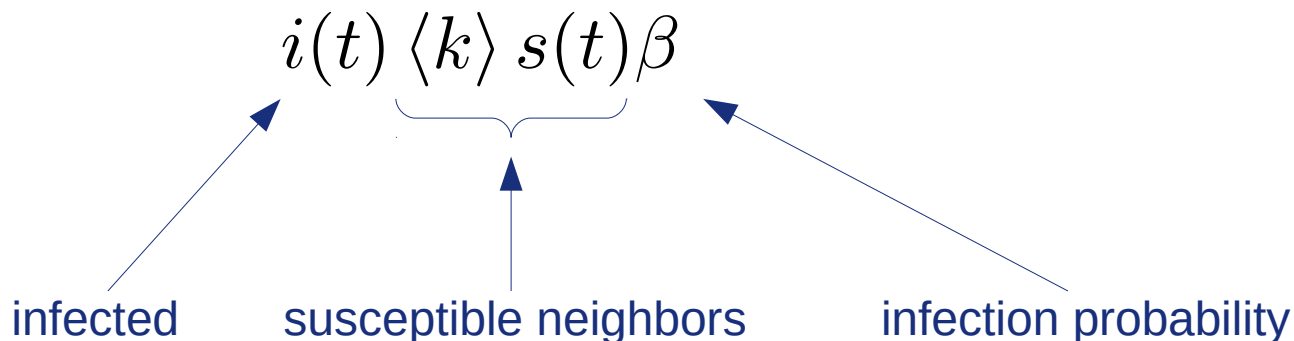
- Number of susceptible $S(t)$
 - Fraction of susceptible $s(t) = S(t) / N$
- Number of infected $I(t)$
 - Fraction of infected $i(t) = I(t) / N$
- $s(t) + i(t) = 1$

How many susceptible neighbors a node has?

$$\langle k \rangle \frac{S(t)}{N} = \langle k \rangle s(t)$$

How many new infections are produced?

(for every infected, iterate through its susceptible neighbors, infect with probability β)



Prove that $i(t) = \frac{i_0 e^{\beta \langle k \rangle t}}{1 - i_0 + i_0 e^{\beta \langle k \rangle t}}$

$$\frac{di(t)}{dt} = i(t) \langle k \rangle (1 - i(t)) \beta$$

Use $\frac{1}{i(1-i)} = \frac{1}{i} + \frac{1}{1-i}$ and integrate from $t = 0$ to t
Denote by $i_0 = i(t = 0)$

$$\int \frac{1}{x} dx = \log x + C$$

$$\int \frac{1}{1-x} dx = -\log(1-x) + C$$

Behavior in the limit $t \rightarrow \infty$

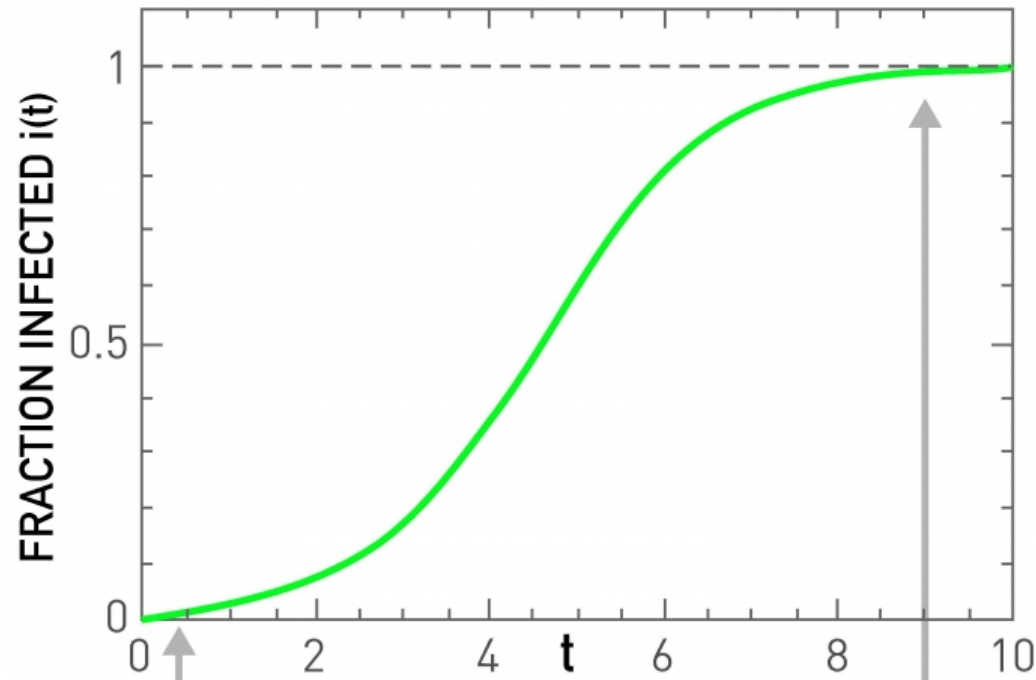
- What is the limit of $i(t) = \frac{i_0 e^{\beta \langle k \rangle t}}{1 - i_0 + i_0 e^{\beta \langle k \rangle t}}$ when $t \rightarrow \infty$?
- Hint: similar to $f(t) = \frac{e^t}{1 + e^t}$

Infected as a function of time (SI)

$$i(t) = \frac{i_0 e^{\beta \langle k \rangle t}}{1 - i_0 + i_0 e^{\beta \langle k \rangle t}}$$

Characteristic time
(to infect $1/e \approx 36\%$ of people):

$$\tau = \frac{1}{\beta \langle k \rangle}$$



exponential
regime

If i is small,

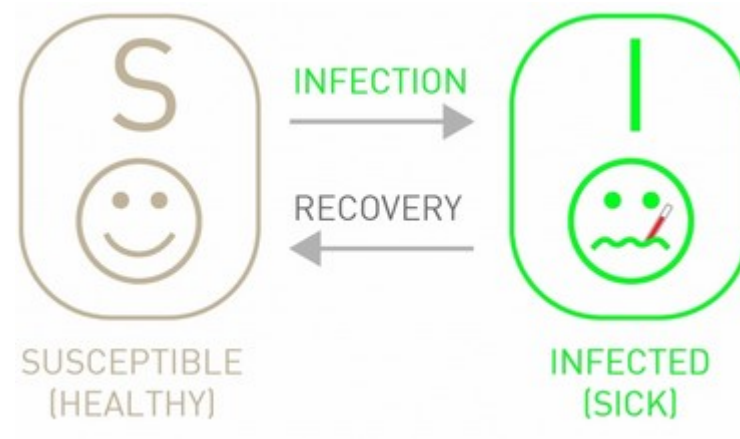
$$i \approx i_0 e^{\beta \langle k \rangle t}$$

saturation
regime

If $i \rightarrow 1$,

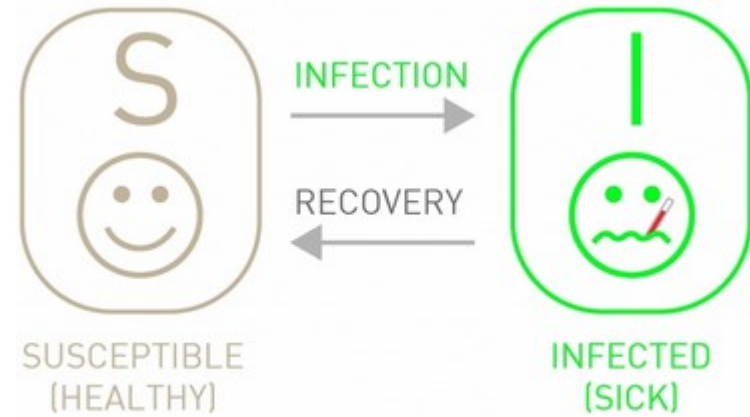
$$\frac{di}{dt} \rightarrow 0$$

The SIS model



The SIS model

- Susceptible:
 - The node can catch the disease
- Infected:
 - The node has the disease and can spread it
 - After some time, it recovers ... but it becomes susceptible again



Infection dynamics

$$\frac{di(t)}{dt} = \beta \langle k \rangle i(t)(1 - i(t)) - \mu i(t)$$

- μ is the recovery rate, i.e., the probability of becoming susceptible again in an unit of time

$$i(t) = \left(1 - \frac{\mu}{\beta \langle k \rangle}\right) \frac{C e^{(\beta \langle k \rangle - \mu)t}}{1 + C e^{(\beta \langle k \rangle - \mu)t}}$$

- C is a constant that depends on i_0

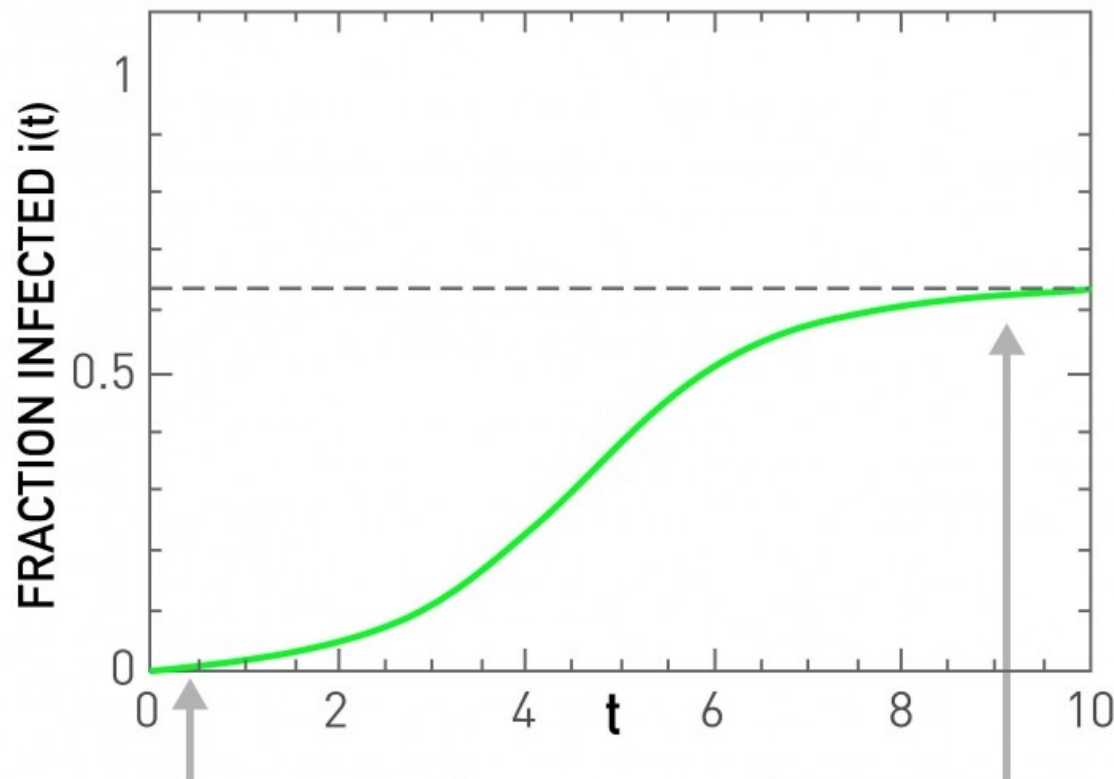
Behavior in the limit $t \rightarrow \infty$

- What is the limit of $i(t) = \left(1 - \frac{\mu}{\beta \langle k \rangle}\right) \frac{C e^{(\beta \langle k \rangle - \mu)t}}{1 + C e^{(\beta \langle k \rangle - \mu)t}}$ when $t \rightarrow \infty$?
- Hint: similar to $f(t) = \alpha \frac{e^t}{1 + e^t}$

Infected as a function of time (SIS)

$$i(t) = \left(1 - \frac{\mu}{\beta \langle k \rangle}\right) \frac{C e^{(\beta \langle k \rangle - \mu)t}}{1 + C e^{(\beta \langle k \rangle - \mu)t}}$$

This is in the case $\mu < \beta \langle k \rangle$



**exponential
outbreak**

If i is small,
 $i \approx i_0 e^{(\beta \langle k \rangle - \mu)t}$

**endemic
state**

$i(\infty) = 1 - \frac{\mu}{\beta \langle k \rangle}$

What happens if $\mu > \beta \langle k \rangle$?

- Remember: $\frac{di(t)}{dt} = \beta \langle k \rangle i(t)(1 - i(t)) - \mu i(t)$

The SIR model



The SIR model



- **Susceptible:**
 - The node can catch the disease
- **Infected:**
 - The node has the disease and can spread it
- **Removed:**
 - The node no longer has the disease, and cannot catch it or propagate it again (could be dead, could be immune)

Infection dynamics in SIR

$$\frac{di(t)}{dt} = \beta \langle k \rangle i(t)(1 - r(t) - i(t)) - \mu i(t)$$

$$\frac{dr(t)}{dt} = \mu i(t)$$

$$\frac{ds(t)}{dt} = -\frac{di(t)}{dt} - \frac{dr(t)}{dt} = -\beta \langle k \rangle i(t)(1 - r(t) - i(t))$$

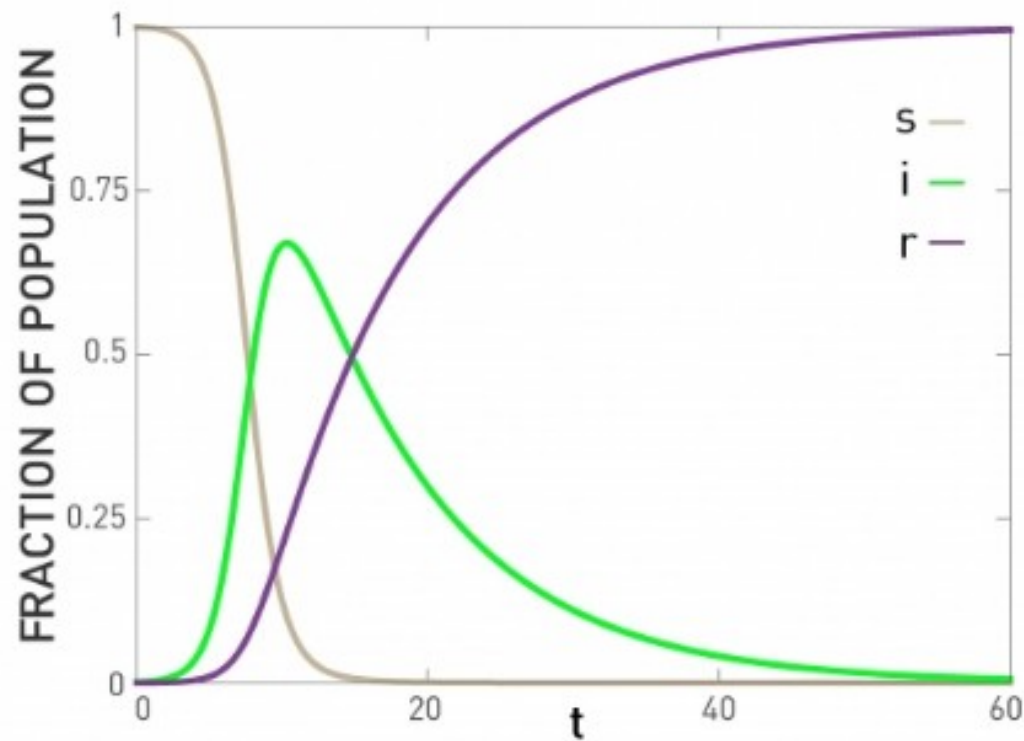
- No closed form solution

Infection dynamics (SIR)

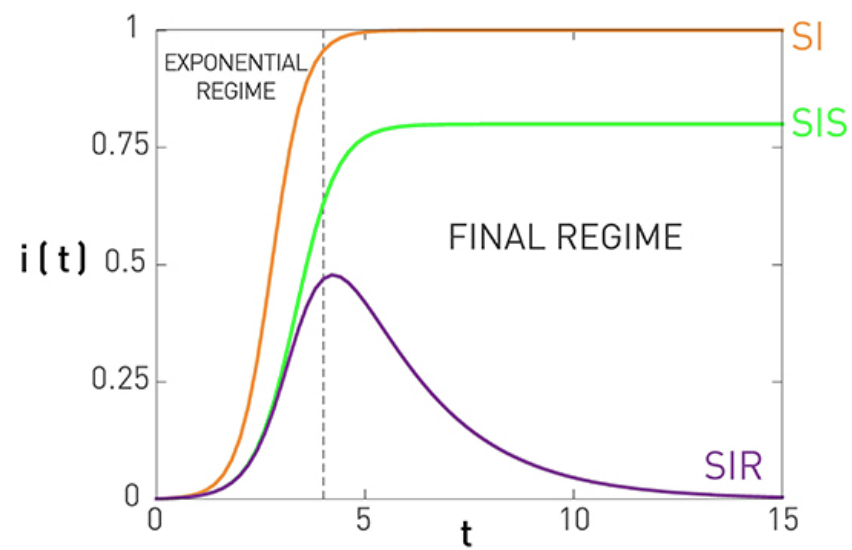
$$\frac{di(t)}{dt} = \beta \langle k \rangle i(t)(1 - r(t) - i(t)) - \mu i(t)$$

$$\frac{dr(t)}{dt} = \mu i(t)$$

$$\frac{ds(t)}{dt} = -\beta \langle k \rangle i(t)(1 - r(t) - i(t))$$



Comparison of $i(t)$



	SI	SIS	SIR
Exponential Regime: Number of infected individuals grows exponentially	$i = \frac{i_0 e^{\beta\langle k \rangle t}}{1 - i_0 + i_0 e^{\beta\langle k \rangle t}}$	$i = \left(1 - \frac{\mu}{\beta\langle k \rangle}\right) \frac{C e^{(\beta\langle k \rangle - \mu)t}}{1 + C e^{(\beta\langle k \rangle - \mu)t}}$	No closed solution
Final Regime: Saturation at $t \rightarrow \infty$	$i(\infty) = 1$	$i(\infty) = 1 - \frac{\mu}{\beta\langle k \rangle}$	$i(\infty) = 0$
Epidemic Threshold: Disease does not always spread	No threshold	$R_0 = 1$	$R_0 = 1$

SI and SIS on a graph

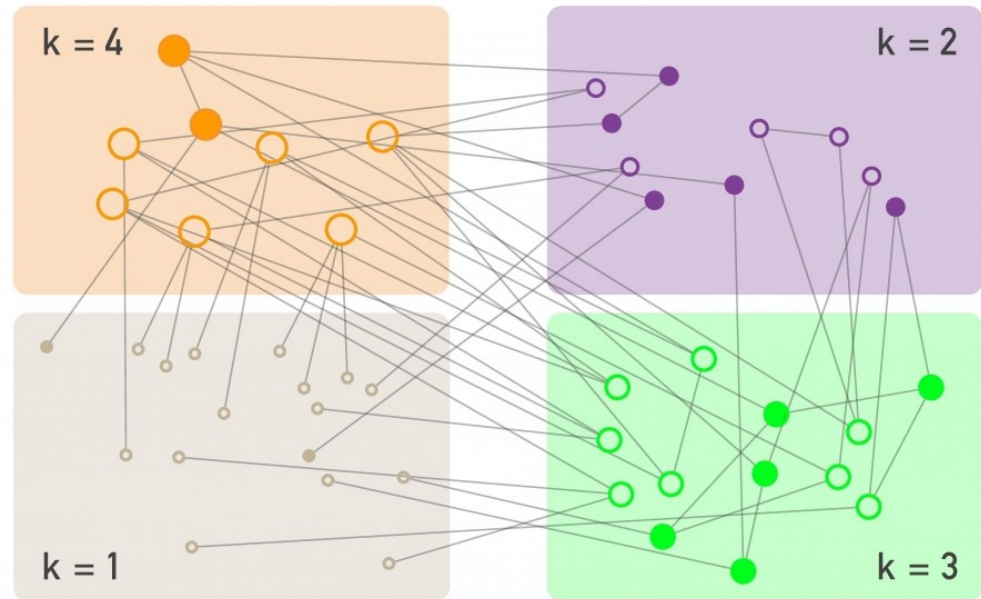
SI dynamics on a graph



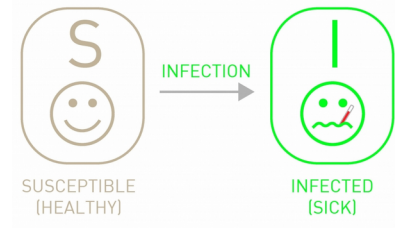
- Degree block approximation: all nodes with the same degree are jointly analyzed

$$i_k(t) = \frac{I_k(t)}{N_k}$$

$$i(t) = \sum_k i_k(t) p_k$$



SI dynamics on a graph



- Degree block approximation: all nodes with the same degree are jointly analyzed

$$\frac{di_k(t)}{dt} = \Theta_k k (1 - i_k(t)) \beta$$

Similar to simple SI model, except:

Θ_k is the fraction of infected neighbors of a susceptible node of degree k

Compare with simple SI model:

$$\frac{di(t)}{dt} = i(t) \langle k \rangle (1 - i(t)) \beta$$

SI model on a graph: infected as a function of time

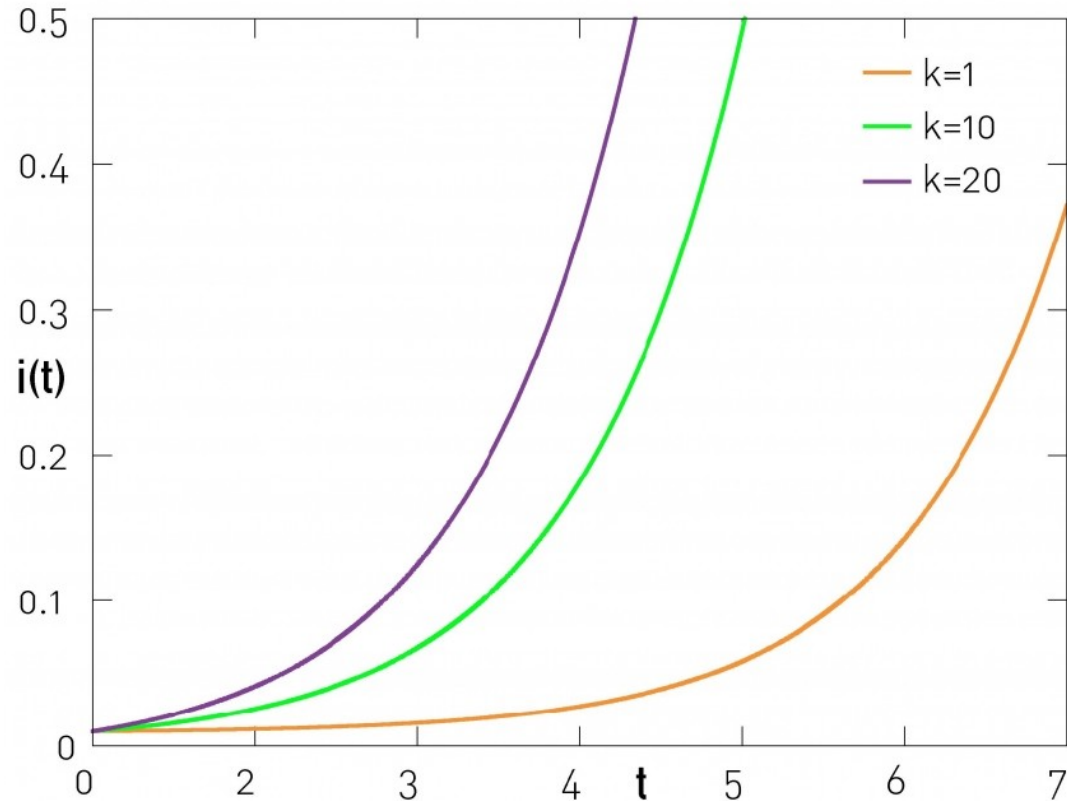
$$i_k(t) \approx i_0 \left(1 + k \frac{\langle k \rangle - 1}{\langle k^2 \rangle - \langle k \rangle} \left(e^{t/\tau^{SI}} - 1 \right) \right)$$

What can you say about $i_k(t)$?

$$\tau^{SI} = \frac{\langle k \rangle}{\beta (\langle k^2 \rangle - \langle k \rangle)}$$

Characteristic time, i.e., the time to infect $1/e \simeq 36\%$ of nodes

Higher degree nodes are more likely to become infected



$$i_k(t) = i_0 \left(1 + \frac{k (\langle k \rangle - 1)}{\langle k^2 \rangle - \langle k \rangle} \left(e^{t/\tau^{SI}} - 1 \right) \right)$$

Characteristic time

(time to infect $1/e \simeq 36\%$ of nodes)

$$\tau^{SI} = \frac{\langle k \rangle}{\beta (\langle k^2 \rangle - \langle k \rangle)}$$

- Random network

$$\langle k^2 \rangle = \langle k \rangle (\langle k \rangle + 1) \Rightarrow$$

- Scale-free network with $\gamma \geq 3$

$$\langle k \rangle, \langle k^2 \rangle \text{ are finite} \Rightarrow$$

- Scale-free network with $\gamma < 3$

$$\langle k^2 \rangle \xrightarrow{N \rightarrow \infty} \infty \Rightarrow$$

Characteristic time

(time to infect $1/e \simeq 36\%$ of nodes)

$$\tau^{SI} = \frac{\langle k \rangle}{\beta (\langle k^2 \rangle - \langle k \rangle)}$$

- Random network

$$\langle k^2 \rangle = \langle k \rangle (\langle k \rangle + 1) \Rightarrow \tau_{ER}^{SI} = \frac{1}{\beta \langle k \rangle}$$

- Scale-free network with $\gamma \geq 3$

$$\langle k \rangle, \langle k^2 \rangle \text{ are finite} \Rightarrow \tau_{BA}^{SI} \text{ is finite}$$

- Scale-free network with $\gamma < 3$

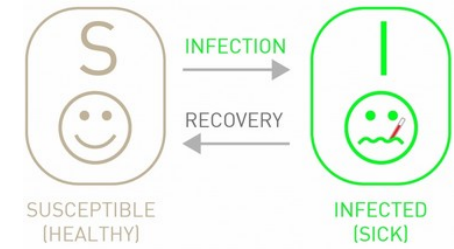
$$\langle k^2 \rangle \xrightarrow{N \rightarrow \infty} \infty \Rightarrow \lim_{N \rightarrow \infty} \tau_{BA}^{SI} = 0$$

Vanishing characteristic time

$$\tau^{SI} = \frac{\langle k \rangle}{\beta (\langle k^2 \rangle - \langle k \rangle)}$$

- If $\lim_{N \rightarrow \infty} \frac{\langle k \rangle}{\langle k^2 \rangle} = 0$ the characteristic time goes to 0
- Networks with skewed degree distributions allow infections with the same β to spread faster

SIS dynamics on a graph



Similar to SI dynamics but allowing recovery

$$\frac{di_k(t)}{dt} = k(1 - i_k(t))\Theta_k\beta - \mu i_k(t) \quad \tau^{SIS} = \frac{\langle k \rangle}{\beta \langle k^2 \rangle - \mu \langle k \rangle}$$

If people recover quickly, $\tau < 0$ and the infection dies out ... unless $\langle k^2 \rangle \rightarrow \infty$

Epidemic threshold

- A key quantity is the spreading rate $\lambda = \frac{\beta}{\mu}$
- The critical spreading rate λ_c called the epidemic threshold, is such that $\tau > 0$
- Spreading rate larger than epidemic threshold:
 - infection becomes endemic
- Spreading rate smaller than epidemic threshold:
 - Infection dies out

Epidemic threshold

- A key quantity is the spreading rate $\lambda = \frac{\beta}{\mu}$
- The critical spreading rate λ_c called the epidemic threshold, is such that $\tau > 0$

Compute the epidemic threshold for an ER graph where $\langle k^2 \rangle = \langle k \rangle (\langle k \rangle + 1)$

$$\tau^{SIS} = \frac{\langle k \rangle}{\beta \langle k^2 \rangle - \mu \langle k \rangle}$$

Epidemic threshold in a scale-free network

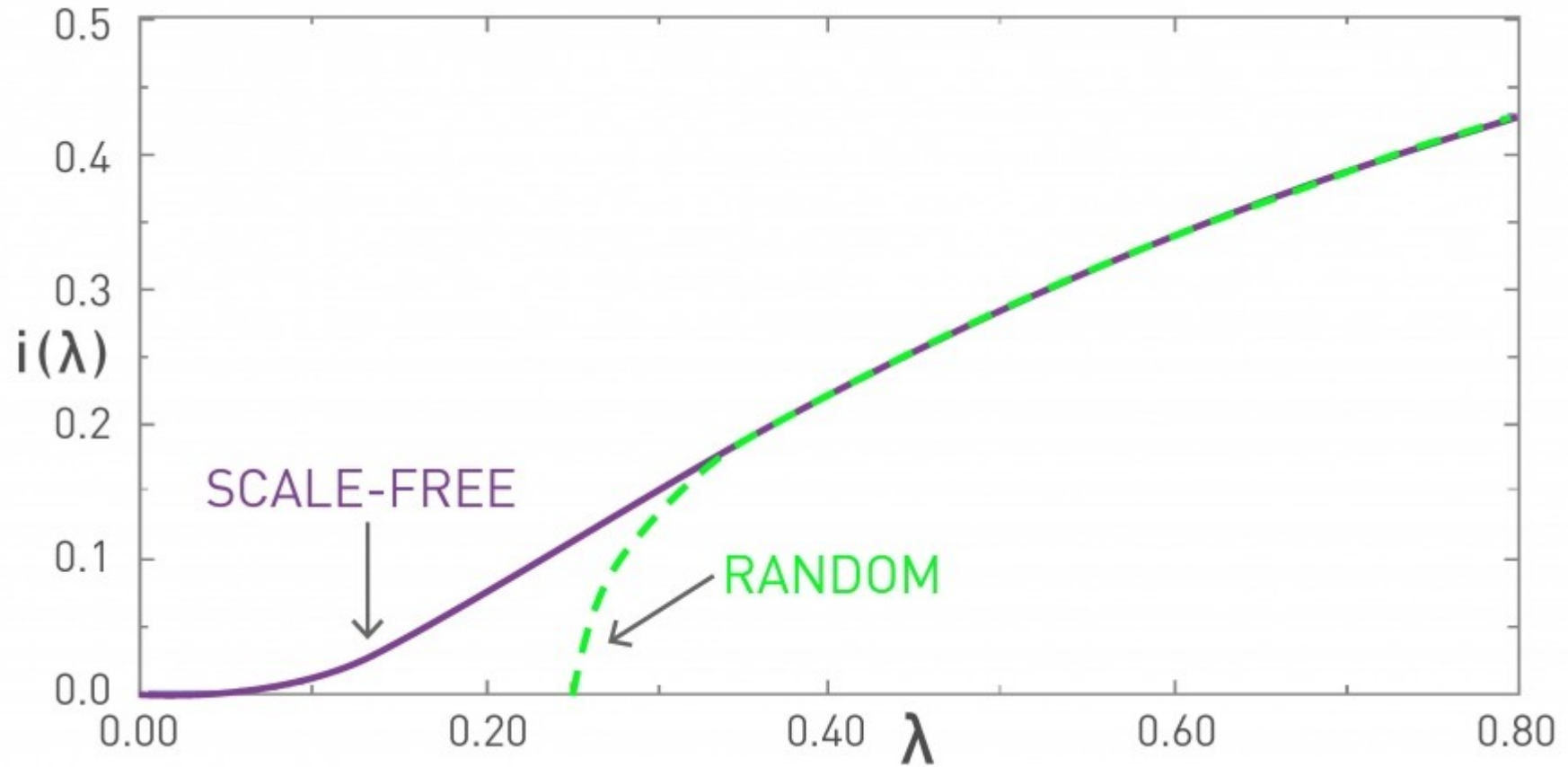
$$\tau^{SIS} = \frac{\langle k \rangle}{\beta \langle k^2 \rangle - \mu \langle k \rangle} > 0 \Rightarrow \frac{\beta}{\mu} > \frac{\langle k \rangle}{\langle k^2 \rangle} = \lambda_c$$

Epidemic threshold in BA graph

- In a scale-free network with $\gamma < 3$

$$\langle k^2 \rangle \xrightarrow{N \rightarrow \infty} \infty \Rightarrow \lim_{N \rightarrow \infty} \tau^{SIS} = 0$$

Infected (in the limit) as a function of the epidemic threshold



Two key results for SI and SIS models on a BA graph

In a large scale-free network with $\gamma < 3$

- An infection may reach everybody in a very short time: $\tau = 0$
- An infection may become endemic even if it is not very contagious and even if people recover fast: $\lambda_c = 0$