Distances in Scale-Free Networks

Introduction to Network Science
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Topic 10

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Contents

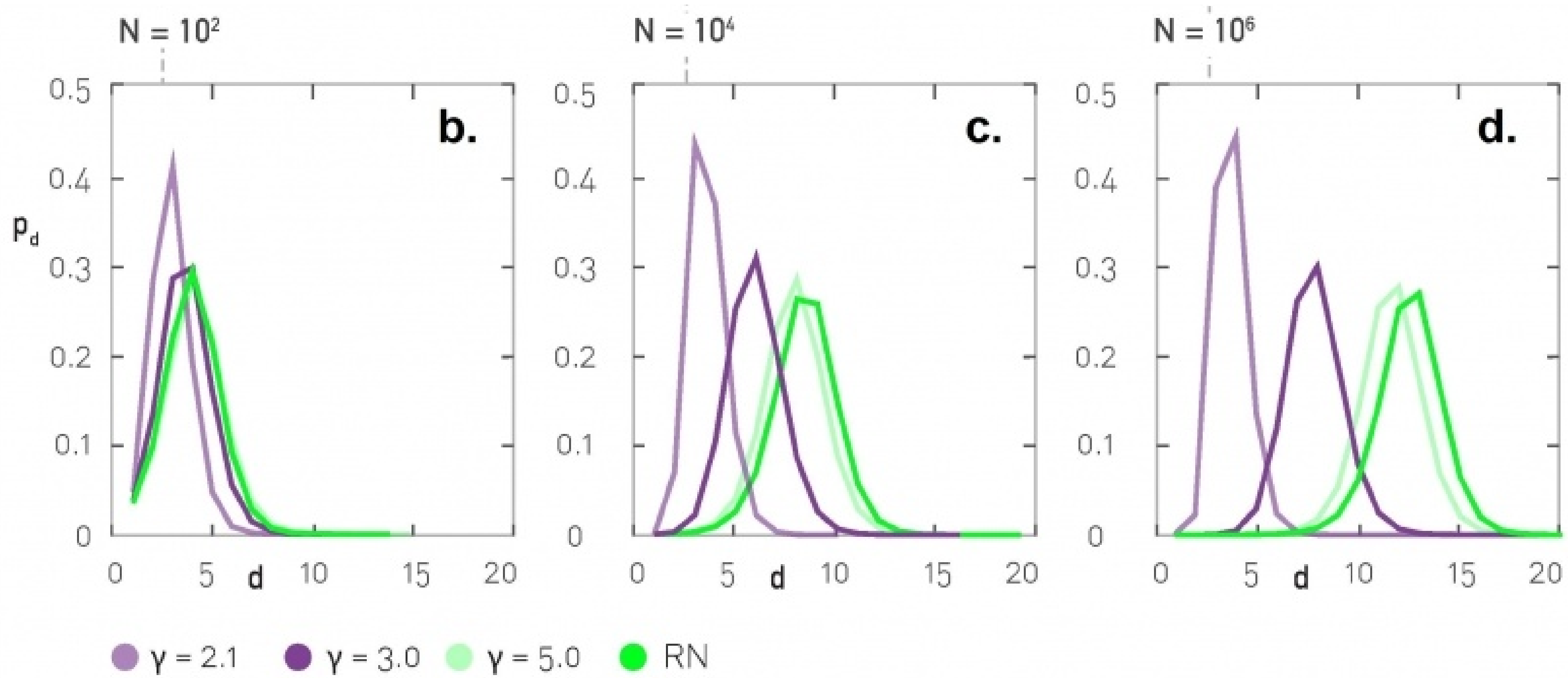
- Distance distribution of scale-free networks

Sources

- Albert László Barabási: Network Science. Cambridge University Press, 2016.
  - Follows almost section-by-section chapter 04
- URLs cited in the footer of specific slides

Distance distributions: simulation results

Scale-free networks of increasing size, <k> = 3



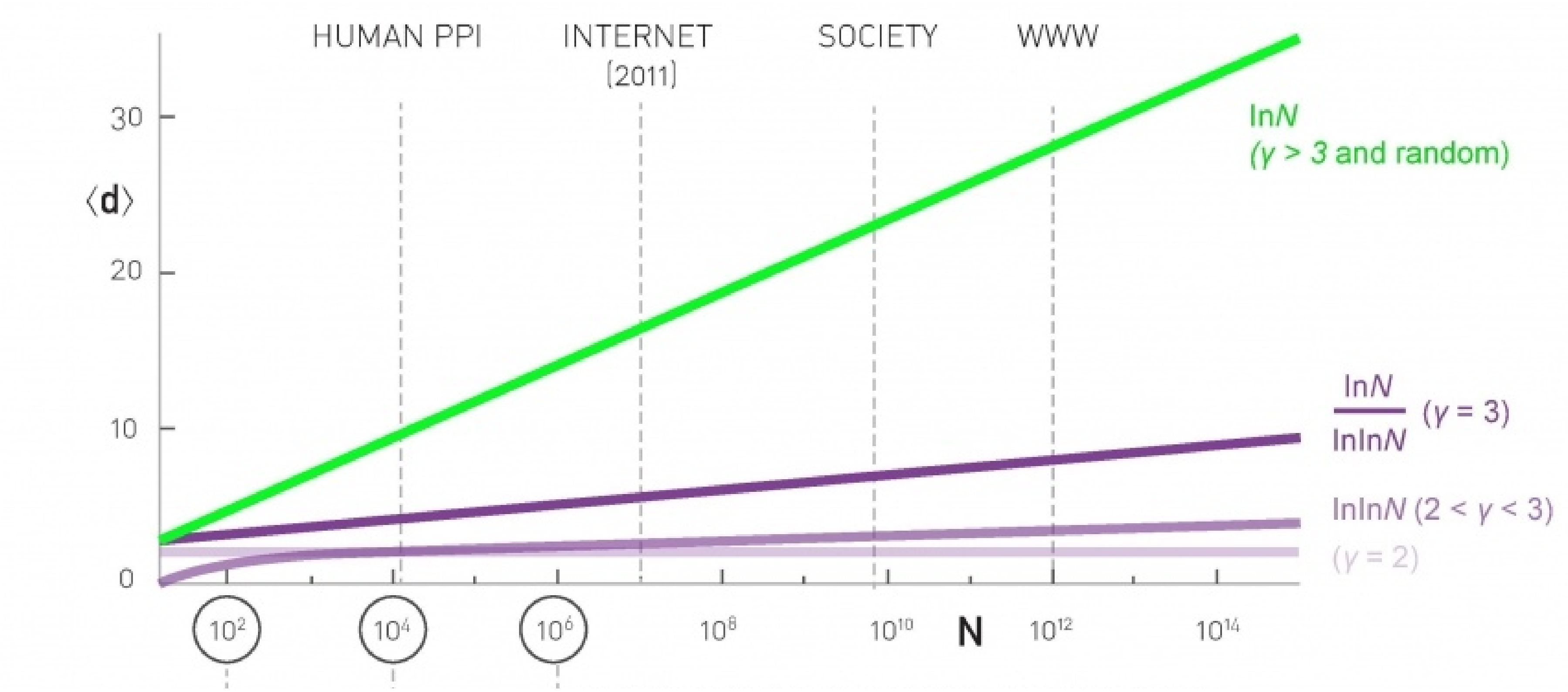
Average distance

- Depends on ɣ and N

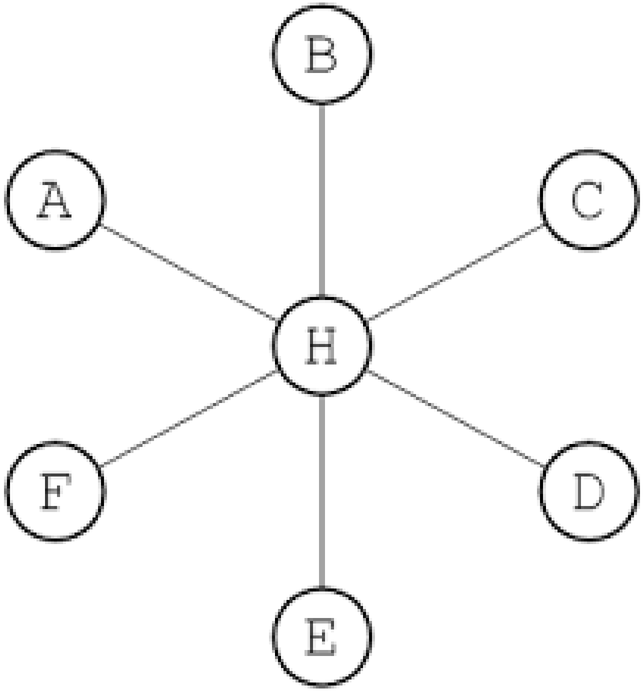
$$\langle d \rangle = \begin{cases} \text{const.} & \text{if } \gamma = 2 \\ \log \log \text{N} & \text{if } 2 < \gamma < 3 \\ \log \text{N} / \log \log \text{N} & \text{if } \gamma = 3 \\ \log \text{N} & \text{if } \gamma > 3 \end{cases}$$

Same as in ER graphs

Average distance and N



Anomalous regime $\gamma = 2$



Ultra-small world $2 < \gamma < 3$

- Average distance follows log(log(N))
- Example (humans):

$$N \approx 7 \times 10^9$$

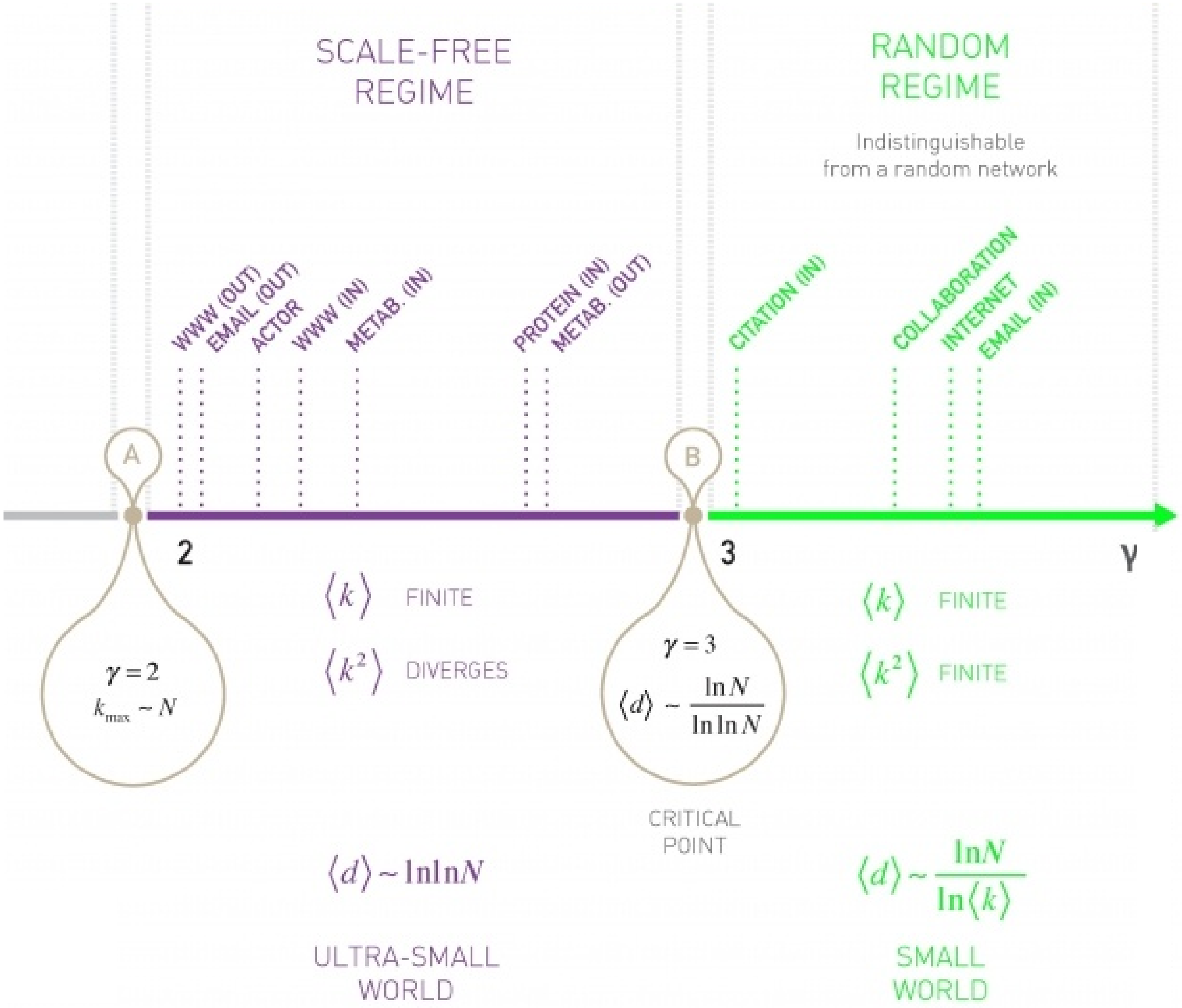
$$\log N \approx 22.66$$

$$\log \log N \approx 3.12$$

Small world $\gamma > 3$

- Average distance follows log(N)
- Similar to ER graphs where it followed log(N)/log(<k>)

The degree distribution exponent plays an important role



When $\gamma > 3$

- In this case it is hard to distinguish this case from an ER graph
- In most real complex networks (but not all)

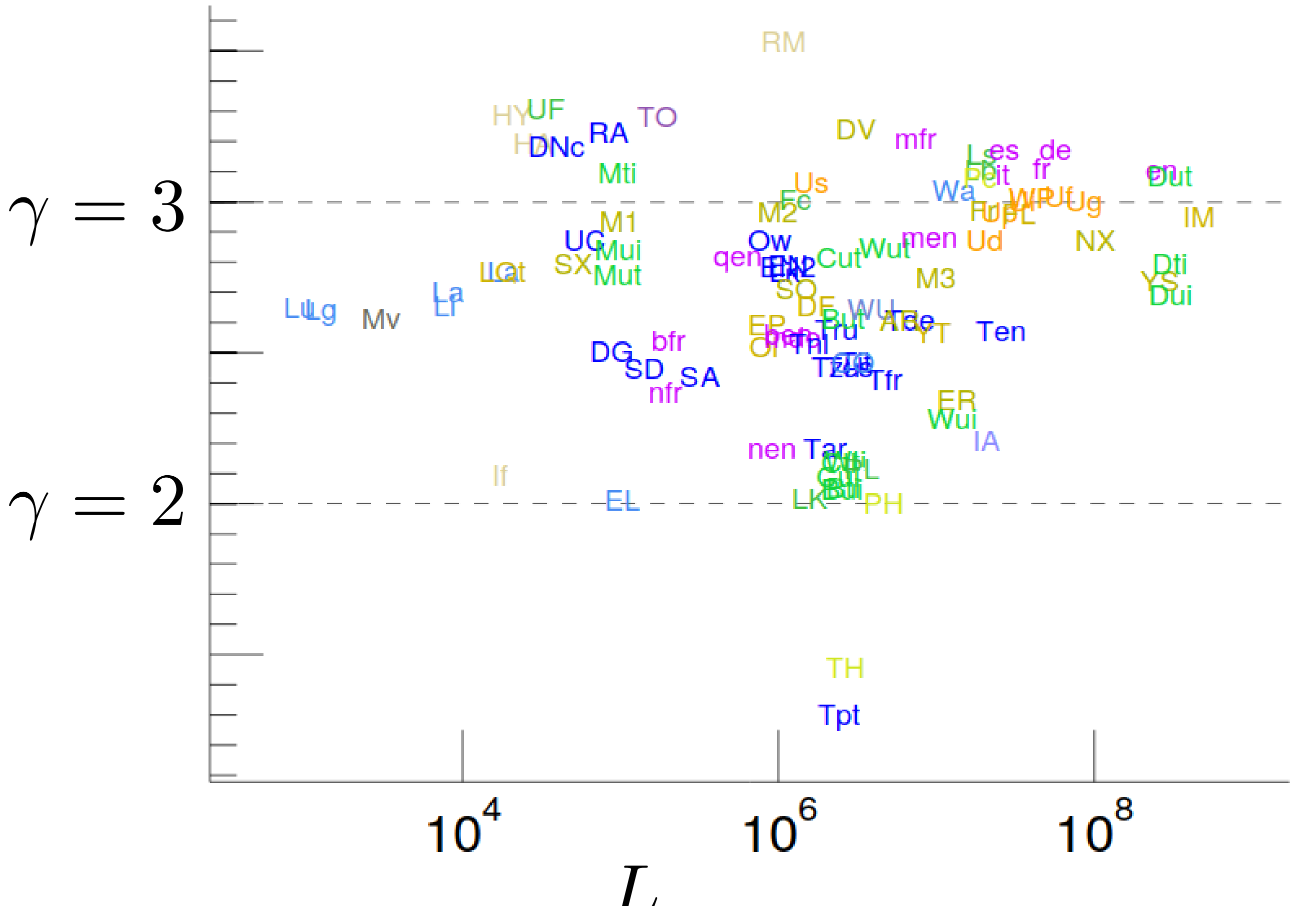
$$2 < \gamma < 3$$

When $\gamma > 3$

- Remember $k_{\max} = k_{\min} N^{\frac{1}{\gamma-1}}$ $\quad N = \left(\frac{k_{\max}}{k_{\min}}\right)^{\gamma-1}$
- Observing the scale-free properties requires that $k_{max} >> k_{min}$, e.g. $k_{max} = 10\ k_{min}$
- Then if $\gamma = 5, N > 10^8$
- Hence we won’t find many such networks

Examples

http://konect.uni-koblenz.de/statistics/prefatt



EL	Wikipedia elections
LK	Linux kernel mailing list threads
Bui	BibSonomy u-i
Bti	BibSonomy t-i
Cui	CiteULike u-i
If	Infectious
PL	Prosper loans
Cti	CiteULike t-i
Wti	Twitter t-i
nen	Wikinews (en)
Tar	Wikipedia talk, Arabic
Wui	Twitter u-i
ER	Epinions
nfr	Wikinews (fr)
Tfr	Wikipedia talk, French
SD	Slashdot
Tzh	Wikipedia talk, Chinese
Tes	Wikipedia talk, Spanish

Etc.

The friendship paradox

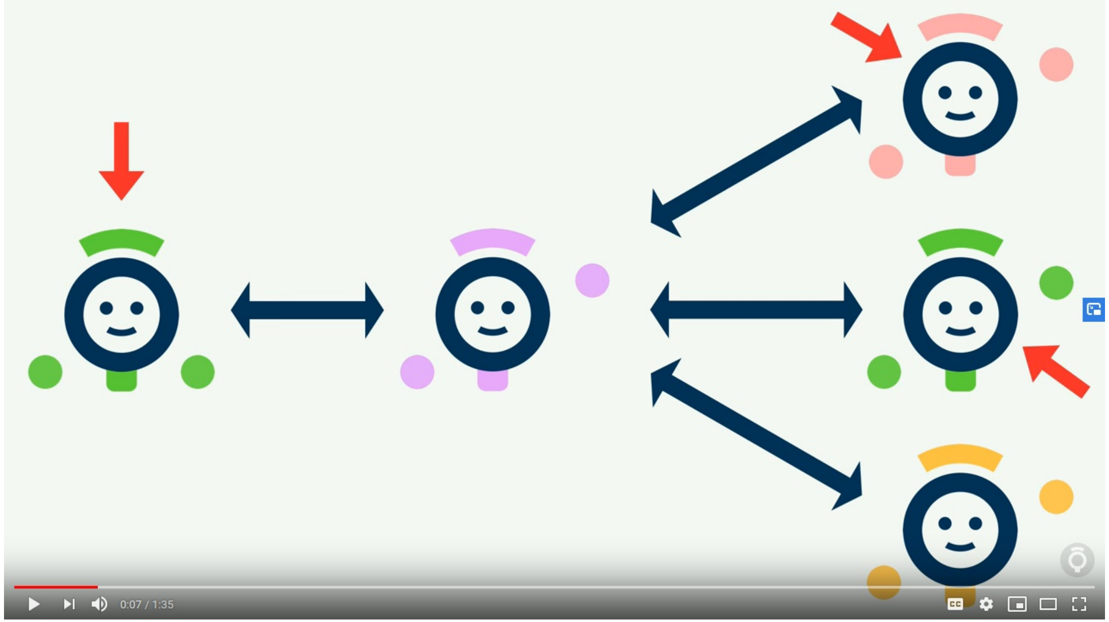
The friendship paradox

- Take a random person x, what is the expected degree of this person? <k>
- Take a random person x, now pick one of x’s neighbors, let’s say y

  What is the expected degree of y?

  It is not <k>

Sampling bias and the friendship paradox (1’35”)



https://www.youtube.com/watch?v=httLvVufAYs

Imagine you're at a random airport on earth

- Is it more likely to be …

  a large airport or a small airport?

- If you take a random flight out of it …

  will it go to a large airport or a small airport?

An example of friendship paradox

- Pick a random airport on Earth
  - Most likely <u>it will be a small airport</u>
- However, no matter how small it is, it **will** have flights to big airports
- On average those airports will have much larger degree

Exercise [B. 2016, Ex. 4.10.2] "Friendship Paradox"

- Remember p_k is the probability that a node has k "friends"
- If we randomly select a link, the probability that a node at any end of the link has k friends is $q_k = C\ k\ p_k$ where C is a normalization factor
  - (a) Find C (the sum of q_k must be 1)

Answer in Nearpod Collaborate
https://nearpod.com/student/
Access to be provided during class

Exercise [B. 2016, Ex. 4.10.2] "Friendship Paradox"

- If we randomly select a link, the probability that a node at any end of the link has k friends is $q_k = C\ k\ p_k$ where C is a normalization factor
- q_k is also the prob. that a randomly chosen node has a neighbor of degree k

(b) Find its expectation $E[q_k]$ which we will call $<k_F>$

Remember $E[X] = \sum_{X_{\min}}^{X_{\max}} x \cdot P(X = x)$

Exercise [B. 2016, Ex. 4.10.2] "Friendship Paradox"

(c) Compute the expected number of friends of a neighbor of a randomly chosen node in the case below

(d) compare with the expected number of friends of a randomly chosen node

$$N = 10000$$

$$\gamma = 2.3$$

$$k_{\min} = 1$$

$$k_{\max} = 1000$$

$$\langle k^n \rangle = C \frac{k_{\max}^{n-\gamma+1} - k_{\min}^{n-\gamma+1}}{n - \gamma + 1}$$

$$C = (\gamma - 1) k_{\min}^{\gamma-1}$$

Code

```python
def degree_moment(kmin, kmax, moment, gamma):
    C = (gamma-1.0)*(kmin**(gamma-1.0))
    numerator = (kmax**(moment-gamma+1.0) - kmin**(moment-gamma+1.0))
    denominator = (moment-gamma+1.0)
    return C * numerator / denominator
```

```python
kavg = degree_moment(kmin=1, kmax=1000, moment=1, gamma=2.3)
print(kavg)
```

```
3.787798988222529
```

```python
ksqavg = degree_moment(kmin=1, kmax=1000, moment=2, gamma=2.3)
print(ksqavg)
```

```
231.94329076177414
```

```python
print(ksqavg / kavg)
```

```
61.23431879119234
```

Summary

Things to remember

- Regimes of distance and connectivity
- The friendship paradox

Practice on your own

- Remember the regimes of a graph given <k>

  (It's useful to know this by heart)

- Estimate degree distributions and distance distributions for some graphs
- Draw a small graph, and sample from that graph until you're convinced $\langle k_F \rangle > \langle k \rangle$