Hierarchical clustering

Introduction to Network Science
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Topic 14

Universitat Pompeu Fabra Barcelona

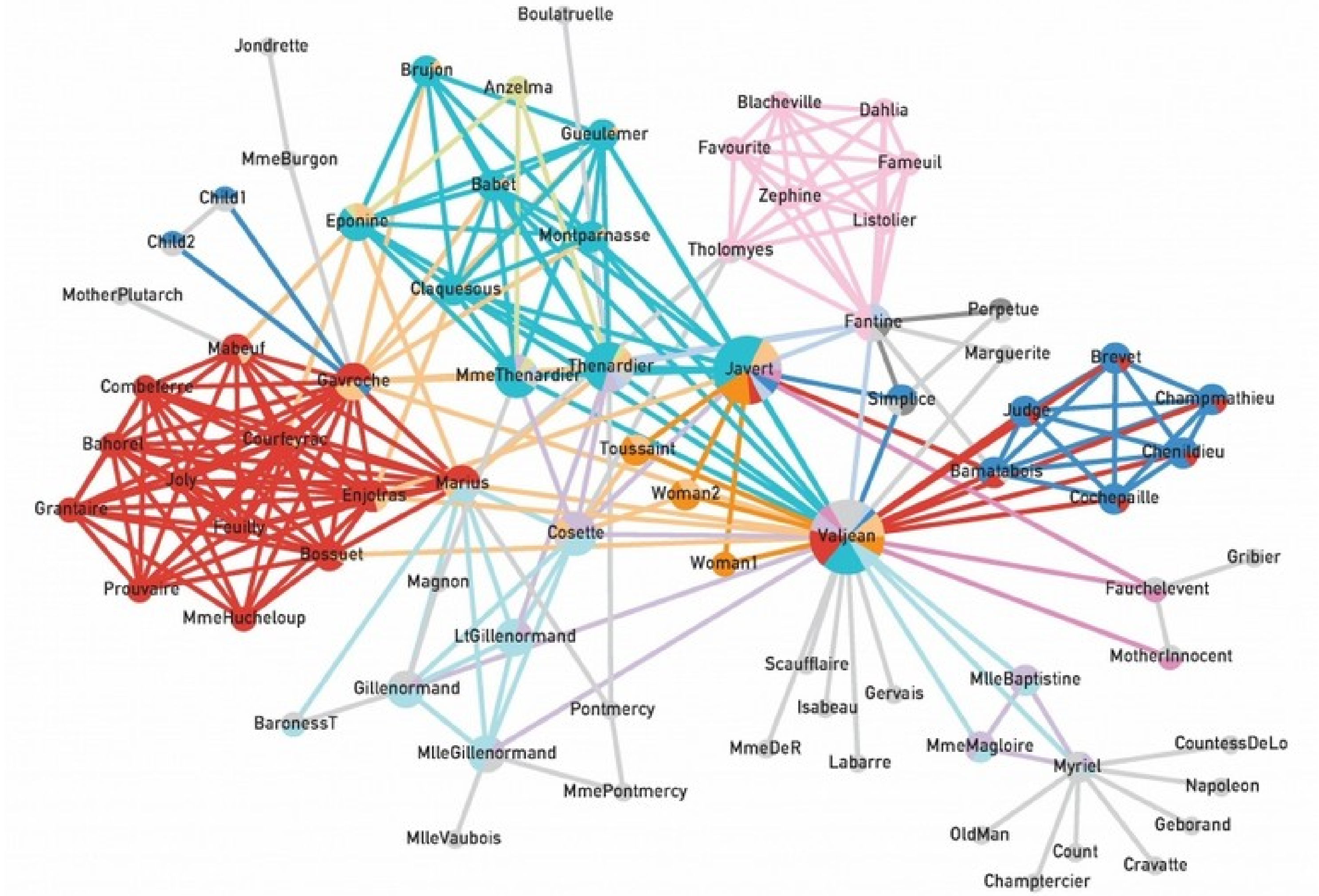
Sources

- Barabási 2016 Chapter 9
- Networks, Crowds, and Markets Ch 3
- C. Castillo (2017) Dense Sub-Graphs and Graph partitioning

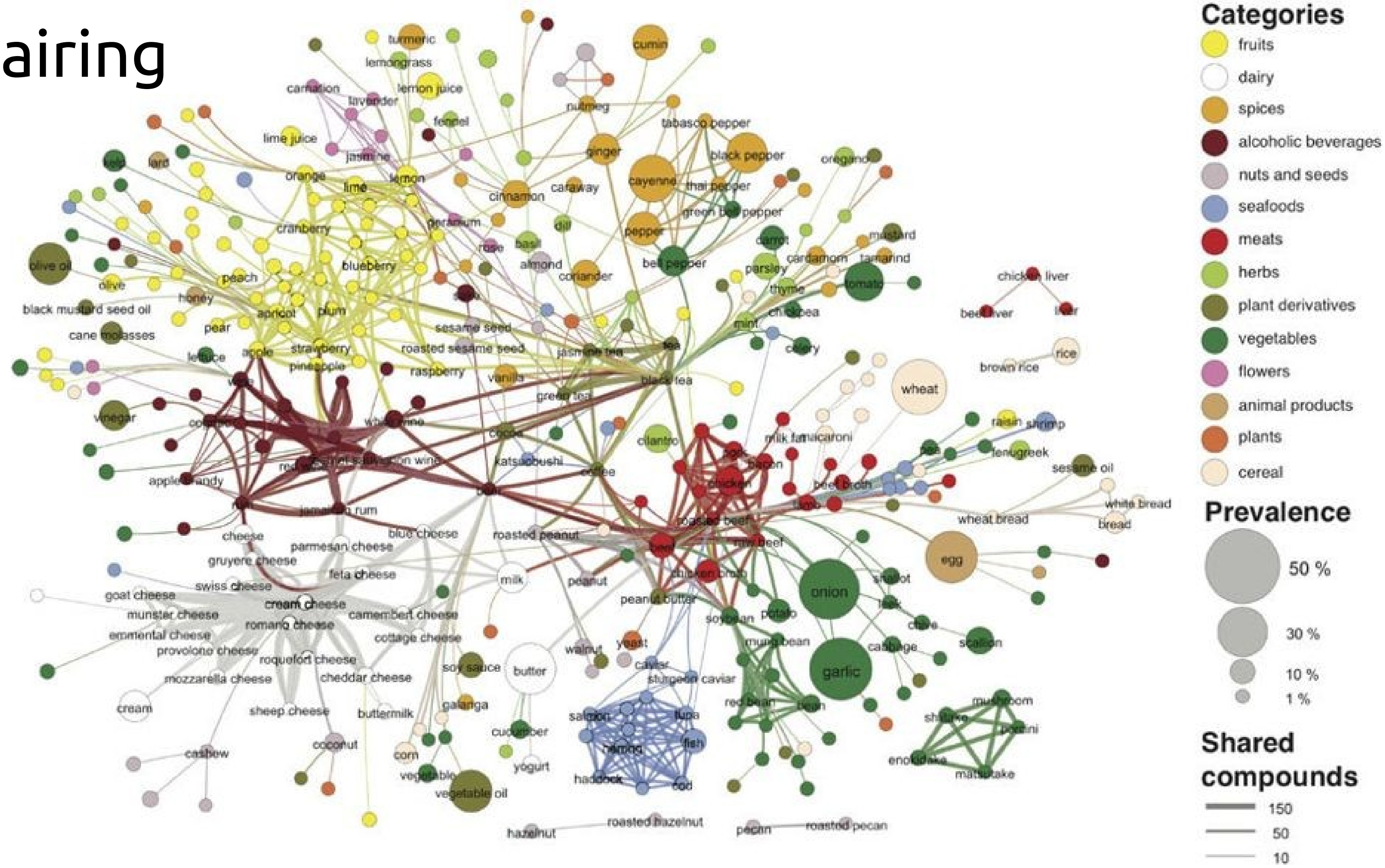
Communities

2 communities	[previous topic]
1 community	[previous topic]
3+ communities	**[this topic]**

Characters in *Les Misérables* (1862)

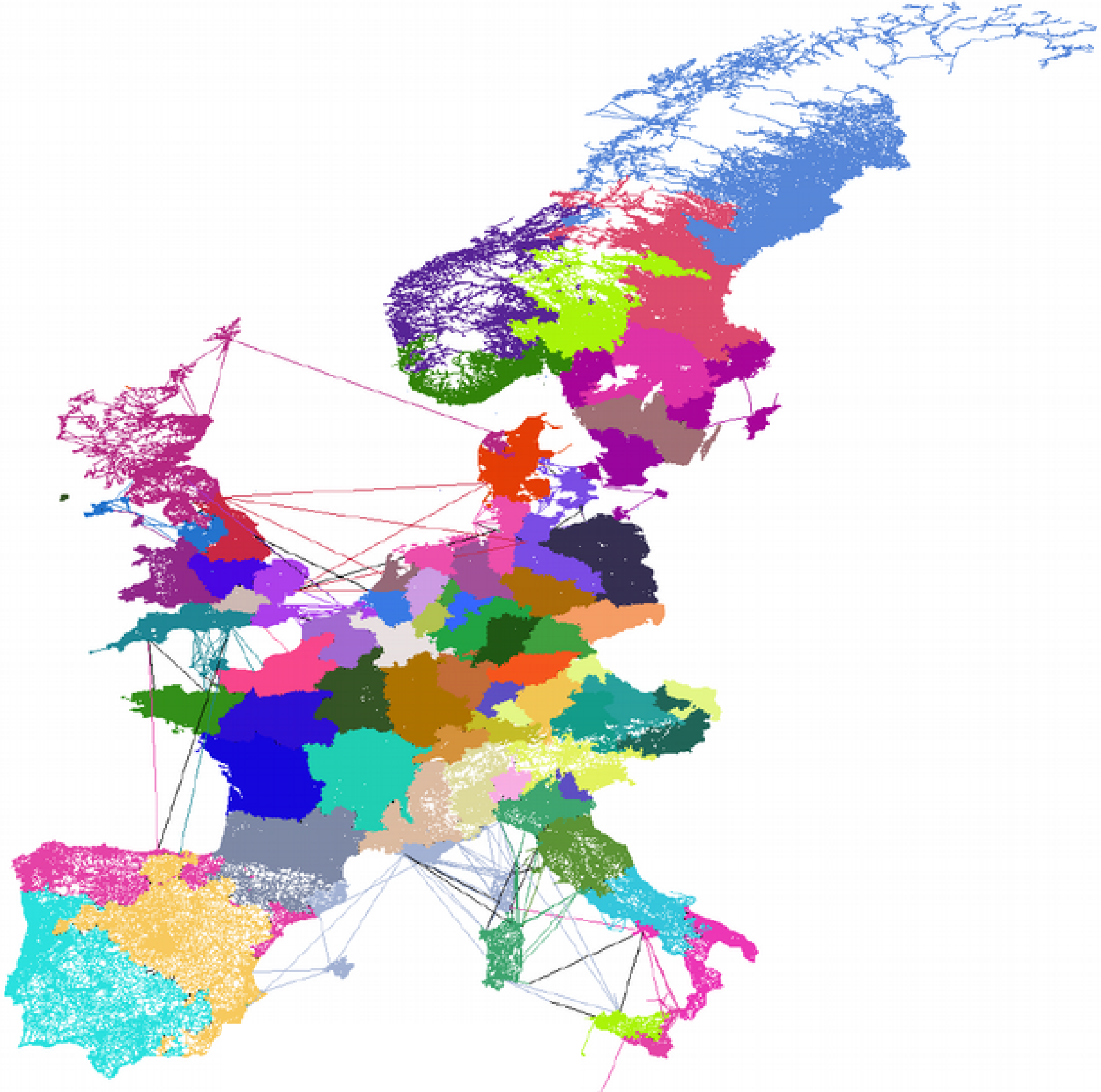


Food pairing



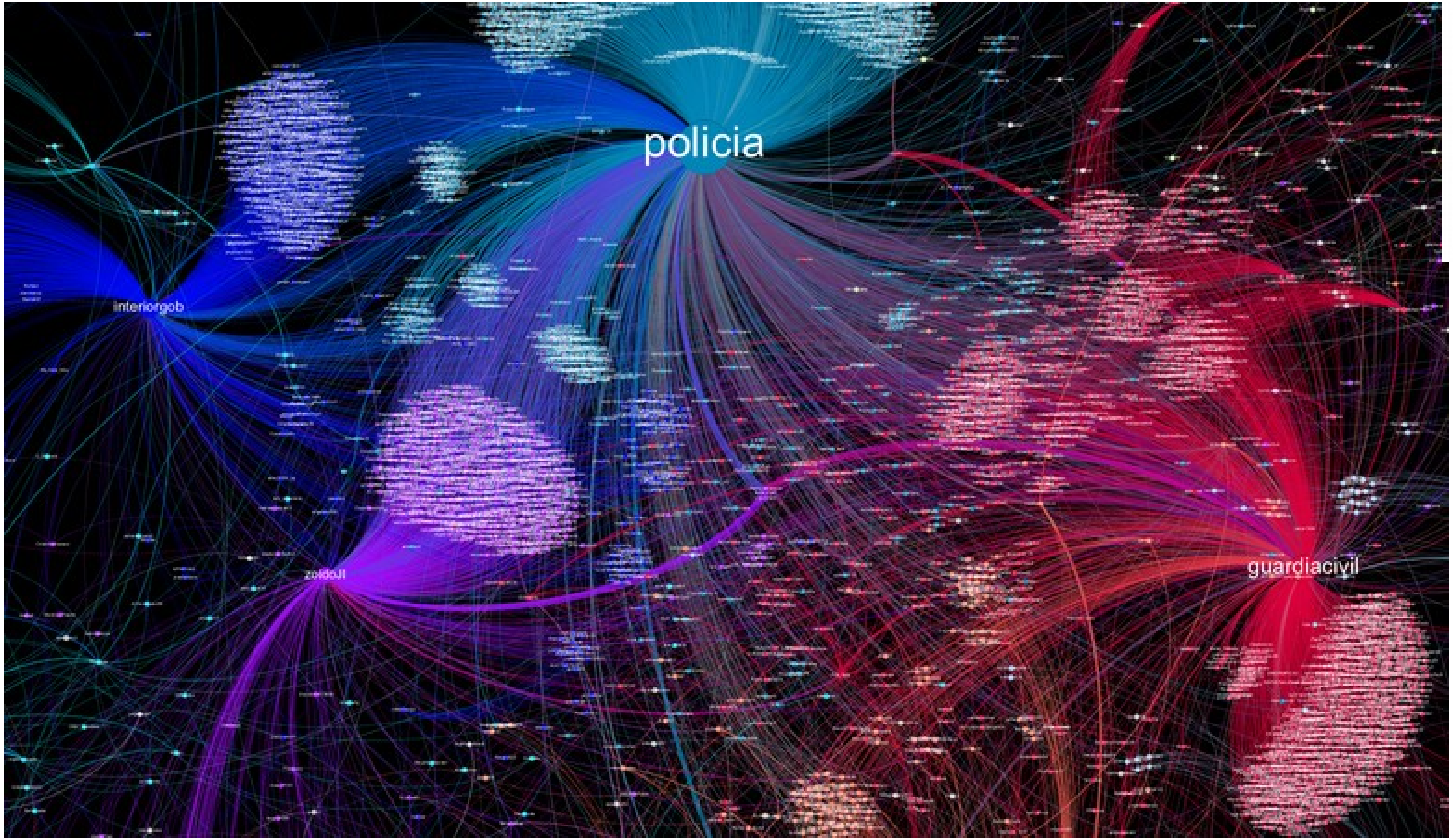
Ahn, Y. Y., Ahnert, S. E., Bagrow, J. P., & Barabási, A. L. (2011). Flavor network and the principles of food pairing. Scientific reports, 1, 196.

Street network in Europe



http://www.par.tuwien.ac.at/teaching/2015w/195.084.psp
slide 353

#estamosporti (Oct 1-3 2017)
A state-sponsored hashtag




Ministerio Interior
@interiorgob

Porque no habrá referéndum
Porque nadie está por encima de la Ley
Porque la unidad de España es indisoluble#EstamosporTI es ya tendencia

3:18 PM - Sep 30, 2017

9,946 6,891 people are talking about this

Because there will be no referendum

Because no one is above the law

Because the unity of Spain is indissoluble

#EstamosporTI is already a trend

Source: Erin Gallagher

Community hypotheses

H1. The community structure of a network is <u>uniquely encoded</u> by its links [discoverable]

H2. A community is a <u>locally dense</u> <u>connected</u> subgraph [dense, connected]

Weak < Strong < Clique

- Let C be a community with N_C nodes
- Given a community C,
  - Let k_i^{int} count links towards C, k_i^{ext} towards V\C
- Clique: $\forall i \in C, k_i^{\text{int}} = N_C$
- Strong community: $\forall i \in C, k_i^{\text{int}} > k_i^{\text{ext}}$
- Weak community: $\sum_{i \in C} k_i^{\text{int}} > \sum_{i \in C} k_i^{\text{ext}}$

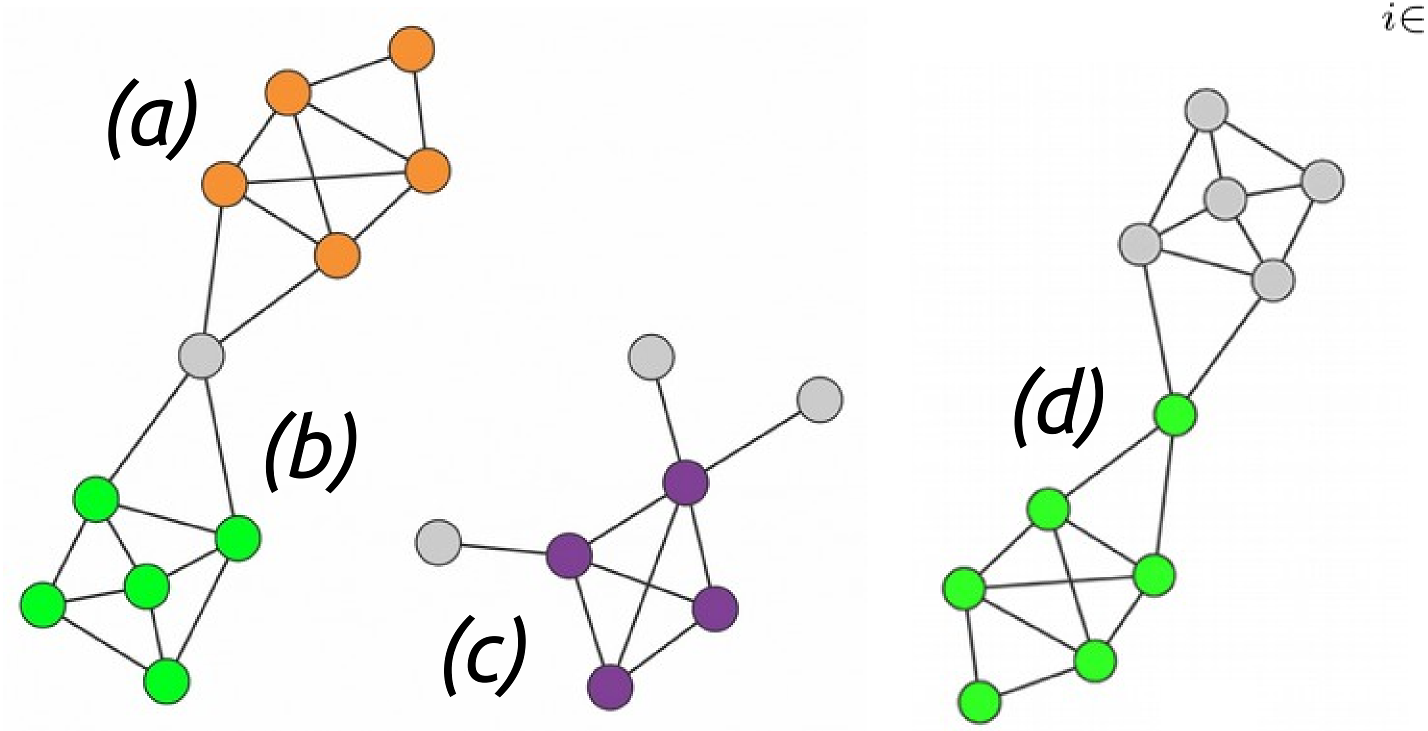
Try it!

Which types of communities are these?

Clique: $\forall i \in C, k_i^{\text{int}} = N_C$

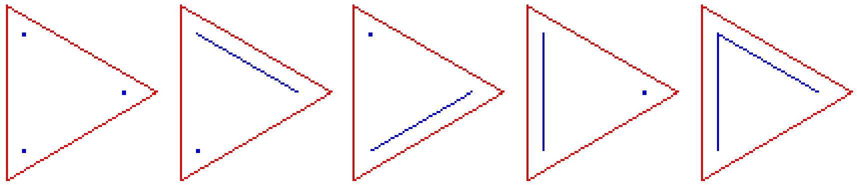
Strong community: $\forall i \in C, k_i^{\text{int}} > k_i^{\text{ext}}$

Weak community: $\sum_{i \in C} k_i^{\text{int}} > \sum_{i \in C} k_i^{\text{ext}}$

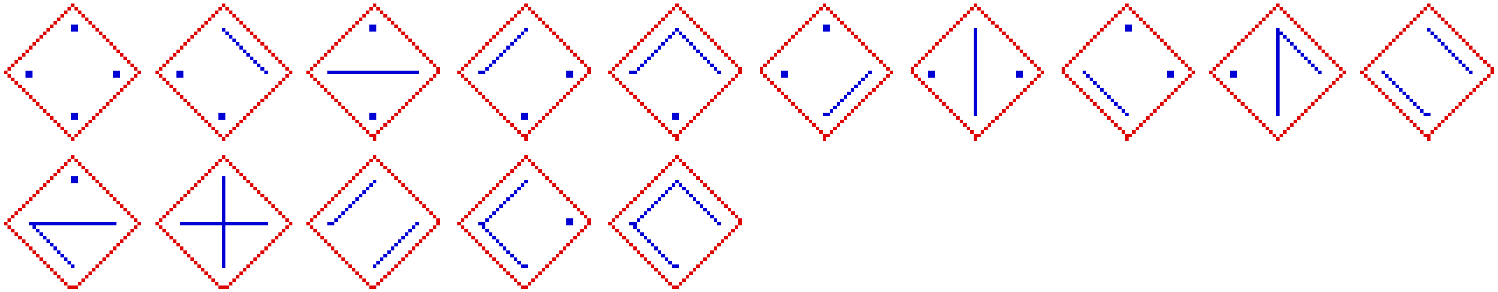


How many possible partitions?

- With 3 nodes: 5 partitions

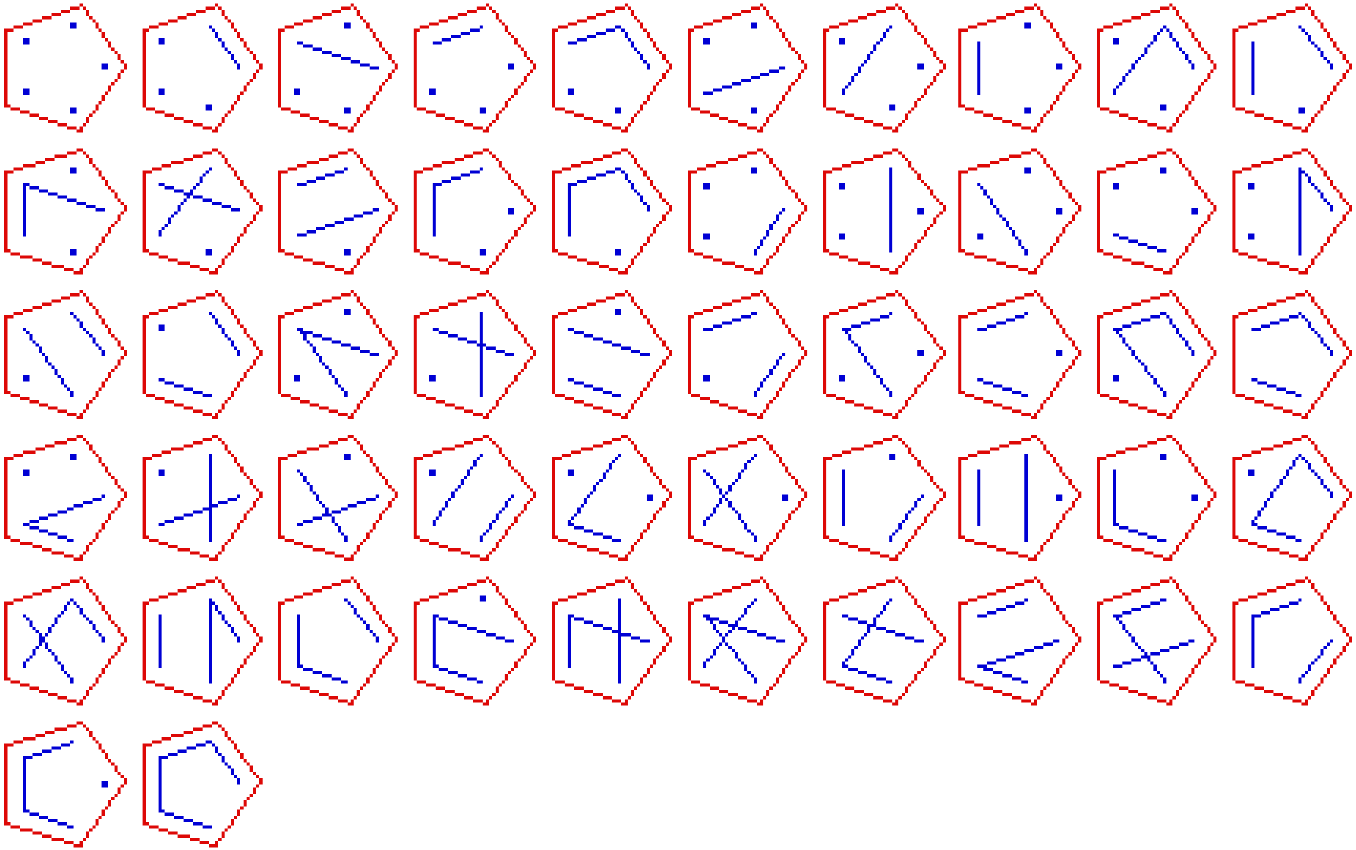


- With 4 nodes: 15 partitions



http://www-history.mcs.st-andrews.ac.uk/Miscellaneous/StirlingBell/bell.html

- With 5 nodes: 52 partitions



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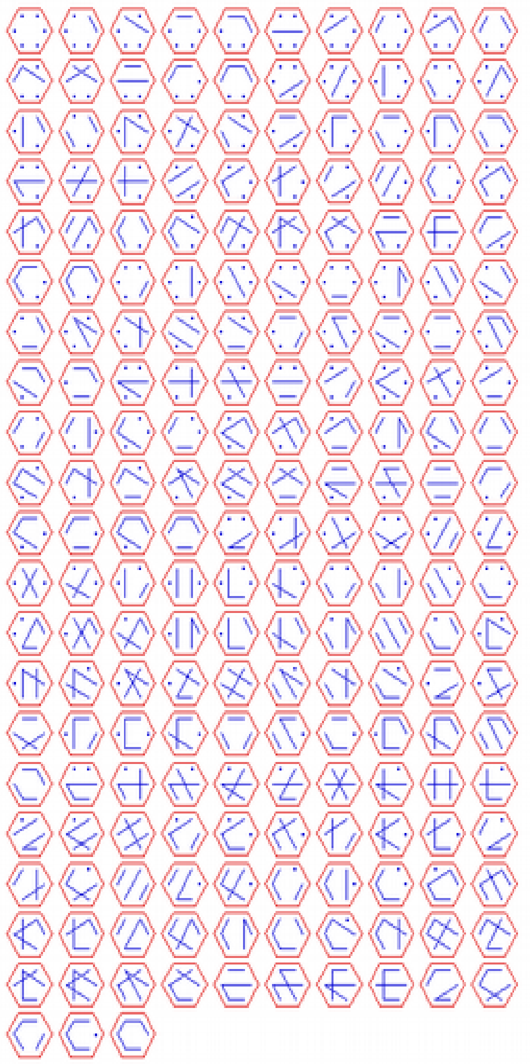
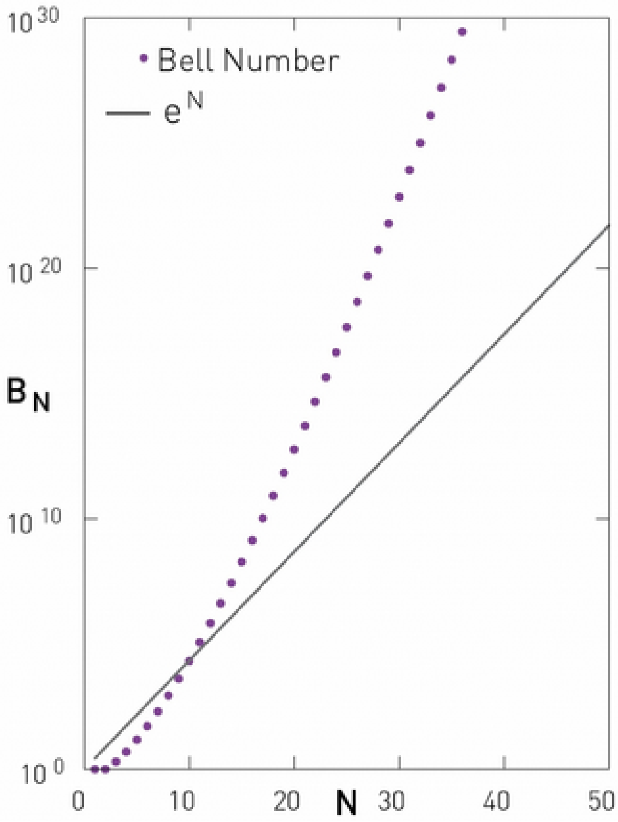


- With 6 nodes: 203 partitions
- With 7 nodes: 877 partitions
- With n nodes: Bell number

$$B_N = \frac{1}{e} \sum_{j=0}^{\infty} \frac{j^n}{j!}$$



http://www-history.mcs.st-andrews.ac.uk/Miscellaneous/StirlingBell/bell.html

Hierarchical algorithms

Recursively:

- Merge nodes that are similar (agglomerative)

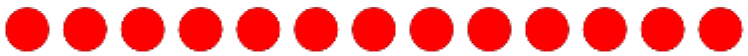
  — OR —

- Split groups that are dissimilar (divisive)

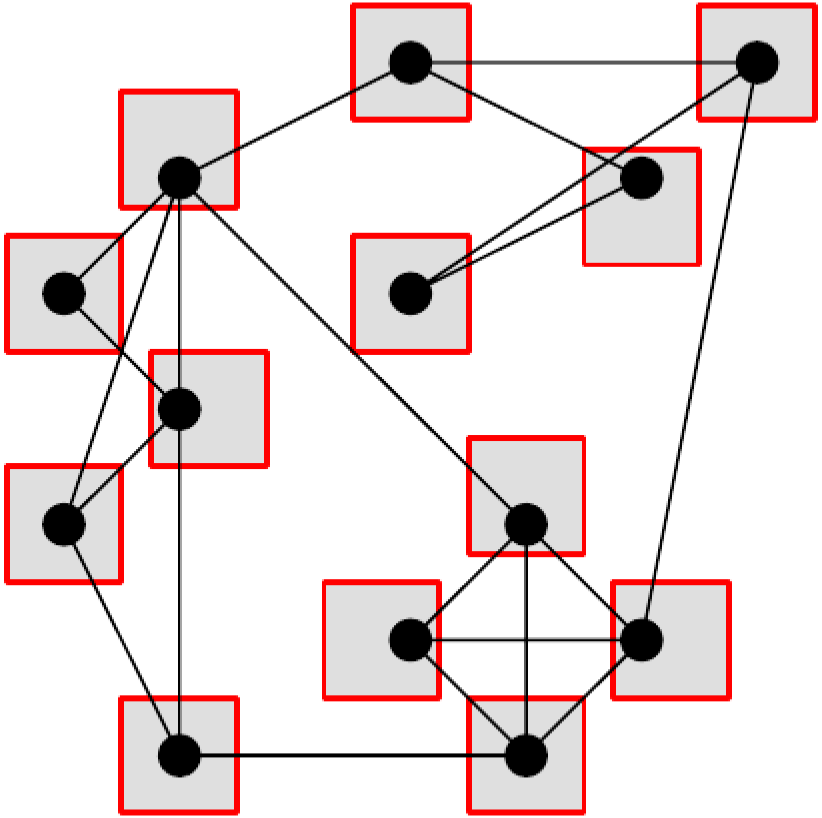
Agglomerative clustering

- Start with every node in a separate community (these are called “singletons”)
- Repeat until all nodes are together:
  - Compute community distances between all pairs
  - Merge the two closest communities

dendrogram

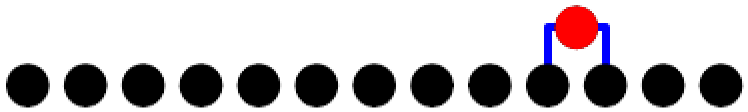


current clustering

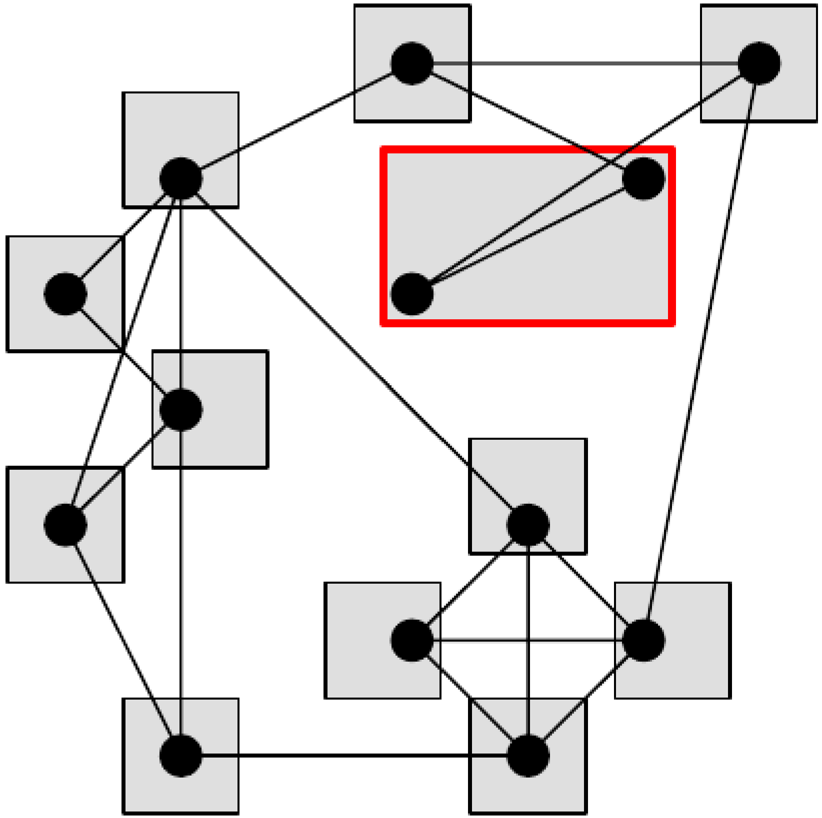


This and next 12: http://www.par.tuwien.ac.at/teaching/2015w/195.084.psp slides 304-306

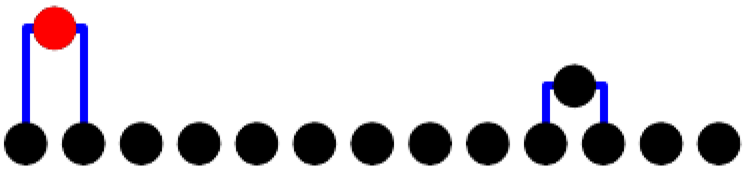
dendrogram



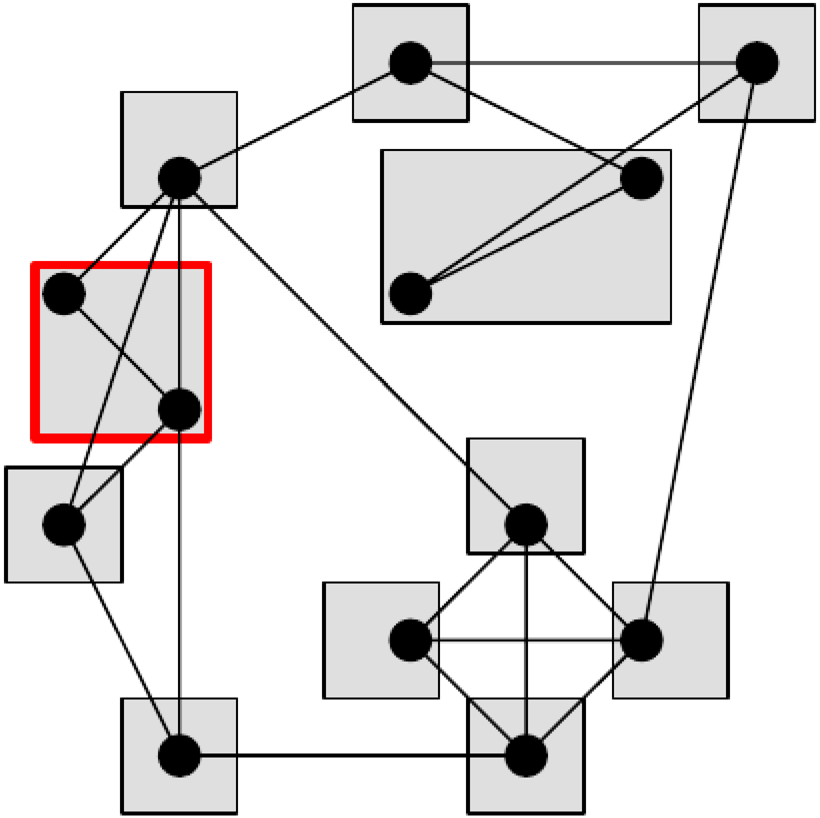
current clustering



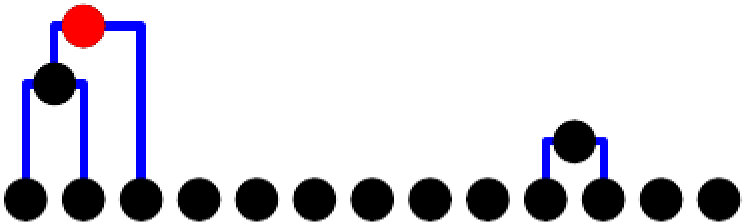
dendrogram



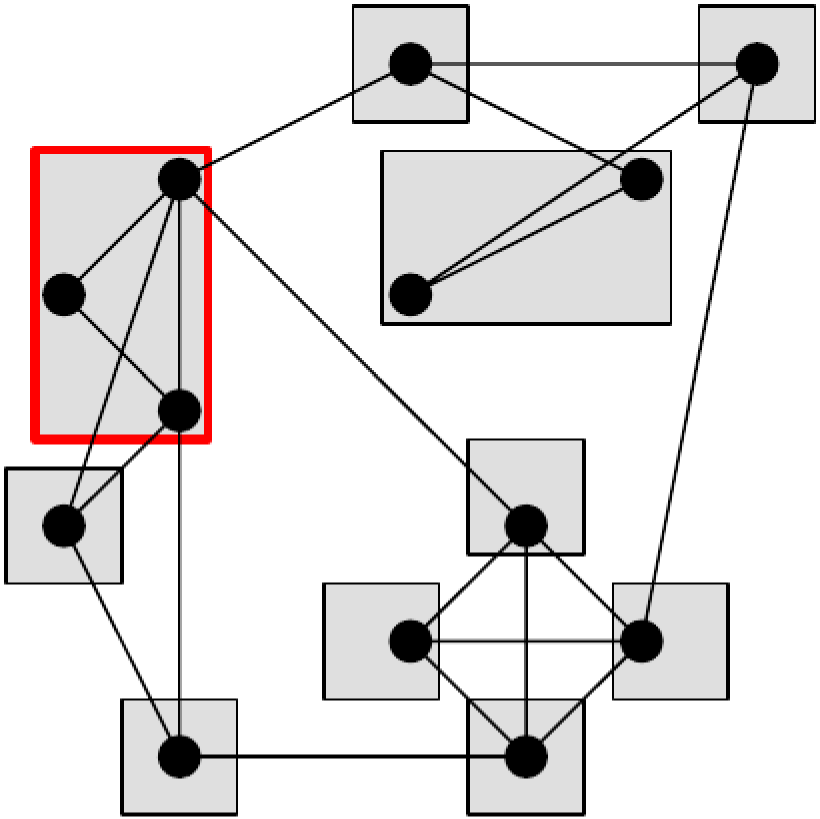
current clustering



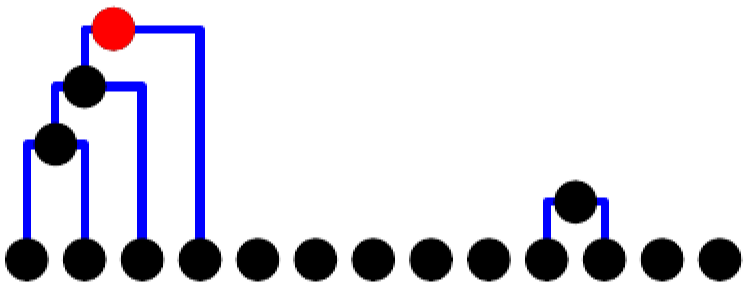
dendrogram



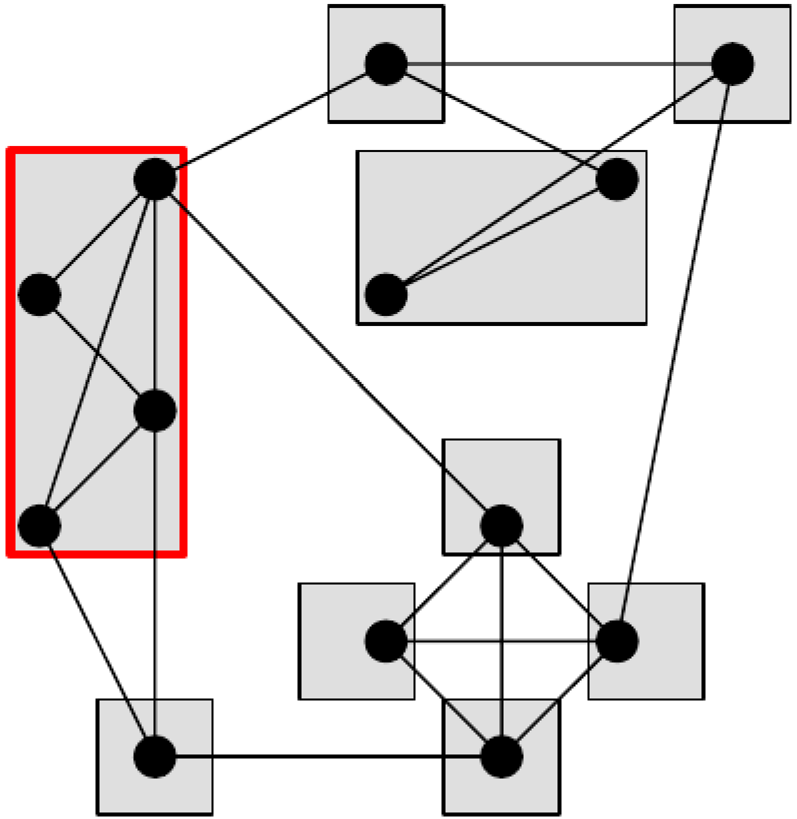
current clustering



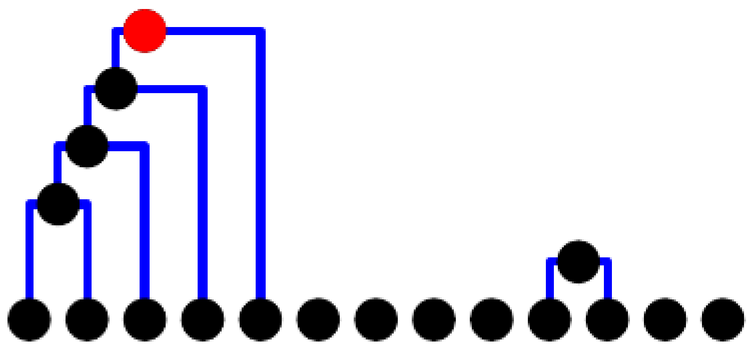
dendrogram



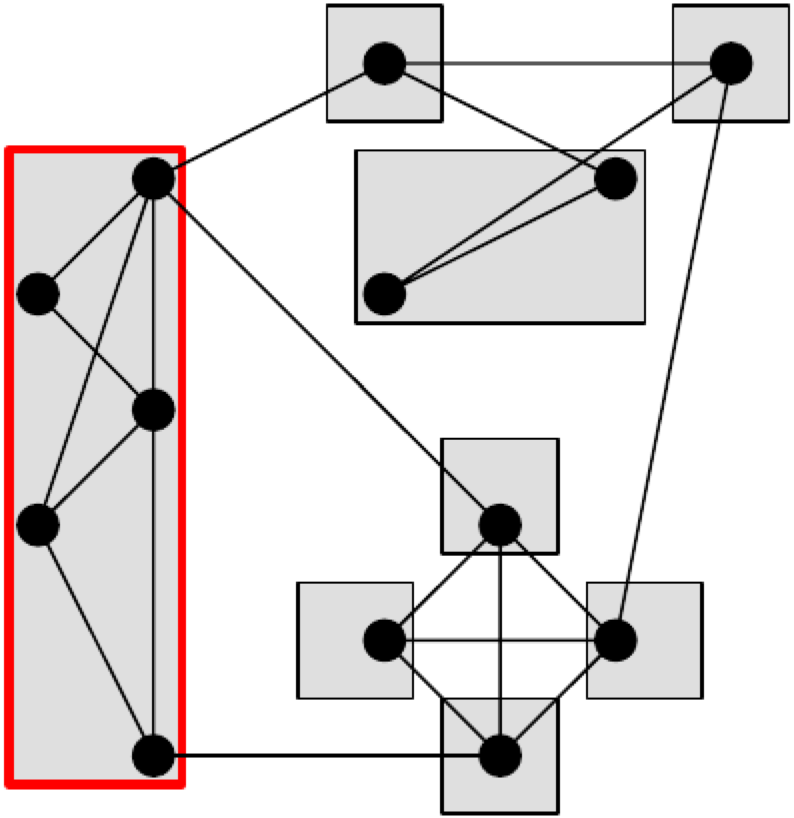
current clustering



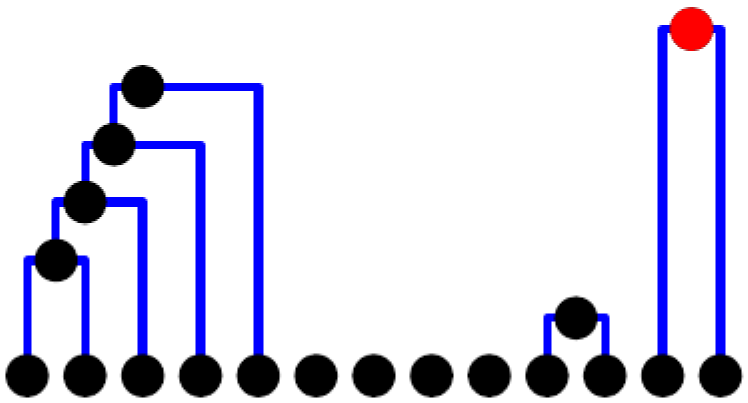
dendrogram



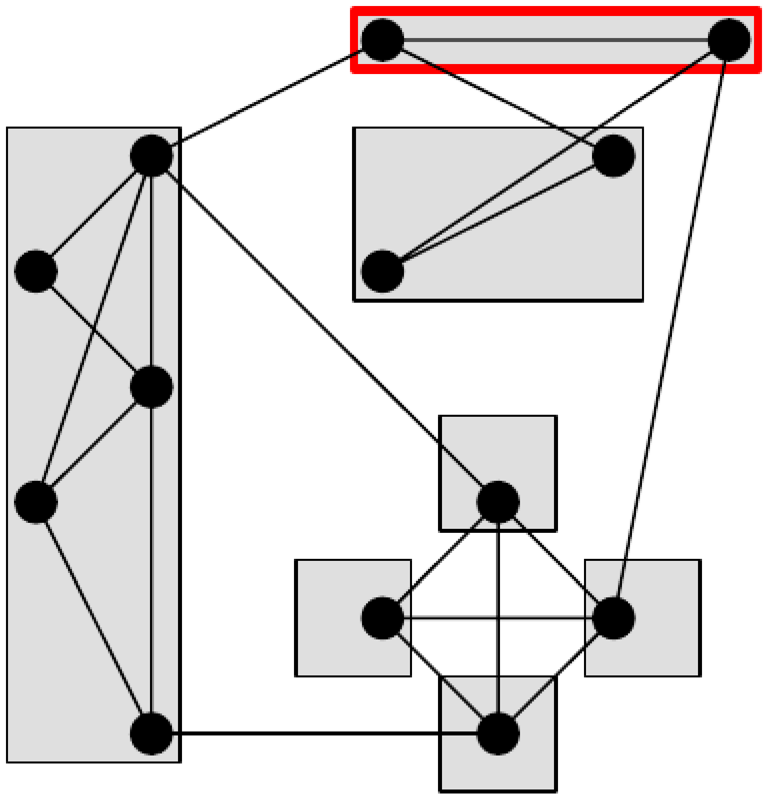
current clustering

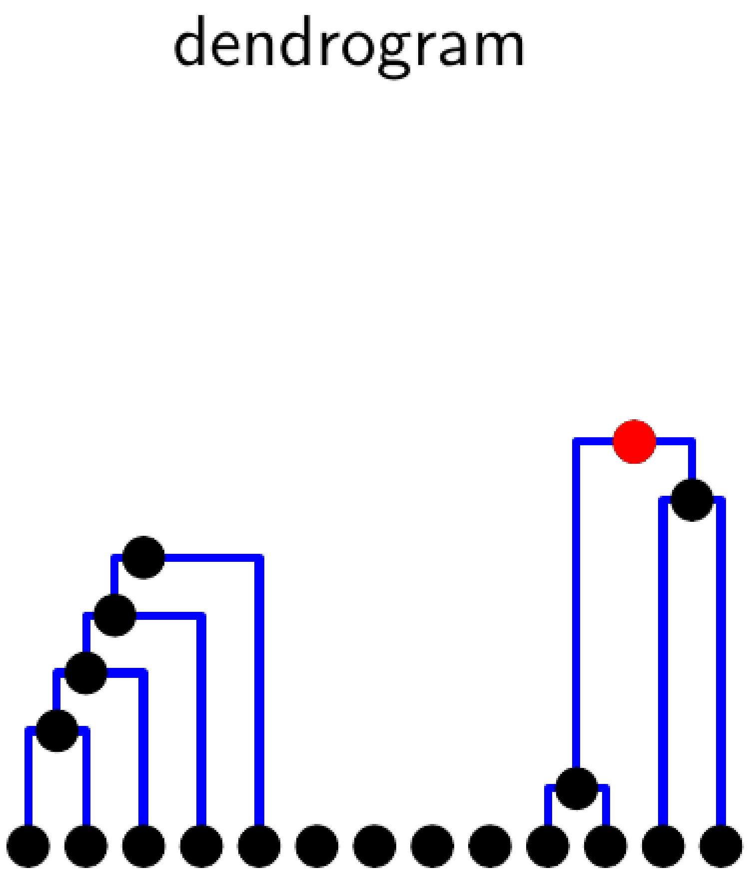


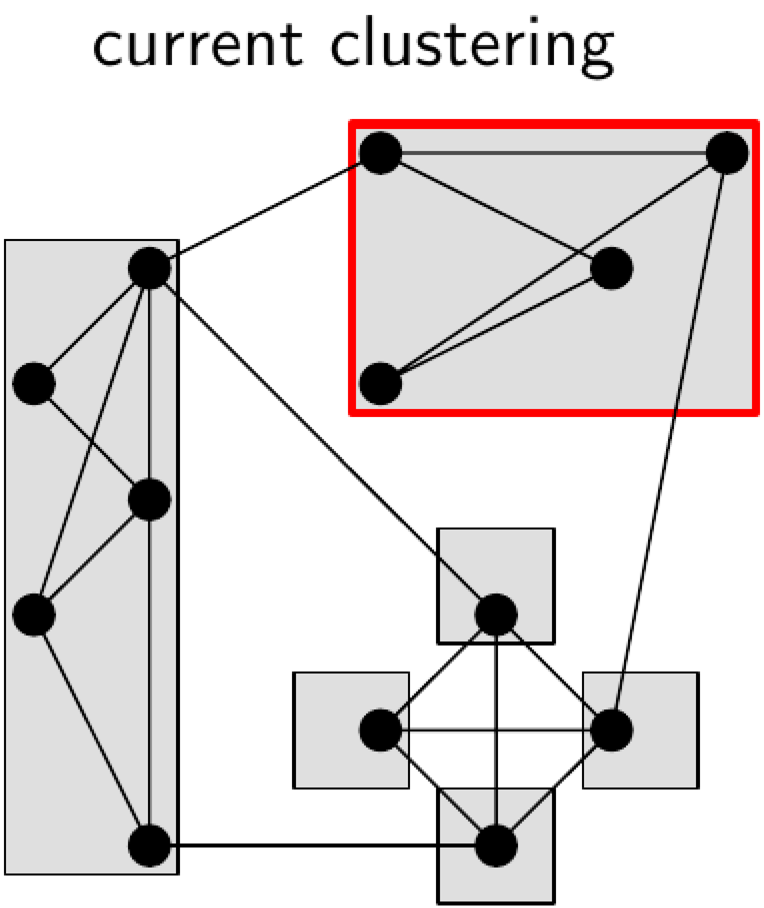
dendrogram



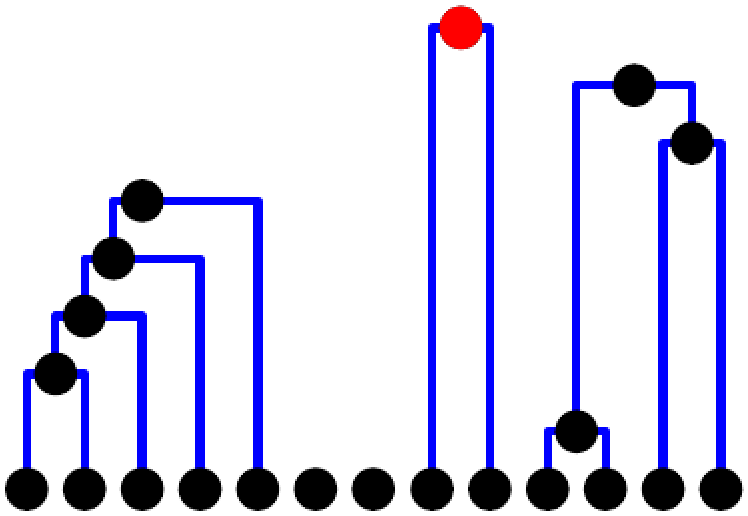
current clustering



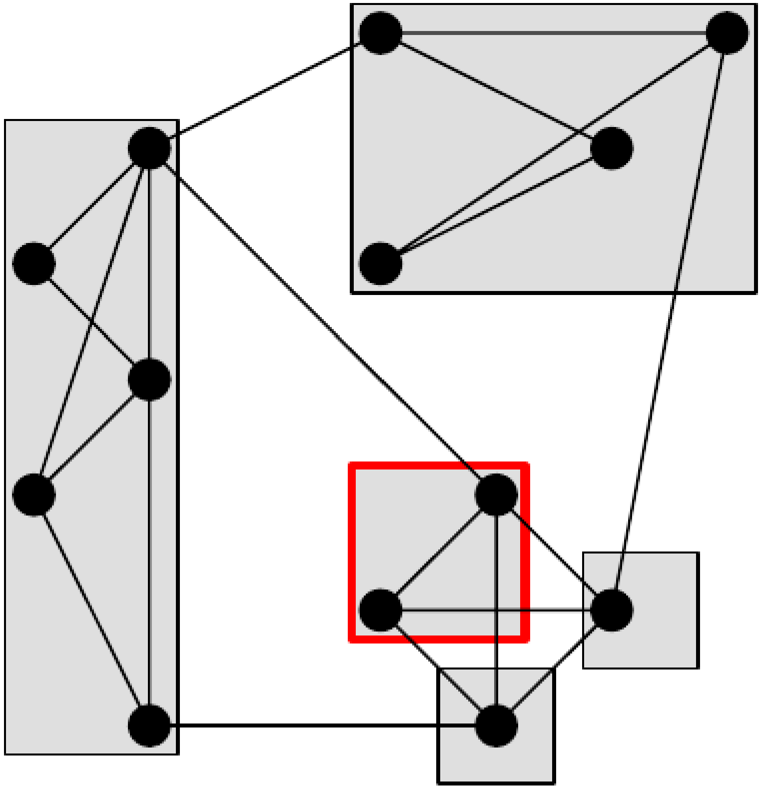

dendrogram


current clustering

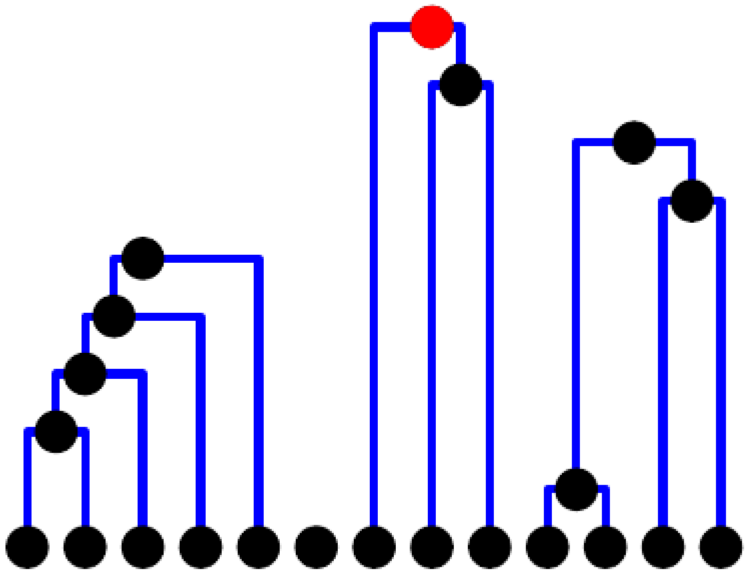
dendrogram



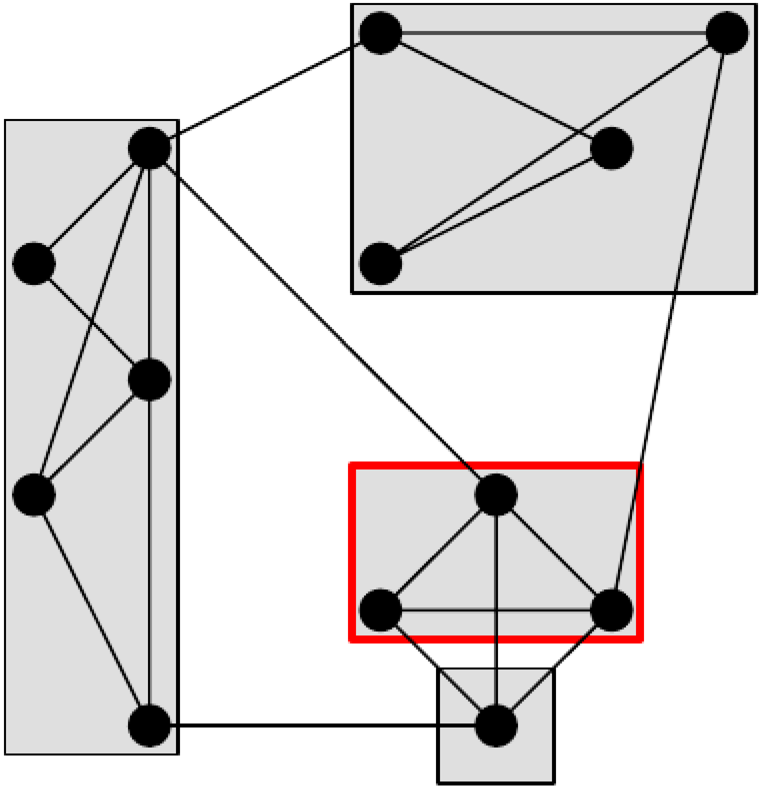
current clustering

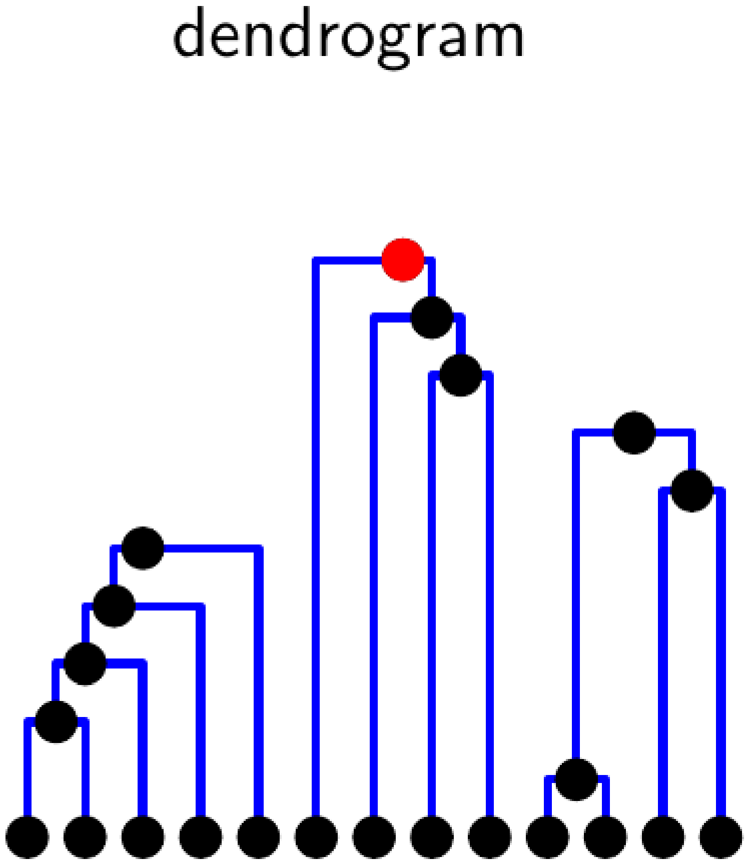


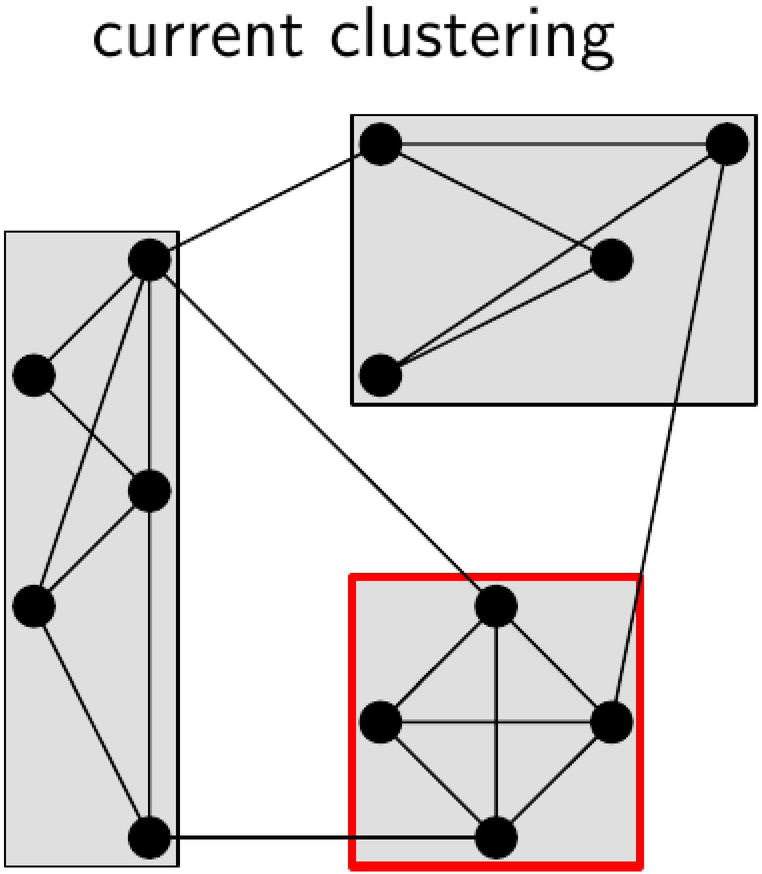
dendrogram



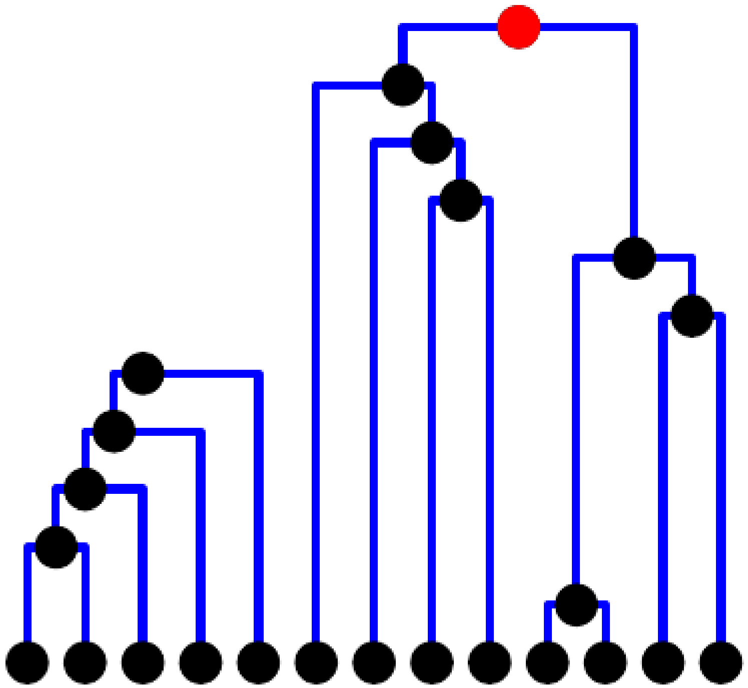
current clustering



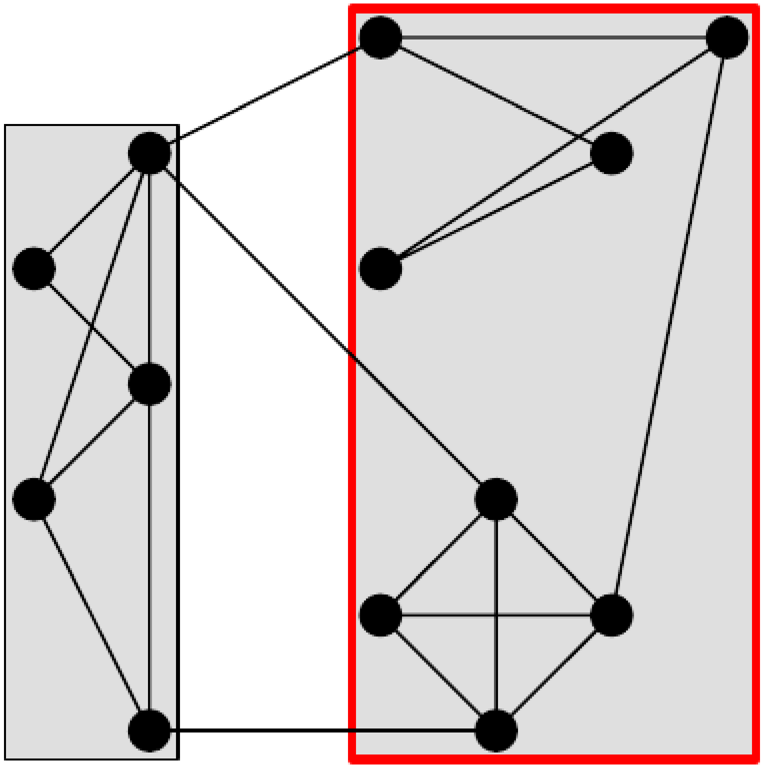

dendrogram


current clustering

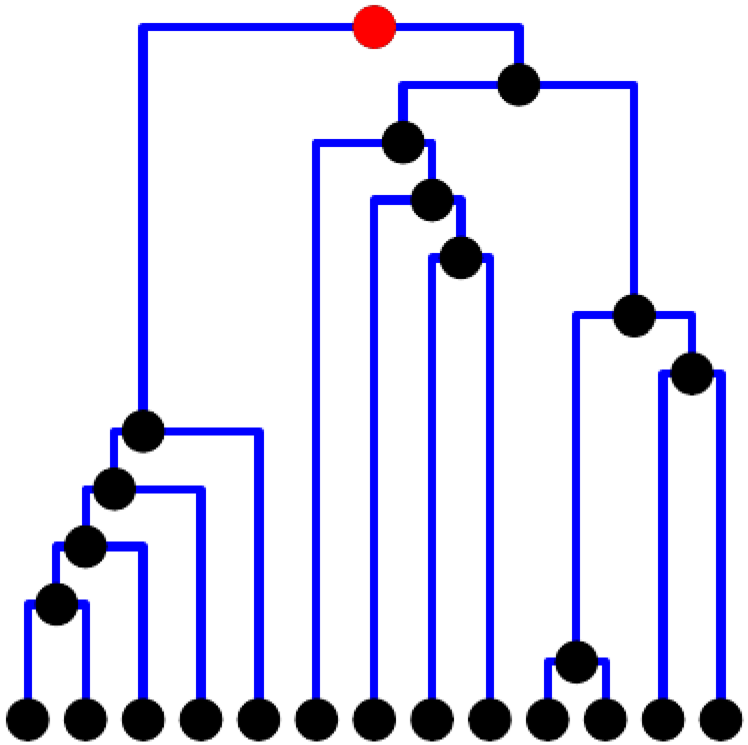
dendrogram



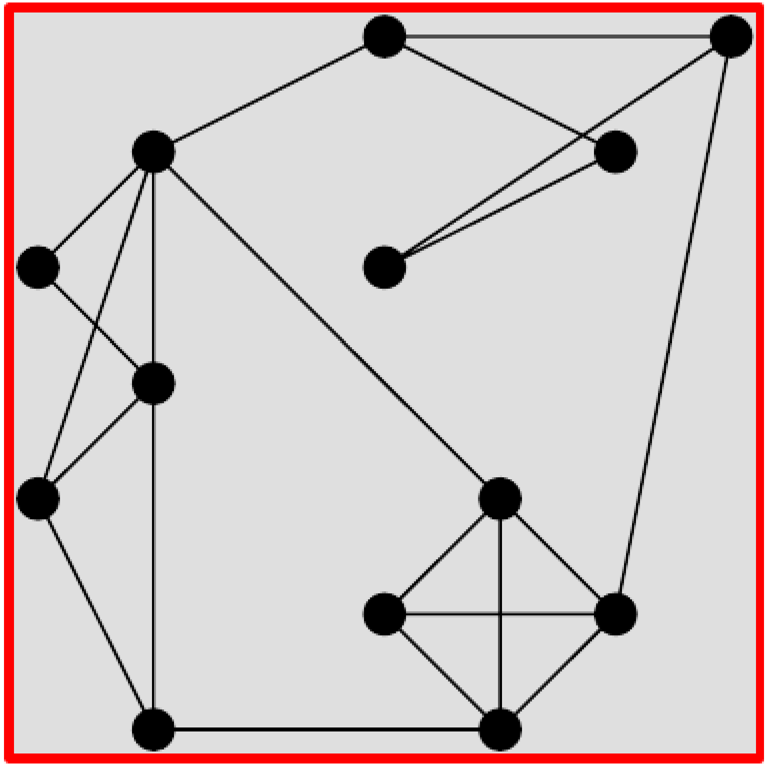
current clustering



dendrogram

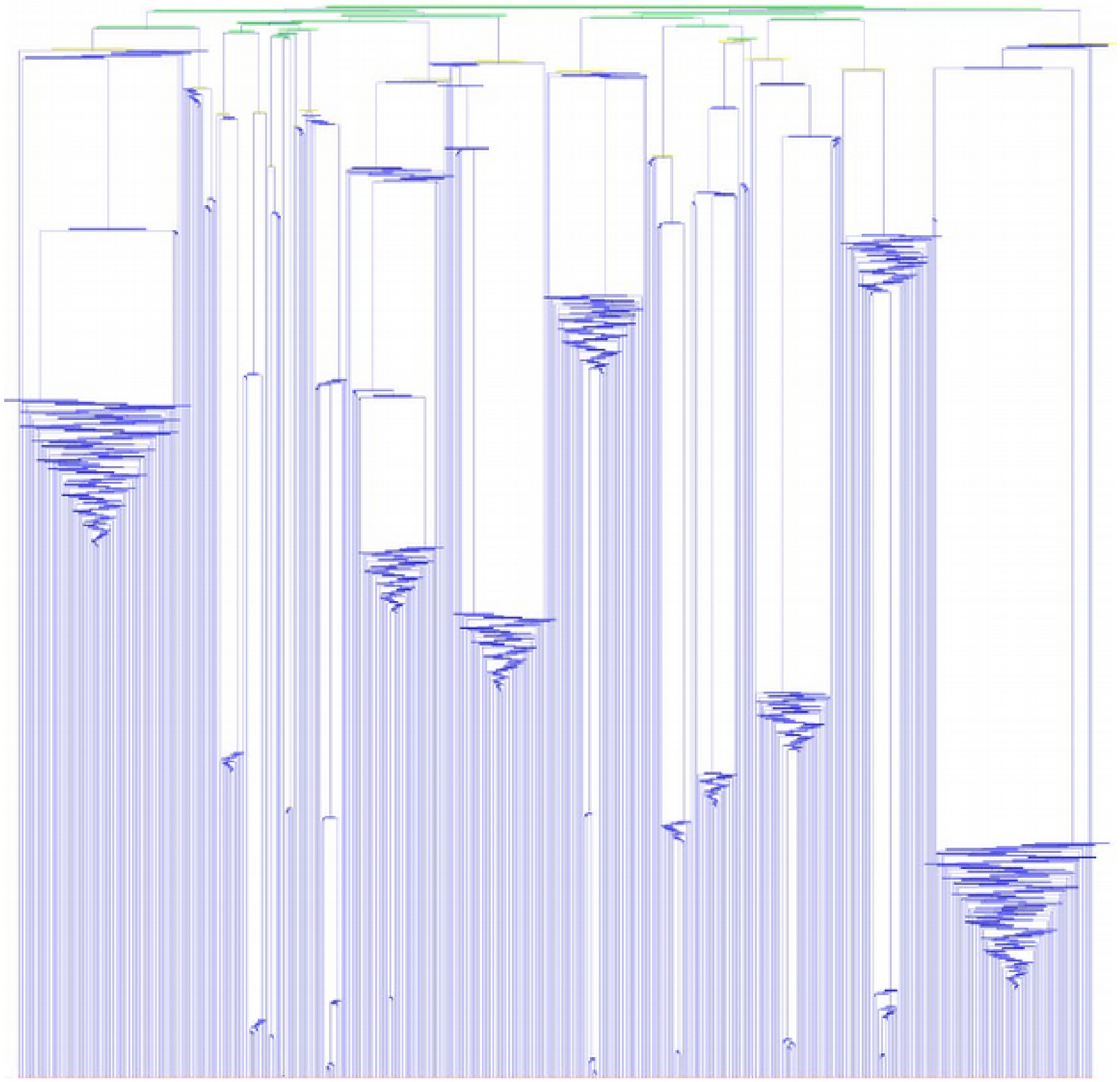


current clustering

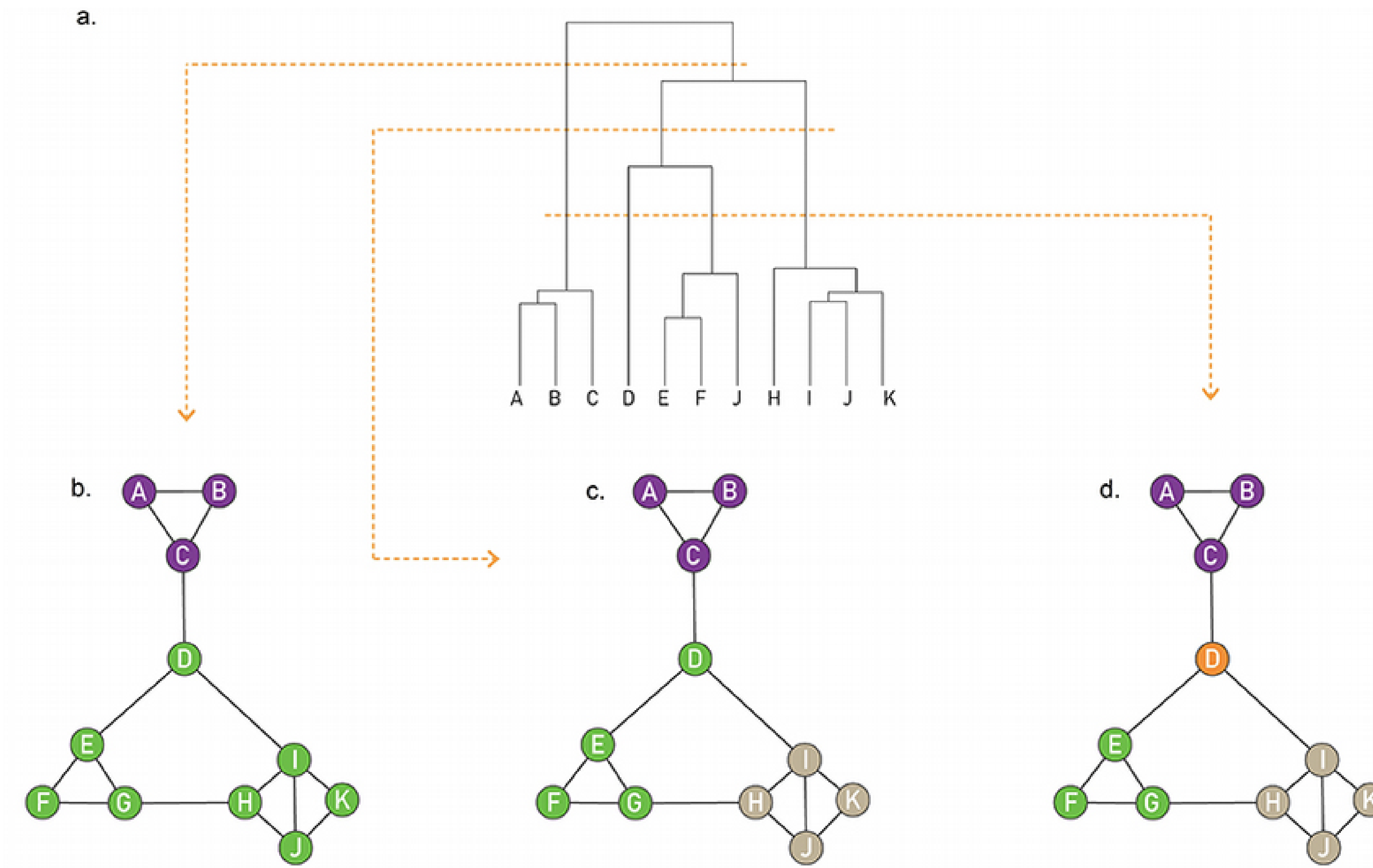


Can handle large graphs

In this example, $|V| \simeq 1000$



Where to cut?

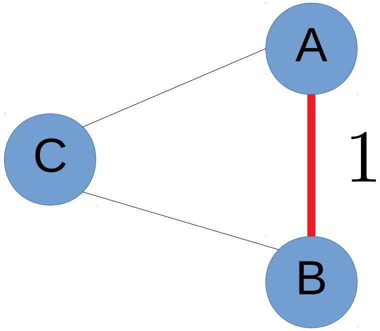


We need to specify some details

- When are two nodes considered similar
- When are two groups of nodes considered similar

Node similarity: topological overlap

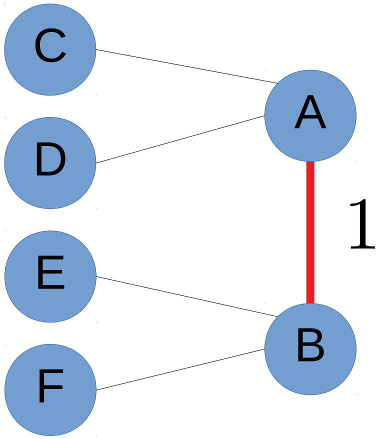
$$x_{ij}^{0} = \frac{|\Gamma(i) \cap \Gamma(j)| + A_{ij}}{\min(k_i, k_j)}$$



$$1 = \frac{|\{C, B\} \cap \{C, A\}| + 1}{\min(2, 2)} = \frac{2}{2}$$

Node similarity: topological overlap

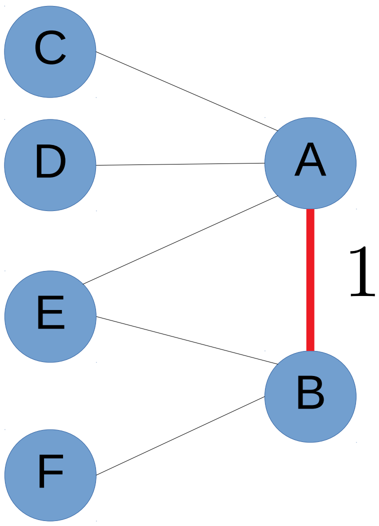
$$x_{ij}^0 = \frac{|\Gamma(i) \cap \Gamma(j)| + A_{ij}}{\min(k_i, k_j)}$$



$$1 = \frac{|\{C, D, B\} \cap \{E, F, A\}| + 1}{\min(3, 3)} = \frac{1}{3}$$

Node similarity: topological overlap

$$x_{ij}^0 = \frac{|\Gamma(i) \cap \Gamma(j)| + A_{ij}}{\min(k_i, k_j)}$$



$$1 = \frac{|\{C, D, E, B\} \cap \{E, F, A\}| + 1}{\min(4, 3)} = \frac{2}{3}$$

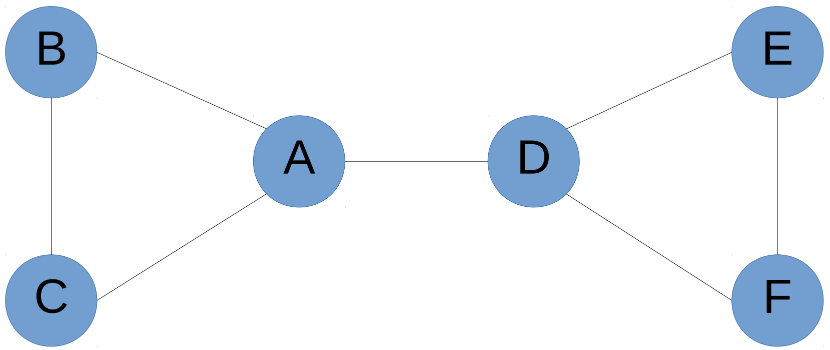
Group similarity of groups U and V

- Single-linkage: $$x_{U,V} = \min_{i \in U, j \in V} x_{ij}$$
- Complete-linkage: $$x_{U,V} = \max_{i \in U, j \in V} x_{ij}$$
- Average-linkage: $$x_{U,V} = \frac{1}{|U||V|} \sum_{i \in U, j \in V} x_{ij}$$

Try it!

Run Kavasz's algorithm on this graph with complete (max) linkage; to save time, compute only over existing edges

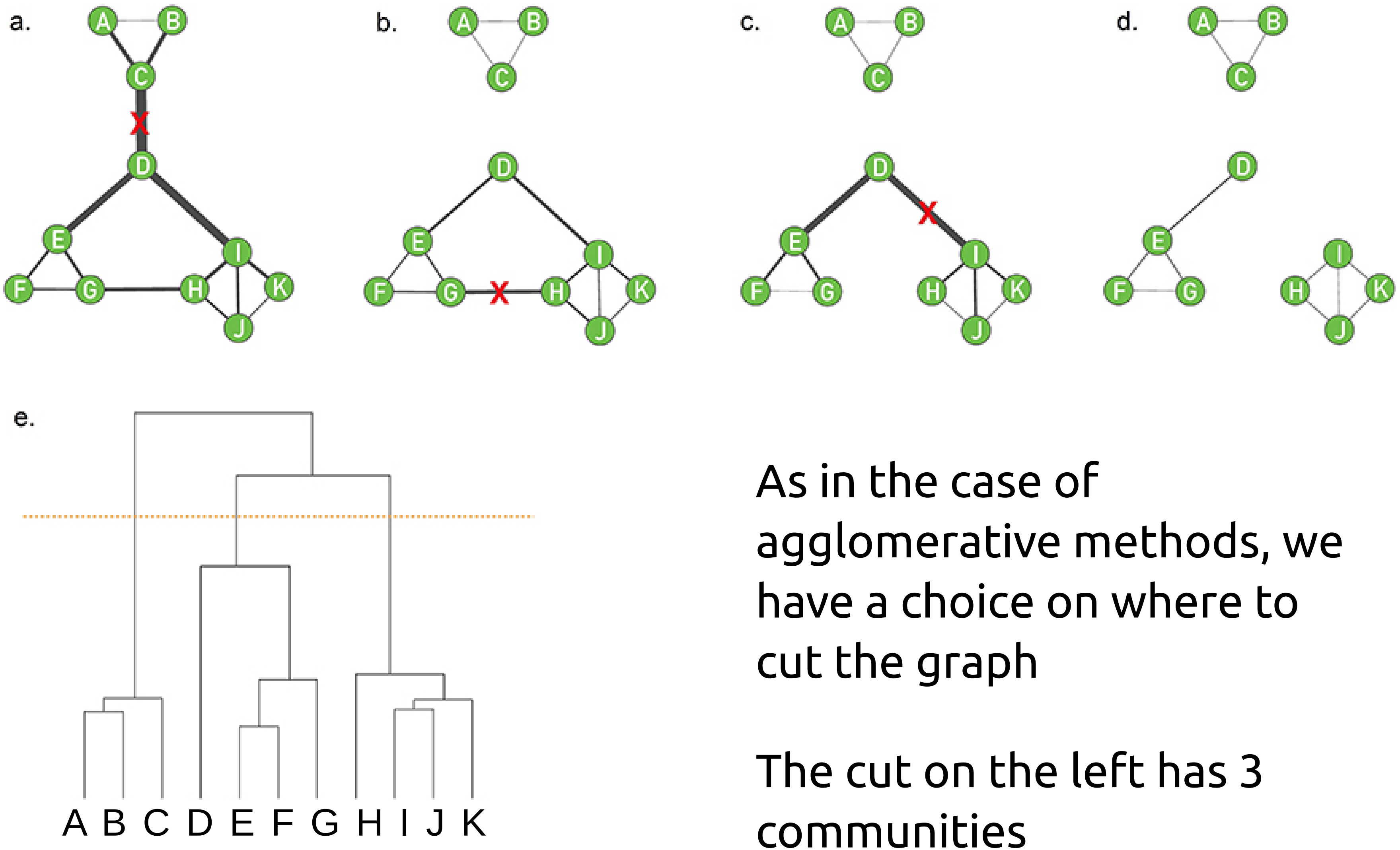
$$x_{ij}^0 = \frac{|\Gamma(i) \cap \Gamma(j)| + A_{ij}}{\min(k_i, k_j)}$$



Divisive clustering

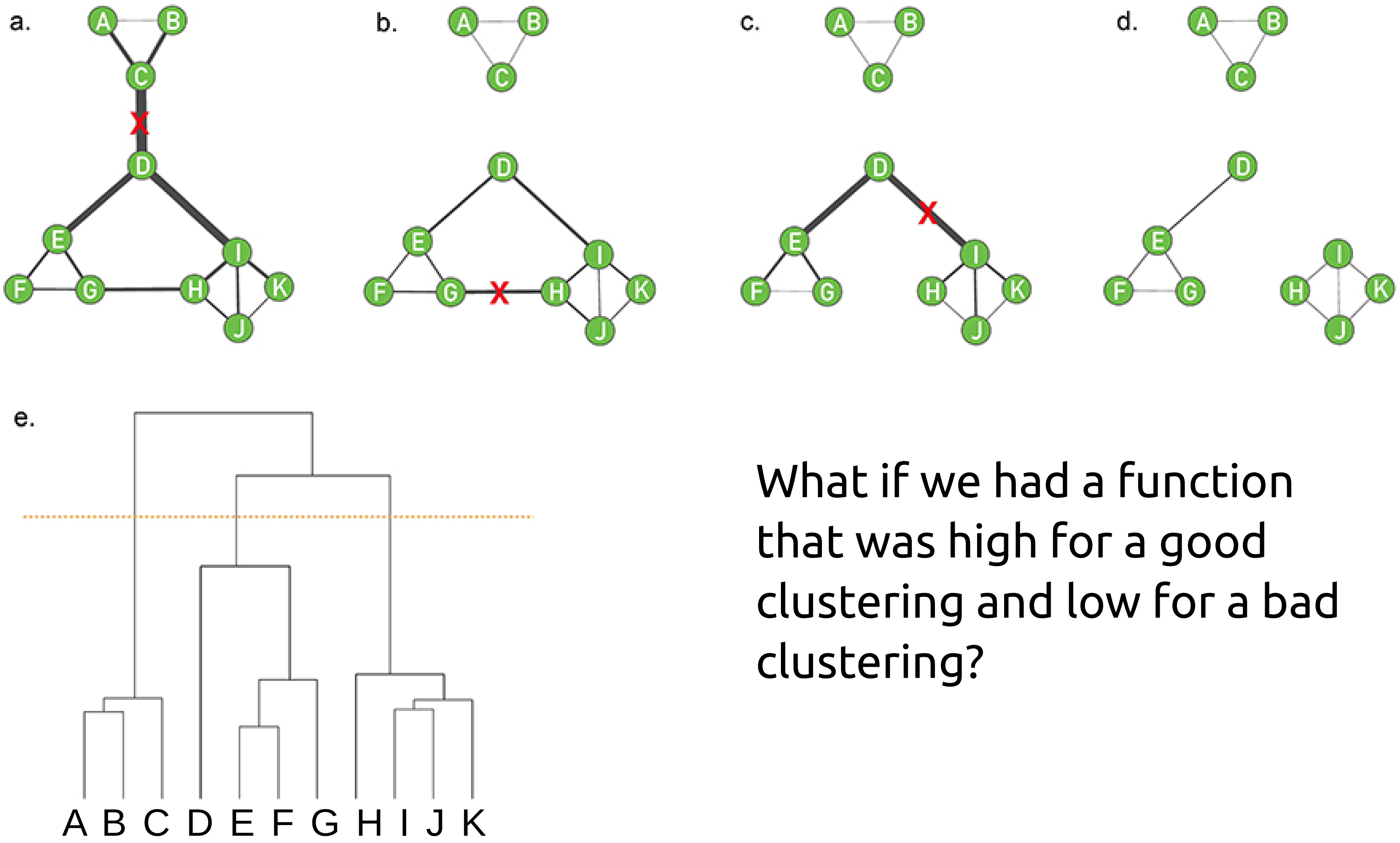
- Start with the original graph
- Repeat until all nodes are separated:
  - Remove a link

- Which link to remove? The one with maximum edge betweenness!

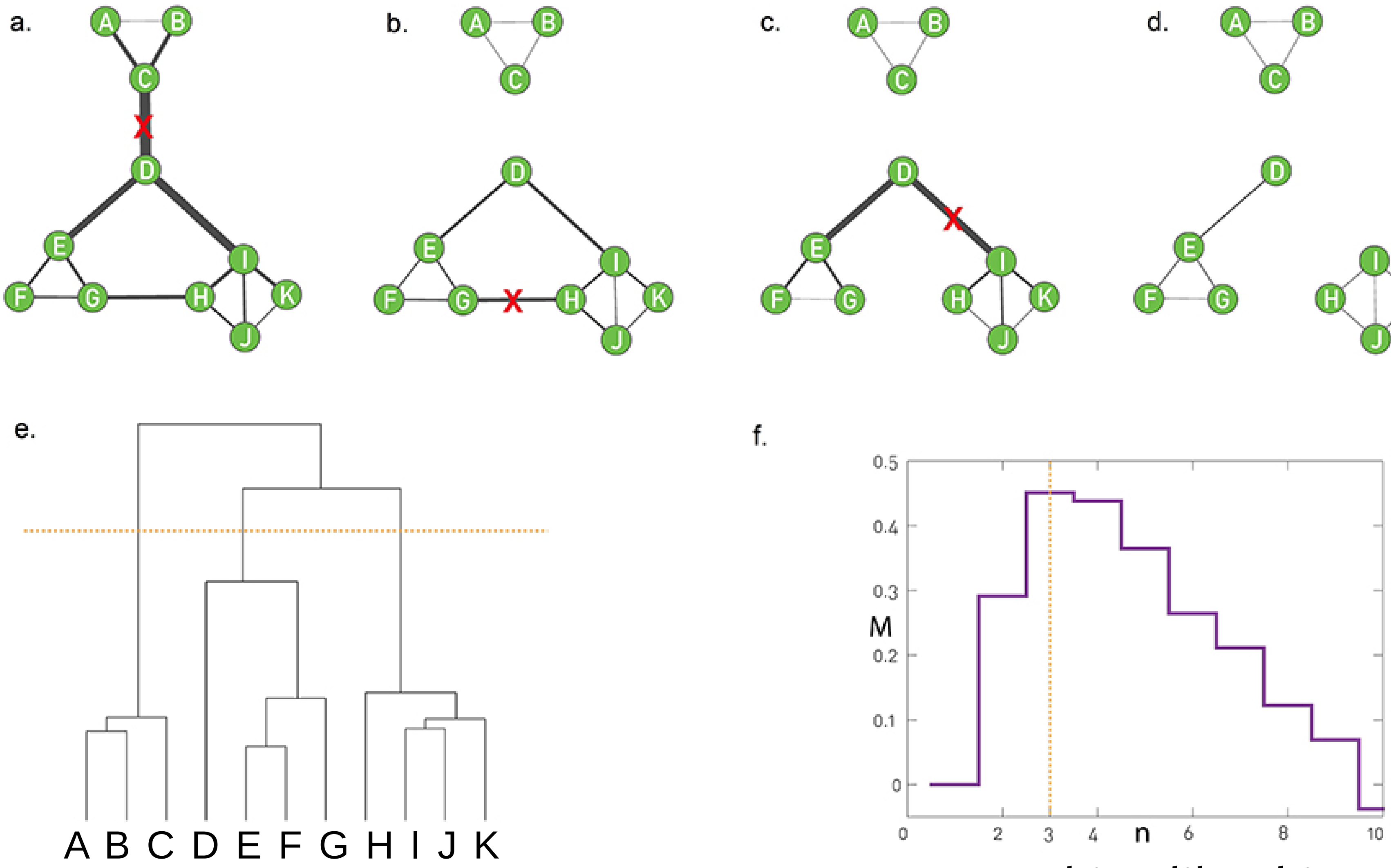


As in the case of agglomerative methods, we have a choice on where to cut the graph

The cut on the left has 3 communities



What if we had a function that was high for a good clustering and low for a bad clustering?



Something like this?

The modularity function

Let e_{ii} = probability that an edge connects a node in community i with another node in community i

$$e_{ii} = \frac{|(u,v) \in E : u \in C_i \wedge v \in C_i|}{|E|}$$

Let a_i = probability that an edge has at least one end in community i

$$a_i = \frac{\sum_{u \in i} k_u}{2|E|}$$

Modularity with n_c communities

$$Q = \sum_{i=1}^{n_c} \left(e_{ii} - a_i^2 \right)$$

$$e_{ii} = \frac{|(u,v) \in E : u \in C_i \wedge v \in C_i|}{|E|}$$

$$a_i = \frac{\sum_{u \in i} k_u}{2|E|}$$

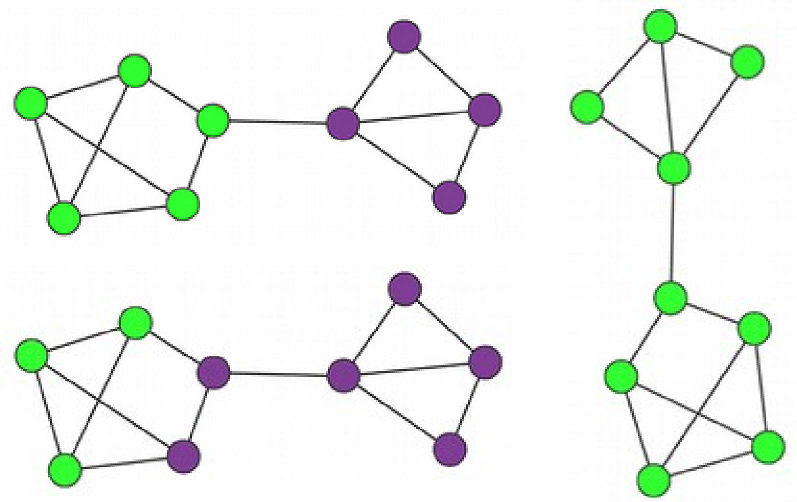
Try it!

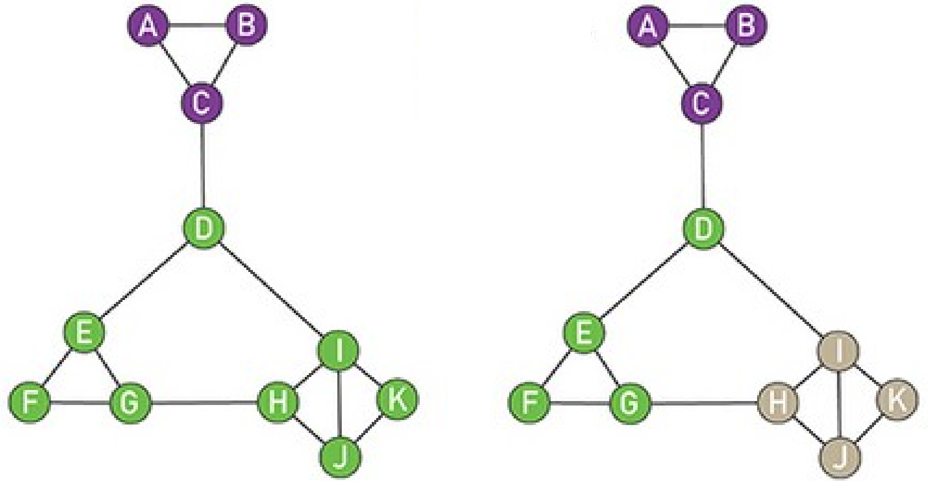
Compute the modularity of these partitions

$$Q = \sum_{i=1}^{n_c} \left(e_{ii} - a_i^2 \right)$$

$$e_{ii} = \frac{|(u,v) \in E : u \in C_i \wedge v \in C_i|}{|E|}$$

$$a_i = \frac{\sum_{u \in i} k_u}{2|E|}$$





Greedy modularity optimization

- Start with each node in one community
- Repeat until all nodes are together:
  - For every pair of communities connected by a link:
    - Compute Q of the networks if those communities were merged
  - Merge the communities that give the larger increase (or the smaller decrease) in Q
- Return the intermediate step at which Q was maximized

Community sizes distribution