

Graph theory basics

Introduction to Network Science

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Topic 04

Contents

- Notation for graphs
- Degree distributions
- Adjacency matrices

Sources

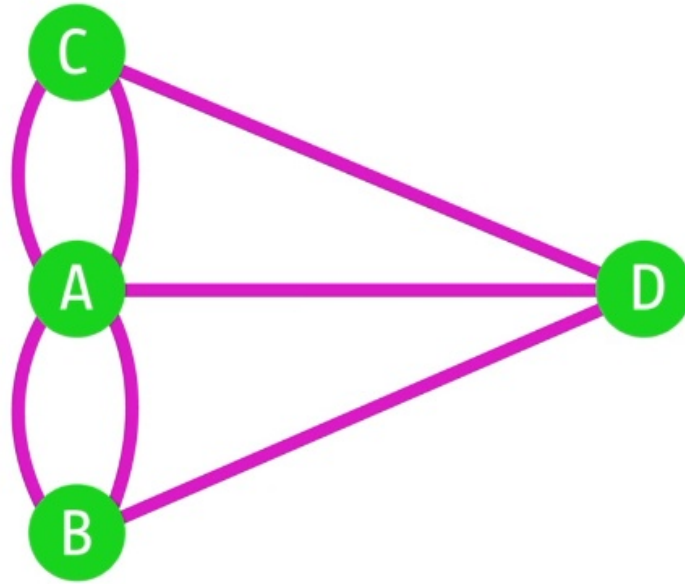
- Albert-László Barabási: Network Science. Cambridge University Press, 2016.
 - Follows almost section-by-section chapter 02
- URLs cited in the footer of specific slides

The seven bridges of Königsberg



<http://networksciencebook.com/images/ch-02/video-2-1.m4v>

Quick Question

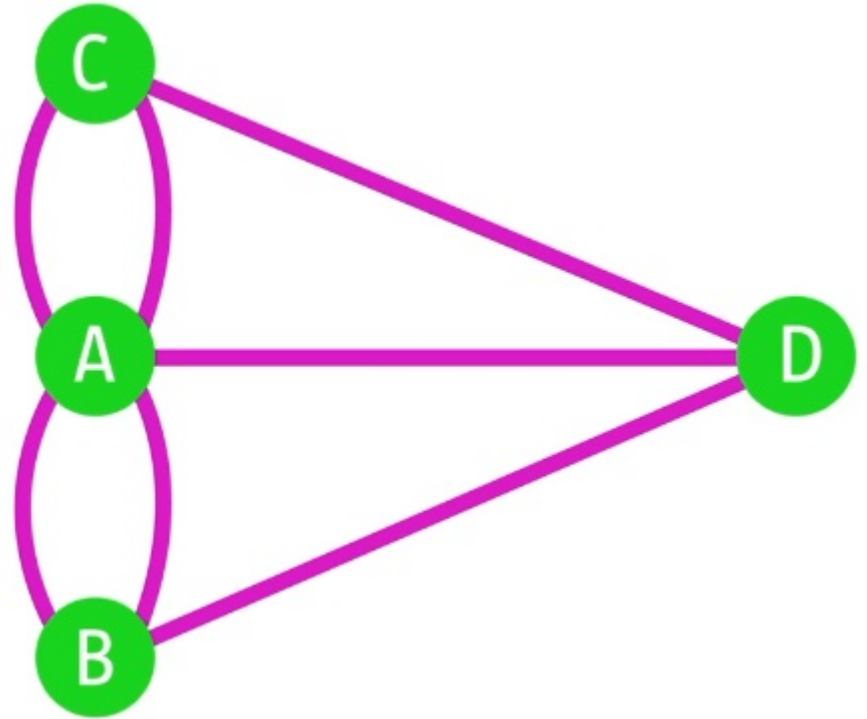


Can one walk across the 7 bridges without crossing the same bridge twice?

Basic concepts

Notation for a graph

- $G = (V, E)$
 - V : nodes or vertices
 - E : links or edges
- $|V| = N$ size of graph
- $|E| = L$ number of links



Typical notation variations

- You may find that G is denoted by (N, A) , this is typical of directed graphs, means “*nodes, arcs*”
- You may find that
 - $|V|$ is denoted by n or N
 - $|E|$ is denoted by m , M , or L

Directed vs undirected graphs

- In an undirected graph
 - E is a symmetric relation
$$(u, v) \in E \Rightarrow (v, u) \in E$$
- In a directed graph, also known as “digraph”
 - E is not a symmetric relation
$$(u, v) \in E \not\Rightarrow (v, u) \in E$$

Example graphs we will use

Network	$ V $	$ E $
Zachary's Karate Club (karate.gml)	34	78
Les Misérables (lesmiserables.gml)	77	254
E-mail exchanges (email-eu-core.csv)	868	25K
US companies ownership	1351	6721
Marvel comics (hero-network.csv)	6K	167K

Degree

- Node i has degree k_i
 - This is the number of links incident on this node
 - The total number of links L is given by $L = \frac{1}{2} \sum_{i=1}^N k_i$
- Average degree $\langle k \rangle = \frac{1}{N} \sum_{i=1}^N k_i = \frac{2L}{N}$

In directed networks

- We distinguish **in-degree** from **out-degree**
 - Incoming and outgoing links, respectively
- Degree is the sum of both $k_i = k_i^{\text{in}} + k_i^{\text{out}}$
- Counting total number of links:

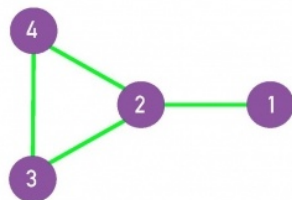
$$L = \sum_{i=1}^N k_i^{\text{in}} = \sum_{i=1}^N k_i^{\text{out}}$$

Degree distribution

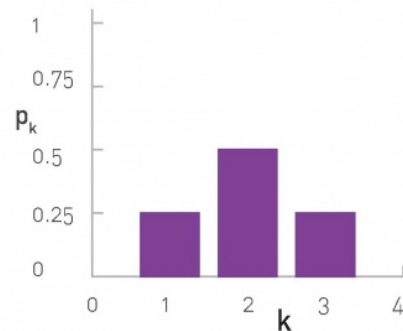
- If there are N_k nodes with degree k
- The **degree distribution** is given by $p_k = \frac{N_k}{N}$
- The average degree is then $\langle k \rangle = \sum_{k=0}^{\infty} k p_k$

Degree distribution; two toy graphs

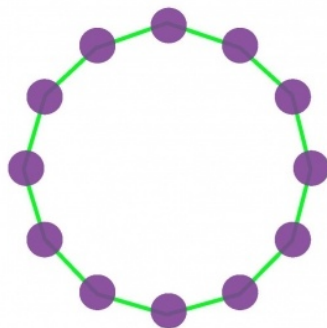
a.



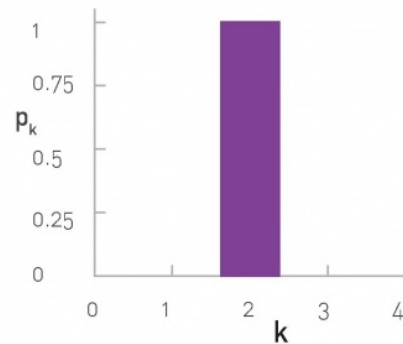
b.



c.

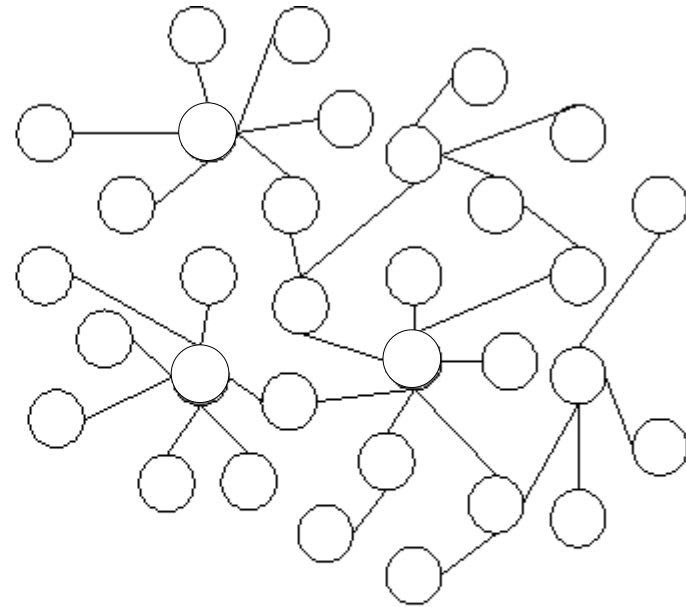
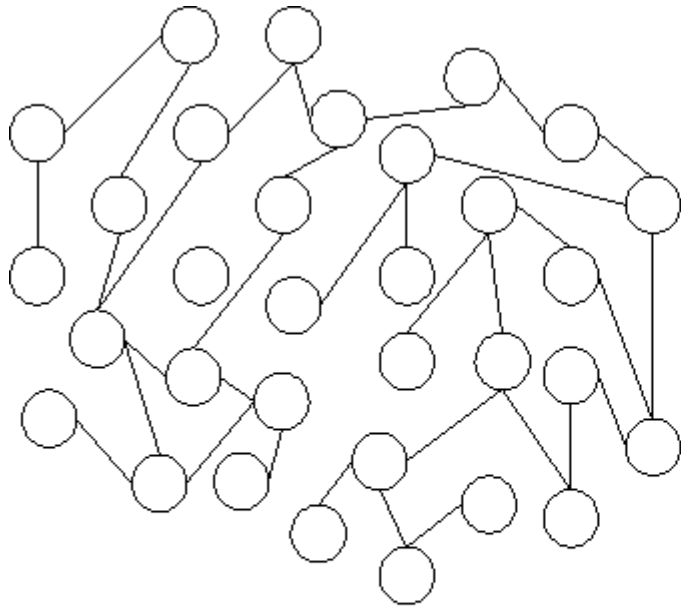


d.



Exercise

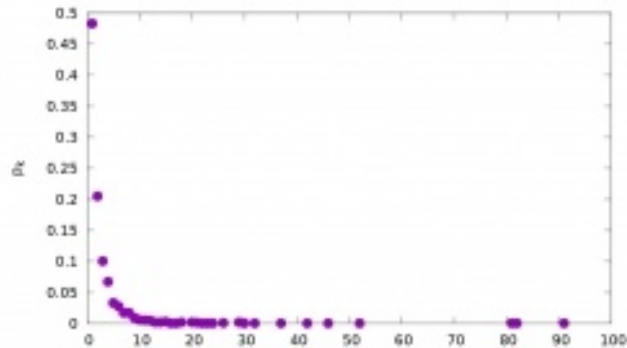
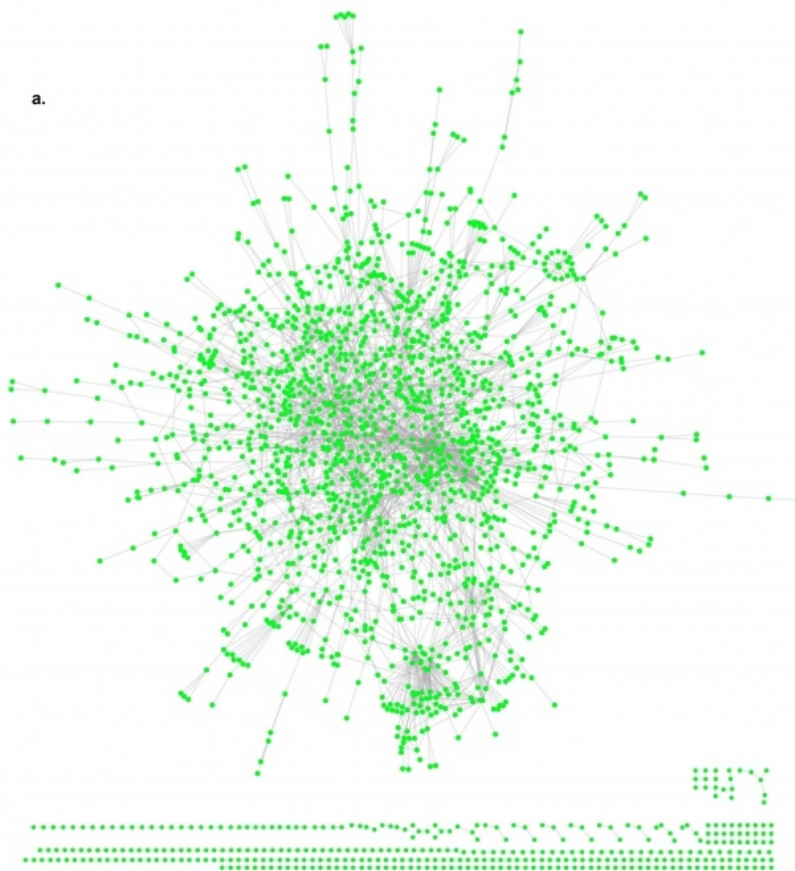
Answer in
Google Spreadsheet



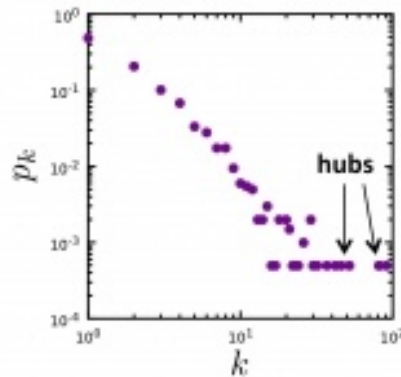
Draw the degree distribution of these graphs

Degree distribution; real graph

a.



Linear
scale



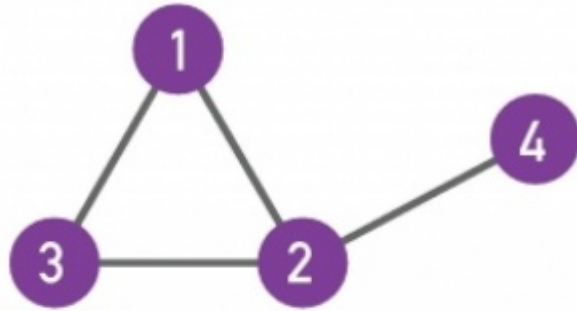
Log-log
scale

Adjacency matrix

What is an adjacency matrix

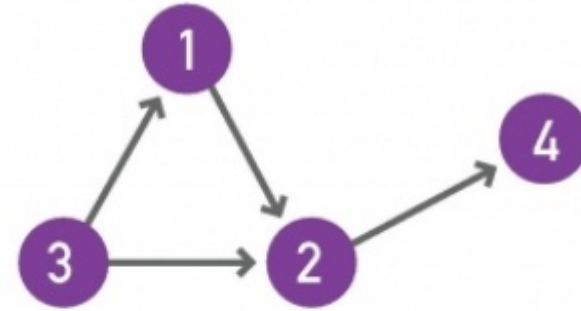
- A is the **adjacency matrix** of $G = (V, E)$ iff:
 - A has $|V|$ rows and $|V|$ columns
 - $A_{ij} = 1$ if $(i,j) \in E$
 - $A_{ij} = 0$ if $(i,j) \notin E$

Examples



Undirected graph

$$A_{ij} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$



Directed graph

$$A_{ij} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Quick Question

- In terms of A , what is the expression for:

$$k_i^{\text{in}} =$$

$$k_i^{\text{out}} =$$

Some “graphology” ...

- G is undirected $\Leftrightarrow A$ is symmetric
- G has a self-loop
 $\Leftrightarrow A$ has a non-zero element in the diagonal
- G is complete $\Leftrightarrow A_{ij} \neq 0$ (except if $i=j$)

Summary

Things to remember

- Definitions:
 - Degree, in-degree, out-degree
- Writing the adjacency matrix of a graph and drawing a graph given its adjacency matrix

Practice on your own

Draw the
indegree,
outdegree,
degree
distribution

Write the
adjacency
matrix

