Graph theory basics

Introduction to Network Science

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Topic 02

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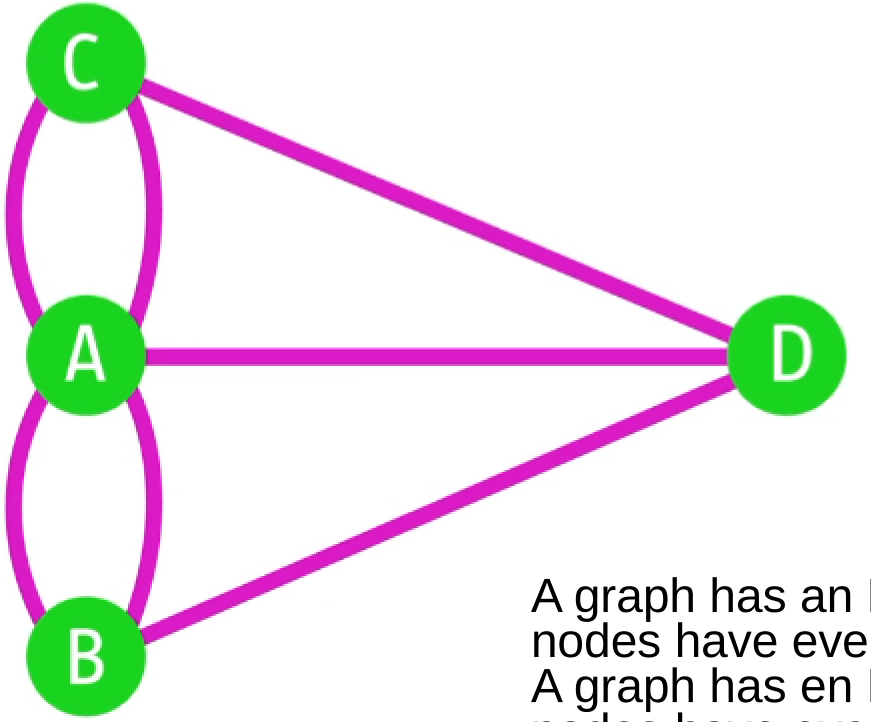
Sources

- Albert László Barabási: Network Science. Cambridge University Press, 2016.
  - Follows almost section-by-section chapter 02
- URLs cited in the footer of specific slides

The seven bridges of Königsberg

http://networksciencebook.com/images/ch-02/video-2-1.m4v

Can one walk across the 7 bridges w/o crossing the same path twice?



No

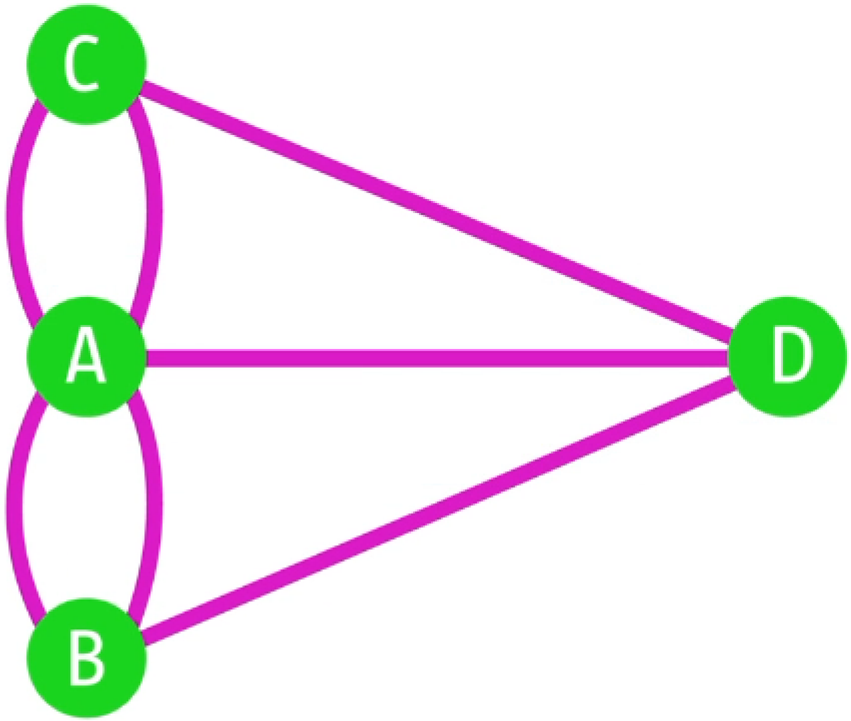
(proven by Euler, 1735)

A graph has an Eulerian circuit iff the graph is connected and all nodes have even degree
A graph has en Eulerian path iff the graph is connected and all nodes have even degree, or only 2 nodes have odd degree

Basic concepts

Notation for a graph

- G = (V,E)
  - V: nodes or vertices
  - E: links or edges
- |V| = N size of graph
- |E| = L number of links



Typical notation variations

- You may find that G is denoted by (N, A), this is typical of directed graphs
- You may find that $|V|$ is denoted by n, $|E|$ is denoted by m

Directed vs undirected graphs

- In a directed graph, also known as “digraph”, E is a symmetric relation
  - $(u,v) \in E \Rightarrow (v,u) \in E$
- In an undirected graph, E is not symmetric

Example graphs we will use

Network	|V|	|E|
Zachary’s Karate Club (karate.gml)	34	78
Les Misérables (lesmiserables.gml)	77	254
E-mail exchanges (email-eu-core.csv)	868	25K
US companies ownership	1351	6721
Marvel comics (hero-network.csv)	6K	570K

https://github.com/chatox/networks-science-course/tree/master/practicum/data

Degree

- Node i has degree k_i
  - This is the number of links incident on this node
  - The total number of links L is given by $$L = \frac{1}{2}\sum_{i=1}^{N} k_i$$
- Average degree $$\langle k \rangle = \frac{1}{N}\sum_{i=1}^{N} k_i = \frac{2L}{N}$$

In directed networks

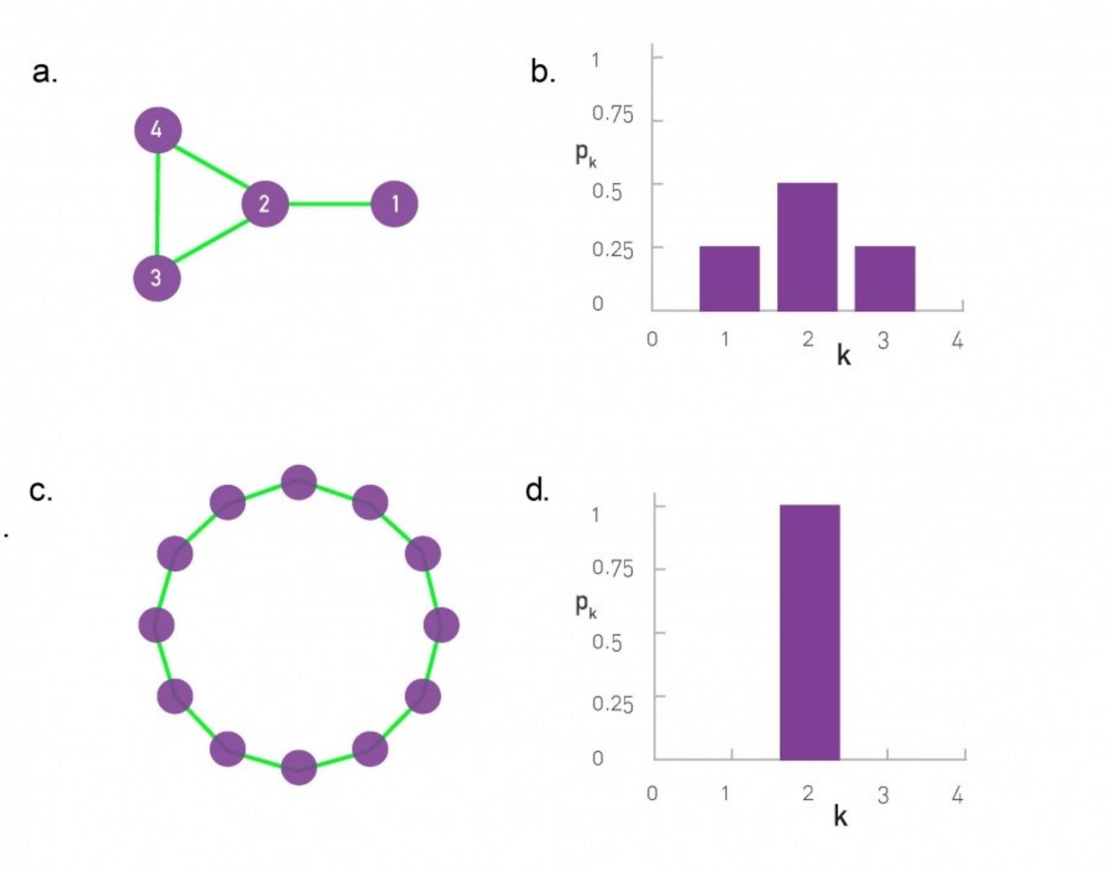
- We distinguish in-degree from out-degree
  - Incoming and outgoing links, respectively
- Degree is the sum of both $k_i = k_i^{\mathrm{in}} + k_i^{\mathrm{out}}$
- Counting total number of links:

$$L = \sum_{i=1}^{N} k_i^{\mathrm{in}} = \sum_{i=1}^{N} k_i^{\mathrm{out}}$$

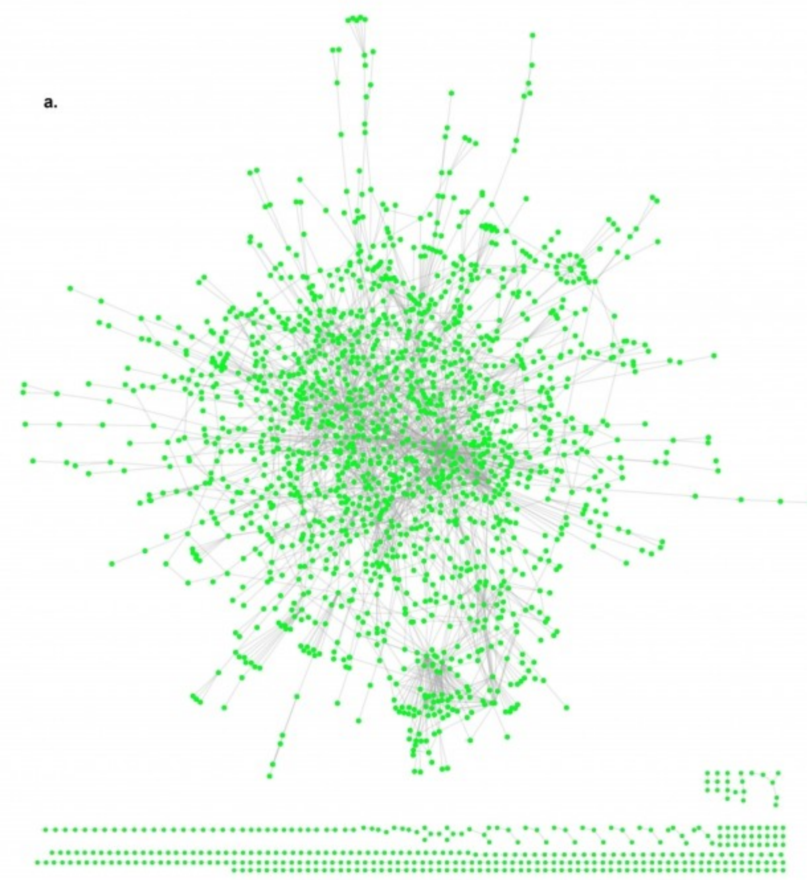
Degree distribution

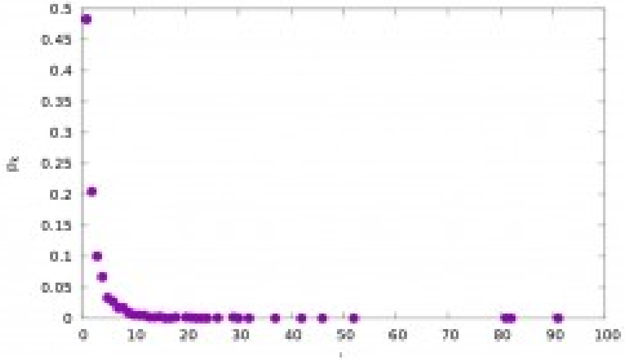
- If there are N_k nodes with degree k
- The degree distribution is given by $p_k = \frac{N_k}{N}$
- The average degree is then $\langle k \rangle = \sum_{k=0}^{\infty} k p_k$

Degree distribution; two toy graphs

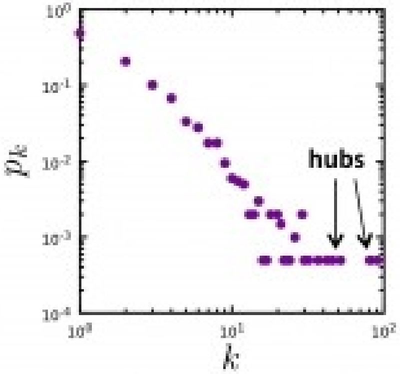


Degree distribution; real graph





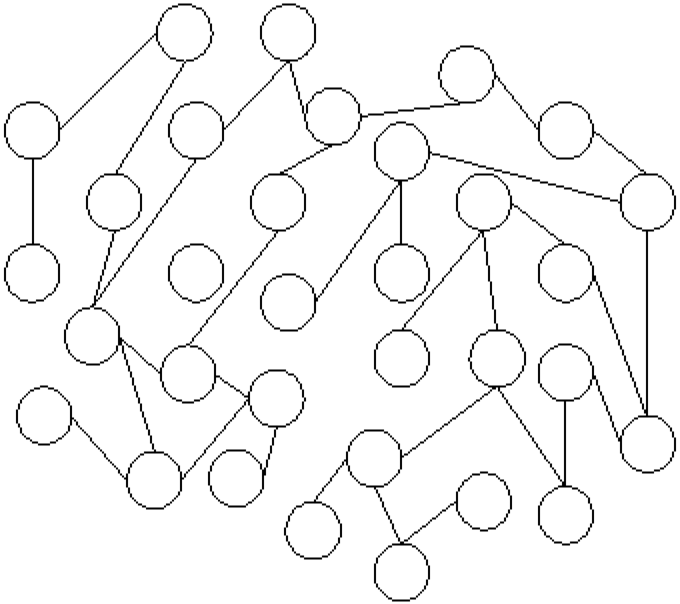
Linear scale

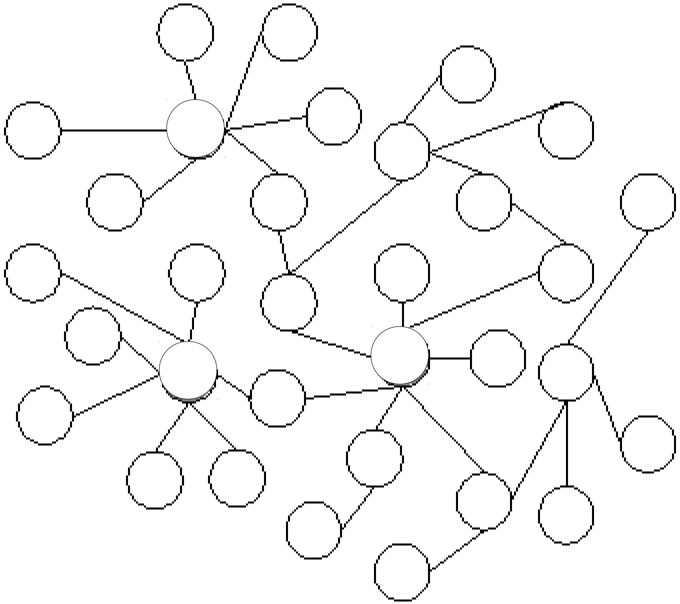


Log-log scale

Exercise

- Draw the degree distribution of these graphs



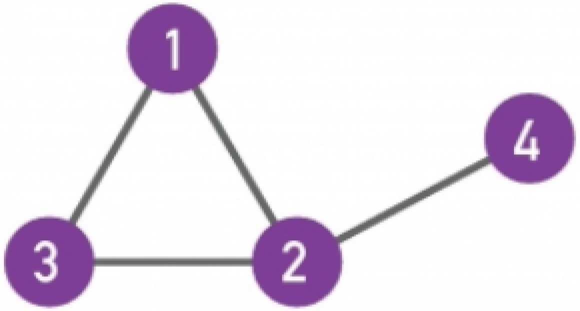


Adjacency matrix

What is an adjacency matrix

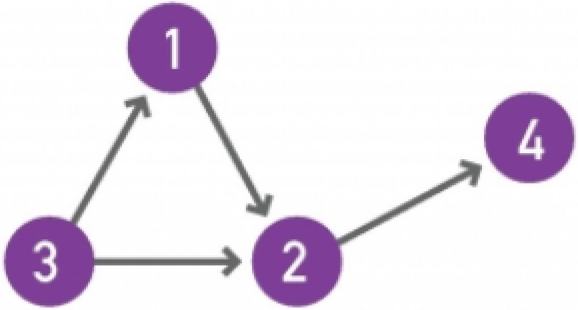
- A is the adjacency matrix of G = (V, E) iff:
  - A has |V| rows and |V| columns
  - $A_{ij} = 1$ if $(i,j) \in E$
  - $A_{ij} = 0$ if $(i,j) \notin E$

Examples



Undirected graph

$$A_{ij} = \begin{matrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$



Directed graph

$$A_{ij} = \begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$

Quick exercise

- In terms of A, what is the expression for:

$$k_i^{\text{in}} =$$

$$k_i^{\text{out}} =$$

Some “graphology” …

- G is undirected ⇔ A is symmetric
- G has a self-loop
  ⇔ A has a non-zero element in the diagonal
- G is complete ⇔ $A_{ij} \neq 0$ (except if i=j)

Real networks are sparse

- Theoretically $L_{\max} = \binom{N}{2} = \frac{N(N-1)}{2}$

- Most real networks are sparse, i.e., $L \ll L_{\max}$

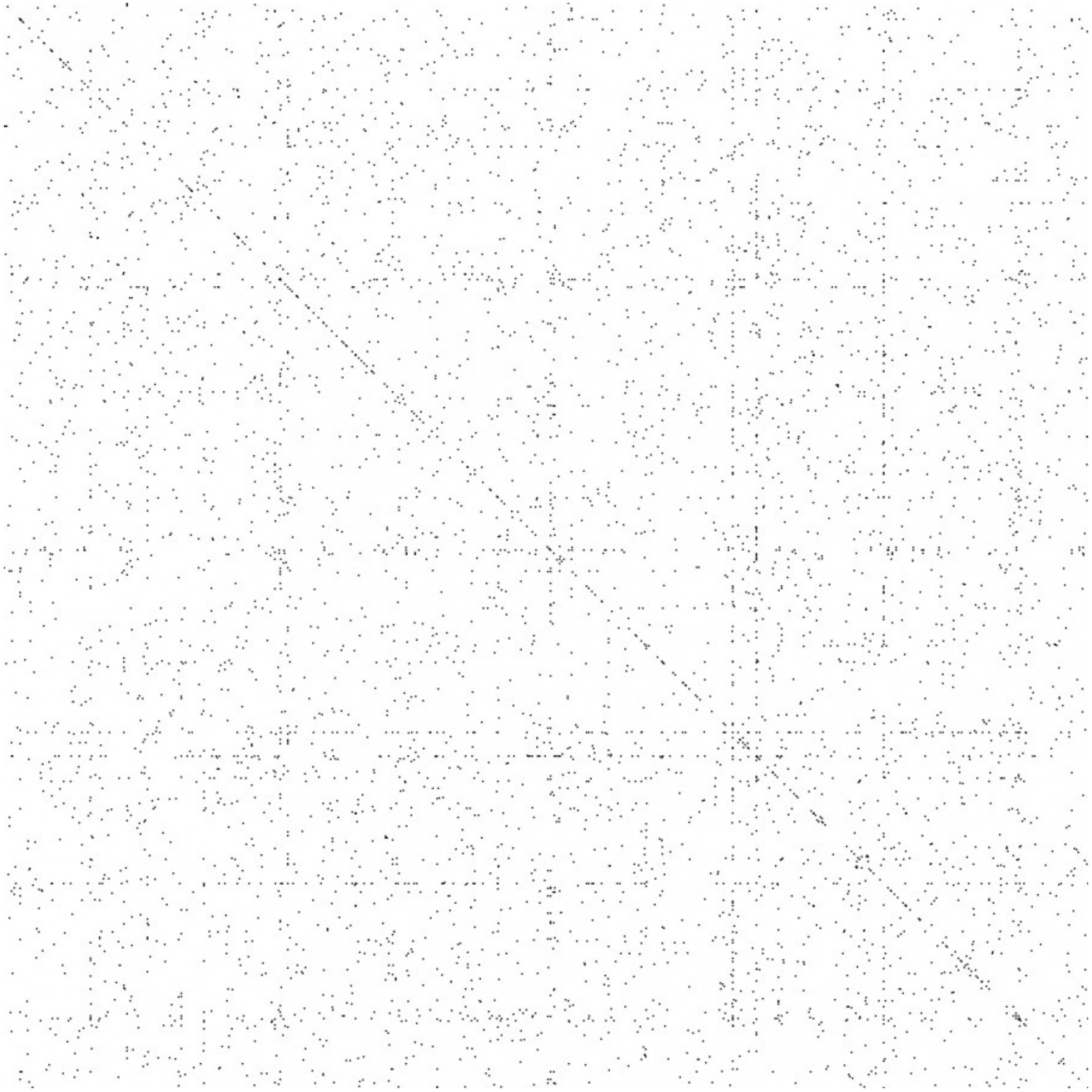
How sparse are some networks?

Network	|V|	|E|	Max |E|
Zachary’s Karate Club	34	78	561
Les Misérables	77	254	2962
E-mail exchanges	868	25K	376K
US companies ownership	1351	6721	911K
Marvel comics	6K	570K	17M

https://github.com/chatox/networks-science-course/tree/master/practicum/data

Example: protein interaction network

(N=2K, L=3K)

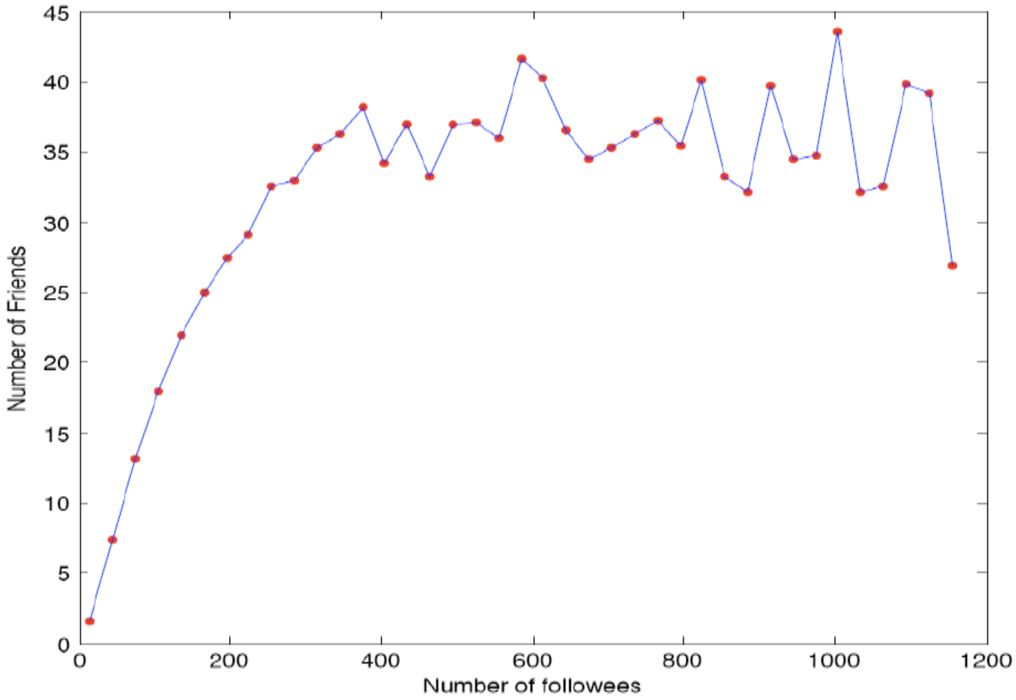


Why are networks sparse?

- Different mechanisms, think about it from the node perspective:
  - How many items **could** the node be connected to
  - Would it be **realistic** to connect to a large fraction of them?
- In social networks, Dunbar's number (≃150)

https://en.wikipedia.org/wiki/Dunbar's_number

Example: actual friends in Twitter vs people you follow in Twitter



Huberman, B., Romero, D. M., & Wu, F. (2008). Social networks that matter: Twitter under the microscope. First Monday, 14(1).

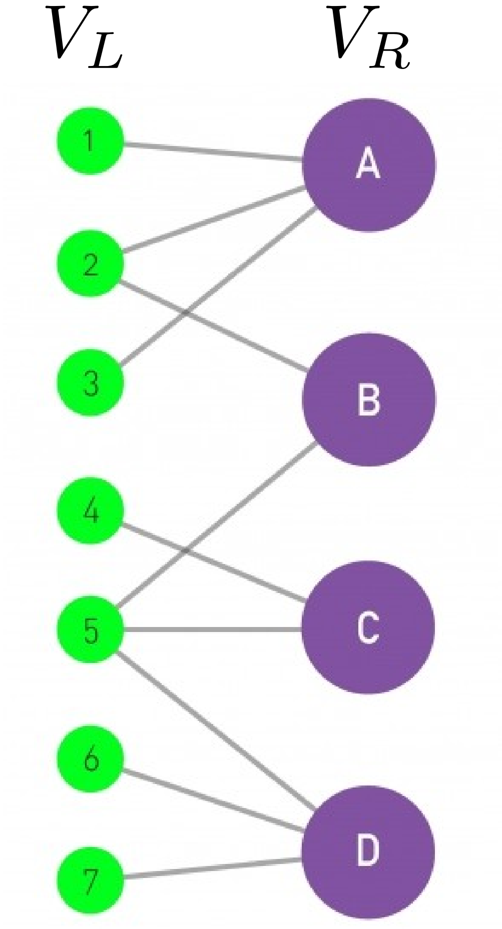
Weighted networks

- In weighted networks, instead of $A_{ij} \in [0, 1]$
- We have that $A_{ij} \in \mathbb{R}$
- Weights may represent different tie strengths

Bipartite networks

- A bipartite graph is a graph G = (V,E) such that

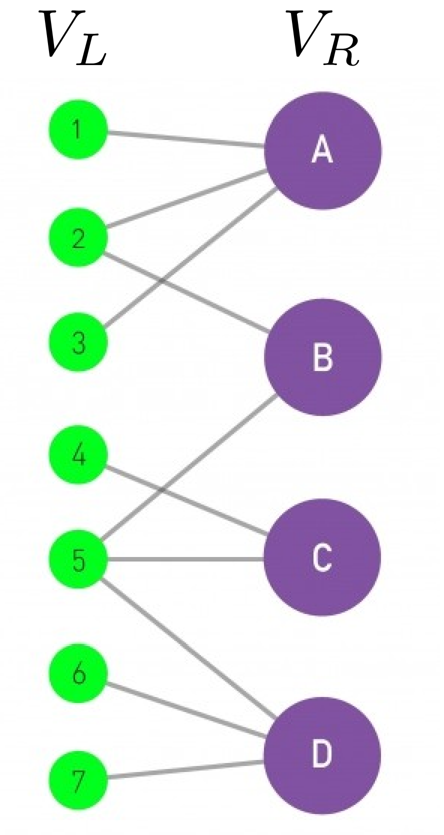
$$V = V_L \cup V_R, V_L \cap V_R = \emptyset, E \subseteq V_L \times V_R$$



Projecting a bipartite network

?

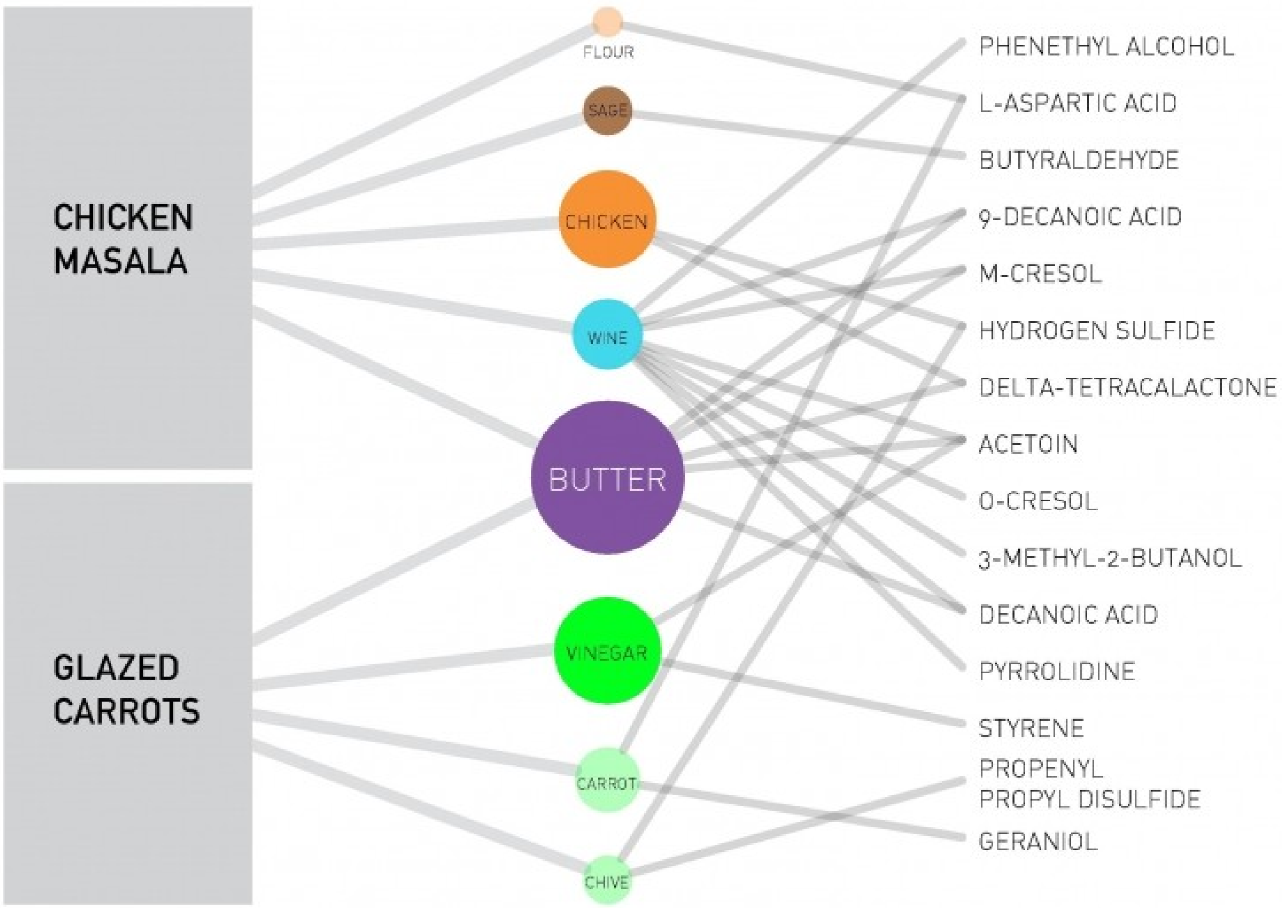
Left projection: graph where nodes are 1, 2, …, 7 and nodes are connected if they share a neighbor



?

Right projection: graph where nodes are A, B, …, D and nodes are connected if they share a neighbor

Tripartite network



Clique and Bi-partite clique

- A **clique** is a complete graph: $E = (V \times V)$
- An **n-clique** is a complete graph of n nodes
- A **bi-partite clique** is such that

$$V = V_1 \cup V_2, V_1 \cap V_2 = \emptyset, E = (V_1 \times V_2)$$

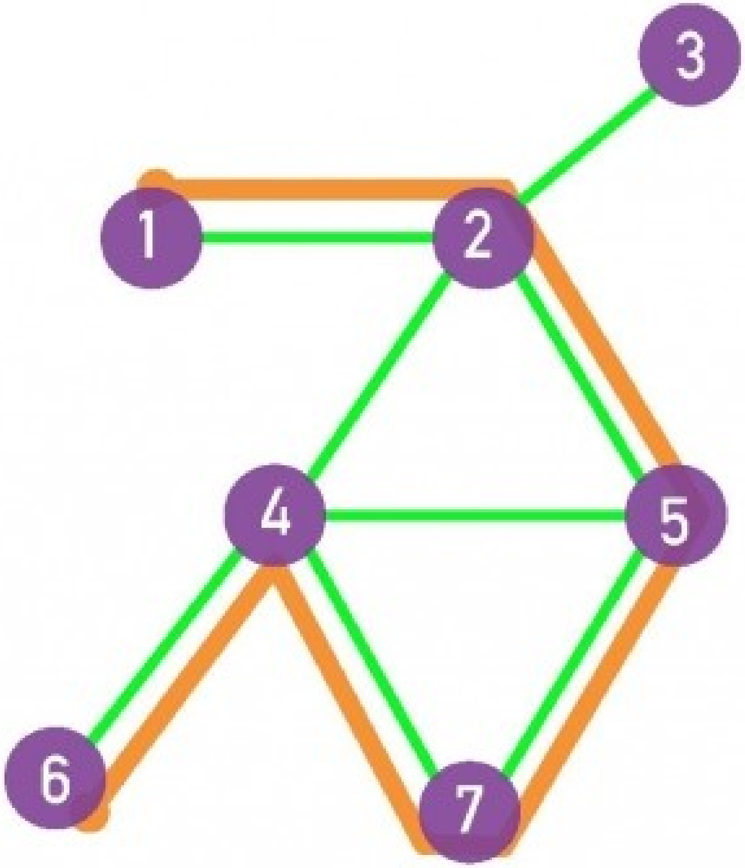
- A **(n_1, n_2)-clique** is a bipartite clique such that

$$|V_1| = n_1, |V_2| = n_2$$

Paths and distances

Paths

- A path is a sequence of edges from E
- The destination of each edge is the origin of the next edge
- The length of the path is the number of edges on it
- Example: a path marked in orange, having length 5



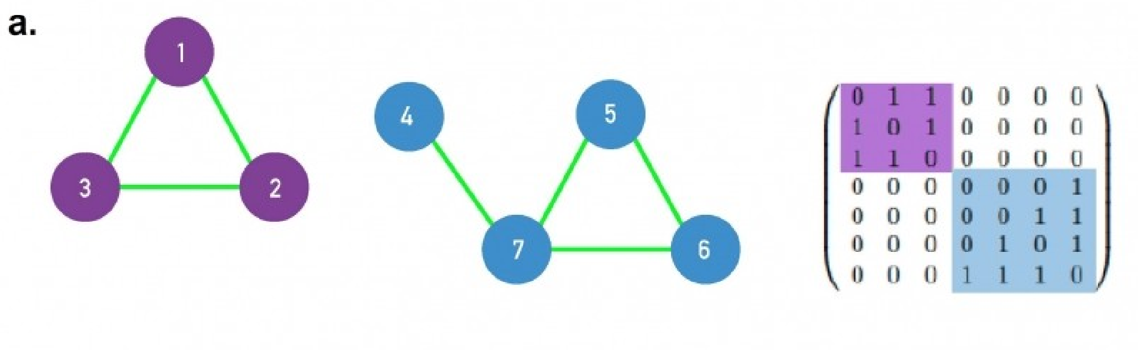
Connectedness

- If a path exists between two nodes i, j:
  - those nodes are part of the same **connected component**
- A graph that has only one connected component is called a **connected graph**

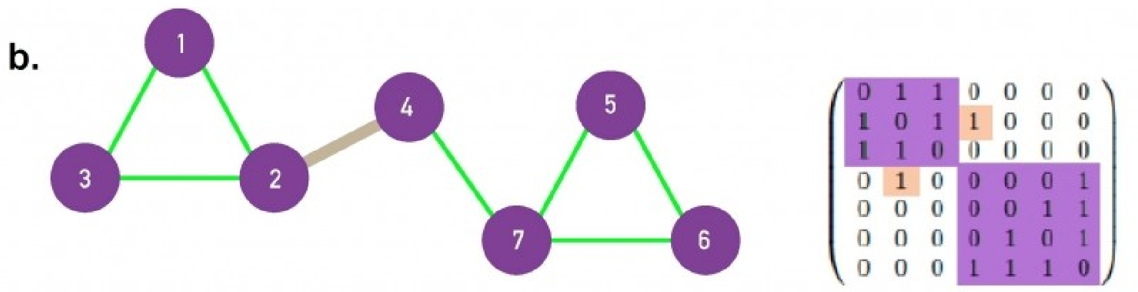
Connected graphs

A disconnected graph has an adjacency matrix that can be arranged in block diagonal form

a. disconnected



b. connected



Distance

- If two nodes i, j are in the same connected component:
  - the distance between i and j, denoted by d_{ij} is the length of the shortest path between them

Diameter

- The **diameter** of a network is the maximum distance between two nodes on it, d_{max}
- The **effective diameter** (or effective-90% diameter) is a number d such that 90% of the pairs of nodes (i,j) are at a distance smaller than d
- The average distance is $\langle d \rangle$, and is measured only for nodes that are in the same connected component

Local clustering coefficient

- The **local clustering coefficient** C_i is a property of a node i
- Let L_i represent the number of links among neighbors of node i

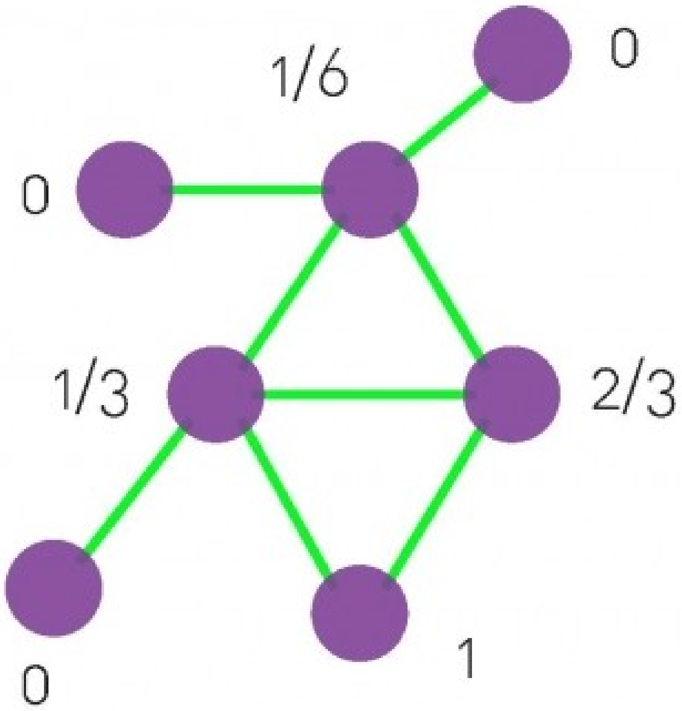
$$C_i = \frac{2L_i}{k_i(k_i - 1)}$$

Average clustering coefficient

- The **average clustering coefficient** is a property of the entire graph

$$\langle C \rangle = \frac{1}{N} \sum_{i=1}^{N} C_i$$

Example C_i
(check to ensure you understood)



More to practice …

- You can practice with exercises in section 2.11 of Barabási (2016)