

Preferential Attachment

Introduction to Network Science

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Topic 05

Sources

- Albert-László Barabási: Network Science. Cambridge University Press, 2016.
 - Follows almost section-by-section chapter 05
- Ravi Srinivasan 2013 Complex Networks Ch 12
- Networks, Crowds, and Markets Ch 18
- URLs cited in the footer of specific slides

Preferential attachment simulation



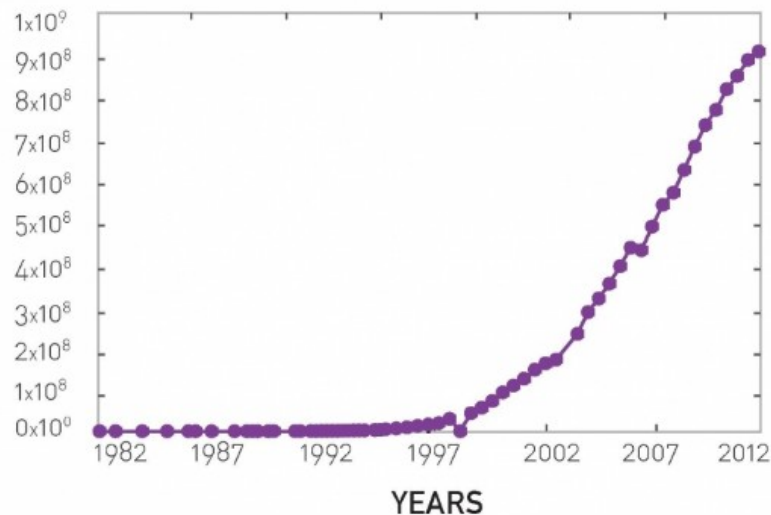
<https://www.youtube.com/watch?v=4GDqJVtPEGg>

We have seen what but not why

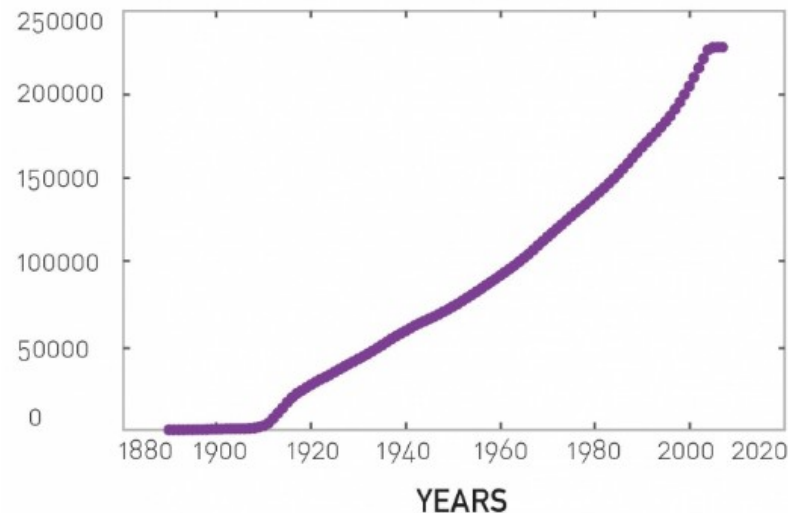
- Power-law degree distributions are prevalent
 - Why?
- Two assumptions in ER networks:
 - There are N nodes that **pre-exist**
 - Nodes connect **at random**
- Let's challenge both assumptions

The number of nodes N increases

Number of hosts in WWW

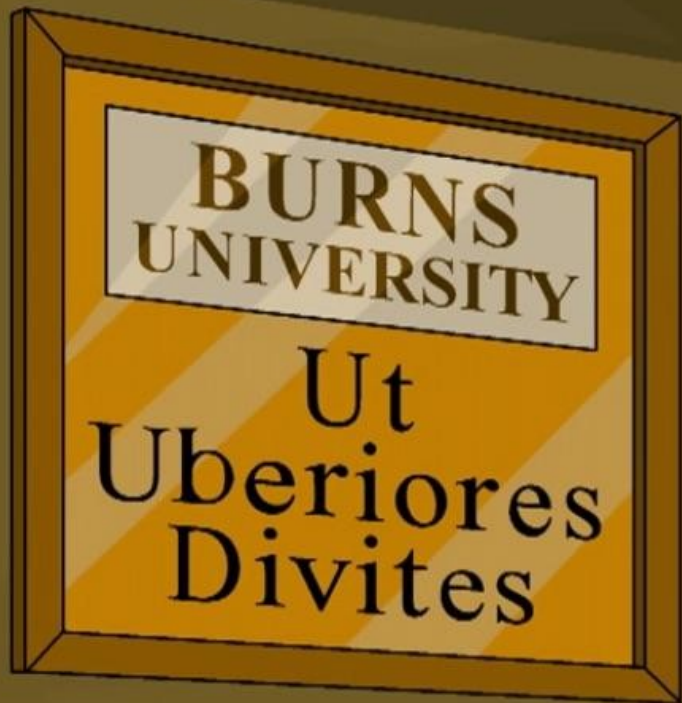


Number of movies in IMDB



Growth

- Suppose there are two web pages on a topic, one with many inlinks the other with few, which one am I most likely to link to?
- Which scientific papers are read? Which are cited?
- Which actors are more sought after for new movies?



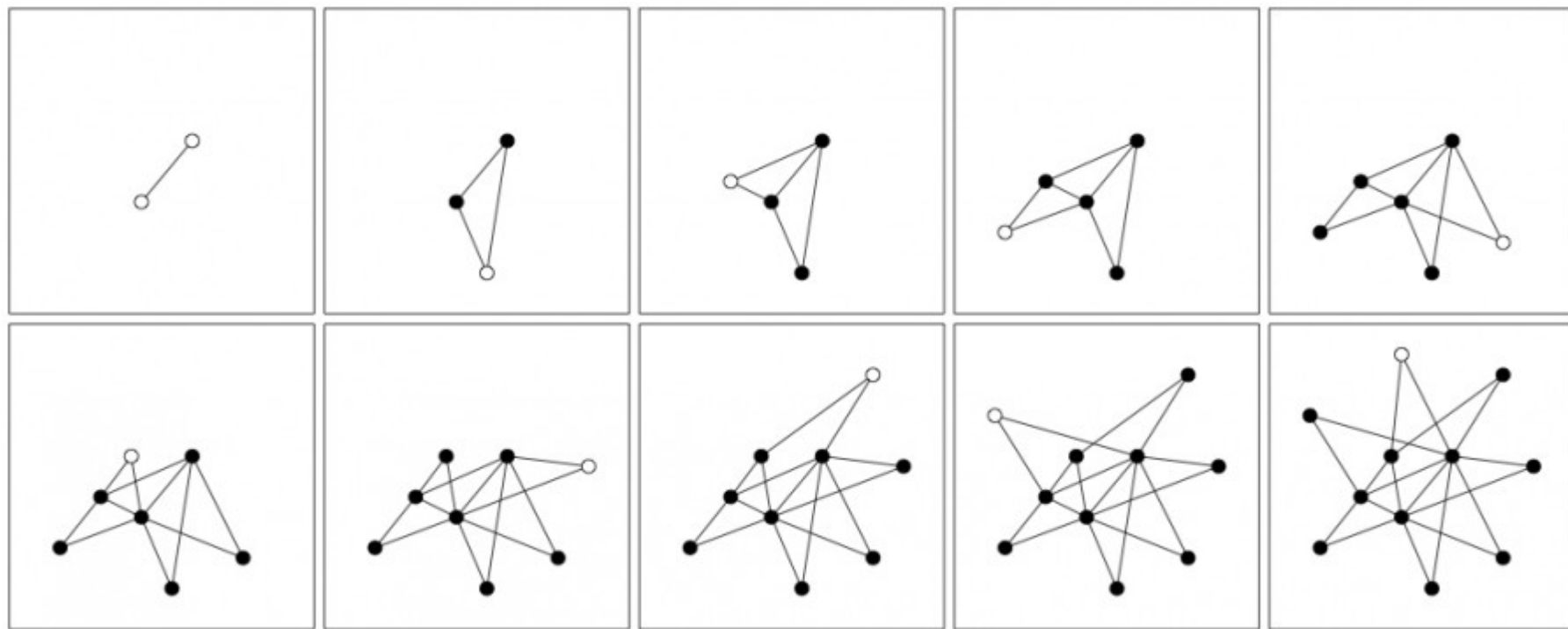
Our motto: *Ut uberiores divites.*

The Barabási-Albert (BA) model

- Network starts with m nodes
- At every time step we add 1 node
- This node will have $m < m_0$ outlinks
- The probability of an existing node to gain a link is

$$\Pi(k_i) = \frac{k_i}{\sum_{j=1}^{N-1} k_j}$$

Example ($m_0 = 2$; $m=2$)



The Barabási-Albert (BA) model

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$$\Pi(k_i) = \frac{k_i}{\sum_{j=1}^{N-1} k_j}$$

Write the formula for $N(t)$ and $L(t)$

Degree $k_i(t)$ as a function of time

$$\frac{dk_i}{dt} = m\Pi(k_i) = m \frac{k_i}{\sum_{j=1}^{N-1} k_j}$$

$$\sum_{j=1}^{N-1} k_j = 2m(t-1) \quad (\text{All nodes minus the current})$$

$$\frac{dk_i}{dt} = \frac{k_i}{2t-2} \approx \frac{k_i}{2t} \quad (\text{For large } t)$$

Degree $k_i(t)$... continued

$$\frac{dk_i(t)}{dt} = \frac{k_i(t)}{2t}$$

$$\frac{1}{k_i(t)} dk_i(t) = \frac{1}{2t} dt$$

$$\int_{t=t_i}^t \frac{1}{k_i(t)} dk_i(t) = \int_{t=t_i}^t \frac{1}{2t} dt \quad (t_i \text{ is the creation time of node } i)$$

$$\log k_i(t) - \log k_i(t_i) = \frac{1}{2} \log t - \frac{1}{2} \log t_i$$

$$\log k_i(t) = \frac{1}{2} \log t - \frac{1}{2} \log t_i + \log m$$

Degree $k_i(t)$... continued

$$\log k_i(t) = \frac{1}{2} \log t - \frac{1}{2} \log t_i + \log m$$

$$k_i(t) = m \left(\frac{t}{t_i} \right)^{\frac{1}{2}}$$

Is the degree growth linear, super-linear, or sub-linear? Intuitively, why?

Degree $k_i(t)$... consequences

$$\log k_i(t) = \frac{1}{2} \log t - \frac{1}{2} \log t_i + \log m$$

$$k_i(t) = m \left(\frac{t}{t_i} \right)^{\frac{1}{2}}$$

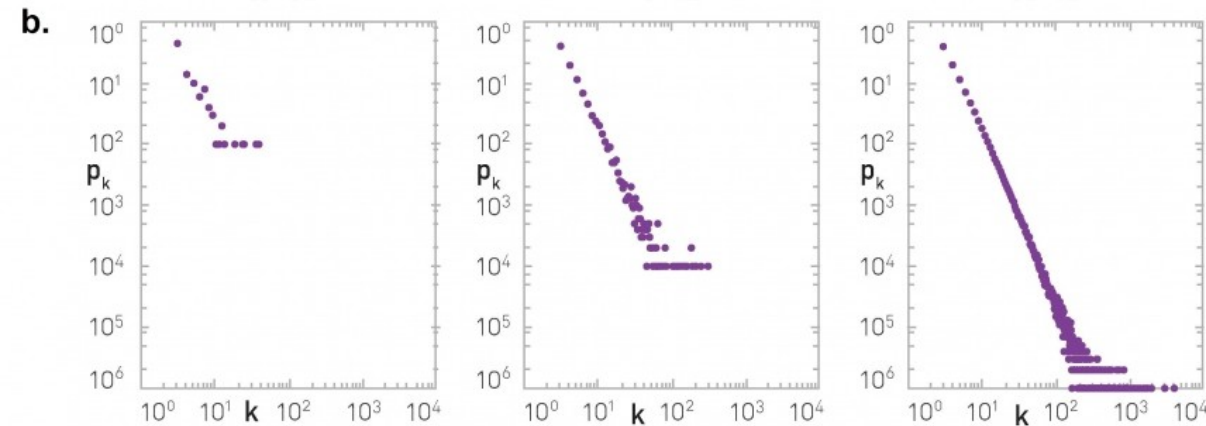
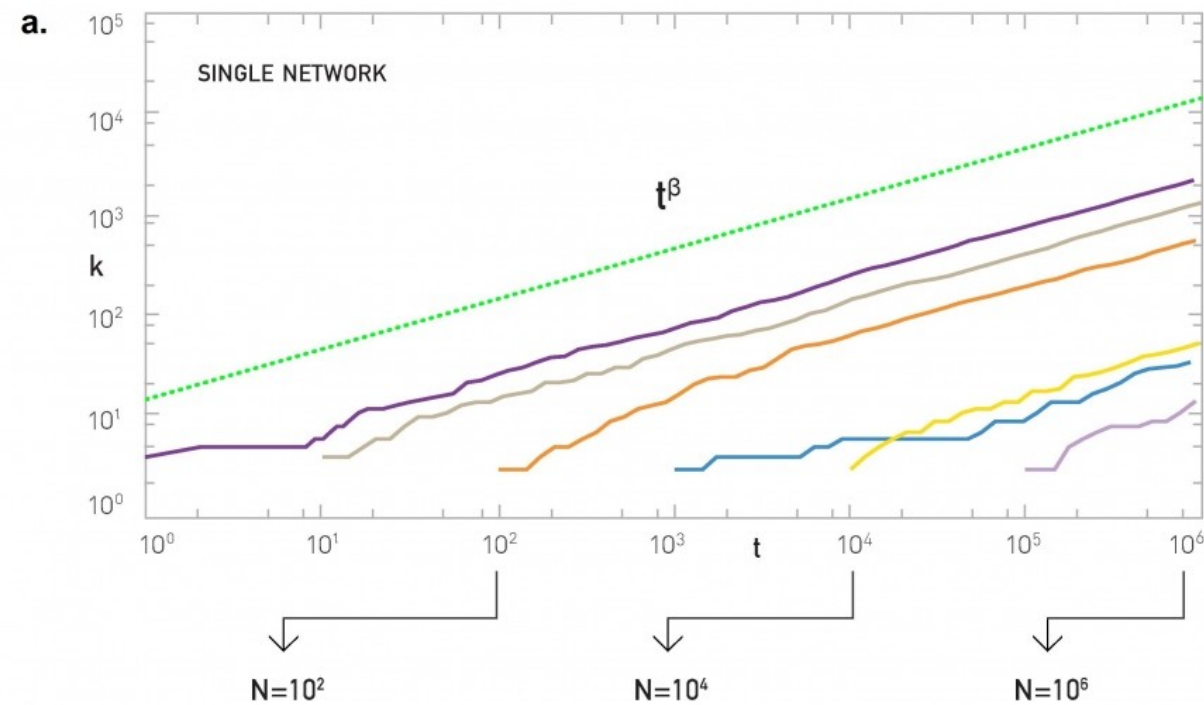
$$\frac{dk_i(t)}{dt} = \frac{k_i(t)}{2t} = \frac{m \left(\frac{t}{t_i} \right)^{\frac{1}{2}}}{2t} = \frac{m}{2(t \cdot t_i)^{\frac{1}{2}}}$$

If $t_i < t_j$ (node i is older than node j), what do we expect of k_i and k_j ?

Simulation results

Model

Nodes with $t_i = 1, 10, 100, 1000, 10000, \dots$



Degree distribution

- Let's calculate the CDF of the degree distribution

$$Pr(k_i \leq k) = 1 - Pr(k_i > k)$$

$$= 1 - Pr\left(m \left(\frac{t}{t_i}\right)^\beta > k\right)$$

$$= 1 - Pr\left(\left(\frac{m}{k}\right)^{1/\beta} > \frac{t_i}{t}\right)$$

$$\frac{t_i}{t} \sim \text{Uniform}(0, 1)$$

$$= 1 - \left(\frac{m}{k}\right)^{1/\beta}$$

Degree distribution

Now let's take the derivative of the CDF to obtain the PDF

$$\begin{aligned} p_k &= \frac{\partial}{\partial k} Pr(k_i \leq k) = \frac{\partial}{\partial k} \left(1 - \left(\frac{m}{k} \right)^{1/\beta} \right) \\ &= -\frac{\partial}{\partial k} \left(\left(\frac{m}{k} \right)^{1/\beta} \right) = -m^{1/\beta} \frac{\partial}{\partial k} \left(\frac{1}{k^{1/\beta}} \right) \\ &= \frac{1}{\beta} \frac{m^{1/\beta}}{k^{1/\beta+1}} \end{aligned}$$

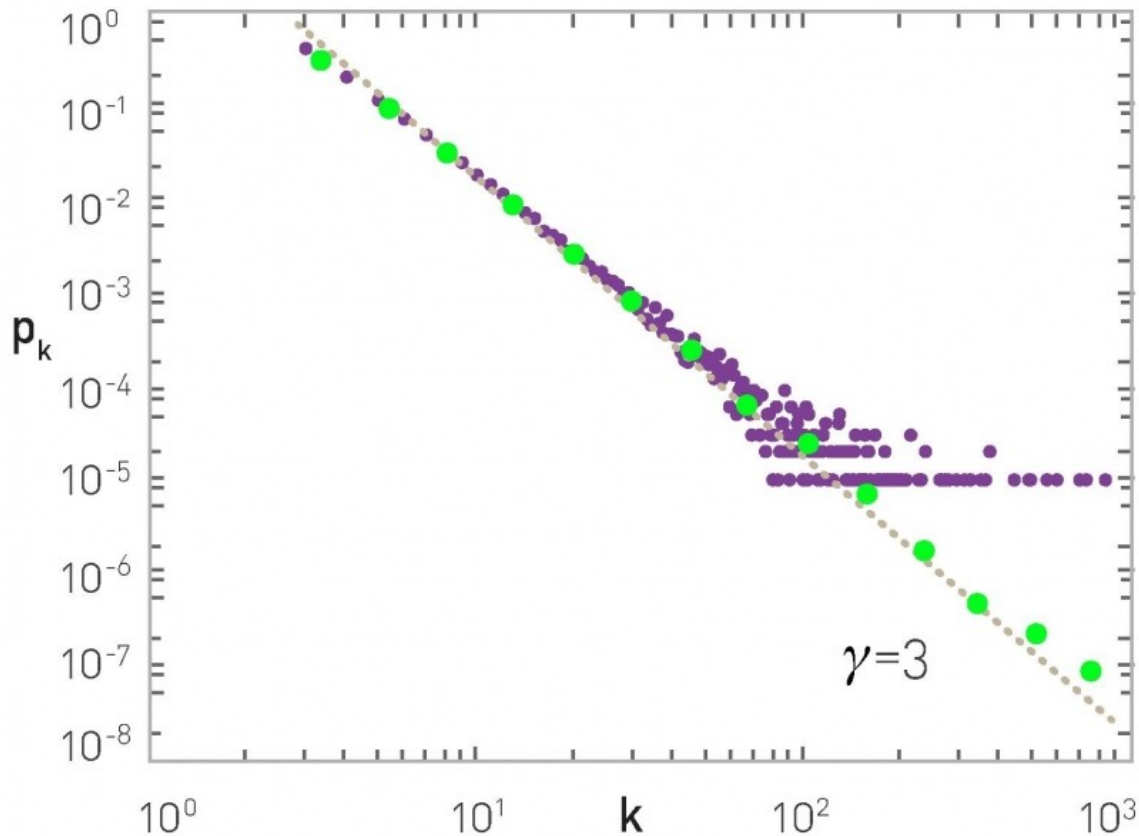
$$p(k) \propto k^{-(1/\beta+1)} = k^{-3}$$

Degree distribution

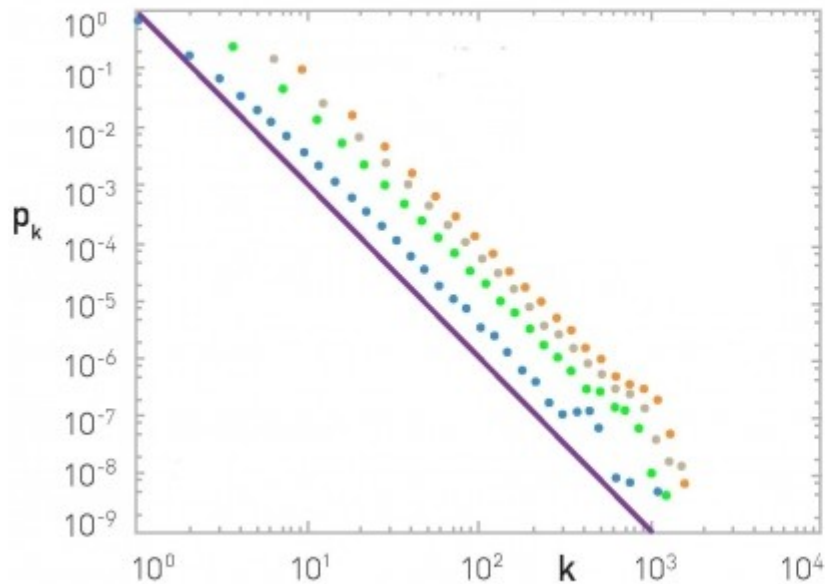
- $\beta = 1/2$ is called the dynamical exponent
- $\gamma = \frac{1}{\beta} + 1 = 3$ is the power-law exponent
- Note that $p(k) \approx 2m^2 k^{-3}$
does not depend on t
hence, it describes a stationary network

Degree distribution, simulation results

$N=100,000$ $m=3$



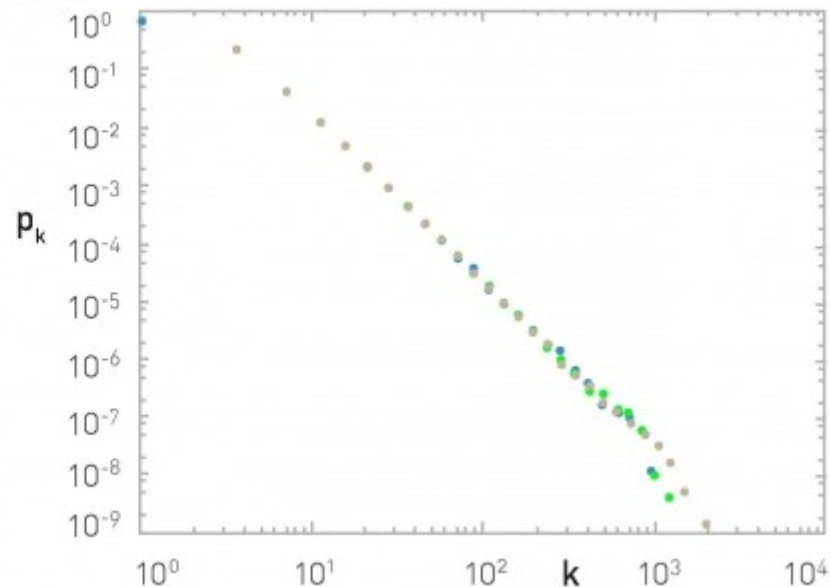
More simulations



$N = 100,000$; $m_0 = m =$
1 (blue), 3 (green), 5 (gray), 7 (orange)

Observe γ is independent of m (and m_0)

The slope of the purple line is -3



$m_0 = m = 3$; $N =$
50K (blue), 100K (green), 200K (gray)

Observe p_k is independent of N

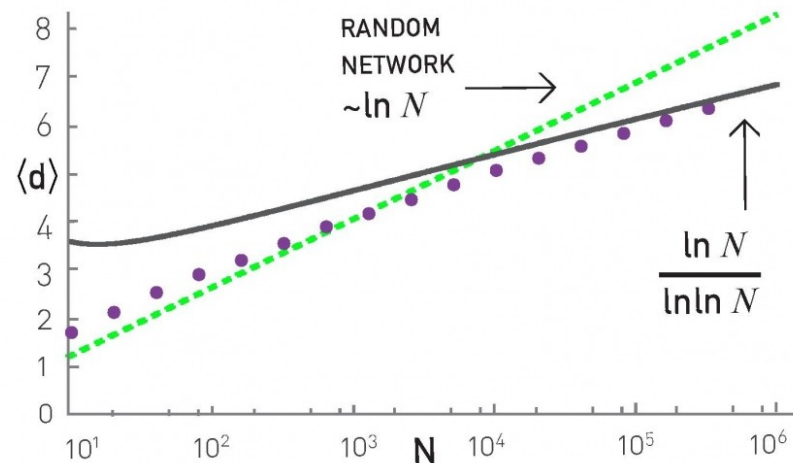
Processes that generate BA networks

- Link-selection model (step)
 - Add one new node v to the network
 - Select an existing link at random and connect v to one of the edges of that existing link
- Copy model (step)
 - Add one new node v to the network
 - Pick a random existing node u
 - With probability p link to u
 - With probability $1-p$ link to a neighbor of u

Average distance

- Distances grow slower than $\log N$

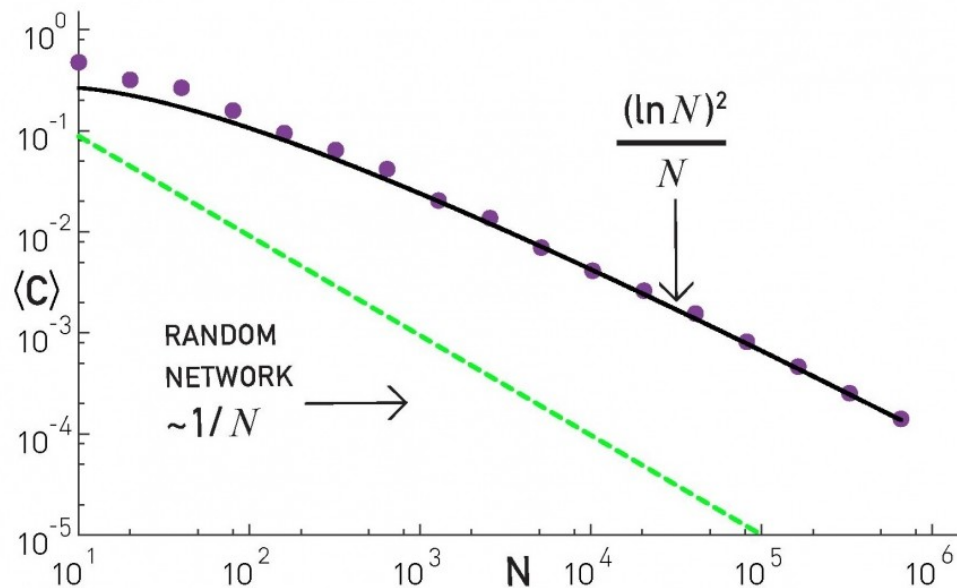
$$\langle d \rangle \sim \frac{\log N}{\log \log N}$$



Clustering coefficient

- BA networks are locally more clustered than ER networks

$$\langle C \rangle \sim \frac{(\log N)^2}{N}$$



Limitations of the BA model

- Predicts a fixed exponent of -3
- Assumes an undirected network, while many real complex networks are directed
- Does not consider node deletions or edge deletions which are common in practice
- Considers that all nodes are equal except for their arrival times

Exercise: the copy model

- In the copy model, at every step t :
 - Add one new node v to the network
 - Pick a random existing node u
 - With probability p link to u
 - With probability $1-p$ link to a neighbor of u
- Simulate it on paper for 5 nodes with $p=0.5$
 - Make sure you understand the model fully!
- What is $N(t)$ and $L(t)$?

In the copy model, at every step t :

- 1) Add one new node v to the network
- 2) Pick a random existing node u
- 3) With probability p link to u
- 4) With probability $1-p$ link to a neighbor of u

- What is k_j^{out} ?
- We will compute $k_j^{\text{in}} = k_j$
- How many links on average gets node i at time t ? In other words, what is ...

$$\frac{d}{dt}k_i(t)$$

- Hint: it has a term with p and a term with $1-p$

- Rearrange terms in $\frac{d}{dt}k_i(t)$
- ... to do something similar to what we did for the BA model, then integrate between t_i and t to obtain an expression for $k_i(t_i)$
- Note that now

$$k_i(t_i) = k_i^{\text{in}}(t_i) = 0$$

- Once you have an expression for $k_i(t_i)$
- Compute $Pr(k_i(t_i) > k)$
- Now write the cumulative distribution function of $k_i(t_i)$
- And compute its derivative to obtain

$$p_k = Pr(k_i(t) = k) = \frac{d}{dk} Pr(k_i(t) \leq k)$$
- It should show exponent $\gamma = \frac{2 - p}{1 - p}$