Closeness and betweenness

Introduction to Network Science
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Topic 09

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Sources

- Networks, Crowds, and Markets Ch 3.6B
- Barabási 2016 Section 9.3.2
- P. Boldi and S. Vigna: Axioms for Centrality in Internet Mathematics 2014.
- Esposito and Pesce: Survey of Centrality 2015.
- C. Castillo: Other centrality slides 2016

Types of centrality measure

- Spectral
  - HITS
  - PageRank
- **Non-spectral**
  - Degree
  - Closeness and harmonic closeness
  - Betweenness

Is u a well-connected person?

- Degree: u has many connections
- Eigenvector: u is connected to the well-connected
- **Closeness:** u is close to many people
  - Average distance from u is small
- **Betweenness**: many connections pass through u
  - Large number of shortest paths pass through u

Closeness

Closeness

- Distance between two nodes is $d(u,v)$
- **Closeness** is the reciprocal of distances

$$\mathrm{closeness}(u) = \frac{1}{\sum_{v \in V, v \neq u} d(u,v)}$$

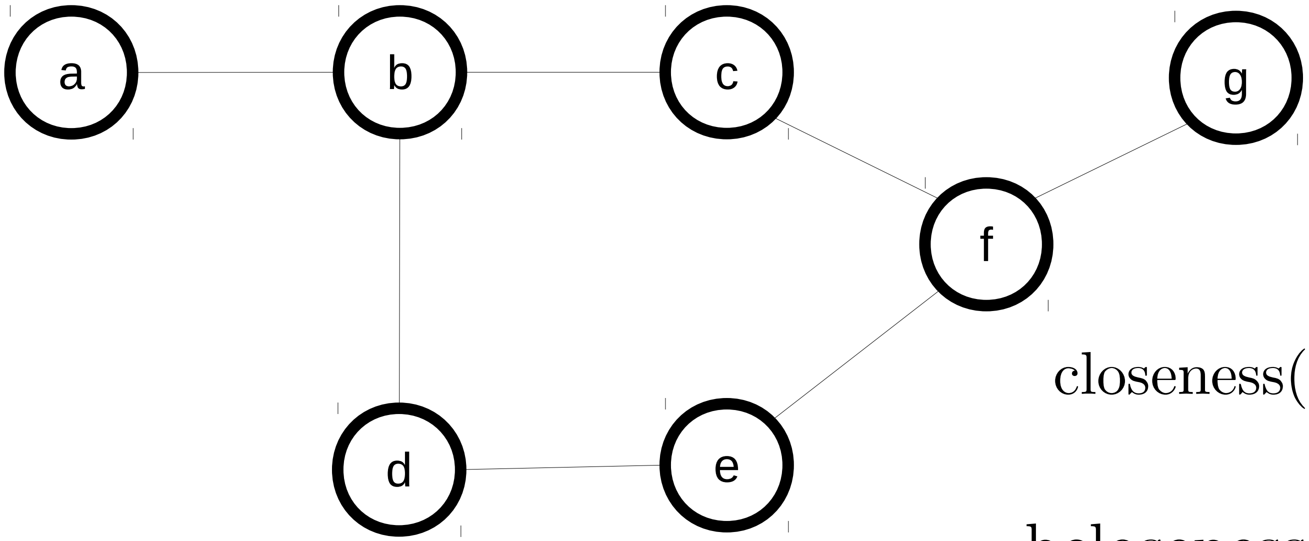
- Some graphs are not connected, in that case $d(u,v)$ can be ∞; assuming $1/\infty = 0$ one can define the **harmonic closeness**:

$$\mathrm{hcloseness}(u) = \sum_{v \neq u} \frac{1}{d(u,v)}$$

Try it!

Compute closeness and harmonic closeness for all the nodes

d(u,v) = 1 if v is a neighbor of u



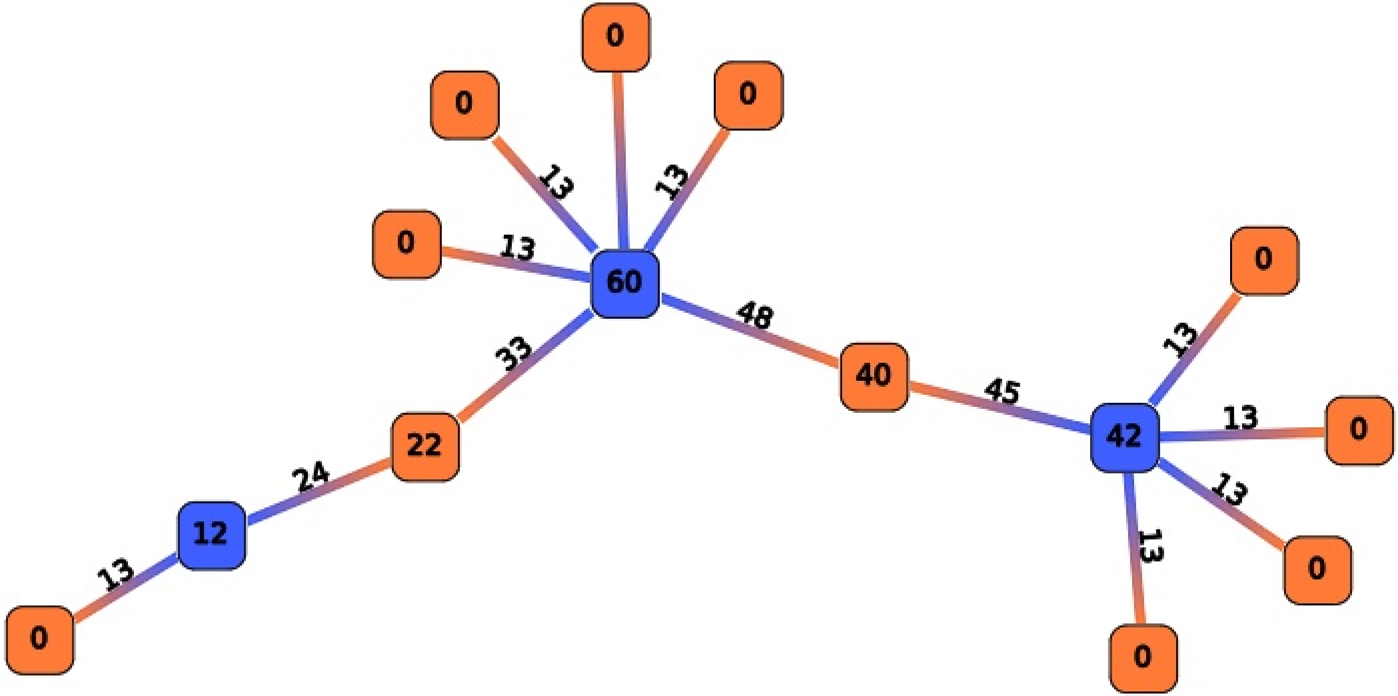
$$\text{closeness}(u) = \frac{1}{\sum_{v \in V, v \neq u} d(u,v)}$$

$$\text{hcloseness}(u) = \sum_{v \in V, v \neq u} \frac{1}{d(u,v)}$$

Betweenness

Node and Edge Betweenness

A node/edge has high betweenness if it participates in many shortest-paths



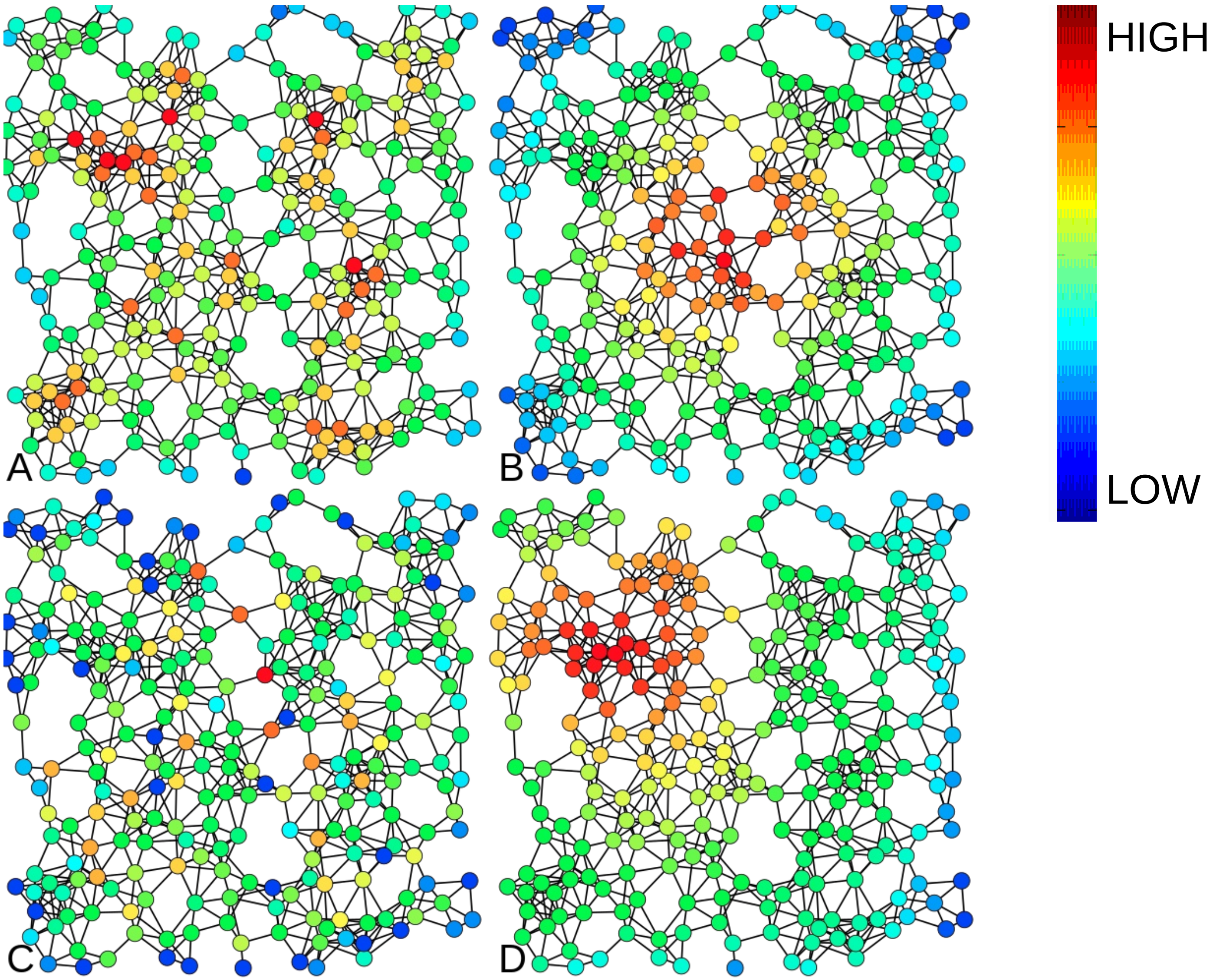
http://rnav.labri.fr/rNAV2.0_public_version/online_doc/_images/ComputeBetweennessCentrality.png

A: Degree

B: Closeness

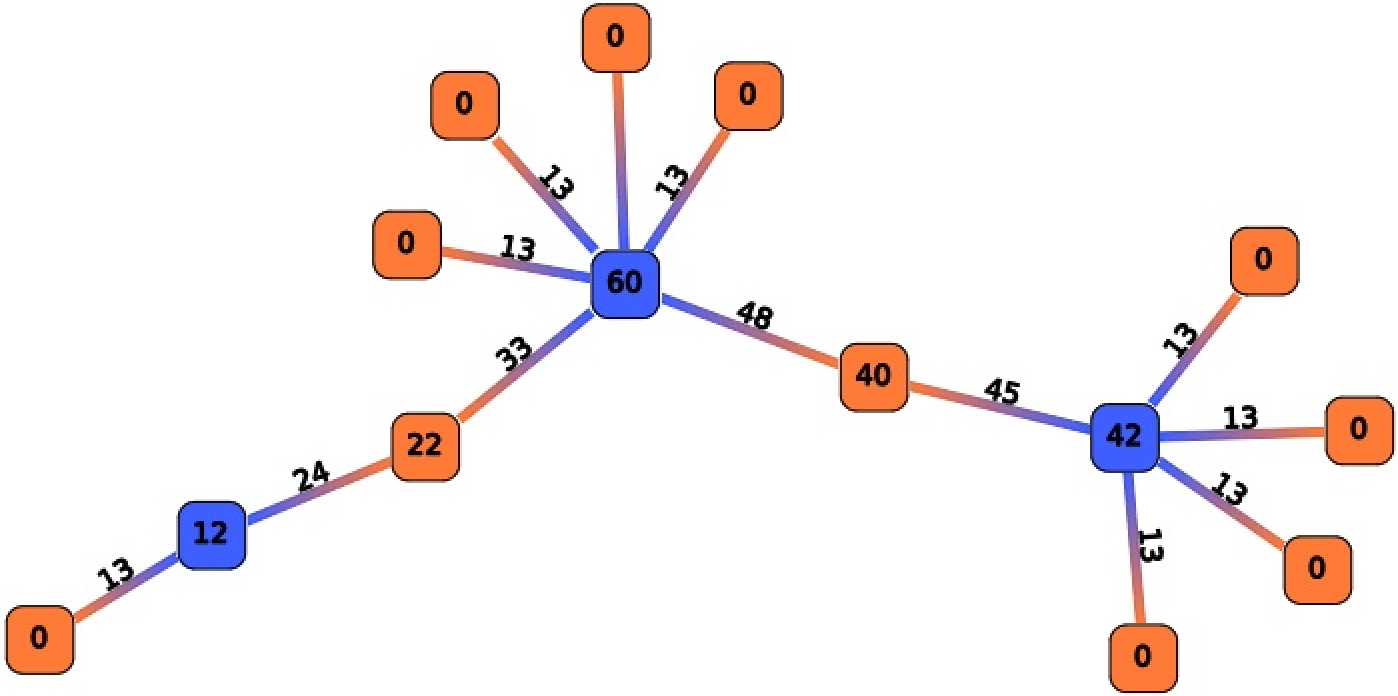
C: Betweenness

D: PageRank



Edge Betweenness

An **edge** has high betweenness if it is part of many shortest-paths … how to compute this efficiently?



http://rnav.labri.fr/rNAV2.0_public_version/online_doc/_images/ComputeBetweennessCentrality.png

Algorithm [Brandes, Newman]

- For every node u in V
  - Layer the graph performing a BFS from u
  - For every node v in V, $v \neq u$, sorted by layer
    - Assign to v a number $s(v)$ indicating how many shortest paths from u arrive to v
  - For every node v in V, $v \neq u$, sorted by reverse layer
    - Score to distribute = 1 + score from children
    - Add score to parent edges in proportion to $s(v)$
- In the end divide all edge scores by two

Ulrich Brandes. A faster algorithm for betweenness centrality. Journal of Mathematical Sociology, 25:163–177, 2001.
Mark E. J. Newman. Scientific collaboration networks: II. Shortest paths, weighted networks, and centrality. Physical Review E, 64:016132, 2001.

Example

For every node u in V

- Layer the graph performing a BFS from u
- For every node v in V, v≠u, sorted by layer
  - Assign to v a number s(v) indicating how many shortest paths from u arrive to v
- For every node v in V, v≠u, sorted by reverse layer
  - Score to distribute = 1 + score from children
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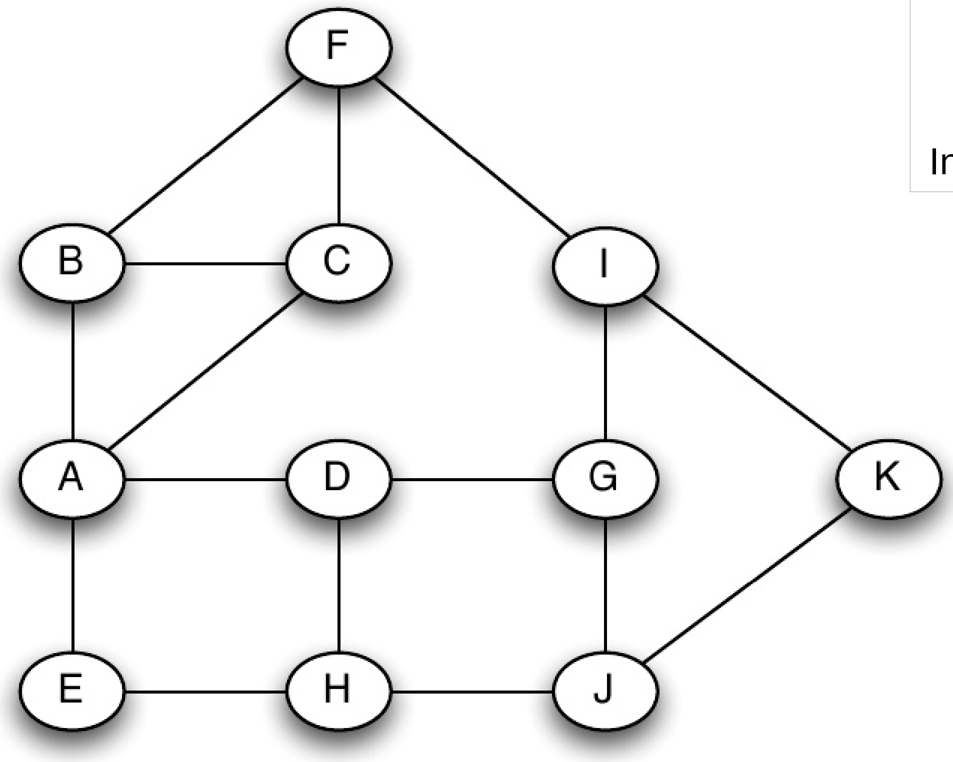

Example

For every node u in V
- Layer the graph performing a BFS from u
- **For every node v in V, v≠u, sorted by layer**
  - **Assign to v a number s(v) indicating how many shortest paths from u arrive to v**
- For every node v in V, v≠u, sorted by reverse layer
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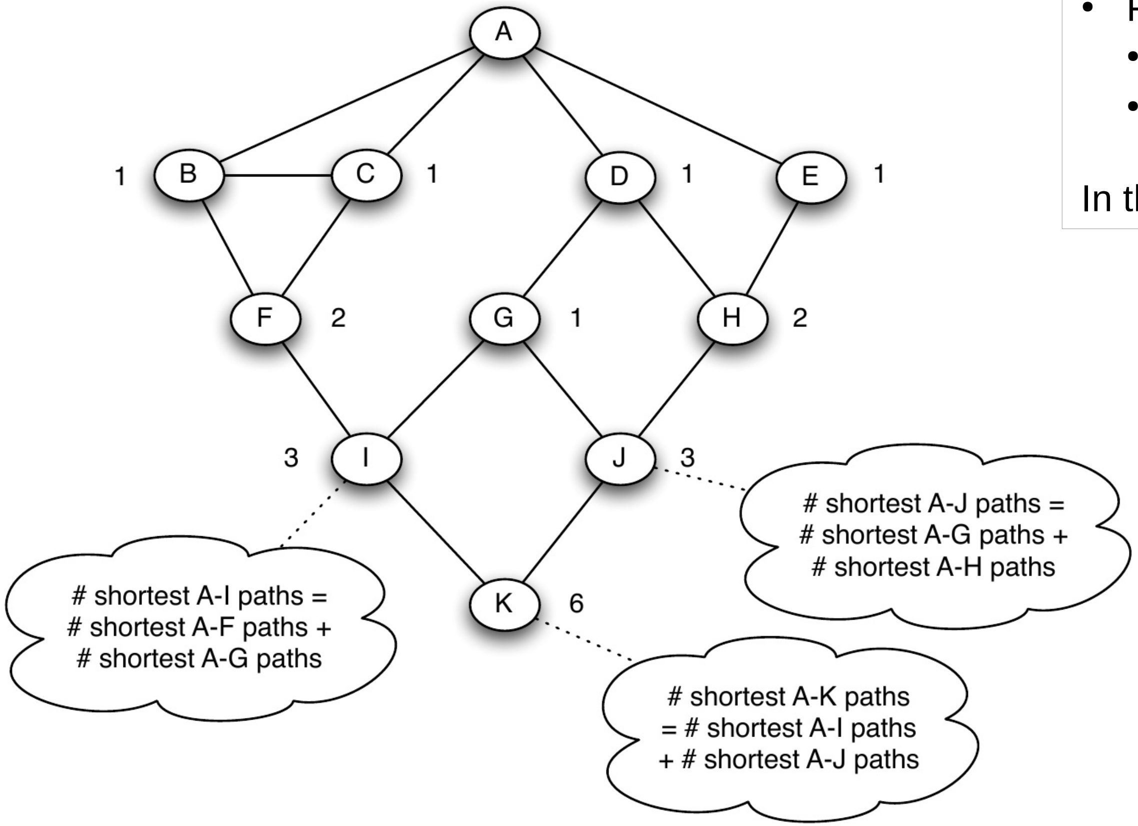


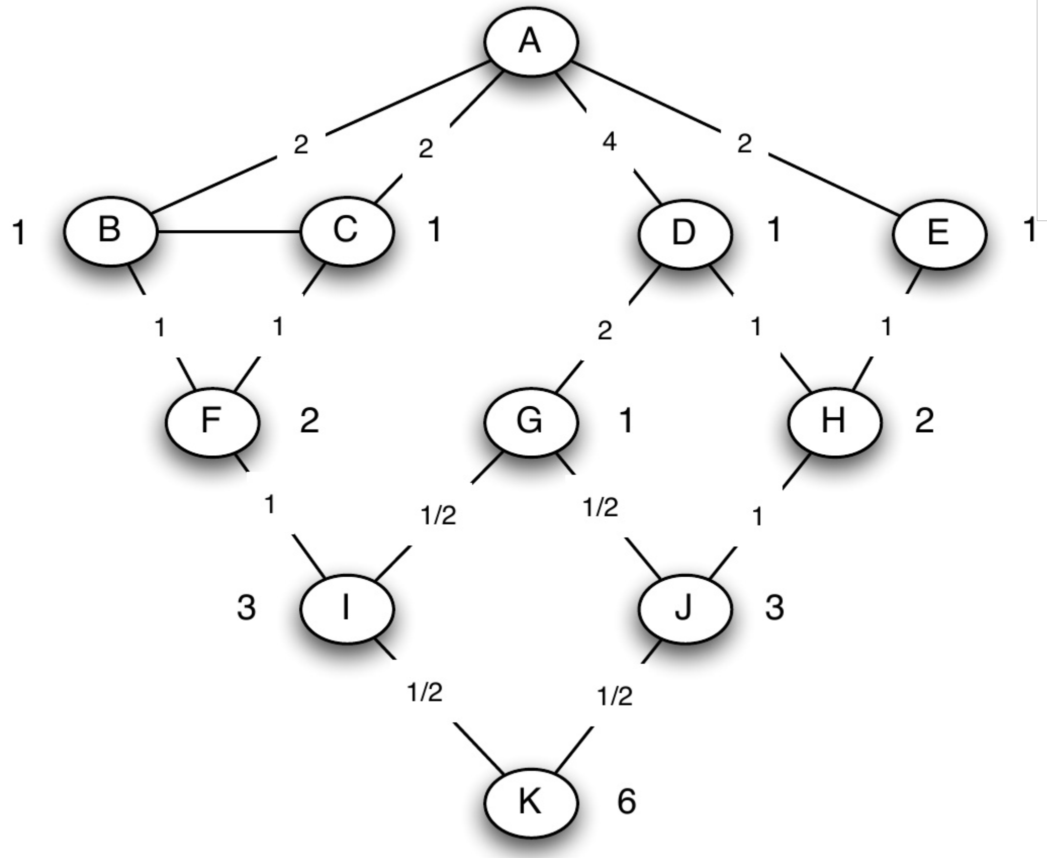
All nodes in layer 1 get s(v)=1

Remaining nodes: simply add s(.) of their parents

Example

For every node u in V
- Layer the graph performing a BFS from u
- For every node v in V, v≠u, sorted by layer
  - Assign to v a number s(v) indicating how many shortest paths from u arrive to v
- **For every node v in V, v≠u, sorted by rev. layer**
  - **Score to distribute = 1 + score from children**
  - **Add score to distribute to parent edges in proportion to s(v)**

In the end divide all edge scores by two



Nodes without children distribute a score of 1

Other nodes distribute 1 + whatever they receive from their children

Result

For every node u in V
- Layer the graph performing a BFS from u
- For every node v in V, v≠u, sorted by layer
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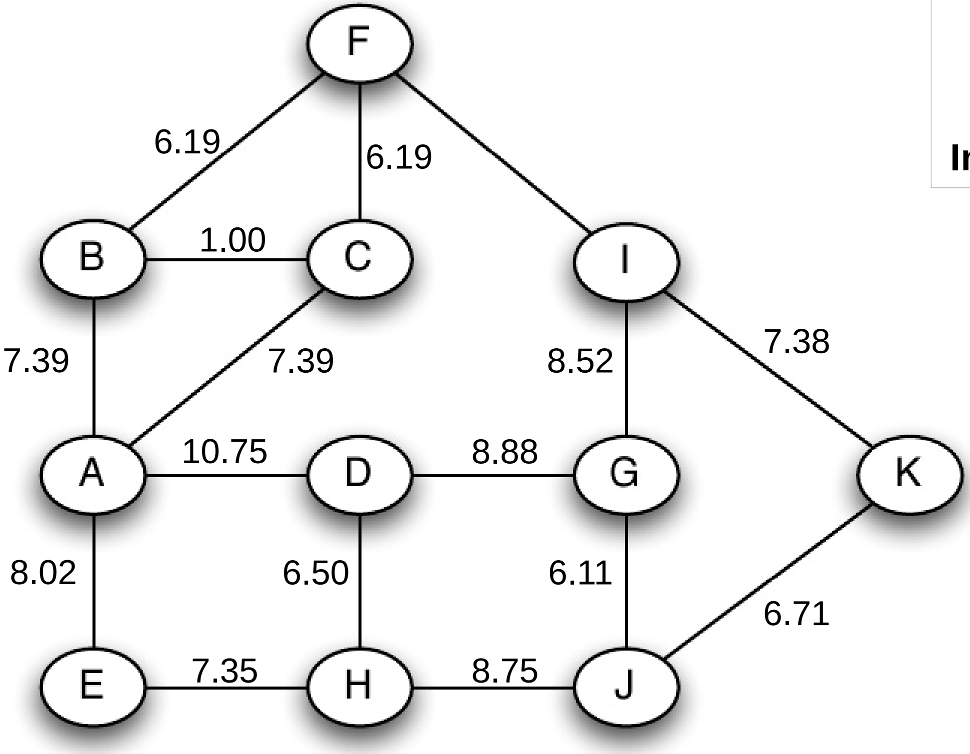
In the end divide all edge scores by two

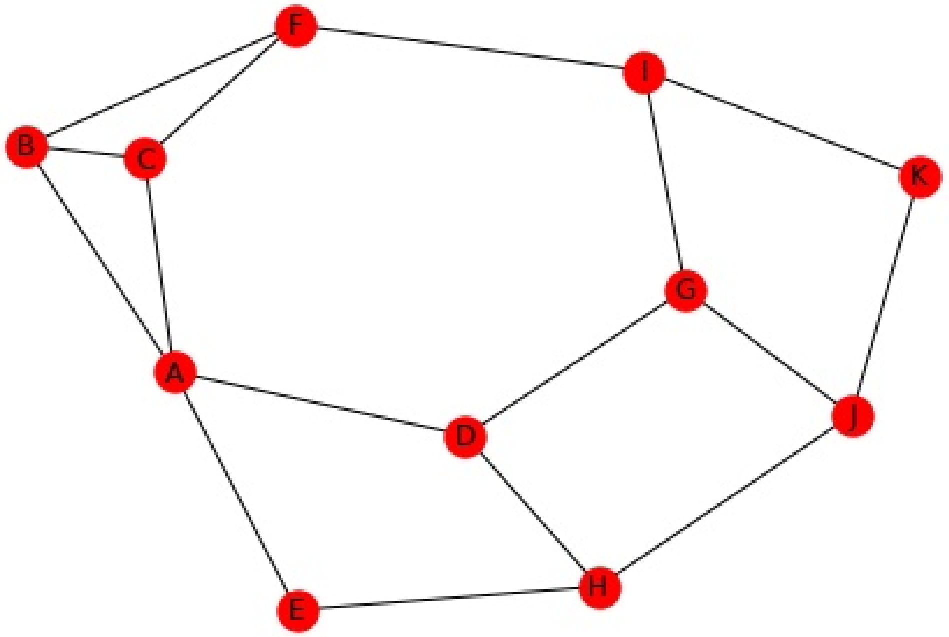


Computed using NetworkX (edge betweenness)

NetworkX code

```
import networkx as nx
g = nx.Graph()
g.add_edge("A", "B")
g.add_edge("A", "C")
g.add_edge("A", "D")
g.add_edge("A", "E")
g.add_edge("B", "C")
g.add_edge("B", "F")
g.add_edge("C", "F")
g.add_edge("D", "G")
g.add_edge("D", "H")
g.add_edge("E", "H")
g.add_edge("F", "I")
g.add_edge("G", "I")
g.add_edge("G", "J")
g.add_edge("H", "J")
g.add_edge("I", "K")
g.add_edge("J", "K")
nx.edge_betweenness(g, normalized=False)
```

```
nx.draw_spring(g, with_labels=True)
```



Try it!

Try to compute it by inspection first

Then use the algorithm;
you should get the same results

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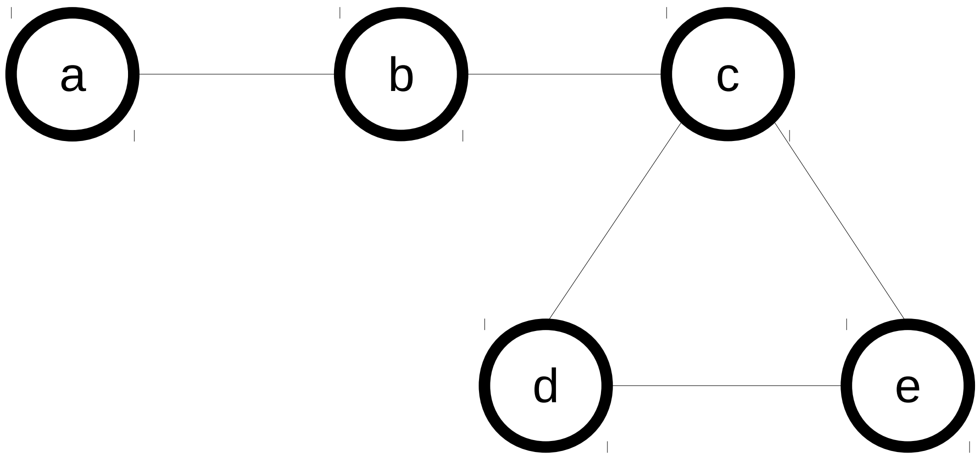
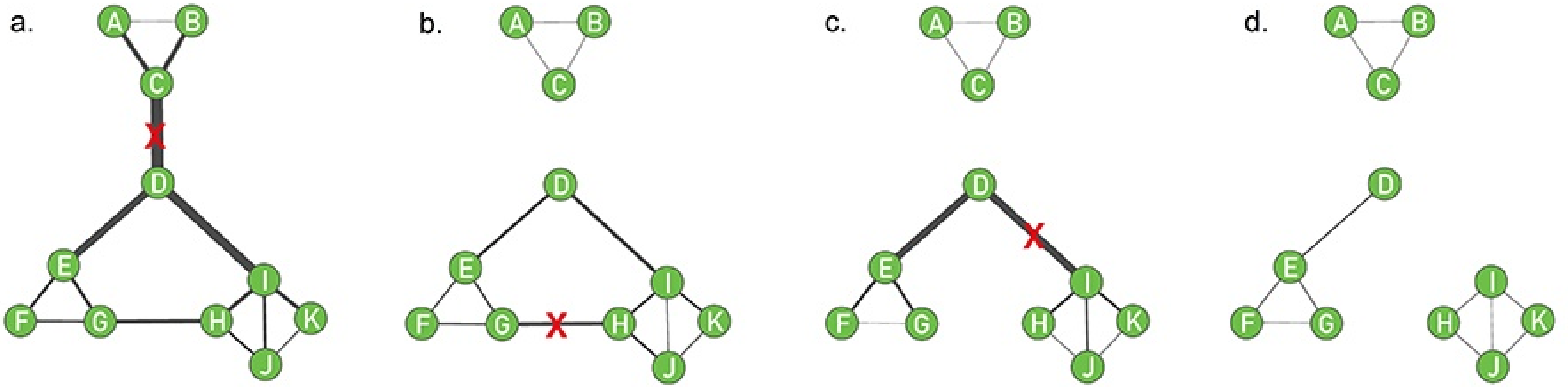


Application: the Girvan-Newman algorithm

- Repeat:
  - Compute edge betweenness
  - Remove edge with larger betweenness



Summary

Things to remember

- Closeness and harmonic closeness
- Node and edge betweenness
- Practice running the Brandes-Newman algorithm on small graphs
- Write code to execute the Brandes-Newman algorithm