

Closeness and betweenness

Introduction to Network Science

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Topic 09

Sources

- [Networks, Crowds, and Markets](#) Ch 3.6B
- Barabási 2016 Section 9.3.2
- P. Boldi and S. Vigna: [Axioms for Centrality](#) in Internet Mathematics 2014.
- Esposito and Pesce: [Survey of Centrality](#) 2015.
- C. Castillo: [Other centrality slides](#) 2016

Types of centrality measure

- Spectral
 - HITS
 - PageRank
- **Non-spectral**
 - Degree
 - Closeness and harmonic closeness
 - Betweenness

Is u a well-connected person?

- Degree: u has many connections
- Eigenvector: u is connected to the well-connected
- **Closeness:** u is close to many people
 - Average distance from u is small
- **Betweenness:** many connections pass through u
 - Large number of shortest paths pass through u

Closeness

Closeness

- Distance between two nodes is $d(u, v)$
- **Closeness** is the reciprocal of distances

$$\text{closeness}(u) = \frac{1}{\sum_{v \in V, v \neq u} d(u, v)}$$

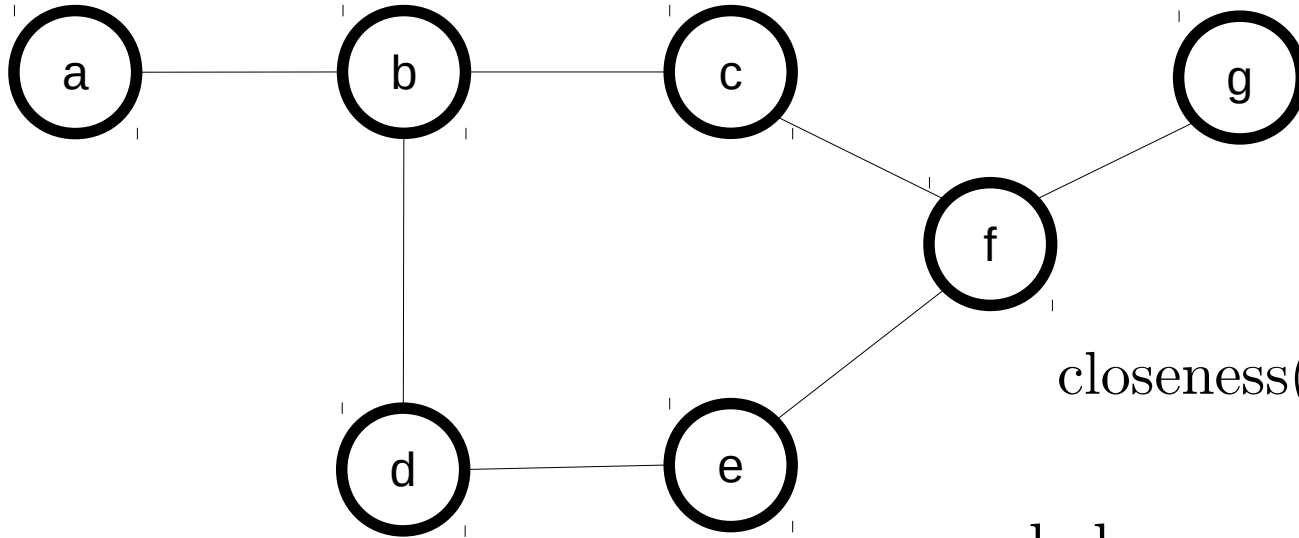
- Some graphs are not connected, in that case $d(u, v)$ can be ∞ ; assuming $1/\infty = 0$ one can define the **harmonic closeness**:

$$\text{hcloseness}(u) = \sum_{v \neq u} \frac{1}{d(u, v)}$$

Try it!

Compute closeness and harmonic closeness for all the nodes

$d(u,v) = 1$ if v is a neighbor of u



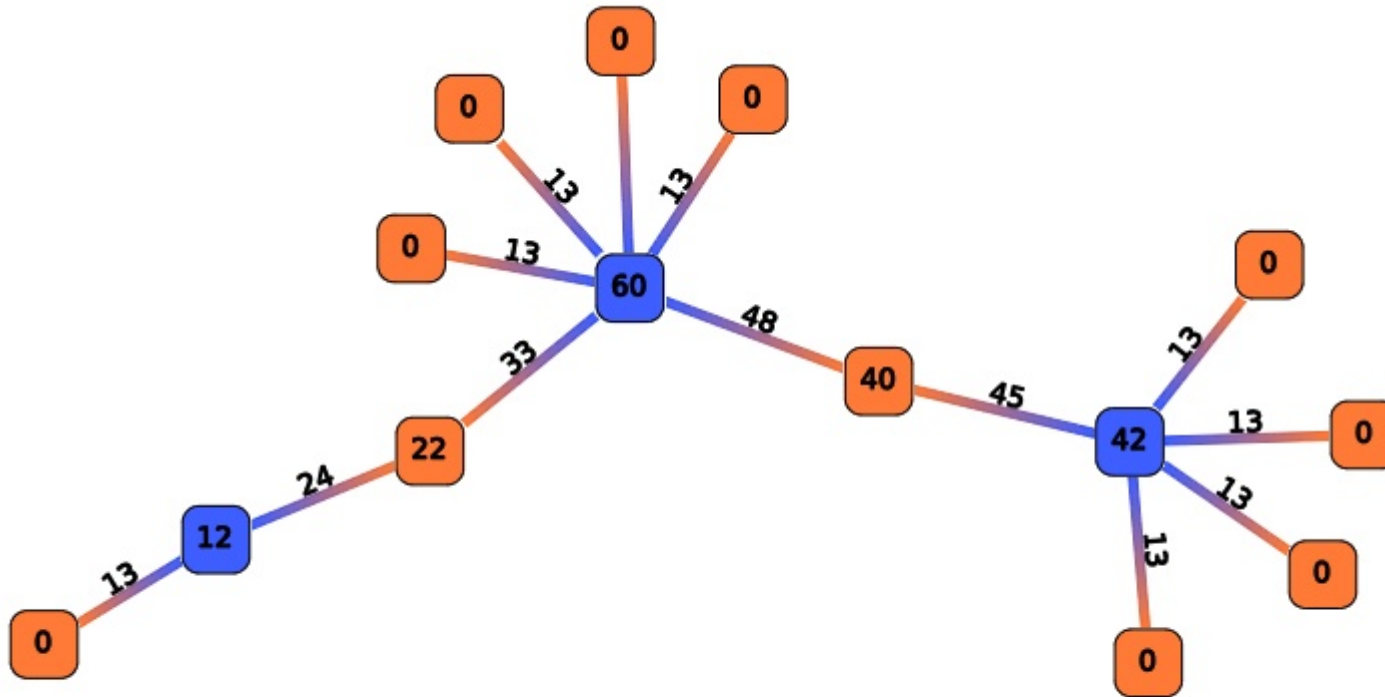
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$$\text{hcloseness}(u) = \sum_{v \in V, v \neq u} \frac{1}{d(u, v)}$$

Betweenness

Node and Edge Betweenness

A node/edge has high betweenness if it participates in many shortest-paths

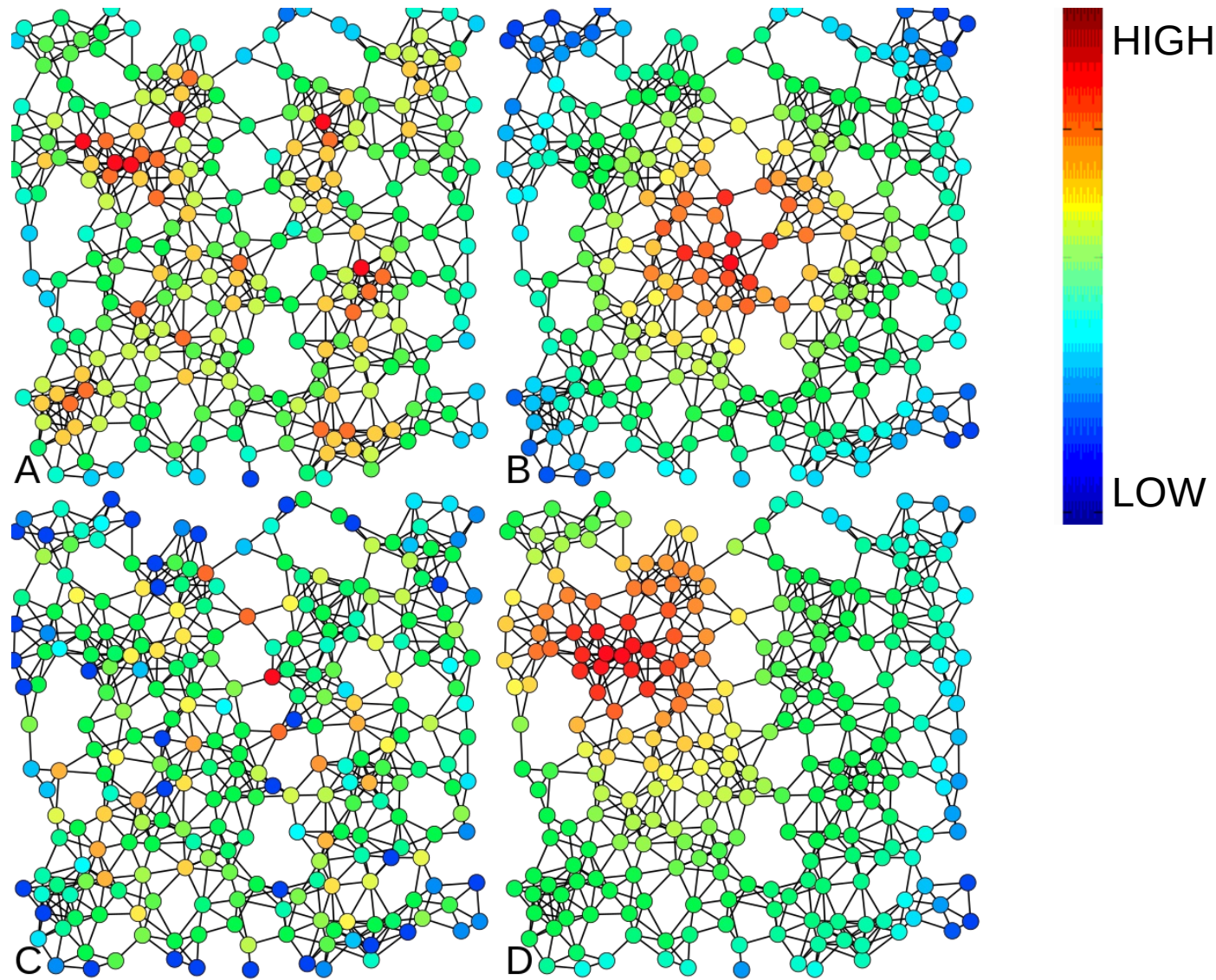


A: Degree

B: Closeness

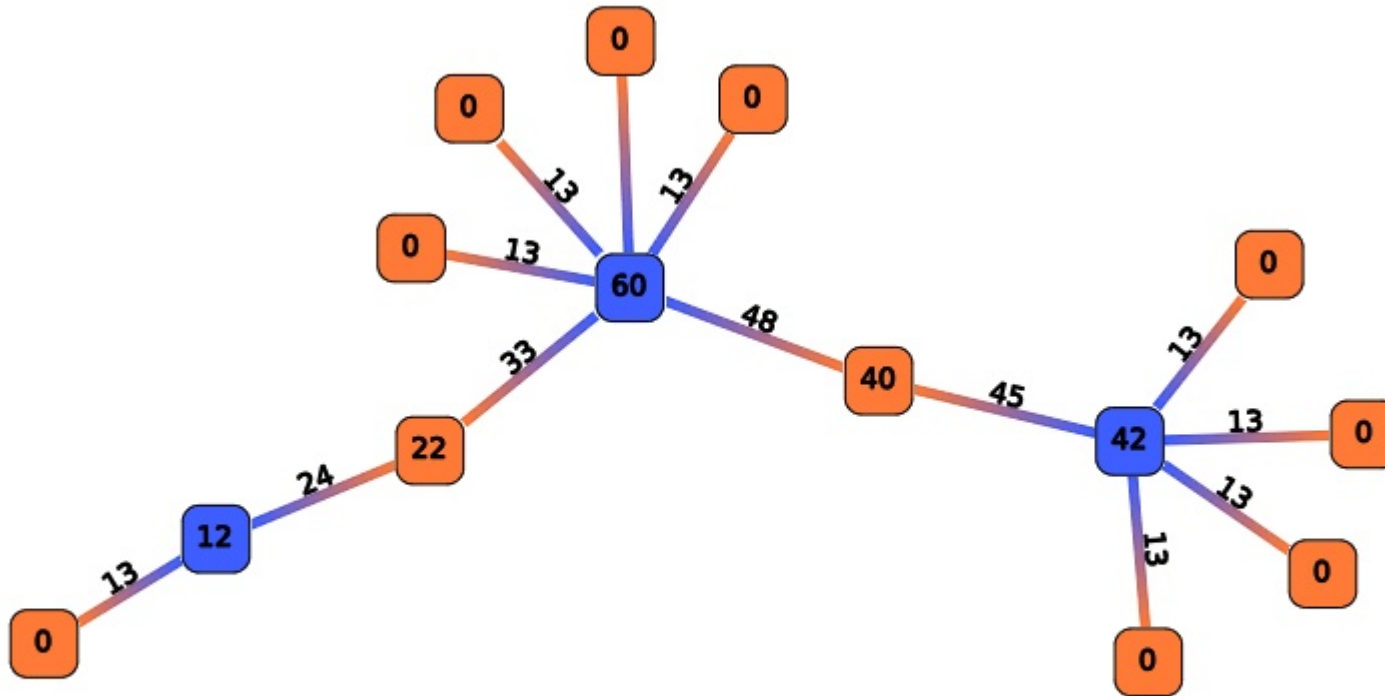
C: Betweenness

D: PageRank



Edge Betweenness

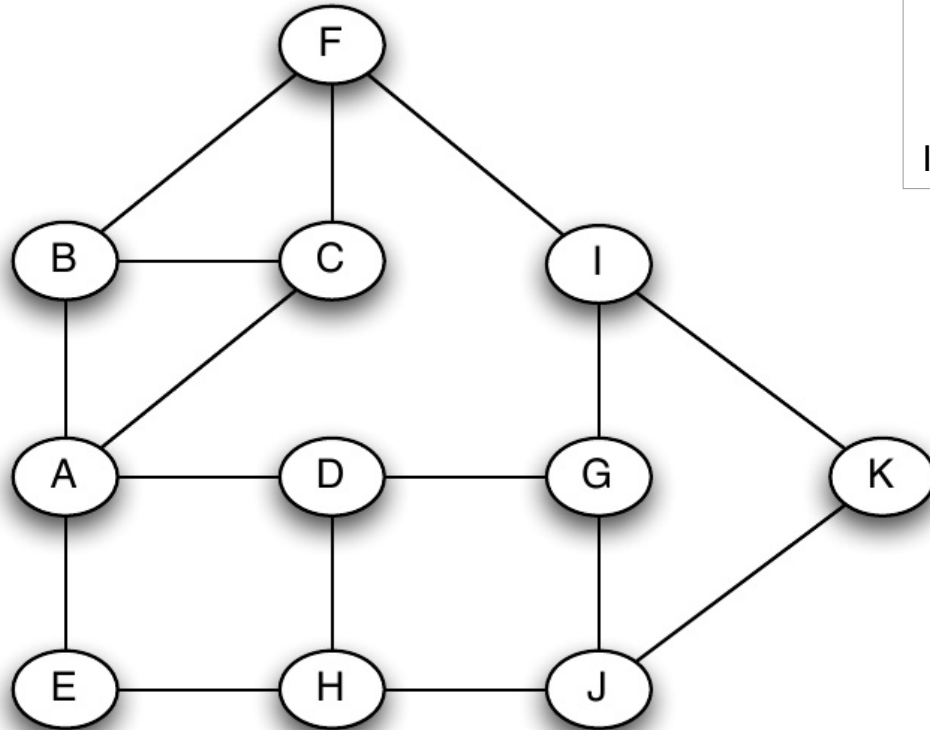
An **edge** has high betweenness if it is part of many shortest-paths ... how to compute this efficiently?



Algorithm [Brandes, Newman]

- For every node u in V
 - Layer the graph performing a BFS from u
 - For every node v in V , $v \neq u$, sorted by layer
 - Assign to v a number $s(v)$ indicating how many shortest paths from u arrive to v
 - For every node v in V , $v \neq u$, sorted by reverse layer
 - Score to distribute = 1 + score from children
 - Add score to parent edges in proportion to $s(v)$
- In the end divide all edge scores by two

Example

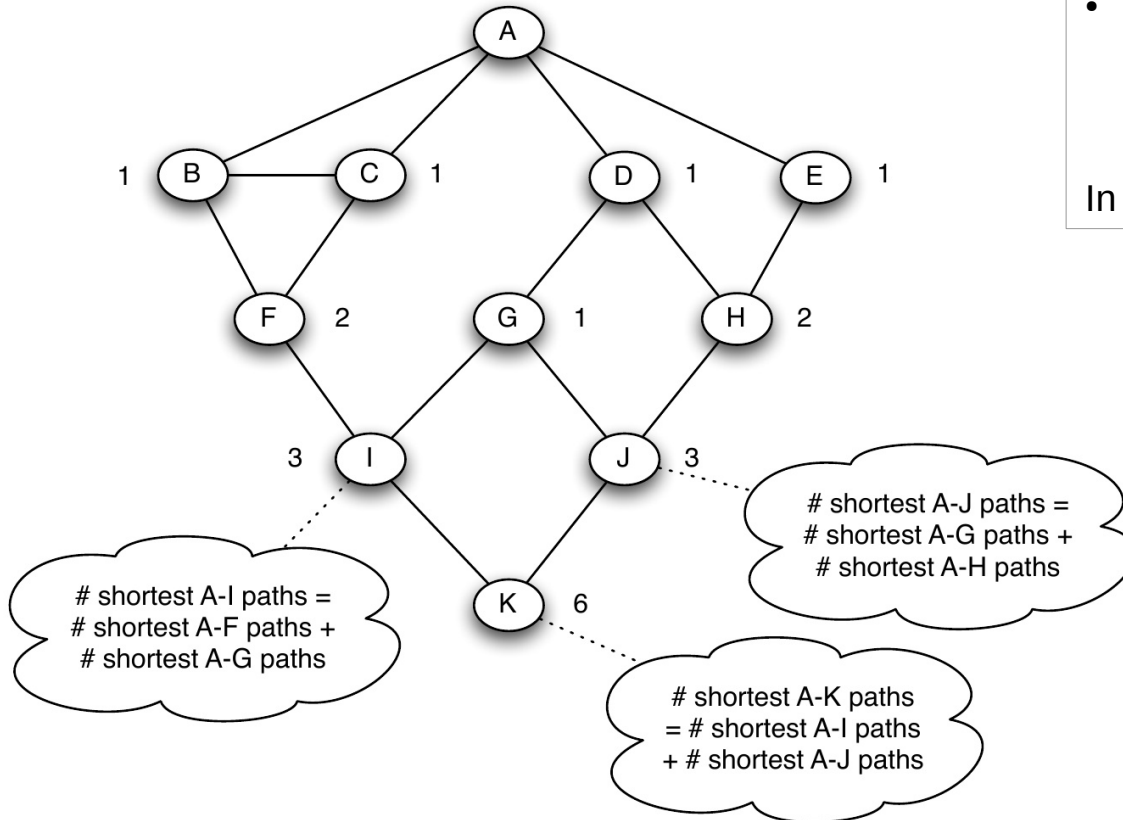


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Example



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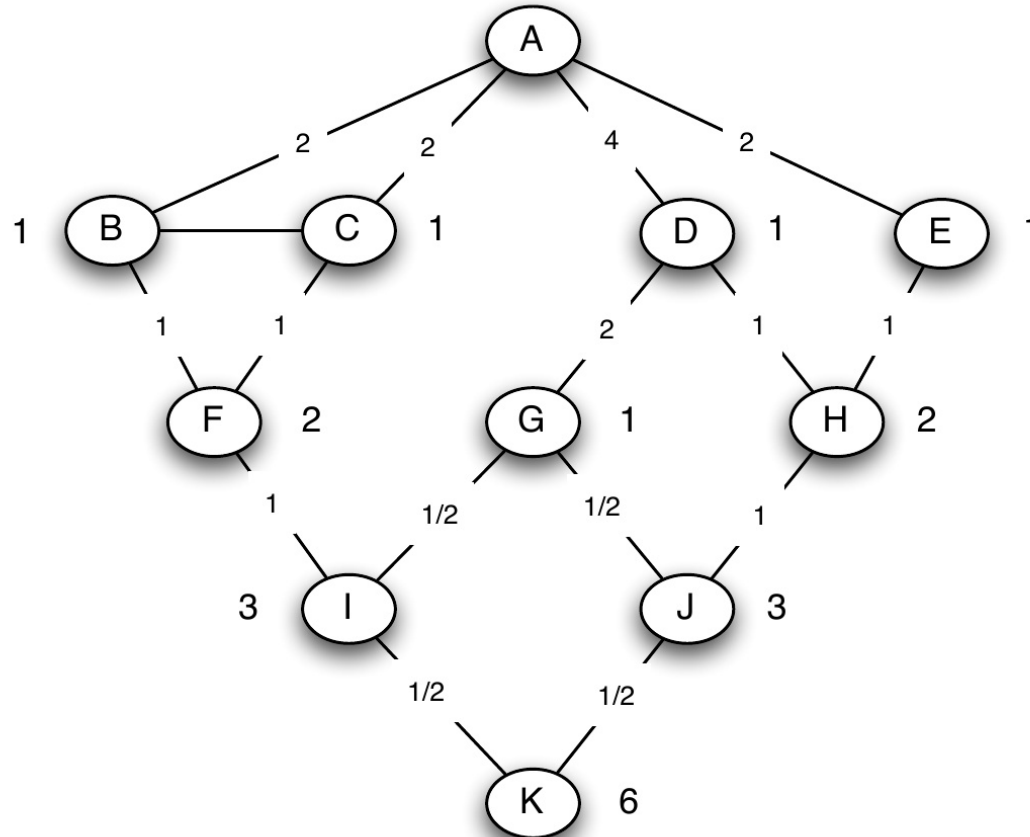
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In the end divide all edge scores by two

All nodes in layer 1 get $s(v)=1$

Remaining nodes: simply add $s(\cdot)$ of their parents

Example



For every node u in V

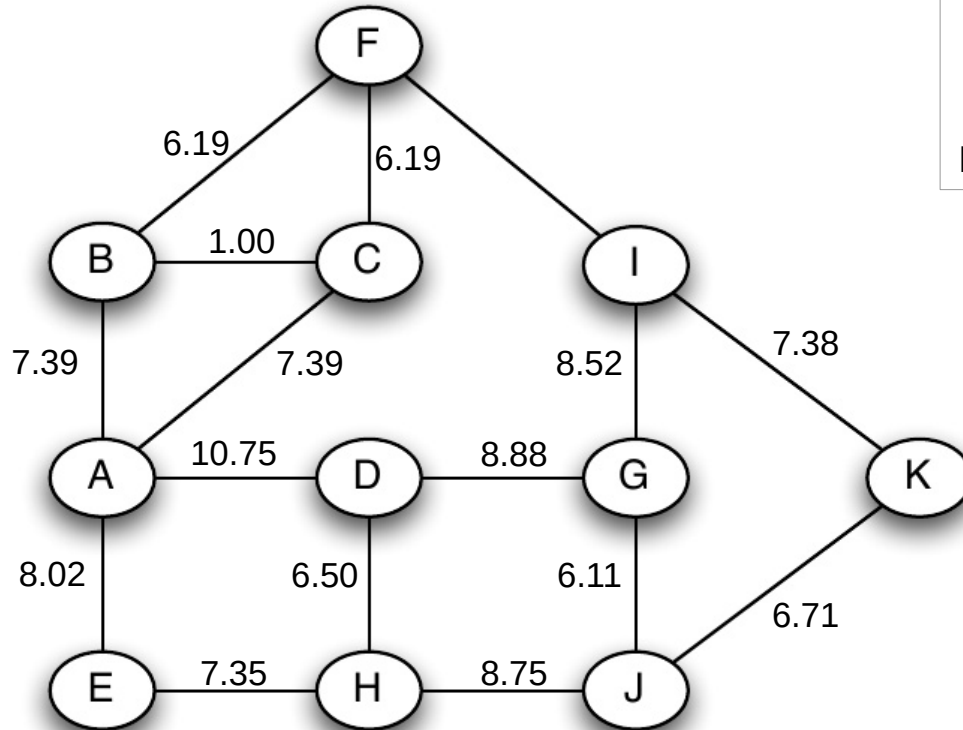
- Layer the graph performing a BFS from u
- For every node v in V , $v \neq u$, sorted by layer
 - Assign to v a number $s(v)$ indicating how many shortest paths from u arrive to v
- **For every node v in V , $v \neq u$, sorted by rev. layer**
 - **Score to distribute = 1 + score from children**
 - **Add score to distribute to parent edges in proportion to $s(v)$**

In the end divide all edge scores by two

Nodes without children distribute a score of 1

Other nodes distribute 1 + whatever they receive from their children

Result



For every node u in V

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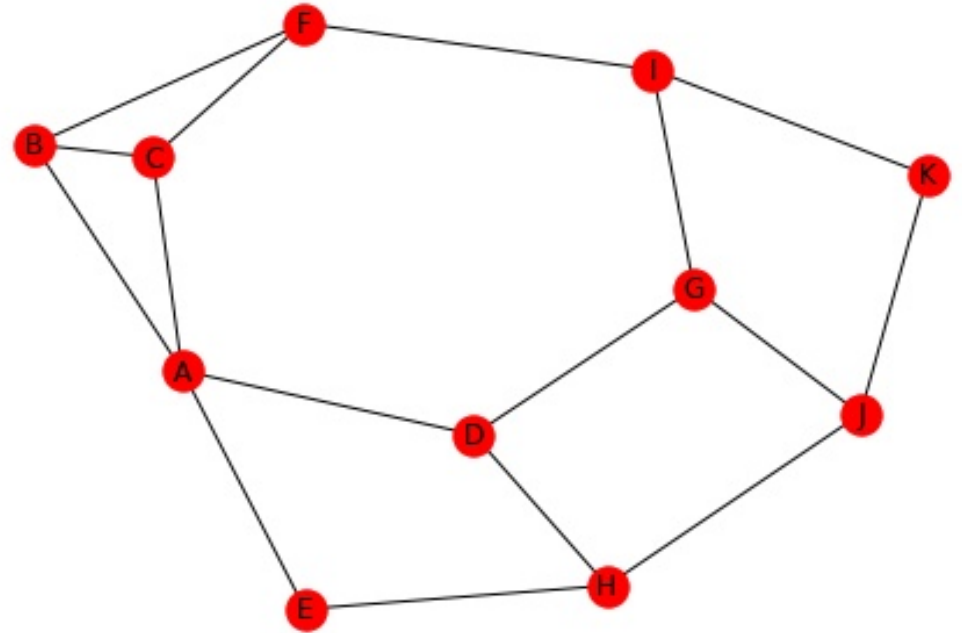
In the end divide all edge scores by two

Computed using NetworkX
(edge betweenness)

NetworkX code

```
import networkx as nx
g = nx.Graph()
g.add_edge("A", "B")
g.add_edge("A", "C")
g.add_edge("A", "D")
g.add_edge("A", "E")
g.add_edge("B", "C")
g.add_edge("B", "F")
g.add_edge("C", "F")
g.add_edge("D", "G")
g.add_edge("D", "H")
g.add_edge("E", "H")
g.add_edge("F", "I")
g.add_edge("G", "I")
g.add_edge("G", "J")
g.add_edge("H", "J")
g.add_edge("I", "K")
g.add_edge("J", "K")
nx.edge_betweenness(g, normalized=False)
```

`nx.draw_spring(g, with_labels=True)`



Try it!

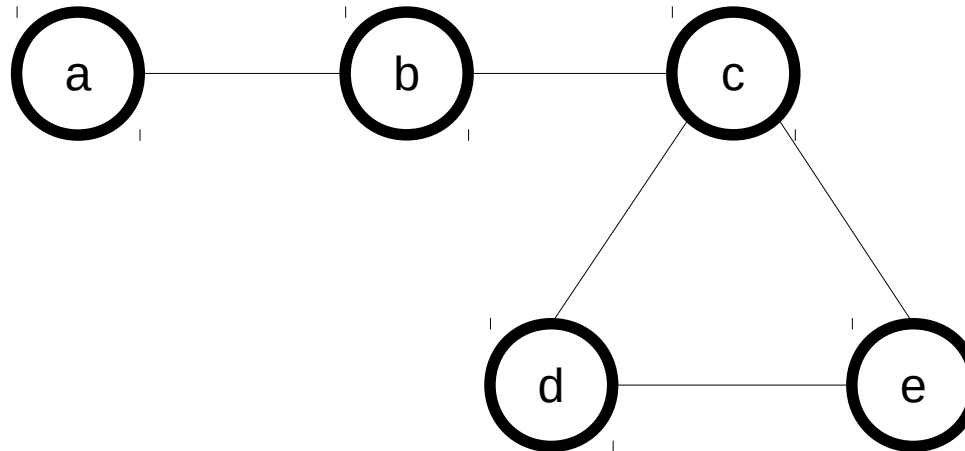
Try to compute it by inspection first

*Then use the algorithm;
you should get the same results*

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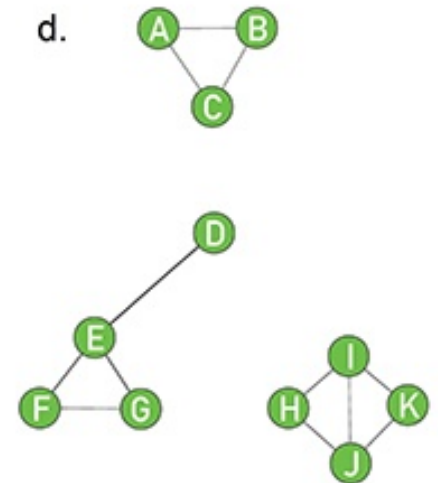
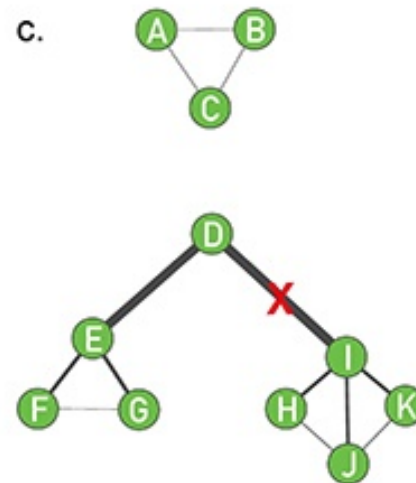
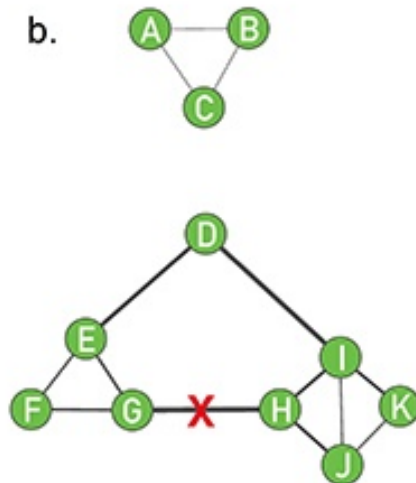
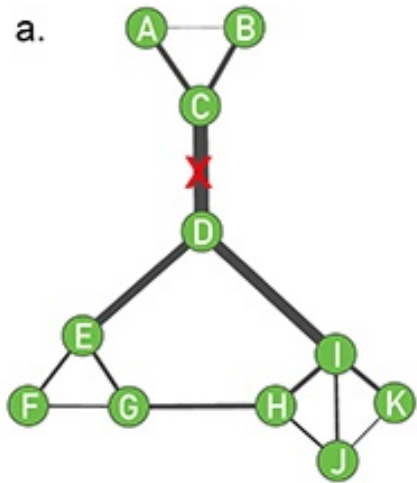


Fractional values?

- In a graph with cycles, you may get fractional values of the edge betweenness for an edge

Application: the Girvan-Newman algorithm

- Repeat:
 - Compute edge betweenness
 - Remove edge with larger betweenness



Summary

Things to remember

- Closeness and harmonic closeness
- Node and edge betweenness
- Practice running the Brandes-Newman algorithm on small graphs
- Write code to execute the Brandes-Newman algorithm