Network flows

Introduction to Network Science
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Topic 12

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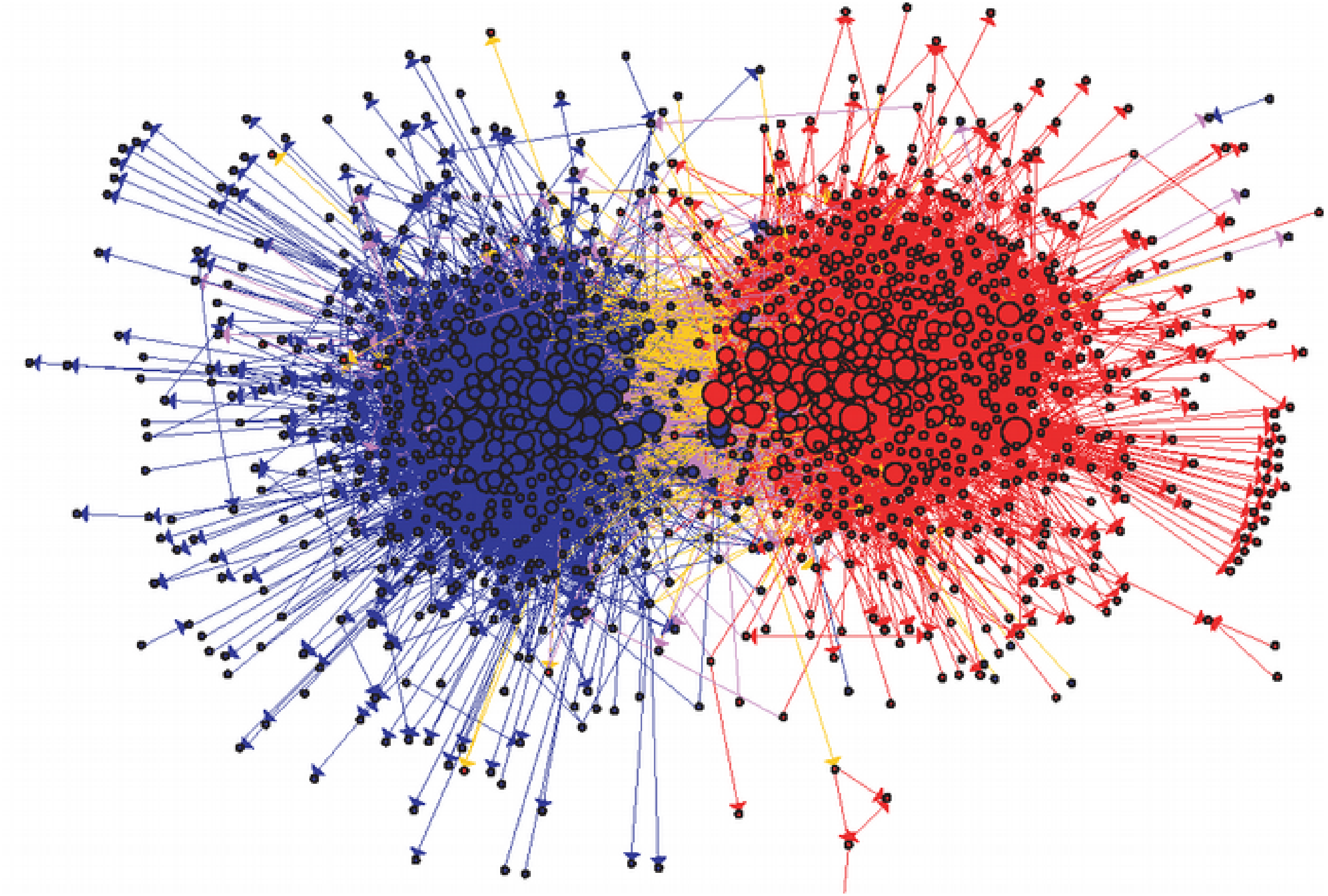
Sources

- Barabási 2016 Chapter 9
- Networks, Crowds, and Markets Ch 3
- C. Castillo: Graph partitioning 2017

US Political Blogs (2004)

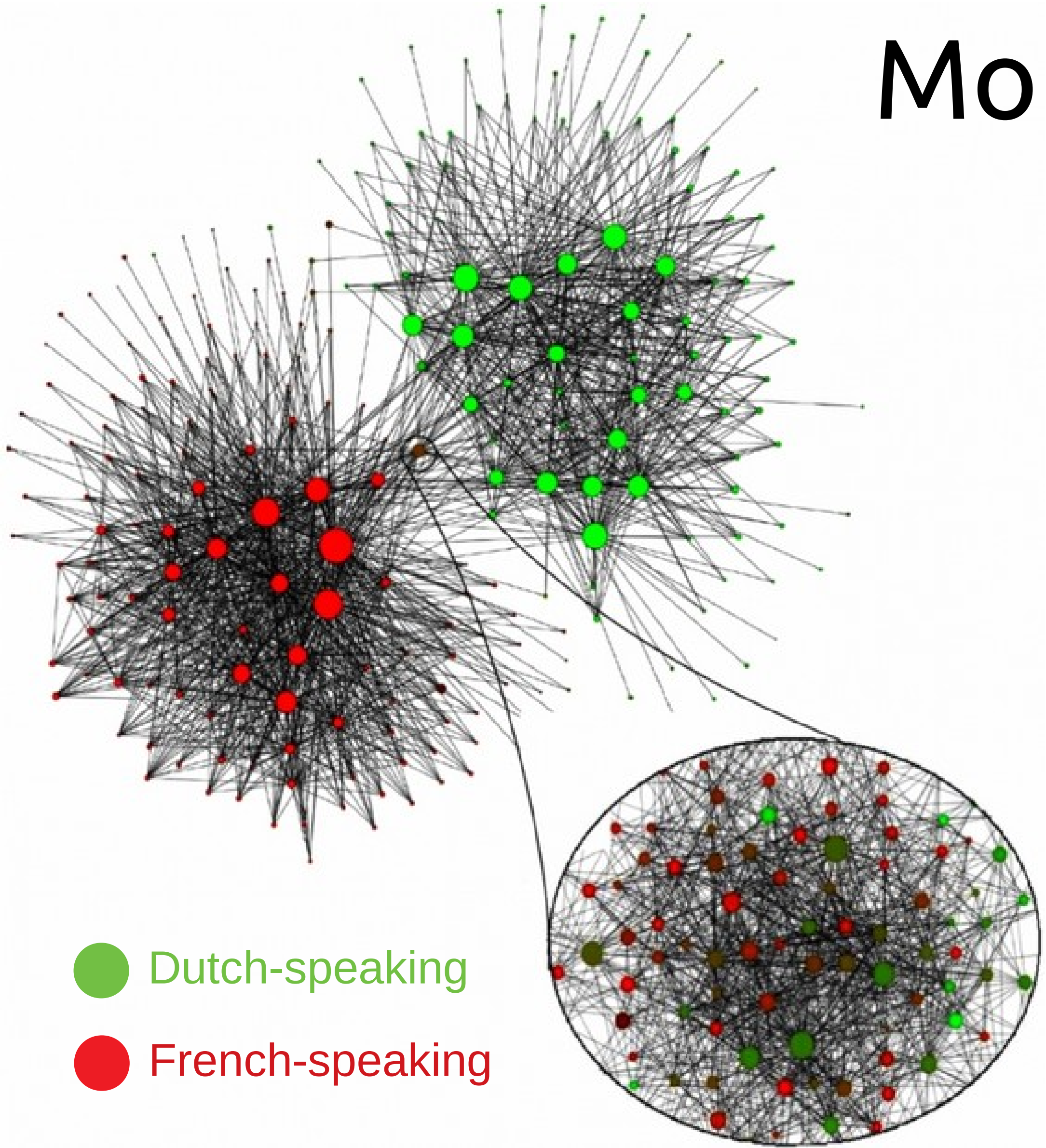
Liberal



Conservative

Adamic, L. A., & Glance, N. (2005, August). The political blogosphere and the 2004 US election: divided they blog. In Proc. LinkKDD (pp. 36-43). ACM.

Mobile phone users in Belgium (2008)

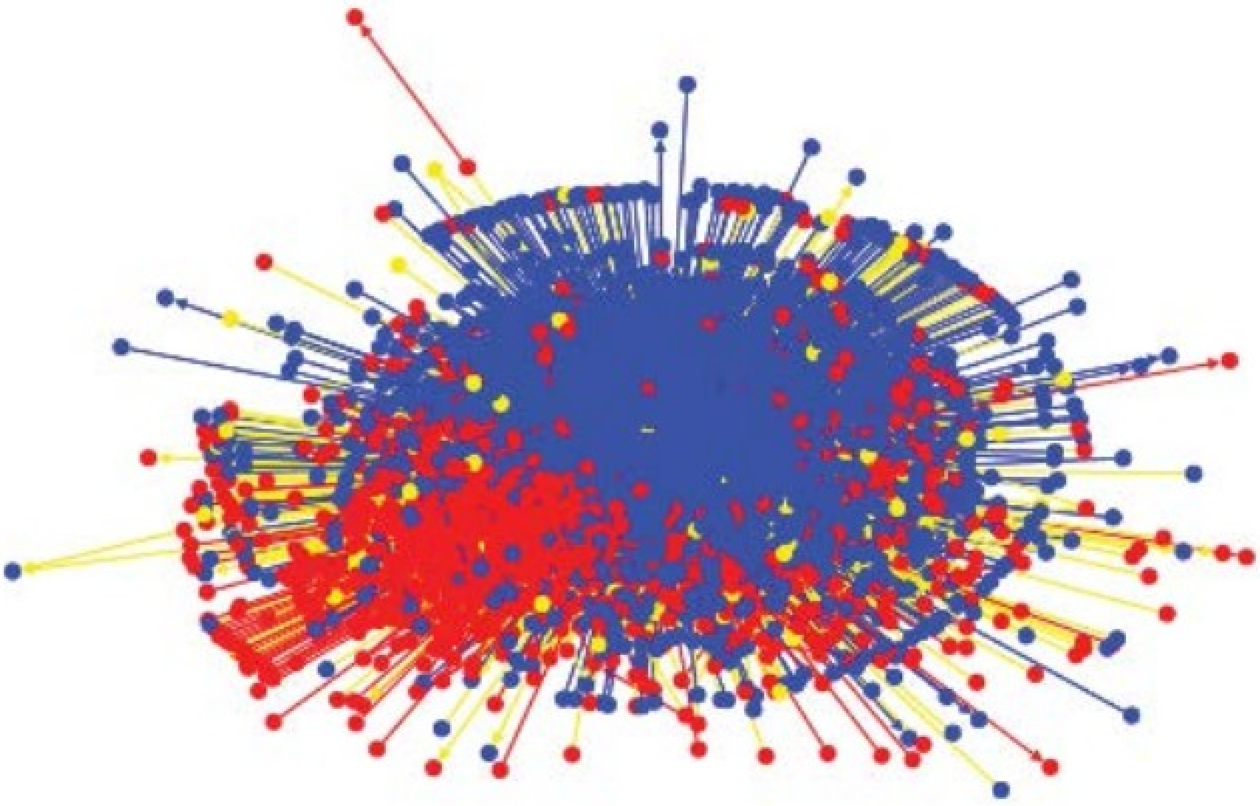


Each node is a community of 100 mobile users or more that tend to call each other

V. D. Blondel, J.-L. Guillaume, R. Lambiotte, and E. Lefebvre. Fast unfolding of communities in large networks. J. Stat. Mech., 2008.

Egyptian Twitter Users (2013)

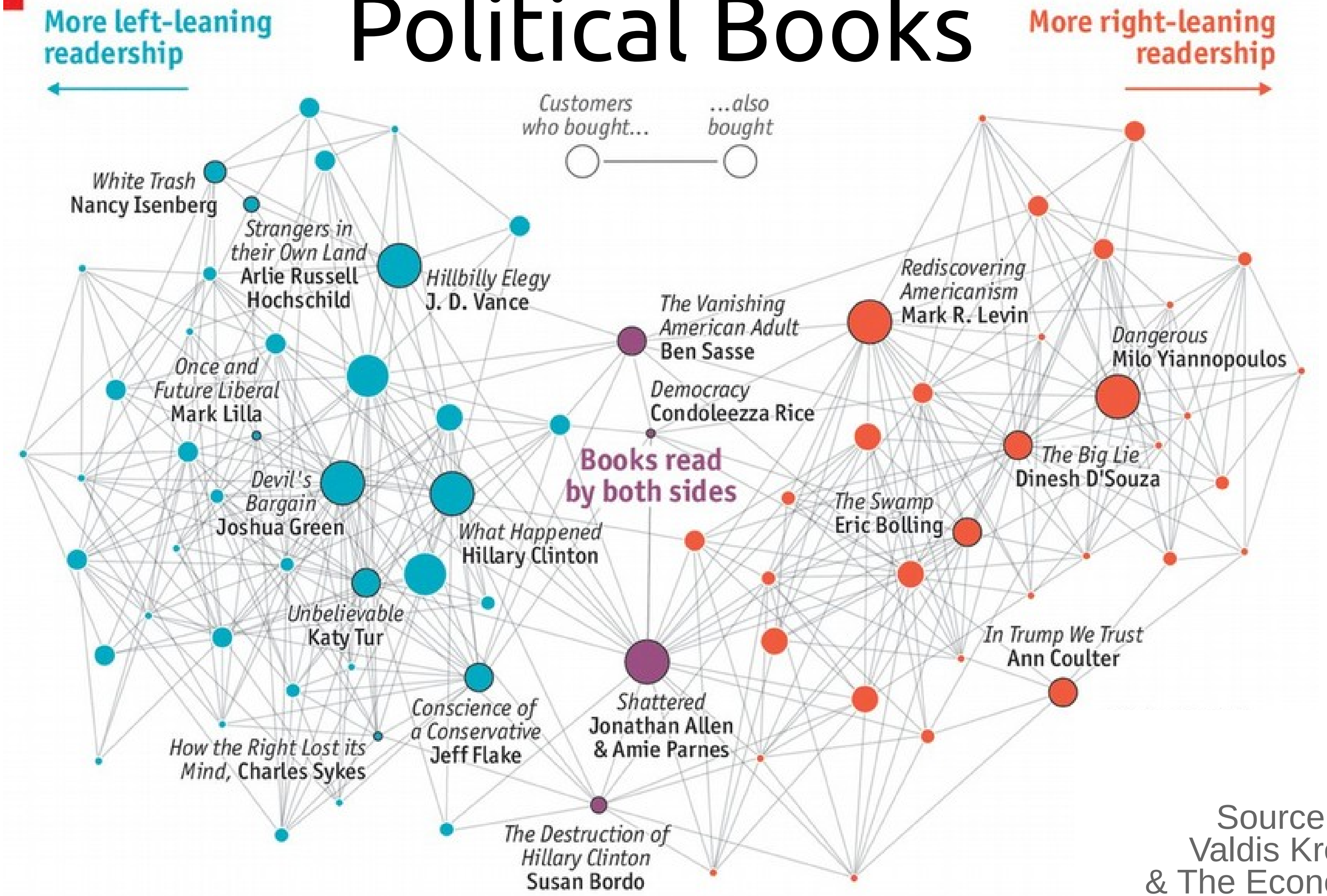
Islamist



Secularist

Weber, I., Garimella, V. R. K., & Batayneh, A. (2013, August). Secular vs. Islamist polarization in egypt on twitter. In Proc. ASONAM (pp. 290-297). ACM.

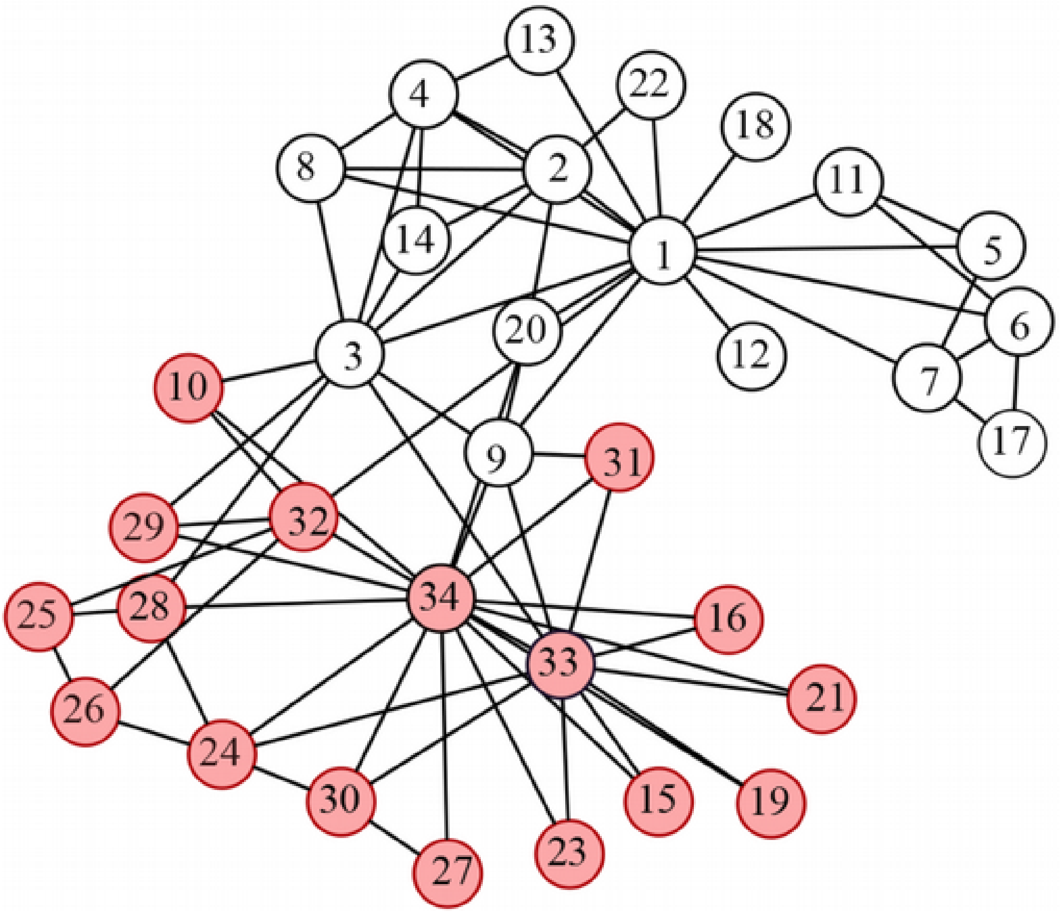
Political Books



Source:
Valdis Kreb
& The Economist

Wayne Zachary’s PhD Thesis (1972)

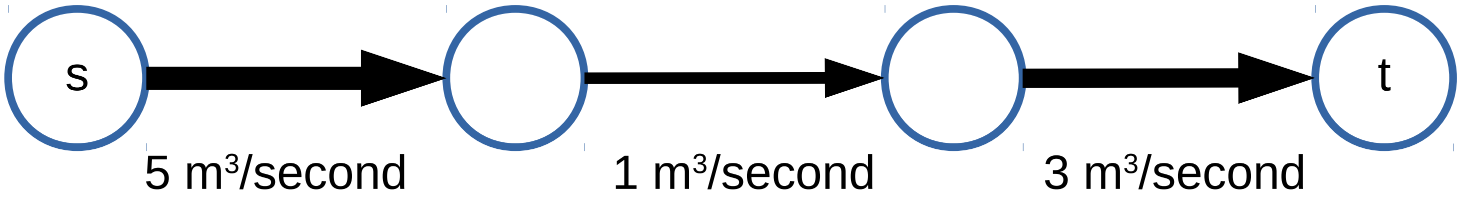
- Studied 34 members of a karate club
- Found 78 links between members who regularly interacted outside the club
- The club splitted in two during the study
- 1=sensei, 34=president



Splitting into two communities:
Max-flow and Min-cut

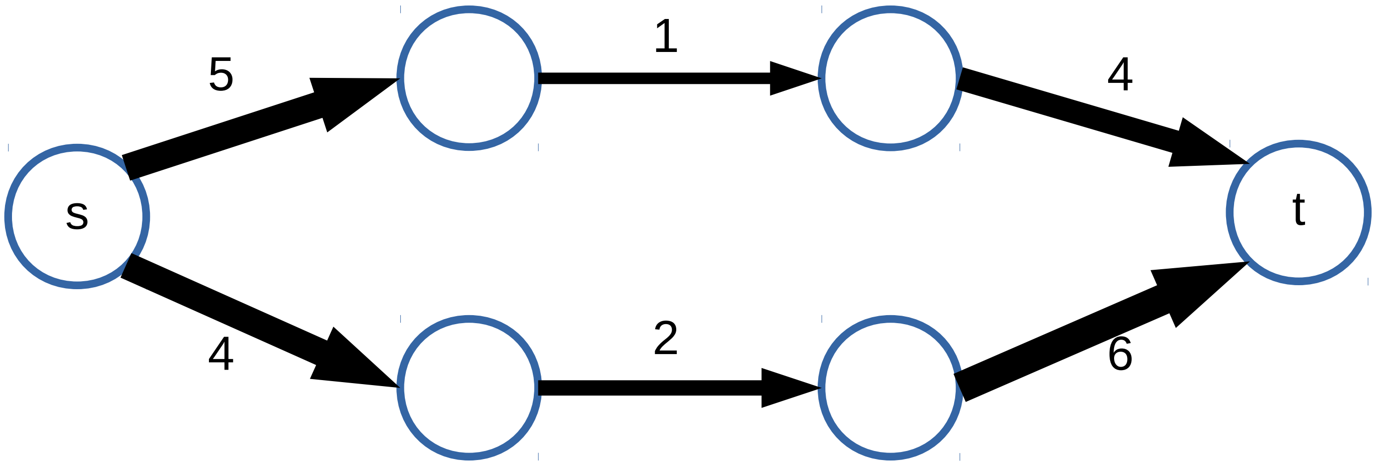
Maximum flow: example 1

- If edge weights were capacities, what is the maximum flow that can be sent from s to t?



Maximum flow: example 2

- If edge weights were capacities, what is the maximum flow that can be sent from s to t?



Maximum flow problem

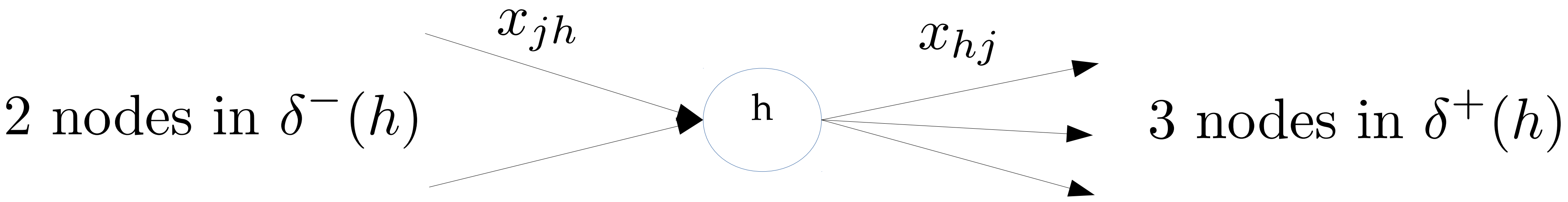
- What is the maximum “flow” that can be carried from s to t?
  - Think of edge weights as capacities (e.g. m^3/s of water)
- What is the flow of an edge?
  - The amount sent through that edge (an assignment)
- What is the net flow of a node?
  - The amount exiting the node minus the amount entering the node

Formulating the max flow problem

- The flow through each edge should be $\leq k_{ij}$
- Net flow node h = OUT(h) – IN(h)
- Node *s* should have positive flow *v*
- Node *t* should have negative flow *-v*
- *What should be the flow of the other nodes?*

Formulating the max flow problem

- Let v be a feasible flow
- Node *s* should have positive flow *v*
- Node *t* should have negative flow *-v*



- *What should be the flow of an arbitrary node h?*

$$\sum_{(h,j)\in\delta^+(h)} x_{hj} - \sum_{(i,h)\in\delta^-(h)} x_{ij} = \ ?$$

Max flow as a linear program

$$\max \quad v \tag{1}$$

$$\sum_{(s,j)\in\delta^+(s)} x_{sj} = v \tag{2}$$

$$-\sum_{(i,t)\in\delta^-(t)} x_{it} = -v \tag{3}$$

$$\sum_{(h,j)\in\delta^+(h)} x_{hj} - \sum_{(i,h)\in\delta^-(h)} x_{ih} = 0, \quad h \in N - \{s,t\} \tag{4}$$

$$x_{ij} \leq k_{ij} \quad (i,j) \in A \tag{5}$$

$$x_{ij} \geq 0 \quad (i,j) \in A \tag{6}$$

http://www.dii.unisi.it/~agnetis/mincutENG.pdf

Writing the dual: each constraint will become a variable

$$
\begin{aligned}
\max \quad & v && && (1)\\
\sum_{(s,j)\in\delta^+(s)} x_{sj} &= v && \text{variable } u_s && (2)\\
-\sum_{(i,t)\in\delta^-(t)} x_{it} &= -v && \text{variable } u_t && (3)\\
\sum_{(h,j)\in\delta^+(h)} x_{hj} - \sum_{(i,h)\in\delta^-(h)} x_{ih} &= 0, \quad h \in N - \{s,t\} && \text{variables } u_j && (4)\\
x_{ij} &\leq k_{ij} \quad (i,j) \in A && \text{variables } y_{ij} && (5)\\
x_{ij} &\geq 0 \quad (i,j) \in A && && (6)
\end{aligned}
$$

Writing the dual

- Remember: the infimum of the solutions of the dual is the supremum of the solutions of primal

$$\min \sum_{(i,j)\in A} k_{ij} y_{ij}$$

$$\begin{aligned} u_i - u_j + y_{ij} &\geq 0, (i,j) \in A \\ -u_s + u_t &= 1 \end{aligned}$$

$$y_{ij} \geq 0$$

- Variables u_i don't enter the objective, only their difference is in the constraints
- We can set them arbitrarily, in particular $u_s = 0$, $u_t = 1$

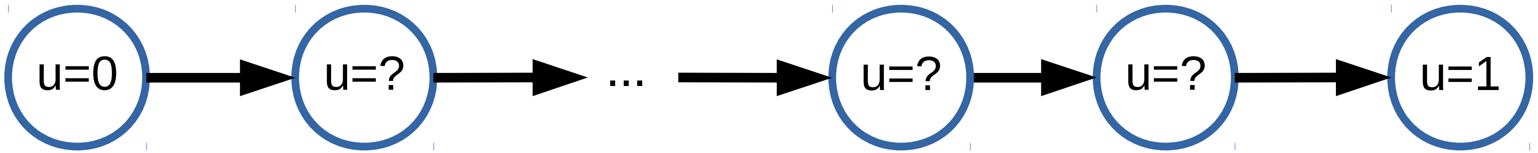
Dual (after simplification)

$$\min \sum_{(i,j)\in A} k_{ij} y_{ij}$$

$$\begin{aligned} u_i - u_j + y_{ij} &\geq 0, (i,j) \in A \\ y_{ij} &\geq 0 \end{aligned}$$

$$u_s = 0, u_t = 1$$

- What happens with the values of u in every simple path going from s to t ?



Dual (after simplification)

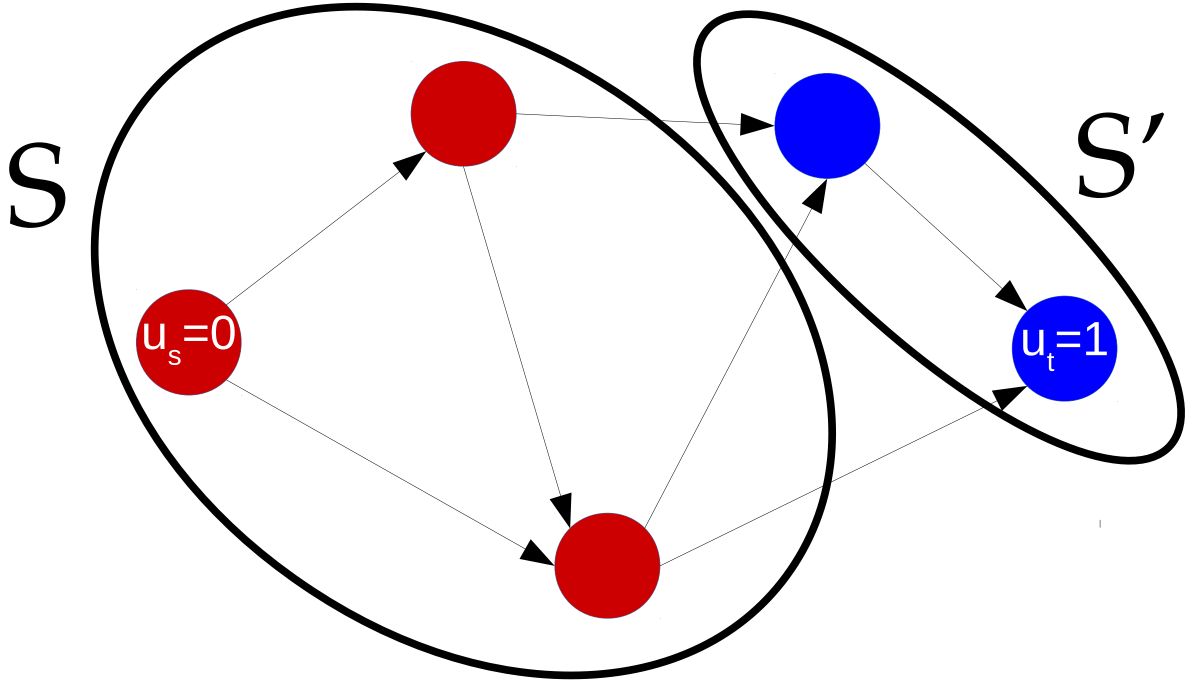
$$min \sum_{(i,j)\in A} k_{ij} y_{ij}$$

$$\begin{aligned} u_i - u_j + y_{ij} &\geq 0, (i,j) \in A \\ y_{ij} &\geq 0 \\ u_s = 0, u_t = 1 & \end{aligned}$$

Every feasible solution represents a cut (S, S')

Dual solutions are cuts

- Every feasible solution of the dual has the form of a cut (S, S')



Dual solutions are cuts

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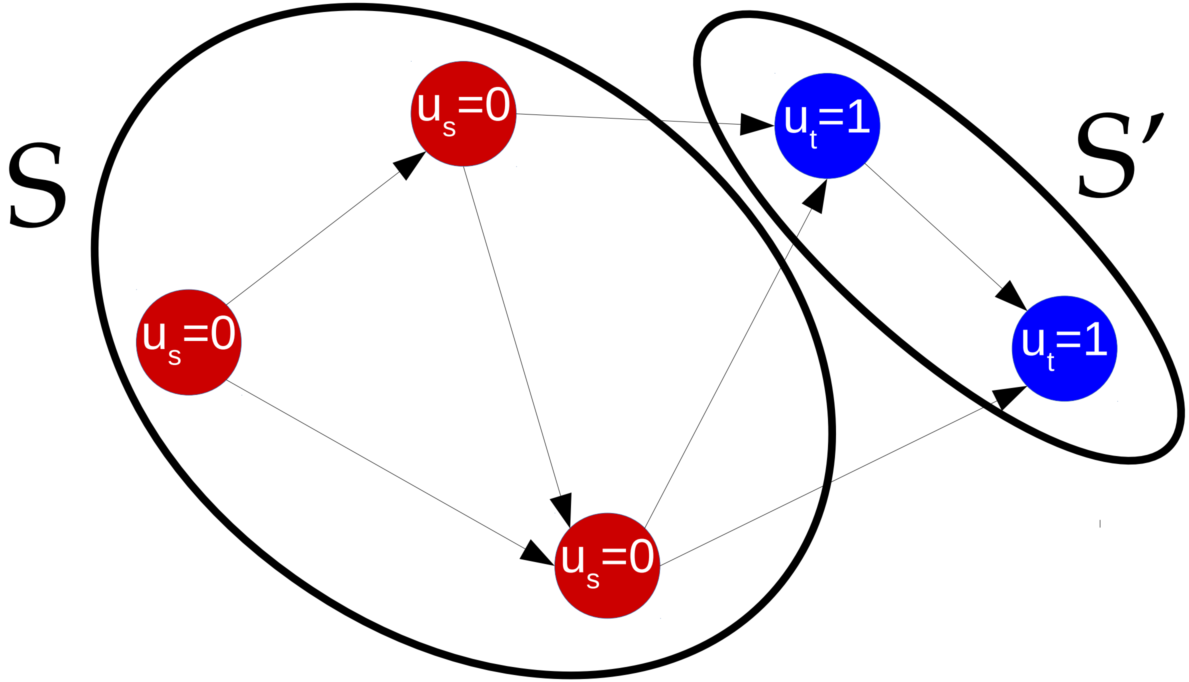
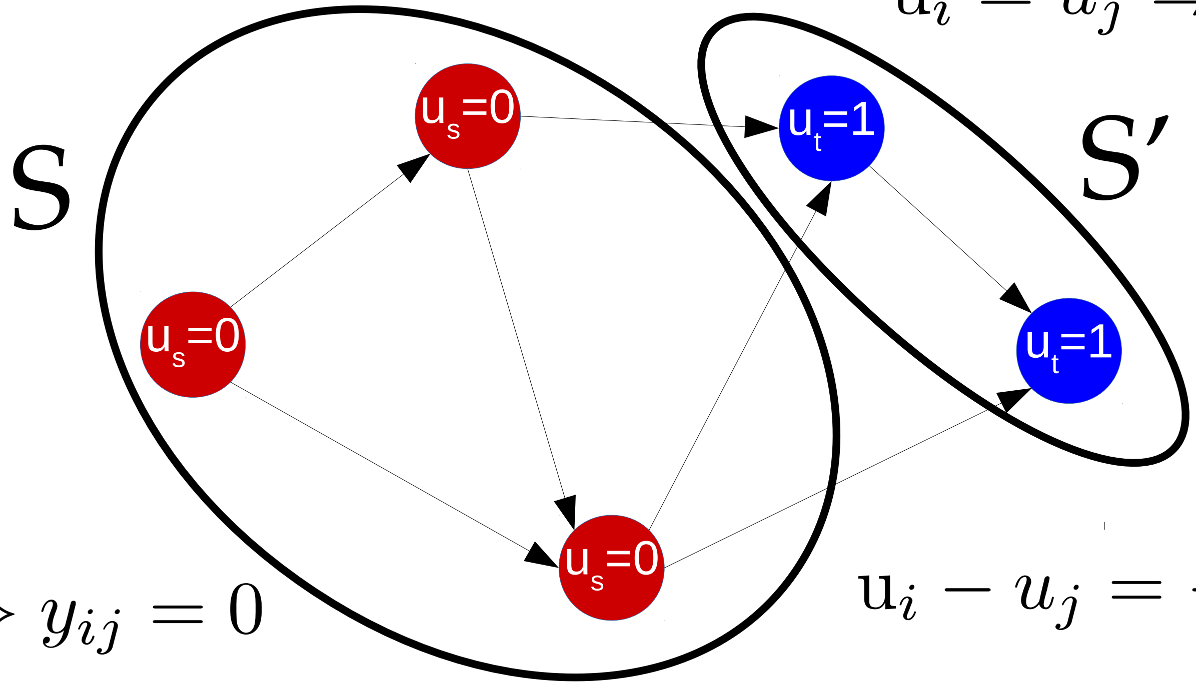


Dual solutions are (s-t)-cuts

$u_i - u_j + y_{ij} \geq 0$ and remember we're trying to minimize $\sum k_{ij} y_{ij}$

$u_i = u_j \Rightarrow y_{ij} = 0$



$u_i = u_j \Rightarrow y_{ij} = 0$

$u_i - u_j = -1 \Rightarrow y_{ij} = 1$

One more thing about the solution

$$
\begin{aligned}
\min \sum_{(i,j)\in A} k_{ij} y_{ij} & & \\
u_i - u_j + y_{ij} &\geq& 0, (i,j) \in A \\
y_{ij} &\geq& 0 \\
u_s = 1, u_t = 0 & &
\end{aligned}
$$

y_{ij} is a dual variable corresponding to primal constraint $x_{ij} \leq k_{ij}$
If y_{ij} is non-zero, then the corresponding constraint is tight
What does it mean for the edges in the cut?

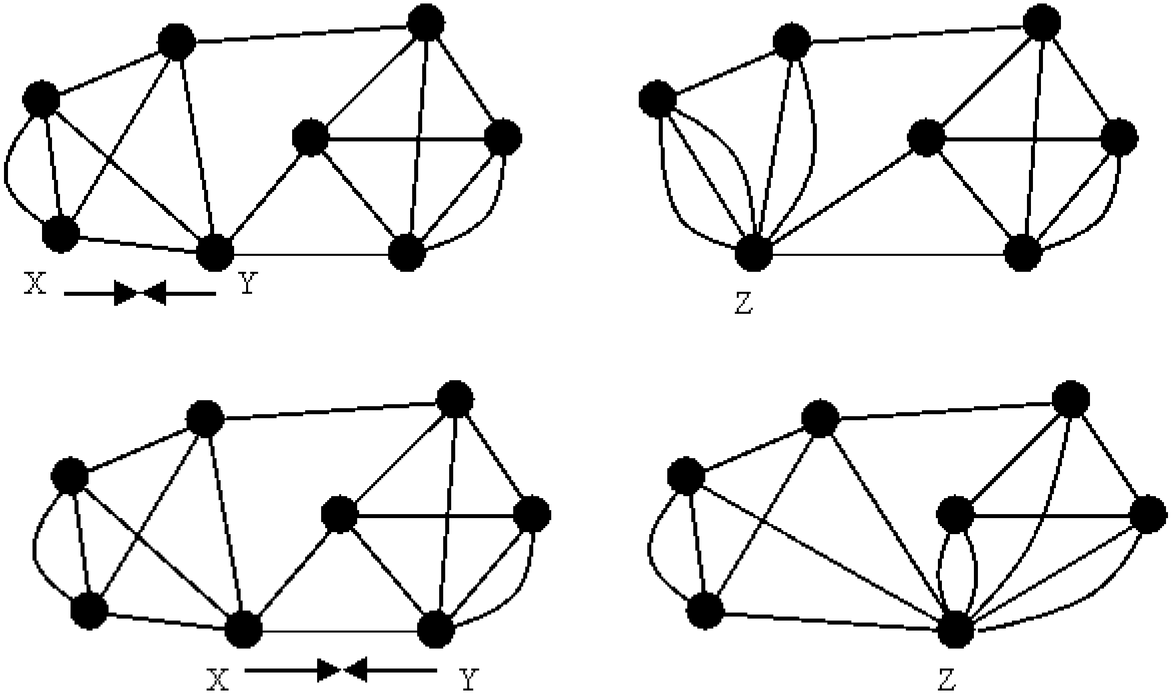
This is an efficient method

- Min-cut and Max-flow are equivalent problems
  - Their solutions are also equal: the value of the maximum flow is equivalent to the minimum cut
- Think of a chain that breaks at the weakest link
- Both can be solved exactly in polynomial time

Randomized algorithm for (s-t)-cuts

- Pick an edge at random (u,v)
- Merge u and v in new vertex uv
- Edges between u and v are removed
- Edges pointing to u or v are added as multi-edges to vertex uv
- When only s and t remain, the multi-edges are a cut, probably the minimum one

Example merges (“contractions”)

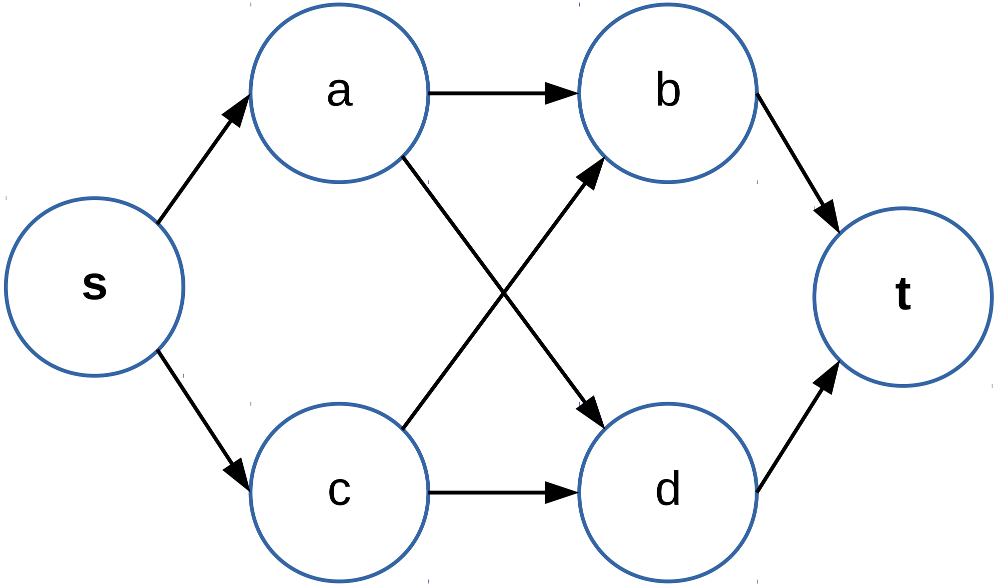


http://www.cs.berkeley.edu/~jfc/cs174lecs/lec18/lec18.html

Try it!

Run the randomized algorithm on this graph

- Pick an edge at random (u,v)
- Merge u and v in new vertex uv
- Edges between u and v are removed
- Edges pointing to u or v are added as multi-edges to vertex uv
- When only s and t remain, the multi-edges are a cut, probably the minimum one



The randomized algorithm might miss the min cut

- Multiple runs are required
- The probability that this finds the min cut in one run is about 1/log(n), so O(log n) iterations are required to find min cut
- Each iteration costs $O(n^2 \log n)$
- $O(n^2 \log^2 n)$ operations needed to find min cut
- Exact algorithm: $O(n^3 + n^2 \log n)$; the n^3 is because of $|V||E|$ operations required

http://www.cs.berkeley.edu/~jfc/cs174lecs/lec18/lec18.html