

Graph theory basics

Introduction to Network Science

Carlos Castillo

Topic 02

Contents

- Degree
- Sparsity
- Bi-partite networks
- Connectedness

Sources

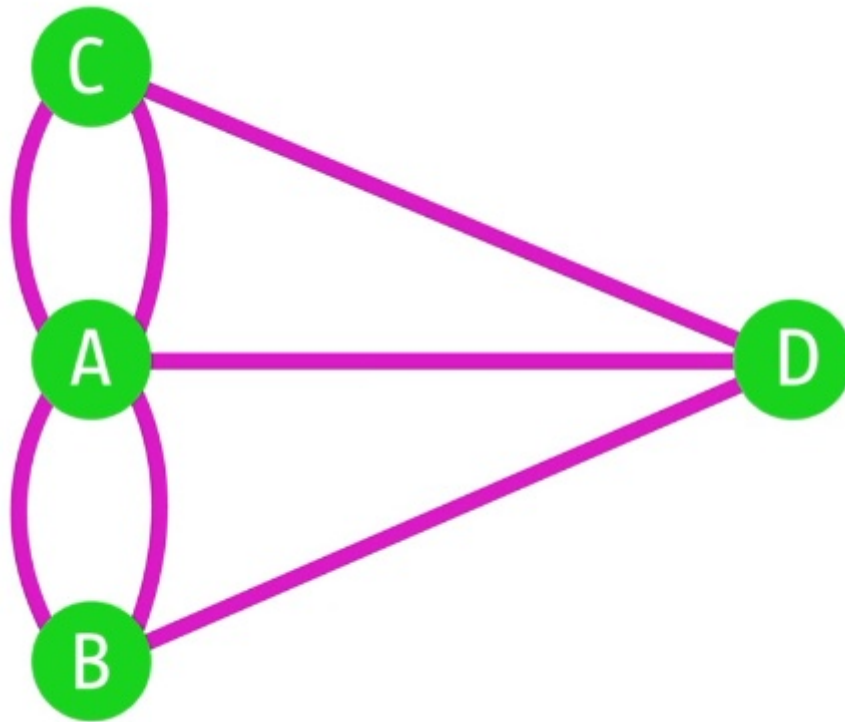
- Albert-László Barabási: Network Science. Cambridge University Press, 2016.
 - Follows almost section-by-section chapter 02
- URLs cited in the footer of specific slides

The seven bridges of Königsberg



<http://networksciencebook.com/images/ch-02/video-2-1.m4v>

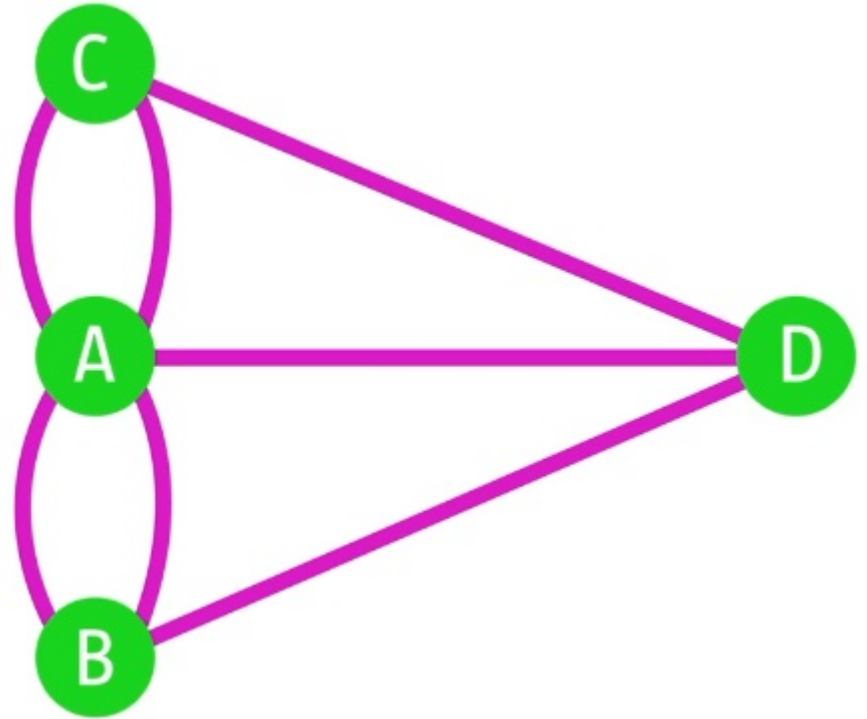
Can one walk across the 7 bridges without crossing the same bridge twice?



Basic concepts

Notation for a graph

- $G = (V, E)$
 - V : nodes or vertices
 - E : links or edges
- $|V| = N$ size of graph
- $|E| = L$ number of links



Typical notation variations

- You may find that G is denoted by (N, A) , this is typical of directed graphs, means “*nodes, arcs*”
- You may find that
 - $|V|$ is denoted by n or N
 - $|E|$ is denoted by m , M , or L

Directed vs undirected graphs

- In an undirected graph
 - E is a symmetric relation
$$(u, v) \in E \Rightarrow (v, u) \in E$$
- In a directed graph, also known as “digraph”
 - E is not a symmetric relation
$$(u, v) \in E \not\Rightarrow (v, u) \in E$$

Example graphs we will use

Network	$ V $	$ E $
Zachary's Karate Club (karate.gml)	34	78
Les Misérables (lesmiserables.gml)	77	254
E-mail exchanges (email-eu-core.csv)	868	25K
US companies ownership	1351	6721
Marvel comics (hero-network.csv)	6K	570K

Degree

- Node i has degree k_i
 - This is the number of links incident on this node
 - The total number of links L is given by
$$L = \frac{1}{2} \sum_{i=1}^N k_i$$
- Average degree
$$\langle k \rangle = \frac{1}{N} \sum_{i=1}^N k_i = \frac{2L}{N}$$

In directed networks

- We distinguish **in-degree** from **out-degree**
 - Incoming and outgoing links, respectively
- Degree is the sum of both $k_i = k_i^{\text{in}} + k_i^{\text{out}}$
- Counting total number of links:

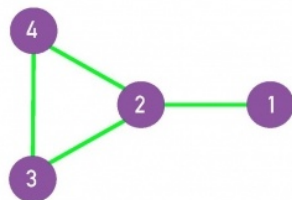
$$L = \sum_{i=1}^N k_i^{\text{in}} = \sum_{i=1}^N k_i^{\text{out}}$$

Degree distribution

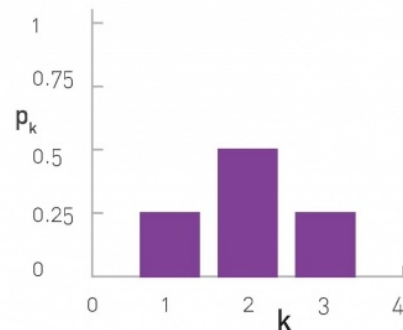
- If there are N_k nodes with degree k
- The **degree distribution** is given by $p_k = \frac{N_k}{N}$
- The average degree is then $\langle k \rangle = \sum_{k=0}^{\infty} k p_k$

Degree distribution; two toy graphs

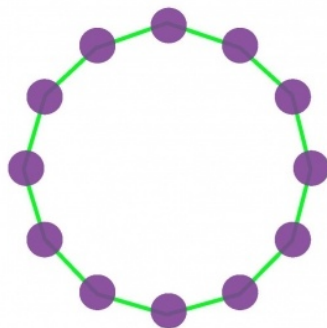
a.



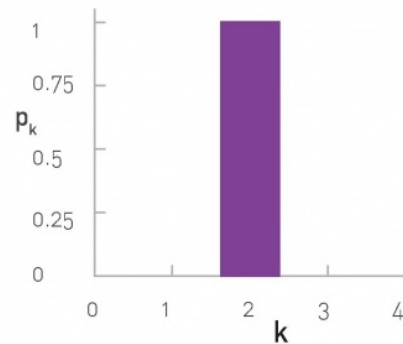
b.



c.

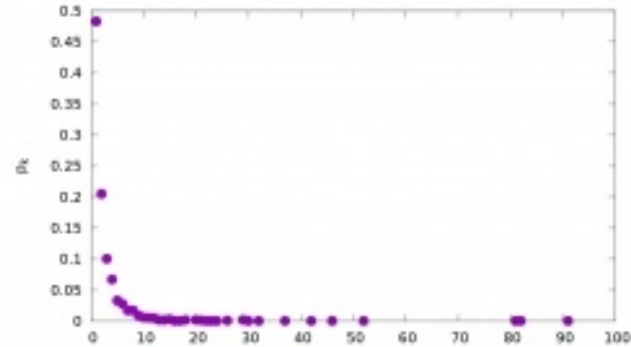
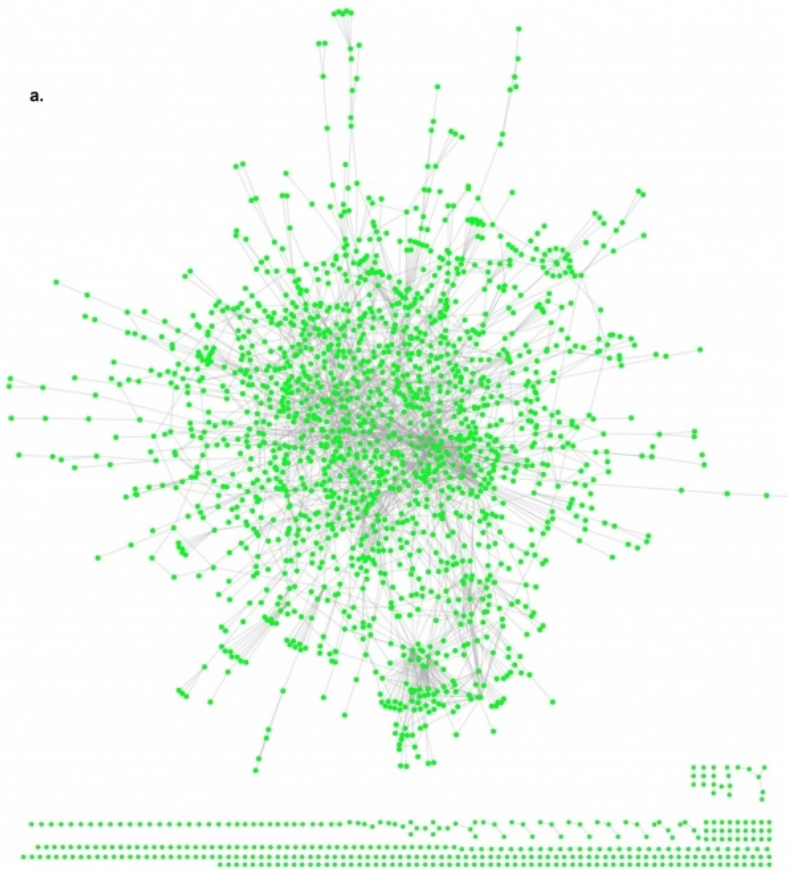


d.

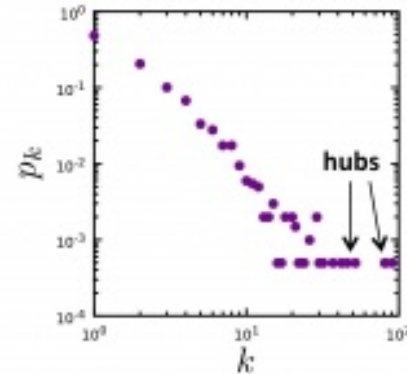


Degree distribution; real graph

a.



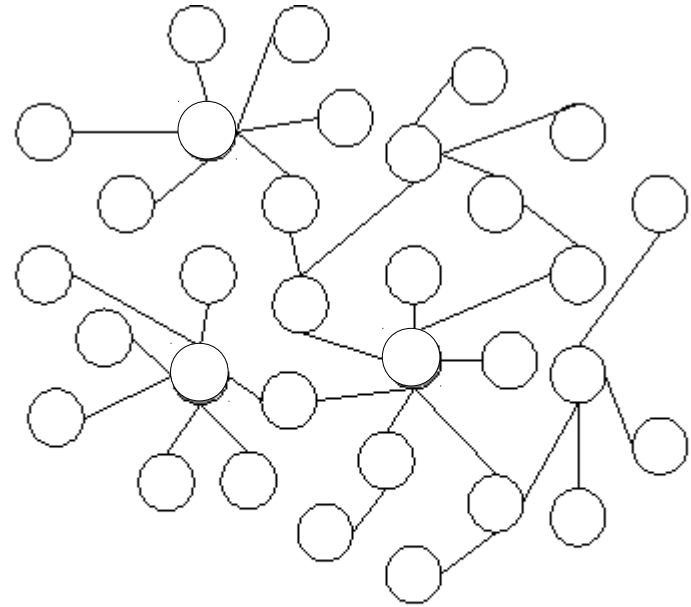
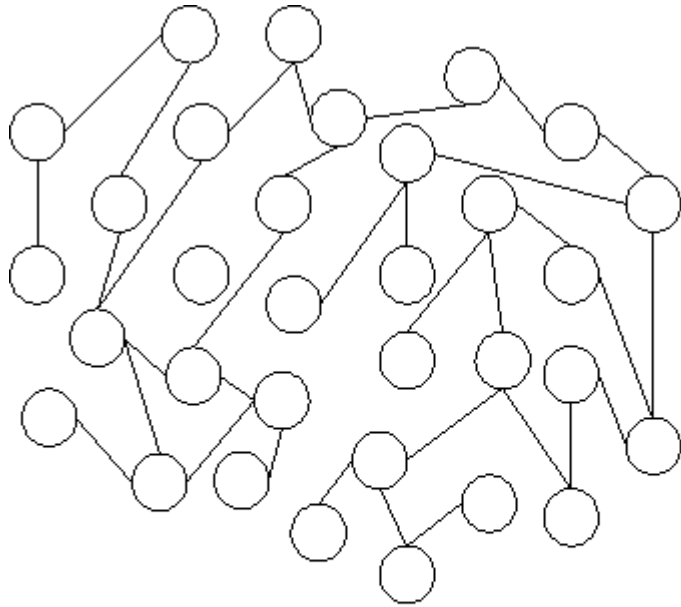
Linear
scale



Log-log
scale

Exercise

- Draw the degree distribution of these graphs

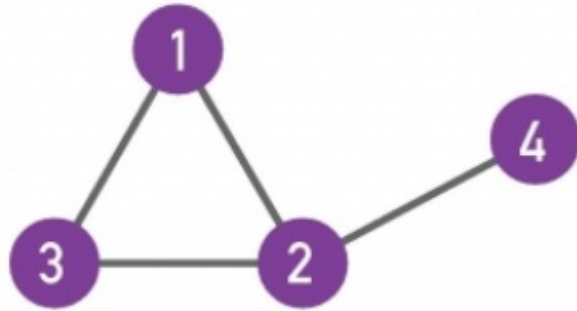


Adjacency matrix

What is an adjacency matrix

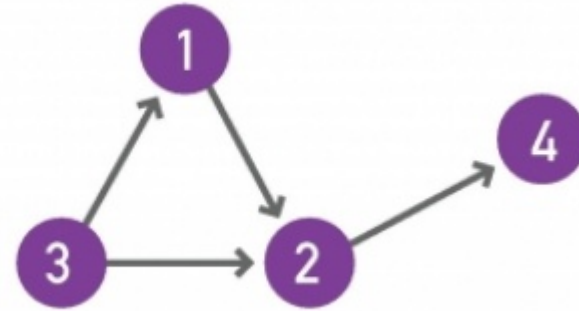
- A is the **adjacency matrix** of $G = (V, E)$ iff:
 - A has $|V|$ rows and $|V|$ columns
 - $A_{ij} = 1$ if $(i,j) \in E$
 - $A_{ij} = 0$ if $(i,j) \notin E$

Examples



Undirected graph

$$A_{ij} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{matrix}$$



Directed graph

$$A_{ij} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

Quick exercise

- In terms of A , what is the expression for:

$$k_i^{\text{in}} =$$

$$k_i^{\text{out}} =$$

Some “graphology” ...

- G is undirected $\Leftrightarrow A$ is symmetric
- G has a self-loop
 $\Leftrightarrow A$ has a non-zero element in the diagonal
- G is complete $\Leftrightarrow A_{ij} \neq 0$ (except if $i=j$)

Real networks are sparse

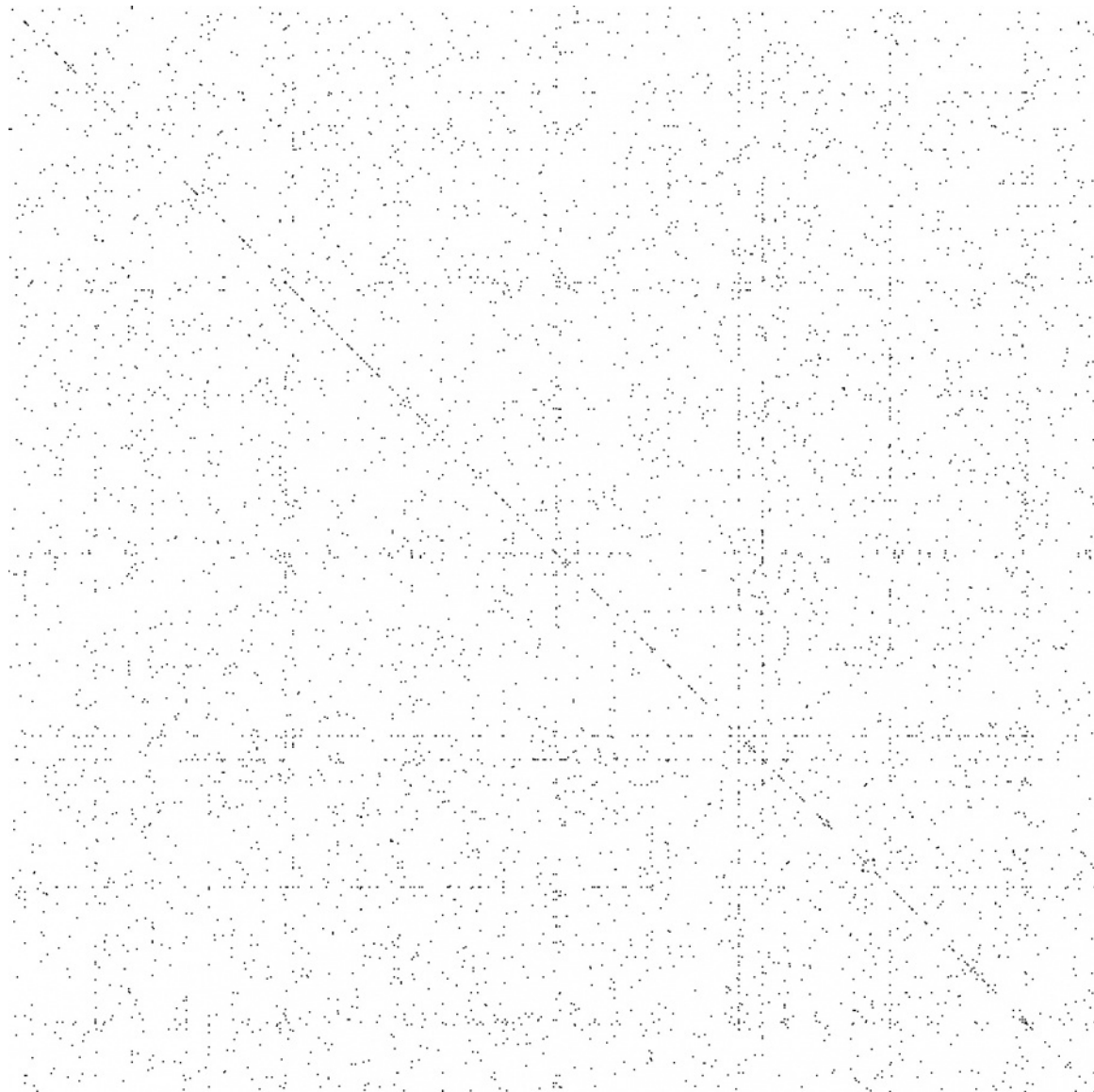
- Theoretically $L_{\max} = \binom{N}{2} = \frac{N(N-1)}{2}$
- Most real networks are sparse, i.e., $L \ll L_{\max}$

How sparse are some networks?

Network	$ V $	$ E $	Max $ E $
Zachary's Karate Club	34	78	561
Les Misérables	77	254	2962
E-mail exchanges	868	25K	376K
US companies ownership	1351	6721	911K
Marvel comics	6K	570K	17M

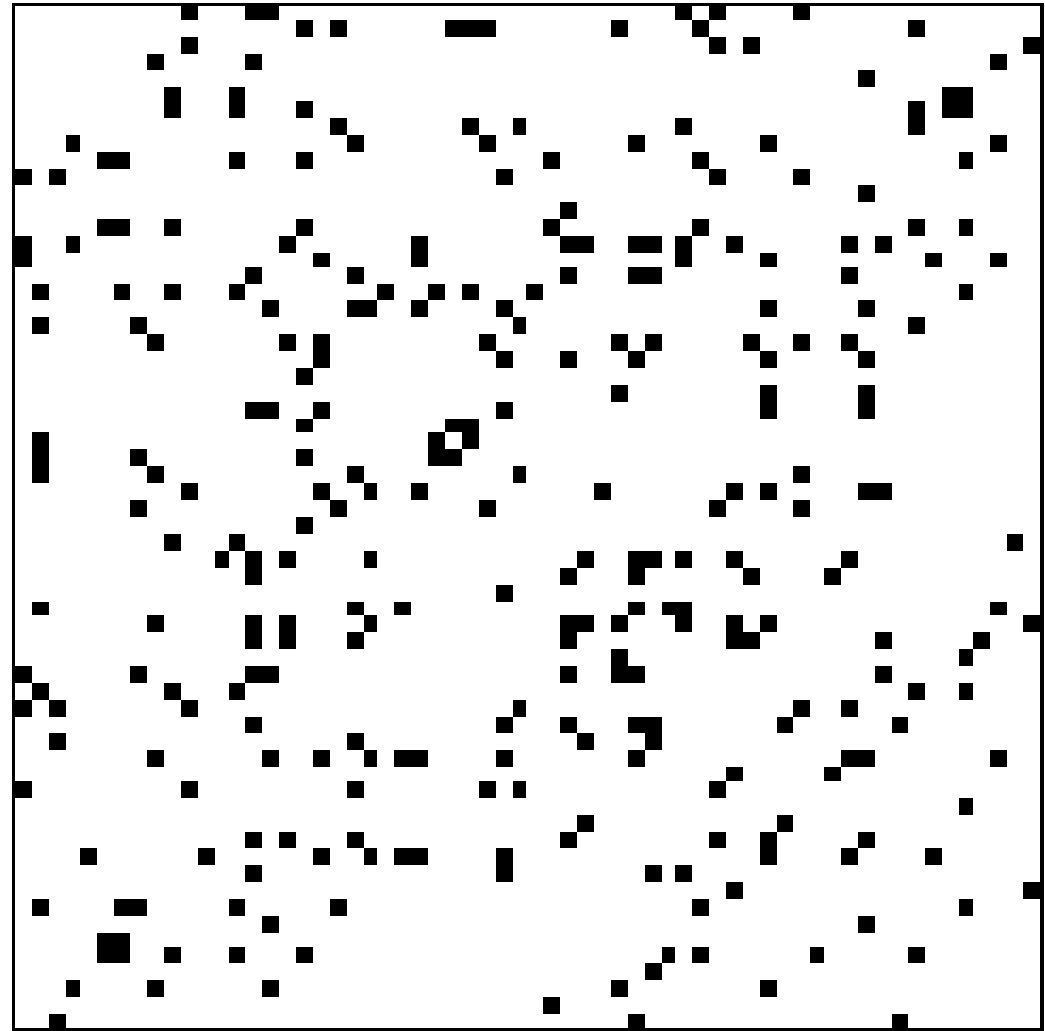
Example: protein interaction network

($N=2K$, $L=3K$)



Example:
dolphins

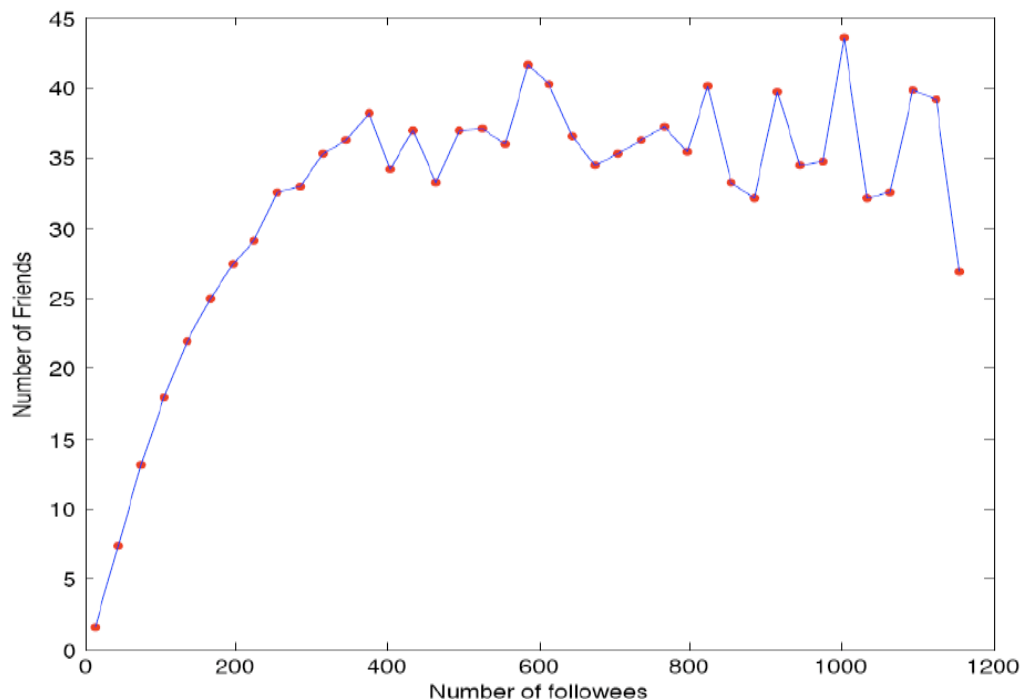
($N=62$, $L=318$)



Why are networks sparse?

- Different mechanisms, think about it from the node perspective:
 - How many items **could** the node be connected to
 - Would it be **realistic** to connect to a large fraction of them?
- In social networks, Dunbar's number (≈ 150)

Example: actual friends in Twitter vs people you follow in Twitter

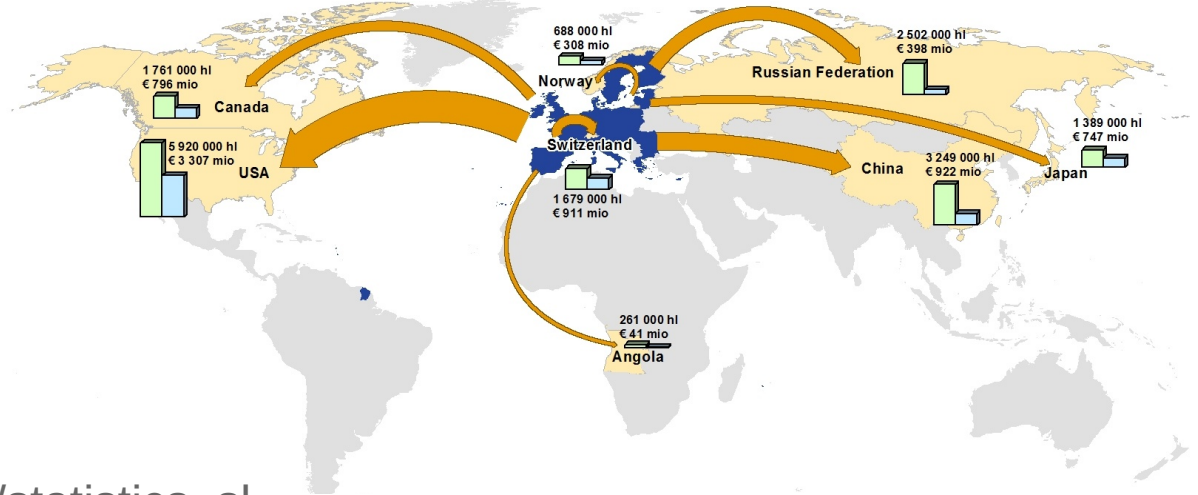
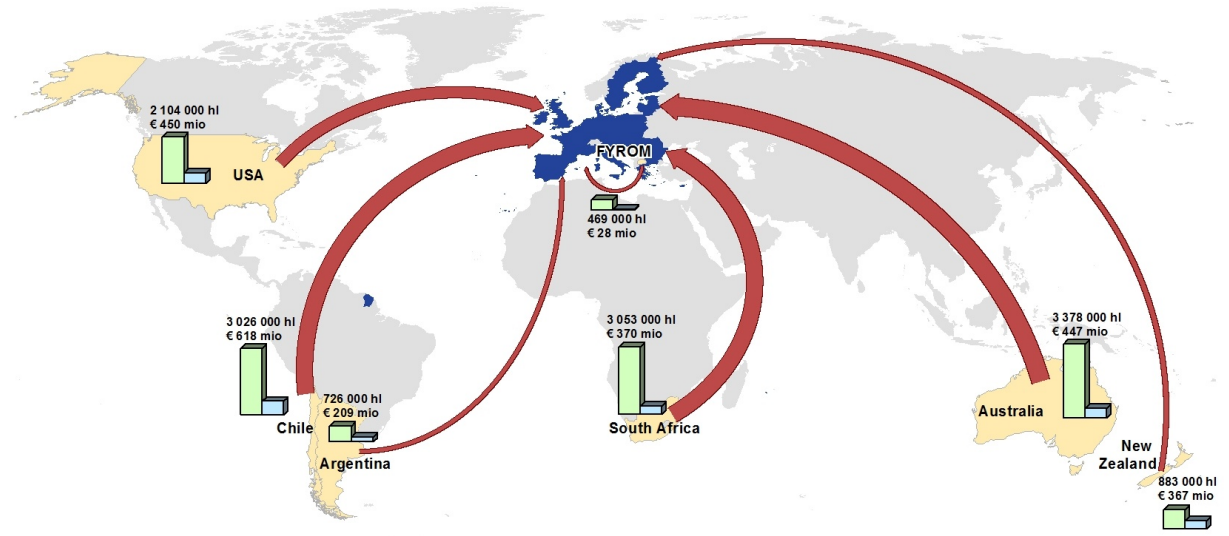


Weighted networks

- In weighted networks, instead of $A_{ij} \in [0, 1]$
- We have that $A_{ij} \in \mathbb{R}$
- Weights may represent different tie strengths

Example: weighted networks

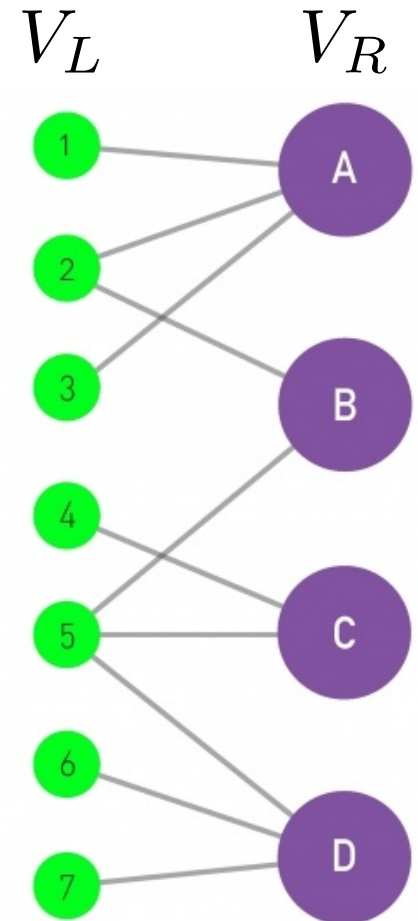
EU imports (top) and
exports (bottom) of wine



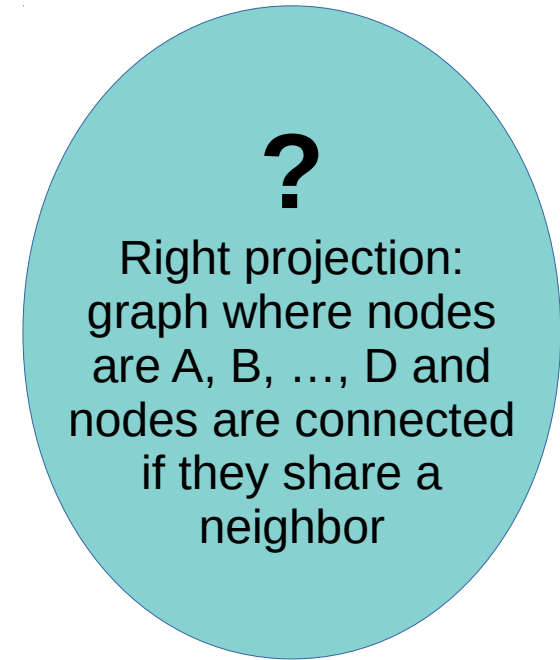
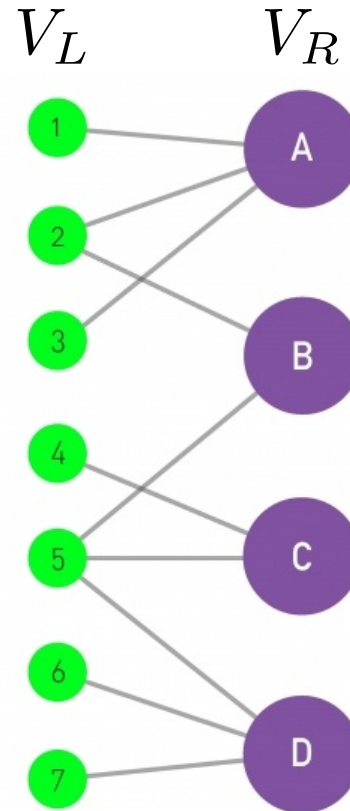
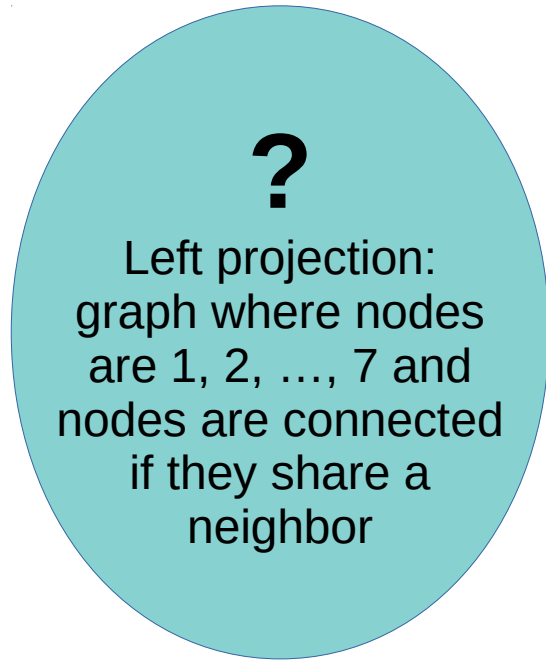
Bipartite networks

- A bipartite graph is a graph $G = (V, E)$ such that

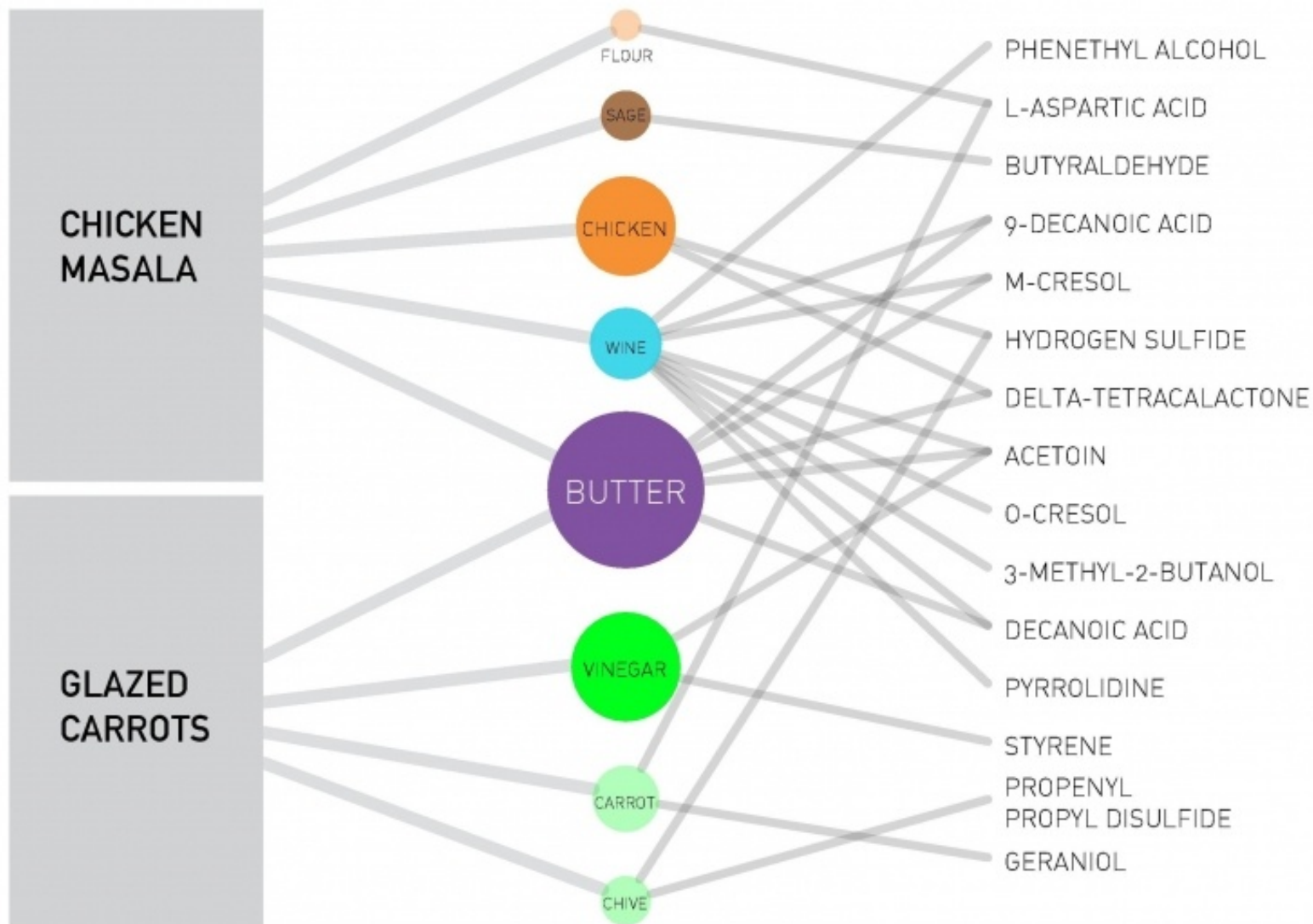
$$V = V_L \cup V_R, V_L \cap V_R = \emptyset, E \subseteq V_L \times V_R$$



Projecting a bipartite network



Tripartite network



Clique and Bi-partite clique

- A **clique** is a complete (sub)graph: $E = (V \times V)$
- An **n-clique** is a complete graph of n nodes
- A **bi-partite clique** is such that

$$V = V_1 \cup V_2, V_1 \cap V_2 = \emptyset, E = (V_1 \times V_2)$$

- A **(n₁, n₂)-clique** is a bipartite clique such that

$$|V_1| = n_1, |V_2| = n_2$$

The word “clique” in popular culture

In some parts of Latin America, a “**clika**” or “**clica**” means a close group of friends, sometimes a gang

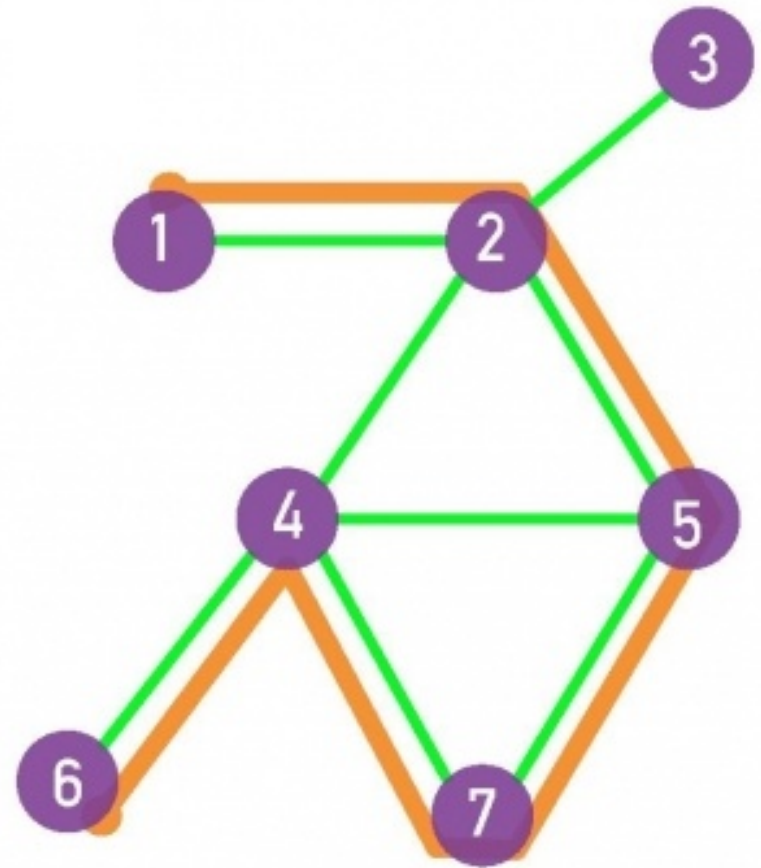


Photo credit: @astro_jr

Paths and distances

Paths

- A path is a sequence of edges from E
- The destination of each edge is the origin of the next edge
- The length of the path is the number of edges on it
- Example: a path marked in orange, having length 5



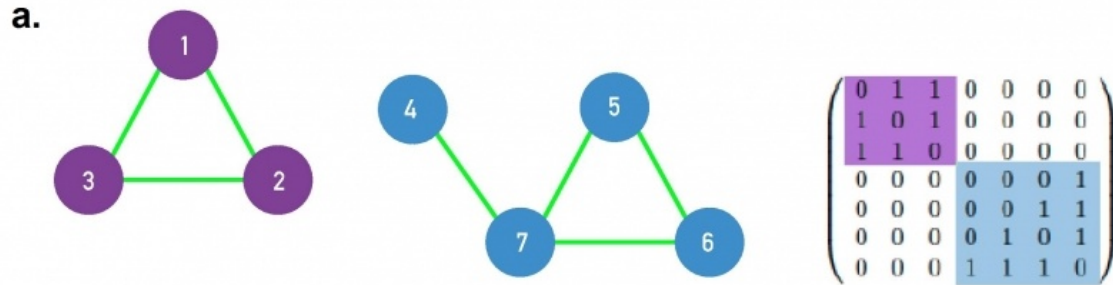
Connectedness

- If a path exists between two nodes i, j :
 - those nodes are part of the same **connected component**
- A graph that has **only one connected component** is called a **connected graph**

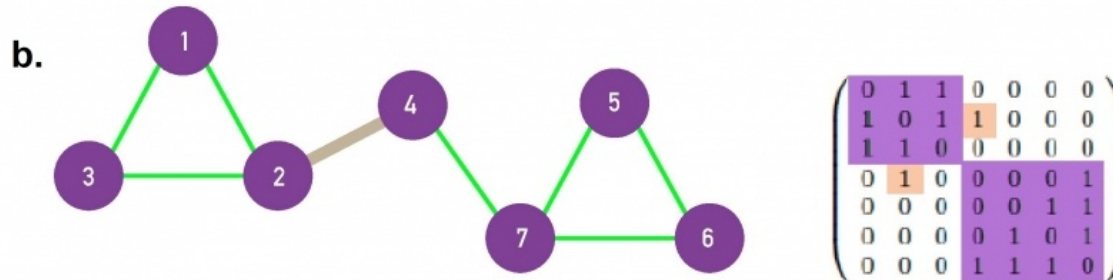
Connected graphs

A **disconnected graph** has an adjacency matrix that can be arranged in block diagonal form

a. disconnected



b. connected



Distance

- If two nodes i, j are in the same connected component:
 - the **distance** between i and j , denoted by d_{ij} is the **length of the shortest path** between them

Diameter

- The **diameter** of a network is the maximum distance between two nodes on it, d_{\max}
- The **effective diameter** (or **effective-90% diameter**) is a number d such that 90% of the pairs of nodes (i,j) are at a distance smaller than d
- The **average distance** is $\langle d \rangle$, and is measured only for nodes that are in the same connected component

Local clustering coefficient

- The **local clustering coefficient** C_i is a property of a node i
- Let L_i represent the number of links among neighbors of node i

$$C_i = \frac{2L_i}{k_i(k_i - 1)} \quad C_i \triangleq 0 \text{ if } k_i \leq 1$$

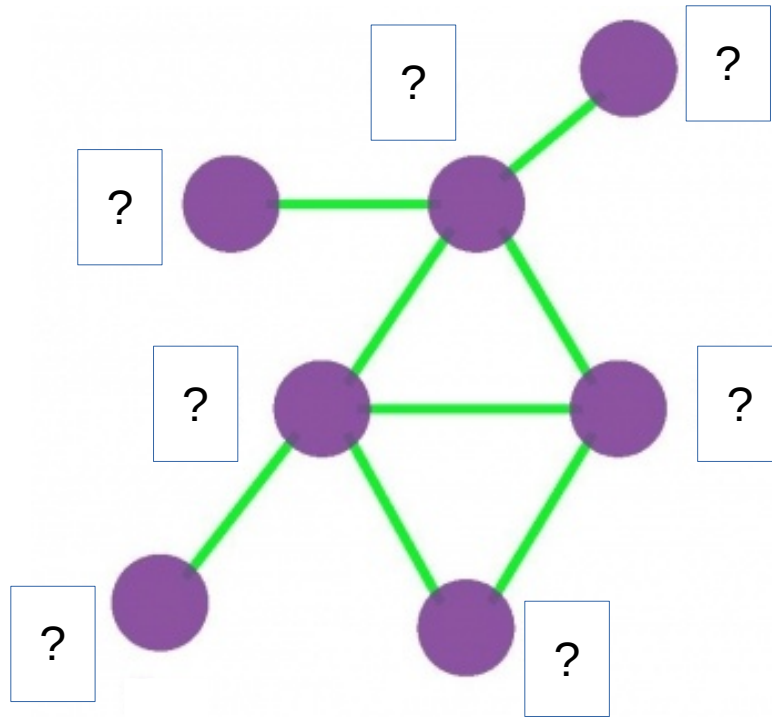
Average clustering coefficient

- The **average clustering coefficient** is a property of the entire graph

$$\langle C \rangle = \frac{1}{N} \sum_{i=1}^N C_i$$

Try it!

What is the local clustering coefficient of each node?



$$C_i = \frac{2L_i}{k_i(k_i - 1)}$$

$$C_i \triangleq 0 \text{ if } k_i \leq 1$$

Summary

Things to remember

- Definitions:
 - Degree, in-degree, out-degree
 - Bi-partite graph, clique
 - Sparse vs dense graph
- Distance, diameter, effective diameter, connected components
- Local and global clustering coefficient
- Writing the adjacency matrix of a graph and drawing a graph given its adjacency matrix

Practice on your own

- Obtain a degree distribution
- Write the adjacency matrix of a graph
- Measure the sparsity of a graph
- Compute the distance between two nodes
- Compute the diameter of a graph
- Identify connected components
- Calculate local clustering coefficient