

# Epidemics

Introduction to Network Science

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Topic 16

# Sources

- Barabási (2016): [Network Science Ch. 10](#)
- Easley and Kleinberg (2010): Networks, Crowds, and Markets [Ch 21](#).

# Examples: human epidemics

- Influenza, measles, STDs
- The “Black Death”  
[next slide]
- Smallpox and other diseases brought by Europeans to America since early 1500s



# The “Black Death” (Bubonic plague)

1300s

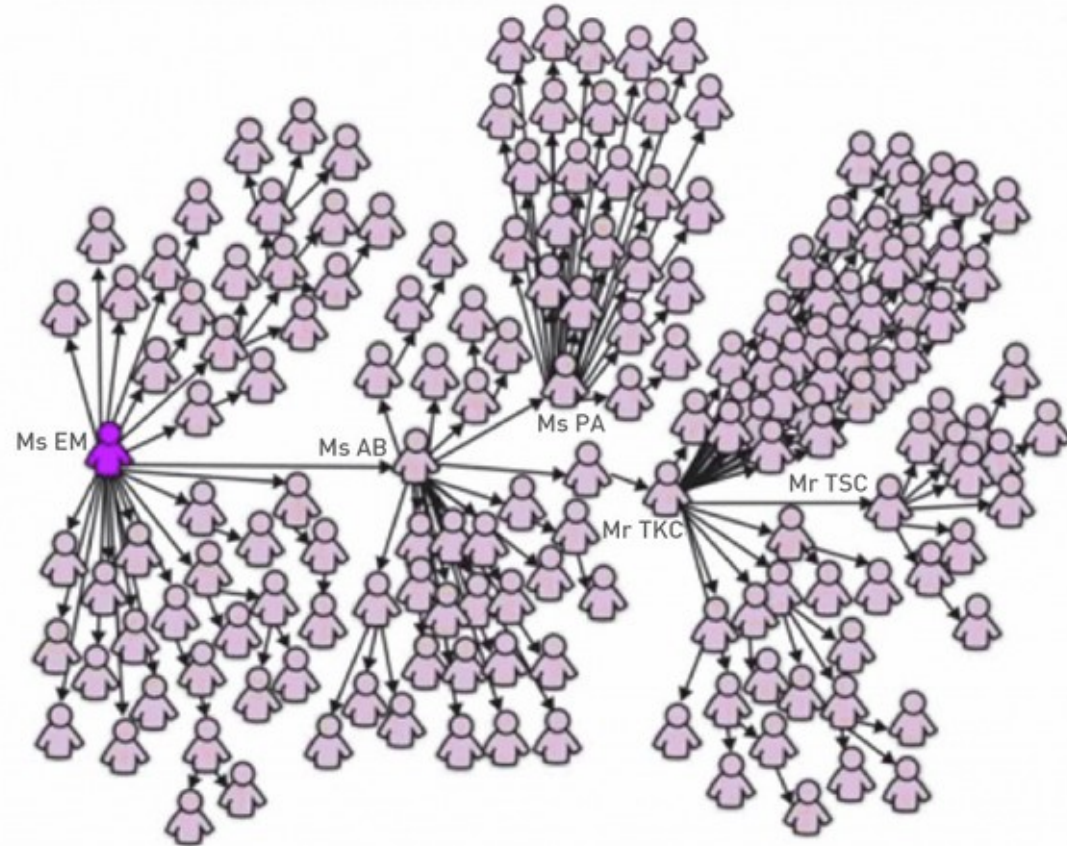
Killed 30%-60% of  
the total population  
of Europe



1346 1347 1348 1349 1350 1351 1352 1353

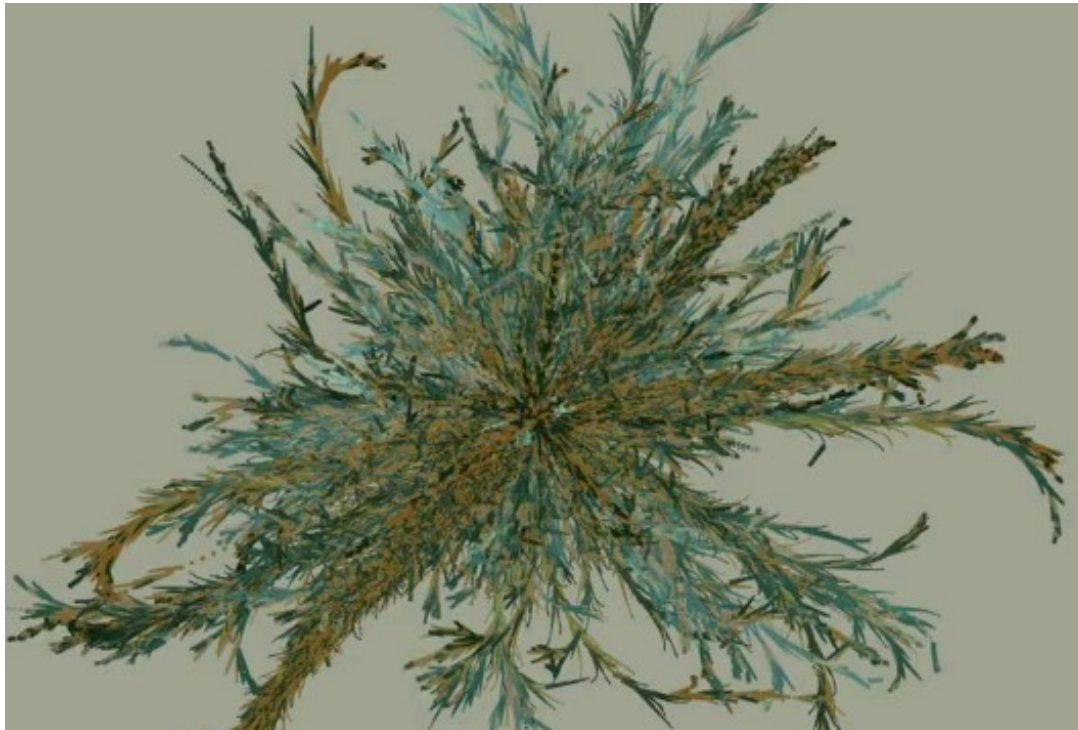
# SARS Outbreak (2003)

- February 21st: Chinese doctor who have been several treating “atypical pneumonia” cases check-ins into hotel in Hong Kong
  - Hospitalized on Feb 22<sup>nd</sup>
  - Died on March 4<sup>th</sup>
- March 1st: “Ms. E. M.” returns to Singapore after visiting Hong Kong
  - Graph depicts 144 out of the first 206 SARS patients in Singapore
  - Ms. E. M. lived, various of her family members died

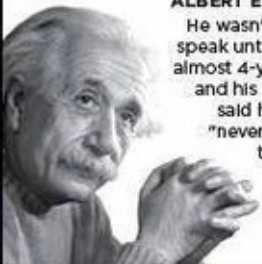
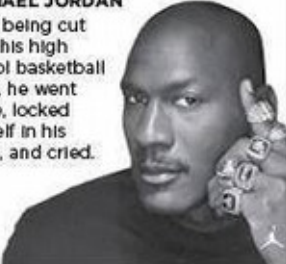

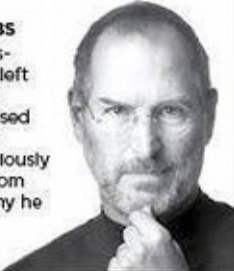






# Spread of a Meme ("Famous Failures")

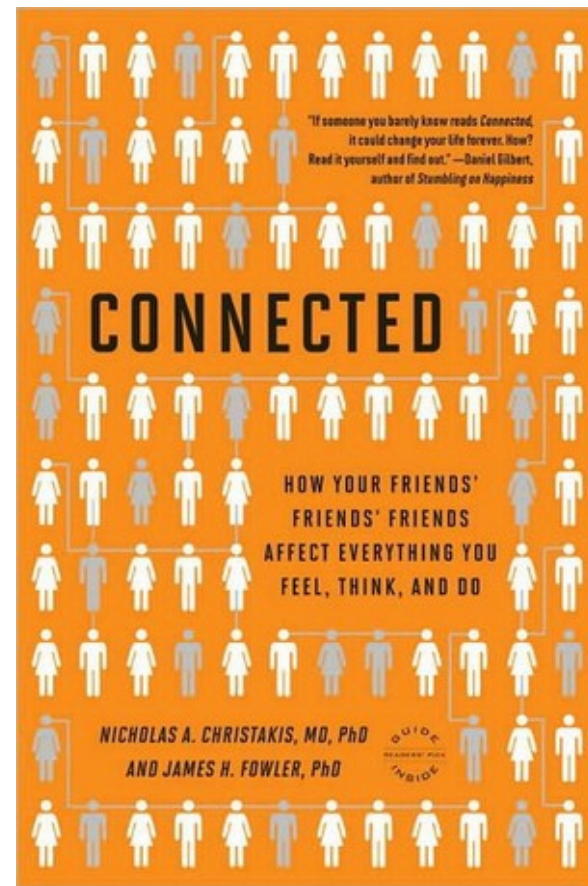


<https://vimeo.com/50730795>

FAMOUS FAILURES	
 <p><b>ALBERT EINSTEIN</b> He wasn't able to speak until he was almost 4-years-old and his teachers said he would "never amount to much"</p>	 <p><b>MICHAEL JORDAN</b> After being cut from his high school basketball team, he went home, locked himself in his room, and cried.</p>
 <p><b>WALT DISNEY</b> Fired from a newspaper for "lacking imagination" and "having no original ideas."</p>	 <p><b>STEVE JOBS</b> At 30-years-old he was left devastated and depressed after being unceremoniously removed from the company he started.</p>
 <p><b>OPRAH WINFREY</b> Was demoted from her job as a news anchor because she "wasn't fit for television."</p>	 <p><b>THE BEATLES</b> Rejected by Decca Recording Studios, who said "We don't like their sound—they have no future in show business."</p>
<b>IF YOU'VE NEVER FAILED, YOU'VE NEVER TRIED ANYTHING NEW</b>	

# Non-trivial examples

- **Back pain:** spread from West to East in Germany after fall of Berlin Wall
- **Suicide:** well known to spread throughout communities on occasion
- **Sexual “scripts”:** expected sequences of behaviors during intimate situations
- **Politics:** the denser your connections, the more intense your convictions



# Simple models for epidemics

- There are many factors:
  - Contagiousness
  - Length of infectious period,
  - Severity
  - ...
- Structure of contacts in a population



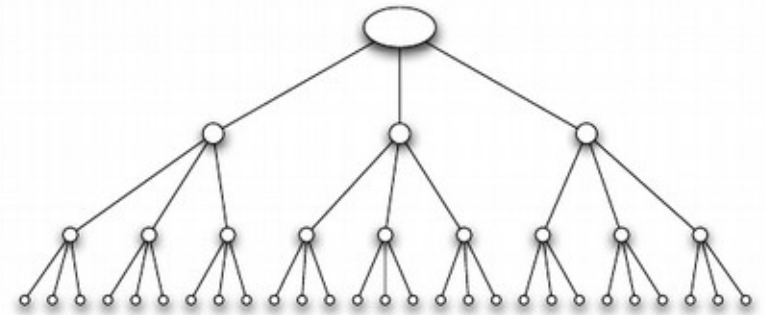
# Diffusion of ideas and diseases

- Adopting a new idea, behavior, fashion, product, taste, may also spread from person to person: “social contagion”
- There is a certain agency of the receiver
- In diffusion of diseases, we assume there is no agency: each contagion is random

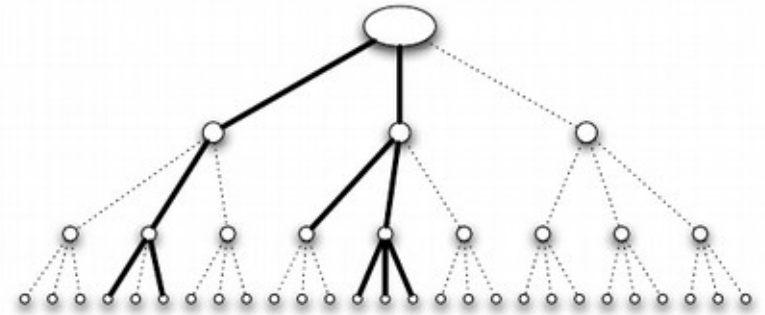
# Simple model: branching process

- Each person interacts with other  $k$  people
- Each interaction ends in infection with probability  $\beta$

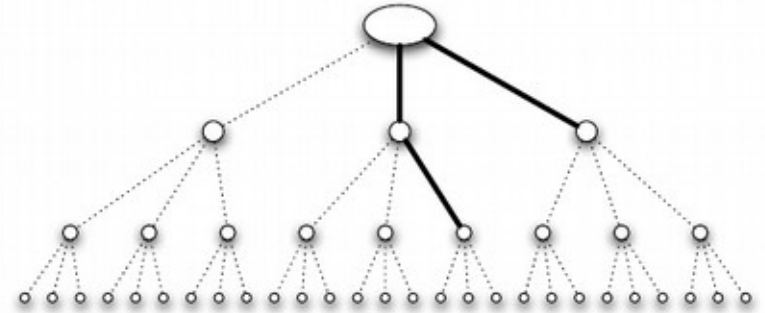
Example:  $k=3$



(a) The contact network for a branching process



(b) With high contagion probability, the infection spreads widely



(c) With low contagion probability, the infection is likely to die out quickly

# Transmission rate or “Basic reproductive number” $R_0$

- Each person interacts with other  $k$  people
- Each interaction ends in infection with probability  $\beta$

- What is the expected number of cases caused by a single individual,  $R_0$ ?
- What do you think happens if  $R_0 < 1$ ?
- What do you think happens if  $R_0 > 1$ ?

Disease	Transmission	$R_0$
Measles	Airborne	12-18
Pertussis	Airborne droplet	12-17
Diphtheria	Saliva	6-7
Smallpox	Social contact	5-7
Polio	Fecal-oral route	5-7
Rubella	Airborne droplet	5-7
Mumps	Airborne droplet	4-7
HIV/AIDS	Sexual contact	2-5
SARS	Airborne droplet	2-5
Influenza (1918 strain)	Airborne droplet	2-3

# Changing $R_0 = \beta k$

- Sanitary practices  
(reduce  $\beta$ )
- Quarantine  
(reduces  $k$ )

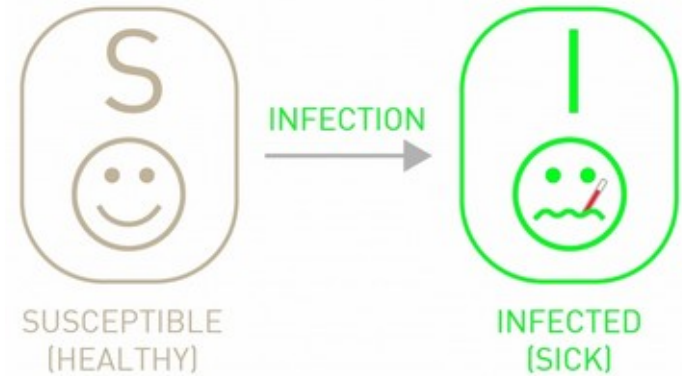
# The SI model





# The SI model

- Susceptible:
  - The node can catch the disease
- Infected:
  - The node has the disease and can spread it
  - It will stay sick forever



# Notation

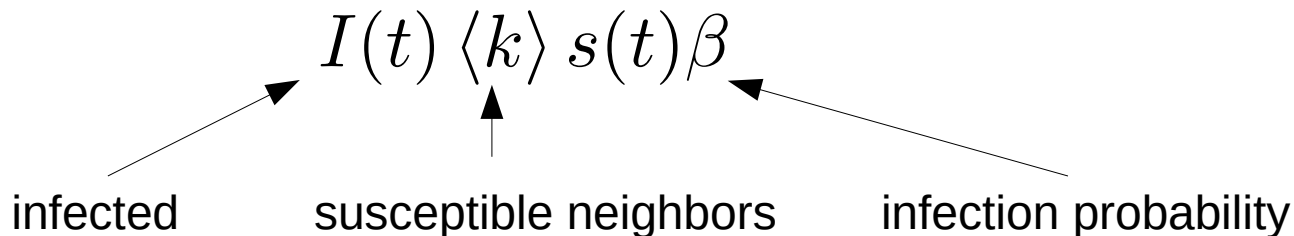
- Number of susceptible  $S(t)$ 
  - Fraction of susceptible  $s(t) = S(t) / N$
- Number of infected  $I(t)$ 
  - Fraction of infected  $i(t) = I(t) / N$
- $s(t) + i(t) = 1$

# How many susceptible neighbors a node has?

$$\langle k \rangle \frac{S(t)}{N} = \langle k \rangle s(t)$$

# How many new infections are produced?

(for every infected, iterate through its susceptible neighbors, infect with probability  $\beta$ )



Prove that  $i(t) = \frac{i_0 e^{\beta \langle k \rangle t}}{1 - i_0 + i_0 e^{\beta \langle k \rangle t}}$

$$\frac{di(t)}{dt} = \beta \langle k \rangle i(t)(1 - i(t))$$

Use  $\frac{1}{i(1-i)} = \frac{1}{i} + \frac{1}{1-i}$  and integrate from  $t = 0$  to  $t$   
Denote by  $i_0 = i(t = 0)$

$$\int \frac{1}{x} dx = \log x + C$$

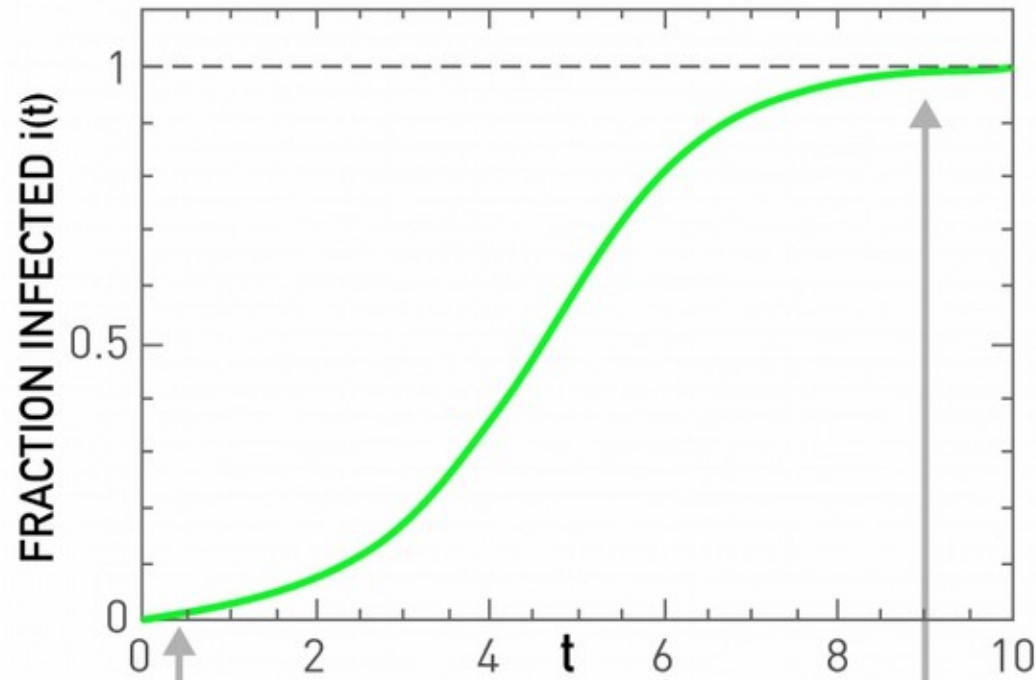
$$\int \frac{1}{1-x} dx = -\log(1-x) + C$$

# Infected as a function of time (SI)

$$i(t) = \frac{i_0 e^{\beta \langle k \rangle t}}{1 - i_0 + i_0 e^{\beta \langle k \rangle t}}$$

Characteristic time  
(to infect  $1/e \approx 36\%$  of people):

$$\tau = \frac{1}{\beta \langle k \rangle}$$



exponential  
regime

If  $i$  is small,

$$i \approx i_0 e^{\beta \langle k \rangle t}$$

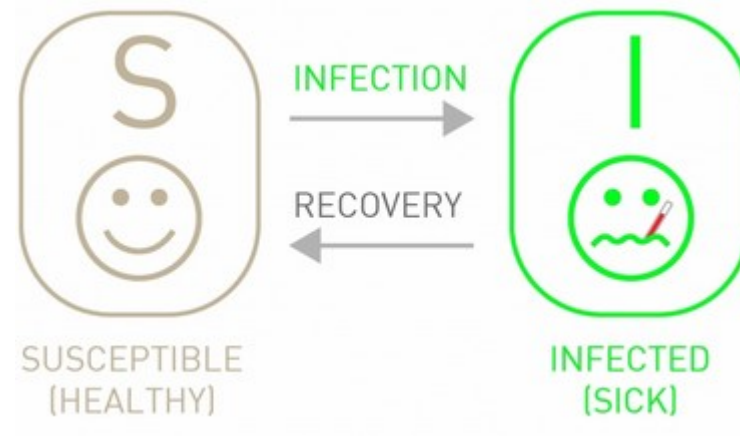
saturation  
regime

If  $i \rightarrow 1$ ,

$$\frac{di}{dt} \rightarrow 0$$



# The SIS model



# The SIS model



- Susceptible:
  - The node can catch the disease
- Infected:
  - The node has the disease and can spread it
  - After some time, it recovers ... but it becomes susceptible again

# Infection dynamics

$$\frac{di(t)}{dt} = \beta \langle k \rangle i(t)(1 - i(t)) - \mu i(t)$$

- $\mu$  is the recovery rate, i.e., the probability of becoming susceptible again in an unit of time

$$i(t) = \left(1 - \frac{\mu}{\beta \langle k \rangle}\right) \frac{C e^{(\beta \langle k \rangle - \mu)t}}{1 + C e^{(\beta \langle k \rangle - \mu)t}}$$

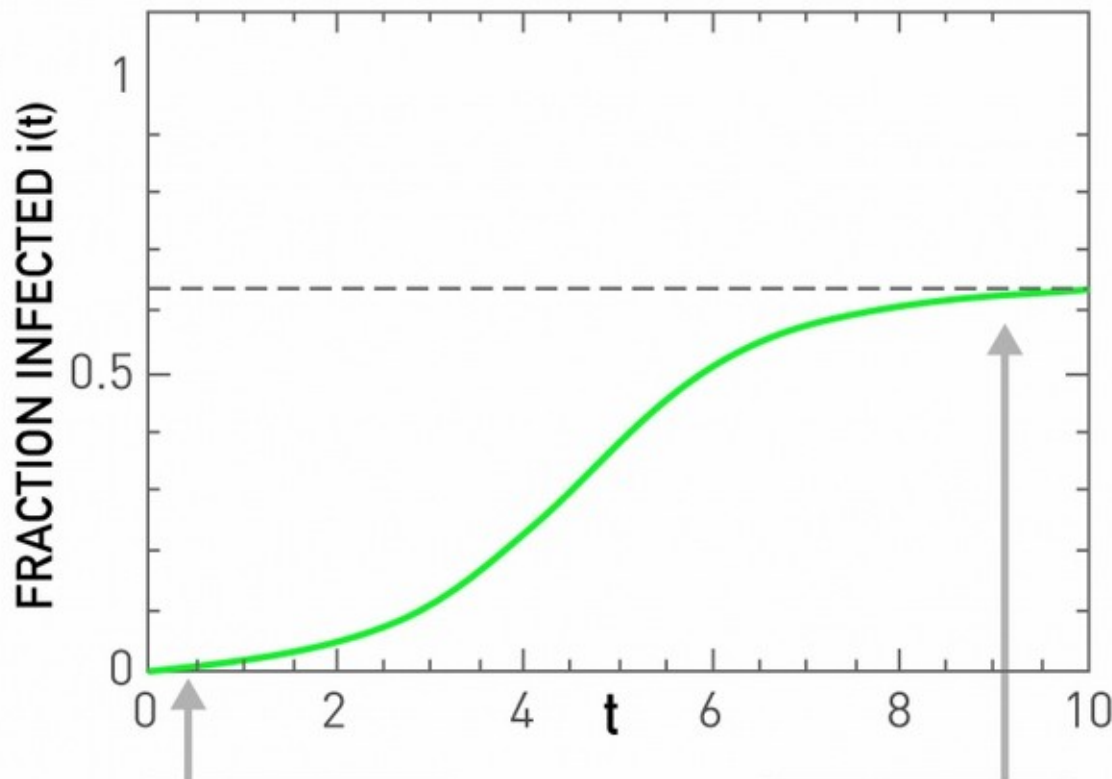
- $C$  is a constant that depends on  $i_0$

# Infected as a function of time (SIS)

$$i(t) = \left(1 - \frac{\mu}{\beta \langle k \rangle}\right) \frac{C e^{(\beta \langle k \rangle - \mu)t}}{1 + C e^{(\beta \langle k \rangle - \mu)t}}$$

This is in the case  $\mu < \beta \langle k \rangle$

In the case  $\mu > \beta \langle k \rangle$  the infection dies out



exponential  
outbreak

If  $i$  is small,  
 $i \approx i_0 e^{(\beta \langle k \rangle - \mu)t}$

endemic  
state

$i(\infty) = 1 - \frac{\mu}{\beta \langle k \rangle}$

# The SIR model





# The SIR model



- Susceptible:
  - The node can catch the disease
- Infected:
  - The node has the disease and can spread it
- Removed:
  - The node no longer has the disease, and cannot catch it or propagate it again (could be dead, could be immune)

# Infection dynamics in SIR

$$\frac{di(t)}{dt} = \beta \langle k \rangle i(t)(1 - r(t) - i(t)) - \mu i(t)$$

$$\frac{dr(t)}{dt} = \mu i(t)$$

$$\frac{ds(t)}{dt} = -\frac{di(t)}{dt} - \frac{dr(t)}{dt} = -\beta \langle k \rangle i(t)(1 - r(t) - i(t))$$

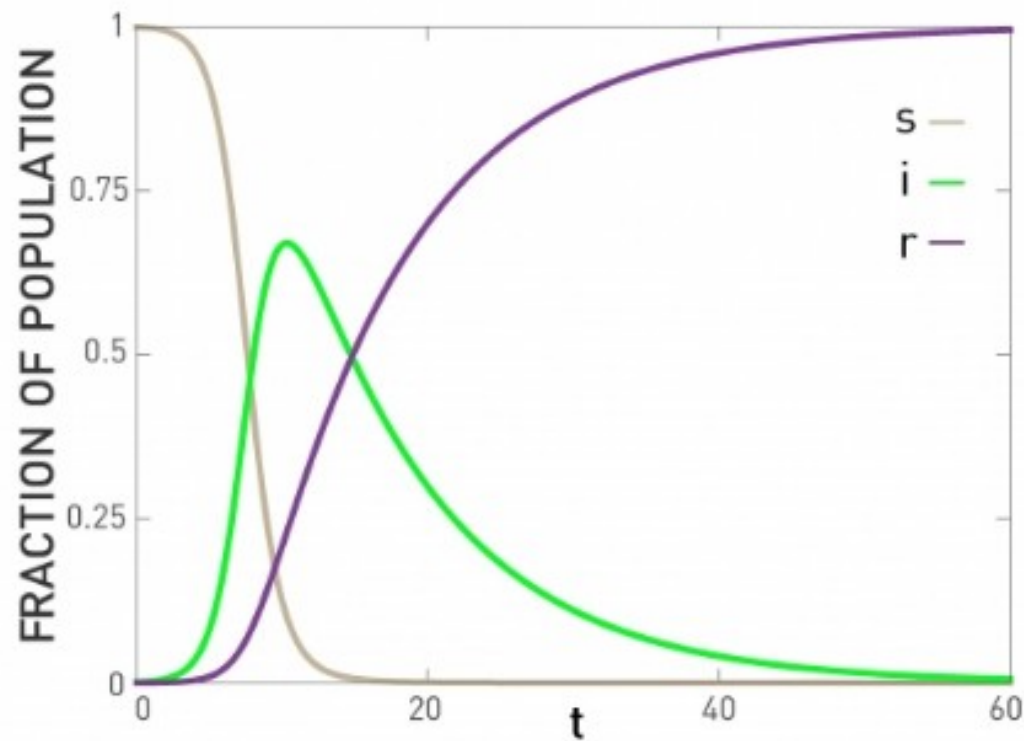
- No closed form solution

# Infection dynamics (SIR)

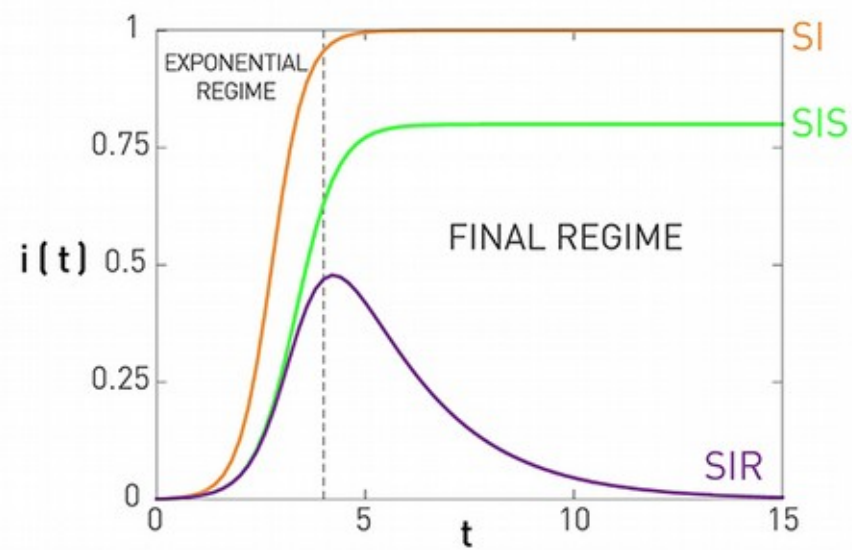
$$\frac{di(t)}{dt} = \beta \langle k \rangle i(t)(1 - r(t) - i(t)) - \mu i(t)$$

$$\frac{dr(t)}{dt} = \mu i(t)$$

$$\frac{ds(t)}{dt} = -\beta \langle k \rangle i(t)(1 - r(t) - i(t))$$



# Comparison of $i(t)$



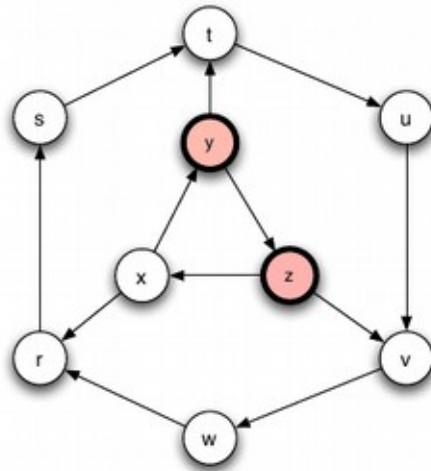
	SI	SIS	SIR
<b>Exponential Regime:</b> Number of infected individuals grows exponentially	$i = \frac{i_0 e^{\beta\langle k \rangle t}}{1 - i_0 + i_0 e^{\beta\langle k \rangle t}}$	$i = \left(1 - \frac{\mu}{\beta\langle k \rangle}\right) \frac{C e^{(\beta\langle k \rangle - \mu)t}}{1 + C e^{(\beta\langle k \rangle - \mu)t}}$	No closed solution
<b>Final Regime:</b> Saturation at $t \rightarrow \infty$	$i(\infty) = 1$	$i(\infty) = 1 - \frac{\mu}{\beta\langle k \rangle}$	$i(\infty) = 0$
<b>Epidemic Threshold:</b> Disease does not always spread	No threshold	$R_0 = 1$	$R_0 = 1$

SI / SIS / SIR on a graph

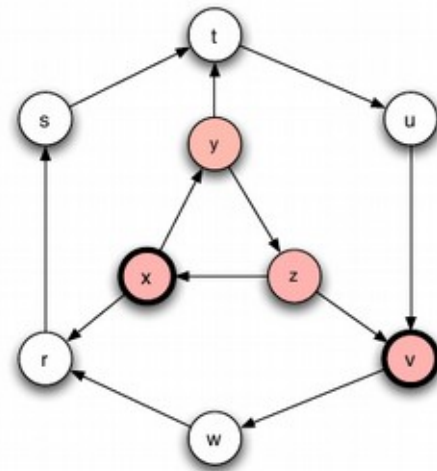


# SIR on a graph

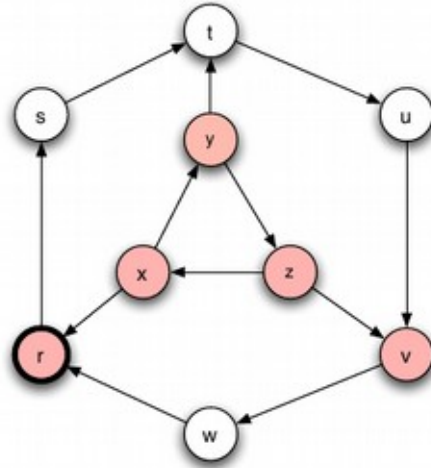
- In this simulation we assume recovery takes one timestep
- Infected nodes have thick borders
- Recovered nodes have thin borders



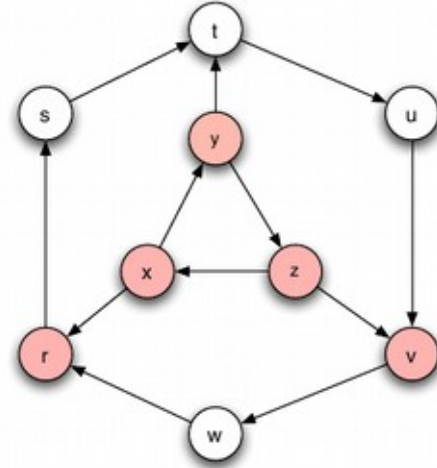
(a)



(b)



(c)



(d)

# SI dynamics on a graph



- Degree block approximation: all nodes with the same degree are jointly analyzed

$$i_k(t) = \frac{I_k(t)}{N_k}$$

$$i(t) = \sum_k i_k(t) p_k$$

$$\frac{di_k(t)}{dt} = k(1 - i_k(t))\Theta_k\beta$$

for every infected,

iterate through its susceptible neighbors,

infect with probability  $\beta$

$\Theta_k$  is the fraction of infected nodes of a susceptible node of degree  $k$

# SI model on a graph: infected as a function of time

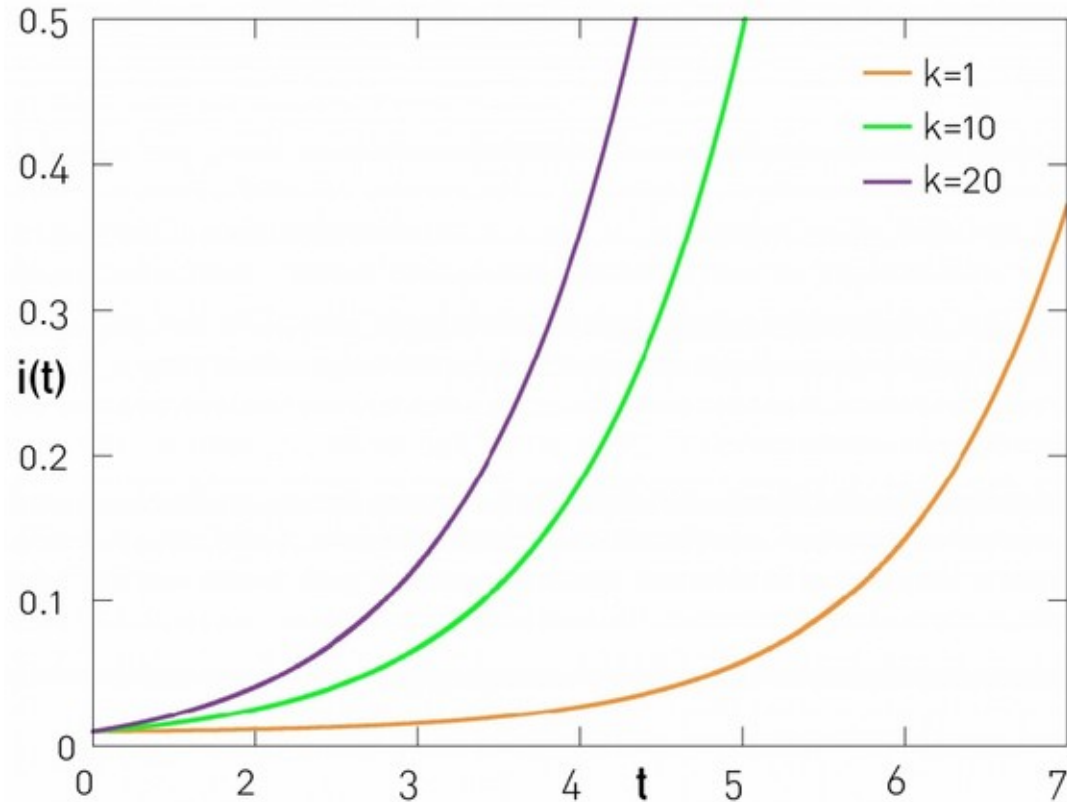
$$i_k(t) \approx i_0 \left( 1 + k \frac{\langle k \rangle - 1}{\langle k^2 \rangle - \langle k \rangle} \left( e^{t/\tau^{SI}} - 1 \right) \right)$$

What can you say about  $i_k(t)$ ?

$$\tau^{SI} = \frac{\langle k \rangle}{\beta (\langle k^2 \rangle - \langle k \rangle)}$$

Characteristic time, i.e., the time to infect  $1/e \approx 36\%$  of nodes

Higher degree nodes are more likely to become infected



$$i_k(t) = i_0 \left( 1 + \frac{k (\langle k \rangle - 1)}{\langle k^2 \rangle - \langle k \rangle} \left( e^{t/\tau^{SI}} - 1 \right) \right)$$

# Characteristic time $\tau^{SI} = \frac{\langle k \rangle}{\beta (\langle k^2 \rangle - \langle k \rangle)}$

- Random network

$$\langle k^2 \rangle = \langle k \rangle (\langle k \rangle + 1) \Rightarrow \tau_{ER}^{SI} = \frac{1}{\beta \langle k \rangle}$$

- Scale-free network with  $\gamma \geq 3$

$$\langle k \rangle, \langle k^2 \rangle \text{ are finite} \Rightarrow \tau^{SI} \text{ is finite}$$

- Scale-free network with  $\gamma < 3$

$$\langle k^2 \rangle \xrightarrow{N \rightarrow \infty} \infty \Rightarrow \lim_{N \rightarrow \infty} \tau^{SI} = 0$$

# Vanishing characteristic time

$$\tau^{SI} = \frac{\langle k \rangle}{\beta (\langle k^2 \rangle - \langle k \rangle)}$$

- If  $\lim_{N \rightarrow \infty} \frac{\langle k \rangle}{\langle k^2 \rangle} = 0$  the characteristic time goes to 0
- Networks with skewed degree distributions allow infections with the same  $\beta$  to spread faster

# SIS dynamics on a graph



Similar to SI dynamics but allowing recovery

$$\frac{di_k(t)}{dt} = k(1 - i_k(t))\Theta_k\beta - \mu i_k(t) \quad \tau^{SIS} = \frac{\langle k \rangle}{\beta \langle k^2 \rangle - \mu \langle k \rangle}$$

If people recover quickly,  $\tau < 0$  and the infection dies out

# Epidemic threshold

- A key quantity is the spreading rate  $\lambda = \frac{\beta}{\mu}$
- The critical spreading rate  $\lambda_c$  called the epidemic threshold, is such that  $\tau > 0$

Compute the epidemic threshold for an ER graph where  $\langle k^2 \rangle = \langle k \rangle (\langle k \rangle + 1)$

$$\tau^{SIS} = \frac{\langle k \rangle}{\beta \langle k^2 \rangle - \mu \langle k \rangle}$$



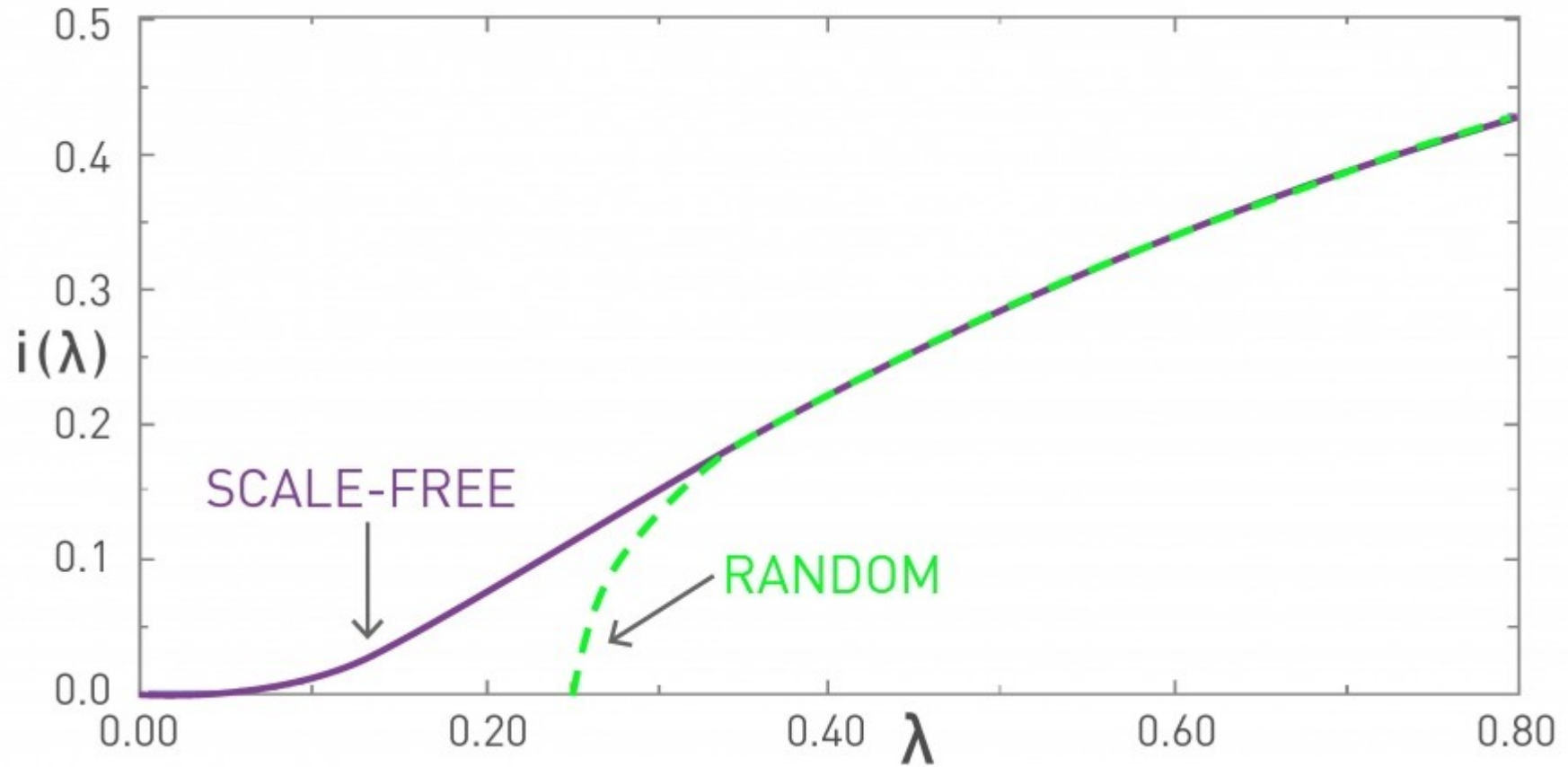
# Epidemic threshold in a scale-free network

$$\tau^{SIS} = \frac{\langle k \rangle}{\beta \langle k^2 \rangle - \mu \langle k \rangle} > 0 \Rightarrow \frac{\beta}{\mu} > \frac{\langle k \rangle}{\langle k^2 \rangle} = \lambda_c$$

- In a scale-free network with  $\gamma < 3$

$$\langle k^2 \rangle \xrightarrow{N \rightarrow \infty} \infty \Rightarrow \lim_{N \rightarrow \infty} \tau^{SIS} = 0$$

# Infected (in the limit) as a function of the epidemic threshold



# Two key results

In a large scale-free network with  $\gamma < 3$

- An infection may reach everybody in a very short time:  $\tau = 0$
- An infection may become endemic even if it is not very contagious and even if people recover fast:  $\lambda_c = 0$