

Epidemic Analysis on Graphs

Introduction to Network Science

Carlos Castillo

Topic 27



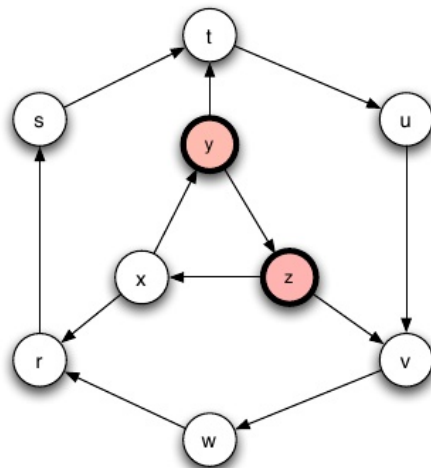
Universitat
Pompeu Fabra
Barcelona

Sources

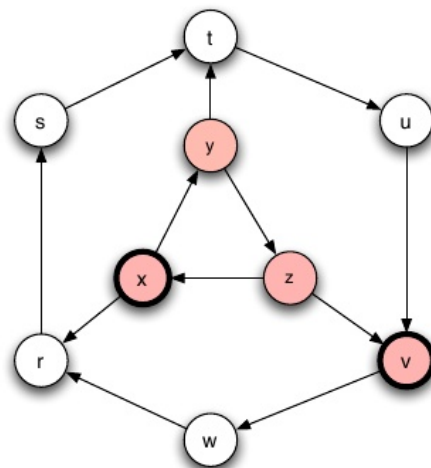
- Barabási (2016): [Network Science Ch. 10](#)
- Easley and Kleinberg (2010): Networks, Crowds, and Markets [Ch 21](#).

SIR on a graph

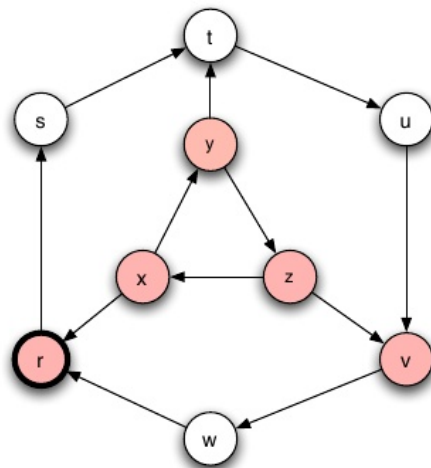
- In this simulation we assume recovery takes one timestep
- Infected nodes have thick borders
- Recovered nodes have thin borders



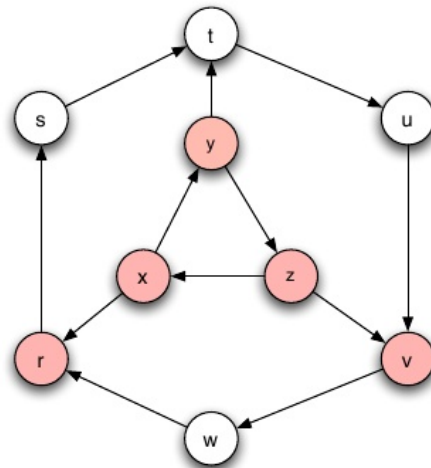
(a)



(b)

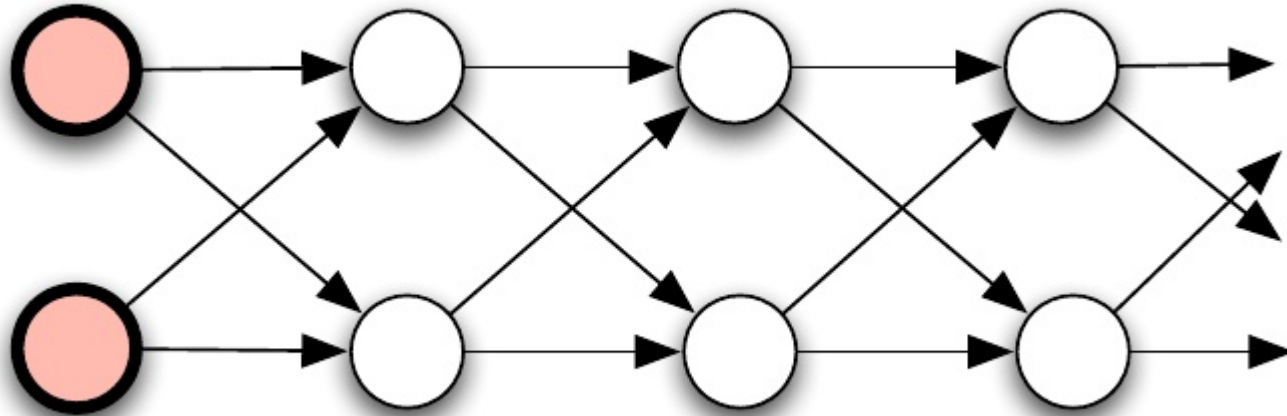


(c)



(d)

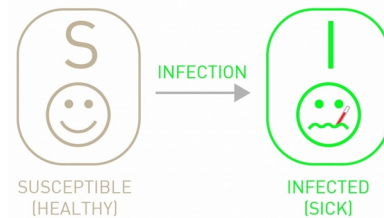
Social distancing



In this network, the epidemic is forced to pass through a narrow “channel” of nodes. In such a structure, even a highly contagious disease will tend to die out relatively quickly (fig 21.3 Easley+Kleinberg)

Computing infection dynamics in known graph models

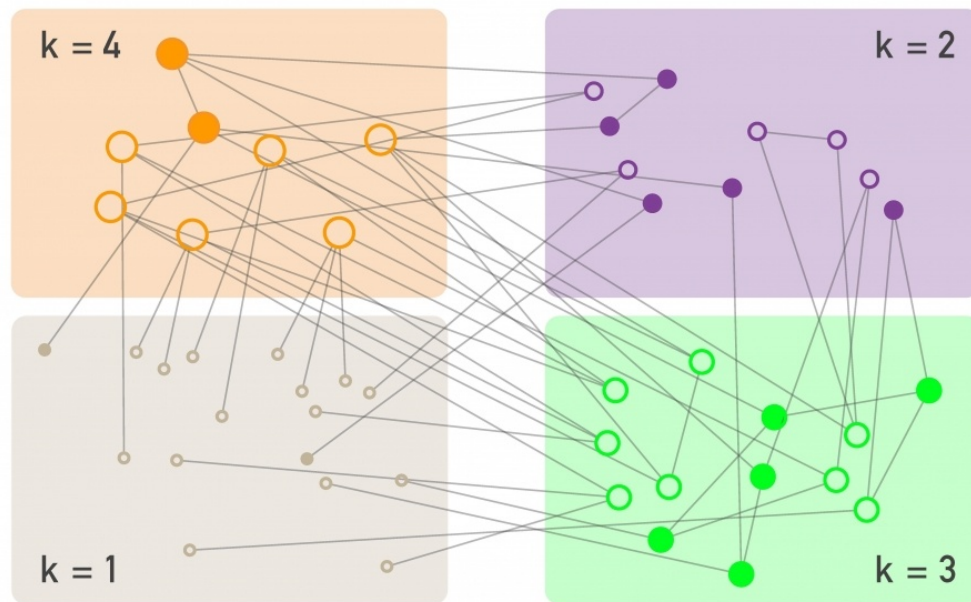
SI dynamics on a graph



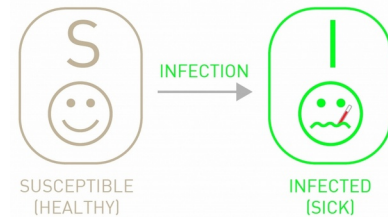
- Degree block approximation: all nodes with the same degree are jointly analyzed

$$i_k(t) = \frac{I_k(t)}{N_k}$$

$$i(t) = \sum_k i_k(t) p_k$$



SI dynamics on a graph



- Degree block approximation: all nodes with the same degree are jointly analyzed

$$\frac{di_k(t)}{dt} = \Theta_k k (1 - i_k(t)) \beta$$

Similar to simple SI model, except:

Θ_k is the fraction of infected neighbors of a susceptible node of degree k

Compare with simple SI model:

$$\frac{di(t)}{dt} = i(t) \langle k \rangle (1 - i(t)) \beta$$

SI model on a graph: infected as a function of time

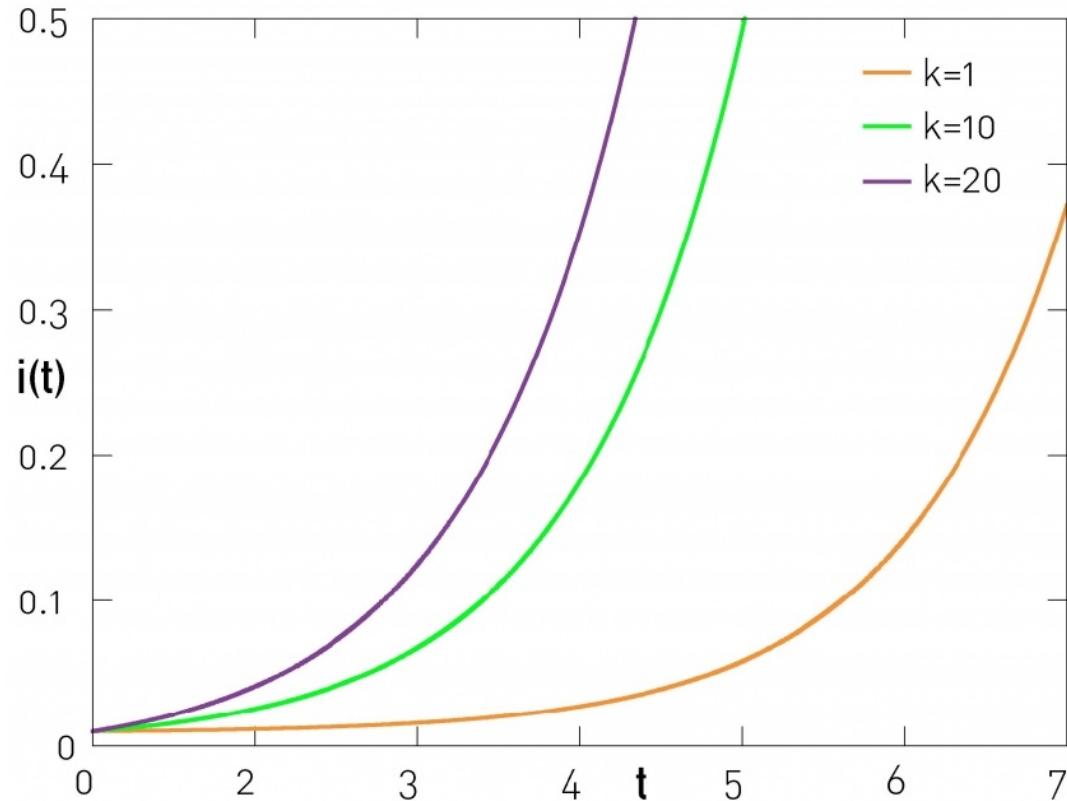
$$i_k(t) \approx i_0 \left(1 + k \frac{\langle k \rangle - 1}{\langle k^2 \rangle - \langle k \rangle} \left(e^{t/\tau^{SI}} - 1 \right) \right)$$

What can you say about $i_k(t)$?

$$\tau^{SI} = \frac{\langle k \rangle}{\beta (\langle k^2 \rangle - \langle k \rangle)}$$

Characteristic time, i.e., the time to infect $1/e \simeq 36\%$ of nodes

Higher degree nodes are more likely to become infected



$$i_k(t) = i_0 \left(1 + \frac{k (\langle k \rangle - 1)}{\langle k^2 \rangle - \langle k \rangle} \left(e^{t/\tau^{SI}} - 1 \right) \right)$$

Characteristic time

(time to infect $1/e \simeq 36\%$ of nodes)

$$\tau^{SI} = \frac{\langle k \rangle}{\beta (\langle k^2 \rangle - \langle k \rangle)}$$

- Random network

$$\langle k^2 \rangle = \langle k \rangle (\langle k \rangle + 1) \Rightarrow$$

- Scale-free network with $\gamma \geq 3$

$$\langle k \rangle, \langle k^2 \rangle \text{ are finite} \Rightarrow$$

- Scale-free network with $\gamma < 3$

$$\langle k^2 \rangle \xrightarrow{N \rightarrow \infty} \infty \Rightarrow$$

Characteristic time

(time to infect $1/e \simeq 36\%$ of nodes)

$$\tau^{SI} = \frac{\langle k \rangle}{\beta (\langle k^2 \rangle - \langle k \rangle)}$$

- Random network

$$\langle k^2 \rangle = \langle k \rangle (\langle k \rangle + 1) \Rightarrow \tau_{ER}^{SI} = \frac{1}{\beta \langle k \rangle}$$

- Scale-free network with $\gamma \geq 3$

$$\langle k \rangle, \langle k^2 \rangle \text{ are finite} \Rightarrow \tau_{BA}^{SI} \text{ is finite}$$

- Scale-free network with $\gamma < 3$

$$\langle k^2 \rangle \xrightarrow{N \rightarrow \infty} \infty \Rightarrow \lim_{N \rightarrow \infty} \tau_{BA}^{SI} = 0$$

Vanishing characteristic time

$$\tau^{SI} = \frac{\langle k \rangle}{\beta (\langle k^2 \rangle - \langle k \rangle)}$$

- If $\lim_{N \rightarrow \infty} \frac{\langle k \rangle}{\langle k^2 \rangle} = 0$ the characteristic time goes to 0
- Networks with skewed degree distributions allow infections with the same β to spread faster

SIS dynamics on a graph



Similar to SI dynamics but allowing recovery

$$\frac{di_k(t)}{dt} = k(1 - i_k(t))\Theta_k\beta - \mu i_k(t) \quad \tau^{SIS} = \frac{\langle k \rangle}{\beta \langle k^2 \rangle - \mu \langle k \rangle}$$

If people recover quickly, $\tau < 0$ and the infection dies out ... unless $\langle k^2 \rangle \rightarrow \infty$

Epidemic threshold

- A key quantity is the spreading rate $\lambda = \frac{\beta}{\mu}$
- The critical spreading rate λ_c called the epidemic threshold, is such that $\tau > 0$
- Spreading rate larger than epidemic threshold:
 - infection becomes endemic
- Spreading rate smaller than epidemic threshold:
 - Infection dies out

Epidemic threshold

- A key quantity is the spreading rate $\lambda = \frac{\beta}{\mu}$
- The critical spreading rate λ_c called the epidemic threshold, is such that $\tau > 0$

Compute the epidemic threshold β/μ for an ER graph where $\langle k^2 \rangle = \langle k \rangle (\langle k \rangle + 1)$

Upload to Nearpod Collaborate
<https://nearpod.com/student/>
Code to be given during class

Begin from: $\tau^{SIS} = \frac{\langle k \rangle}{\beta \langle k^2 \rangle - \mu \langle k \rangle}$ and use that $\tau^{SIS} > 0$

Epidemic threshold in a scale-free network

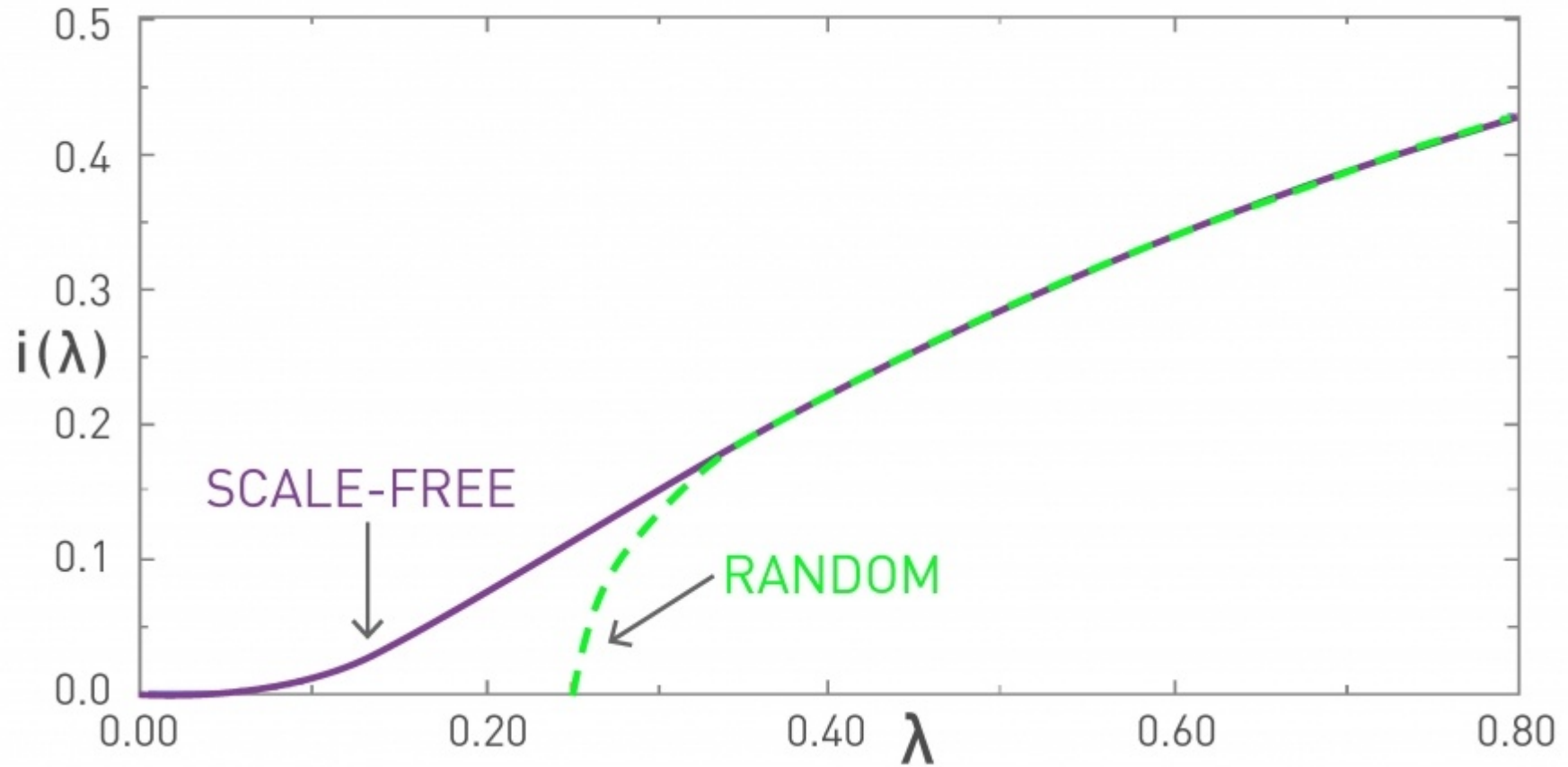
$$\tau^{SIS} = \frac{\langle k \rangle}{\beta \langle k^2 \rangle - \mu \langle k \rangle} > 0 \Rightarrow \frac{\beta}{\mu} > \frac{\langle k \rangle}{\langle k^2 \rangle} = \lambda_c$$

Epidemic threshold in BA graph

- In a scale-free network with $\gamma < 3$

$$\langle k^2 \rangle \xrightarrow{N \rightarrow \infty} \infty \Rightarrow \lim_{N \rightarrow \infty} \tau^{SIS} = 0$$

Infected (in the limit) as a function of the epidemic threshold



Two key results for SI and SIS models on a BA graph

In a large scale-free network with $\gamma < 3$

- An infection may reach everybody in a very short time: $\tau = 0$
- An infection may become endemic even if it is not very contagious and even if people recover fast: $\lambda_c = 0$

Summary

Things to remember

- Degree block approximation
- Epidemic thresholds in scale-free networks