Epidemics

Introduction to Network Science

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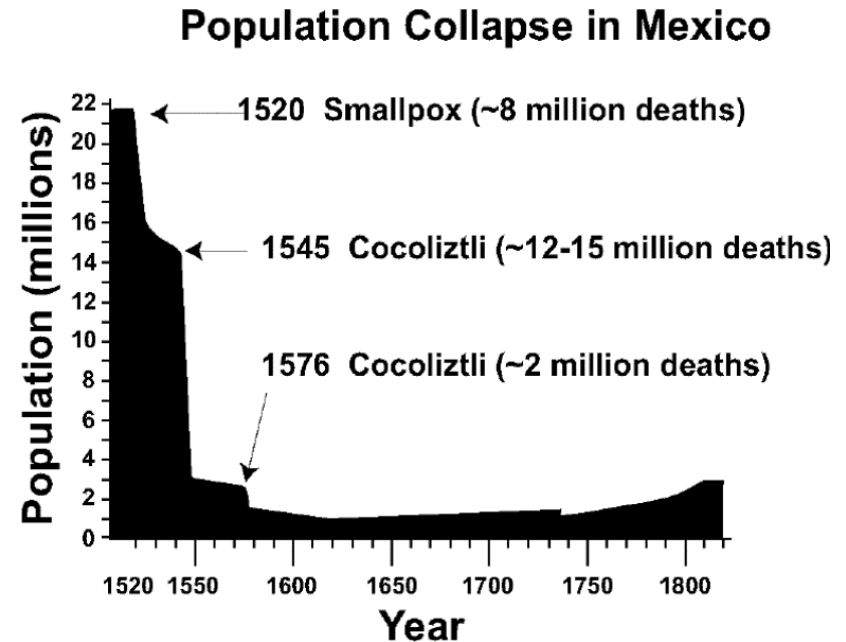
Topic 26

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Sources

- Barabási (2016): Network Science Ch. 10
- Easley and Kleinberg (2010): Networks, Crowds, and Markets Ch 21.

Examples: human epidemics

- Influenza, measles, STDs

- The "Black Death"
  [next slide]

- Smallpox and other diseases brought by Europeans to America since early 1500s

Population Collapse in Mexico

1520 Smallpox (~8 million deaths)

1545 Cocoliztli (~12-15 million deaths)

1576 Cocoliztli (~2 million deaths)

https://en.wikipedia.org/wiki/Cocoliztli_Epidemic_of_1545-1548

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The "Black Death" (Bubonic plague)

1300s

Killed 30%-60% of the total population of Europe

1346 1347 1348 1349 1350 1351 1352 1353

SARS Outbreak (2003)

- February 21st: Chinese doctor who have been several treating "atypical pneumonia" cases check-ins into hotel in Hong Kong
  - Hospitalized on Feb 22nd
  - Died on March 4th
- March 1st: "Ms. E. M." returns to Singapore after visiting Hong Kong
  - Graph depicts 144 out of the first 206 SARS patients in Singapore
  - Ms. E. M. lived, various of her family members died

https://en.wikipedia.org/wiki/Timeline_of_the_SARS_outbreak
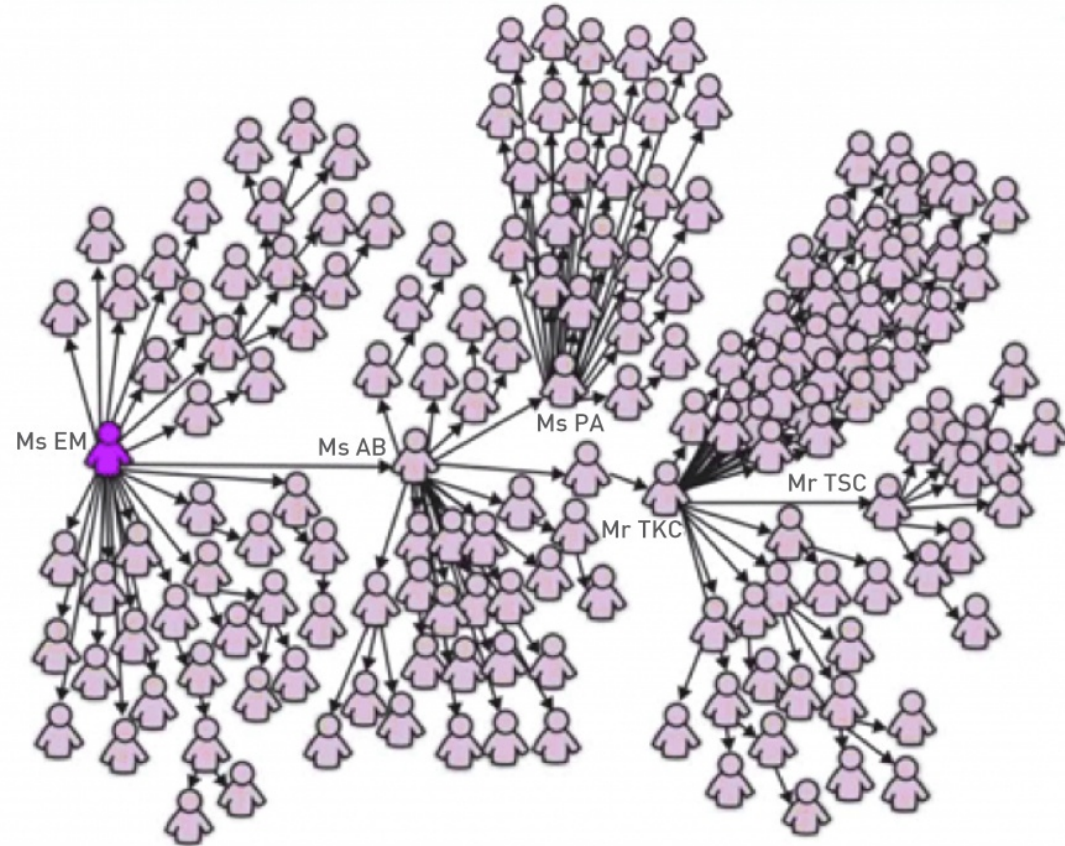
COVID-19

Why this curve?

How can one make this kind of forecast?
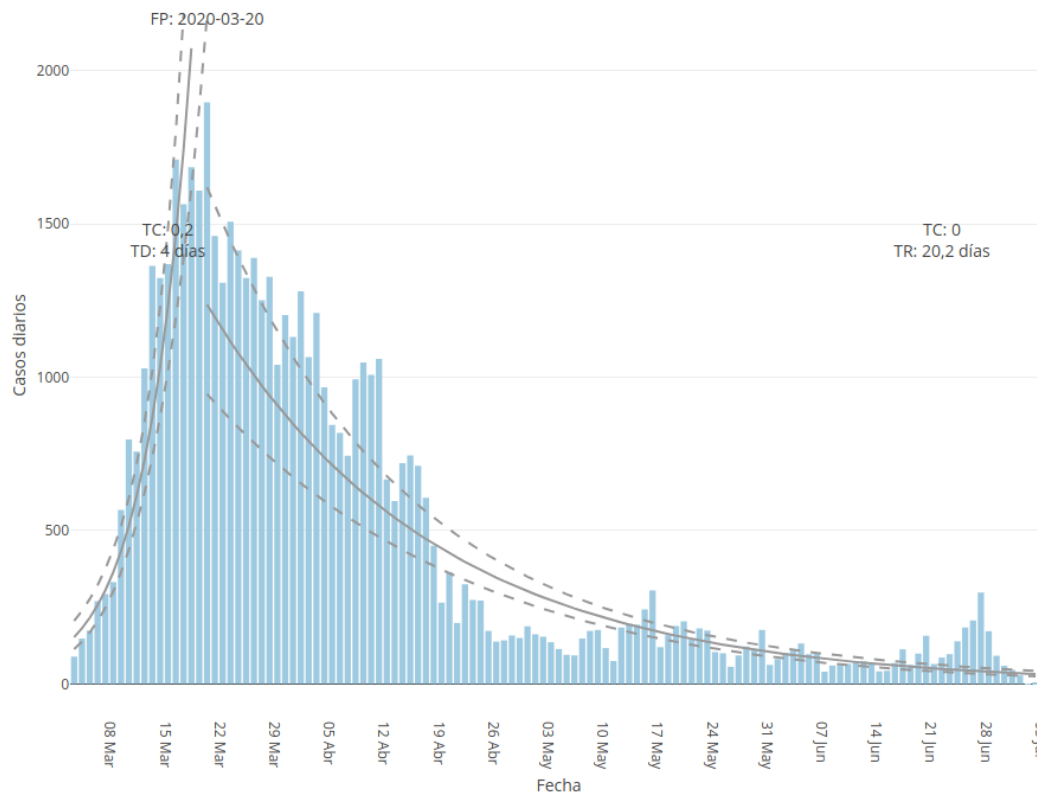
COVID-19 cases in Catalonia as of July 2020 - https://cnecovid.isciii.es/covid19/#ccaa

Diffusion of ideas vs diseases

- Adopting a new idea, behavior, fashion, product, taste, may also spread from person to person: "social contagion"

- There is a certain agency of the receiver

- In diffusion of diseases, we assume **there is no agency: each contagion is random**

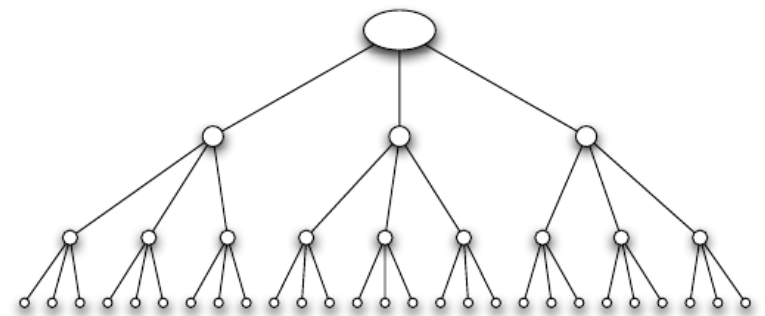# Simple model: branching process

Modeling epidemics

- There are many factors:
  - Contagiousness
  - Length of infectious period,
  - Severity
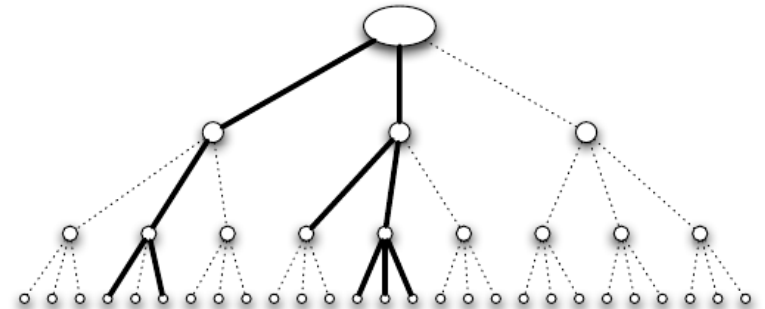 - ...
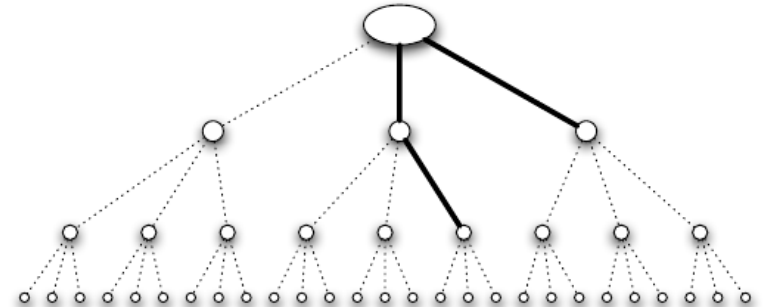- Structure of contacts in a population

Simple model: branching process

(a) *The contact network for a branching process*

(b) *With high contagion probability, the infection spreads widely*

(c) *With low contagion probability, the infection is likely to die out quickly*

- Each person interacts with other k people

- Each interaction ends in infection with probability β

Example: k=3

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Transmission rate or "Basic reproductive number" R_0

- Each person interacts with other k people
- Each interaction ends in infection with probability β

- What is the expected number of cases caused by a single individual, R_0?

- What do you think happens if $R_0 < 1$?

- What do you think happens if $R_0 > 1$?

Disease	Transmission	R_0
Measles	Airborne	12-18
Pertussis	Airborne droplet	12-17
Diptheria	Saliva	6-7
Smallpox	Social contact	5-7
Polio	Fecal-oral route	5-7
Rubella	Airborne droplet	5-7
Mumps	Airborne droplet	4-7
HIV/AIDS	Sexual contact	2-5
SARS	Airborne droplet	2-5
Influenza (1918 strain)	Airborne droplet	2-3

Changing $R_0 = \beta k$

- Sanitary practices (reduce β)
- Quarantine (reduces k)
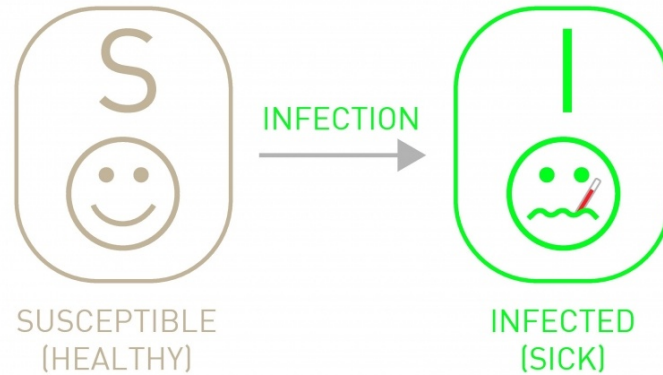
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The SI model

The SI model

SUSCEPTIBLE (HEALTHY) → INFECTION → INFECTED (SICK)

- Susceptible:
  - The node can catch the disease

- Infected:
  - The node has the disease and can spread it
 - It will stay sick forever
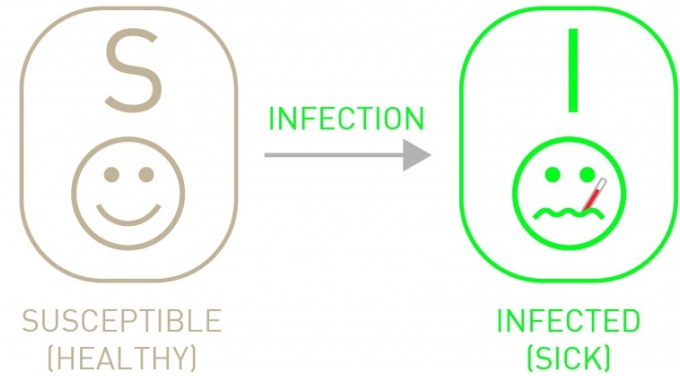
Notation

- Number of susceptible S(t)
  - Fraction of susceptible $s(t) = S(t) / N$
- Number of infected I(t)
  - Fraction of infected $i(t) = I(t) / N$
- $s(t) + i(t) = 1$

How many susceptible neighbors a node has?

$$\langle k \rangle \, \frac{S(t)}{N} = \langle k \rangle \, s(t)$$

How many new infections are produced?
(for every infected, iterate through its susceptible neighbors, infect with probability β)

$$i(t) \, \langle k \rangle \, s(t) \beta$$

infected   susceptible neighbors   infection probability
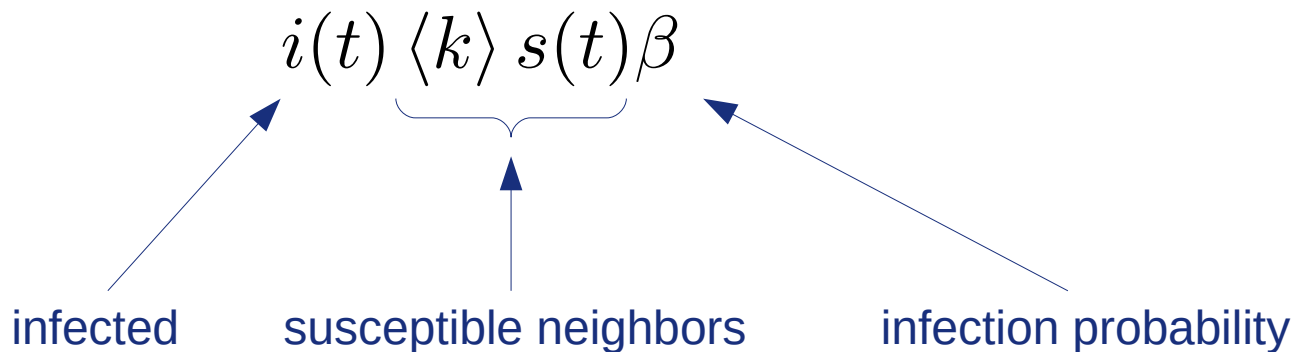
Prove that $i(t) = \dfrac{i_0 e^{\beta \langle k \rangle t}}{1 - i_0 + i_0 e^{\beta \langle k \rangle t}}$

Begin from: $\dfrac{di(t)}{dt} = i(t) \langle k \rangle (1 - i(t)) \beta$

First, place all terms with $i(t)$ on the left side

Second, use $\dfrac{1}{x \cdot (1 - x)} = \dfrac{1}{x} + \dfrac{1}{1 - x}$

Third, integrate from $t = 0$ to t and denote by $i_0 = i(t = 0)$

$$\int \frac{1}{x} dx = \log x + C \qquad \int \frac{1}{1 - x} dx = -\log(1 - x) + C$$
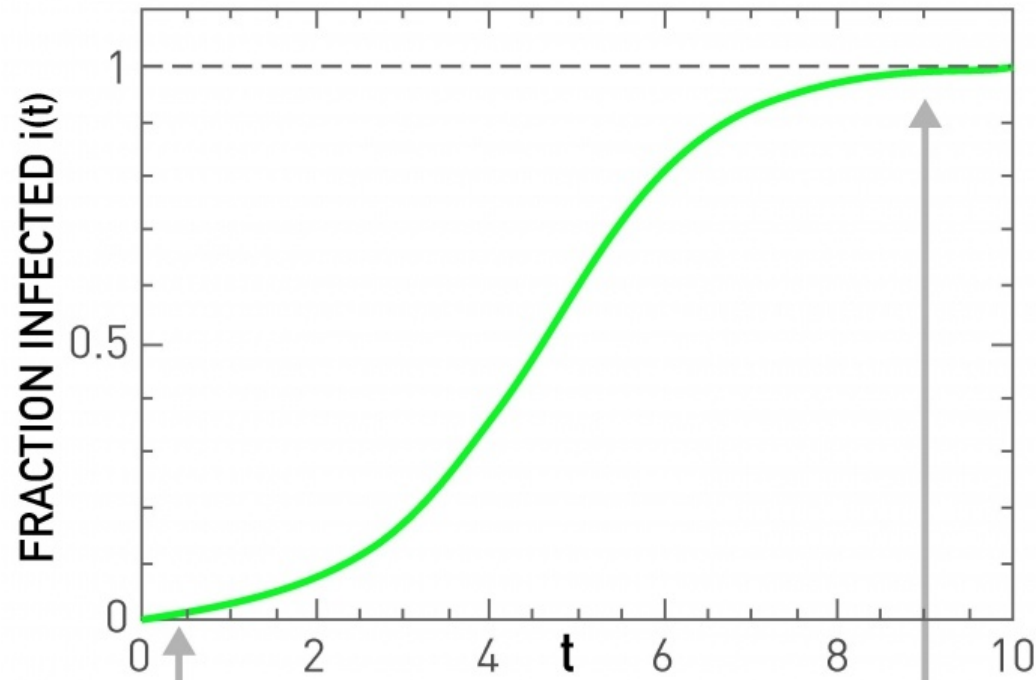
Behavior in the limit $t \rightarrow \infty$

- What is the limit of $i(t) = \dfrac{i_0 e^{\beta \langle k \rangle t}}{1 - i_0 + i_0 e^{\beta \langle k \rangle t}}$ when $t \rightarrow \infty$?

- Hint: similar to $f(t) = \dfrac{e^t}{1 + e^t}$

Infected as a function of time (SI)

$$i(t) = \frac{i_0 e^{\beta \langle k \rangle t}}{1 - i_0 + i_0 e^{\beta \langle k \rangle t}}$$

Characteristic time
(to infect 1/e ≃ 36% of people):

$$\tau = \frac{1}{\beta \langle k \rangle}$$

exponential regime

If i is small,

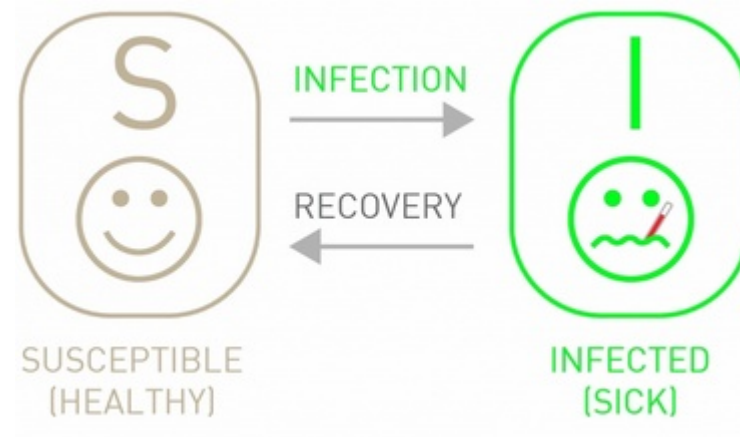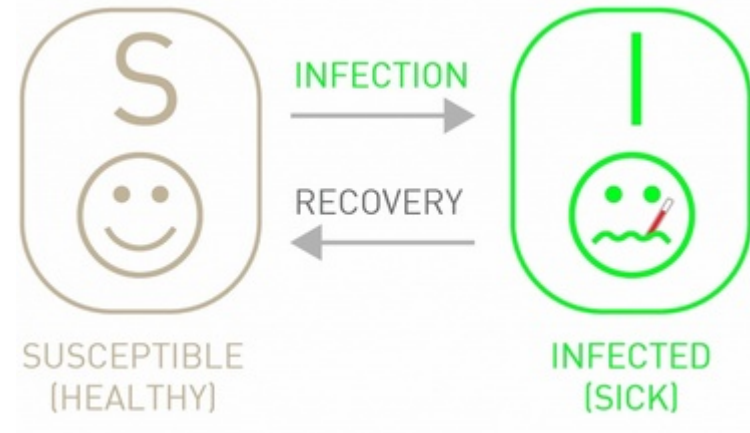$i \approx i_0 e^{\beta \langle k \rangle t}$

saturation regime

If $i \to 1$,

$\dfrac{di}{dt} \to 0$

The SIS model

The SIS model

- # Susceptible:
  - ## The node can catch the disease

- # Infected:
  - ## The node has the disease and can spread it
  - ## After some time, it recovers ... but it becomes susceptible again

Infection dynamics

$$\frac{di(t)}{dt} = \beta \langle k \rangle \, i(t)(1 - i(t)) - \mu i(t)$$

- μ is the recovery rate, i.e., the probability of becoming susceptible again in an unit of time

$$i(t) = \left(1 - \frac{\mu}{\beta \langle k \rangle} \right) \frac{Ce^{(\beta \langle k \rangle - \mu)t}}{1 + Ce^{(\beta \langle k \rangle - \mu)t}}$$

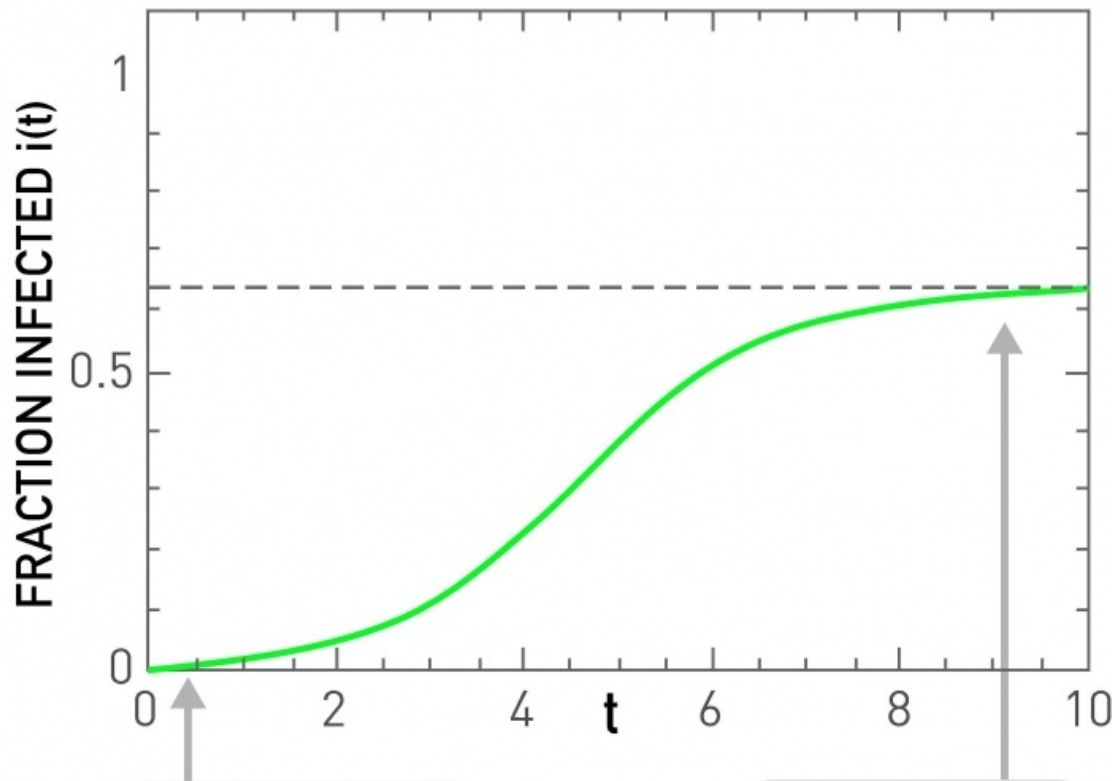- C is a constant that depends on i_0

Behavior in the limit $t \to \infty$

- What is the limit of $i(t) = \left(1 - \dfrac{\mu}{\beta \langle k \rangle} \right) \dfrac{Ce^{(\beta \langle k \rangle - \mu)t}}{1 + Ce^{(\beta \langle k \rangle - \mu)t}}$ when $t \to \infty$?

- Hint: similar to $f(t) = \alpha \dfrac{e^t}{1 + e^t}$

Infected as a function of time (SIS)

$$i(t) = \left(1 - \frac{\mu}{\beta\langle k\rangle}\right)\frac{Ce^{(\beta\langle k\rangle-\mu)t}}{1+Ce^{(\beta\langle k\rangle-\mu)t}}$$

This is in the case $\mu < \beta\langle k\rangle$

FRACTION INFECTED i(t)

exponential outbreak

If i is small,

$i \approx i_0 e^{(\beta\langle k\rangle t - \mu)t}$

endemic state

$i(\infty) = 1 - \dfrac{\mu}{\beta\langle k\rangle}$

What happens if $\mu > \beta \langle k \rangle$?

- Remember: $\dfrac{di(t)}{dt} = \beta \langle k \rangle i(t)(1 - i(t)) - \mu i(t)$

The SIR model

The SIR model

SUSCEPTIBLE (HEALTHY) → INFECTION → INFECTED (SICK) → REMOVAL → REMOVED (IMMUNE/DEAD)

- Susceptible:
  - The node can catch the disease

- Infected:
  - The node has the disease and can spread it

- Removed:
  - The node no longer has the disease, and cannot catch it or propagate it again (could be dead, could be immune)

Infection dynamics in SIR

$$\frac{di(t)}{dt} = \beta \langle k \rangle i(t)(1 - r(t) - i(t)) - \mu i(t)$$

$$\frac{dr(t)}{dt} = \mu i(t)$$

$$\frac{ds(t)}{dt} = -\frac{di(t)}{dt} - \frac{dr(t)}{dt} = -\beta \langle k \rangle i(t)(1 - r(t) - i(t))$$
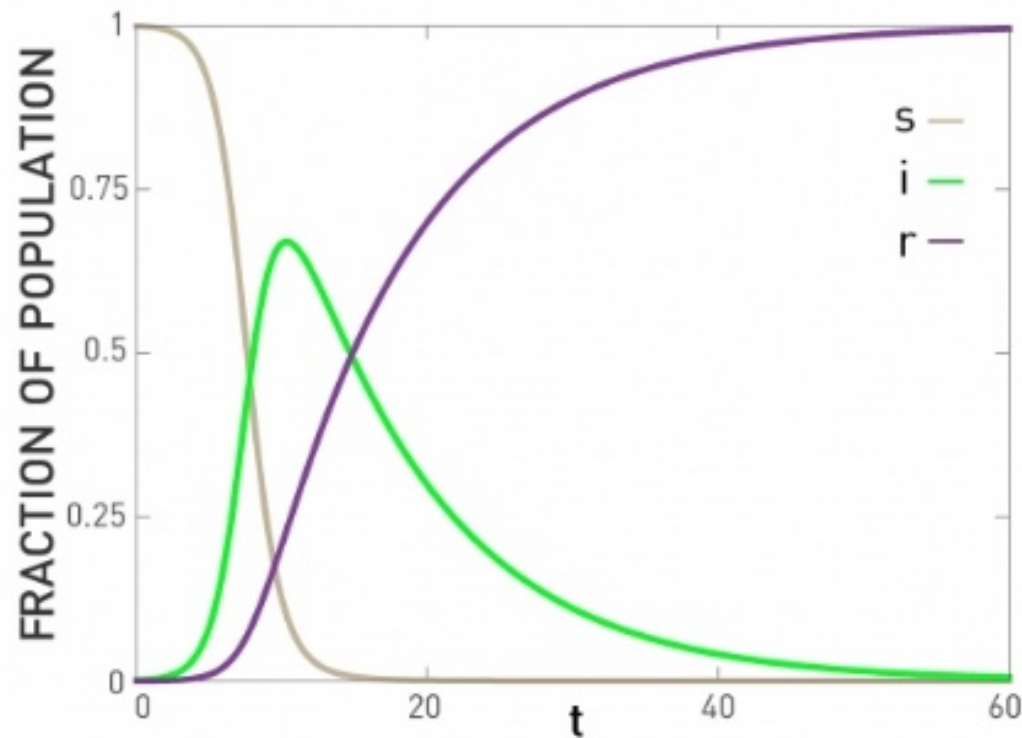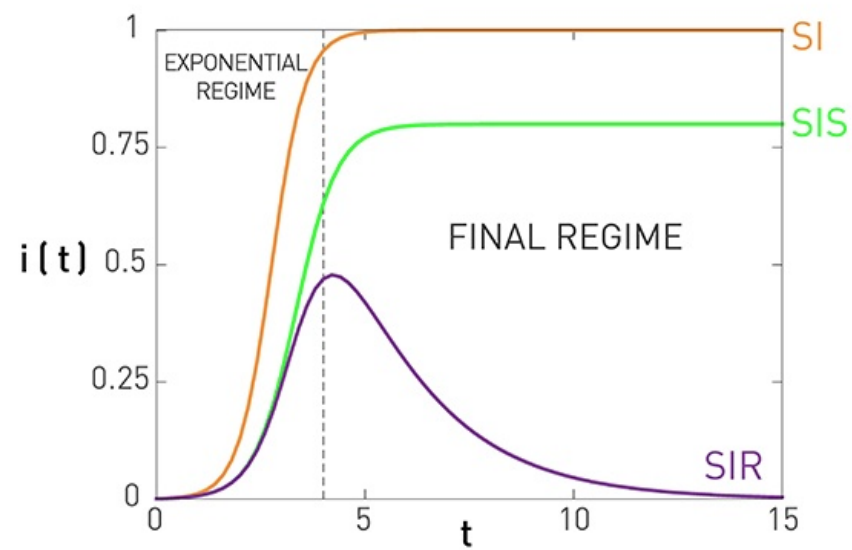
- No closed form solution

Infection dynamics (SIR)

$$\frac{di(t)}{dt} = \beta \langle k \rangle \, i(t)(1 - r(t) - i(t)) - \mu i(t)$$

$$\frac{dr(t)}{dt} = \mu i(t)$$

$$\frac{ds(t)}{dt} = -\beta \langle k \rangle \, i(t)(1 - r(t) - i(t))$$

Comparison of *i(t)*

	SI	SIS	SIR
Exponential Regime: Number of infected individuals grows exponentially	$i = \dfrac{i_0 e^{\beta\langle k\rangle t}}{1-i_0+i_0 e^{\beta\langle k\rangle t}}$	$i = \left(1-\dfrac{\mu}{\beta\langle k\rangle}\right)\dfrac{Ce^{(\beta\langle k\rangle-\mu)t}}{1+Ce^{(\beta\langle k\rangle-\mu)t}}$	No closed solution
Final Regime: Saturation at t→ =∞	$i(\infty)=1$	$i(\infty)=1-\dfrac{\mu}{\beta\langle k\rangle}$	$i(\infty)=0$
Epidemic Threshold: Disease does not always spread	No threshold	$R_0=1$	$R_0=1$

Things to remember

- SI, SIS, SIR models

- Which are the states in each process and which are the possible transitions

- Equations for number of nodes in each state

- Regimes under different parameters

- Practice executing by hand and write code if it helps you remember better each process

Practice on your own

Under the **SIS** model, $i(t) = \left(1 - \dfrac{\mu}{\beta \langle k \rangle}\right) \dfrac{Ce^{(\beta\langle k \rangle - \mu)t}}{1 + Ce^{(\beta\langle k \rangle - \mu)t}}$

1. When $\mu < \beta \langle k \rangle$ what is the limit of $i(t)$?

2. How is this state called?

3. What happens when $\mu > \beta \langle k \rangle$?

4. What conditions lead to large values of $i(t)$?

Practice on your own (cont.)

- In the **SIRS** epidemic model, there are three possible states for a node: susceptible, infected, and recovered. Susceptible nodes can become infected, infected nodes can become recovered, and *recovered nodes can become susceptible again.*

- During one unit of time, with probability β an infected node can infect one of its contacts, with probability μ, an infected node can recover, and with probability σ, a recovered node can become susceptible again.

- Let s(t) be the fraction of susceptible nodes, i(t) be the fraction of infected nodes, r(t) the fraction of recovered nodes, and <k> the average degree of the graph. Write the equations, simplifying them appropriately, for:

$$1.\frac{di(t)}{dt} \quad 2.\frac{dr(t)}{dt} \quad 3.\frac{ds(t)}{dt}$$

4. Is σ > μ sufficient to say that the recovered will tend to zero in the long run?