

# Graph theory basics

Introduction to Network Science

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Topic 04

# Contents

- Notation for graphs
- Degree distributions
- Adjacency matrices

# Sources

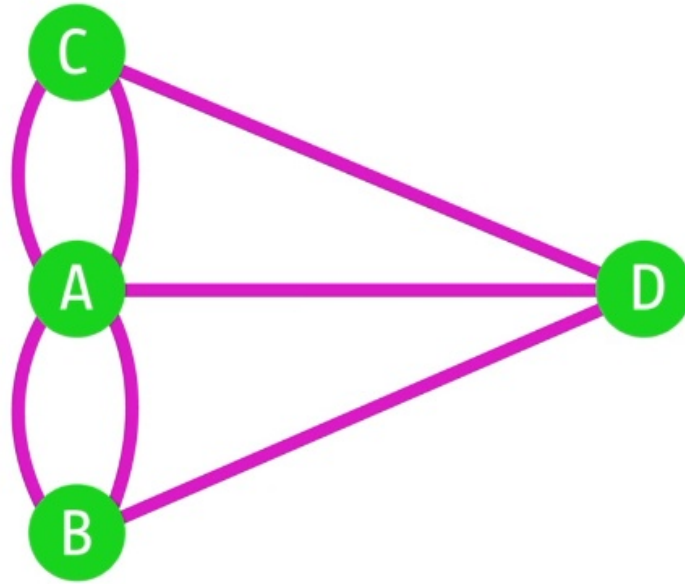
- Albert-László Barabási: Network Science. Cambridge University Press, 2016.
  - Follows almost section-by-section chapter 02
- URLs cited in the footer of specific slides

# The seven bridges of Königsberg



<http://networksciencebook.com/images/ch-02/video-2-1.m4v>

# Quick Question

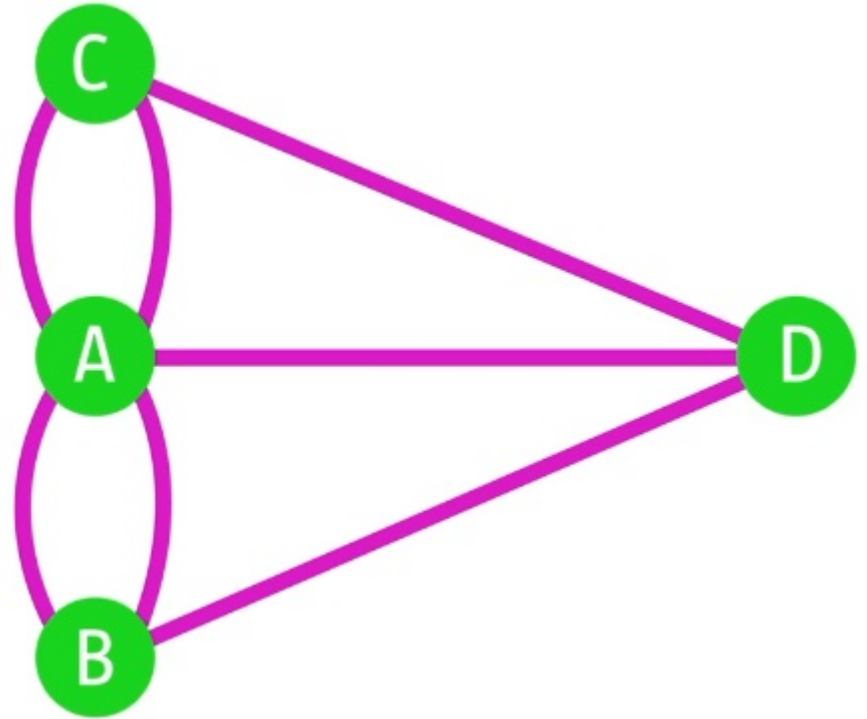


Can one walk across the 7 bridges without crossing the same bridge twice?

# Basic concepts

# Notation for a graph

- $G = (V, E)$ 
  - $V$ : nodes or vertices
  - $E$ : links or edges
- $|V| = N$  size of graph
- $|E| = L$  number of links



# Typical notation variations

- You may find that  $G$  is denoted by  $(N, A)$ , this is typical of directed graphs, means “*nodes, arcs*”
- You may find that
  - $|V|$  is denoted by  $n$  or  $N$
  - $|E|$  is denoted by  $m$ ,  $M$ , or  $L$



# Directed vs undirected graphs

- In an undirected graph
  - $E$  is a symmetric relation
$$(u, v) \in E \Rightarrow (v, u) \in E$$
- In a directed graph, also known as “digraph”
  - $E$  is not a symmetric relation
$$(u, v) \in E \not\Rightarrow (v, u) \in E$$

# Example graphs we will use

Network	$ V $	$ E $
Zachary's Karate Club (karate.gml)	34	78
Les Misérables (lesmiserables.gml)	77	254
E-mail exchanges (email-eu-core.csv)	868	25K
US companies ownership	1351	6721
Marvel comics (hero-network.csv)	6K	167K

# Degree

- Node  $i$  has degree  $k_i$ 
  - This is the number of links incident on this node
  - The total number of links  $L$  is given by 
$$L = \frac{1}{2} \sum_{i=1}^N k_i$$
- Average degree 
$$\langle k \rangle = \frac{1}{N} \sum_{i=1}^N k_i = \frac{2L}{N}$$

# In directed networks

- We distinguish **in-degree** from **out-degree**
  - Incoming and outgoing links, respectively
- Degree is the sum of both  $k_i = k_i^{\text{in}} + k_i^{\text{out}}$
- Counting total number of links:

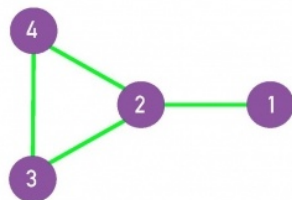
$$L = \sum_{i=1}^N k_i^{\text{in}} = \sum_{i=1}^N k_i^{\text{out}}$$

# Degree distribution

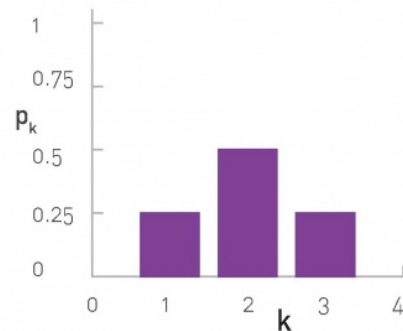
- If there are  $N_k$  nodes with degree  $k$
- The **degree distribution** is given by  $p_k = \frac{N_k}{N}$
- The average degree is then  $\langle k \rangle = \sum_{k=0}^{\infty} k p_k$

# Degree distribution; two toy graphs

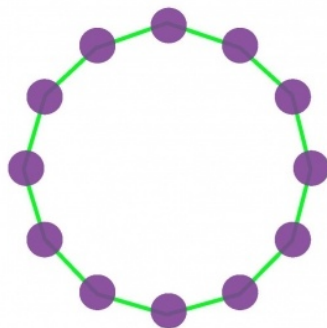
a.



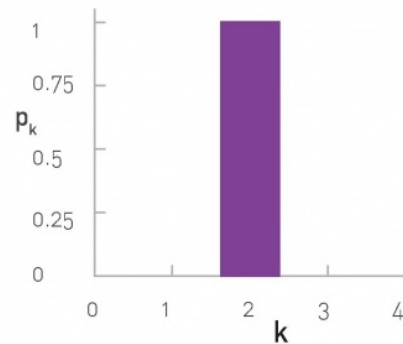
b.



c.

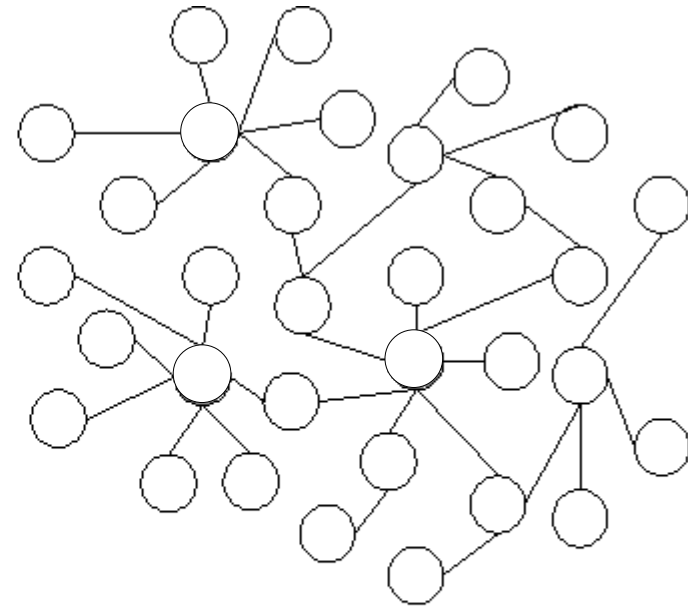
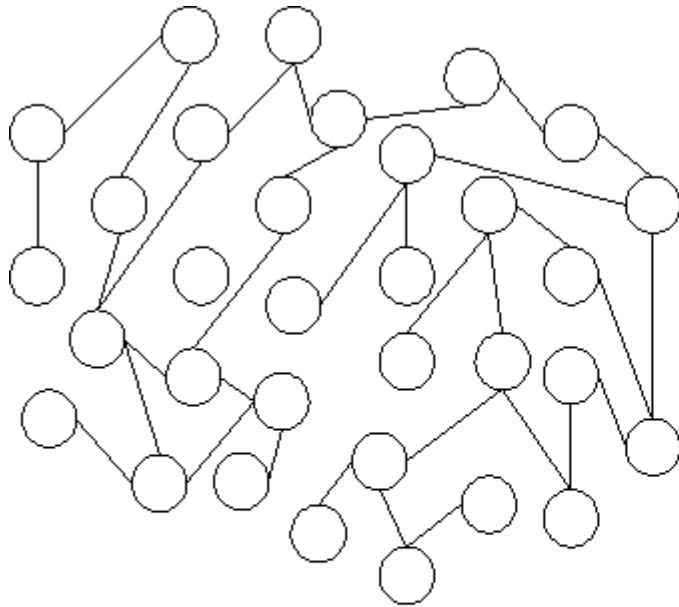


d.



# Exercise

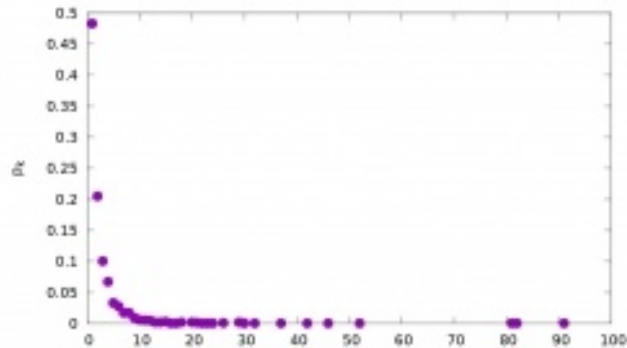
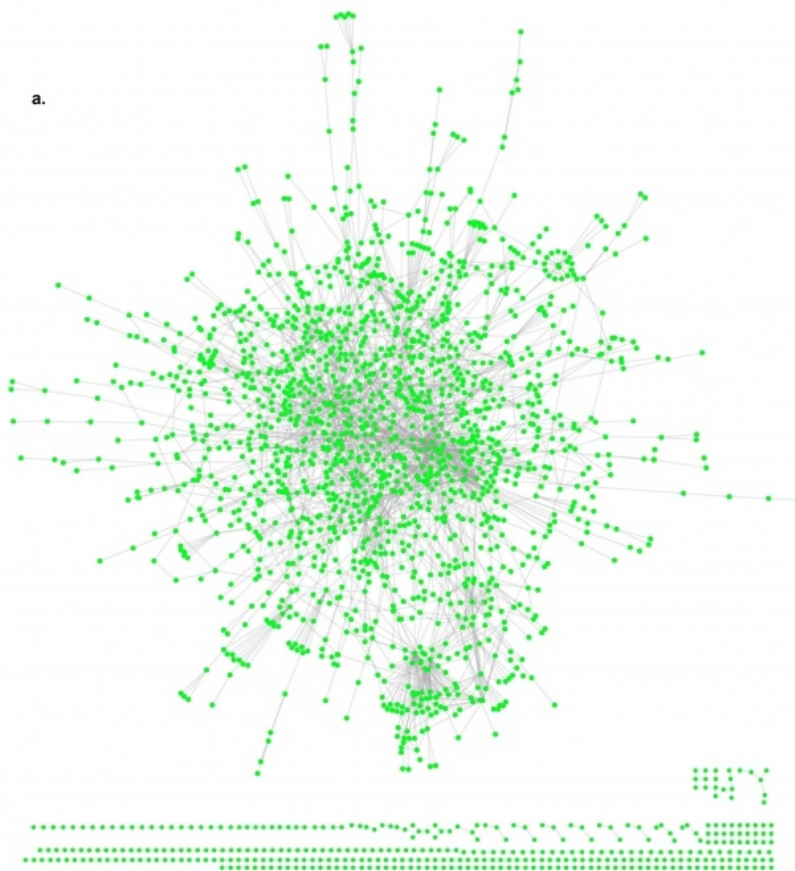
Answer in  
Google Spreadsheet



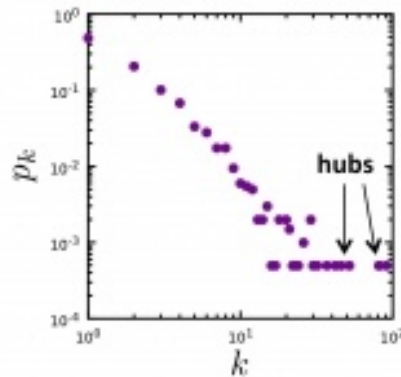
Draw the degree distribution of these graphs

# Degree distribution; real graph

a.



Linear  
scale



Log-log  
scale

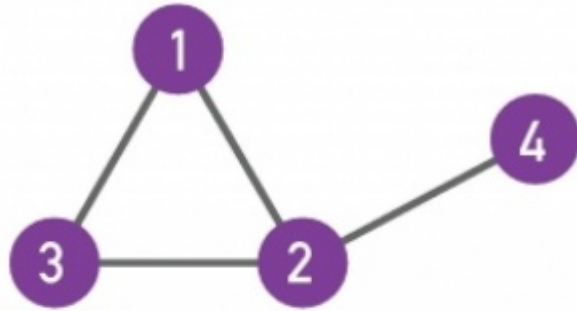


# Adjacency matrix

# What is an adjacency matrix

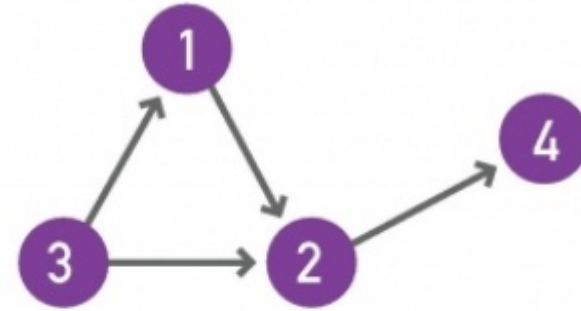
- A is the **adjacency matrix** of  $G = (V, E)$  iff:
  - A has  $|V|$  rows and  $|V|$  columns
  - $A_{ij} = 1$  if  $(i,j) \in E$
  - $A_{ij} = 0$  if  $(i,j) \notin E$

# Examples



Undirected graph

$$A_{ij} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$



Directed graph

$$A_{ij} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# Quick Question

- In terms of  $A$ , what is the expression for:

$$k_i^{\text{in}} =$$

$$k_i^{\text{out}} =$$

# Some “graphology” ...

- $G$  is undirected  $\Leftrightarrow A$  is symmetric
- $G$  has a self-loop  
 $\Leftrightarrow A$  has a non-zero element in the diagonal
- $G$  is complete  $\Leftrightarrow A_{ij} \neq 0$  (except if  $i=j$ )

# Summary

# Things to remember

- Definitions:
  - Degree, in-degree, out-degree
- Writing the adjacency matrix of a graph and drawing a graph given its adjacency matrix

# Practice on your own

Draw the  
indegree,  
outdegree,  
degree  
distribution

Write the  
adjacency  
matrix

