Scale-free networks

Introduction to Network Science
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Topic 04

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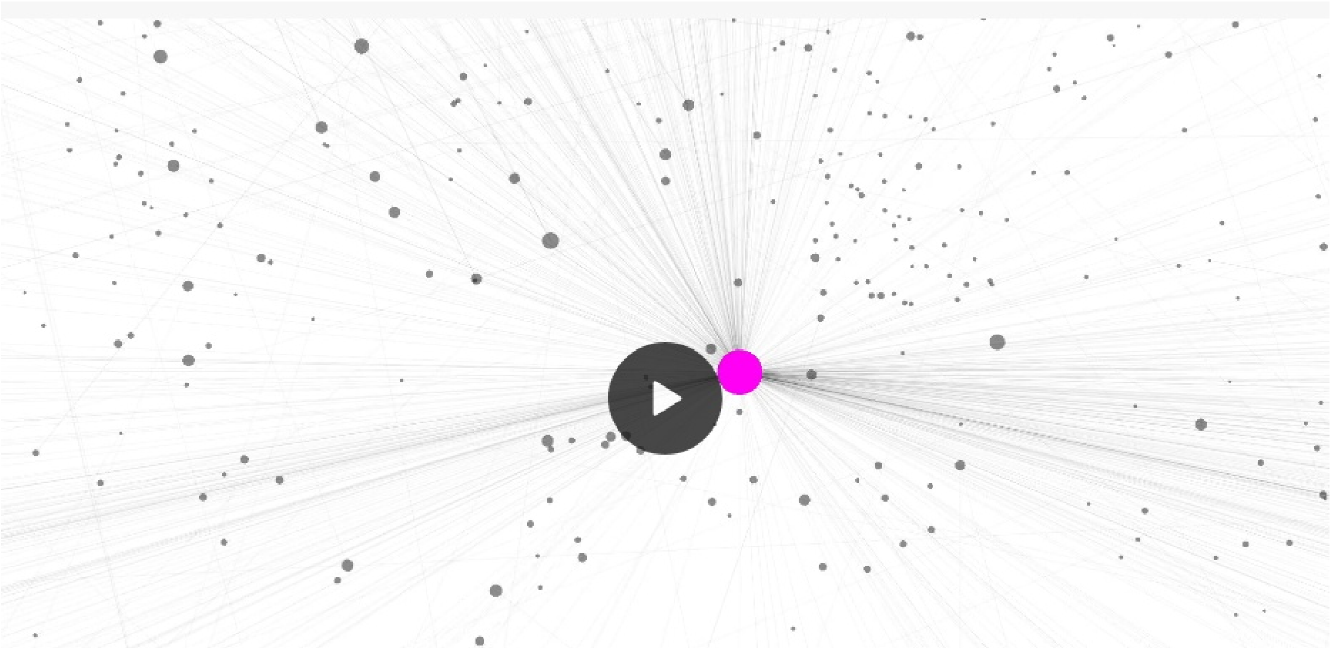
Contents

- Characteristics of scale-free networks
- Degree distribution of scale-free networks
- Distance distribution of scale-free networks

Sources

- Albert László Barabási: Network Science. Cambridge University Press, 2016.
  - Follows almost section-by-section chapter 04
- URLs cited in the footer of specific slides

nd.edu in 1998 (N=300K, L=1.5M)
nd1998

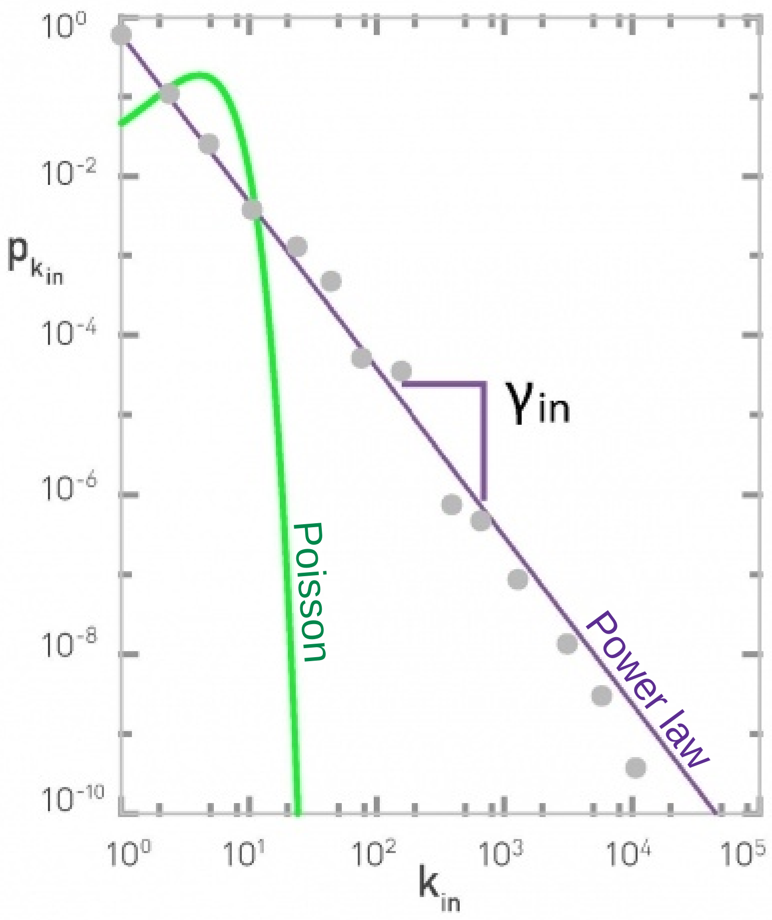


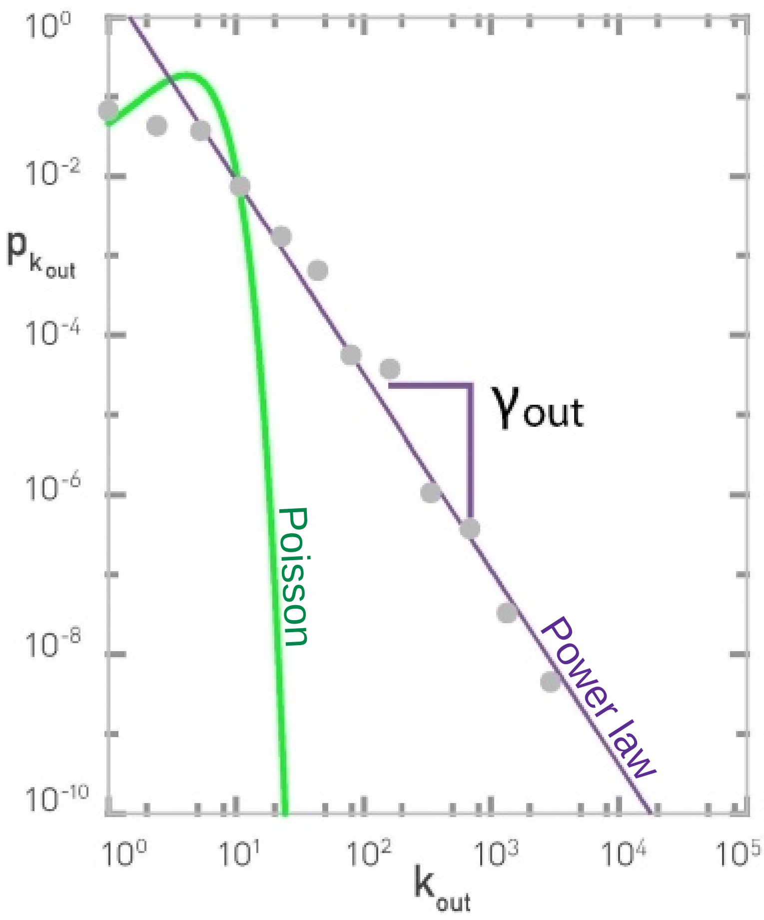
http://networksciencebook.com/images/ch-04/video-4-1.mov

What the Web Graph has but random networks don't have

- Large "hubs"
  - Nodes with a very high degree
  - Very unlikely in a random (ER) graph
- We have already seen the Poisson distribution is a bad approximation of the observed degree distribution

Degree distributions in nd1998





A good approximation of degree in real networks

- Straight descending line in log-log plot

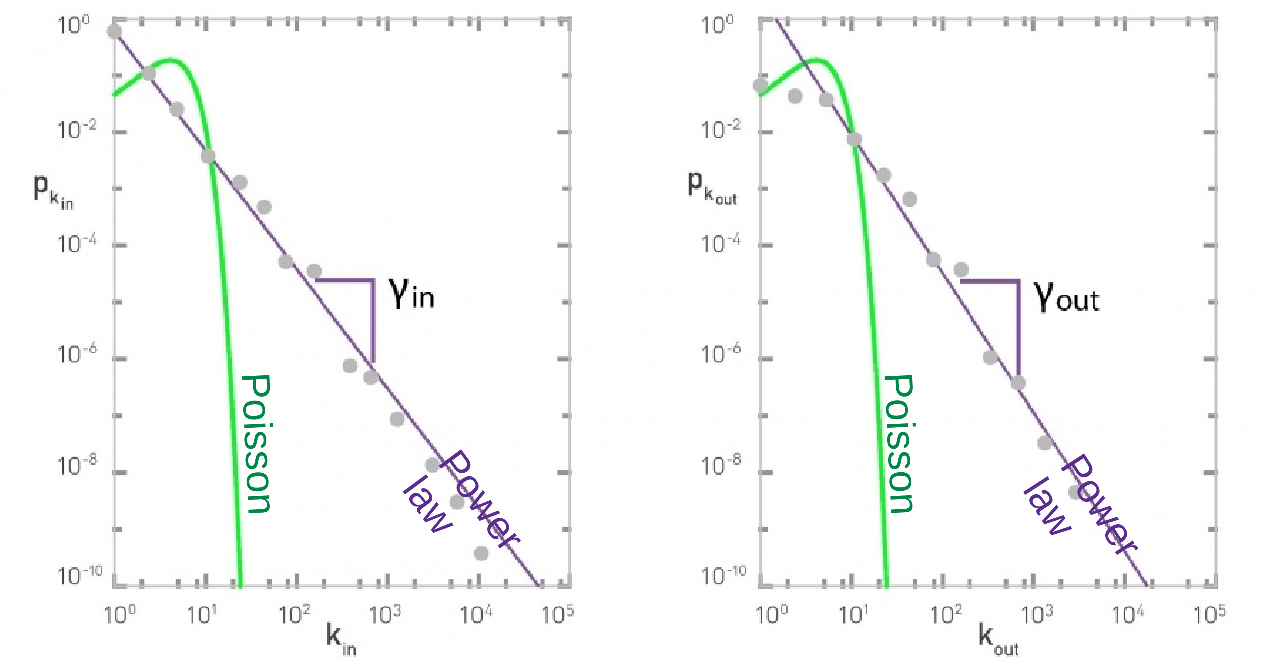
$$\log p_k \sim -\gamma \log k$$

$$p_k \sim k^{-\gamma}$$

- Parameter ɣ is the exponent of the power law

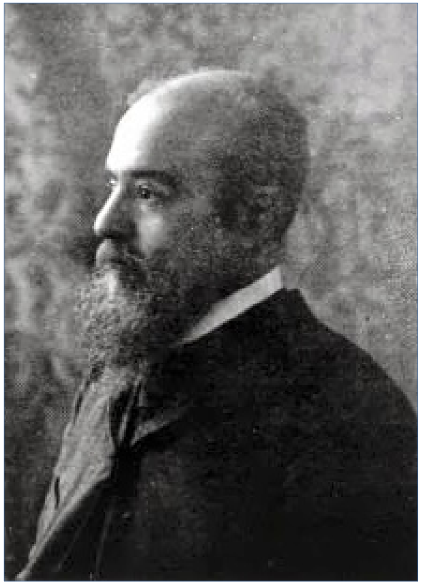
A scale-free network is a network whose degree distribution follows a power law

Degree distributions in nd1998



What kind of values of gamma reduce the “long tail” of the power law?

Parenthesis: Pareto



- Italian economist Vilfredo Pareto in the 19th century noted 80% of money was earned by 20% of people

- More recently …
  - 80 percent of links on the Web point to only 15 percent of pages;
  - 80 percent of citations go to only 38 percent of scientists;
  - 80 percent of links in Hollywood are to 30 percent of actors
- A debate that is still open: the wealth of the 1% and the 0.1%

In directed networks ...

- Each node has two degrees: k_{in} and k_{out}
- In general they may differ, hence

  $p_{k_{in}} \sim k^{-\gamma_{in}}$

  $p_{k_{out}} \sim k^{-\gamma_{out}}$

- In nd1998, $\gamma_{in} \simeq 2.1$, $\gamma_{out} \simeq 2.4$

Formally (discrete)

$$p_k = Ck^{-\gamma}$$

$$\sum_{k=1}^{\infty} p_k = 1 \longrightarrow C = \frac{1}{\sum_{k=1}^{\infty} k^{-\gamma}} = \frac{1}{\zeta(\gamma)}$$

Riemann's zeta

$$p_k = \frac{k^{-\gamma}}{\zeta(\gamma)}$$

This formalism assumes there are no nodes with degree zero

Formally (continuous)

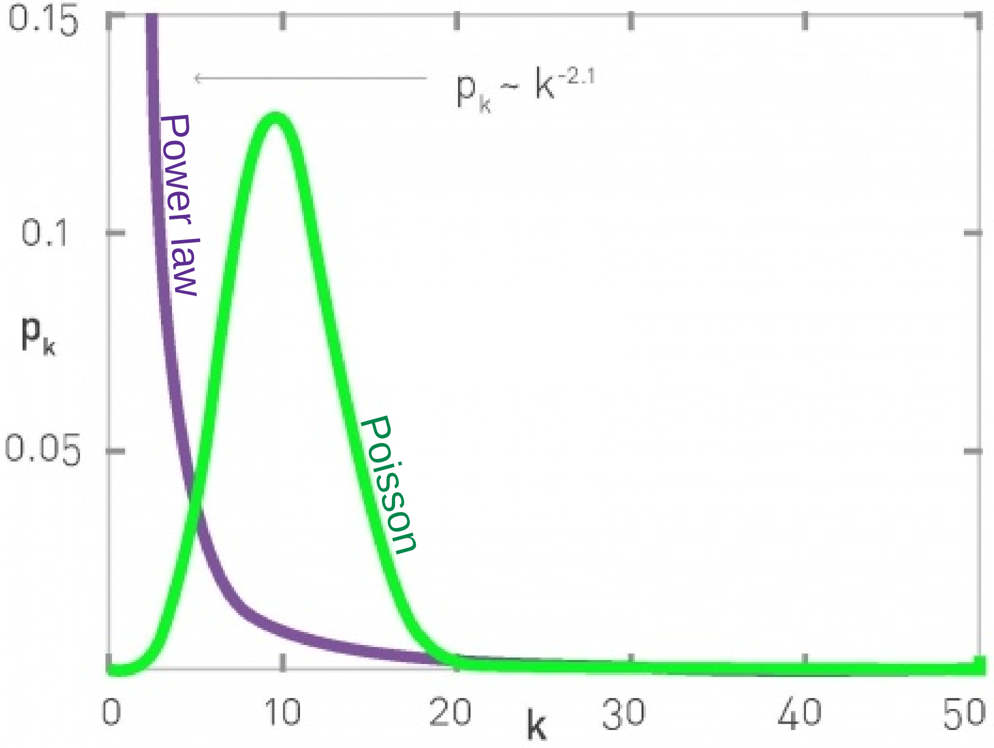
$$p_k = Ck^{-\gamma}$$

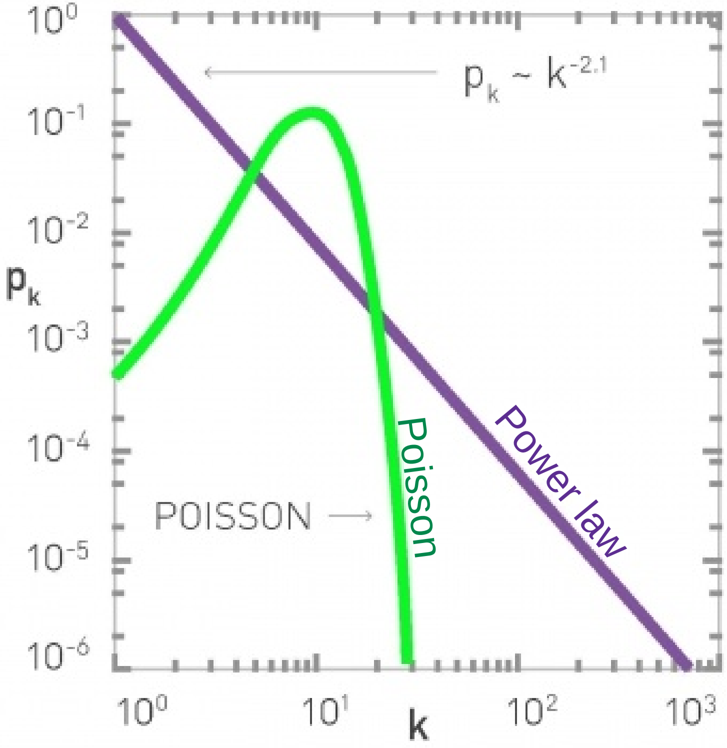
$$\int_{k=k_{\min}}^{\infty} p_k = 1 \longrightarrow C = \frac{1}{\int_{k=k_{\min}}^{\infty} k^{-\gamma}} = (\gamma - 1)k_{\min}^{\gamma-1}$$

$$p_k = (\gamma - 1)k_{\min}^{\gamma-1}k^{-\gamma}$$

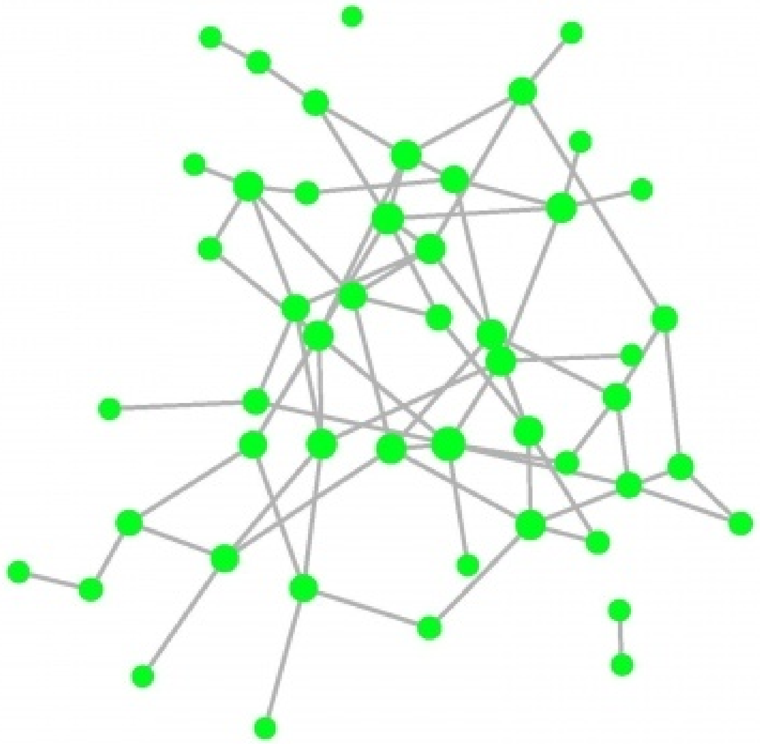
k_{min} is the smaller degree found in the network

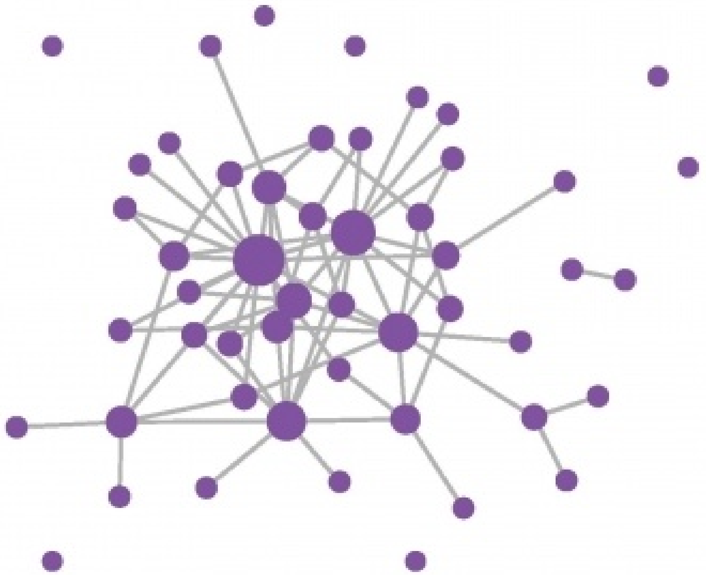
Comparing Poisson to power law





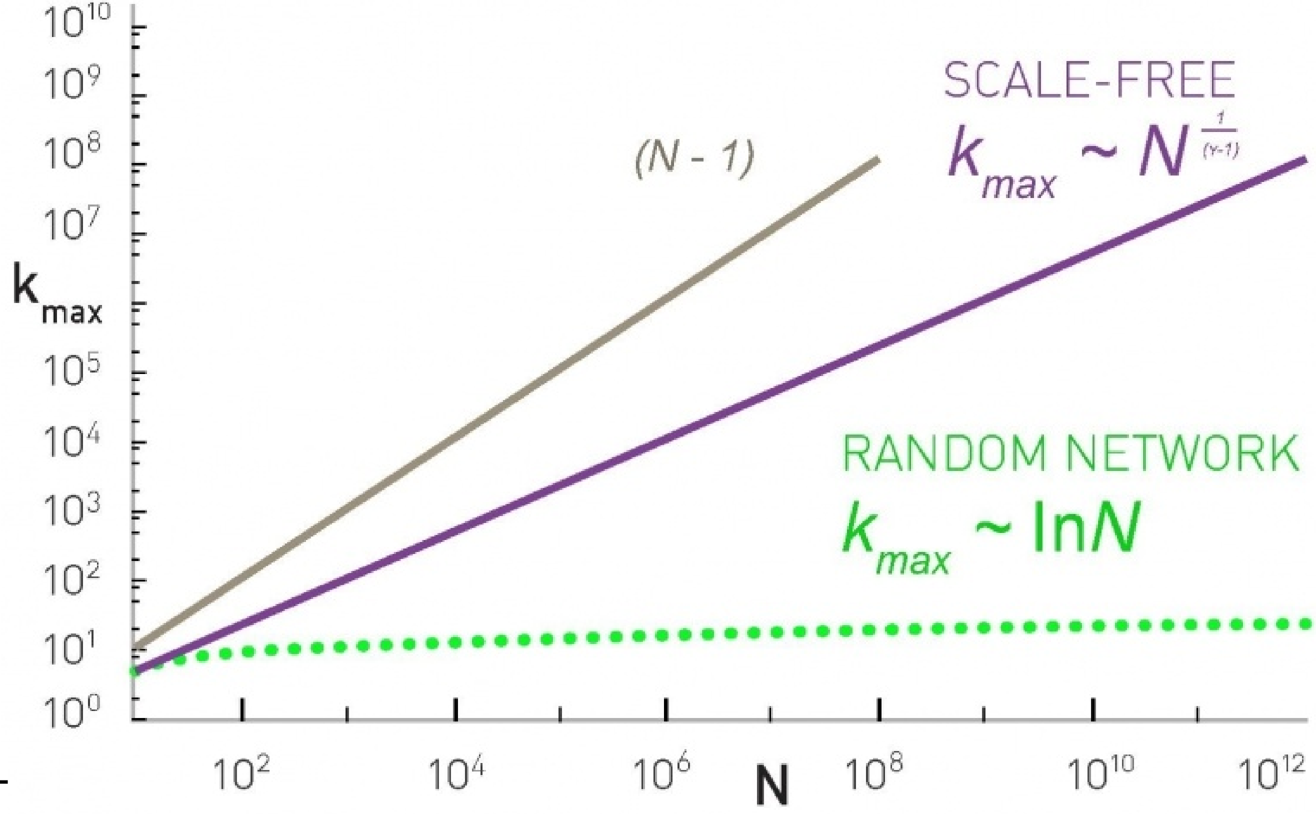
Comparing Poisson to power law





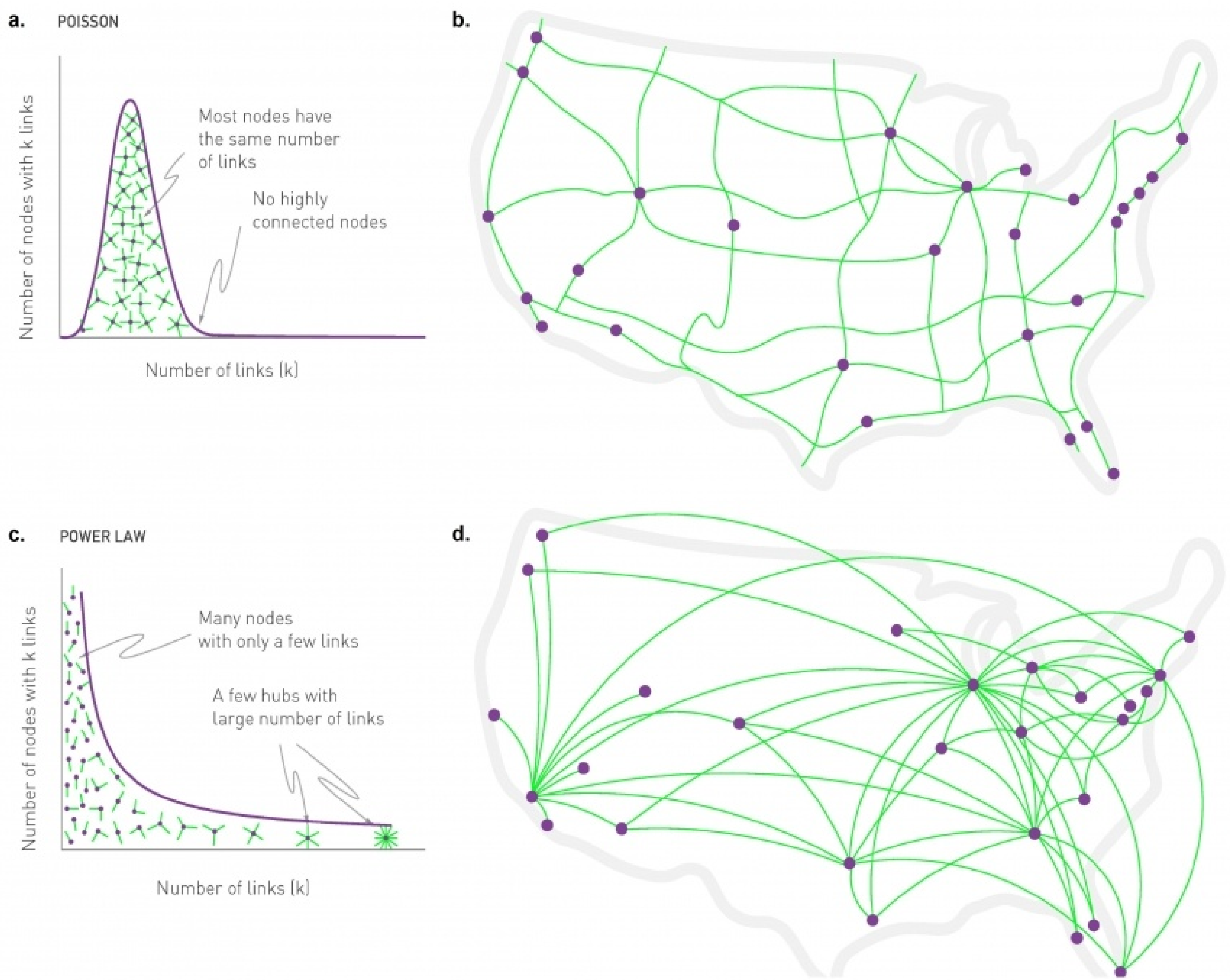
The natural cut-off of the degree

The largest hub cannot have more than N-1 connections



$$k_{\max} = k_{\min} N^{\frac{1}{\gamma - 1}}$$

Random vs scale-free networks



Ground transportation

Air transportation

What does it mean "scale-free"?

- A distribution has a "scale" if values are close to each other, for instance in a random network $\sigma_k = \langle k \rangle^{1/2}$
- Hence, most nodes are in the range $\langle k \rangle \pm \langle k \rangle^{1/2}$
- However in scale-free networks …

What does it mean "scale-free"?

- Moments of degree distribution

$$\langle k^n \rangle = \int_{k_{\min}}^{k_{\max}} k^n p_k dk = C \frac{k_{\max}^{n-\gamma+1} - k_{\min}^{n-\gamma+1}}{n-\gamma+1}$$

$$C = (\gamma - 1) k_{\min}^{\gamma-1}$$

What does it mean “scale-free”?

$$\sigma_k^2 = \langle k^2 \rangle - \langle k \rangle^2$$

- In a scale-free network

$$\langle k^2 \rangle = \int_{k_{\min}}^{k_{\max}} k^2 p_k dk = C \frac{k_{\max}^{3-\gamma} - k_{\min}^{3-\gamma}}{3-\gamma}$$

- This diverges as $k_{\max} \to \infty$ if $\gamma < 3$
- Hence there is no “typical” scale

What does it mean "scale-free"?

$$\sigma_k^2 = \langle k^2 \rangle - \langle k \rangle^2$$

- In a scale-free network

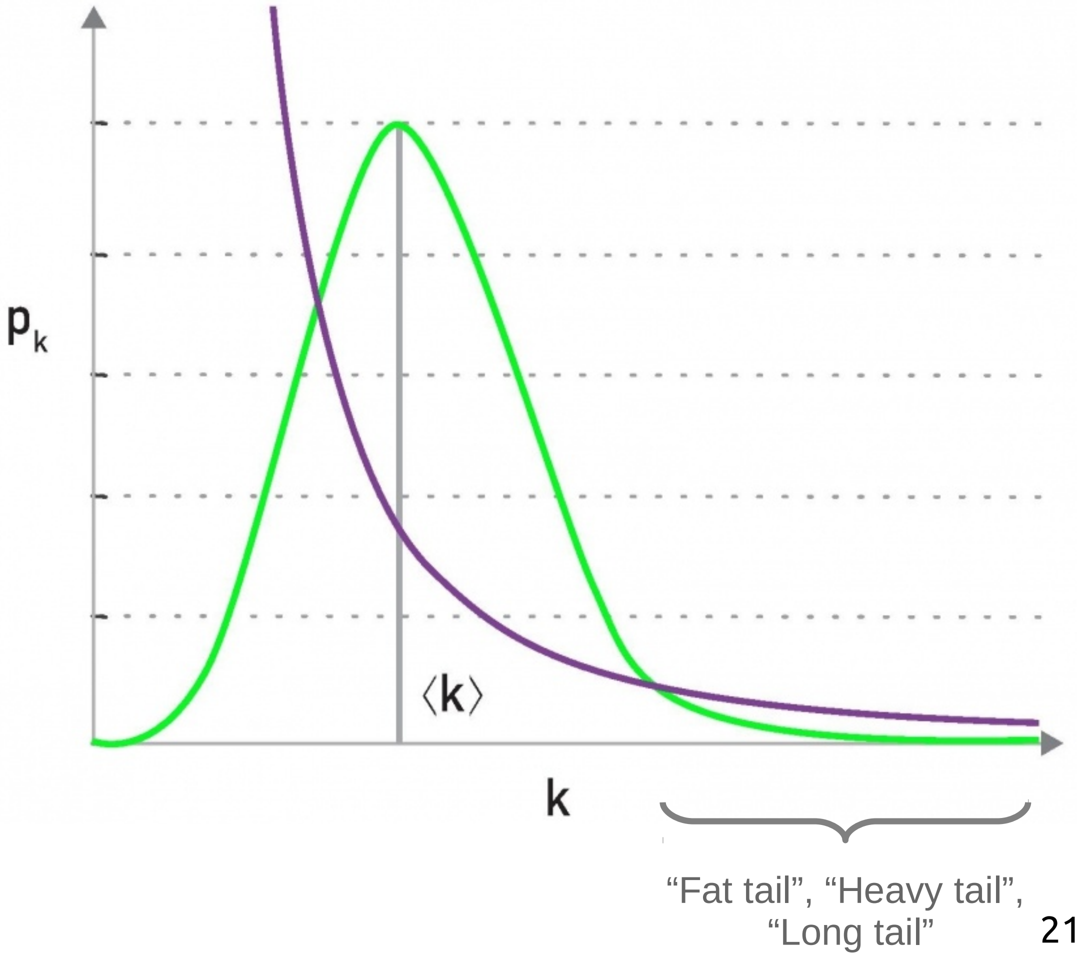
$$\langle k^2 \rangle = \int_{k_{\min}}^{k_{\max}} k^2 p_k dk = C \frac{k_{\max}^{3-\gamma} - k_{\min}^{3-\gamma}}{3-\gamma}$$

- What happens with the variance of the degree for networks with high max degree?

Example: nd1998

$$k_{\text{in}} = 4.60 \pm 1546$$

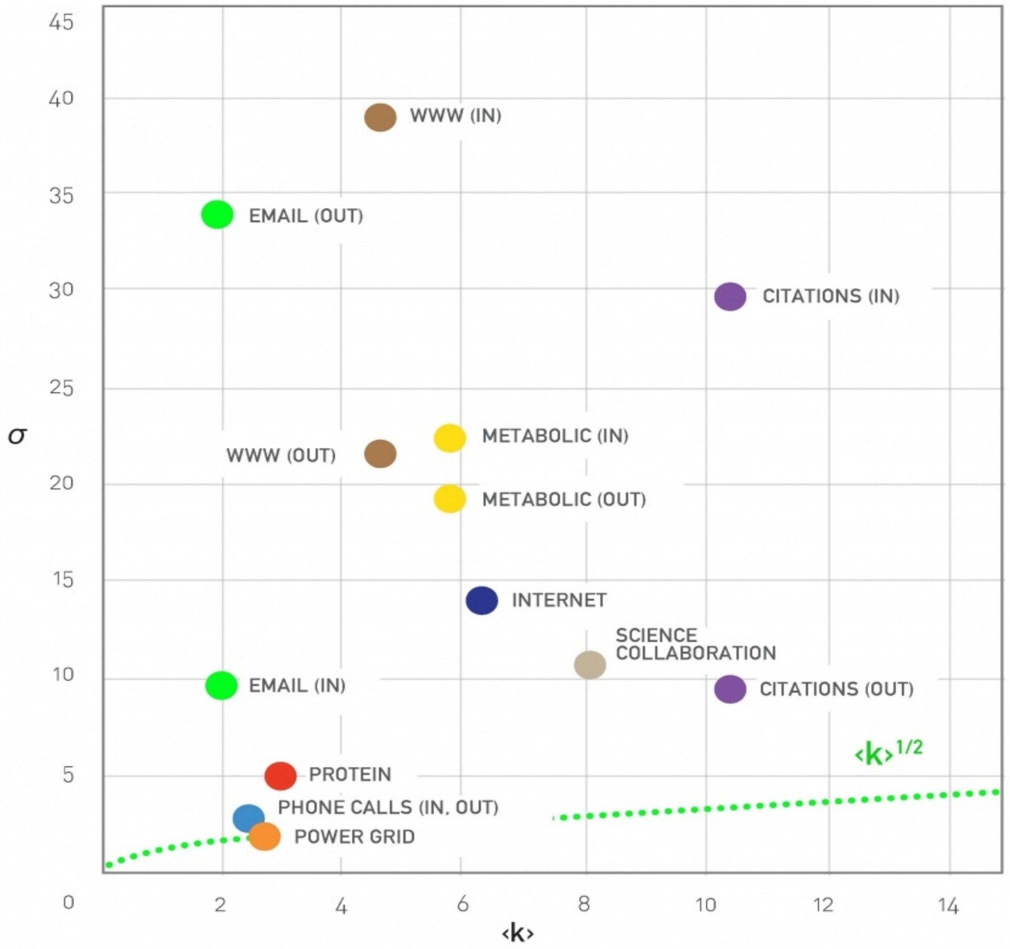
In general, the average degree is not very informative in scale-free networks



Real network examples

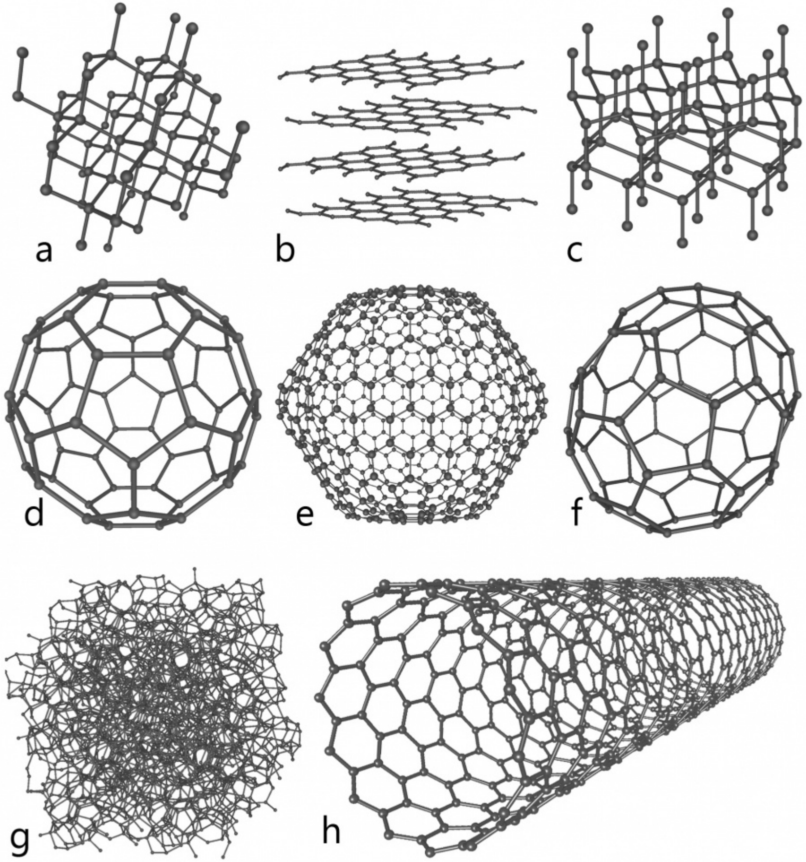
Network	N	L	$\langle k \rangle$	$\langle k_{in}^2 \rangle$	$\langle k_{out}^2 \rangle$	$\langle k^2 \rangle$	γ_{in}	γ_{out}	γ
Internet	192,244	609,066	6.34	-	-	240.1	-	-	3.42*
WWW	325,729	1,497,134	4.60	1546.0	482.4	-	2.00	2.31	-
Power Grid	4,941	6,594	2.67	-	-	10.3	-	-	Exp.
Mobile-Phone Calls	36,595	91,826	2.51	12.0	11.7	-	4.69*	5.01*	-
Email	57,194	103,731	1.81	94.7	1163.9	-	3.43*	2.03*	-
Science Collaboration	23,133	93,437	8.08	-	-	178.2	-	-	3.35*
Actor Network	702,388	29,397,908	83.71	-	-	47,353.7	-	-	2.12*
Citation Network	449,673	4,689,479	10.43	971.5	198.8	-	3.03*	4.00*	-
E. Coli Metabolism	1,039	5,802	5.58	535.7	396.7	-	2.43*	2.90*	-
Protein Interactions	2,018	2,930	2.90	-	-	32.3	-	-	2.89*-

Real network examples



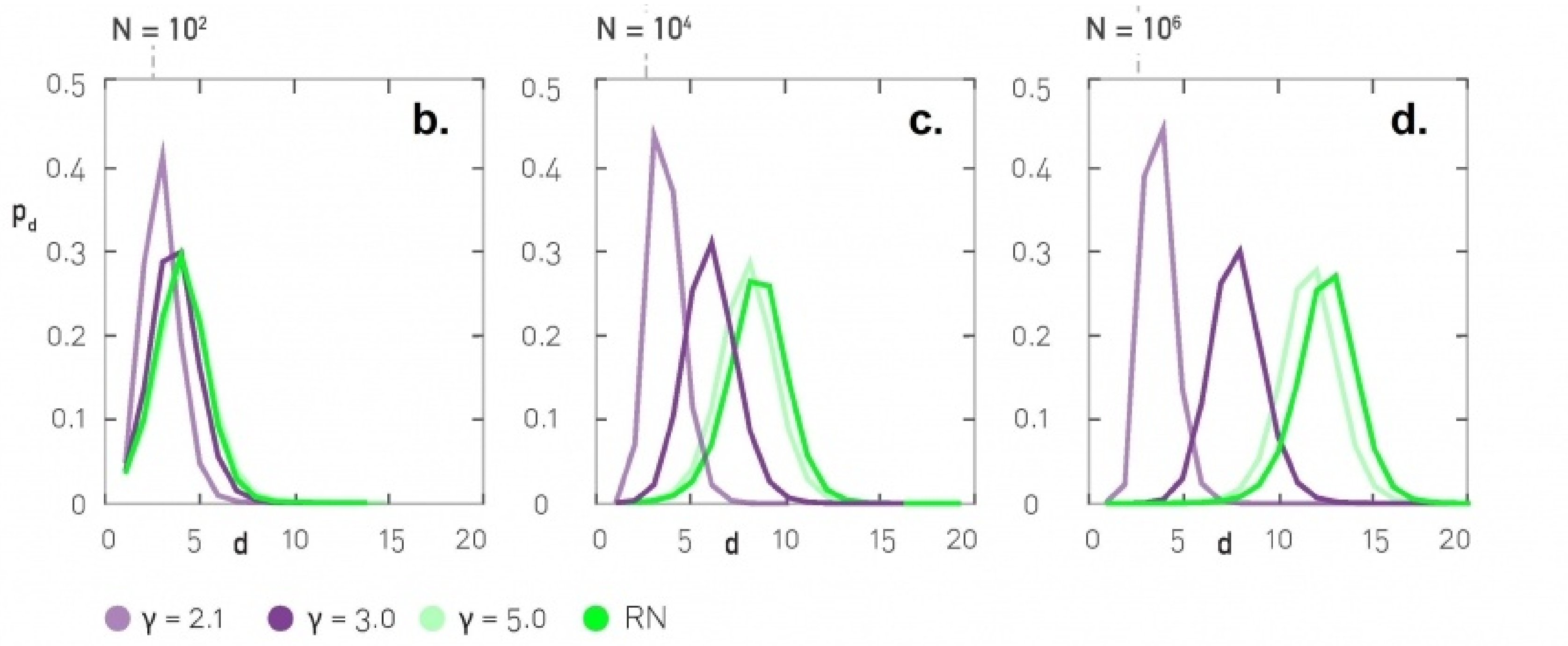
When you don't observe the scale-free property

- In general, when there is a limit to k_{max}
- Out-degree in some social networks
- Materials networks



Distance distributions: simulation results

Scale-free networks of increasing size, <k> = 3



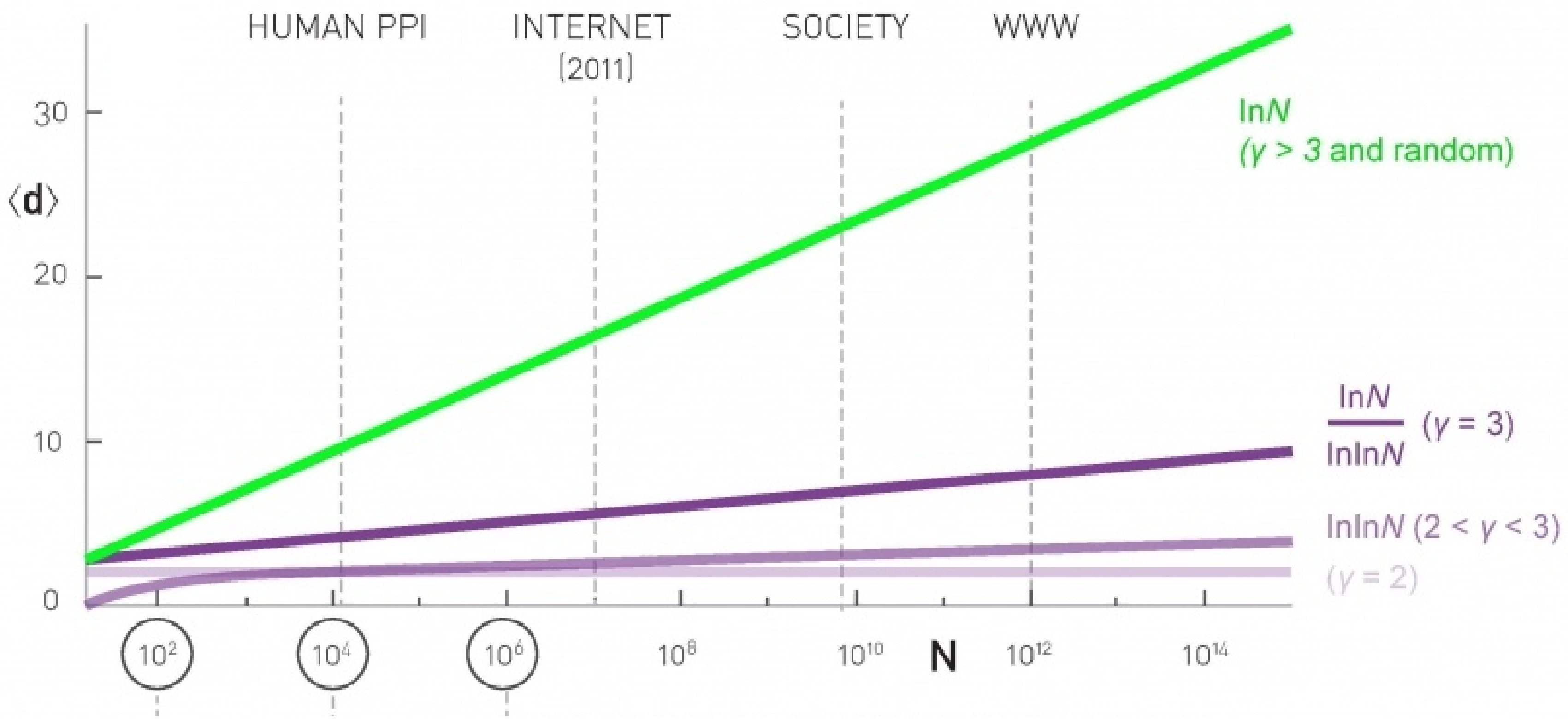
Average distance

- Depends on ɣ and N

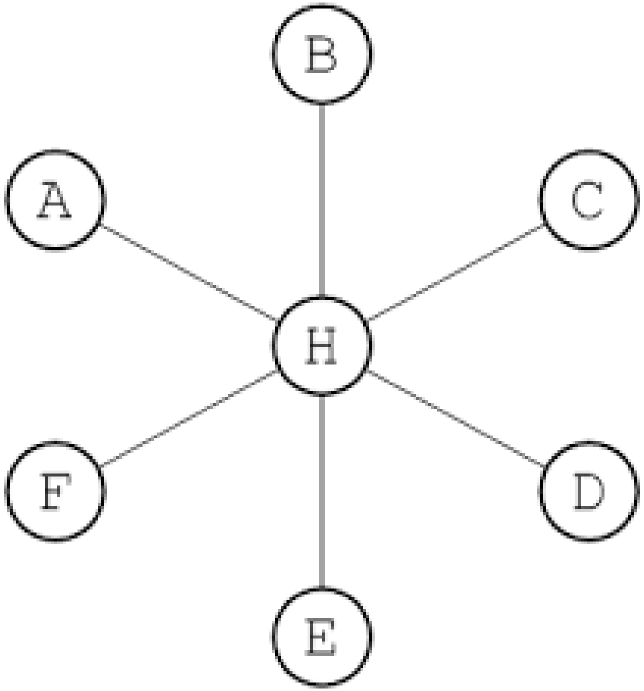
$$\langle d \rangle = \begin{cases} \text{const.} & \text{if } \gamma = 2 \\ \log\log \mathrm{N} & \text{if } 2 < \gamma < 3 \\ \log \mathrm{N} / \log\log \mathrm{N} & \text{if } \gamma = 3 \\ \log \mathrm{N} & \text{if } \gamma > 3 \end{cases}$$

Same as in ER graphs

Average distance and N



Anomalous regime $\gamma = 2$



Ultra-small world $2 < \gamma < 3$

- Average distance follows log(log(N))
- Example (humans):

$$N \approx 7 \times 10^9$$

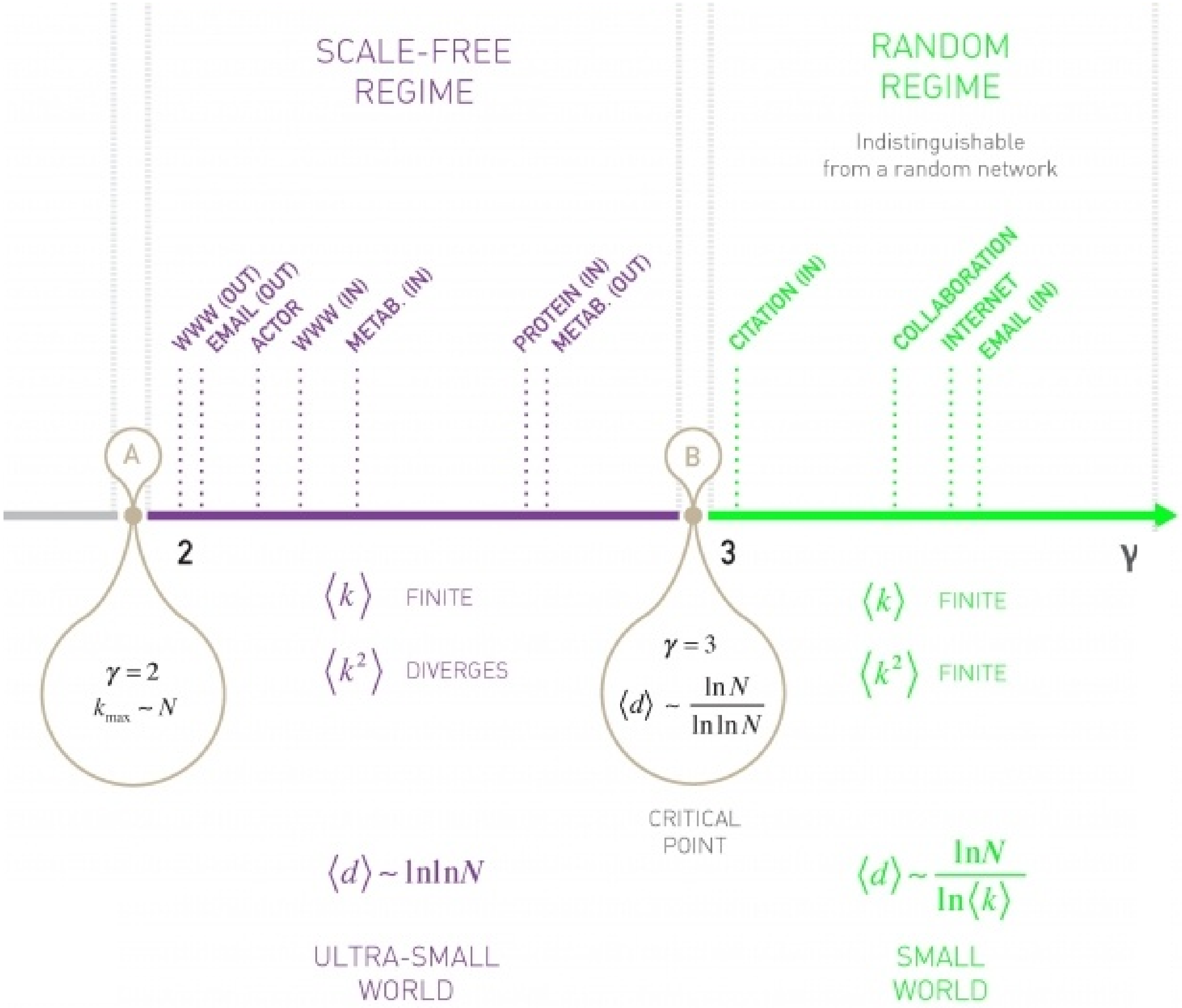
$$\log N \approx 22.66$$

$$\log \log N \approx 3.12$$

Small world $\gamma > 3$

- Average distance follows log(N)
- Similar to ER graphs where it followed log(N)/log(<k>)

The degree distribution exponent plays an important role



When $\gamma > 3$

- In this case it is hard to distinguish this case from an ER graph
- In most real complex networks (but not all)

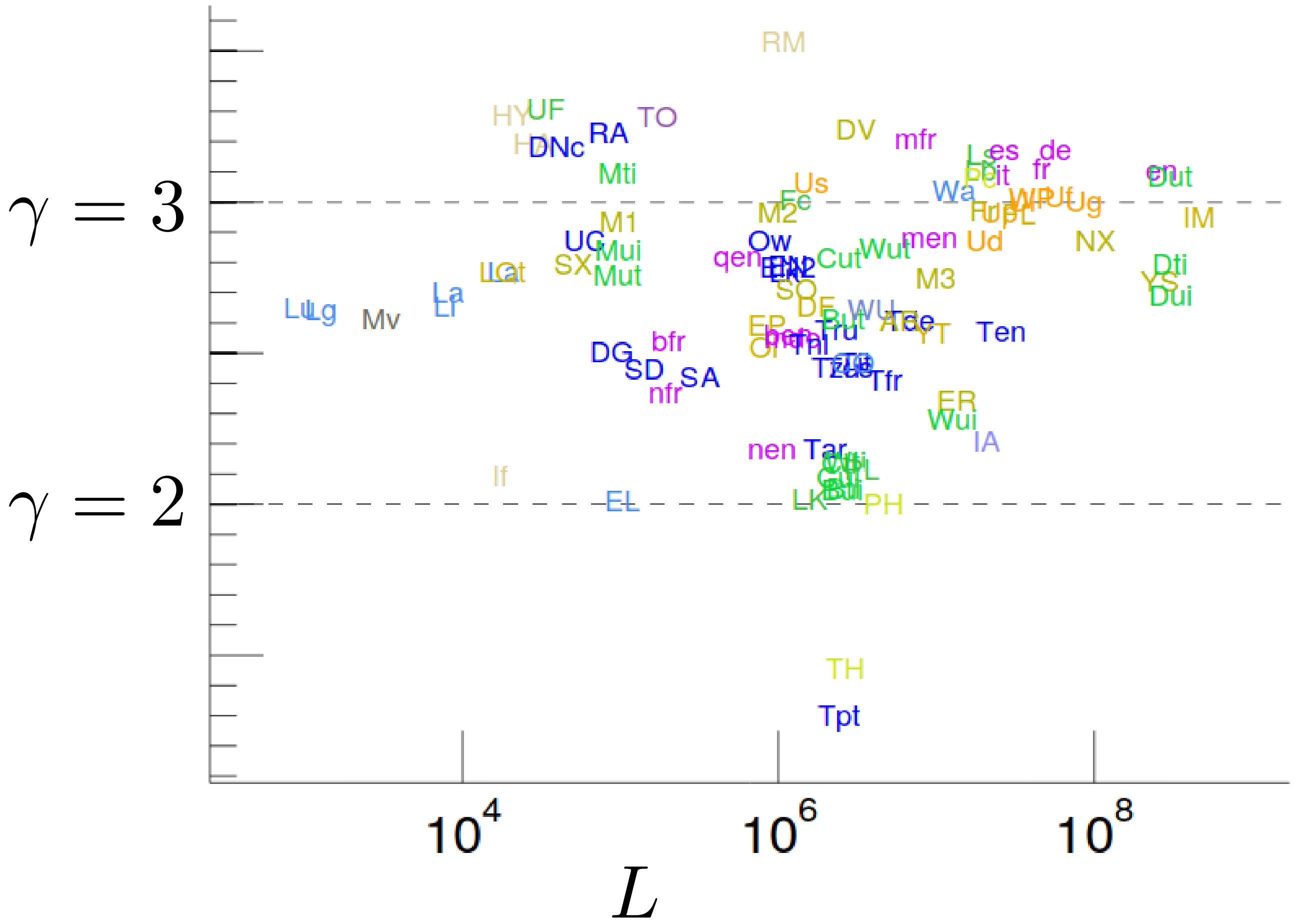
$$2 < \gamma < 3$$

When $\gamma > 3$

- Remember $k_{\max} = k_{\min} N^{\frac{1}{\gamma-1}}$ $N = \left(\frac{k_{\max}}{k_{\min}}\right)^{\gamma-1}$
- Observing the scale-free properties requires that $k_{max} >> k_{min}$, e.g. $k_{max} = 10\ k_{min}$
- Then if $\gamma = 5, N > 10^8$
- Hence we won't find many such networks

Examples

http://konect.uni-koblenz.de/statistics/prefatt



EL	Wikipedia elections
LK	Linux kernel mailing list threads
Bui	BibSonomy u-i
Bti	BibSonomy t-i
Cui	CiteULike u-i
If	Infectious
PL	Prosper loans
Cti	CiteULike t-i
Wti	Twitter t-i
nen	Wikinews (en)
Tar	Wikipedia talk, Arabic
Wui	Twitter u-i
ER	Epinions
nfr	Wikinews (fr)
Tfr	Wikipedia talk, French
SD	Slashdot
Tzh	Wikipedia talk, Chinese
Tes	Wikipedia talk, Spanish

Etc.

Exercise [B. 2016, Ex. 4.10.2] ”Friendship Paradox”

- Remember p_k is the probability that a node has k “friends”
- If we randomly select a link, the probability that a node at any end of the link has k friends is $q_k = C\, k\, p_k$ where C is a normalization factor
  - (a) Find C (the sum of q_k must be 1)

Exercise [B. 2016, Ex. 4.10.2] "Friendship Paradox"

- If we randomly select a link, the probability that a node at any end of the link has k friends is $q_k = C\, k\, p_k$ where C is a normalization factor
  - (b) q_k is also the prob. that a randomly chosen node has a neighbor of degree k; find its average

Exercise [B. 2016, Ex. 4.10.2] "Friendship Paradox"

(c-d) Compute the expected number of friends of a neighbor of a randomly chosen node; compare with the expected number of friends of a randomly chosen node when

$$N = 10000$$

$$\gamma = 2.3$$

$$k_{\min} = 1$$

$$k_{\max} = 1000$$

$$\langle k^n \rangle = C \frac{k_{\max}^{n-\gamma+1} - k_{\min}^{n-\gamma+1}}{n-\gamma+1}$$

$$C = (\gamma - 1) k_{\min}^{\gamma-1}$$

Code

```python
def degree_moment(kmin, kmax, moment, gamma):
    C = (gamma-1.0)*(kmin**(gamma-1.0))
    numerator = (kmax**(moment-gamma+1.0) - kmin**(moment-gamma+1.0))
    denominator = (moment-gamma+1.0)
    return C * numerator / denominator
```

```python
kavg = degree_moment(kmin=1, kmax=1000, moment=1, gamma=2.3)
print(kavg)
```

```
3.787798988222529
```

```python
ksqavg = degree_moment(kmin=1, kmax=1000, moment=2, gamma=2.3)
print(ksqavg)
```

```
231.94329076177414
```

```python
print(ksqavg / kavg)
```

```
61.23431879119234
```

An example of friendship paradox

- Pick a random airport on Earth
  - Most likely it will be a small airport
- However, no matter how small it is, it **will** have flights to big airports
- On average those airports will have much larger degree

Summary

Things to remember

- Definition of scale-free
- Power law
- Regimes of distance and connectivity
- The friendship paradox

Practice on your own

- Remember the regimes of a graph given <k>

  (It's useful to know this by heart)
- Estimate degree distributions and distance distributions for some graphs
- Apply the friendship paradox to some graphs