§ 2.1-19. LAPLACE TRANSFORMS OF SOME SPECIAL FUNCTIONS

(1) **The Sine Integral Function.** It is denoted by $Si(t)$ and is defined by

$$Si(t) = \int_0^t \frac{\sin u}{u}\, du$$

Now
$$L\{Si(t)\} = L\left\{\int_0^t \frac{\sin u}{u}\, du\right\} = L\left\{\int_0^t \frac{\sin t}{t}\, dt\right\}$$

$$= \frac{1}{s} \cot^{-1} s \qquad \text{[see Ex. 15 of § 2.1-15]}$$

Alternative Method :

$$L\{Si(t)\} = L\left\{\int_0^t \frac{\sin u}{u}\, du\right\} = L\left\{\int_0^t \left(1 - \frac{u^2}{3\,!} + \frac{u^4}{5\,!} - \frac{u^6}{7\,!} + \ldots\right) du\right\}$$

$$= L\left\{t - \frac{t^3}{3\,(3\,!)} + \frac{t^5}{5\,(5\,!)} - \frac{t^7}{7\,(7\,!)} + \ldots\right\}$$

$$= \frac{1\,!}{s^2} - \frac{3\,!}{s^4} \cdot \frac{1}{3\,(3\,!)} + \frac{5\,!}{s^6} \cdot \frac{1}{5\,(5\,!)} - \ldots = \frac{1}{s}\left[\frac{1}{s} - \frac{1}{3}\left(\frac{1}{s}\right)^3 + \frac{1}{5}\left(\frac{1}{s}\right)^5 - \ldots\right]$$

$$= \frac{1}{s} \tan^{-1}\left(\frac{1}{s}\right)$$ [from Gregory's Ser...

$$= \frac{1}{s} \cot^{-1} s.$$

(2) The Cosine Integral Function. It is denoted by $Ci(t)$ and is defined by

$$Ci(t) = \int_t^\infty \frac{\cos u}{u}\, du.$$

Now $$L\{Ci(t)\} = L\left\{\int_t^\infty \frac{\cos u}{u}\, du\right\}.$$

Let $$F(t) = \int_t^\infty \frac{\cos u}{u}\, du = -\int_\infty^t \frac{\cos u}{u}\, du$$

$\therefore$ $$F'(t) = -\frac{\cos t}{t} \Rightarrow t\,F'(t) = -\cos t.$$

$\therefore$ $$L\{t\,F'(t)\} = L\{-\cos t\}$$

$\Rightarrow$ $$-\frac{d}{ds} L\{F'(t)\} = -\frac{s}{s^2+1}$$

$\Rightarrow$ $$\frac{d}{ds}[s\,f(s) - F(0)] = \frac{s}{s^2+1}$$ [where $f(s) = L\{F(t)\}$

$\Rightarrow$ $$\frac{d}{ds}[s\,f(s)] = \frac{s}{s^2+1},$$ [$\because$ $F(0)$ is constant

Integrating, $$s\,f(s) = \tfrac{1}{2}\log(s^2+1) + C$$

where C is a constant of integration to be determined.

From (1), it follows that

$$\lim_{t\to\infty} F(t) = 0.$$

Now from final value theorem [see § 2.1-18]

$$\lim_{s\to 0} s\,f(s) = \lim_{t\to\infty} F(t) = 0$$ [use (3)

Therefore taking lim as $s \to 0$ of both sides of (2), we get

$$0 = 0 + C \Rightarrow C = 0.$$

Hence from (2), we have

$$s\,f(s) = \tfrac{1}{2}\log(s^2+1) \Rightarrow f(s) = \frac{1}{2s}\log(s^2+1)$$

$\Rightarrow$ $$L\{Ci(t)\} = \frac{1}{2s}\log(s^2+1).$$ Ans

(3) The Exponential Integral function. It is denoted by $Ei(t)$ and is defined by

$$Ei(t) = \int_t^\infty \frac{e^{-u}}{u}\, du.$$

Now $$L\{Ei(t)\} = L\left\{\int_t^\infty \frac{e^{-u}}{u}\, du\right]$$

Let $$F(t) = \int_t^\infty \frac{e^{-u}}{u}\, du = -\int_\infty^t \frac{e^{-u}}{u}\, du$$ ...(1)

$$F'(t) = -e^{-t}/t \Rightarrow tF'(t) = -e^{-t} \Rightarrow L\{tF'(t)\} = -L\{e^{-t}\}$$

$\therefore$
$$-\frac{d}{ds} L\{F'(t)\} = -\frac{1}{s+1}$$

$\Rightarrow$
$$\frac{d}{ds}\{s f(s) - F(0)\} = \frac{1}{s+1} \qquad \text{[where } f(s) = L\{F(t)\}]$$

$\Rightarrow$
$$\frac{d}{ds}\{s f(s)\} = \frac{1}{s+1} \qquad [\because F(0) \text{ is constant}]$$

$\Rightarrow$ Integrating, $\quad s f(s) = \log(s+1) + C$ ...(2)

From (1), $\quad \lim_{t \to \infty} F(t) = 0.$ ...(3)

Now from final value theorem [see § 2.1-18]

$$\lim_{s \to 0} s f(s) = \lim_{t \to \infty} F(t) = 0. \qquad \text{[from (3)]}$$

Taking lim as $s \to 0$ of (2), we have

$$0 = 0 + C \Rightarrow C = 0.$$

Hence from (2), we have

$$s f(s) = \log(s+1) \Rightarrow f(s) = \frac{1}{s}\log(s+1)$$

$\Rightarrow$
$$L\{Ei(t)\} = (1/s) \,.\, \log(s+1).$$ **Ans.**

(4) **Unit Step Function (or Heaviside's Unit Function).** It us usually denoted by $H(t-a)$ and is defined by

$$H(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}.$$

Now
$$L\{H(t-a)\} = \int_0^\infty e^{-st} H(t-a)\, dt = \int_t^a e^{-st}(0)\, dt + \int_a^\infty e^{-st} \,.\, 1\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-as}}{s}.$$ **Ans.**

(5) **Dirac Delta Function (or Unit Impulse Function).** **[R.G.T.U. Dec. 2003; Jan. 2006]**

It is denoted by $F_\varepsilon(t)$ and is defined by

$$F_\varepsilon(t) = \begin{cases} 1/\varepsilon, & 0 \le t \le \varepsilon \\ 0, & t > \varepsilon. \end{cases} \quad \text{where } \varepsilon > 0.$$

Now
$$L\{F_\varepsilon(t)\} = \int_0^\infty e^{-st} F_\varepsilon(t)\, dt = \int_0^\varepsilon e^{-st} \,.\left(\frac{1}{\varepsilon}\right) dt + \int_\varepsilon^\infty e^{-st} \,.\, 0\, dt$$

$$= \frac{1}{\varepsilon}\left[\frac{e^{-st}}{-s}\right]_0^\varepsilon = \frac{1}{s\varepsilon}[1 - e^{-s\varepsilon}].$$ **Ans.**

(6) **Error Function.** It is denoted by erf $(\sqrt{t})$ and is defined by

$$\text{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2}\, du.$$

Now
$$L\{\text{erf}(\sqrt{t})\} = L\left\{\frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2}\, du\right\}.$$

By definition of error function, we have

$$\text{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}\,du = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}\left(1 - \frac{u^2}{1!} + \frac{u^4}{2!} - \frac{u^6}{3!} + \dots\right)du$$

$$= \frac{2}{\sqrt{\pi}}\left[u - \frac{u^3}{3} + \frac{u^5}{5(2!)} - \frac{u^7}{7(3!)} + \dots\right]_0^{\sqrt{t}} = \frac{2}{\sqrt{\pi}}\left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5(2!)} - \frac{t^{7/2}}{7(3!)} + \dots\right.$$

$$\Rightarrow \quad L\{\text{erf}\sqrt{t}\} = \frac{2}{\sqrt{\pi}}\left[L\{t^{1/2}\} - \frac{1}{3}L\{t^{3/2}\} + \frac{1}{5(2!)}L\{t^{5/2}\} - \frac{1}{7(3!)}L\{t^{7/2}\} + \dots\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{\Gamma(\frac{3}{2})}{s^{3/2}} - \frac{1}{3}\frac{\Gamma(\frac{5}{2})}{s^{5/2}} + \frac{1}{5(2!)}\cdot\frac{\Gamma(\frac{7}{2})}{s^{7/2}} - \frac{1}{7(3!)}\cdot\frac{\Gamma(\frac{9}{2})}{s^{9/2}} + \dots\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2}\cdot\frac{1}{s^{3/2}} - \frac{\sqrt{\pi}}{2}\cdot\frac{1}{2}\frac{1}{s^{5/2}} + \frac{\sqrt{\pi}}{2}\cdot\frac{1.3}{2.4}\frac{1}{s^{7/2}} - \frac{\sqrt{\pi}}{2}\cdot\frac{1.3.5}{2.4.6}\frac{1}{s^{9/2}} + \dots\right]$$

$$= \frac{1}{s^{3/2}}\left[1 - \frac{1}{2}\cdot\frac{1}{s} + \frac{1.3}{2.4}\frac{1}{s^2} - \frac{1.3.5}{2.4.6}\frac{1}{s^3} + \dots\right]$$

$$= \frac{1}{s^{3/2}}\left(1 + \frac{1}{s}\right)^{-1/2} = \frac{1}{s^{3/2}}\left(\frac{s}{s+1}\right)^{1/2} = \frac{1}{s\sqrt{(s+1)}}.$$

Some Deductions. Let $\text{erf}(\sqrt{t}) = F(t)$ and $L\{F(t)\} = f(s)$.

(i) Since $\quad L\{F(at)\} = \frac{1}{a}f\left(\frac{s}{a}\right)$

$$\Rightarrow \quad L\{\text{erf}(2\sqrt{t})\} = L\{\text{erf}\sqrt{(4t)}\} = \frac{1}{4}\cdot\frac{1}{\frac{s}{4}\sqrt{\left(\frac{s}{4}+1\right)}} = \frac{2}{s\sqrt{(s+4)}}.$$

(ii) $$L\{t\,.\,\text{erf}(2\sqrt{t})\} = -\frac{d}{ds}L\{\text{erf}(2\sqrt{t})\} = -\frac{d}{ds}\left[\frac{2}{s\sqrt{(s+4)}}\right] = \frac{3s+8}{s^2(s+4)^{3/2}}.$$

(iii) $$L\{e^{at}\,\text{erf}\sqrt{t}\} = f(s-a)$$ [by first shifting theore

$$= \frac{1}{(s-a)\sqrt{(s-a+1)}}.$$

$$\therefore \quad L\{e^{4t}\,\text{erf}\sqrt{t}\} = \frac{1}{(s-4)\sqrt{(s-3)}}.$$

(7) Complementary Error Function. It is denoted by $\text{erfc}(\sqrt{t})$ and is defined by

$$\text{erfc}(\sqrt{t}) = 1 - \text{erf}(\sqrt{t})$$

by $$\text{erfc}(\sqrt{t}) = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}\,du = \frac{2}{\sqrt{\pi}}\left[\int_0^{\infty} e^{-u^2}\,du + \int_{\sqrt{t}}^{0} e^{-u^2}\,du\right] = \frac{2}{\sqrt{\pi}}\int_{\sqrt{t}}^{\infty} e^{-u^2}$$

Now $$L\{\text{erfc}(\sqrt{t})\} = L\{1 - \text{erf}\sqrt{t}\}$$

$$= L\{1\} - L\{\text{erf}(\sqrt{t})\} = \frac{1}{s} - \frac{1}{s\sqrt{(s+1)}} = \frac{\sqrt{(s+1)} - 1}{s\sqrt{(s+1)}}$$

$$= \frac{\{\sqrt{(s+1)} - 1\}\{\sqrt{(s+1)} + 1\}}{s\sqrt{(s+1)}\{\sqrt{(s+1)} + 1\}} = \frac{1}{s\sqrt{(s+1)}\{\sqrt{(s+1)} + 1\}}.$$

(8) Bessel Function. Bessel function of order n is denoted by $J_n(t)$ and is defined by

$$J_n(t) = \frac{t^n}{2^n\,\Gamma(n+1)}\left[1 - \frac{t^2}{2\,.\,(2n+2)} + \frac{t^4}{2.4.(2n+2)\,(2n+4)} - \dots\right]$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r\,!\,\Gamma(n+r+1)}\cdot\left(\frac{t}{2}\right)^{n+2r} \qquad \dots(1)$$

Bessel function of order zero is denoted by $J_0(t)$ and is defined by

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2\,.\,4^2} - \frac{t^6}{2^2\,.\,4^2\,.\,6^2} + \dots.$$

(I) Now $\quad L\{J_0(t)\} = L\left\{1 - \frac{t^2}{2^2} + \frac{t^4}{2^2\,.\,4^2} - \frac{t^6}{2^2\,.\,4^2\,.\,6^2} + \dots\right\}$

$$\Rightarrow \quad L\{J_0(t)\} = L\{1\} - \frac{1}{2^2}L\{t^2\} + \frac{1}{2^2\,.\,4^2}L\{t^4\} - \frac{1}{2^2\,.\,4^2\,.\,6^2}L\{t^6\} + \dots$$

$$= \frac{1}{s} - \frac{1}{2^2}\cdot\frac{2\,!}{s^3} + \frac{1}{2^2\,.\,4^2}\cdot\frac{4\,!}{s^5} - \frac{1}{2^2\,.\,4^2\,.\,6^2}\,\frac{6\,!}{s^7} + \dots$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1\,.\,3}{2\,.\,4}\left(\frac{1}{s^2}\right)^2 - \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\left(\frac{1}{s^2}\right)^3 + \dots\right]$$

$$= \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-1/2} = \frac{1}{s}\left(\frac{s^2}{s^2+1}\right)^{1/2} = \frac{1}{\sqrt{(1+s^2)}}.$$

Some Deductions. Let $L\{F(t)\} = f(s)$

(i) Since $\quad L\{F(at)\} = \frac{1}{a}f\left(\frac{s}{a}\right)$,

$$\Rightarrow \quad L\{J_0(at)\} = \frac{1}{a}\cdot\frac{1}{\sqrt{[1+(s/a)^2]}} = \frac{1}{\sqrt{(s^2+a^2)}}.$$

(ii) $$L\{t\,J_0(at)\} = -\frac{d}{ds}L\{J_0(at)\} = -\frac{d}{ds}\left[\frac{1}{\sqrt{(s^2+a^2)}}\right] = \frac{s}{(s^2+a^2)^{3/2}}.$$

(iii) Since $L\{e^{-at}F(t)\} = f(s+a)$ where $f(s) = L\{F(t)\}$,

$$\Rightarrow \quad L\{e^{-at}J_0(at)\} = \frac{1}{\sqrt{[(s+a)^2+a^2]}} = \frac{1}{\sqrt{(s^2+2as+2a^2)}}.$$

(iv) By definition, $\quad L\{J_0(t)\} = \int_0^{\infty} e^{-st}\,J_0(t)\,dt = \frac{1}{\sqrt{(1+s^2)}}.$

Taking limit as $s \to 0$, we have

$$\int_0^{\infty} J_0(t)\,dt = 1.$$

(II) Putting $n = 1$ in (1), we have

$$J_1(t) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\,(r+1)\,!}\cdot\left(\frac{t}{2}\right)^{2r+1} = \frac{t}{2} - \frac{1}{2^2}\,\frac{t^3}{4} + \frac{1}{2^2\,.\,4^2}\cdot\frac{t^5}{6} - \dots$$

$$\Rightarrow \quad L\{J_1(t)\} = \frac{1}{2}\,.\,L\{t\} - \frac{1}{2^2\,.\,4}\,.\,L\{t^3\} + \frac{1}{2^2\,.\,4^2\,.\,6}\,.\,L\{t^5\} - \dots$$

$$\Rightarrow \quad = \frac{1}{2} \cdot \frac{1!}{s^2} - \frac{1}{2^2 . 4} \cdot \frac{3!}{s^4} + \frac{1}{2^2 . 4^2 . 6} \cdot \frac{5!}{s^6} + \dots$$

$$= 1 - \left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) - \frac{1 . 3}{2 . 4}\left(\frac{1}{s^2}\right)^2 - \frac{1 . 3 . 5}{2 . 4 . 6}\left(\frac{1}{s^2}\right)^3 + \dots\right]$$

$$= 1 - \left(1 + \frac{1}{s^2}\right)^{-1/2} = 1 - \frac{s}{\sqrt{(s^2+1)}}.$$

Also, $$L\{tJ_1(t)\} = -\frac{d}{ds} L\{J_1(t)\} = -\frac{d}{ds}\left(1 - \frac{s}{\sqrt{s^2+1}}\right) = \frac{1}{(s^2+1)^{3/2}}.$$

Alternative Method. We know that

$$J_1(t) = -J_0'(t)$$

$$\therefore \quad L\{J_1(t)\} = -L\{J_0'(t)\} = -[sL\,J_0(t) - J_0(0)] = -\left[s \cdot \frac{1}{\sqrt{(s^2+1)}} - 1\right]$$

[$\because$ $J_0(0) = 1$ and use § 2.1-19, 8 (I)]

$$= 1 - \frac{s}{\sqrt{(s^2+1)}}.$$

(III) Putting $n = 0$ and $a\sqrt{t}$ for t in (1), we get

$$J_0(a\sqrt{t}) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2}\left(\frac{a\sqrt{t}}{2}\right)^{2r} = 1 - \frac{a^2 t}{2^2} + \frac{a^4 t^2}{2^2 . 4^2} - \frac{a^6 t^3}{2^2 . 4^2 . 6^2} + \dots$$

$$\Rightarrow \quad L\{J_0(a\sqrt{t})\} = L\{1\} - \frac{a^2}{2^2} L\{t\} + \frac{a^4}{2^2 . 4^2} L\{t^2\} - \dots$$

$$= \frac{1}{s} - \frac{a^2}{2^2} \cdot \frac{1!}{s^2} + \frac{a^4}{2^2 . 4^2} \cdot \frac{2!}{s^3} - \dots$$

$$= \frac{1}{s}\left[1 - \frac{1}{1!}\left(\frac{a^2}{4s}\right) + \frac{1}{2!}\left(\frac{a^2}{4s}\right)^2 - \frac{1}{3!}\left(\frac{a^2}{4s}\right)^3 + \dots\right]$$

$$= \frac{1}{s} \cdot e^{-(a^2/4s)}.$$

Ans.

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