Solution of non-linear equations:

1) **Bisection method:** This method consists in locating the root of the equation $f(x) = 0$ between a and b.

Conditions:

1) If $f(x)$ is continuous between a and b.

2) $f(a)$ and $f(b)$ are of opposite signs

Then there is a root between a and b.

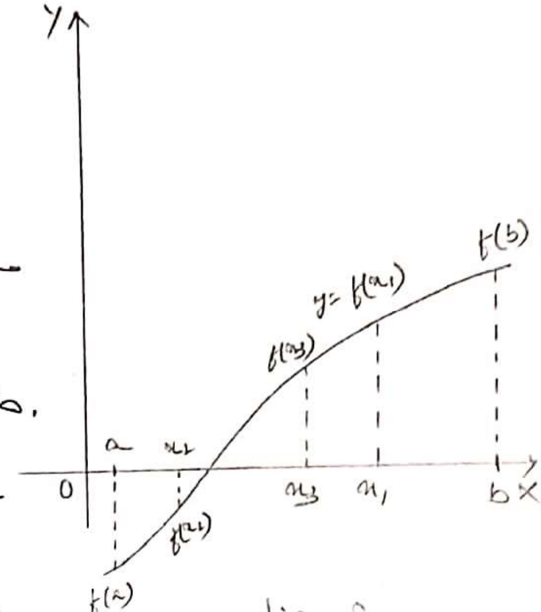

fig – I

Let $f(a)$ be negative and $f(b)$ be positive.

Then the first approximation to the root is $x_1 = \frac{1}{2}(a+b)$.

If $f(x_1) = 0$, then x_1 is a root of $f(x) = 0$, otherwise, the root lies between a and x_1 or x_1 and b according as $f(x_1)$ is positive or negative.

Then we bisect the interval as before and continue the process until the root is found to desired accuracy

Note:- In the fig-I, $f(x_1)$ is positive, so the root lies between a and x_1. Therefore, the second approximation to the root is $x_2 = \frac{1}{2}(a+x_1)$.

Again, $f(x_2)$ is negative, therefore, the root lies between x_1 and x_2 and the third approximation to the root is $x_3 = \frac{1}{2}(x_1+x_2)$ and so on.

Q→ Find a root of the equation $x^3 - 4x - 9 = 0$ using bisection method correct to three decimal places.

Sol: Let $f(x) = x^3 - 4x - 9$

Now, $f(0) = -9 < 0$

$f(1) = -12 < 0$

$f(2) = -9 < 0$

$f(3) = 6 > 0$

Since $f(2)$ is -ve and $f(3)$ is +ve, the root lies between 2 and 3.

$\therefore \; x_1 = \frac{1}{2}(2+3) = 2.5$

and $f(x_1) = (2.5)^3 - 4 \times 2.5 - 9$

$= -3.375 < 0$

Since $f(x_1) < 0$ and $f(3) > 0$, the root lies between x_1 and 3.

$\therefore \; x_2 = \frac{1}{2}(x_1 + 3) = \frac{1}{2}(2.5 + 3) = 2.75$

and $f(x_2) = (2.75)^3 - 4 \times 2.75 - 9 = 0.7969 > 0$

Since $f(x_2) > 0$ and $f(x_1) < 0$, the root lies between x_1 and x_2.

$\therefore \; x_3 = \frac{1}{2}(x_1 + x_2) = \frac{1}{2}(2.5 + 2.75) = 2.625$

and $f(x_3) = (2.625)^3 - 4 \times 2.625 - 9 = -1.4121 < 0$

Since $f(x_3) < 0$ and $f(x_2) > 0$, the root lies between x_2 and x_3.

$\therefore \; x_4 = \frac{1}{2}(x_2 + x_3) = \frac{1}{2}(2.75 + 2.625) = 2.6875$

and $f(x_4) = (2.6875)^3 - 4 \times 2.6875 - 9 = -0.3391 < 0$

Since $f(x_4) < 0$ and $f(x_2) > 0$, the root lies between x_2 and x_4. ⓑ

$\therefore x_5 = \frac{1}{2}(x_2 + x_4) = \frac{1}{2}(2.75 + 2.6875) = 2.71875$

and $f(x_5) = (2.71875)^3 - 4 \times 2.71875 - 9 = 0.2209 > 0$

Since $f(x_5) > 0$ and $f(x_4) < 0$, the root lies between x_4 and x_5.

$\therefore x_6 = \frac{1}{2}(x_4 + x_5) = \frac{1}{2}(2.6875 + 2.71875) = 2.70313$

and $f(x_6) = (2.70313)^3 - 4 \times 2.70313 - 9 = -0.0609 < 0$

Since $f(x_6) < 0$ and $f(x_5) > 0$, the root lies between x_5 and x_6.

$\therefore x_7 = \frac{1}{2}(x_5 + x_6) = \frac{1}{2}(2.71875 + 2.70313) = 2.71094$

and $f(x_7) = (2.71094)^3 - 4 \times 2.71094 - 9 = 0.07947 > 0$

Since $f(x_7) > 0$ and $f(x_6) < 0$, the root lies between x_6 and x_7.

$\therefore x_8 = \frac{1}{2}(x_6 + x_7) = \frac{1}{2}(2.70313 + 2.71094)$

$= 2.70703$

and $f(x_8) = (2.70703)^3 - 4 \times 2.70703 - 9$

$= 0.03594 > 0$

Since $f(x_8) > 0$ and $f(x_6) < 0$, the root lies between x_6 and x_8.

$\therefore x_9 = \frac{1}{2}(x_6 + x_8) = \frac{1}{2}(2.70313 + 2.70703)$

$= 2.70508$

and $f(x_9) = (2.70508)^3 - 4 \times 2.70508 - 9$

$= -0.0260 < 0$

Since $f(x_9) < 0$ and $f(x_8) > 0$, the root lies between x_8 and x_9.

$$\therefore x_{10} = \frac{1}{2}(x_8 + x_9) = \frac{1}{2}(2.70703 + 2.70508)$$

$$= 2.70605$$

and $$f(x_{10}) = (2.70605)^3 - 4 \times 2.70605 - 9$$

$$= -0.00858 < 0.$$

Since $f(x_{10}) < 0$ and $f(x_8) > 0$, the root lies between x_8 and x_{10}.

$$\therefore x_{11} = \frac{1}{2}(x_8 + x_{10}) = \frac{1}{2}(2.70703 + 2.70605)$$

$$= 2.70654$$

Since x_{10} and x_{11} are the same upto three decimal places, the root is 2.706

Q→ Find the root of the following equations correct upto four decimal places

1) $x^3 + 3x - 1 = 0$

2) $x^3 + x^2 - 1 = 0$

3) $x^4 - x^3 - 2x^2 - 6x - 4 = 0$

4) $3x = \sqrt{1 + \sin x}$

5) $3x = \cos x + 1$

6) $x - \cos x = 0$

7) $x e^x = 1$

8) $x \log_{10} x = 12$ lying between 2 and 3.

2) Regula-falsi method (Method of false position)

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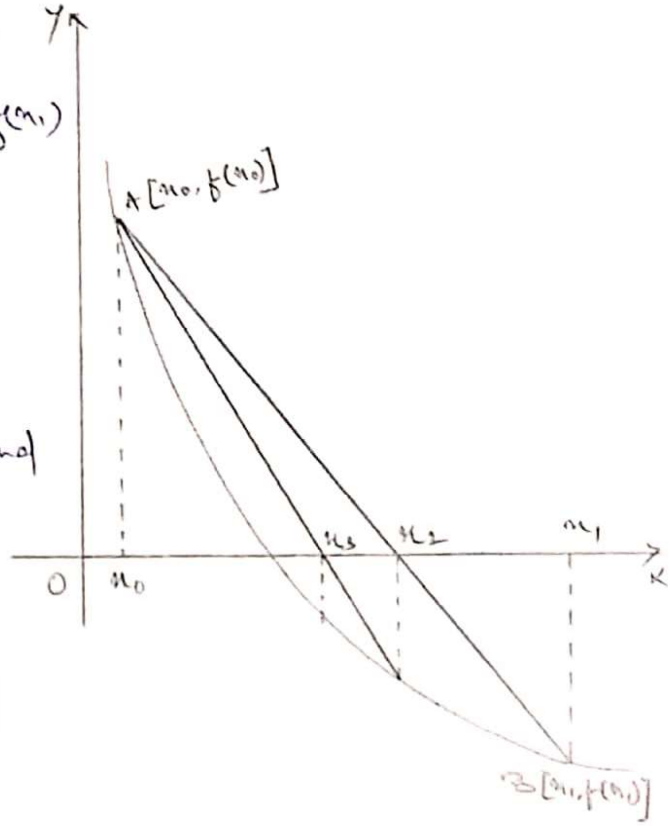
Here, we choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs i.e. the graphs of $y = f(x)$ crosses the x-axis between these points. This indicates that a root lies between x_0 and x_1 and $f(x_0) \cdot f(x_1) < 0$.



Now, equation of the chord joining the points $A[x_0, f(x_0)]$ and $B[x_1, f(x_1)]$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} (x - x_0) \quad \text{——— (1)}$$

This method consists in replacing the curve AB by means of the chord AB and taking the point of intersection of the chord with the x-axis as an approximation to the root. So, the abscissa of the point where the chord cuts the x-axis $(y=0)$ is given by

$$\boxed{x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)} \quad \text{——— (2)}$$

$$\boxed{= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}}$$

Now, if $f(x_0)$ and $f(x_2)$ are of opposite signs, then the root lies between x_0 and x_2. So, replacing x_1 by x_2 in (2), we obtain the next approximation x_3 and this procedure is repeated till the root is found to desired accuracy.

Q→ Find a real root of the equation $x \log_{10} x = 1.2$ by regula-falsi method correct upto four decimal places.

Sol: Let $f(x) = x \log_{10} x - 1.2$

Now, $f(1) = 1 \log_{10} 1 - 1.2 = -1.2 < 0$

$f(2) = 2 \log_{10} 2 - 1.2 = -0.59794 < 0$

$f(3) = 3 \log_{10} 3 - 1.2 = 0.23136 > 0$

Since $f(2) < 0$ and $f(3) > 0$, therefore, the root lies between 2 and 3.

Taking $x_0 = 2$ and $x_1 = 3$,

$f(x_0) = -0.59794$ and $f(x_1) = 0.23136$

$$\therefore x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{2 \times 0.23136 - 3 \times (-0.59794)}{0.23136 - (-0.59794)}$$

$$= 2.72102$$

and $f(x_2) = 2.72102 \log_{10} (2.72102) - 1.2$

$= -0.01709 < 0$

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Since $f(x_2) < 0$ and $f(x_1) > 0$, the root lies between x_1 and x_2.

$$\therefore x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$$= \frac{(2.72102)(0.23136) - 3(-0.01709)}{0.23136 - (-0.01709)}$$

$$= 2.74021$$

and $f(x_3) = 2.74021 \log_{10}(2.74021) - 1.2$

$$= -3.80325 \times 10^{-4} < 0$$

Since $f(x_3) < 0$ and $f(x_1) > 0$, the root lies between x_3 and x_1.

$$\therefore x_4 = \frac{x_3 f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)}$$

$$= \frac{(2.74021)(0.23136) - 3(-0.0003803)}{0.23136 - (-0.0003803)}$$

$$= 2.74063$$

and $f(x_4) = (2.74063) \log_{10}(2.74063) - 1.2$

$$= -1.4038 \times 10^{-5} < 0$$

Since $f(x_4) < 0$ and $f(x_1) > 0$, the root lies between x_4 and x_1.

$$\therefore x_5 = \frac{x_4 f(x_1) - x_1 f(x_4)}{f(x_1) - f(x_4)}$$

$$= \frac{(2.74063)(0.23136) - 3(-0.0000140)}{0.23136 - (-0.000014)}$$

$= \cancel{2.74064} = 2.74064$

Since x_4 and x_5 are the same upto four decimal places, therefore, the root is ~~2~~ 2.7406.

Q-> use regula-falsi method to find the root of $2x - \log_{10} x = 7$, which lies between 3.5 and 4, correct upto five places of decimal.

Q-> using regula-falsi method, find the root of

(a) $x^3+x+1=0$ correct upto three decimal places

(b) $xe^x = 3$ correct upto three decimal places

(c) $e^{-x} - \sin x = 0$ correct upto three decimal places.

(d) $xe^x = \cos x$ correct upto three decimal places.

3) Newton-Raphson method:

Let x_0 be an approximate root of the equation $f(x)=0$.
If $x_1 = x_0 + h$ be the exact root, then $f(x_1) = 0$

$\therefore$ Expanding $f(x_0+h)$ by Taylor's series,

$$f(x_0+h) = 0$$

$$\Rightarrow f(x_0) + hf'(x_0) + \frac{h^2}{\underline{|2}} f''(x_0) + \cdots\cdots = 0$$

Since h is small, neglecting h^2 and higher powers of h, we get

$$f(x_0) + hf'(x_0) = 0$$

$$\Rightarrow h = -\frac{f(x_0)}{f'(x_0)}$$

$\therefore$ a closer approximation to the root is given by

$$x_1 = x - \frac{f(x_0)}{f'(x_0)}$$

Similarly, starting with x_1, a still better approximation x_2 is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general,

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

which is known as the Newton-Raphson formula or Newton's iteration formula.

Geometrical interpretation:

Let x_0 be a point near the root α of the equation $f(x)=0$ (fig-1)

Then the equation of the tangent at $A_0[x_0, f(x_0)]$ is

$$y - f(x_0) = f'(x_0)(x-x_0)$$

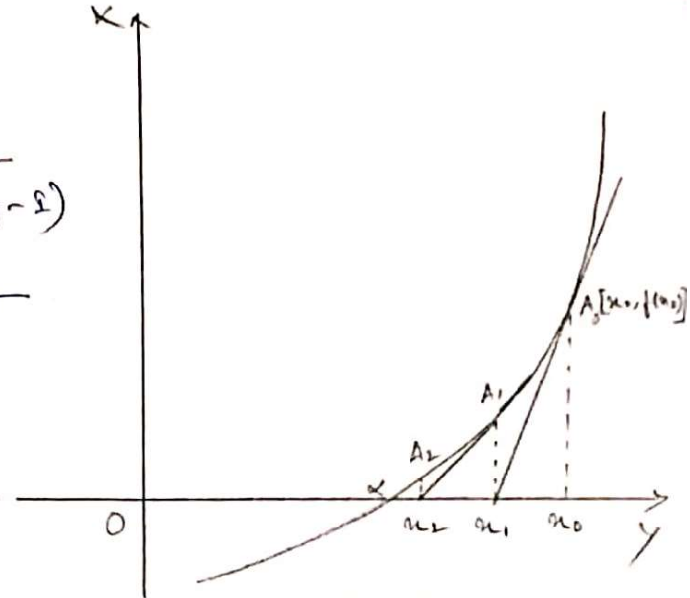

fig-1

It cuts the x-axis at $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

which is the first approximation to the root α.

If A_1 be the point corresponding to x_1 on the curve, then the tangent at A_1 will cut the x-axis at x_2, which is nearer to α and is therefore a second approximation to the root. Repeating this process, we approach to the root α quite rapidly.

Q→ Find by Newton-Raphson method, the real root of the equation $3x = \cos x + 1$ correct to four decimal places.

Sol^n: Let $f(x) = 3x - \cos x - 1$

(Note: angles should be changed to degree first $1 \text{ radian} \approx 57.2958°$)

Now, $f(0) = -1 - 1 = -2 < 0$

$f(1) = 3 - \cos(57.2958) - 1$

$= 1.4597 > 0$

So a root of $f(x) = 0$ lies between 0 and 1. It is nearer to 1.

Let us take $x_0 = 0.6$.

Also, $f'(x) = 3 + \sin x$

∴ Newton-Raphson formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$

$$= \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n} \quad \text{——— (1)}$$

Putting $n = 0$ in (1), the first approximation x_1 is given by

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = \frac{0.6 \sin(0.6) + \cos(0.6) + 1}{3 + \sin(0.6)} = 0.6071$$

Putting $n = 1$ in (1), we get

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} = \frac{0.6071 \sin(0.6071) + \cos(0.6071) + 1}{3 + \sin(0.6071)} = 0.6071$$

Since the values of x_1 and x_2 are the same, therefore, the reqd. root is 0.6071.

Q→ Find the positive root of $x^4 - x = 10$ correct upto three decimal places, using Newton-Raphson method.

Q→ Using Newton-Raphson method, find the root of

(a) $x \log_{10} x = 1.2$ correct upto five decimal places.

(b) $x^3 + 2x^2 + 10x - 20$ upto 10 iterations

(c) $x^4 - x - 9 = 0$ correct upto four places of decimal.

Q→ Find the iterative formula to find $\frac{1}{N}$.

Soln:- Let $x = \frac{1}{N}$

$$\Rightarrow \frac{1}{x} - N = 0$$

Taking $f(x) = \frac{1}{x} - N$

$$\Rightarrow f'(x) = -x^{-2}$$

∴ Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\left(\frac{1}{x_n} - N\right)}{-\left(\frac{1}{x_n}\right)^2} = x_n + x_n - Nx_n^2 = x_n(2 - Nx_n)$$

$$\therefore x_{n+1} = x_n(2 - Nx_n).$$

Q→ Find $\frac{1}{31}$ using Newton's method.

Taking $N = 31$ in the above formula, we get

$\cancel{x_{n+1} = x_n(2 - 31x_n)}$

$$x_{n+1} = x_n(2 - 31x_n)$$

Since an approximate value of $\frac{1}{31}$ is 0.03, we take $x_0 = 0.03$

Then, $x_1 = x_0(2 - 31x_0) = 0.0321$

$$x_2 = x_1(2 - 31x_1) = 0.032257$$

$x_3 = x_2(2 - 31x_2) = 0.03226$

Since x_2 and x_3 are the same upto four places of decimal, therefore, the reqd. solution is 0.0322.

Q→ using Newton-Raphson formula, find the iterative formula to find

(a) $\sqrt{N}$

(b) $1/\sqrt{N}$

(c) $\sqrt[3]{N}$

Q→ using Newton-Raphson method, find

(a) $\sqrt{5}$

(b) $1/\sqrt{14}$

(c) $\sqrt[3]{24}$