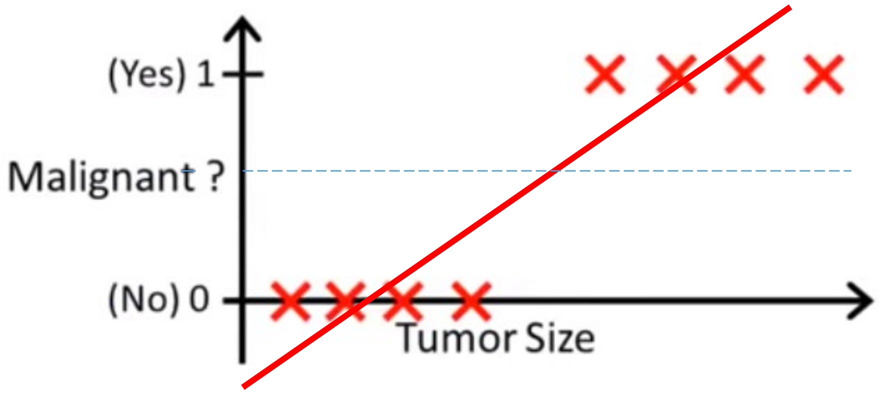
Logistic Regression

Classification problem



Use linear regression? $h_\theta(x) = x^T\theta$

Threshold classifier output $h_\theta(x)$ at 0.5:

If $h_\theta(x) \geq 0.5$, predict "y = 1"

If $h_\theta(x) < 0.5$, predict "y = 0"

Inconvenience of linear regression:

Classification: y = 0 or 1

$h_\theta(x)$ can be > 1 or < 0

Introduce:

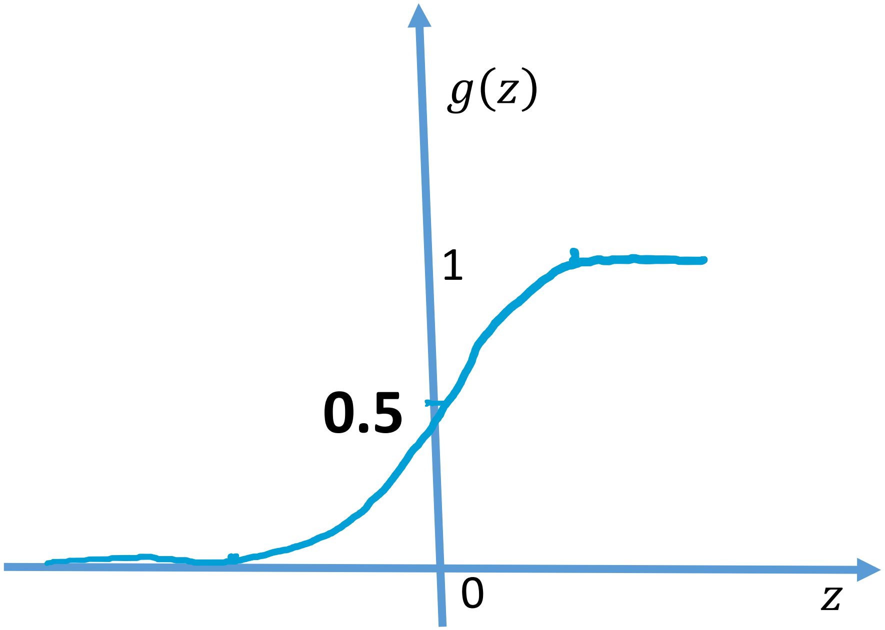
Logistic Regression: $0 \leq h_\theta(x) \leq 1$

Logistic regression model:

Goal: $0 \leq h_\theta(x) \leq 1$

$$h_\theta(x) = g(x^T\theta)$$

where $g(z) = \dfrac{1}{1+e^{-z}}$



Sigmoid/logistic function

$h_\theta(x)$: estimated probability that $y = 1 \; or \; 0$ given input x

$$h_\theta(x) = P(y|x,\theta)$$

$$h_\theta(x) = P(y|x, \theta)$$

- Probability of predicting y, given x, parameterized by θ

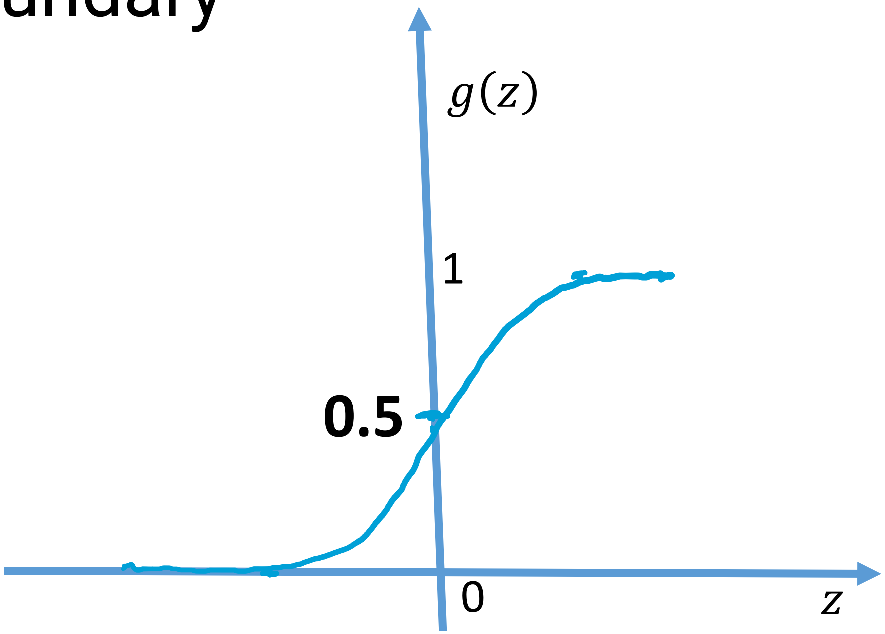
$$\text{Predict } y = 1 \text{ if } h_\theta(x) \geq 0.5$$
$$\text{Predict } y = 0 \text{ if } h_\theta(x) < 0.5$$

- For example, $h_\theta(x) = 0.85$, tell patient that 85% of chance of tumor being cancerous

Decision Boundary

$$h_\theta(x) = g(x^T\theta)$$

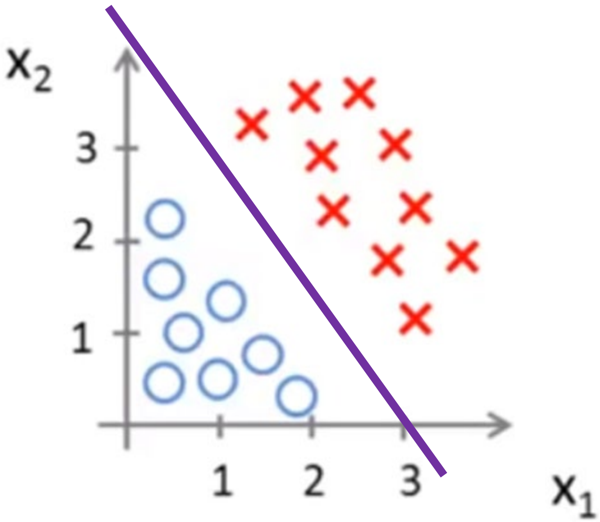
(Let $z = x^T\theta$)



Predict $y = 1$ if $h_\theta(x) \geq 0.5$ or $z \geq 0$
Predict $y = 0$ if $h_\theta(x) < 0.5$ or $z < 0$

Decision boundary: $x^T\theta = 0$

Decision Boundary

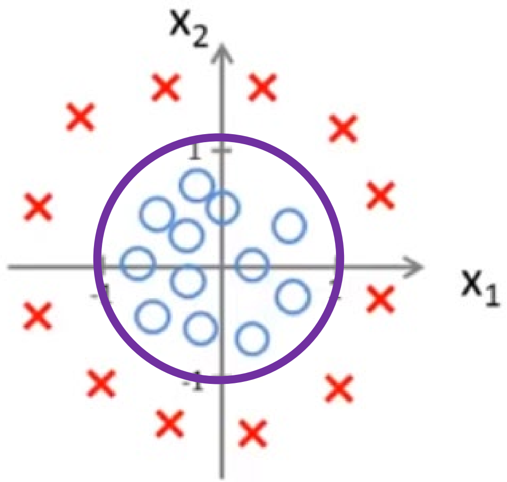


$$h_\theta(x) = g(-3 + x_1 + x_2)$$

Predict $y = 1$ if $h_\theta(x) \geq 0.5$ $or (-3 + x_1 + x_2) \geq 0$
Predict $y = 0$ if $h_\theta(x) < 0.5$ $or (-3 + x_1 + x_2) < 0$

Decision boundary:
$\boldsymbol{x_1 + x_2 = 3}$

Nonlinear decision boundaries



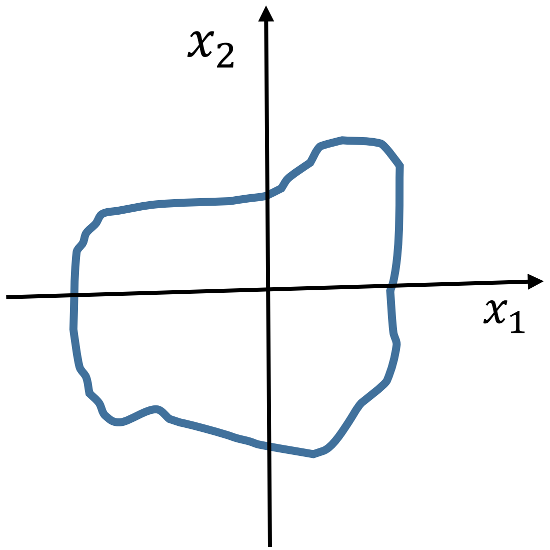
$$h_\theta(x) = g(\theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_1^2 + \theta_4 x_2^2)$$

θ=[-1, 0, 0, 1, 1]

Decision boundary:

$$\boldsymbol{x_1}^2 + \boldsymbol{x_2}^2 = 1$$

More complicated nonlinear decision boundary



$$h_\theta(x) = g\Big(\theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_1^2 + \theta_4 x_1^2 x_2 + \theta_5 x_1^2 x_2^2 + \theta_6 x_1^3 x_2 + \cdots\Big)$$

Decision boundary:

$$\theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_1^2 + \theta_4 x_1^2 x_2 + \theta_5 x_1^2 x_2^2 + \theta_6 x_1^3 x_2 + \cdots = 0$$

Given training set for supervised learning:

$$\left\{\left(x^{(1)}, y^{(1)}\right), \left(x^{(2)}, y^{(2)}\right), \cdots, \left(x^{(m)}, y^{(m)}\right)\right\}$$

where

$$x \in \begin{bmatrix} x_0 \\ x_1 \\ \cdots \\ x_n \end{bmatrix} \qquad x_0 = 1, y \in \{0,1\}$$

In a logistic regression model,

$$h_\theta(x) = \frac{1}{1 + e^{-\theta^T x}}$$

How to choose parameter θ?

Cost function consideration:

If we use the cost formulation as in linear regression,

$$J(\theta) = \frac{1}{m}\sum_{i=1}^{m} \frac{1}{2}\left(h_\theta\left(x^{(i)}\right) - y^{(i)}\right)^2$$

Recall:

$$h_\theta(x) = g(x^T\theta) = \frac{1}{1+e^{-(x^T\theta)}}$$

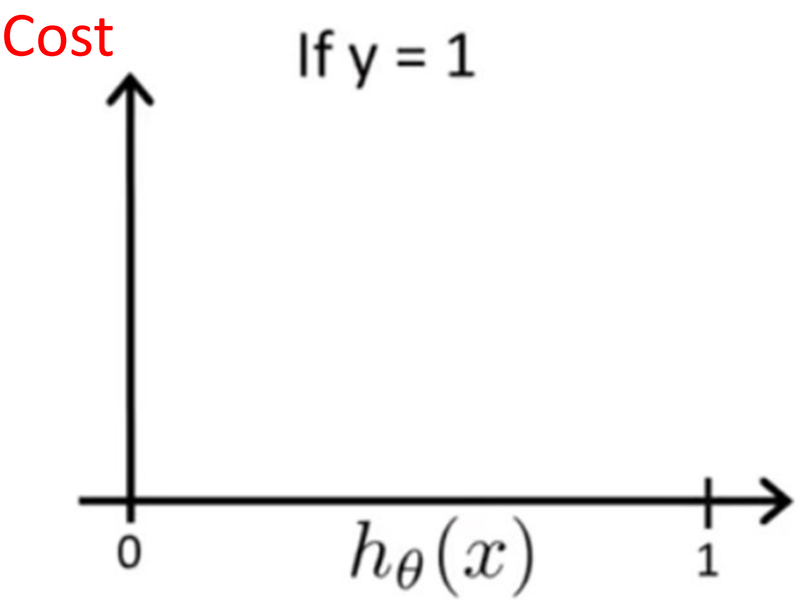
g is a sigmoid/logistic function

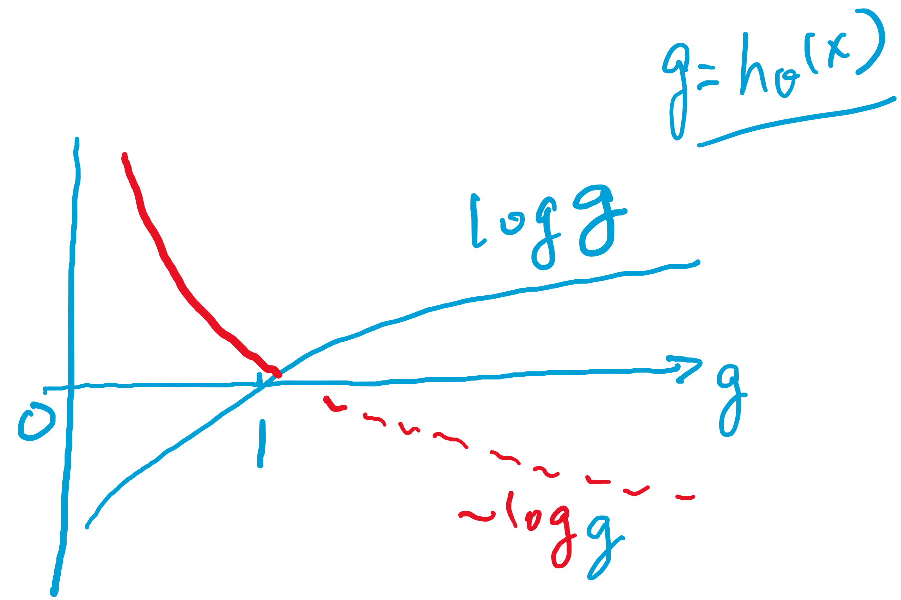
Look at the term:

$$\frac{1}{2}\left(h_\theta\left(x^{(i)}\right) - y^{(i)}\right)^2 \Longrightarrow \text{Nonconvex, thus local minima}$$

Propose “logistic regression cost function” as follows:

$$\mathrm{Cost}(h_\theta(x), y) = \begin{cases} -\log\left(h_\theta(x)\right) & if\ y = 1 \\ -\log\left(1 - h_\theta(x)\right) & if\ y = 0 \end{cases}$$



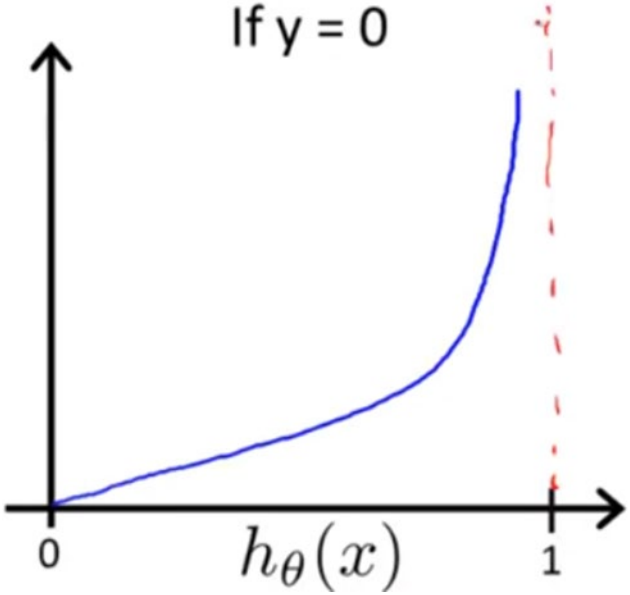


$$Cost\ = 0\ if\ y = 1, h_\theta(x) = 1$$

$$Cost\ \to \infty\ \ if\ y = 1, but\ \ h_\theta(x) \to 0$$

$$\mathrm{Cost}(h_\theta(x), y) = \begin{cases} -\log(h_\theta(x)) & if\ y = 1 \\ -\log(1 - h_\theta(x)) & if\ y = 0 \end{cases}$$

Cost



$$Cost\ = 0\ if\ y = 0, h_\theta(x) = 0$$

$$Cost\ \to \infty\ \ if\ y = 0, but\ \ h_\theta(x) \to 1$$

Logistic regression cost function

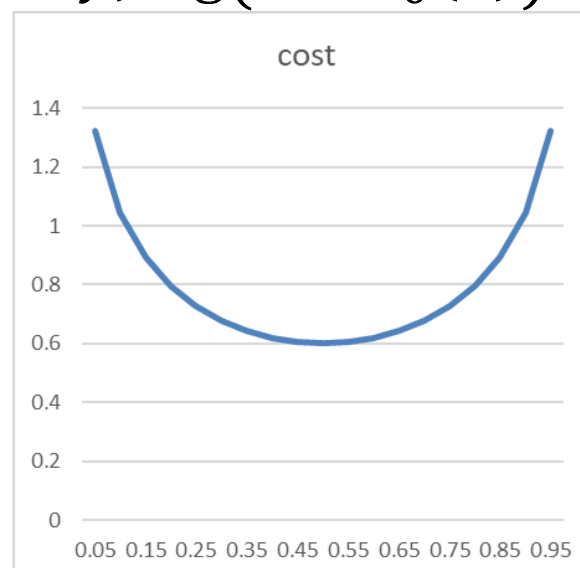
$$J(\theta) = \frac{1}{m}\sum_{i=1}^{m} \mathrm{Cost}\left(h_\theta\left(x^{(i)}\right), y^{(i)}\right)$$

$$\mathrm{Cost}(h_\theta(x), y) = \begin{cases} -\log\left(h_\theta(x)\right) & \text{if } y = 1 \\ -\log\left(1 - h_\theta(x)\right) & \text{if } y = 0 \end{cases}$$

Note: $y = 0$ or 1

Simplification:

$$\Rightarrow \quad \mathrm{Cost}(h_\theta(x), y) = -y\log\left(h_\theta(x)\right) - (1 - y)\log\left(1 - h_\theta(x)\right)$$



Logistic regression cost function

$$J(\theta) = \frac{1}{m}\sum_{i=1}^{m} \mathrm{Cost}\left(h_\theta\left(x^{(i)}\right), y^{(i)}\right)$$

$$= -\frac{1}{m}\sum_{i=1}^{m} y^{(i)} \log\left(h_\theta\left(x^{(i)}\right)\right) - \left(1 - y^{(i)}\right)\log\left(1 - h_\theta\left(x^{(i)}\right)\right)$$

To fit parameters θ:

$$\min_{\theta} J(\theta)$$

To get parameters

$$\boldsymbol{\theta} = (\theta_0, \theta_1, \dots, \theta_n)$$

To make a prediction given new x:

Compute output:

$$h_\theta(x) = \frac{1}{1 + e^{-\theta^T x}}$$

Apply gradient descent to reduce the cost measure

$$J(\theta) = -\frac{1}{m}\sum_{i=1}^{m} y^{(i)} \log\left(h_\theta\left(x^{(i)}\right)\right) - \left(1 - y^{(i)}\right)\log\left(1 - h_\theta\left(x^{(i)}\right)\right)$$

Update parameters $\theta_0\ \theta_1 \ldots \theta_n$ using gradient descent until convergence

Simultaneously update θ_j , $j = 0, 1, \ldots, n$, according to

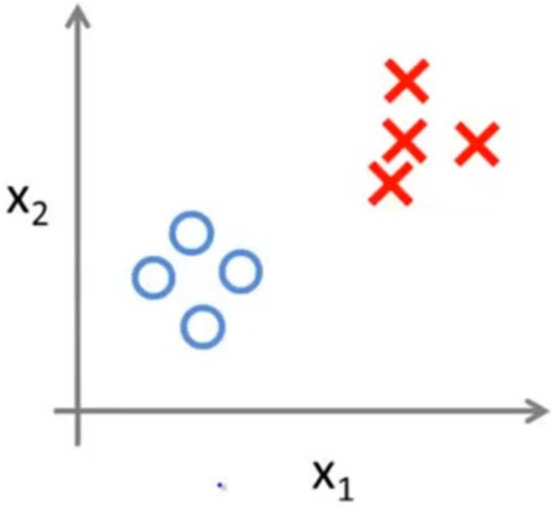
$$\theta_j := \theta_j - \eta \frac{\partial}{\partial \theta_j} J(\theta_0, \theta_1, \ldots, \theta_n)$$

$$\frac{\partial}{\partial \theta_j} J(\theta_0, \theta_1, \ldots, \theta_n) = \frac{1}{m}\sum_{i=1}^{m}\left(h_\theta\left(x^{(i)}\right) - y^{(i)}\right) x_j{}^{(i)}$$

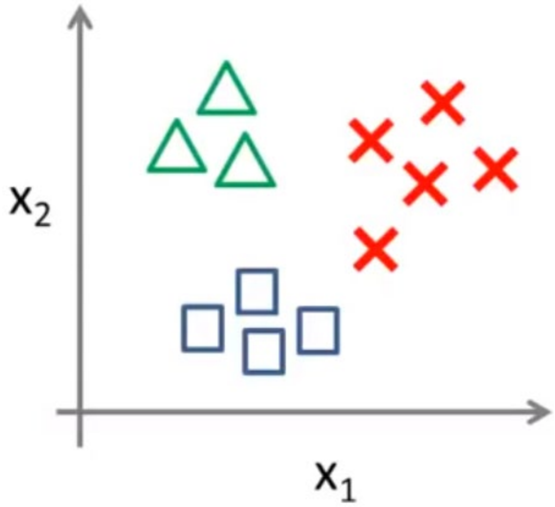
One-vs-all for multi-class classification

Train a logistic regression classifier $h_\theta^{(i)}(x)$ for each class i to predict the probability that $y = i$.

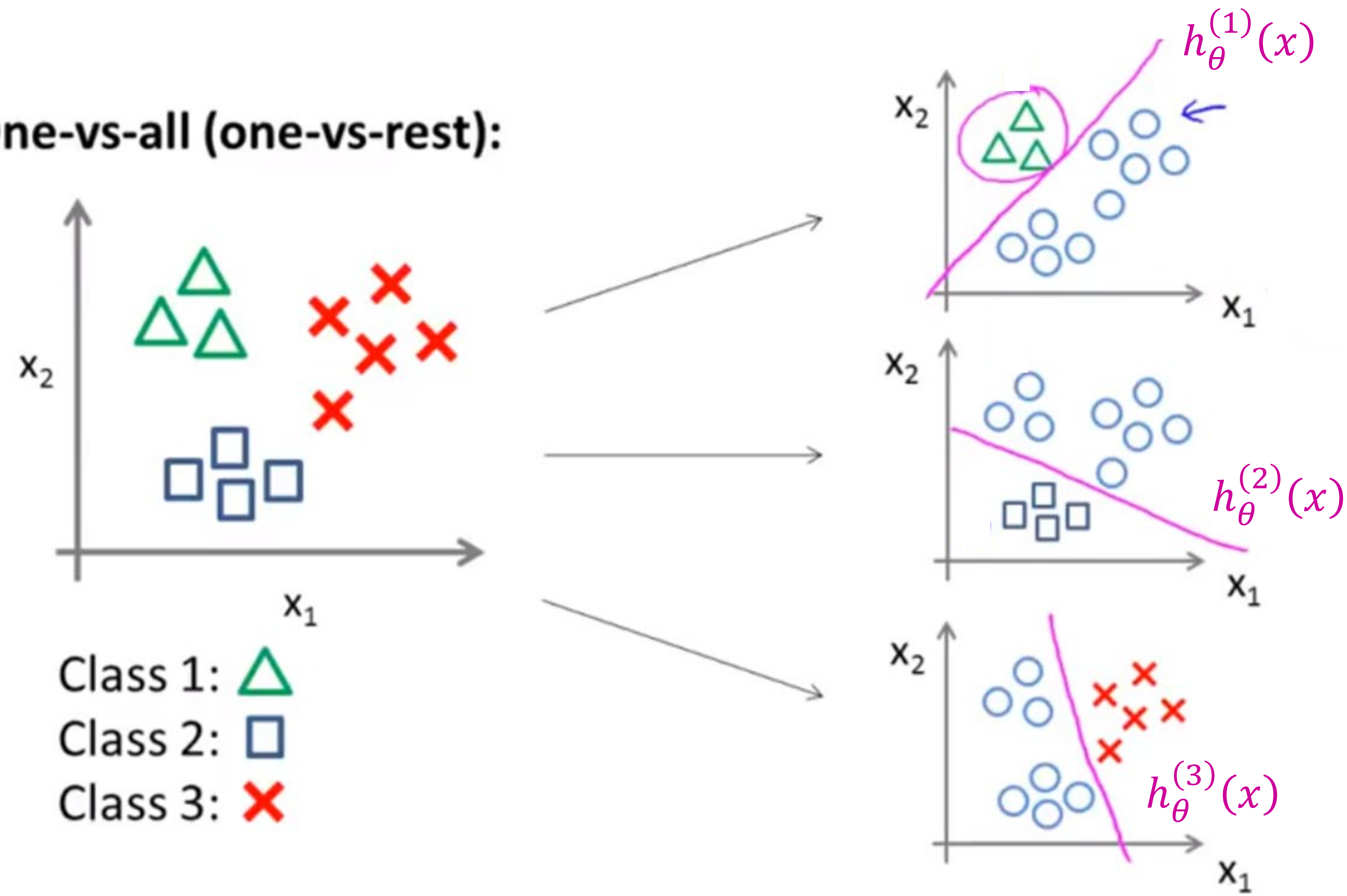
Binary classification:



Multi-class classification:



One-vs-all (one-vs-rest):



$$h_\theta^{(i)}(x) = P(y = i|x;\theta) \qquad (i = 1, 2, 3)$$