Derivation of Nonlinear Mathematical Model of Two-Wheeled Inverted Pendulum

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***Abstract*—This paper begins with a useful presentation of known theoretical mechanics concepts for obtaining the equations of motion for systems of rigid bodies with and without non-holonomic constraints, using D'Alembert's principle. Then the equations of motion of a two-wheeled inverted pendulum on a flat surface with non-sliding wheels are obtained, by using the presented theory. Finally the nonlinear dynamical system is linearized and a linear state feedback control law for multi-input systems is obtained. The nonlinear system is simulated and stabilized using the linear control law.**

***Index Terms*—two-wheeled inverted pendulum; D'Alembert's principle; system modelling**

I. Introduction

In the recent years two-wheeled inverted pendulum platforms were increasingly used for design of personal transport vehicles and robots. In the case of robots as well as in the case of transport vehicle one might want to be in position of setting a predefined trajectory for the device. This paper provides a full nonlinear mathematical model for the two-wheeled inverted pendulum assuming a flat surface and non-sliding wheels, through the use of non-holonomic velocity constraints. The paper begins with a section with a good synthesis of some theoretical mechanics concepts useful for mathematical modeling of systems of rigid bodies with and without non-holonomic constraints. The next section gives in great detail the derivation of the full nonlinear model of the vehicle. The following section provides the linear model and some information about the controllability of the system. Also a didactic linear state feedback is obtained. The nonlinear model is simulated and stabilized with this state feedback. The paper ends with a Conclusion and Future work section.

As for the work done by other researchers, the Figure 1, taken from [5] shows the state of the art for this kind of models.

II. D'Alembert's principle for a system of rigid bodies

In the following subsections we recall some results from [3] and [4] and present them in a convenient form.

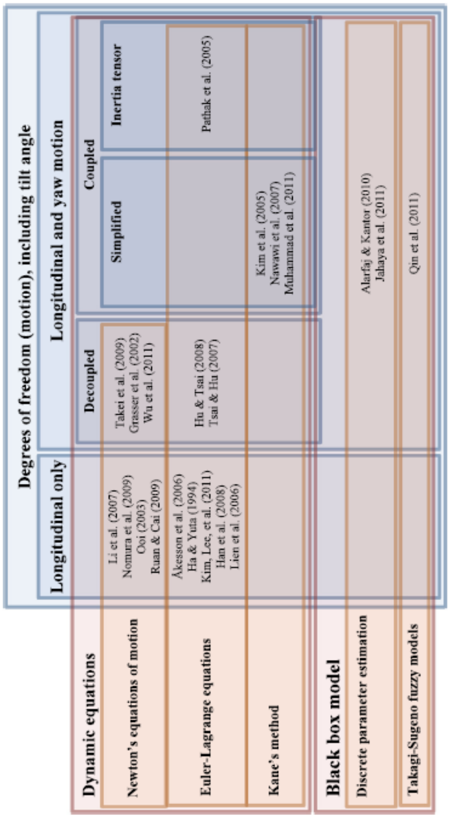

Fig. 1. State of the art. From [5]

A. Without non-holonomic constraints

Consider an unconstrained system of N rigid bodies with n degrees of freedom. Let $\vec{v}_1(t), \dots, \vec{v}_N(t)$ be the velocities of the center of mass for each body, $\vec{\omega}_1(t), \dots, \vec{\omega}_N(t)$ the axis of rotation for each body, $\vec{F}_1(t), \dots, \vec{F}_N(t)$ the resultant of the applied forces for each body and $\vec{T}_1(t), \dots, \vec{T}_N(t)$ the resultant applied torque about the center of mass for each body. Let $q_1, \dots, q_n$ denote the independent generalized coordinates in which the geometry of the vehicle is expressed. Also, let δW denote the virtual work, and δq_i denote a virtual displacement. It can be shown that the principle of D'Alembert for a system of rigid bodies states:

$$\delta W = \sum_{j=1}^{n}\left(\sum_{i=1}^{N}\left(\vec{F}_i - \dot{\vec{P}}_i\right)^T \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} + \sum_{i=1}^{N}\left(\vec{T}_i - \dot{\vec{L}}_i\right)^T \frac{\partial \vec{\omega}_i}{\partial \dot{q}_j}\right)\delta q_j \equiv 0 \quad (1)$$

where $\vec{P}_i$ and $\vec{L}_i$ are the linear and angular momentum of the i'th rigid body. Because the virtual displacements are independent one has:

$$\begin{cases} \sum_{i=1}^{N}\left(\vec{F}_i - \dot{\vec{P}}_i\right)^T \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_1} + \sum_{i=1}^{N}\left(\vec{T}_i(t) - \dot{\vec{L}}_i\right)^T \cdot \frac{\partial \vec{\omega}_i}{\partial \dot{q}_1} = 0 \\ \vdots \\ \sum_{i=1}^{N}\left(\vec{F}_i(t) - \dot{\vec{P}}_i\right)^T \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_n} + \sum_{i=1}^{N}\left(\vec{T}_i - \dot{\vec{L}}_i\right)^T \cdot \frac{\partial \vec{\omega}_i}{\partial \dot{q}_n} = 0 \end{cases} \quad (2)$$

This is a system of n equations fully describing the motion of the system of rigid bodies. All the forces, moments and velocities are expressed in world frame. Therefore, in the following we are interested in expressing the linear and angular velocity and momentum as well as the applied forces and torques in the world frame using the generalized coordinates $q = [q_1, \dots, q_n]^T$

Remark II.1. One can recognize in Equation (2) the, so called, generalized force Q_j associated with the virtual displacement δq_j:

$$Q_j = \sum_{i=1}^{N}\left(\vec{F}_i(t)\right)^T \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} + \sum_{i=1}^{N}\left(\vec{T}_i(t)\right)^T \cdot \frac{\partial \vec{\omega}_i}{\partial \dot{q}_j} \quad (3)$$

and the so called, generalized inertia force Q_j^*:

$$Q_j^* = -\left(\sum_{i=1}^{N}\left(\dot{\vec{P}}_i(t)\right)^T \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} + \sum_{i=1}^{N}\left(\dot{\vec{L}}_i(t)\right)^T \cdot \frac{\partial \vec{\omega}_i}{\partial \dot{q}_j}\right) \quad (4)$$

Therefore $\delta W = \sum_{j=1}^{n}\left(Q_j + Q_j^*\right)\delta q_j = 0$ and from here, because the generalised coordinates q_j are independent follow the equations of motion $Q_j + Q_j^* = 0 \ \forall j \in \{1, \dots, n\}$

B. With non-holonomic constraints

Suppose there are m non-holonomic constraints written in the following form [1]:

$$A(q) \cdot \dot{q} = O_{m\times 1} \quad (5)$$

In this case the generalized coordinates q are no longer independent, hence we can not simply write $Q_j + Q_j^* = 0$ $\forall j$. Suppose that $rank(A(q)) = m$ that is the constraints are independent. From [1] we define $\lambda \in \mathbb{R}^m$ and the equations of motion are:

$$Q_j + Q_j^* = \lambda^T \cdot A_j(q) \qquad \forall j \in \{1, \dots, n\} \quad (6)$$

where $A_j(q)$ is the $j'th$ column of $A(q)$. One can rewrite Equation (6) in the following form:

$$\begin{bmatrix} Q_1 + Q_1^* \\ \vdots \\ Q_n + Q_n^* \end{bmatrix} = A(q)^T \cdot \lambda \quad (7)$$

From [2], let $W(q) \in \mathbb{R}^{n\times(n-m)}$ be a matrix whose columns form a base for the null space of $A(q)$ hence $A(q) \cdot W(q) = O_{m\times(n-m)}$. Upon multiplying the left side of Equation (7) with $W^T(q)$, one obtains the equations of motion:

$$W^T(q) \cdot \begin{bmatrix} Q_1 + Q_1^* \\ \vdots \\ Q_n + Q_n^* \end{bmatrix} = W^T(q) \cdot A(q)^T \cdot \lambda = O_{(n-m)\times m} \cdot \lambda = O_{(n-m)\times 1} \quad (8)$$

Equation (8) and (5) form together, a system of n equations with n unknowns.

In the following we will apply this well known theory to obtain the nonlinear model of a two-wheeled inverted pendulum, suitable for simulation and control.

III. The Derivation of Nonlinear Mathematical Model

A. The geometry of the vehicle

The two-wheeled inverted pendulum we are modeling here is composed of three rigid bodies: two wheels and the main body, as in the following figure. The depicted axes are $\vec{i}, \vec{j}, \vec{k}$

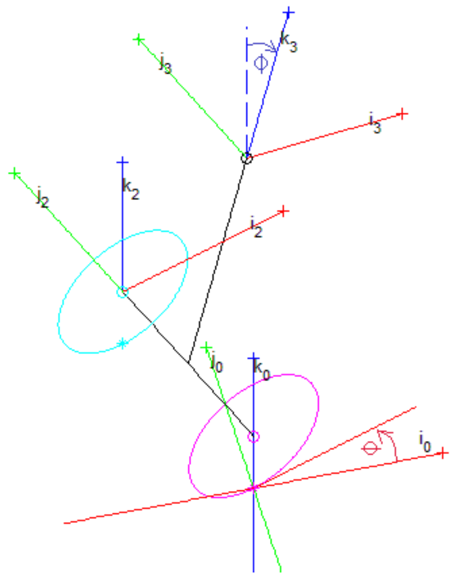

Fig. 2. The two-wheeled inverted pendulum

with red, green and blue, respectively. The generalized coordinates are x, y the position of the right wheel (with magenta) contact point, θ, ϕ the rotation about $\vec{k}$ and $\vec{j}$ axis respectively. Then there are ϕ_1 and ϕ_2 the angles of the first respectively second wheel. Hence $q = [x, y, \theta, \phi, \phi_1, \phi_2]^T$ We will denote $\{\vec{i}_0, \vec{j}_0, \vec{k}_0\}$ the versors of the world frame, $\{\vec{i}_1, \vec{j}_1, \vec{k}_1\}$ the versors of the first wheel, $\{\vec{i}_2, \vec{j}_2, \vec{k}_2\}$ the versors of the second wheel and $\{\vec{i}_3, \vec{j}_3, \vec{k}_3\}$ the versors of the main body. About these we have

$$\begin{bmatrix}\vec{i}_2\\ \vec{j}_2\\ \vec{k}_2\end{bmatrix} = \begin{bmatrix}\vec{i}_1\\ \vec{j}_1\\ \vec{k}_1\end{bmatrix} = \begin{bmatrix}\cos(\theta) & \sin(\theta) & 0\\ -\sin(\theta) & \cos(\theta) & 0\\ 0 & 0 & 1\end{bmatrix} \cdot \begin{bmatrix}\vec{i}_0\\ \vec{j}_0\\ \vec{k}_0\end{bmatrix} = R_{k\theta} \begin{bmatrix}\vec{i}_0\\ \vec{j}_0\\ \vec{k}_0\end{bmatrix} \tag{9}$$

and

$$\begin{bmatrix}\vec{i}_3\\ \vec{j}_3\\ \vec{k}_3\end{bmatrix} = \begin{bmatrix}\cos(\phi) & 0 & -\sin(\phi)\\ 0 & 1 & 0\\ \sin(\phi) & 0 & \cos(\phi)\end{bmatrix} \cdot \begin{bmatrix}\vec{i}_1\\ \vec{j}_1\\ \vec{k}_1\end{bmatrix} = R_{j\phi} \cdot \begin{bmatrix}\vec{i}_1\\ \vec{j}_1\\ \vec{k}_1\end{bmatrix} \tag{10}$$

Denote P_{c1} the first wheel contact point, C_1 the center of mass of the first wheel, C_2 the center of mass of the second wheel, C_3 the center of mass of the main body, P_{c2} the second wheel contact point and S the middle of $\overrightarrow{C_1C_2}$, then

$$\overrightarrow{OP_{c1}} = \begin{bmatrix}x\\ y\\ 0\end{bmatrix} \qquad \overrightarrow{C_1P_{c1}} = \begin{bmatrix}0\\ 0\\ -r\end{bmatrix}$$
$$\overrightarrow{C_1C_2} = R_{k\theta}^T \begin{bmatrix}0\\ l\\ 0\end{bmatrix} = \begin{bmatrix}-l\sin(\theta)\\ l\cos(\theta)\\ 0\end{bmatrix} \tag{11}$$

$$\overrightarrow{C_2P_{c2}} = \begin{bmatrix}0\\ 0\\ -r\end{bmatrix} \qquad \overrightarrow{C_1S} = \frac{1}{2}\begin{bmatrix}-l\sin(\theta)\\ l\cos(\theta)\\ 0\end{bmatrix}$$
$$\overrightarrow{C_3S} = R_{k\theta}^T R_{j\phi}^T \begin{bmatrix}0\\ 0\\ -L\end{bmatrix} \overrightarrow{C_3S} = -L \cdot \begin{bmatrix}\cos(\theta)\sin(\phi)\\ \sin(\theta)\sin(\phi)\\ \cos(\phi)\end{bmatrix} \tag{12}$$

From here

$$\overrightarrow{OC_1} = \overrightarrow{OP_{c1}} - \overrightarrow{C_1P_{c1}}; \qquad \overrightarrow{OC_2} = \overrightarrow{OC_1} + \overrightarrow{C_1C_2}$$
$$\overrightarrow{OS} = \overrightarrow{OC_1} + \overrightarrow{C_1S}; \qquad \overrightarrow{OC_3} = \overrightarrow{OS} - \overrightarrow{C_3S} \tag{13}$$

B. Linear and angular momentum

For the derivatives of linear and angular momentum we need the linear and angular velocities and their derivatives.

1) Linear velocity, acceleration and momentum: For the linear velocities of the rigid bodies, we have:

$$v_j = \dot{\overrightarrow{OC_j}} = \sum_{i=1}^{6} \frac{\partial \overrightarrow{OC_j}}{\partial q_i}\dot{q}_i \qquad \forall j \in \{1,2,3\}$$
$$a_j = \ddot{\overrightarrow{OC_j}} = \sum_{i=1}^{6} \frac{\partial v_j}{\partial q_i}\dot{q}_i + \sum_{i=1}^{6} \frac{\partial v_j}{\partial \dot{q}_i}\ddot{q}_i \qquad \forall j \in \{1,2,3\} \tag{14}$$

and from here we have

$$\dot{P}_1 = m_1 \cdot a_1 \qquad \dot{P}_2 = m_2 \cdot a_2 \qquad \dot{P}_3 = m_3 \cdot a_3 \tag{15}$$

2) Angular velocity, acceleration and momentum: The angular velocity of a rigid body can be obtain given the rotation matrix, from the relation $\dot{R}^T = [\omega]R^T$, where $[\omega]$ is the skew symmetric form of the vector ω. Hence

$$\omega_1 = \begin{bmatrix}0\\ 0\\ \dot{\theta}\end{bmatrix} + R_{k\theta}^T \begin{bmatrix}0\\ \dot{\phi}_1\\ 0\end{bmatrix} = \begin{bmatrix}-\sin(\theta)\dot{\phi}_1\\ \cos(\theta)\dot{\phi}_1\\ \dot{\theta}\end{bmatrix} \tag{16}$$

Similarly

$$\omega_2 = \begin{bmatrix}-\sin(\theta)\dot{\phi}_2\\ \cos(\theta)\dot{\phi}_2\\ \dot{\theta}\end{bmatrix} \qquad \omega_3 = \begin{bmatrix}-\sin(\theta)\dot{\phi}\\ \cos(\theta)\dot{\phi}\\ \dot{\theta}\end{bmatrix} \tag{17}$$

For the angular acceleration we have:

$$\dot{\omega}_j = \sum_{i=1}^{6} \frac{\partial \omega_j}{\partial q_i}\dot{q}_i + \sum_{i=1}^{6} \frac{\partial \omega_j}{\partial \dot{q}_i}\ddot{q}_i \qquad \forall j \in \{1,2,3\} \tag{18}$$

The derivative of angular momentum is calculated as follows

$$\dot{L}_j = \omega_j \times (I_j \cdot \omega_j) + I_j \cdot \dot{\omega}_j \qquad \forall j \in \{1,2,3\} \tag{19}$$

where I_j is the inertia matrix of the rigid body while moving, hence

$$\begin{aligned} I_1 &= R_{k\theta}^T R_{j\phi_1}^T I_{01} R_{j\phi_1} R_{k\theta}\\ I_2 &= R_{k\theta}^T R_{j\phi_2}^T I_{02} R_{j\phi_2} R_{k\theta}\\ I_3 &= R_{k\theta}^T R_{j\phi}^T I_{03} R_{j\phi} R_{k\theta} \end{aligned} \tag{20}$$

where

$$I_{01} = I_{02} = m \cdot r^2 \begin{bmatrix}0.5 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0.5\end{bmatrix} \qquad I_{03} = O_{3\times3} \tag{21}$$

are the inertia matrix in body frame. The first and second rigid bodies are approximated by circles, of equal mass and radius, rotating about $\vec{j}$ axis, and the third body is a dot (hence $I_{03} = O_{3\times3}$)

C. Applied forces and torques

The following forces are applied on the each rigid body:

$$F_1 = \begin{bmatrix}0\\ 0\\ -m_1 g\end{bmatrix} \qquad T_1 = R_{k\theta}^T \begin{bmatrix}0\\ T_{d1}\\ 0\end{bmatrix} = \begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\\ 0\end{bmatrix} T_{d1} \tag{22}$$

$$F_2 = \begin{bmatrix}0\\ 0\\ -m_2 g\end{bmatrix} \qquad T_2 = R_{k\theta}^T \begin{bmatrix}0\\ T_{d2}\\ 0\end{bmatrix} = \begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\\ 0\end{bmatrix} T_{d2} \tag{23}$$

$$F_3 = \begin{bmatrix}0\\ 0\\ -m_3 g\end{bmatrix} \qquad T_3 = -R_{k\theta}^T \begin{bmatrix}0\\ T_{d1}+T_{d2}\\ 0\end{bmatrix} = -\begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\\ 0\end{bmatrix}(T_{d1}+T_{d2}) \tag{24}$$

D. Non-holonomic constraints

These constraints have arisen from the fact that the wheels are not allowed to slide, hence for each contact point we have the following velocity constraints:

$$\overrightarrow{\dot{OP}_{c1}} = \begin{bmatrix}\dot{x}\\ \dot{y}\\ 0\end{bmatrix} = R_{k\theta}^T \begin{bmatrix} r\cdot\dot{\phi}_1\\ 0\\ 0\end{bmatrix} = \begin{bmatrix} r\cos(\theta)\dot{\phi}_1\\ r\sin(\theta)\dot{\phi}_1\\ 0\end{bmatrix} \tag{25}$$

and for the second contact point:

$$\overrightarrow{\dot{OP}_{c2}} = \begin{bmatrix}\dot{x} - l\cos(\theta)\dot{\theta}\\ \dot{y} - l\sin(\theta)\dot{\theta}\\ 0\end{bmatrix} = R_{k\theta}^T \begin{bmatrix} r\cdot\dot{\phi}_2\\ 0\\ 0\end{bmatrix} = \begin{bmatrix} r\cos(\theta)\dot{\phi}_2\\ r\sin(\theta)\dot{\phi}_2\\ 0\end{bmatrix} \tag{26}$$

We write these in the form of Equation (5) as follows:

$$A(q)\cdot\dot{q} = \\ = \begin{bmatrix} 1 & 0 & 0 & 0 & -r\cos(\theta) & 0\\ 0 & 1 & 0 & 0 & -r\sin(\theta) & 0\\ 1 & 0 & -l\cos(\theta) & 0 & 0 & -r\cos(\theta)\\ 0 & 1 & -l\sin(\theta) & 0 & 0 & -r\sin(\theta)\end{bmatrix} \begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\phi}\\ \dot{\phi}_1\\ \dot{\phi}_2\end{bmatrix} \\ = O_{4\times 1} \tag{27}$$

One can see that $rank(A(q)) = 3$ hence we will consider just the first three lines of the matrix $A(q)$. That is, if the first contact point does not slide in $\vec{j}$ direction, then the second contact point cannot slide either, hence:

$$A(q) = \begin{bmatrix} 1 & 0 & 0 & 0 & -r\cos(\theta) & 0\\ 0 & 1 & 0 & 0 & -r\sin(\theta) & 0\\ 1 & 0 & -l\cos(\theta) & 0 & 0 & -r\cos(\theta)\end{bmatrix} \tag{28}$$

We compute $W(q)$, whose columns form a basis for the null space of $A(q)$, and obtain

$$W(q) = \begin{bmatrix} 0 & r\cos(\theta) & 0\\ 0 & r\sin(\theta) & 0\\ 0 & \frac{r}{l} & -\frac{r}{l}\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \tag{29}$$

From Equation (28) one can obtain

$$\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 0 & -r\cos(\theta)\end{bmatrix} \begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\phi}_2\end{bmatrix} = \\ = -\begin{bmatrix} 0 & 0 & -r\cos(\theta)\\ 0 & 0 & -r\sin(\theta)\\ -l\cos(\theta) & 0 & 0\end{bmatrix} \begin{bmatrix}\dot{\theta}\\ \dot{\phi}\\ \dot{\phi}_1\end{bmatrix} \tag{30}$$

hence for $\theta \neq \frac{pi}{2} + k\pi$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\phi}_2\end{bmatrix} = \begin{bmatrix} r\cos(\theta)\dot{\phi}_1\\ r\sin(\theta)\dot{\phi}_1\\ \dot{\phi}_1 - \frac{l}{r}\dot{\theta}\end{bmatrix} \quad \begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{\phi}_2\end{bmatrix} = \frac{d}{dt}\begin{bmatrix} r\cos(\theta)\dot{\phi}_1\\ r\sin(\theta)\dot{\phi}_1\\ \dot{\phi}_1 - \frac{l}{r}\dot{\theta}\end{bmatrix} \tag{31}$$

E. The equations of motion

From the above section we have

$$Q_1 + Q_1^* = \begin{bmatrix} \frac{\partial v_1}{\partial \dot{x}} & \frac{\partial v_2}{\partial \dot{x}} & \frac{\partial v_3}{\partial \dot{x}} & \frac{\partial \omega_1}{\partial \dot{x}} & \frac{\partial \omega_2}{\partial \dot{x}} & \frac{\partial \omega_3}{\partial \dot{x}}\end{bmatrix}\cdot \\ \cdot \begin{bmatrix} F_1 - \dot{P}_1\\ F_2 - \dot{P}_2\\ F_3 - \dot{P}_3\\ T_1 - \dot{L}_1\\ T_2 - \dot{L}_2\\ T_3 - \dot{L}_3\end{bmatrix} \tag{32}$$

The other equations are obtain in a similar way. Then

$$W(q)^T \cdot \begin{bmatrix} Q_1 + Q_1^*\\ \vdots\\ Q_6 + Q_6^*\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \tag{33}$$

From Equation (33) and (31) we obtain:

$$B_2(q)\begin{bmatrix}\ddot{\theta}\\ \ddot{\phi}\\ \ddot{\phi}_1\end{bmatrix} + B_{11}(q)\begin{bmatrix}\dot{\theta}^2\\ \dot{\theta}\dot{\phi}\\ \dot{\theta}\dot{\phi}_1\\ \dot{\phi}^2\\ \dot{\phi}\dot{\phi}_1\\ \dot{\phi}_1^2\end{bmatrix} + B_1(q)\begin{bmatrix}\dot{\theta}\\ \dot{\phi}\\ \dot{\phi}_1\end{bmatrix} + \\ + B_0(q) + B_c\begin{bmatrix}T_{d1}\\ T_{d2}\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \tag{34}$$

where

$$B_2(q) = \begin{bmatrix} B_2^1(q) & B_2^2(q) & B_2^3(q)\end{bmatrix} \tag{35}$$

with B_2^1, B_2^2, B_2^3 being some column vectors given below:

$$B_2^1(q) = \begin{bmatrix} \frac{1}{2}Llm_3\cos(\phi)\\ \frac{1}{4}lm_3r - \frac{r}{4l}(4m_3L^2\sin(\phi)^2 + 4mr^2)\\ \frac{1}{l}(m_3L^2r\sin(\phi)^2 + mr^3) + l(mr + m_3r\frac{1}{4})\end{bmatrix} \\ B_2^2(q) = \begin{bmatrix} L^2m_3\\ -Lm_3r\cos(\phi)\frac{1}{2}\\ -Lm_3r\cos(\phi)\frac{1}{2}\end{bmatrix} \\ B_2^3(q) = \begin{bmatrix} 0\\ -mr^2\\ 0\end{bmatrix} \tag{36}$$

The next matrix, $B_{11}(q)$, has the following form:

$$B_{11}(q) = \begin{bmatrix} B_{11}^1(q) & B_{11}^2(q) & B_{11}^3(q) & B_{11}^4(q) & O_{3\times 2}\end{bmatrix} \tag{37}$$

where

$$B_{11}^1(q) = \begin{bmatrix} \frac{1}{2}L^2 m_3 \sin(2\phi) \\ L m_3 r \sin(\phi) \\ 0 \end{bmatrix}$$

$$B_{11}^2(q) = \begin{bmatrix} 0 \\ -L^2 m_3 r \sin(2\phi)\frac{1}{l} \\ L^2 m_3 r \sin(2\phi)\frac{1}{l} \end{bmatrix}$$

$$B_{11}^3(q) = \begin{bmatrix} 0 \\ -\frac{1}{l} L m_3 r^2 \sin(\phi) \\ \frac{1}{l} L m_3 r^2 \sin(\phi) \end{bmatrix}$$

$$B_{11}^4(q) = \begin{bmatrix} 0 \\ \frac{1}{2} L m_3 r \sin(\phi) \\ \frac{1}{2} L m_3 r \sin(\phi) \end{bmatrix} \tag{38}$$

The next matrices $B_1(q), B_0(q)$ and B_c are:

$$B_1(q) = \begin{bmatrix} 0 & 0 & -L m_3 r \cos(\phi) \\ 0 & 0 & \frac{-1}{2} r^2 (2m + m_3) \\ lmr & 0 & \frac{-1}{2} r^2 (4m + m_3) \end{bmatrix} \tag{39}$$

$$B_0(q) = \begin{bmatrix} L g m_3 \sin(\phi) \\ 0 \\ 0 \end{bmatrix} \qquad B_c = \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{40}$$

Here the calculation were performed using the symbolic tool from $MATLAB^{®}$

IV. Simulation and Stabilization of the Non-Linear Model

In the following the non-linear model is stabilized using a linear state feedback. The linear model can be easily obtained from the above:

$$B_{2l} \begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \\ \ddot{\phi}_1 \end{bmatrix} + B_{1l} \begin{bmatrix} \dot{\theta} \\ \dot{\phi} \\ \dot{\phi}_1 \end{bmatrix} + B_{0l} \begin{bmatrix} \theta \\ \phi \\ \phi_1 \end{bmatrix} + B_c \begin{bmatrix} T_{d1} \\ T_{d2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{41}$$

where

$$B_{2l} = \begin{bmatrix} \frac{1}{2} L l m_3 & -L^2 m_3 & 0 \\ \frac{1}{4} l m_3 r - \frac{1}{l} m r^3 & \frac{-1}{2} L m_3 r & -m r^2 \\ l(mr + \frac{m_3 r}{4}) + \frac{m r^3}{l} & \frac{-1}{2} L m_3 r & 0 \end{bmatrix} \tag{42}$$

$$B_{1l} = \begin{bmatrix} 0 & 0 & -L m_3 r \\ 0 & 0 & -\frac{r^2}{2}(2m + m_3) \\ lmr & 0 & -\frac{r^2}{2}(2m + m_3) \end{bmatrix} \quad B_{0l} = \begin{bmatrix} 0 & L g m_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{43}$$

Define $X = [\theta, \phi, \phi_1, \dot{\theta}, \dot{\phi}, \dot{\phi}_1]^T$ and obtain the linear model:

$$\dot{X} = \begin{bmatrix} O_{3\times3} & I_3 \\ -B_{2l}^{-1} B_{0l} & -B_{2l}^{-1} B_{1l} \end{bmatrix} X + \begin{bmatrix} O_{3\times2} \\ -B_{2l}^{-1} B_c \end{bmatrix} \cdot \begin{bmatrix} T_{d1} \\ T_{d2} \end{bmatrix} =$$
$$= A_p X + B_p \begin{bmatrix} T_{d1} \\ T_{d2} \end{bmatrix} \tag{44}$$

We considered

$$r = 0.4; \quad l = 1; \quad L = 1.4;$$
$$m_1 = m_2 = m = 1.3; \quad m_3 = 90; \quad g = 9.81; \tag{45}$$

and obtained

$$A_p = \begin{bmatrix} 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 \\ 0 & 292.7387 & 0 & -0.8621 & 0 & 0.6897 \\ 0 & 111.5567 & 0 & -0.3079 & 0 & -0.0394 \\ 0 & -966.0378 & 0 & 0.3448 & 0 & -1.2759 \end{bmatrix}$$

$$B_p = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -0.2368 & -1.8947 \\ -0.0903 & -0.6823 \\ 5.5892 & 1.4447 \end{bmatrix} \tag{46}$$

Let V be a matrix whose columns are left eigenvectors of A_p. Then one can verify that $B_p^T \cdot V$ does not have any null column, therefore the pair (A_p, B_p) is controllable. The eigenvalues of Ap are $\{0, 0, \pm 10.4, -0.8262 \pm 1.1021i\}$, hence the system is open loop unstable. The linear system was stabilized by setting the eigenvalues of the closed loop with a linear state feedback. The closed loop eigenvalues are didactically chosen $\{-2, -3, -4, -, 5, -6, -7\}$. The state feedback was obtained using the *place* function from $MATLAB^{®}$

$$K = \begin{bmatrix} -1.65 & -106.07 & 6.13 & -2.9250 & 13.34 & 1.54 \\ 17.84 & -335.47 & -11.62 & 22.16 & -85.10 & 0.48 \end{bmatrix} \tag{47}$$

The nonlinear model was implemented in

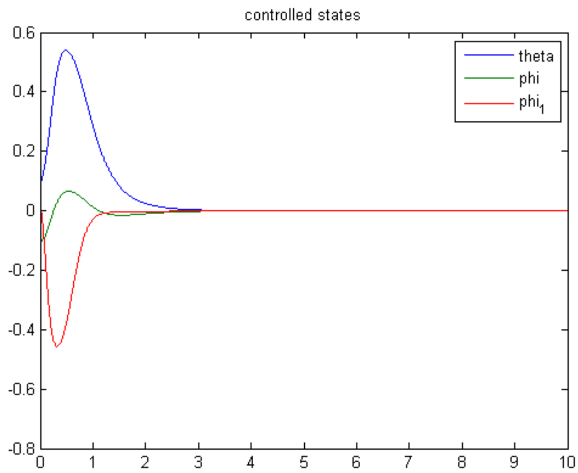

Fig. 3. The states of the nonlinear system

$Simulink\ MATLAB^{®}$ and stabilized with the above linear state feedback control law.

One can see in Figure 3 that, the heading angle θ, the pitch angle ϕ and the wheel angle ϕ_1, converge to zero from some random non-zero, but small, initial state.

V. Conclusions and future work

The purpose of this article was to provide a full nonlinear model, of the two-wheeled inverted pendulum, suitable for simulation and control. The proposed model is easy to simulate and linearize. Although the control of the system is not the main focus of this paper, a linear state feedback was obtained, capable of stabilizing the system.

Naturally, a future work is concerned with the control of the two-wheeled inverted pendulum. Here, one can be interested in some control techniques like state feedback liniarization together with robust control treating nonliniarities, like $sin(\cdot)$, as uncertainties. Also, geometric control techniques are a good option to be studied.

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