Studying the Optimal Control of a Non-Holonomic System using an Inverted Pendulum

Team 12

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PROBLEM STATEMENT

- An inverted pendulum on a cart is an essential optimal control problem because it is a classic example of a non-linear and unstable system that can be stabilized using control theory. It is widely used in engineering, robotics, and control systems as a benchmark for testing and evaluating control algorithms.

- The challenge lies in finding the right balance between keeping the pendulum upright while keeping the cart stable.

- The solution to this problem requires precise controls in real-time to maintain balance, which makes it a good project for future optimal control developments.

MATHEMATICAL MODELLING

- Nonlinear system of equations:

$$m\ddot{x}cos(\theta) + ml\ddot{\theta} = mgsin(\theta)$$

$$(M+m)\ddot{x} - mlsin(\theta)\dot{\theta}^2 + mlcos(\theta)\ddot{\theta} = u$$

- State-space model:

$$\frac{d}{dt}\delta x = A\delta x + B\delta u$$

$$A = J_x(x_0, u_0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(M+m)g}{Ml} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-mg}{M} & 0 & 0 & 0 \end{bmatrix}$$

$$B = J_u(x_0, u_0) = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial u} \\ \frac{\partial f_4}{\partial u} \end{bmatrix}_{(x_0, u_0)} = \begin{bmatrix} 0 \\ \frac{-1}{Ml} \\ 0 \\ \frac{1}{M} \end{bmatrix}$$

$$y = \begin{bmatrix} \theta \\ x \end{bmatrix} = Cx = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix}$$

MATHEMATICAL MODELLING

- Optimal Control using LQR is designed using the state space model defined in the previous slide to minimize the cost function,

$$J = \int (X^T Q X + u^T R u) dt$$

- The LQR gain is obtained by,

$$K = R^{-1} B^T P$$

- Here, P is found from the solution of the matrix algebraic Riccatii equation,

$$A^T P + PA - PBR^{-1} B^T P + Q = 0$$

SIMULATION

- The model depicted in Figure 1 determines the position and angle of the Inverted Pendulum, which is transmitted to the LQR function block in Figure 2 for further processing.

- Linear analysis of the model presented in Figure 1 demonstrates that the inverted pendulum is inherently unstable and relies on the cart for stabilization.

- The reference for the upward position of the inverted pendulum is established using constant blocks.

- The LQR output and PID controls are employed to keep the pendulum in an upright position.

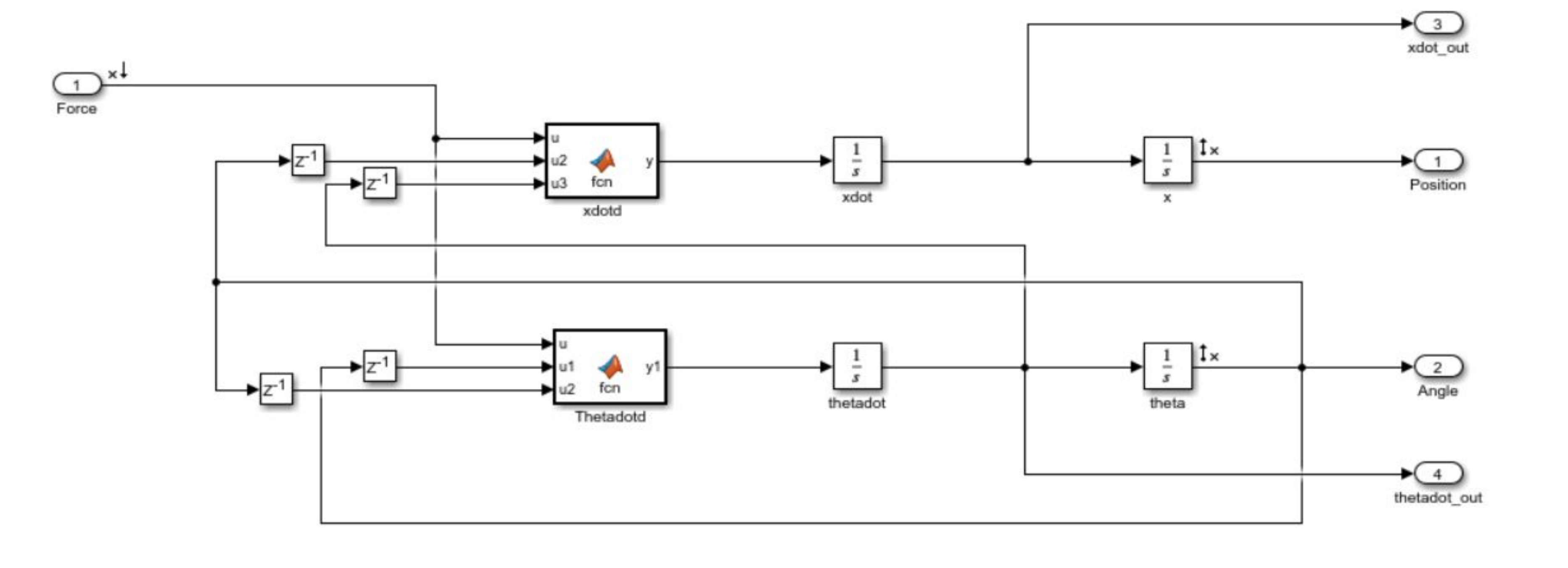

Figure 1: Simulink Model of the Inverted Pendulum

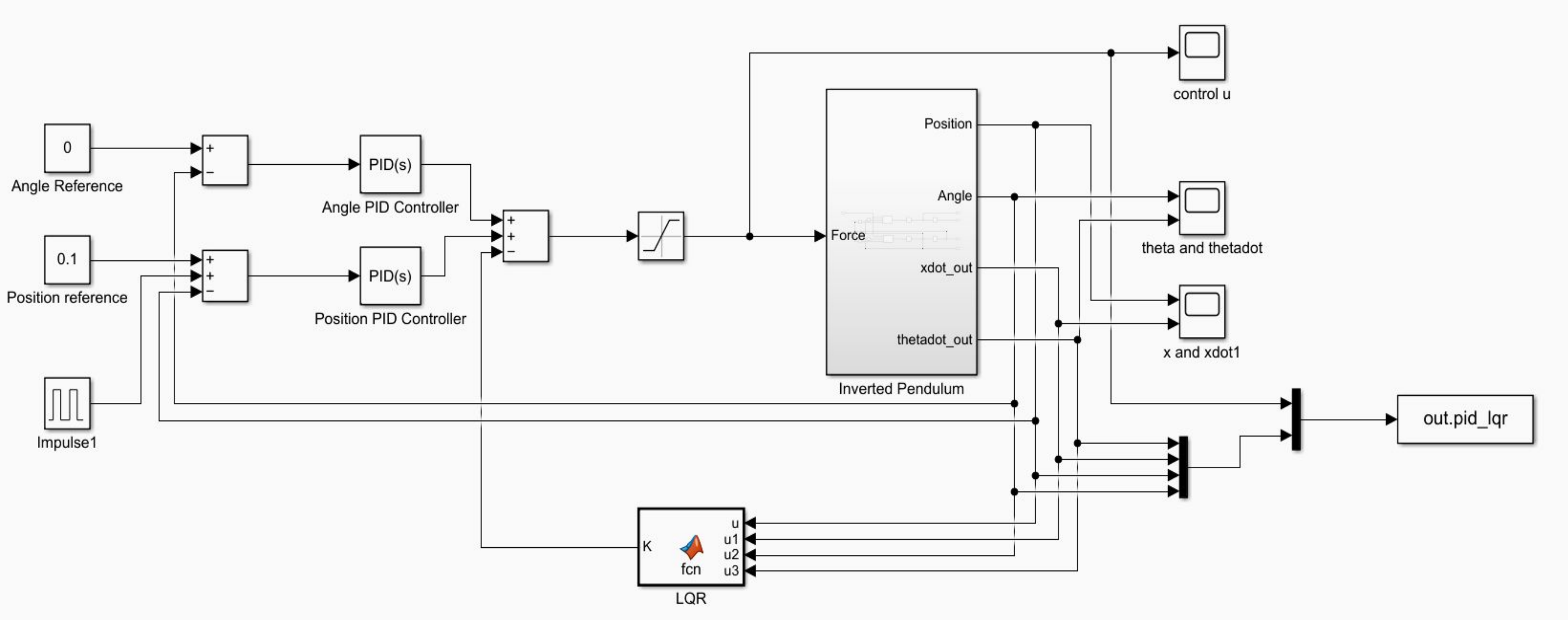

Figure 2: Simulink Model for the Inverted Pendulum-Cart System with PID and LQR

FUTURE WORK

- Simulation:
  - Fix bugs to get values of Q and R to generate the K vector
  - The K vector will serve as the feedback for the two PID controllers in our robot.
  - A dynamic simulation of the robot can be created to fine-tune the PID error coefficients.

- Implementation:
  - Manufacturing of the physical cart is completed, the electronics and firmware are in progress.
  - The hardware components include a controller, motor driver, and IMU sensor, and if time permits, we plan to add an ultrasonic sensor to prevent collisions.
  - We will use a timer-based scheduler on an Arduino platform to sample the sensors at a constant time interval $\Delta(t)$.

STUDYING THE OPTIMAL CONTROL OF A NON-HOLONOMIC SYSTEM USING AN INVERTED PENDULUM

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ABSTRACT

This report discusses the team's progress to date in the study of the optimal control of a non-holonomic system using an inverted pendulum. In this progress check, we study the control technique that combines the Linear Quadratic Regulator (LQR) and Proportional, Integral, and Derivative (PID) applied to a highly non-linear inverted pendulum-cart system. This report presents a detailed mathematical description of the problem statement, the MATLAB-SIMULINK simulation results of the defined model, and the current hardware implementation. This project refers to the paper by L.B. Prasad, et al. [1].

1. INTRODUCTION

An inverted pendulum on a cart is an essential optimal control problem because it is a classic example of a non-linear and unstable system that can be stabilized using control theory It is widely used in engineering, robotics, and control systems as a benchmark for testing and evaluating control algorithms. The challenge lies in finding the right balance between keeping the pendulum upright and keeping the cart stable. The solution to this problem requires precise controls in real-time to maintain balance, which makes it a significant research area for future development in optimal control.

2. MATHEMATICAL MODELLING

2.1. System of equations defining the Inverted Pendulum

Referring to the free-body diagram of an inverted pendulum mounted on a cart, shown in Fig. 1[1], and assuming the pendulum-rod is mass-less and the hinge is frictionless, the system of equations defining this model is derived. The coordinated system, referenced at point O is shown in Fig. 1. The mass of the cart and the ball at the upper end of the pendulum rod is denoted by M and m, respectively. The length of the pendulum rod is denoted as l. An external force, applied to the cart, along the x-axis is denoted as $u(t)$. Along with this, a gravitational force acts on the point mass at all times. The cart position and the tilt angle of the pendulum rod are functions of time denoted by $x(t)$ and $\theta(t)$, respectively.

The time-dependent center of gravity of the point mass is given by the coordinates (x_G, y_G). Referring to Fig. 1., the location of the center of gravity of the pendulum mass on the x-y plane is, $x_G = x + lsin(\theta)$ and $y_G = lcos(\theta)$.

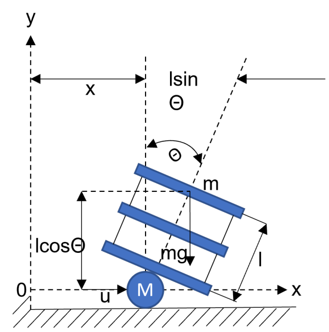

Fig. 1: Free-body diagram of the inverted pendulum-cart system that is driven by a motor. The mass at the upper end of the pendulum rod is denoted by m, and the cart's mass is denoted by M. The length of the pendulum rod is denoted as l. An external force in the x-direction, applied to the cart is denoted by u.

Balancing the forces acting on the model, in the x-axis, by considering Newton's second law of motion, we derive the following expression,

$$M\frac{d^2}{dt^2}x + m\frac{d^2}{dt^2}x_G = u \tag{1}$$

Substituting the location of the center of gravity of the pendulum mass into (1),

$$M\frac{d^2}{dt^2}x + m\frac{d^2}{dt^2}[x + lsin(\theta)] = u \tag{2}$$

Further simplifying eqn.(2),

$$(M+m)\ddot{x} - mlsin(\theta)\dot{\theta}^2 + mlcos(\theta)\ddot{\theta} = u \tag{3}$$

Similarly, balancing the torque on the mass due to the acceleration force with the torque on the mass due to the gravity results in,

$$(F_x cos\theta)l - (F_y sin\theta)l = (mgsin\theta)l \tag{4}$$

In eqn. (4), the force components are defined as,

$$\begin{aligned} F_x &= m\frac{d^2}{dt^2}x_G = m[\ddot{x} - lsin(\theta)\dot{\theta}^2 + lcos(\theta)\ddot{(\theta)}] \\ F_y &= m\frac{d^2}{dt^2}y_G = -m[lcos(\theta)\dot{\theta}^2 + lsin(\theta)\ddot{(\theta)}] \end{aligned} \tag{5}$$

Substituting the force components F_x and F_y from eqn. (5) into eqn. (4) and simplifying results in,

$$m\ddot{x}cos(\theta) + ml\ddot{\theta} = mgsin(\theta) \tag{6}$$

Considering the derived system of nonlinear equations from eqn.(3) and (6), the following section discusses the standard state-space model of these two non-linear equations.

2.2. Non-linear system of equations defining the Inverted Pendulum

The state-space model, describing the system of non-linear equations derived in the previous section, is required to numerically simulate the inverted pendulum-cart dynamic system. In this section, the system of non-linear equations is shown in eqn. (3) and (6) will be represented in the following state-space form,

$$\frac{d}{dt}x = f(x, u, t)$$

To achieve this, the equations must be manipulated to have a single double-derivative term, i.e., $\ddot{x}$ and $\ddot{\theta}$. Manipulating eqn. (6) and substituting it into eqn. (3),

$$(M + m - mcos^2\theta)\ddot{x} = u + mlsin\theta\dot{\theta}^2 - mgcos\theta sin\theta \quad (7)$$

Similarly, we also derive the equation,

$$[mlcos^2\theta - (M + m)l]\ddot{\theta} = ucos\theta - (M + m)gsin\theta + mlcos\theta sin\theta\dot{\theta}^2 \quad (8)$$

Therefore,

$$\ddot{x} = \frac{u + ml(sin\theta)\dot{\theta}^2 - mgcos\theta sin\theta}{M + m - mcos^2\theta} \quad (9)$$

$$\ddot{\theta} = \frac{ucos\theta - (M + m)gsin\theta + ml(cos\theta sin\theta)\dot{\theta}}{mlcos^2\theta - (M + m)l} \quad (10)$$

Converting eqn. (9) and (10) into state space form by considering the following state variables,

$$x_1 = \theta \quad x_2 = \dot{x} = \dot{\theta} \quad x_3 = x \quad x_4 = \dot{x} = \dot{x_3}$$

$$\frac{d}{dt}x = \frac{d}{dt}\begin{bmatrix} x_1 \\ x2 \\ x_3 \\ x_4 \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \quad (11)$$

The output of the state-space model may be written as,

$$y = \begin{bmatrix} \theta \\ x \end{bmatrix} = Cx = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} \quad (12)$$

2.3. Linear system of equations defining the Inverted Pendulum

Since the system is described by non-linear equations and the goal of this system is to keep the inverted pendulum in the upright position, around $\theta = 0$, the non-linear system from eqn. (12) is simplified and linearized under the condition that $x_0 = 0$ and $u_0 = 0$ as follows,

$$\frac{d}{dt}\delta x = J_x(x_0, u_0)\delta x + J_u(x_0, u_0)\delta u \quad (13)$$

Therefore,

$$\frac{d}{dt}\delta x = A\delta x + B\delta u$$

The components of the Jacobian matrices are considered as follows,

$$\frac{\partial f_i}{\partial x_1}, \frac{\partial f_i}{\partial x_2}, \frac{\partial f_i}{\partial x_3}, \quad \text{and} \quad \frac{\partial f_i}{\partial x_4}$$

Hence,

$$A = J_x(x_0, u_0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(M+m)g}{Ml} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-mg}{M} & 0 & 0 & 0 \end{bmatrix} \quad (14)$$

Similarly for $J_u(x_0, u_0)$,

$$B = J_u(x_0, u_0) = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial u} \\ \frac{\partial f_4}{\partial u} \end{bmatrix}_{(x_0, u_0)} = \begin{bmatrix} 0 \\ \frac{-1}{Ml} \\ 0 \\ \frac{1}{M} \end{bmatrix} \quad (15)$$

Therefore, the state-space model for the system is defined by eqn. (12) and (13).

3. CONTROL METHODS

The control system introduced to stabilize the inverted pendulum-cart system to the aforementioned reference point, we use a combination of PID and LQR control methods. The instantaneous states of the system, including pendulum angle, angular velocity, cart position, and cart velocity, are considered available for measurement and fed to the LQR, which is designed using the linear state-space model of the system. The optimal control value of LQR is combined with the PID control value to obtain a resultant optimal control. The tuning of the PID controllers is done through trial and error method by observing the responses achieved to be optimal.

3.1. Optimal Control using Linear Quadratic Regulator

Linear Quadratic Regulator considers the state space of the system and the control input to make optimal control decisions. The objective of the LQR is to minimize the cost function defined as,

$$J = \int (X^T Q X + u^T R u)dt \quad (16)$$

in eqn. (16), Q and R are positive semi-definite and positive definite symmetric matrices, respectively, that we will assume in a later section. The LQR gain vector K is given as,

$$K = R^{-1}B^T P \quad (17)$$

In eqn. (17), P is a positive definite symmetric matrix obtained from the algebraic reccatti equation,

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \qquad (18)$$

4. SIMULATIONS

The simulations for this project are done using MATLAB Simulink. At this point in time, the Simulink model for the Inverted Pendulum has been designed and simulated [4]. And while the model for the Inverted Pendulum on a cart has been designed, it is yet to be simulated. There are some errors that need to be addressed in the upcoming month to simulate the project in its entirety.
We assume dummy values of the mass at the end of the pendulum rod m and the mass of the cart M as $0.23 \quad kg$ and $2.4 \quad kg$, respectively. The length of the pendulum l is taken to be $0.36m$ and the length of the cart L is $\pm 0.5m$. The matrices used to design the LQR are designed using these values [1][2].

4.1. Inverted Pendulum model and simulation

The inverted pendulum model was created as depicted in Fig. 2. The two degrees of freedom $\ddot{x}$ and $\ddot{\theta}$ are represented by equations in the function blocks. Following the integration of the function blocks for the degrees of freedom, the model's outputs of the angle and position, x, θ and $\dot{x}$, $\dot{\theta}$, of the inverted pendulum are produced, and connections are created as depicted in the figure.

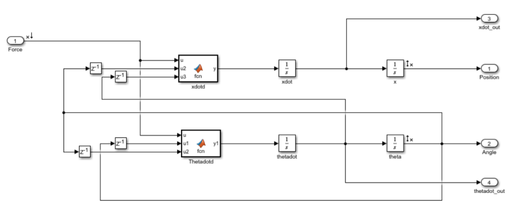

Fig. 2: Simulink model for the inverted pendulum

This model is then applied as a subsystem to replicate the behavior of the inverted pendulum, with the force applied serving as the input and the position and angle of the inverted pendulum serving as the output. The simulations were performed using dummy values which will be updated once the hardware specifications are finalized.

4.2. Inverted Pendulum on a Cart Model

The inverted pendulum on a cart is modeled as shown in Fig. 3. The constant blocks serve as a reference input for the pendulum's position and angle when stable. When an impulse is applied to the cart, the PID controllers are employed to keep the pendulum oriented vertically. The inverted pendulum's position and the angle are fed back to the LQR to update the PID controllers and make the necessary adjustments to maintain the system's stability.

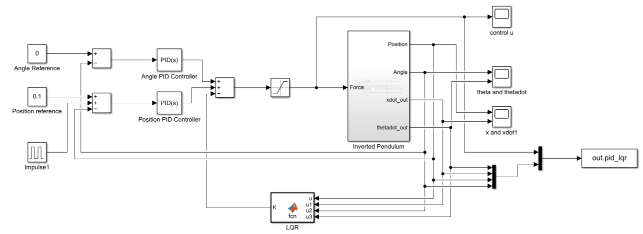

Fig. 3: Completed Model for the simulation of an Inverted Pendulum-Cart System

5. FUTURE WORK

5.1. Simulation

We have identified a few bugs in the simulation code that need to be fixed to move forward with our project. Once we have a functioning simulation, we will be able to determine the appropriate values for Q and R to generate the K vector. This K vector will serve as the feedback for the two PID controllers in our robot. Additionally, a dynamic simulation of the robot will allow us to fine-tune the PID error coefficients based on the robot's physical parameters.

5.2. Implementation

We have completed the manufacturing of the physical cart, but the electronics and firmware are still in progress. The hardware components include a controller, motor driver, and IMU sensor, and if time permits, we plan to add an ultrasonic sensor to prevent collisions. To ensure that the feedback from the sensors is consistent, we will use a timer-based scheduler on an Arduino platform to sample the sensors at a constant time interval $\Delta(t)$. By implementing this system in the real world, we will be able to visualize the performance of LQR+PID control.

6. REFERENCES

[1] Lal Bahadur Prasad, Barjeev Tyagi, Hari Om Gupta. "Modelling Simulation for Optimal Control of Nonlinear Inverted Pendulum Dynamical System using PID Controller LQR". 2012 Sixth Asia Modelling Symposium

[2] Vinayak Kumar, Ruchi Agarwal. "Modeling and Control of Inverted Pendulum cart system using PID-LQR based Modern Controller".2022 IEEE Students Conference on Engineering and Systems (SCES), July 01-03, 2022, Prayagraj, India

[3] "Inverted pendulum: Simulink modeling," Control Tutorials for MATLAB and Simulink - Inverted Pendulum: Simulink Modeling. [Online].