Modeling and Control of Inverted Pendulum cart system using PID-LQR based Modern Controller

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***Abstract*—The paper proposes a modern controller i.e. combination of Linear quadratic regulator (LQR) and Proportional, integral and derivative (PID) to control a highly non-linear inverted pendulum cart system. The detailed mathematical modeling of the inverted pendulum cart system as well as the modern controller are discussed in this paper. The performance of the proposed controller is compared with the conventional PID controller. The comparative results are shown using several simulated waveforms as well as in terms of time response specifications such as rise time, settling time, and peak overshoot and undershoot under MATLAB/Simulated environment.**

***Keywords*—— IP (Inverted pendulum), PID controller, LQR, PID-LQR, Inverted pendulum cart system**

I. INTRODUCTION

The Inverted Pendulum (IP) is a highly unstable mechanism as it contains fewer control inputs than the number of independent variables or parameters in the cart system, therefore the control is difficult and this system falls into the category of under-actuated systems [1], [2]. The inverted pendulum system is widely used in number of applications such as Segway, human posture systems, and rocket launching systems etc. As it is most popular in real-world applications, therefore the control of inverted pendulums has attracted the attention of researchers in the field of control engineering [3], [4].

Many modern and classical control strategies may be designed, tested, evaluated, and compared in a highly unstable inverted pendulum system. The control challenge involves creating dynamic models of systems for achieving the required response and performance of the system, therefore control law and dynamic model are employed. Non-linear control systems such as robotic systems, missile systems and many other non-linear control systems have been controlled using linear control methods such as PID control [6], [7].

PID control is a straightforward and efficient control approach and it is widely used in industrial control systems due to its easy application [8], [9]. However, conventional PID controller doesn't give satisfactory results under the sudden transient condition in many non-linear applications. Therefore, some other option needs to incorporate with classical approach for controlling of the non-linear system [5].

Linear quadratic regulator (LQR) is modern control approach and shows optimum results in various applications. LQR takes optimum control decisions based on the control parameters and state of the dynamic system [12]. The optimal approach LQR seeks control instructions and acts for exceeding the system's performance criteria within physical constraints [3]. A feedback output controller is proposed for the stabilization of the inverted pendulum on a cart in the presence of uncertainties [8]-[11].

Therefore, an attempt is made to propose a PID-LQR based modern controller for inverted pendulum cart system application. The performance of the proposed controller is compared with the conventional PID controller.

II. MODELLING OF INVERTED PENDULUM-CART SYSTEM

Fig.1 shows a free body schematic diagram of inverted pendulum cart system. The nomenclature x and θ indicate cart position and tilt angle variation in a vertically upward respectively. The pendulum rod is assumed massless and the hinge is considered as frictionless in the calculation. All the forces, that acted on the cart system are depicted in the diagram.

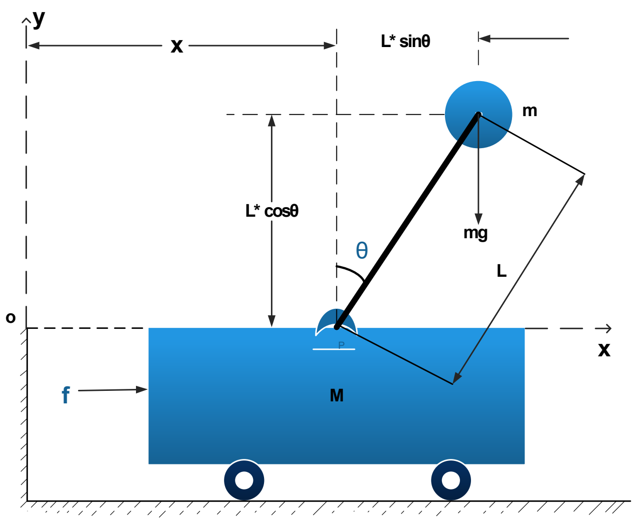

Fig.1. Schematic diagram of inverted pendulum-cart system

The ball mass's time-dependent center of gravity (COG) is provided by coordinates (x_{COG}, y_{COG}). The balance of force on the system are represented in horizontal x- direction as given in Eq. below, [3]

$$M\frac{d^2x}{dt^2} + m\frac{d^2x_{COG}}{dt^2} = f \tag{1}$$

The position of the pendulum mass's center of gravity is

$$x_{COG} = \mathrm{x} + L\sin\theta\,, y_{COG} = L\cos\theta \tag{2}$$

Where L = length of pendulum rod

Eq. 3 is obtained from Eq.1 and Eq. 2,

$$(M+m)\ddot{x} - mL\sin\theta\dot{\theta}^2 + mL\cos\theta\ddot{\theta} = f \tag{3}$$

Fig. 2 shows vector force components, operating on the cart system. The resultant torque can be expressed as [3]

$$(F_x\cos\theta)L - (F_y\sin\theta)L = (mg\sin\theta)L \tag{4}$$

Where,

Force in the x-direction: $F_x = m\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_{COG}$

Force in the y-direction: $F_y = m\frac{\mathrm{d}^2}{\mathrm{d}t^2}y_{COG}$

From Eq. (4),

$$m\ddot{x}\cos\theta + mL\ddot{\theta} = mg\sin\theta \tag{5}$$

From Eq. 3 and Eq. 5, The cart modeling is represented using Eq. (6-7)

$$\ddot{x} = \frac{f + mL(\sin\theta)\dot{\theta}^2 - mg\cos\theta sin\theta}{M + m - m\cos^2\theta} \tag{6}$$

$$\ddot{\theta} = \frac{f\cos\theta - (M+m)gsin\theta + mL(\cos\theta sin\theta)\dot{\theta}^2}{mL\cos^2\theta - (M+m)\,\mathrm{L}} \tag{7}$$

The Cart system is represented using nonlinear eq. (6-7). The state-space modeling is employed and it is reduced to set of linear equations.

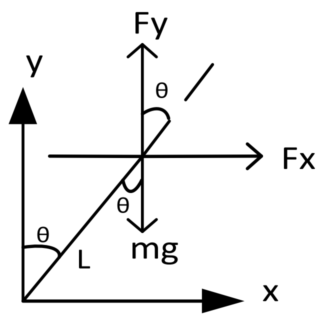

Fig.2. Vector diagram for balancing torque.

III. STATE SPACE MODELLING & LINEARIZATION

The modeling of the inverted cart system is linearized using state space modeling.
The standard state space equation is given in eq. 8

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{g}(x, f, t) \tag{8}$$

By considering state variables

$$x_1 = \theta,$$

$$x_2 = \dot{\theta} = \dot{x}_1,$$

$$x_3 = \mathrm{x},$$

$$x_4 = \dot{x} = \dot{x}_3 \tag{9}$$

The state space equation is as follows:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix} \tag{10}$$

The value of g_1, g_2, g_3, and g_4 are obtained from Eq. 6 and Eq. 7.

$$g_1 = x_2 \tag{11}$$

$$g_2 = \frac{\mathrm{f}\cos x_1 - (M+m)\,\mathrm{g}sin\,x_1 + mL(\cos\mathrm{x}_1\sin\mathrm{x}_1)x_2^2}{mL\cos^2 x_1 - (\mathrm{M}+\mathrm{m})\mathrm{L}} \tag{12}$$

$$g_3 = x_4 \tag{13}$$

$$g_4 = \frac{\mathrm{f} + ml(\sin x_1)x_2^2 - mg\cos x_1\sin x_1}{M + m - m\cos^2 x_1} \tag{14}$$

The output of the inverted pendulum system is represented in Eq. 15.

$$\boldsymbol{y} = \mathrm{C}\mathbf{x}$$

$$y = \begin{bmatrix} \theta \\ x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} \tag{15}$$

The objective of the control system is to keep the inverted pendulum at an equilibrium position i.e., θ=0. The system, represented in Eq. 10 is linearized by jacobian matrices.

$$\frac{\mathrm{d}\delta\mathbf{x}}{\mathrm{d}t} = \boldsymbol{J}_x(\boldsymbol{x}_0, \mathrm{f}_0)\delta\mathbf{x} + \boldsymbol{J}_f(\mathbf{x}_0, f_0)\delta\mathrm{f} \tag{16}$$

The reference state of the pendulum is considered stationary i.e. $x_0 = 0$ and upright direction i.e. $f_0 = 0$ without input force. Consequently, the component of jacobian matrices is decided by

$$\left.\frac{\partial f_i}{\partial x_1}\right|_{x_0, f_0}, \left.\frac{\partial f_i}{\partial x_2}\right|_{x_0, f_0}, \left.\frac{\partial f_i}{\partial x_3}\right|_{x_0, f_0}, \text{ and } \left.\frac{\partial f_i}{\partial x_4}\right|_{x_0, f_0},$$

The Jacobian terms (J_x and J_u) are obtained as mentioned in Eq. 17 and Eq. 18.

$$J_x(x_0, f_0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(M+m)g}{ML} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{mg}{M} & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

Taking the derivative of Eq. 17 concerning f.

$$J_u(x_0,f_0) = \begin{bmatrix} \frac{\partial g_1}{\partial f} \\ \frac{\partial g_2}{\partial f} \\ \frac{\partial g_3}{\partial f} \\ \frac{\partial g_4}{\partial f} \end{bmatrix}_{x_0,f_0} = \begin{bmatrix} 0 \\ \frac{\cos x_1}{mL\cos^2 x_1 - (M+m)L} \\ 0 \\ \frac{1}{M+m-m\cos^2 x_1} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{ML} \\ 0 \\ \frac{1}{M} \end{bmatrix}_{x_0,f_0} \quad (18)$$

After simplification, Eq. (16) may be written as Eq. 19.

$$\frac{\mathrm{d}\delta\boldsymbol{x}}{\mathrm{dt}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(M+m)g}{ML} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{mg}{M} & 0 & 0 & 0 \end{bmatrix} \delta\boldsymbol{x} + \begin{bmatrix} 0 \\ -\frac{1}{ML} \\ 0 \\ \frac{1}{M} \end{bmatrix} \delta f \quad (19)$$

Eq. (20) represents a linearized model of the inverted pendulum cart system.

$$\frac{\mathrm{d}\delta\boldsymbol{x}}{\mathrm{dt}} = \boldsymbol{A}\delta\boldsymbol{x} + \boldsymbol{B}\delta f \quad (20)$$

The equation for linear state space is

$$\dot{x} = Ax + Bu \quad (21)$$

Where $\boldsymbol{x} = [\theta,\ \dot{\theta},\ \mathrm{x},\ \dot{x}]^{\mathrm{T}}$.

State feedback $\quad \mathrm{U} = -\mathrm{K}\,\mathrm{x}$

$$\dot{\mathrm{x}} = (\mathrm{A} - \mathrm{BK})\mathrm{x} \quad (22)$$

The value of K is obtained from the minimization of the cost function, mentioned in Eq. (23).

$$J = \int (\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x} + u^{\mathrm{T}}\boldsymbol{R}u)\mathrm{d}t \quad (23)$$

Where Q and R are positive semi-definite and positive definite symmetric matrices respectively.

$$A^{\mathrm{T}}P + PA - PBR^{-1}B^{\mathrm{T}}P + Q = 0 \quad (24)$$

Where P is positive definite symmetric constant matrix algebraic Riccati equation (ARE).

Therefore, the gain vector K is calculated using Eq. (24)

$$\mathrm{K} = \mathbf{R}^{-1}\mathrm{B}^{\mathrm{T}}\mathrm{P} \quad (25)$$

IV. CONTROL SCHEMES

Conventional PID controllers and Modern Controllers are applied to the inverted pendulum cart system simultaneously. Both control schemes are applied to the system simultaneously. Fig. 3 and Fig. 4 show the simulated diagram using a conventional PID control scheme and a modern LQR-based PID control scheme respectively [6][7].

A. Conventional PID controller

The inverted pendulum cart system is controlled by two PID controllers. The position of the cart system is controlled by one PID controller whereas the angle position of the inverted pendulum is maintained by another PID controller.

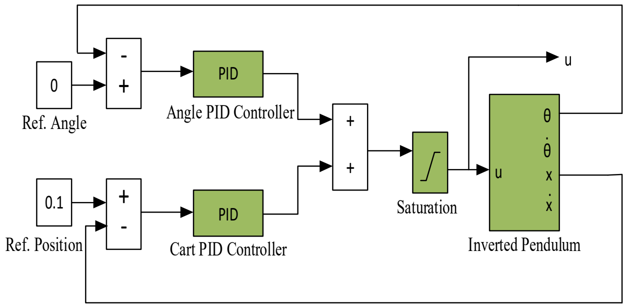

Fig.3. Conventional PID control scheme to inverted pendulum-cart system

The PID control equations for angle (u_θ) of the inverted pendulum and cart position (u_x) are given in Eq. 27 and Eq. 28 respectively.

$$u_\theta = K_{p\theta}e_\theta(t) + K_{i\theta}\int e_\theta(t)\mathrm{d}t + K_{d\theta}\frac{\mathrm{d}e_\theta(t)}{\mathrm{d}t} \quad (26)$$

$$u_x = K_{px}e_x(t) + K_{ix}\int e_x(t)\mathrm{d}t + K_{dx}\frac{\mathrm{d}e_x(t)}{\mathrm{d}t} \quad (27)$$

Where $e_\theta(t)$ and $e_x(t)$ are angle and cart position errors respectively. The PID controller gains i.e. $K_{p\theta}$, , $K_{d\theta}$ are used for inverted pendulum's angle whereas the PID controller gains i.e. K_{px}, K_{ix}, K_{dx} are used for cart position.

B. Modern PID-LQR based Controller

The modern controller is combined features of PID and LQR. In this modern control scheme, two PID controllers are used for controlling of cart position and angle of the inverted pendulum system as same in the conventional control scheme. The feedback is given with LQR as shown in Fig. 4.

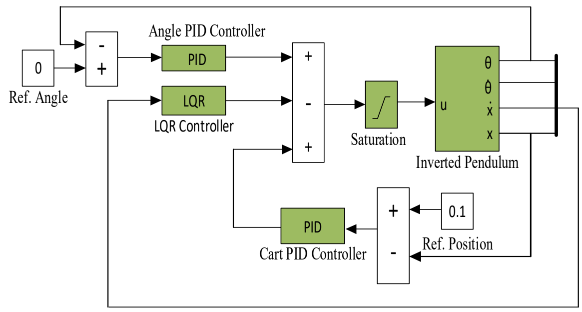

Fig .4. Modern PID-LQR based control scheme to inverted pendulum-cart system

IV. PERFORMANCE ANALYSIS OF CONVENTIONAL AND MODERN PID-LQR BASED CONTROLLER

A. Comparative analysis of both controller

The system parameters i.e. the weight of the cart, pendulum mass, and length of rod mentioned in the Appendix are considered to calculate matrix A, B, and C using Eq. 19 are used to get values of matrix A, B, C and matrix D by using data e.g. weight of cart the pendulum mass, length of the rod as mentioned in Table IV.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 23.369 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.94 & 0 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ -0.9057 \\ 0 \\ 0.4167 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\quad \boldsymbol{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The value of matrix Q and matrix R matrix are taken as [3]

$$\boldsymbol{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 450 & 0 \\ 0 & 0 & 0 & 255 \end{bmatrix} \quad \& \quad \boldsymbol{R} = [1]$$

So gain vector is obtained from Eq. 25,

$$\mathrm{K} = \quad [-180 \;\; -30 \;\; -26 \;\; -32]$$

Table 1 shows PID control parameters for conventional and modern controller gains.

TABLE I: CONTROLLER GAIN -CONVENTIONAL AND MODERN CONTROLLER

Control schemes	***Angle controthe l of inverted pendulum***			***Position control of cart system***		
	$K_{p\theta}$	$K_{i\theta}$	$K_{d\theta}$	K_{px}	K_{ix}	K_{dx}
Conventional PID Controller gain	-41.2	0.1	-6	-0.9	- 0.003	-2.34
Modern PID Controller gain	5	2	2	1.5	-7.52	5.01
Modern (LQR) Controller Gain (K)	K= [-180 -30 -26 -32]					

Fig. 5 and Fig. the 6 represent comparative performance of conventional controller for the controlling of cart position and angle of inverted pendulum system respectively. Table II represents time specification response of conventional and modern controller for the controlling of cart position. The reference is taken 0.1m for the cart position. The modern controller depicts improved time specification response in terms of rise time, settling time; maximum overshoot and minimum undershoot as compared to conventional PID controller. The inverted pendulum angle is maintained to be 0^0 i.e. vertical positions. Fig. 6 shows simulated output for conventional and modern controller. Table III represents comparative results of both control schemes.

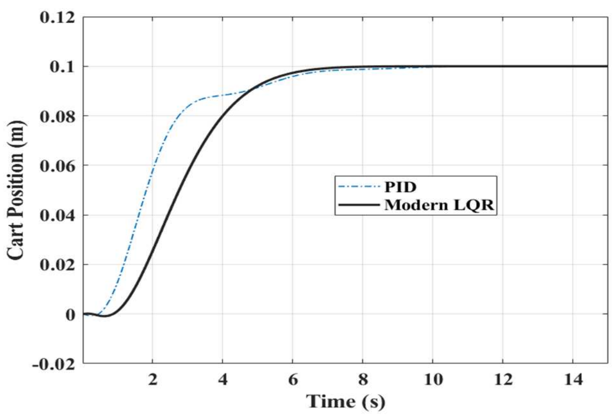

Fig.5. Control of cart position using conventional and modern controller

TABLE II: COMPARATIVE RESULTS FOR CART POSITION USING CONVENTIONAL AND MODERN CONTROLLER

Time Response Specification	***Modern Controller (LQR+PID)***	***Conventional Controller(PID)***
Rise Time (s)	3.2952	3.7591
Settling Time (s)	6.239	6.9017
Maximum overshoot	0.1000	0.1201
Minimum undershoot	0.0900	0.0961

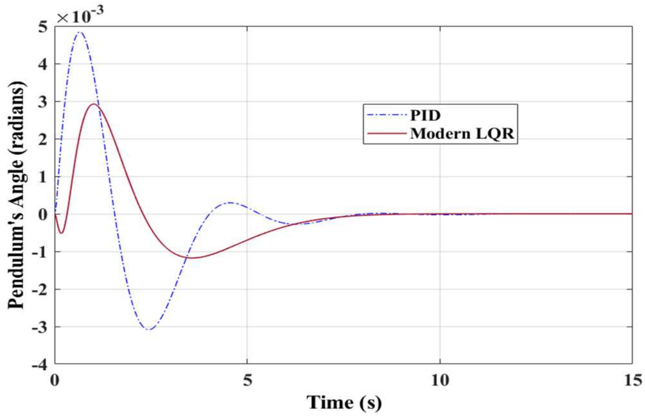

Fig. 6. Control of pendulm Angular response using conventional and modern controller

TABLE III: COMPARATIVE RESULTS FOR PENDULUM ANGULAR RESPONSE USING CONVENTIONAL AND MODERN CONTROLLER

Time Response Specification	***PID+LQR***	***PID***
Rise Time (s)	0.0039	0.0031
Settling Time (s)	7.69	7.73
Maximum Overshoot	0.0029	0.0048
Minimum undershoot	-0.0012	-0.0031

B. Responses after disturbance with PID-LQR based Controller

For all system variable parameters, we can monitor the responses of different parameters after causing disturbances at specific times t = 10 sec and t = 40 sec. Fig. 7 and Fig. 8 shows the responses of inverted pendulum angle and inverted pendulum cart position with PID-LQR based modern controller.

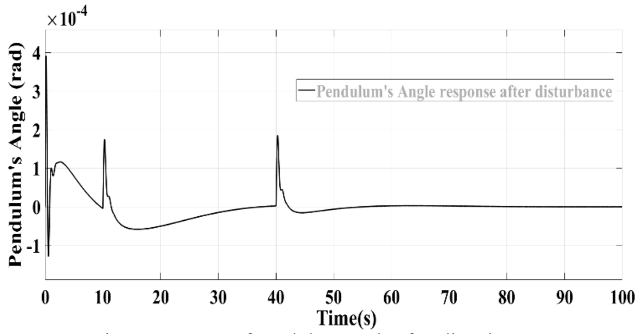

Fig. 7 Response of pendulum angle after disturbances

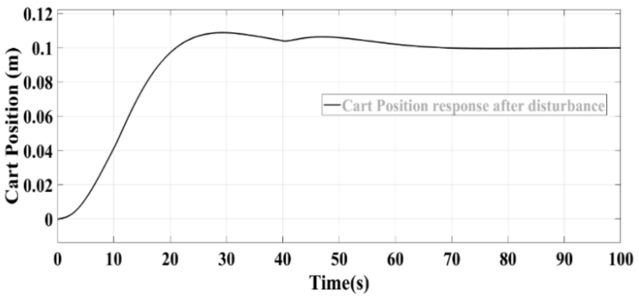

Fig. 8 Response of pendulum cart position after disturbances

Fig. 7 depicts the inverted pendulum angle response to a disturbance of magnitude 0.0005 at times t =10 and t =40 seconds. Fig. 8 displays the inverted pendulum's cart position response for various disturbances. After a disruption, the PID-LQR based modern controller stabilizes in the shortest period.

VI. Conclusion

The modern controller i.e. combination of PID and LQR has been proposed to control inverted pendulum cart system. The modeling cart system as well as modern controller has been presented in this paper. The performance of the proposed controller has been evaluated and compared with the conventional PID controller. It has been observed that modern controller gives better results in terms of rise time, settling time, peak overshoot and undershoot as compared to the conventional PID controller for the non-linear application i.e. inverted pendulum cart system. As detailed modelling of the proposed modern controller has been discussed, therefore the proposed paper shall be helpful for researchers, academicians and control engineers who are working in the field of non-linear applications such as control of missile.

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Appendix

Symbol	***Parameter***	***Value***
M	Weight of cart	2400 gm
m	Mass of pendulum	230 gm
L	The length of rod	0.46 m
g	Gravity	9.81 m/s^2