STUDYING THE OPTIMAL CONTROL OF A NON-HOLONOMIC SYSTEM USING AN INVERTED PENDULUM

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ABSTRACT

This report discusses the team's progress to date in the study of the optimal control of a non-holonomic system using an inverted pendulum. In this progress check, we study the control technique that combines the Linear Quadratic Regulator (LQR) and Proportional, Integral, and Derivative (PID) applied to a highly nonlinear and unstable system, the inverted pendulum. The LQR controller, which is designed using a linear state-space model, is used to control the nonlinear system states. The objective is to stabilize the inverted pendulum in an upright position and move the cart to the desired position. This report presents a detailed mathematical description of the problem statement, the MATLAB-SIMULINK simulation results of the defined model, and the current hardware implementation. This project refers to the paper by L.B. Prasad, et al. (1).

1. INTRODUCTION

The Inverted Pendulum is a highly nonlinear and unstable mechanical system that belongs to the class of under-actuated systems, making it a challenging benchmark for testing and evaluating control techniques. Stabilizing the inverted pendulum to keep the pendulum in an upright position during movements is a common research interest in control engineering, as it represents a class of altitude control problems. The control problem involves obtaining dynamic models of the system and determining control laws or strategies to achieve the desired system response and performance, which can be challenging for highly nonlinear systems. The control of such systems, including the inverted pendulum, is an important and challenging task for real-world applications. It is widely used in engineering, robotics, and control systems as a benchmark for testing and evaluating control algorithms.

The goal is to achieve optimal performance in controlling dynamical systems. Various optimization and optimal control techniques have been documented in literature for both linear and nonlinear dynamical systems(2).

The Proportional-Integral-Derivative (PID) control method is a straightforward and effective solution to various practical control problems. Its three components (P, I, and D) provide a comprehensive approach to addressing both the short-term and long-term responses of the system.

This project has implemented both linear quadratic regulator (LQR), which is an optimal control technique, and PID control, which are commonly used for linear dynamical systems, to control the nonlinear inverted pendulum-cart dynamical system. Although advanced control methods are becoming popular in recent trends, the authors have chosen these methods as they are effective and robust, despite being relatively simple.

This report is organized in 8 sections. Beginning with the mathematical modeling of the system, we then discuss the control methods that we have decided to explore for the purpose of this project. We subsequently discuss the simulation of the system on MATLAB and SIMULINK. Further, we show the hardware implementation of the system.

2. MATHEMATICAL MODELLING

Referring to the free-body diagram of an inverted pendulum mounted on a cart, shown in Fig. 1, and assuming the pendulum-rod is mass-less and the hinge is frictionless, the system of equations defining this model is derived(3).

$$(M+m)\ddot{x} - mlsin(\theta)\dot{\theta}^2 + mlcos(\theta)\ddot{\theta} = u \qquad (1)$$

This equation represents the balance of forces in the x-direction. Here, M is the mass of the cart, m is the mass of the ball-point at the upper end of the pendulum, l is the length to the ball-point mass at the end of the pendulum rod, $u(t)$ is the external force applied to the system. The coordinated system, referenced at point O, is shown in Fig. 1. Along with this, a gravitational force acts on the point mass at all times. The cart position and the tilt angle of the pendulum rod are functions of time denoted by $x(t)$ and $\theta(t)$, respectively.

Similarly, the torque balance on the system is mathematically formulated as follows,

$$m\ddot{x}cos(\theta) + ml\ddot{\theta} = mgsin(\theta) \qquad (2)$$

The goal of the inverted pendulum-cart system is to maintain the inverted pendulum upright around $\theta = 0$.

2.1. Non-linear system of equations defining the Inverted Pendulum

The state-space model, describing the system of non-linear equations derived in the previous section, is required to simulate the inverted pendulum-cart dynamic system numerically(4). In this section, the system of non-linear equations, as derived

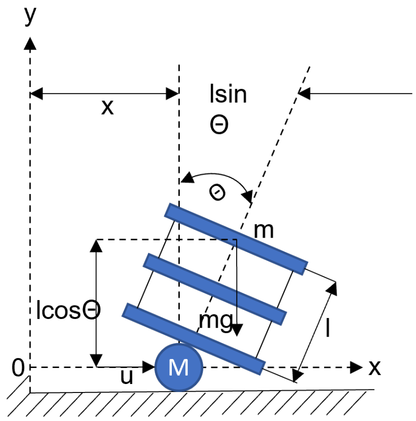

Fig. 1: Free-body diagram of the inverted pendulum-cart system that is driven by a motor. The mass at the upper end of the pendulum rod is denoted by m, and the cart's mass is denoted by M. The length of the pendulum rod is denoted as l. An external force in the x-direction, applied to the cart, is denoted by u.

in the previous section, will be represented in the following state-space form,

$$\frac{d}{dt}x = f(x, u, t) \tag{3}$$

$$\frac{d}{dt}x = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \tag{4}$$

The output equation is written as,

$$y = Cx \quad \text{or} \quad y = \begin{bmatrix} \theta \\ x \end{bmatrix} \tag{5}$$

$$y = \begin{bmatrix} \theta \\ x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} \tag{6}$$

Since the system is described by non-linear equations and the goal of this system is to keep the inverted pendulum in the upright position, around $\theta = 0$, the non-linear system from eqn. (12) is simplified and linearized under the condition that $x_0 = 0$ and $u_0 = 0$ as follows,

$$\frac{d}{dt}\delta x = J_x(x_0, u_0)\delta x + J_u(x_0, u_0)\delta u = A\delta x + B\delta u \tag{7}$$

Therefore,

$$\frac{d}{dt}\delta x = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(M+m)g}{Ml} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-mg}{M} & 0 & 0 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{-1}{Ml} \\ 0 \\ \frac{1}{M} \end{bmatrix} \delta u \tag{8}$$

3. CONTROL METHODS

The control system introduced to stabilize the inverted pendulum-cart system to the aforementioned reference point, we use a combination of PID and LQR control methods. The instantaneous states of the system, including pendulum angle θ, cart position x, angular velocity $\dot{\theta}$, and cart velocity $\dot{x}$, are considered available for measurement and fed to the LQR, which is designed using the linear state-space model of the system. The optimal control value of LQR is combined with the PID control value to obtain a resultant optimal control. The tuning of the PID controllers is done through a trial and error method by observing the responses achieved to be optimal.

3.1. Optimal Control using Linear Quadratic Regulator

Linear Quadratic Regulator considers the state space of the system and the control input to make optimal control decisions. The objective of the LQR is to minimize the cost function defined as,

$$J = \int (X^T Q X + u^T R u) dt \tag{9}$$

in eqn. (16), Q and R are positive semi-definite and positive definite symmetric matrices, respectively, that we will assume in a later section. The LQR gain vector K is given as,

$$K = R^{-1} B^T P \tag{10}$$

In eqn. (17), P is a positive definite symmetric matrix obtained from the algebraic Ricatti equation,

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \tag{11}$$

4. SIMULATIONS

The simulations for this project are done using MATLAB. At this point in time, the entire system has been designed, modeled, and simulated. The simulations yielded promising results and showed that the system was controllable.

While attempting to fix the errors that were encountered during the first progress check, it was found that the code would not work even when perfect. Upon conducting further research it was found that the state space matrices specified in the key reference paper were in fact flawed, which meant that new state space matrices were required in order to simulate the control of the inverted pendulum.

We have taken values of the mass at the end of the pendulum rod m and the mass of the cart M as 98g and 63g, respectively. The length of the pendulum l is taken to be $0.1m$ and the height of the cart L is $0.5m$. The matrices used to design the LQR are designed using these values (1)(5).

4.1. Inverted Pendulum on a Cart Model

The inverted pendulum on a cart is modeled as shown in Fig. 3. The constant blocks serve as a reference input for the pendulum's position and angle when stable. When an impulse is applied to the cart, the PID controllers are employed to keep the pendulum oriented vertically. The inverted pendulum's position and the angle are fed back to the LQR to update the PID controllers and make the necessary adjustments to maintain the system's stability.

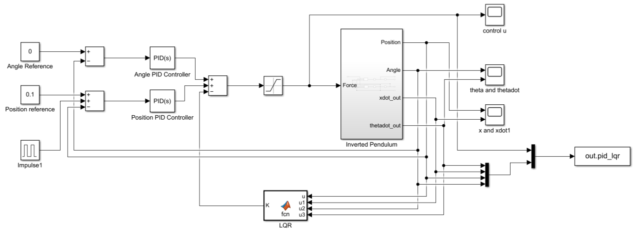

Fig. 2: Completed Model for the simulation of an Inverted Pendulum-Cart System

4.2. MATLAB Code Simulation and Results

The MATLAB code was written to find open and closed-loop responses of the system and plot graphs for feedback control with and without the LQR feed-forward gain and graphs for the LQR response itself. All the plotted graphs show the position of the cart and the angle of the pendulum that results from the movement of the cart.

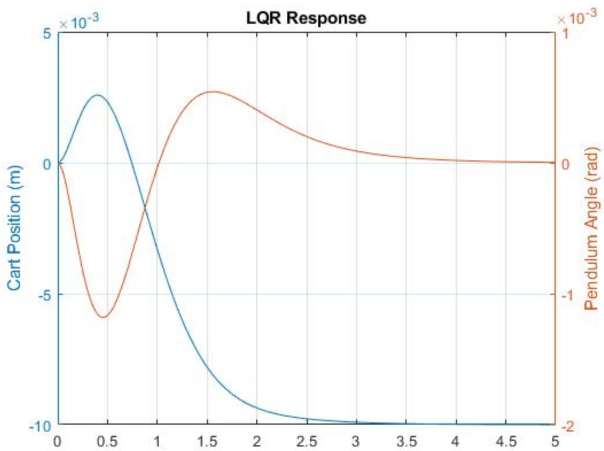

Fig. 3: LQR Response: It may be observed that the inverted pendulum is stabilized to 0 using the LQR solely as the control method after two overshoots, while the cart position is seen to reach $10m$ from the original position.

As we can see from the above graphs, the LQR Response with and without the feed-forward gain, shown in Fig. 3 and Fig. 4, show that the angle of the pendulum is kept at 0 after being balanced by the movement of the cart. The results of the simulation are verified with Observer-Based State-Feedback Control shown in Fig. 5 and Fig. 6. LQR is a type of State-Feedback controller so the graphs being consistent across both shows that the LQR Response is functioning as it should.

The Observer-Based State-Feedback controller is a dynamic two-stage feedback controller designed such that the controller first estimates the state variables of the system, using a measured output and known input, which is called the state observer for the system. This estimate is treated as the state of the system and used by a feedback controller to then control the system.

The next milestone for our simulation result was to show the stability and the movement that occurs during the stabilization of the system. In Fig. 5 we can see that the control law, u, is stabilized at zero over time which shows that the Inverted Pendulum system is stable and no more adjustments are necessary. It is essential to note that Fig. 5 provides a snapshot of the control process for the inverted pendulum system and represents just one instance of its control. In practice, the control process would need to be continuously repeated to maintain the stability of the pendulum. This is because any disturbance to the system can cause it to lose stability. The control algorithm employed in our project is designed to detect any instability in the system and initiate the control to bring the pendulum back to an upright position.

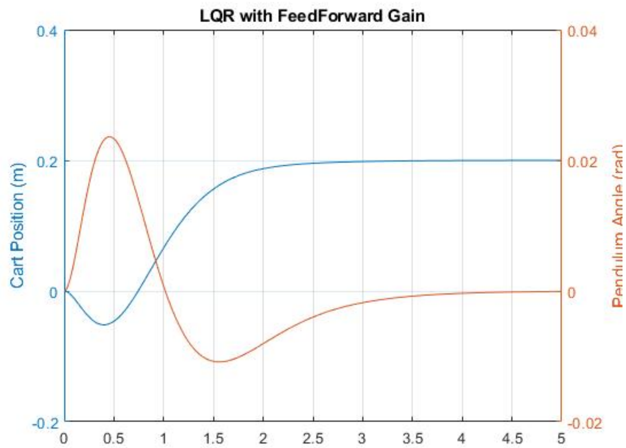

Fig. 4: LQR Response with Feed-Forward Gain: This figure shows the simulation of the inverted pendulum-cart system with the hybrid LQR-PID control method. Comparing these results with that obtained in Fig. 3, it may be observed that the pendulum stabilizes after two overshoots and the cart is $0.2m$ from the original position.

Fig. 6 demonstrates the stabilization of the individual components of the system over time. The graph displays four variables: x is the position of the cart, θ is the angle of the inverted pendulum, v is the velocity of the cart at any given time, and ω is the angular velocity of the cart at any given time. The graph illustrates the dynamic behavior of the system, where the position of the cart and the angle of the inverted pendulum are varied over time to achieve stability. The graph shows that the cart starts from a stationary position and then moves to compensate for the angle of the pendulum until it reaches a state of equilibrium. The graph also displays the changes in velocity and angular velocity of the cart, which initially increase due to the movement required to stabilize the pendulum. However, once the system reaches stability, these values return to their original state. As shown in the graph, the system reaches full stabilization around the four-second mark, and the position of the cart and the angle of the pendulum remain constant. It can be seen that the stabilization process is not immediate, and the system undergoes some initial oscillations before converging to a stable state.

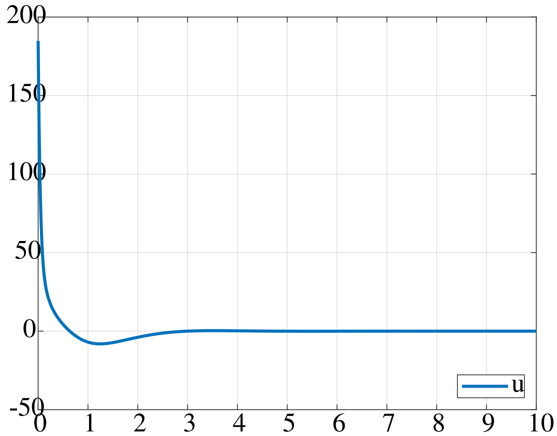

Fig. 5: Control Law Stability: We can see that the control law u, is stabilized at zero over time which shows that the inverted pendulum system is stable and no more adjustments are necessary. This means that the control law has achieved a steady-state response, where the output of the system remains constant over time. In this case, the output is the angle of the pendulum and the position of the cart. It is important to note that this figure shows one an instance of control of the inverted pendulum. If the system loses its stability, the control law would need to start again to stabilize the system. Therefore, the stability of the system depends on the ability of the control law to react to changes in the system and maintain its stability over time.

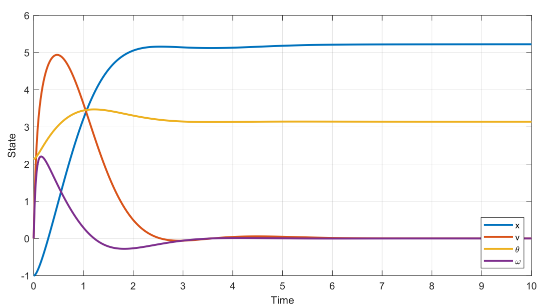

Fig. 6: Stability Simulation: This graph shows the various states of the system over time. The blue line shows the position of the cart, yellow is the angle of the inverted pendulum, orange is the velocity of the cart and purple is the angular velocity of the cart.

5. HARDWARE IMPLEMENTATION

To understand the real-world applications of our PID and LQR control algorithms, we undertook a hardware implementation of our model. This implementation allowed us to visualize the performance of our control systems in a physical setting and assess their effectiveness in real-time.

The hardware prototype was designed with the following specifications. The physical cart has a pendulum mass, m, of 63g and the mass of the cart, M, is 98g. The length of the pendulum, l, is taken to be $0.1m$ and the height of the cart L is $0.05m$. The length of the track on which the cart can move is $1m$ The hardware components used to control the cart include a controller, a motor driver, and an IMU sensor to measure the acceleration and velocity.

An Arduino Uno board with a time-based scheduler is being used to control the components and compensate for the change in the angle of the inverted pendulum in a timely manner. The proportional gain, k_p; integral gain, k_i, and derivative gain k_d values that were obtained from the simulation are used as parameters to control the system.

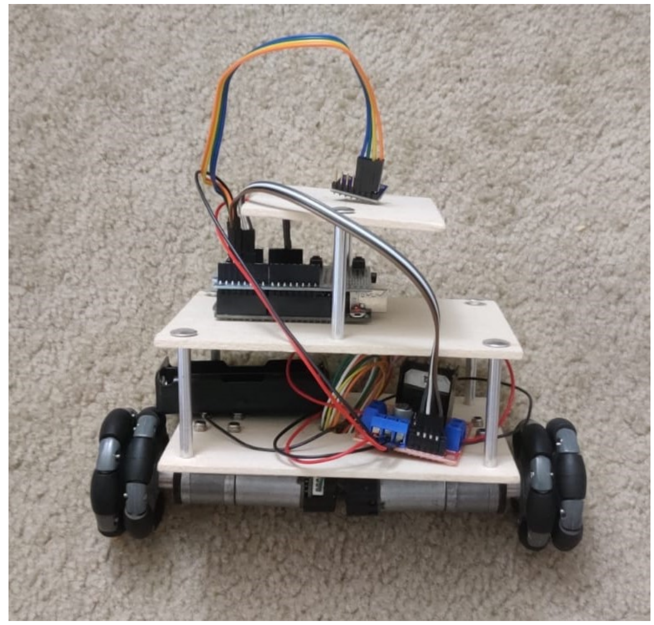

Fig. 7: Inverted pendulum-cart hardware implementation: The physical cart has been built alongside the development of the MATLAB-SIMULINK implementation of the model. This includes the Arduino board with the time-based scheduler to control the components and compensate for the change in the angle of the inverted pendulum in a timely manner.

6. ARDUINO CODE

For the hardware implementation of our project, we utilized the powerful and versatile Arduino Uno board. This control board is built around the ATmega328p microcontroller and boasts an impressive array of features that make it a popular choice for a variety of electronics, robotics, and coding-based projects. With 14 digital input/output pins and 6 analog pins, the board offers a high degree of customization and can be programmed to perform a wide range of tasks. We selected the Arduino Uno board for our project based on its reputation for reliability and ease of use. The development of the Arduino code involves writing programs using the Arduino Integrated Development Environment (IDE) software. This software is designed to facilitate the coding process by offering a range of tools, including a C++ code editor, various libraries, and pre-built functions that can be used to program the microcontroller effectively. The Arduino IDE's integration with Arduino boards eliminates the need for external hardware or software, making it a convenient and simple platform to program the microcontroller. To make the Arduino code easier to comprehend and more efficient, we divided it into multiple files, each with a specific function. Some of these files are main.ino, Timer.ino, IMU.ino, MotorController.ino, and PIDController.ino.

The main.ino file serves as the entry point at first, completes all initialization, and activates the scheduler. Next, the Timer.ino file describes how this scheduler uses the internal timer to measure time with an accuracy of up to 1 ms. The scheduler calls all other functions at regular intervals to ensure consistent feedback and control.

The IMU.ino software then takes data from the inertial measurement unit through an i2c channel. This is then decoded and converted to the required units. The actual acceleration was measured using data from the accelerometer and gyroscope. The IMU is our primary feedback sensor and provides vital information on the angle of the pendulum and the orientation of the cart.

The control then transfers to the PIDController.ino file, where we proceed to compute the required velocity and direction to compensate for the desired angle and the IMU sensor error. This is then delivered to the MotorController.ino file. The MotorController.ino file controls the direction and speed of the motors, which in turn controls the angle of the pendulum and the orientation of the cart.

The MPU-9250 sensor, which is a combination of an accelerometer and a gyroscope, was connected to the Arduino, and the sensor data was obtained. Fig. 8 shows the plot for the sensor data that exhibits the control of the system that eventually brings the system to stability.

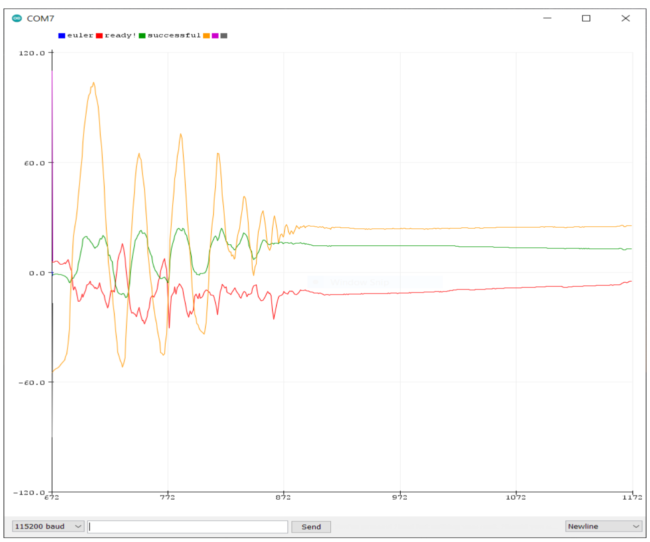

Fig. 8: This figure shows the acceleration values on all three axes starting with the unstable cart that gradually becomes stable.

7. MILESTONES

The first milestone of this project was to complete the research and choose a paper that would serve as our key reference paper. Once that was completed, we started working on the mathematical modeling of the state-space system.

Once the paper was selected and the mathematical modeling was complete, the next step was to start modeling the Inverted Pendulum on a Cart system itself. MATLAB Simulink was selected for this purpose. We coded the simulation of the LQR system on MATLAB and designed the Simulink model that was described in the key reference paper. Both simulations yielded promising results and showed that the system could be stabilized. Yet to be completed is the fine-tuning of these simulations to simulate the working of the system to our liking.

For the hardware implementation, we started by searching for the hardware components we deemed necessary for the design and model that was envisioned. The physical cart was then designed and built as per the specifications mentioned in Section 5. We then coded the Arduino board to control the components and the system. Yet to be completed is the fine-tuning of the working of the cart to efficiently and accurately stabilize the system.

8. CONCLUSIONS AND DISCUSSIONS

The project "Studying the Optimal Control of a Non-Holonomic System using an Inverted Pendulum" successfully achieved its objectives of investigating and analyzing the optimal control strategies for a non-holonomic system using an inverted pendulum. The study revealed that using the optimal control strategy of a combination of a Proportional Integral Derivative (PID) controller and a Linear Quadratic Regulator (LQR) with a state feedback controller could effectively control the inverted pendulum system, and the obtained results showed that the system's stability and robustness were significantly improved.

The study highlighted the significance of understanding the dynamics of a non-holonomic system and its constraints to design effective control strategies. The use of simulation software and mathematical modeling tools like MATLAB and Simulink proved to be effective in analyzing the system's behavior and evaluating the control strategies.

However, the hardware implementation of the inverted pendulum system was not completed as planned due to errors found in the mathematical model and Simulink model presented in the key reference paper. Unfortunately, our various efforts to try and correct these mistakes in order to complete the hardware simulation as planned were in vain.

Future work should focus on rectifying the errors in the mathematical and Simulink models, and implementing the control strategy on the hardware prototype. Further research could also explore the use of other control strategies and evaluate their effectiveness on the non-holonomic system. The simulations can also be modified to be more accurate in their control such as perfecting the angle of stability and making sure the cart returns to its original position once the system is stable.

Additionally, the impact of external disturbances on the system's stability and robustness should also be explored to ensure its usability in real-world scenarios. The project's findings have practical applications for various future works in robotics, control systems, and transportation engineering, where non-holonomic systems are commonly used. Overall, this project contributed to advancing the understanding of optimal control strategies for non-holonomic systems, and it provides a foundation for future research in this area.

In conclusion, the project provides valuable insights into the optimal control of non-holonomic systems using an inverted pendulum and highlights the need for accuracy and precision in the mathematical and simulation models. The successful implementation of the control strategy on the hardware platform will further enhance the understanding of non-holonomic systems and their practical applications.

9. STATEMENT OF CONTRIBUTION

The replication of the paper(1) is a collaborative effort between Aniruddha Anand Damle, Vishal Rajesh Vasisht, and Prakriti Biswas.

We, Team 12, consisting of Prakriti Biswas, Aniruddha Anand Damle, and Vishal Rajesh Vasisht, hereby state our individual contributions to the project as follows:

Prakriti Biswas conducted extensive research on the project topic and contributed to the development of the project proposal, developed and implemented the mathematical model of the system, worked on debugging and fine-tuning the simulation code, contributed to the analysis and interpretation of simulation results, drafted and wrote sections 2 and 3 of the report, and contributed to the final editing.
Cross-signed by: AAD, VRV

Aniruddha Anand Damle engaged in extensive research on the project topic and played a vital role in shaping the project proposal, designed and constructed the hardware prototype of the inverted pendulum system, worked on debugging and fine-tuning the simulation code, contributed to the analysis, interpretation, and verification of simulation results, drafted and wrote sections 5 and 6 of the report, and contributed to the final editing.
Cross-signed by: PB, VRV

Vishal Rajesh Vasisht conducted a comprehensive study on the project topic and made valuable contributions towards the formulation and development of the project proposal, worked on the implementation of the code for the simulation of the inverted pendulum system, worked on debugging and fine-tuning the simulation code, contributed to the analysis and interpretation of simulation results, drafted and wrote sections 4 and 8 of the report, and contributed to the final editing.
Cross-signed by: AAD, PB

We hereby acknowledge and appreciate each team member's contributions to the project, which significantly contributed to its success.

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