Technische Universität München
Lehrstuhl für Informatik V
Univ.-Prof. Dr. M. Bader
Carsten Uphoff, Nikola Tchipev

SS 2018
Sheet 2
27.04.2018

Scientific Computing II

Iterative Solvers

Exercise 1: Fourier Analysis for Two-Grid Iteration

We stick to the Poisson example:

$$\begin{aligned} -\frac{d^2u}{dx^2} &= f(x), \quad x \in (0,1), \\ u(0) = u(1) &= 0. \end{aligned}$$

This time, however, we want to consider the more general case of having $N+1$ grid points, where $N := 2K$, and a resulting mesh size $h := 1/N$. A step towards multigrid methods consists in the *two-grid method* based on *coarse-grid correction*. In this particular case, we formulate the algorithm as follows:

- After some iteration steps using a smoother (like Jacobi), compute residual $r^{old} := b^h - A_h u^h$. The residual and the error e^{old} fulfill the residual equation $r^{old} = Ae^{old}$.
- Restrict the residual to the coarse grid using an operator R. This yields a (residual-like) vector r^{2h} on the coarse grid. The coarse grid is assumed to have $K+1$ points.
- Solve the residual equation $A_{2h}e^{2h} = r^{2h}$ on the coarse grid. A_{2h} corresponds to the coarse-grained system that resembles A_h on the fine grid.
- Interpolate the resulting coarse-grid approximation of the error e^{2h} to the fine grid and correct the fine-grid error $e^{new} := e^{old} - Pe^{2h}$. The interpolation operator shall be denoted by P.

Furthermore, we define vectors q_h^m with components $(q_h^m)_i := \sin(m\pi hi)$. With respect to the fine grid, we have a low frequency if $m < N/2$, and a high frequency if $m \geq N/2$.
In the following, we want to step through the Fourier analysis for this method and investigate the convergence behaviour, considering one cycle of this two-grid algorithm.

(a) Give a closed expression for e^{new} which only depends on e^{old} and not on e^{2h} anymore. You may use the operators R, P, A_{2h} and A_h from above.

(b) Define the restriction operator R by injection (see lecture slides). Show that the restriction of any error frequency Rq_h^m yields a low frequency, that is

$$Rq_h^m = \begin{cases} q_{2h}^m & \text{if} \quad m < N/2 \\ -q_{2h}^{N-m} & \text{if} \quad m \geq N/2. \end{cases}$$

(c) For interpolation, we want to use the linear interpolation scheme:

$$(Pq_{2h}^m)_i = \begin{cases} \underbrace{(q_{2h}^m)_{i/2}}_{= (q_h^m)_i} & \text{for} \quad i = 2, 4, \ldots, N-2 \\ \frac{1}{2}\big(\underbrace{(q_{2h}^m)_{(i-1)/2}}_{= (q_h^m)_{i-1}} + \underbrace{(q_{2h}^m)_{(i+1)/2}}_{= (q_h^m)_{i+1}}\big) & \text{for} \quad i = 1, 3, \ldots, N-1. \end{cases}$$

Use the function definitions

$$\begin{aligned} \tfrac{1}{2}(\cos(\pi i) + 1) &= \begin{cases} 1 & \text{for } i = 2, 4, \ldots, N-2 \\ 0 & \text{for } i = 1, 3, \ldots, N-1 \end{cases} \\ \tfrac{1}{2}(-\cos(\pi i) + 1) &= \begin{cases} 0 & \text{for } i = 2, 4, \ldots, N-2 \\ 1 & \text{for } i = 1, 3, \ldots, N-1 \end{cases} \end{aligned}$$

and

$$(q_h^{N-m})_i = \sin((N-m)\pi h i) = \underbrace{\sin(N\pi h i)}_{=0} \cos(m\pi h i) - \cos(N\pi h i) \underbrace{\sin(m\pi h i)}_{=(q_h^m)_i} = -\cos(\pi i)(q_h^m)_i$$

to re-write the interpolated coarse grid frequency Pq_{2h}^m as

$$Pq_{2h}^m = a_m \cdot q_h^m + b_m \cdot q_h^{N-m}$$

with (frequency-dependent) constants a_m, b_m. What does the latter equation tell us about the frequency of the interpolated function Pq_{2h}^m?

(d) Putting the results from prolongation and restriction together, we can derive by further computation (not discussed in the exercise) that

$$\begin{aligned} PA_{2h}^{-1}RA_h q_h^m &= \begin{cases} q_h^m - \frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} q_h^{N-m} & \text{if} \quad m < \frac{N}{2} \\ -\frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} q_h^{N-m} + q_h^m & \text{if} \quad m \geq \frac{N}{2} \end{cases} \\ &= q_h^m - \frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} q_h^{N-m} \end{aligned}$$

Assume that the initial error is a linear combination of two sines: $e^{old} = x_1 q_h^m + x_2 q_h^{N-m}$ with $m < N/2$. Express $PA_{2h}^{-1}RA_h e^{old}$ in terms of q_h^m and q_h^{N-m}, a 2×2-matrix B_m, and the vector $(x_1, x_2)^T$. That is, find a representation

$$PA_{2h}^{-1}RA_h e^{old} = \begin{pmatrix} q_h^m & q_h^{N-m} \end{pmatrix} B_m \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

With this in mind, we can simplify the analysis with a change of basis. Let

$$Q = (q_h^1, q_h^{N-1}, q_h^2, q_h^{N-2}, \ldots, q_h^{K-1}, q_h^{N-K+1}, q_h^K).$$

As this matrix is invertible, we can find a vector x^{new} with $e^{new} = Qx^{new}$ and a vector x^{old} with $e^{old} = Qx^{old}$. The entry $(x^{old})_i$ can be interpreted as the weight of the frequency in the i-th column of Q. If we left-multiply e^{new} with Q^{-1} we have

$$x^{new} = Q^{-1}e^{new} = Q^{-1}e^{old} - Q^{-1}PA_{2h}^{-1}RA_h e^{old} = x^{old} - Cx^{old} = (I - C)x^{old}.$$

That is, we have found a frequency mapping $x \mapsto (I - C)x$ and may directly work with the error's frequency components instead with the error itself.

Determine the matrix C. (C will be block diagonal and consist of blocks B_m.)

(e) In order to remove high frequency errors, we apply two Jacobi iterations before (*pre-smoothing*) and after (*post-smoothing*) the coarse-grid correction scheme. Following the Fourier analysis (see Sheet 1) for the basis vectors $(q_h^m)_{m=1,\dots,N-1}$, one weighted Jacobi iteration ($\omega = 1/2$) corresponds to the update rule (again working on the error's frequency components with the same ordering as in (d))

$$x \mapsto M_f x.$$

I.e. for the frequency pair m and $N - m$ we obtain

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} \frac{1}{2}(1 + \cos(m\pi h)) & 0 \\ 0 & \frac{1}{2}(1 - \cos(m\pi h)) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Because C is a block diagonal matrix and M_f is diagonal, we may limit the discussion to frequency pairs. The general results then follow easily.

Give an estimate for the maximum eigenvalue of the overall solver operations, that is pre-smoothing $\to$ restriction $\to$ coarse-grid correction $\to$ interpolation $\to$post-smoothing.