Technische Universität München
Lehrstuhl für Informatik V
Univ.-Prof. Dr. M. Bader
Carsten Uphoff

SS 2018
Sheet 3
04.05.2018

Scientific Computing II

Multigrid Methods

Exercise 1: Galerkin Construction of Coarse Grid Operator

We consider the discrete Poisson system with Dirichlet boundary conditions and 7 grid points,

$$\begin{aligned} \frac{-u_{i-1}+2u_i-u_{i+1}}{h^2} &= f_i, \quad i=1\ldots 6 \\ u_0=u_6 &= 1, \quad \text{(inhomogeneous)} \end{aligned} \tag{1}$$

where the mesh size is $h := \frac{1}{6}$.

(a) Formulate the system of equations in matrix form, i.e. $A_h u = b_h$. What do you need to do in order to re-use the (reduced) operator A_h from the first worksheet?

(b) Define a linear mapping $R : \mathbb{R}^5 \to \mathbb{R}^2$ according to the full weighting scheme. Set up the respective matrix R.

(c) Define a linear mapping $P : \mathbb{R}^2 \to \mathbb{R}^5$ which prolongates a solution vector from the coarse to the fine grid using linear interpolation. Set up the respective matrix P. How are R and P related?

(d) Based on the matrices R, P and A_h, compute the coarse grid operator $A_{2h} := RA_hP$. Compare A_{2h} to the operator that you obtain by discretising the problem on the coarse grid. What do you observe?

(e) What happens if you restrict a constant vector, e.g. $r = \vec{1}$. Can you "mend" the restriction operator such that a constant vector is invariant under restriction? Does it matter?

(f) Say you are given routines for interpolation, restriction, and applying the fine grid operator. How could you compute RA_hP?

Exercise 2: Multigrid for Convection-Diffusion

Consider the one-dimensional convection-diffusion equation

$$-\epsilon u_{xx} + u_x = f(x) \tag{2}$$

and its discrete representation

$$\frac{\epsilon}{h^2}(-u_{i-1} + 2u_i - u_{i+1}) + \frac{1}{h}(u_i - u_{i-1}) = f_i, \quad i = 1, ..., N-1 \tag{3}$$

with Dirichlet conditions $u_0 = u_N = 0$ and $\epsilon \geq 0$. In this exercise, we assume a vanishing right hand side $f(x) = 0$. The solution of the differential equation is thus given by $u(x) = 0$.

(a) Show that for $\epsilon \ll 1$, the Jacobi method applied to Eq. (3) converges very slowly whereas the Gauss-Seidel method is expected to converge very fast.
Hint: Insert an error frequency $u_n = \sin(\pi k i h)$ *into the respective iterative scheme and analyse the behaviour for* $\epsilon \to 0$.

(b) In programming assignment 3, we want to use matrix-dependent restriction and interpolation to set up multigrid for solving Eq. (3). Show that the Galerkin coarsening $A_{2h} := RA_hP$ applied to an arbitrary three-point stencil $[s_l \quad s_c \quad s_r]$ with $s_c = -(s_l + s_r)$ on the fine grid yields the coarse grid stencil

$$\frac{1}{s_c}[-s_l^2 \quad (s_l^2 + s_r^2) \quad -s_r^2]. \tag{4}$$