# Assignment

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1. (i) Answer :- Random Variable $x$ is uniformly distributed in $[-2, 2]$

$\Rightarrow$ Probablity density function

$$f(x) = \begin{cases} 1/4 & x \in [-2, 2] \\ 0 & \text{elsewhere} \end{cases}$$

Now,

$\Rightarrow |x-1| \geq 0.5$

$\Rightarrow x-1 \geq 0.5$ or $x-1 \leq 0.5$

$\Rightarrow x \geq 1.5$ or $x \leq 0.5$

$\Rightarrow P(|X-1| \geq 0.5) = P(X \geq 1.5) + P(X \leq 0.5)$

$$= \int_{1.5}^{\infty} f(x)\,dx + \int_{-\infty}^{0.5} f(x)\,dx$$

$$= \int_{1.5}^{2} \frac{1}{4}\,dx + \int_{0.2}^{0.5} \frac{1}{4}\,dx$$

$\Rightarrow \frac{0.5}{4} + \frac{2.5}{4}$

$\Rightarrow 3/4$

$P(|X-1| \geq 0.5) = 0.75$ Ans

1. (iii) Answer :-

The following problem is an example of Binomial Distribution with $n = 2000$, $p = 1/2$

Let, $X \sim B(2000, 1/2)$

Now. $\bar{x} = np = 2000 \times \frac{1}{2} = 1000$

$\sigma^2 = npq = 2000 \times \frac{1}{2} \times \frac{1}{2} = 500$

Now, according to chebyshev's inequality

$P(|x-\bar{x}| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$

Let, $\epsilon = 100$

$P(|x-1000| \geq 100 \leq \frac{500}{100^2}$

$$P(X-1000 \geq 100) + P(X-1000 \leq -100) \leq \frac{1}{20}$$

$$P(X \geq 1100) + P(X \leq 900) \leq \frac{1}{20}$$

Now,

$$P(900 < X < 1100) = 1 - (P(X \geq 1100) + P(X \leq 900)) \geq 1 - \frac{1}{20}$$

$$P(900 < X < 100) \geq \frac{19}{20}$$

Hence proved

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Q2. (i) Answer:- Given

$$f(x) = \frac{1}{2\lambda} e^{-|\frac{x-\mu}{\lambda}|} ; \quad -\infty < x < \infty, \lambda > 0$$

Mean $\alpha_1 = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-\infty}^{\infty} \frac{x}{2\lambda} e^{-|\frac{x-\mu}{\lambda}|} dx$

$$\Rightarrow \alpha_1 = \int_{-\infty}^{\mu} \frac{x}{2\lambda} e^{\frac{x-\mu}{\lambda}} dx + \int_{\mu}^{\infty} \frac{x}{2\lambda} e^{\frac{\mu-x}{\lambda}} dx$$

Let,

$$I_1 = \int_{-\infty}^{\mu} \frac{x}{2d} e^{\frac{x-\mu}{d}} dx \quad \text{and}$$

$$I_2 = \int_{\mu}^{\infty} \frac{x}{2d} e^{\frac{\mu-x}{d}} dx$$

Let,

$$\frac{x-\mu}{d} = z \Rightarrow x = \mu + dz$$

$$dx = d\,dz$$

$$\Rightarrow I_1 = \int_{-\infty}^{0} \frac{1}{2d} (\mu + dz) e^{z} d\, dz$$

$$= \frac{\mu}{2} \int_{-\infty}^{\infty} e^{z} dx + \frac{d}{2} \int_{-\infty}^{0} z e^{z} dz$$

$$\Rightarrow \frac{\mu}{2} + \frac{d}{2} \left[ 0 - \int_{-\infty}^{0} e^{z} dz \right]$$

$$\Rightarrow \frac{\mu}{2} - \frac{d}{2}$$

$$I_1 = \frac{\mu}{2} - d/2$$

Now, Let, $\frac{\mu-x}{d} = k \Rightarrow x = \mu - dk$

$$dx = -d\,dk$$

$$I_2 = \int_0^{\infty} \left(\frac{\mu - \lambda u}{2\lambda}\right) e^{u} (-\lambda)\, du = \frac{\mu}{2} \int_{-\infty}^{0} e^{u}\, du$$

$$- \frac{\lambda}{2} \int_{-\infty}^{0} u e^{u}\, du$$

$$= \frac{\mu}{2} - \frac{\lambda}{2}(-1) = \frac{\mu}{2} + \frac{\lambda}{2}$$

Now, $\alpha_1 = I_1 + I_2 = \mu$

$\boxed{\text{Mean} = \mu}$ Ans

Variance. $\mu_2 = E[(x-m)^2]$ where $m$ is mean

$$\Rightarrow \mu_2 = E(x^2 - 2mx + m^2)$$

$$= E(x^2) - 2m E(x) + m^2.$$

$$= \alpha_2 - 2m \cdot m + m^2$$

$$= \alpha_2 - m^2$$

$$= \alpha_2 - \alpha_1^2$$

$$\alpha_2 = E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx + \int_{\infty}^{\mu} \frac{x^2}{2\lambda} e^{\frac{x-\mu}{\lambda}}\, dx$$

$$+ \int_{\mu}^{\infty} \frac{x^2}{2\lambda} e^{\frac{\mu - x}{\lambda}}\, dx$$

Let, $I_1 = \int_{-\infty}^{\mu} \frac{x^2}{2\lambda} e^{\frac{x-\mu}{\lambda}} dx$

and $I_2 = \int_{\mu}^{\infty} \frac{x^2}{2\lambda} e^{\frac{\mu-x}{\lambda}} dx$

Let, $\frac{x-\mu}{\lambda} = z \Rightarrow x = \mu + \lambda z$

$dx = \lambda dz$

$$\Rightarrow I_1 = \int_{-\infty}^{0} \frac{(\mu^2 + 2\mu\lambda z + \lambda^2 z^2)}{2\cancel{\lambda}} \times e^{z} \times dz$$

$$= \frac{\mu^2}{2}\int_{-\infty}^{0} e^{z} dz + \mu\lambda \int_{-\infty}^{0} z e^{z} dz$$

$$+ \frac{\lambda^2}{2}\int_{-\infty}^{0} z^2 e^{z} dz$$

$$= \frac{\mu^2}{2} - \mu\lambda + \frac{\lambda^2}{2}\left[0 - 2\int_{-\infty}^{0} z e^{z} dz\right]^{\infty}$$

$$I_1 = \frac{\mu^2}{2} - \mu\cdot\lambda + \lambda^2$$

Let, $\frac{\mu-x}{\lambda} = z \Rightarrow x = \mu - \lambda z$

$\Rightarrow dx = -\lambda dz$

$$\alpha_2 = I_1 + I_2 = 2d^2 + \mu^2$$

$$\mu_2 = \alpha_2 - \alpha_1^2 = 2d^2 + \mu^2 - \mu^2 = 2d^2$$

$$\boxed{\sigma = \sqrt{\mu_2} = \sqrt{2}\,d}$$

Ans

$$\mu_3 = E[(x-m)^3]$$

$$= E(x^3 - 3x^2 m + 3m^2 x - m^3)$$

$$= \alpha_3 - 3m\alpha_2 + 3m^2 m - m^3$$

$$= \alpha_3 - 3\alpha_1 \alpha_2 + 2\alpha_1^3$$

$$\alpha_3 = E(x^3) = \int_{-\infty}^{\infty} \frac{x^3}{2d} e^{-\frac{|x-\mu|}{d}} dx$$

$$= \int_{-\infty}^{\mu} \frac{x^3}{2d} e^{\frac{x-\mu}{d}} dx + \int_{\mu}^{\infty} \frac{x^3}{2d} e^{\frac{\mu-x}{d}} dx$$

let, $I_1 = \int_{-\infty}^{\mu} \frac{x^3}{2d} e^{\frac{x-\mu}{d}} dx$ and

$$I_2 = \int_{\mu}^{\infty} \frac{x^3}{2d} e^{\frac{\mu-x}{d}} dx$$

let, $\frac{x-\mu}{d} = z \Rightarrow x = \mu + dz \quad dx = d\,dz$

$$I_1 = \int_{-\infty}^{0} \left( \frac{\mu^2 + 2h\alpha\mu + h^2\alpha^2}{2\alpha} \right) e^{h} \alpha \, dh$$

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$$= \frac{\mu^2}{2} \int_{-\infty}^{0} e^{h} \, dh + \mu\alpha \int_{-\infty}^{0} h e^{h} \, dh + \frac{\alpha^2}{2} \int_{-\infty}^{0} h^2 e^{h} \, dh \quad dz$$

$$\Rightarrow \frac{\mu^2}{2} - \mu\alpha + \frac{\alpha^2}{2} \left[ 0 - 2 \int_{-\infty}^{0} h e^{h} \, dh \right]$$

$$\Rightarrow \frac{\mu^2}{2} - \mu\alpha + \frac{\alpha^2}{2} \left[ -2(-1) \right]$$

$$I_1 = \frac{\mu^2}{2} - \mu\alpha + \alpha^2$$

let, $\frac{\mu - x}{\alpha} = z \Rightarrow x = \mu - \alpha z$

$dx = -\alpha \, dz$

$$\alpha_2 = I_1 + I_2 = 2\lambda^2 + \mu^2$$

$$\mu_2 = \alpha_2 - \alpha_1^2 = 2\lambda^2 + \mu^2 - \mu^2 = 2\lambda^2$$

$$\boxed{\sigma = \sqrt{\mu_2} = \sqrt{2}\lambda}$$

Ans

$$\mu_3 = E[(x-m)^3]$$

$$= E(x^3 - 3x^2 m + 3m^2 x - m^3)$$

$$= \alpha_3 - 3m\alpha_2 + 3m^2 m - m^3$$

$$= \alpha_3 - 3\alpha_1\alpha_2 + 2\alpha_1^3$$

$$\alpha_3 = E(x^3) = \int_{-\infty}^{\infty} \frac{x^3}{2\lambda} e^{-\frac{|x-\mu|}{\lambda}} dx$$

$$= \int_{-\infty}^{\mu} \frac{x^3}{2\lambda} e^{\frac{x-\mu}{\lambda}} dx + \int_{\mu}^{\infty} \frac{x^3}{2\lambda} e^{\frac{-\mu-x}{\lambda}} dx$$

let, $I_1 = \int_{-\infty}^{\mu} \frac{x^3}{2\lambda} e^{\frac{x-\mu}{\lambda}} dx$ . and

$$I_2 = \int_{\mu}^{\infty} \frac{x^3}{2\lambda} e^{\frac{\mu-x}{\lambda}} dx$$

let, $\frac{x-\mu}{\lambda} = z \Rightarrow x = \mu + \lambda z \quad dx = \lambda dz$

$$I_1 = \int_{-\infty}^{0} \left( \frac{\mu^2 + 2h\alpha\mu + h^2\alpha^2}{2\alpha} \right) e^{h} \alpha \, dh$$

$$= \frac{\mu^2}{2} \int_{-\infty}^{0} e^{h} \, dh + \mu\alpha \int_{-\infty}^{0} h e^{h} \, dh + \frac{\alpha^2}{2} \int_{-\infty}^{0} h^2 e^{h} \, dh$$

$$\Rightarrow \frac{\mu^2}{2} - \mu\alpha + \frac{\alpha^2}{2} \left[ 0 - 2 \int_{-\infty}^{0} h e^{h} \, dh \right]$$

$$\Rightarrow \frac{\mu^2}{2} - \mu\alpha + \frac{\alpha^2}{2} \left[ -2(-1) \right]$$

$$I_1 = \frac{\mu^2}{2} - \mu\alpha + \alpha^2$$

let, $\frac{\mu - x}{\alpha} = z \Rightarrow x = \mu - \alpha z$

$$dx = -\alpha \, dz$$

$$I_2 = \int_0^{\infty} \frac{[\mu^3 - 3\mu^2 \alpha z + 3\mu \alpha^2 z^2 - \alpha^3 z^3] e^{z}}{2\alpha} (-\alpha)\, dz$$

$$= \frac{\mu^3}{2} \int_{-\infty}^{0} e^{z}\, dz - \frac{3\mu^2 \alpha}{2} \int_{-\infty}^{0} z e^{z} dz + \frac{3\mu \alpha^2}{2} \int_{-\infty}^{0} z^2 e^{z} dz$$

$$- \frac{\alpha^3}{2} \int_{-\infty}^{0} z^3 e^{z} dz \,.$$

$$= \frac{\mu^3}{2} + \frac{3\mu^2 \alpha}{2} + 3\mu\alpha^2 + \alpha^3$$

$$\alpha_3 = I_1 + I_2 = \mu^3 + 6\alpha^2 \mu$$

$$\mu_3 = \alpha_3 - 3\alpha_1 \alpha_2 - 2\alpha_1^3$$

$$= \mu^3 + 6\alpha^2 \mu - 3\mu(2\alpha^2 + \mu^2) + 2\mu^3$$

$$= \mu^3 + 6\alpha^2 \mu - 6\mu\alpha^2 - 3\mu^3 + 2\mu^3$$

$$= 0$$

coefficient of skewness

$$\left[ \gamma_1 = \frac{\mu_3}{\sigma^3} = 0 \right] \text{ Ans}$$

2. (ii.) Answer :-

$$f(x) = \frac{\alpha}{\pi} \frac{1}{(x-\mu)^2+\alpha^2} \qquad -\infty < x < \infty$$

Now,

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} \frac{\alpha}{\pi} \frac{1}{(x-\mu)^2+\alpha^2}\,dx$$

$$= \frac{\alpha}{\pi} \times \frac{1}{\alpha} \left[ \tan^{-1}\left(\frac{x-\mu}{\alpha}\right) \right]_{-\infty}^{x}$$

$$= \frac{1}{\pi} \left[ \tan^{-1}\left(\frac{x-\mu}{\alpha}\right) + \frac{\pi}{2} \right]$$

$$F(z_{3/4}) = 3/4$$

$$= \frac{1}{\pi} \left[ \tan^{-1}\left(\frac{x-\mu}{\alpha}\right) + \frac{\pi}{2} \right] = 3/4$$

$$= \tan^{-1}\left(\frac{x-\mu}{\alpha}\right) = \frac{\pi}{4}$$

$$= \frac{x-\mu}{\alpha} = 1$$

$$x = \alpha + \mu$$

$$z_{3/4} = \alpha + \mu$$

$$F(z_{1/4}) = 1/4$$

$$\Rightarrow \frac{1}{\pi}\left[\tan^{-1}\left(\frac{x-\mu}{\lambda}\right)+\frac{\pi}{2}\right]=1/4$$

$$\Rightarrow \tan^{-1}\left(\frac{x-\mu}{\lambda}\right)=-\frac{\pi}{4}$$

$$\frac{x-\mu}{\lambda}=-1$$

$$x=\mu-\lambda$$

$$\xi_{1/4}=\mu-\lambda$$

Now,

$$\text{quartile deviation}=\xi_{3/4}-\xi_{1/4}$$

$$=\frac{\mu+\lambda-\mu+\lambda}{2}=\lambda \text{ Ans}$$

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3.) (i.) Let, $X \sim \chi^2$

$$f(x)=\frac{e^{-x/2}\left(\frac{x}{2}\right)^{n/2-1}}{2\sqrt{(n/2)}} \quad 0<x<\infty$$

$$=0 \quad \text{elsewhere}$$

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characteristic function $\cdot\ \chi(t) = E(e^{itx})$

$$= \int_{-\infty}^{\infty} e^{itx} f(x)\, dx$$

$$= \int_0^{\infty} \frac{e^{itx}\, e^{-x/2}\, x^{n/2 - 1}}{2^{n/2} \sqrt{(n/2)}}$$

$$= \frac{1}{2^{n/2} \cdot \sqrt{n/2}} \int_0^{\infty} e^{itx}\, e^{-x/2}\, x^{n/2 - 1}\, dx$$

$$e^{itx} = \sum_{k=0}^{\infty} \frac{1}{\lfloor k} (itx)^k$$

$$\chi(t) = \frac{1}{2^{n/2} \sqrt{n/2}} \int_0^{\infty} \left( \sum_{k=0}^{\infty} \frac{1}{\lfloor k} (itx)^k \right) e^{-x/2}\, x^{n/2 - 1}\, dx$$

$$= \frac{1}{2^{n/2} \sqrt{(n/2)}} \left( \sum_{k=0}^{\infty} \frac{1}{\sqrt{k}} (it)^k \int_0^{\infty} e^{-x/2}\, x^{k + n/2 - 1}\, dx \right.$$

$$= \frac{1}{2^{n/2} \sqrt{n/2}} \left( \sum_{k=0}^{\infty} \frac{1}{\lfloor k} (it)^k\, 2^{n/2 + k} \sqrt{(n/2 + k)} \int_0^{\infty} \frac{e^{-x/2}\, x^{k + n/2 - 1}}{2^{n/2 + k} \sqrt{n/2 + k}}\, dx \right)$$

now

$$\frac{e^{-x/2} \cdot x^{k+n/2-1}}{2^{k+n/2} \sqrt{k+n/2}}$$ is a probability density function

$\chi^2$ - distribution with $2k+n$ degree of freedom

$$\int_0^\infty \frac{e^{-x/2}\, x^{k+n/2-1}}{2^{k+n/2} \sqrt{(k+n/2)}} = 1$$ { Integral of probability density function over $(-\infty, \infty)$ is 1

$$\phi(t) = \frac{1}{2^{n/2} \sqrt{n/2}} \sum_{k=0}^{\infty} \frac{1}{\lfloor k} (it)^k \, 2^{n/2+k} \sqrt{n/2+k}$$

$$= \sum_{k=0}^{\infty} \frac{1}{\lfloor k} (it)^k \, 2^k \, \frac{\sqrt{(n/2+k)}}{\sqrt{n/2}}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{1}{\lfloor k} (2it)^k \frac{\sqrt{n/2+k}}{\Gamma(n/2)}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{1}{\lfloor k} (2it)^k \cdot \prod_{j=0}^{k-1} \left(\frac{n}{2} + j\right)$$

$$\Rightarrow 1 + \frac{(2it)^1}{\lfloor 1} \times \frac{n}{2} + \frac{(2it)^2}{\lfloor 2} \times \left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right) + \dots$$

$$\chi(t) = (1-2it)^{-n/2} \text{ Ans}$$

$\alpha_1 =$ coeff of $it = n$

$\alpha_2 =$ coeff of $\frac{(it)^2}{\lfloor 2} = 2^2\left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right) = n(n+2)$

$\sigma^2 = \alpha_2 - \alpha_1^2 = n^2 + 2n - n^2 = 2n$ Ans

3.(iii) Ans:-

$$X \sim F(m,n)$$

$$\Rightarrow f(x) = \frac{m^{m/2}\, n^{n/2} \cdot x^{m/2-1}}{B_2\left(\frac{m}{2}, \frac{n}{2}\right)(mx+n)^{m+n}} \qquad x > 0$$

$$= 0 \qquad x < 0$$

$$Y = \frac{1}{X}$$

let, $y = 1/x$. ( Monotonic on $x \in (0, \infty)$)

$$\frac{dy}{dx} = \frac{-1}{x^2} \Rightarrow \frac{dx}{dy} = -\frac{1}{y^2}$$

$$f(x) = \frac{m^{m/2}\, n^{n/2}\, x^{m/2-1}}{B_2\left(\frac{m}{2}, n/2\right)(mx+n)^{(m+n)/2}}$$

$$f(y) = f(x) = \frac{m^{m/2}\, n^{n/2} \left(\frac{1}{y}\right)^{m/2-1}}{B_2\left(m/2, \frac{n}{2}\right)\left(m/y + n\right)^{m+n/2}} \times \frac{1}{y^2}$$

$$= \frac{m^{m/2}\, n^{n/2} \left(\frac{1}{y}\right)^{-n/2-1}}{B_2\left(\frac{n}{2}, m/2\right)(ny+m)^{\frac{m+n}{2}}} \times \frac{1}{y^2}$$

$$= \frac{m^{m/2}\, n^{n/2} \left(\frac{1}{y}\right)^{-n/2-1}}{\beta_2\left(\frac{n}{2}, \frac{m}{2}\right)(ny+m)^{\frac{m+n}{2}}} \times \frac{1}{y^2}$$

$$= \frac{n^{n/2}\, m^{m/2} \cdot y^{n/2-1}}{\beta_2\left(\frac{n}{2}, \frac{m}{2}\right)(ny+m)^{n+m/2}} \quad \left\{ \because \beta_2(m/n) = \beta_2(n, m) \right.$$

$\Rightarrow \quad Y \sim F(n/m)$ Hence proved

9. (ii) Given $n = 20$

Sample mean $= \bar{x} = 16.9$

standard deviation of population $= \sigma = 5.2$

Now, $p(-u_\epsilon < U < u_\epsilon) = 1-\epsilon$

Now, $1-\epsilon = 0.95$

$= \epsilon = 0.05$

$\frac{\epsilon}{2} = 0.025$

$$\frac{1}{\sqrt{2\pi}} \int_0^{1.96} e^{-t^2/2}\, dt = 0.4750$$

Now, $\frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-t^2/2}\, dt = \sqrt{\pi/2} \times \frac{1}{\sqrt{2\pi}} \times \sqrt{2}$

$= \frac{1}{2} = 0.5$

$$\therefore \frac{1}{\sqrt{2\pi}} \int_{1.96}^{\infty} e^{-t^2/2}\, dt = 0.025 = \epsilon/2$$

$\Rightarrow p(U > 1.96) = 0.025 = \epsilon/2$

Now if $P(-u_\epsilon < u < u_\epsilon) = 1-\epsilon$

$P(|u| < u_\epsilon) = 1-\epsilon$

$P(|u| > u_\epsilon) = \epsilon$

$P(u > u_\epsilon) = \epsilon/2$

Therefore the confidence interval of

$$\left(\bar{x} - \frac{\sigma u_\epsilon}{\sqrt{n}}, \bar{x} + \frac{\sigma u_\epsilon}{\sqrt{n}}\right) = (14.52, 13.18)$$

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4) (ii) Ans →

Now

$$r_{xy} = \frac{\text{cov}(x,y)}{\sigma_x \sigma_y} \rightarrow (1)$$

Now, $\sigma_x^2 = E[(x-\bar{x})^2]$

$\sigma_y^2 = E[(y-\bar{y})^2]$

Let $\text{cov}(x,y) = E[(x-\bar{x})(y-\bar{y})]$

$\text{var}(x+y) = E[\{E(x+y) - E(x+y)\}^2]$

$= E[\{x+y-(\bar{x}+\bar{y})\}^2]$

$$\Rightarrow E[\{(x-\bar{x})+(y-\bar{y})\}^2]$$

$$= E[(x-\bar{x})^2]+E[(y-\bar{y})^2]+2E(x-\bar{x})$$

$$= \sigma_x^2+\sigma_y^2+2\,Cov(x,y)$$

$$var(X+Y) = \sigma_x^2+\sigma_y^2+2\rho_{xy}\sigma_x\sigma_y$$

Hence prove

$$var(X-Y) = E[\{(x-y)-E(x-y)\}^2]$$

$$= E[\{x-y-(\bar{x}-\bar{y})\}^2]$$

$$= E[\{(x-\bar{x})-(y-\bar{y})\}^2]$$

$$= E[(x-\bar{x})^2]+E[(y-\bar{y})^2]$$

$$-2E\{(x-\bar{x})(y-\bar{y})\}$$

$$= \sigma_x^2+\sigma_y^2-2\,Cov(x,y)$$

$$var(x-y) = \sigma_x^2+\sigma_y^2-2\rho_{xy}\sigma_x\sigma_y$$ prove

Hence prove