Real Functions

$$f : A \subset \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto y = f(x)$$

$$\forall x \in A \ \exists! \ y = f(x) \in \mathbb{R}$$

Exists one and only one

$$A \overset{\text{def}}{=} \text{Domain}$$

WE LOOK INTO ELEMENTARY FUNCTIONS

	f	D_f
Most simple	$P_n(x)$	$\mathbb{R}$
Inverse of some Polyn.	$\sqrt{x}$	$[0, +\infty)$
	$\sqrt[3]{x}$	$\mathbb{R}$
Composed by Polyn.	P_n / Q_n	$\{x \in \mathbb{R} : Q_n(x) \neq 0\}$
Exp	e^x	$\mathbb{R}$
Inverse	$\log x$	$(0, +\infty)$
Trig.	$\sin x$	$\mathbb{R}$
Inverse	$\arcsin x$	$[-1, 1]$
Trig	$\cos x$	$\mathbb{R}$
Compound	$\operatorname{tg} x$	$\{x \in \mathbb{R} : x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$

Exercise 2.1 (v)

$$f(x) = \frac{1}{1 - \log x}$$

Domain

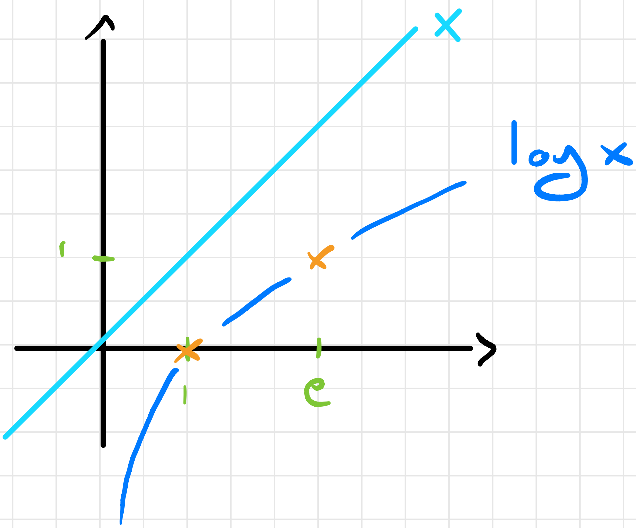
when log works   when fraction works

$$D_f = \{ x \in \mathbb{R} : x > 0 \text{ and } 1 - \log x \neq 0 \}$$

$$1 - \log x \neq 0 \iff 1 \neq \log x \iff x \neq e$$

$$\boxed{\begin{aligned} \log e &= 1 \\ \log 1 &= 0 \end{aligned}}$$

$$D_f = (0, e) \cup (e, \infty)$$



EXERCISE 2.1 (iii)

$$f(x) = \frac{1}{x - \sqrt{1-x^2}}$$ DOMAIN ?

$$D_f = \{ x \in \mathbb{R} : \underbrace{\overbrace{1-x^2 \geq 0}^{\text{when root exist}}}_{A} \text{ and } \underbrace{\overbrace{x - \sqrt{1-x^2} \neq 0}^{\text{when denominator exist}}}_{B} \}$$

$$A \Rightarrow 1 \geq x^2 \Rightarrow -1 \leq x \leq 1$$

$B \Rightarrow$ DROP OUT POINTS SUCH THAT

$$x = \sqrt{1-x^2} \Rightarrow 2x^2 = 1$$

$$\Rightarrow x_{\pm} = \pm \sqrt{\frac{1}{2}}$$

OSS $x = \sqrt{\cdots} \geq 0$

$$\sqrt{x} : [0, +\infty) \to [0, +\infty)$$

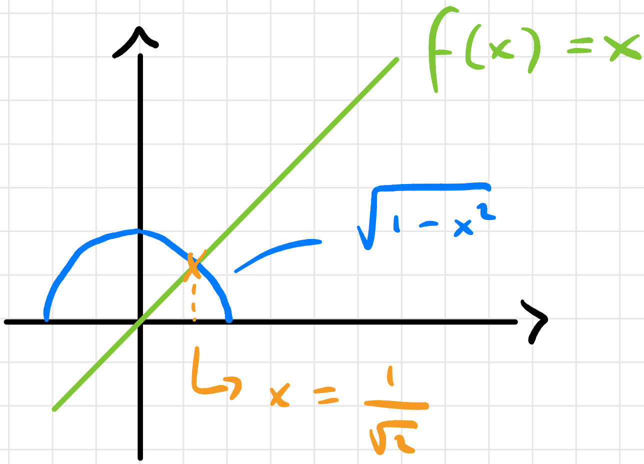
$$X_+ = \frac{1}{\sqrt{2}}$$

$$B \Rightarrow X_+ \neq \frac{1}{\sqrt{2}}$$

$$D_f = \left[-1, \frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right]$$

$$x = -\frac{1}{\sqrt{2}}$$

$$-\frac{1}{\sqrt{2}} - \sqrt{1 - \frac{1}{2}} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \neq 0$$



EXERCISE 2.1 (vii)

$$f(x) = \frac{\sqrt{5-x}}{\log x}$$

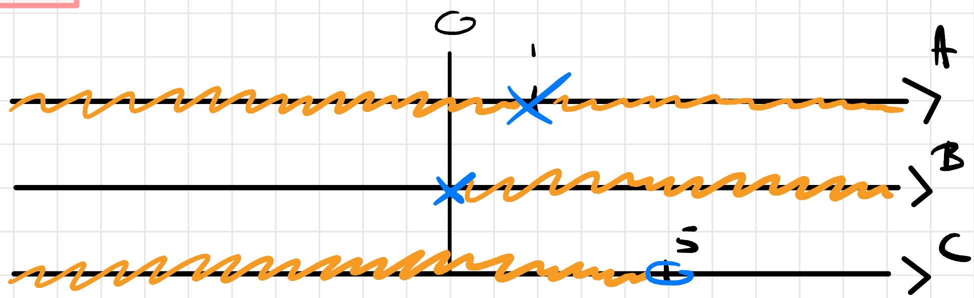
Joins Root and log

$$D_f = \{x \in \mathbb{R} : \underbrace{\overbrace{\log x \neq 0}^{A}}_{\text{when rational function works}} \text{ and } \underbrace{\overbrace{x > 0}^{B}}_{\text{when log works}} \text{ and } \underbrace{\overbrace{5-x \geq 0}^{C}}_{\text{when root works}} \}$$

$$\boxed{\log 1 = 0}$$

$x \neq 1$

$x > 0$



$$D_f = (0, 1) \cup (1, 5]$$

EXERCISE 2.1 (viii)

$$f(x) = \arcsin(\log x)$$

$$D_f = \{x \in \mathbb{R} : x > 0 \text{ and } -1 \leq \log x \leq 1\}$$

$$-1 \leq \log x \leq 1 \Rightarrow e^{-1} \leq e^{\log x} \leq e^{1}$$

↳ e monotone increasing

$$x \leq y \Rightarrow e^x \leq e^y$$

$$D_f = \left[\frac{1}{e}, e\right]$$

EXERCISE 2.1 (iv)

$$f(x) = \sqrt{1-\sqrt{4-x^2}}$$ DOMAIN?

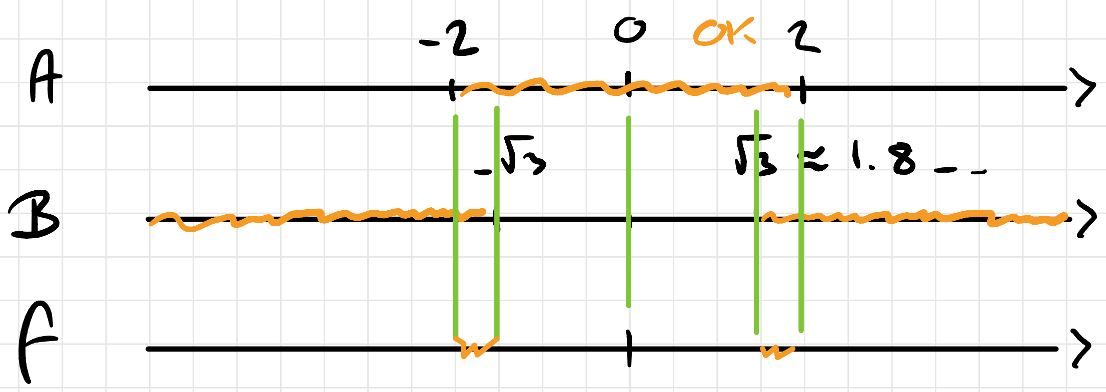
$$D_f = \{x \in \mathbb{R} : \underbrace{4-x^2 \geq 0}_{A} \text{ and } \underbrace{1-\sqrt{4-x^2} \geq 0}_{B}\}$$

$$A \Rightarrow -2 \leq x \leq 2$$

$$B \Rightarrow \sqrt{4-x^2} \leq 1$$ (BOTH SIDES ARE POSITIVES)

$$\Rightarrow 4-x^2 \leq 1 \Rightarrow 3 \leq x^2$$

$$\Rightarrow x \geq \sqrt{3} \text{ or } x \leq -\sqrt{3}$$



$$D_f = [-2, -\sqrt{3}] \cup [\sqrt{3}, 2]$$ sup D_f ...

ANOTHER PROPERTY --

ODD / EVEN

$f(-x) = f(x)$ EVEN

$f(-x) = -f(x)$ ODD

$f(-x) \neq \pm f(x)$ NEITHER

EXERCISE 2.2 (a)

f, g ODD

$h = f + g$?

$k = f \cdot g$?

$u = f \circ g$?

$$h(-x) = f(-x) + g(-x) =$$ f, g ODD

$$= -f(x) - g(x)$$

$$= -(f+g) = -h$$ (DISTRIBUTIVE)

ODD

$$K(-x) = f(-x)\,g(-x) =$$ f, g ODD

$$= [-f(x)][-g(x)]$$

$$= fg = K$$

EVEN (LIKE NUMBERS)

$$u(-x) = f(g(-x))$$

$$= f(-g(x))$$ g ODD

$$= -f(g(x)) = u$$ f ODD

ODD

EXERCISE (very SIMPLE)

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\cosh(-x) = \frac{e^{-x} + e^{x}}{2} = \cosh x$$ EVEN

$$\sinh(-x) = \frac{e^{-x} - e^{x}}{2}$$

$$= -\left(\frac{e^{x} - e^{-x}}{2}\right) = -\sinh x$$ ODD

DOMAIN $\mathbb{R}$

EXERCISE 2.3 (iv)

$$f(x) = \cos(x^3)\sin(x^2)e^{-x^4}$$ ODD/EVEN

$$f(-x) = \cos\left[\underbrace{(-x)^3}_{-x^3}\right]\sin\left[\underbrace{(-x)^2}_{x^2}\right]e^{-\underbrace{(-x)^4}_{x^4}}$$

$$= \underbrace{\cos(-x^3)}_{\cos(x^3)}\sin(x^2)\,e^{-x^4}$$

$$= f(x)$$ EVEN

EXERCISE 2.3 (vi)

$$f(x) = \log\left(\sqrt{x^2+1} - x\right)$$

EVEN, ODD, NEITHER

$$f(-x) = \log\left(\sqrt{x^2+1} + x\right)$$

$$= \log\left[\left(\sqrt{x^2+1} + x\right)\cdot 1\right]$$

$$1 = \frac{\sqrt{x^2+1} - x}{\sqrt{x^2+1} - x} \qquad (a+b)(a-b) = a^2 - b^2$$

$$f(-x) = \log\left[\frac{(\overbrace{\sqrt{x^2+1}}^{a} + \overbrace{x}^{b})(\overbrace{\sqrt{x^2+1}}^{a} - \overbrace{x}^{b})}{\sqrt{x^2+1} - x}\right]$$

$$= \log\left[\frac{\cancel{x^2}+1 - \cancel{x^2}}{\sqrt{x^2+1} - x}\right] = \log\left[\left(\sqrt{x^2+1} - x\right)^{-1}\right]$$

$$= -\log\left(\sqrt{x^2+1} - x\right) = -f(x)$$

$$\log a^b = b \log a$$

$$\log(ab) = \log a + \log b$$

$$\log\left(\frac{a}{b}\right) = \log a + \underbrace{\log\left(\frac{1}{b}\right)}_{\log(b^{-1})}$$

$$= \log a - \log b$$

$$\log\left(\frac{a}{b}\right) = -\log\left(\frac{b}{a}\right)$$

$$\log\left(\sqrt[m]{x}\right) = \log\left[(x)^{\frac{1}{m}}\right] = \frac{\log x}{m}$$

EXERCISE 2.9

$$\begin{cases} x^y = y^x & \text{(i)} \\ y = 3x & \text{(ii)} \end{cases}$$

(i) $x^y = y^x$

$\Rightarrow \log x^y = \log y^x$

$\Rightarrow y \log x = x \log y$

$\Rightarrow 3x \log x = x \log 3 + x \log x$

(ii)

$\Rightarrow 2\cancel{x} \log x = \cancel{x} \log 3$

$\Rightarrow \log x = \dfrac{\log 3}{2} = \log \sqrt{3}$

$\Rightarrow x = \sqrt{3}$

def $f : A \subset \mathbb{R} \rightarrow \mathbb{R}$

$$f \text{ INJECTIVE} \iff \forall x_1, x_2 \in A$$

$$\underbrace{x_1 \neq x_2}_{Hy} \Rightarrow \underbrace{f(x_1) \neq f(x_2)}_{Th}$$

EXERCISE 2.6 (i)

$$f(x) = 7x - 4$$

$$x_1 \neq x_2 \Rightarrow x_1 - x_2 \neq 0$$

$$f(x_1) = 7x_1 - 4$$

$$f(x_2) = 7x_2 - 4$$

$$f(x_1) - f(x_2) = 7\underbrace{(x_1 - x_2)}_{Hy} \neq 0$$

Another point of view

Check the number of solution of

$$y = 7x - 4$$

1 solution $\Rightarrow$ INJECTIVE

$$x = \frac{y + 4}{7}$$

EXERCISE 2.6 (ii)

$f(x) = \sin(7x - 4)$ INJECTIVE?

$\forall x_1, x_2 \in \mathbb{R} \quad x_1 \neq x_2 :$

$7x_1 - 4 = 7x_2 - 4 + 2n\pi \quad (n \in \mathbb{Z})$

$$\left.\begin{array}{l} 7x^2 - 3x + 1 \ (1) \\ 4x + 1 \\ 8x^3 + 7 \end{array}\right\} \text{Polinomio}$$

$$P_n(x) = \sum_{k=0}^{n} a_k x^n$$

(1) $n = 2$ $\qquad a_k = 0 \quad k > 0$

$a_2 = 7$

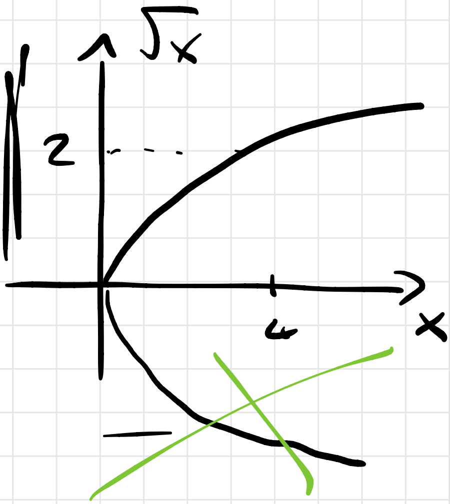
$a_1 = -3$

$a_0 = 1$

$$\sqrt{4} = 2, \cancel{-2}$$

$$\forall x \quad \sqrt{x} \geq 0$$

$$\sqrt{x}$$



$$\sqrt{x^2} = |x|$$