TAYLOR SERIES

$$f(x) = \underbrace{\sum_{n=0}^{\infty} a_n (x-a)^n}_{< \infty \ \forall x \in (-\rho, \rho)} + \underbrace{\lim_{k} R_{k,a}(x)}_{\longrightarrow 0}$$

(f diff)

THEOREM

POWER SERIES of f IS UNIQUE

Ex

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \rho = \infty$$

$$\hookrightarrow e = \lim_{n} \left(1 + \frac{1}{n}\right)^n = \sum_{n} \frac{1}{n!}$$

$$\underbrace{\sin(x)}_{\text{ODD}} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \rho = \infty$$

ODD

THEOREM

Hy) f DIFF. n TIMES

$$f(x) = \sum_{n}^{\infty} a_n (x-a)^n \quad , \quad |x-a| < \rho$$

$$\Rightarrow f'(x) = \sum_{n}^{\infty} n a_n (x-a)^n \quad |x-a| < \rho$$

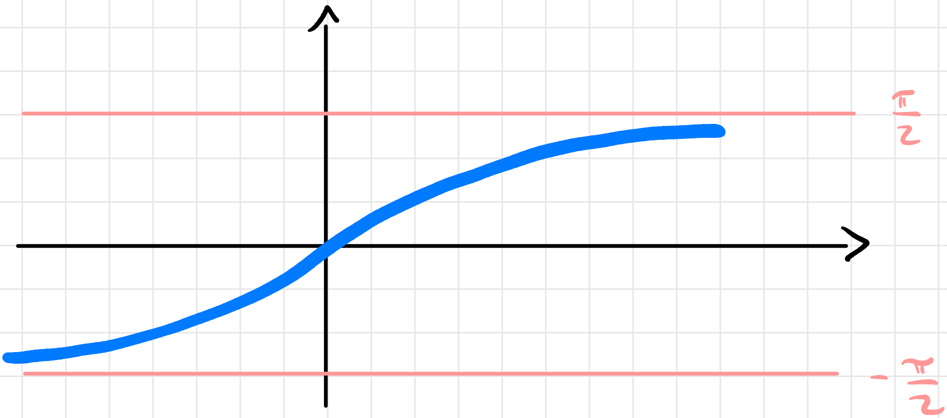
SAME CONVERGENT RADIUS

Ex $$\underbrace{\cos(x)}_{\text{even}} = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \underbrace{\rho = \infty}_{\text{LIKE SIN}}$$

even

Ex $f(x) = \underbrace{\arctan(x)}_{\text{ODD}}$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} a_n x^{2n+1} \text{/ODD}$$



$$\lim_{x \to \pm\infty} \arctan(x) = \pm \frac{\pi}{2}$$

$$\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2} = \underbrace{\frac{1}{1-(-x^2)}}_{\text{Geometric Series}}$$

$$\sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \underline{\sum_{n=0}^{\infty} (2n+1) a_n x^{2n}}$$

$$\rho = 1$$

$$\Rightarrow a_n = \frac{(-1)^n}{2n+1}$$

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

$\rho = 1$ (Because of THEOREM Before)

$$\frac{1}{\rho} = \lim_n \left| \frac{a_{n+1}}{a_n} \right| = \lim_n \frac{2n+1}{2n+3} = 1$$

TAYLOR

↳ APPLICATIONS

- LIMITS
- NUM. APP
- SERIES
- CONCAVITY / CONVEXITY

DEF f CONVEX $a \in I_a \iff \forall x \in I_a \quad \underbrace{f(x)}_{\text{FUNCTION}} \geq \underbrace{f(a) + f'(a)(x-a)}_{\text{TANGENT } a}$

↳ Lies above

CONCAVE " " $\leq$

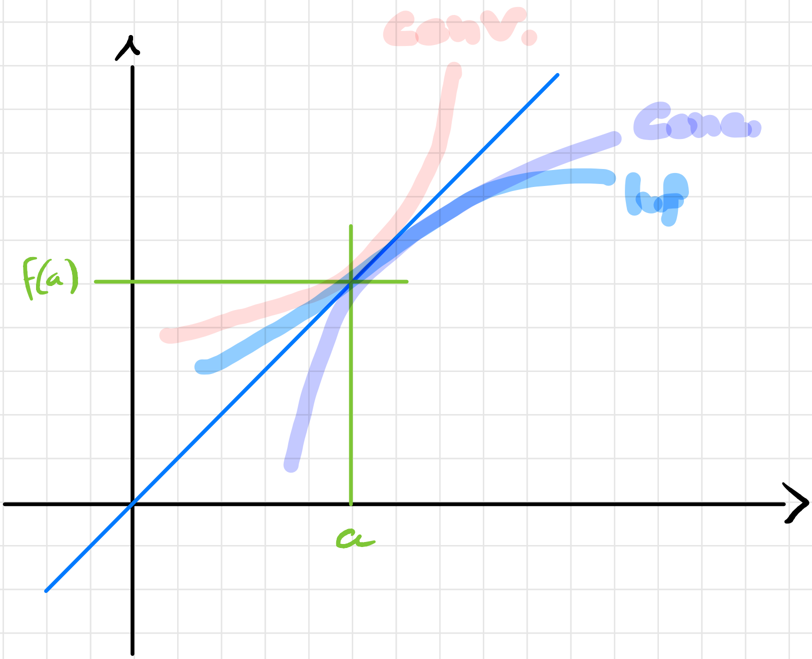
(Lies Below)

DEF a INFLECTION POINT

↳ $x > a \Rightarrow f(x) \gtrless f(a) + f'(a)(x-a)$

and

↳ $x < a \Rightarrow$ " $\lessgtr$ "



f convex $a \iff \underbrace{f(x) - f(a) - f'(a)(x-a)} > 0$

Remainder $\dfrac{f^{(n)}(c)\,(x-a)^n}{n!}$

First non-zero derivative

convexity $\longrightarrow$ $\operatorname{sign}\{ f^{(n)}(c)(x-a)^n \}$

$=$ 〃 (a) 〃

n ODD $\begin{cases} (x-a)^n > 0 & x > a \\ \text{〃} < 0 & < \end{cases}$ | sign $f^{(n)}$ does not matter

Inflection

n EVEN $\Rightarrow (x-a)^n \geq 0 \quad \forall x \in I_a$

$\hookrightarrow$ SIGN $f^{(n)}(a)$

- $f^{(n)}(a) > 0$ conv.
- 〃 < 0 conc.

n	$f^{(n)}(a)$	PROPERTY
ODD	+ / −	INFLECTION
EVEN	+	CONVEX
EVEN	−	CONCAVE

All THE TOOLS TO PROVIDE SKETCH of a fun.

Ex

$$f(x) = \frac{3x^2 + x + 1}{x + 2}$$

$$D_f = \mathbb{R} - \{-2\}$$

CONT / DIFF ? $\forall x \in D_f$

SIGNS ? $\forall x \in D_f \quad 3x^2 + x + 1 > 0$

$$x_\pm = \frac{-1 \pm \sqrt{1 - 12}}{6} \notin \mathbb{R}$$

$$\Rightarrow \forall x > -2 \,, \quad f(x) > 0$$

$$\text{''} < \text{''} \qquad \text{''} < \text{''}$$

GROWTH

$$f'(x) = \frac{(x+2)(6x+1) - (3x^2+x+1)}{(x+2)^2}$$

$$= \frac{6x^2+12x + x + 2 - 3x^2 - x - 1}{(x+2)^2} = \frac{3x^2+12x+1}{(x+2)^2}$$

$$x_\pm = \frac{-12 \pm \sqrt{144 - 12}}{6} = -2 \pm \sqrt{\frac{132}{36}}$$

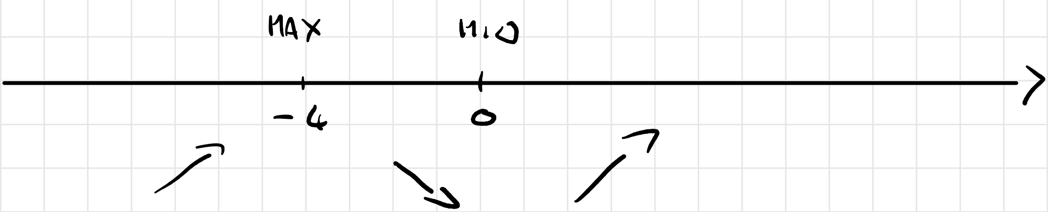
$$= -2 \pm \sqrt{\frac{11}{3}}$$

divide by 24

$$\approx -2 \pm \sqrt{\frac{12}{3}} = \underbrace{-2 \pm 2}_{0,\ -4}$$

$$x \in \left[-2 - \sqrt{\frac{11}{3}}, -2 + \sqrt{\frac{11}{3}}\right], \quad f'(x) \leq 0$$

$$\text{"} \quad [\ \quad \text{"} \quad]^c, \quad \text{"} \geq \text{"}$$



$$f''(x) = \frac{(x+2)^2(6x+12) - 2(x+2)(3x^2+12x+1)}{(x+2)^4}$$

$$= \frac{(x+2)(6x+12) - 6x^2 - 24x - 2}{(x+2)^3}$$

$$= \frac{6\cancel{x^2} + 12\cancel{x} + \cancel{12x} + 24 - 6\cancel{x^2} - 2\cancel{4x} - 2}{(x+2)^3} = \frac{22}{(x+2)^3}$$

$x > -2$ CONVEX

$x < -2$ CONCAVE

ASYMPTOTE

$$\lim_{x \to 2^{\pm}} f(x) = \pm\infty$$ VERTICAL AS.

$$\lim_{x \to \pm\infty} f(x) = \pm\infty$$ NO HORIZ. AS

How?

BEHAVIOR AT $\pm\infty$

$$f(x) = g(x) + \underset{x \to \pm\infty}{o(1)}$$

$$\underbrace{3x^2 + x + 1}_{P(x)} \longrightarrow P_{2,-2}(x)$$

$$P_{2,-2}(x) = \frac{P''(-2)}{2!}(x+2)^2 + P'(-2)(x+2) + P''(-2)$$

$$= 11 - 11(x+2) + 3(x+2)^2$$

$$\left(f(x) = \frac{11}{x+2} - 11 + 3(x+2)\right.$$

Second derivative much more easy

$$= \underset{x \to \pm\infty}{o(1)} + \underbrace{3(x+2) - 11}_{3x + 6 - 11 = 3x - 5}$$

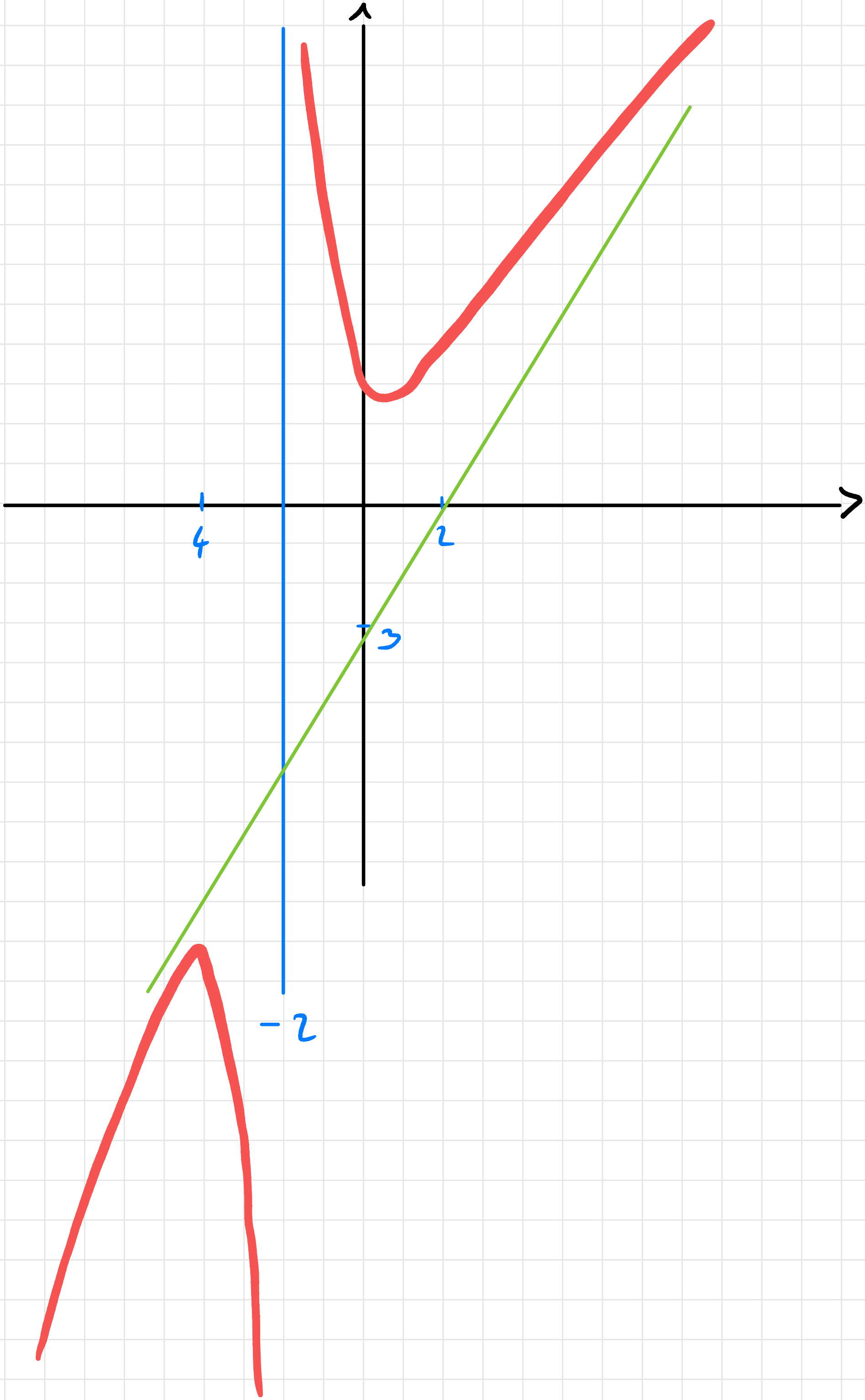
def $y = mx + b$ INCLUDED AS. f

$$m = \lim_{x \to \pm\infty} \frac{f(x)}{m} = \lim_{x \to \pm\infty} \frac{3x^2 + x + 1}{x^2 + 2x} = 3$$

$$b = \lim_{x \to \pm\infty} [f(x) - mx]$$

$$= \lim_{x \to \pm\infty} \frac{3x^2 + x + 1}{x + 2} - 3x$$

$$= \quad '' \quad \frac{-5x + 1}{x + 2} = -5$$


4
2
3
-2

$$f(x) = \frac{11}{x+2} - 11 + 3(x+1)$$

$$f'(x) = -\frac{11}{(x+2)^2} + 3$$

$$f''(x) = +\frac{22}{(x+2)^3}$$