# Ex 9.1 (vii)

$$\int dx \, \frac{1+\sqrt{1-\sqrt{x}}}{\sqrt{x}}$$

HINT : IMMEDIATE or ALMOST SO - -

$$\boxed{\int f'(g(x))\, g'(x)\, dx = f(g(x)) + c}$$

$$f, g = ?$$

OBSERVE $\displaystyle \frac{d}{dx}\left(1-\sqrt{x}\right) = -\underbrace{\frac{1}{2\sqrt{x}}}_{\text{GOOD CANDIDATE for } g}$

$$\int dx \, \frac{1-\sqrt{1-\sqrt{x}}}{\sqrt{x}} = -2\int \left(1+\sqrt{1-\sqrt{x}}\right)\left(1-\sqrt{x}\right)' dx$$

$$= -\underbrace{2\int \left(1-\sqrt{x}\right)' dx}_{2(1-\sqrt{x})+c} - \underbrace{2\int \sqrt{1-\sqrt{x}}\,\left(1-\sqrt{x}\right)' dx}_{\text{IMMEDIATE}}$$

$$= -2\left(1-\sqrt{x}\right) - \frac{4}{3}\left(1-\sqrt{x}\right)^{\frac{3}{2}} + c$$

$$= 2\sqrt{x} - \frac{4}{3}\left(1-\sqrt{x}\right)^{\frac{3}{2}} + c'$$

$g = 1-\sqrt{x}$

$f' = \sqrt{x}$

$\Rightarrow f = \frac{2}{3} x^{\frac{3}{2}}$

Ex 9.3 (xix)

$$\int \sqrt{2+e^x}\, dx$$ HINT: CHANGE of VARIABLE

$t = e^x$, $\dot{t} = t\, dx$, $dx = \frac{dt}{t}$

$$\int \sqrt{2+t}\, \frac{dt}{t}$$ TRICKY

FIRST ATTEMPT

$z = \sqrt{2+e^x}$, $dz = \frac{e^x}{2\sqrt{2+e^x}}\, dx$

$e^x = z^2 - 2$, $dx = \frac{2z}{z^2-2}\, dz$

$$\int \sqrt{2+e^x}\, dx = 2\int \frac{z^2}{z^2-2}\, dz = 2\int \frac{z^2-2+2}{z^2-2}\, dz$$

$$= 2\int \left(1 + \frac{2}{z^2-2}\right) dz = 2\left[z + c + \int \frac{dz}{\left(\frac{z}{2}\right)^2 - 1}\right]$$

$\bar{z} = \frac{z}{\sqrt{2}}$, $dz = \sqrt{2}\, d\bar{z}$

$$2\left[z + c + \sqrt{2}\int \frac{d\bar{z}}{\bar{z}^2 - 1}\right] = 2z - 2\sqrt{2}\int \frac{d\bar{z}}{\bar{z}^2-1} + c'$$

$$= 2z - 2\sqrt{2}\int \frac{d}{d\bar{z}} \operatorname{arctanh}(\bar{z})\, dz$$

$$= 2z - 2\sqrt{2}\operatorname{arctanh}\left(\frac{z}{\sqrt{2}}\right) + c'$$

$$= 2\sqrt{2+e^x} - 2\sqrt{2}\operatorname{arctanh}\left(\sqrt{1+\frac{e^x}{2}}\right) + c'$$

$$= 2\sqrt{2+e^x} - \sqrt{2}\log\left(\frac{\sqrt{2}+\sqrt{1+e^x}}{\sqrt{2}-\sqrt{1+e^x}}\right) + c'$$

$$\operatorname{arctanh} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

Ex 9.6 (iii)

$$\int dx \, \frac{x^2}{(1-x^2)^{\frac{3}{2}}}$$

Hint: CHANGE OF $\underbrace{\text{VARIABLE}}$ → WHICH ONE?

$$x = \sin(t) \, , \quad dx = \cos(t)\, dt$$

$$\int \frac{\sin^2(t)}{\cos^3(t)} \cos(t)\, dt = \underbrace{\int tg^2(t)\, dt}_{\text{EX 9.4 (vii)}}$$

NOTE THAT

$$\frac{d}{dx}\left( tg\, x - x \right) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} - 1 = \frac{1-\cos^2 x}{\cos^2 x}$$

$$= \frac{\sin^2 x}{\cos^2 x} = tg^2 x$$

$$\int tg^2 t \, dt = \int \frac{d}{dt}(tg\, t - t)\, dt = tg\, t - t + c$$

$$= \frac{\sin(t)}{\cos(t)} - t = \frac{x}{\sqrt{1-x^2}} - \arcsin(x) + C$$

## Ex 9.5 (xi)

$$\int \frac{x \log x}{(1+x^2)^2}\, dx$$ Hint. By Parts

$$\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{-2x}{(1+x^2)^2} \Rightarrow \frac{x}{(1+x^2)^2} = -\frac{1}{2}\frac{d}{dx}\left(\frac{1}{1+x^2}\right)$$

$$\int \frac{x \log x}{(1+x^2)^2} = -\frac{1}{2}\left(\frac{\log x}{1+x^2}\right) + \underbrace{\frac{1}{2}\int \frac{dx}{x(1+x^2)}}_{\text{RATIONAL FUN.}}$$

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$1 = A(x^2+1) + Bx^2 + Cx$$

$$= A + Cx + (A+B)x^2$$

$$\begin{cases} A = 1 \\ C = 0 \\ A+B = 0 \end{cases} \Rightarrow A = 1,\ C = 0,\ B = -1$$

IT FOLLOWS

$$\int \frac{dx}{x(1+x^2)} = \underbrace{\int \frac{dx}{x}}_{\log|x|} - \underbrace{\int \frac{x}{1+x^2}}\, dx$$

$$t = x^2, \quad dt = 2x\,dx$$

$$dx = \frac{dt}{2x}$$

$$= \log|x| - \frac{1}{2}\int \frac{dt}{1+t}$$

$$= \log|x| - \frac{1}{2}\log(1+t^2)$$

$$\int \frac{x \log x}{(1+x^2)^2}\, dx = -\frac{1}{2}\left(\frac{\log x}{1+x^2}\right) + \frac{1}{2}\log x - \frac{1}{4}\log(1+x^2) + c$$

Ex 9.7 (iv)

$$I_m = \int (\tan x)^m \, dx \qquad (m \in \mathbb{N})$$

$$\int \tan^2 x \, dx = \tan x - x + c$$

RECURRENCE $I_m = f(I_{m-k})$

$$(\tan x)^m = (\tan x)^{m-2} (\tan x)^2$$

$$= (\tan x)^{m-2} [\underbrace{\tan^2 x + 1}_{(\tan x)'} - 1]$$

$$\frac{d}{dx}(\tan^2 x - x) = (\tan x)' - 1 = \tan^2 x$$

$$= (\tan x)^{m-2} (\tan x)' - (\tan x)^{m-2}$$

$$I_m = \int (\tan x)^{m-2} (\tan x)' - I_{m-2}$$

$$\int (\tan x)^{m-2} (\tan x)' \, dx$$

BY PARTS

$$= (\tan x)^{m-1} - \int (m-2)(\tan x)^{m-3} \underbrace{(\tan x)'}_{(\tan x)^2+1} \tan x \, dx$$

$$= (\tan x)^{m-1} - \int (m-2)(\tan x)^{m} \, dx - \int (m-2)(\tan x)^{m-2} \, dx$$

$$= (\tan x)^{m-1} - (m-2)\left[ I_m + I_{m-2} \right]$$

$$I_m = (\tan x)^{m-1} - (m-2)\left[ I_m + I_{m-2} \right] - I_{m-2}$$

$$(m-1) I_m = (\tan x)^{m-1} - (m-1) I_{m-1}$$

*

$$\boxed{I_m = \frac{(\tan x)^{m-1}}{m-1} - I_{m-2}}$$

$$* \quad I_m = (\tan x)^{m-1} - (m-2)\left[ I_m + I_{m-2} \right] - I_{m-2}$$

$$= (\tan x)^{m-1} - \underline{(m-2) I_m} - \underbrace{(m-2+1)}_{m-1} I_{m-2}$$

$$I_m + (m-2) I_m = (\tan x)^{m-1} - (m-1) I_{m-2} \quad //$$

$$I_m \underbrace{(1+m-2)}_{m-1} =$$

## Ex

$$\int \arctan\sqrt{x}\, dx = x \arctan\sqrt{x} - \int \frac{x}{1+x} \frac{1}{2\sqrt{x}}\, dx$$

$$= x \arctan\sqrt{x} - \overbrace{\frac{1}{2}\int \frac{\sqrt{x}}{1+x}\, dx}^{*}$$

$$* \quad \frac{1}{2}\int \frac{\sqrt{x}}{1+x}\, dx = \frac{1}{2}\int \frac{1}{\sqrt{x}} \frac{x}{1+x}\, dx$$

$$= \int (\sqrt{x})' \left(1 - \frac{1}{1+x}\right) dx$$

$$= \sqrt{x} - \int \underbrace{\frac{(\sqrt{x})'}{1+x}}_{\frac{d}{dx}\arctan\sqrt{x}}\, dx = \sqrt{x} - \arctan\sqrt{x}$$

$$\int \arctan\sqrt{x} = (x+1)\arctan\sqrt{x} - \sqrt{x}$$

## Ex

$$I_m = \int (\log x)^m \, dx$$

$$I_2 = \int (\log x)^2 \, dx$$

$$= x \log x - 2 \int \cancel{x} \log x \frac{1}{\cancel{x}} \, dx$$

$$= x \log x - 2 \int \log x \, dx$$

$$I_m = \int (\log x)^m \, dx$$

$$= x (\log x)^m - \int m x (\log x)^{m-1} \frac{dx}{m}$$

$$= x (\log x)^m - m I_{m-1}$$

Ex 9.3 (iii)

$$\int \cos(\log x)\, dx \qquad z = \log x \;,\; dz = \frac{dx}{e^z}$$

$$\int e^z \cos(z)\, dz = e^z \cos(z) + \int e^z \sin(z)\, dz$$

$$= e^z \cos(z) + e^z \sin(z) - \int e^z \cos(z)\, dz$$

$$\Rightarrow \int e^z \cos(z) = \frac{e^z}{2}\left( \cos(z) + \sin(z) \right) + c$$

$$\int \cos(\log x)\, dx = \frac{x}{2}\left( \cos(\log x) + \sin(\log x) \right) + c$$