Summary

W1 $\mathbb{N}$, $\mathbb{R}$ (SET of NUMBERS)

$\mathbb{N} \downarrow$ BASIC

$\mathbb{R} \downarrow$ RICHEST

W2 REAL FUN.

W3-4 SEQUENCES

↳ W5 SERIES

$\underbrace{\hspace{10em}}$

LIMIT

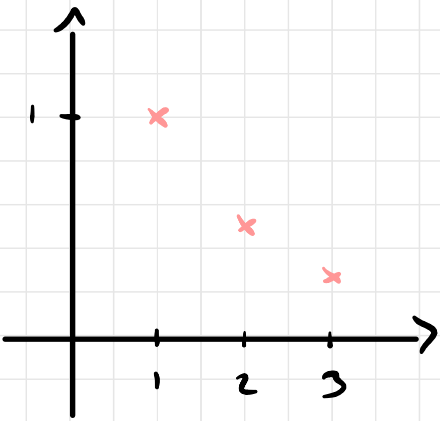
W6 LIMIT REAL FUNCTIONS

Idea ?

↳ Extend what we learnt for sequences

$f : \mathbb{N} \to \mathbb{R}$

Ex $n \longmapsto a_n = \frac{1}{n}$

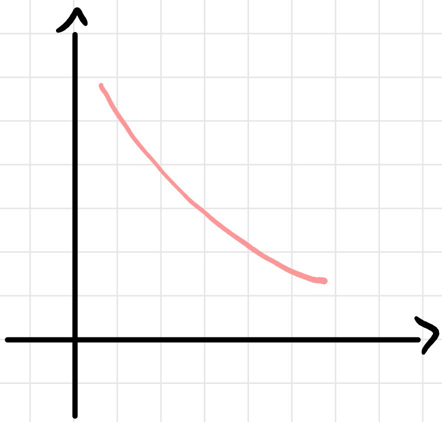


List of points

$$\lim_{n \to +\infty} \frac{1}{n} = 0$$

$f : \mathbb{R} \to \mathbb{R}$

Ex $x \longmapsto f(x) = \frac{1}{x}$



Line

$$\underbrace{\lim_{x \to +\infty} \frac{1}{x} = 0}$$

What does it mean?

Any difference?

$$\underbrace{\lim_{x \to +\infty}}\ f(x)$$

↳ TRIVIAL EXTENSION of $\lim_{n \to +\infty}$

<u>OBS</u> $\mathbb{R}$ MUCH RICHER THAN $\mathbb{N}$

↳ $\lim_{x \to -\infty} f(x)$

↳ TRIVIAL EXTENSION of PREVIOUS CASE

- CONTINUOUS SET of POINTS

$$\underbrace{\lim_{x \to x_0} f(x)}$$

WHAT DOES IT MEAN?

LIMIT → OPERATIONS

↳ FUNCTIONS → NOT NUMBER

↓ TREND

↓ REPLACE NUMBER?

Ex SEE YOU AT 13h IN THE DINING ROOM

↳ AS FAR AS MY CLOCK APPROACHES 13h I GET CLOSE TO THE DINING ROOM

↳ $$\lim_{t \to t_0} f(t) = x_d$$

$$t_0 = 13h$$

x_d = D.R. Position

$f(t)$ = MY POSITION AS A FUNCTION OF TIME

$$\underbrace{t \to t_0}\quad \Longrightarrow \quad \underbrace{|t - t_0| < \overbrace{\delta}^{\text{ARBITRARY SMALL}}}_{\text{difference}}, \quad t \in \underbrace{(t_0 - \delta, t_0 + \delta)}_{\text{ENVIRONMENT}}$$

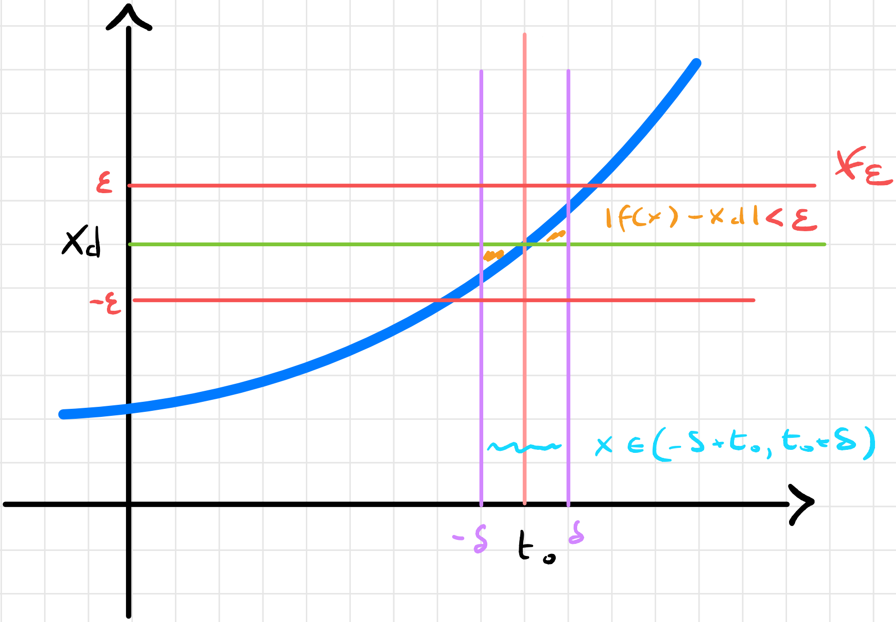
MY CLOCK APPROACHES 13h

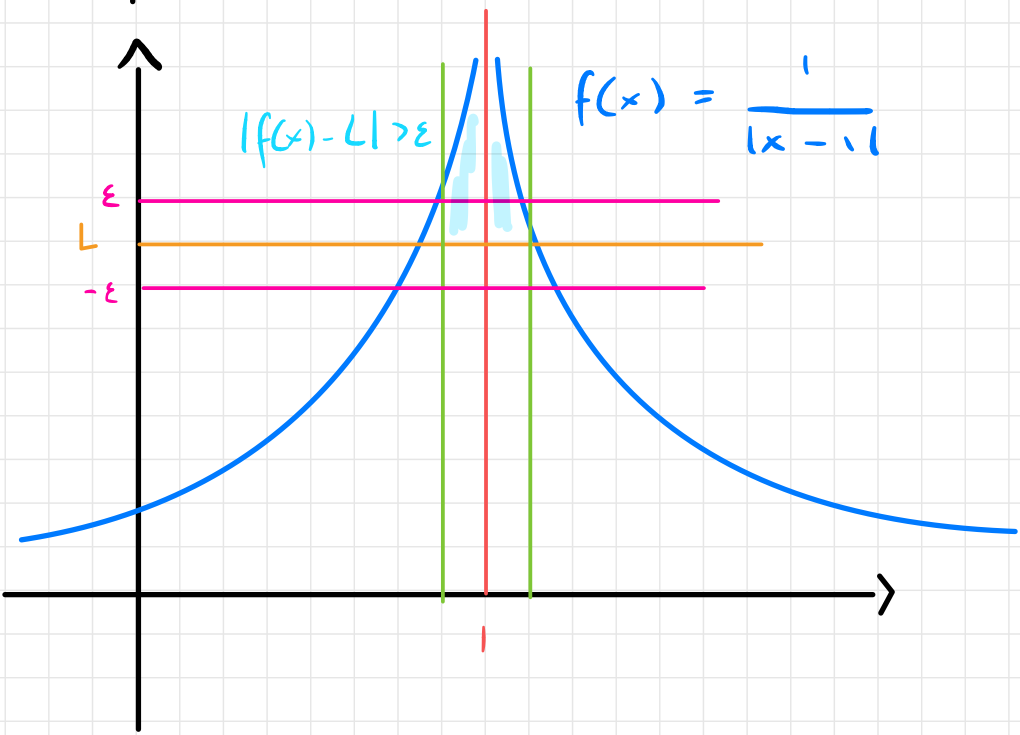
$$\overbrace{t \to t_0} \quad \Longrightarrow \quad \overbrace{f(x) \to x_d}^{\text{I GET CLOSE TO DINING ROOM}}$$

$$\hookrightarrow \underbrace{|f(x) - x_d|}_{\text{DISTANCE between DINING ROOM and ME}} < \overbrace{\varepsilon}^{\text{SMALL NUMBER}}$$

REGARDLESS I SELECT ε

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \ \forall t \in D_f \ |t - t_0| < \delta \Rightarrow |f(t) - x_d| < \varepsilon$$


∀ε
ε
Xd
|f(x) − xd| < ε
−ε
x ∈ (−δ + t₀, t₀ + δ)
−δ
t₀
δ


f(x) = 1 / |x − 1|
|f(x) − L| > ε
ε
L
−ε
1

As well as of sequences, we do not use definition

- Elementary limits
- Properties

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$$

$$\lim_{x \to 0} \frac{2}{x^2}(1 - \cos x) = 1$$

Equivalence

$$\lim_{x \to +\infty} \frac{\log(x)}{x} = 0$$

Negligibility

ALGEBRAIC RULES

$$\lim_{x \to x_0} f(x) = L_f$$

$$\lim_{x \to x_0} g(x) = L_g$$

$$\lim_{x \to x_0} f(x) + g(x) = L_f + L_g$$

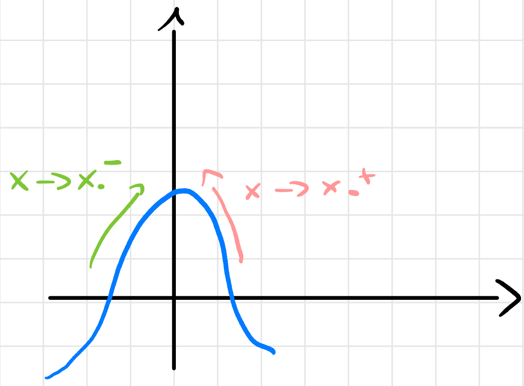
Similarly for OTHER OPERATIONS

- SANDWICH

LET'S FOCUS ON DIFFERENCE

ONE SIDE LIMIT

$$\lim_{x \to x_0^+} f(x) = L$$



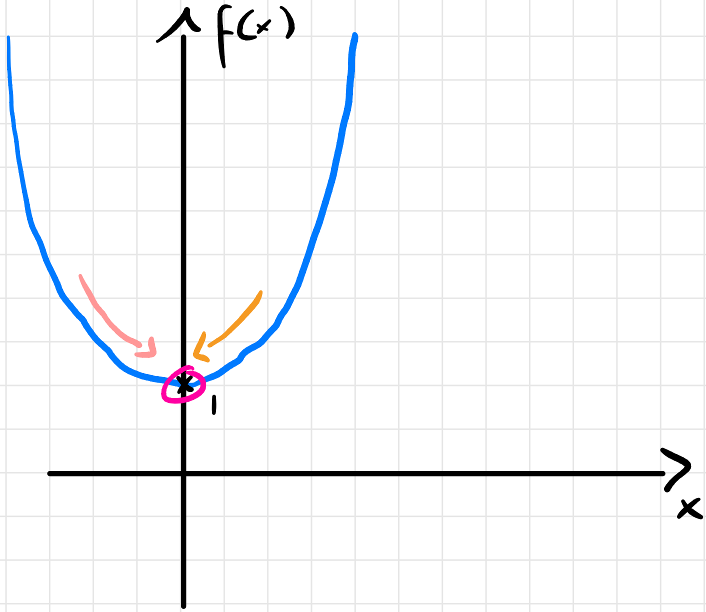
FUNCTION TREND when x get close x_0 from a certain DIRECTION

Prop

$$\underbrace{\lim_{x \to x_0} f(x) = L}_{\text{LIMIT EXISTS}} \iff \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = L$$

Ex $f(x) = x^2 + 1$

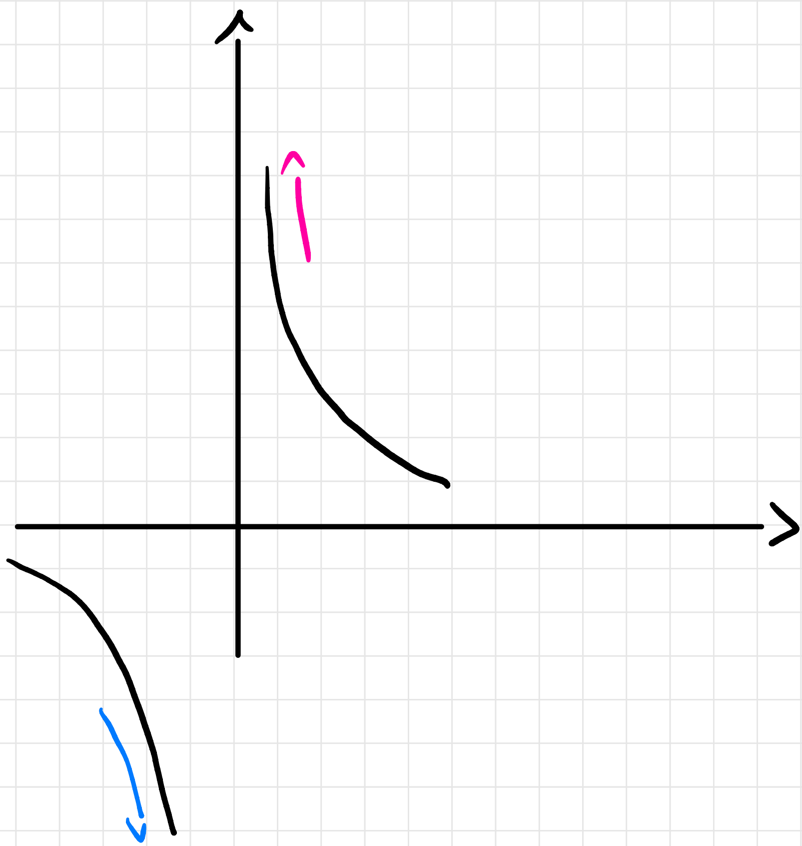
$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 1$$



Ex $f(x) = \frac{1}{x}$

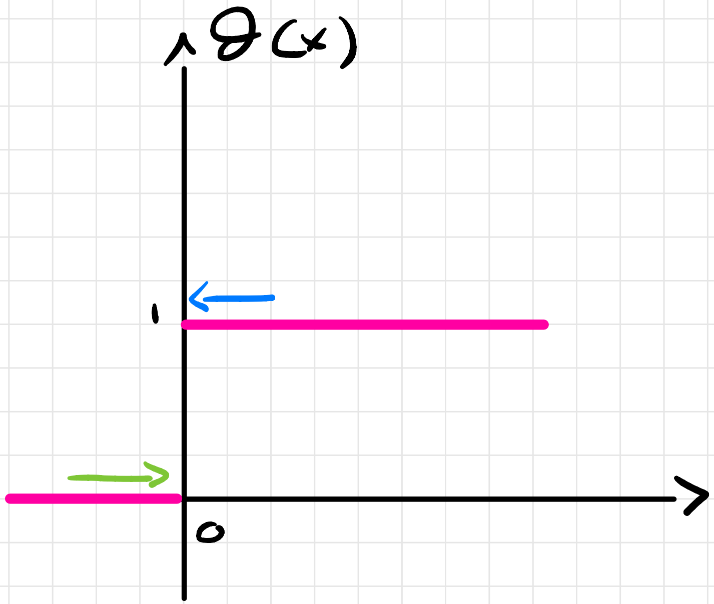
$$\lim_{x \to 0^-} f(x) = -\infty$$

$$\lim_{x \to 0^+} f(x) = +\infty$$



Ex HEAVISIDE

$$\vartheta(x) \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases} = \frac{x + |x|}{2}$$



$$\lim_{x \to 0^-} \vartheta(x) = 0$$

$$\lim_{x \to 0^+} \vartheta(x) = 1$$

CONTINUITY

$f : A \subset \mathbb{R} \to \mathbb{R}$

f CONT. $x_0 \iff \lim_{x \to x_0} f(x) = f(x_0)$

IT MAY BE EXTENDED

f CONT $[a, b) \iff \forall x \in (a,b)$ f CONT x

$\lim_{x \to a^+} f(x) = f(a)$

ROUGHLY SPEAKING . . .

CONTINUOUS FUNCTIONS MAY BE DRAWN WITHOUT REMOVING PEN FROM THE SHEET

Ex $\sin(x)$, $\cos(x)$ CONT. $\mathbb{R}$

Ex Polynomials cont $\mathbb{R}$

Ex HEAVISIDE non CONT.

Prop

Hy) f, g cont. x_0

$\Rightarrow$ (Th) $f \pm g$ cont x_0

fg " x_0

f/g " " $(g(x_0) \neq 0)$

$f \circ g$ " " (f cont $g(x_0)$)

f^{-1} " " (f inv)

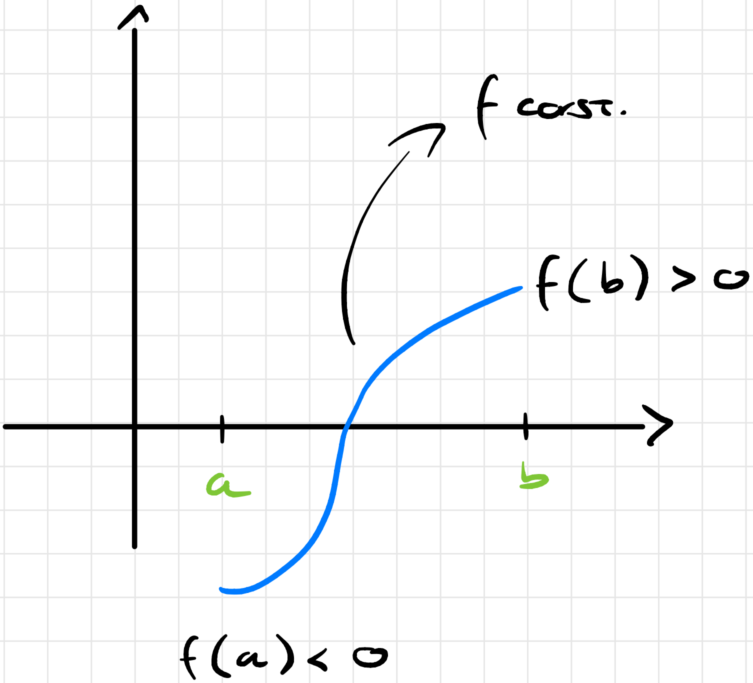
Theorem (Bolzano)

Hy) f cont $[a,b]$

Hy) $f(a)f(b)<0$

$\Rightarrow$ (Th) $\exists c \in (a,b) : f(c)=0$

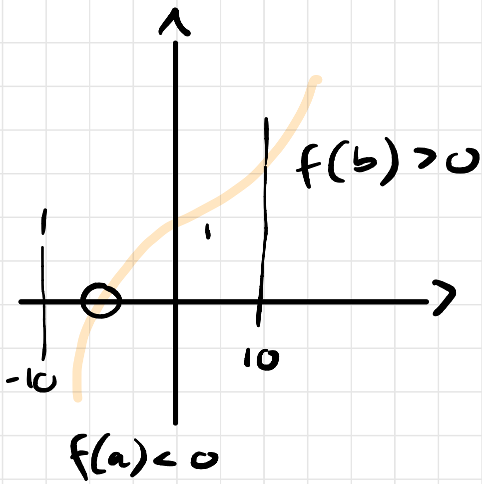
Ex



SOLUTION of EQUATION

$$x^3 + 2 = 1$$

$$f(x) = 0 \quad : \quad f(x) = x^3 + 2 - 1 = x^3 + 1$$



$$f \text{ cont } [a, b]$$

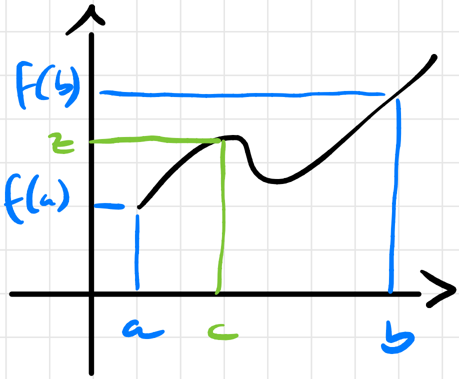
EXISTS AT LEAST ONE SOLUTION

COROLLARY (INTERMEDIATE VALUES THEOREM)

<u>Hy</u> f cont $[a,b]$

<u>Hy</u> $\min\{f(a), f(b)\} < z < \max\{f(a), f(b)\}$

$\Rightarrow \exists c \in (a,b) \quad f(c) = z$



A CONTINUOUS FUNCTION TAKES ALL INTERMEDIATE VALUE

THEOREM

f cont $[a,b]$

$\Rightarrow \exists\ x_m,\ x_M \in [a,b]$

$$f(x_m) \leq f(x) \leq f(x_M)\ \forall x \in [a,b]$$