TODAY LECTURE

✓ energy

↳ ORGANIZE A PROOF

↳ THERE IS NOT A RECEIPE

→ WE CAN RECOGNIZE A PATTERN

$$\underbrace{Hy)}_{\text{WHAT YOU KNOW}} \quad \underbrace{\Longrightarrow}_{\text{ALGEBRAIC RULES, OTHER THEOREMS}} \quad \underbrace{Th)}_{\text{WHAT YOU WANT TO SHOW}}$$

IF THIS IS TRUE

THIS IS ALSO TRUE

EXERCISE 1.1 (b)

$\alpha, \beta, \gamma \in \mathbb{R}$

Hy) $0 < \alpha < \beta, \quad c > 0$) STARTING POINT

$\Rightarrow$ (Th) $\dfrac{\alpha}{\beta} < \dfrac{\alpha + c}{\beta + c}$

$$\alpha < \beta \Rightarrow \underline{c}\alpha < \underline{c}\beta$$

↳ $c > 0$

$$\Rightarrow c\alpha + \alpha\beta < c\beta + \alpha\beta$$

(COMM) $\alpha c + \alpha\beta < c\beta + \alpha\beta$

(DIST) $\alpha(c + \beta) < (c + \alpha)\beta$

(INV) $\dfrac{\alpha}{\beta}\,\dfrac{(\cancel{c+\beta})}{(\cancel{c+\beta})} < \dfrac{(c+\alpha)}{(c+\beta)}\,\dfrac{\cancel{\beta}}{\cancel{\beta}}$

(COMM) $\dfrac{\alpha}{\beta} < \dfrac{\alpha + c}{\beta + c} \quad \square$

EXERCISE 1.2

$$|\alpha + \beta| = |\alpha| + |\beta| \quad (Hy)$$

$$\Rightarrow \alpha\beta > 0 \quad (Th)$$

def

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{"} \ \ x < 0 \end{cases} = \max\{-x, x\}$$

Properties

1) $|x| \geq 0$
2) $|x| = 0 \Rightarrow x = 0$
3) $|xy| = |x||y|$
4) $|x+y| \leq |x| + |y|$
5) $||x| - |y|| \leq |x - y|$

OF COURSE → def MULTIPLICATION.

$$x^2 = |x^2| = |x\,x| = |x||x| = |x|^2$$

$\hookrightarrow x^2 \geq 0$ $\qquad\qquad$ $\hookrightarrow (3)$

$$\Rightarrow x^2 = |x|^2$$

$$\text{Hy)}\quad \underbrace{|\alpha+\beta|^2}_{A} = \underbrace{(|\alpha|+|\beta|)^2}_{B}$$

Previous Property

$$A = |\alpha+\beta|^2 = (\alpha+\beta)^2 = \alpha^2+\beta^2+2\alpha\beta$$

↳ def $(\alpha+\beta)^2$

$$B = |\alpha|^2 + |\beta|^2 + 2|\alpha||\beta|$$
$$= \alpha^2 + \beta^2 + 2|\alpha\beta|$$

$$A = B \Rightarrow \cancel{\alpha^2}+\cancel{\beta^2}+2\alpha\beta = \cancel{\alpha^2}+\cancel{\beta^2}+2|\alpha\beta|$$
$$\Rightarrow 2\alpha\beta = 2|\alpha\beta| > 0$$
$$\Rightarrow \alpha\beta > 0$$

What about the opposite?

$$\underbrace{\alpha\beta > 0}_{Hy} \Rightarrow \underbrace{|\alpha+\beta| = |\alpha| + |\beta|}_{Th}$$

$$Hy \begin{cases} \alpha, \beta > 0 & (\text{CASE I}) \\ \alpha, \beta < 1 & \text{'' \quad II} \end{cases}$$

CASE I

$\to \alpha, \beta > 0$

$$|\alpha+\beta| = \alpha+\beta = |\alpha| + |\beta|$$

$\hookrightarrow \alpha+\beta > 0$

CASE II

$\to$ DISTRIBUTIVE

$$|\alpha+\beta| = -(\alpha+\beta) = -\alpha-\beta = |\alpha| + |\beta|$$

$\hookrightarrow \alpha,\beta<0$ $\qquad$ $\hookrightarrow \alpha,\beta<0$

OVERALL

if and only if

$$|\alpha+\beta| = |\alpha| + |\beta| \iff \alpha\beta > 0$$

PREVIOUS EXERCISE : $\mathbb{R}$

↳ METHOD $\mathbb{N}$

↳ INDUCTION

Problem 1.7 e

Prove $\underbrace{n < 2^n}_{p(n)} \quad \forall n \in \mathbb{N}$

Is $p(n)$ TRUE for each NATURAL NUMBER

FIRST STEP $\quad p(n=0)$ TRUE ?

$0 < 2^0 = 1$ OK ! $\quad$ JUST REPLACE $n = 1$

Hy $\quad \exists n \in \mathbb{N} : p(n)$ TRUE $\quad$) STARTING POINT

WE ASSUME $n < 2^n$ TRUE

CAN WE OBTAIN $p(n+1)$ TRUE

$n+1 < 2^{n+1}$ $\quad$) END POINT

$$n < 2^n \Rightarrow n+1 < 2^n + 1$$

$$\hookrightarrow a < b \Rightarrow a + c < b + c$$

why $+1$? $\Rightarrow$ Because I get close to $p(n+1)$

$$n+1 < \underbrace{2^n + 1}_{\text{I want to work with this}} = 2^n + 2^0 < 2^n + 2^n$$

$$\hookrightarrow 2^0 < 2^n$$

$$n+1 < 2^n + 2^n = 2 \cdot 2^n = 2^{n+1}$$

$$\Rightarrow \underbrace{n+1 < 2^{n+1}}_{p(n+1)}$$

$p(n)$ TRUE $\Rightarrow$ $p(n+1)$ TRUE

IF IT IS TRUE FOR A CERTAIN n, IT IS ALSO TRUE for the next one,

But we proved THAT IT IS TRUE for $n=0$

SO IT IS TRUE FOR $n=1, 2, 3$

$$\forall n \in \mathbb{N} \quad n < 2^n$$

EXERCISE 1.7(f)

Prove by Induction $2^n < n! \ \forall n \geq 4$

$$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdots \cdot 1$$

Ex $3! = 3 \cdot 2 \cdot 1 = 6$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

definition by RECURRENCE

$$n! = \begin{cases} n \cdot (n-1)! & n > 1 \\ 1 & n = 1 \end{cases}$$

$p(n) \quad 2^n < n!$

FIRST STEP $\quad p(4) \quad \underset{16}{\underset{\shortparallel}{2^4}} < \underset{24}{\underset{\shortparallel}{4!}} \quad$ OK

Hy) $\exists\, n \geq 4$: $p(n)$ TRUE → STARTING POINT

$$2^n < n!$$

?

$\Rightarrow$ $p(n+1)$ TRUE (Th) → END POINT

$$2^{n+1} < (n+1)!$$

$$\underbrace{2^n < n!}_{Hy} \Rightarrow \underbrace{2 \cdot 2^n}_{\substack{\parallel \\ 2^{n+1}}} < 2n! \overset{2<n+1\ \forall n \geq 4}{<} \underbrace{(n+1)\,n!}_{(n+1)!}$$

$$\Rightarrow \underbrace{2^{n+1} < (n+1)!}_{p(n+1)}$$

$$\forall n \geq 4 \quad 2^n < n!$$

EXERCISE 1.8 (c)

Prove by INDUCTION

$$\underbrace{\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}}_{P(n)} \qquad \forall n > 1$$

FIRST STEP $P(2)$

$$\sum_{i=1}^{2} \frac{1}{\sqrt{i}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = \frac{1+\sqrt{2}}{\sqrt{2}} \overset{?}{>} \sqrt{2}$$

$$\frac{\sqrt{2}}{\sqrt{2}} \; \frac{1+\sqrt{2}}{\sqrt{2}} = \frac{2+\sqrt{2}}{2} > \frac{\sqrt{2}+\sqrt{2}}{2} = \sqrt{2} \quad \checkmark$$

$2 > \sqrt{2}$

Hy) $\exists\, n > 1 \quad p(n)$ TRUE

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}$$

$\Rightarrow p(n+1)$ ALSO TRUE ?

$$\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} > \sqrt{n+1}$$

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n} \qquad (hy)$$

$$\Rightarrow \underbrace{\sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}}}_{= \sum_{i=1}^{n+1} \frac{1}{\sqrt{i}}} > \sqrt{n} + \frac{1}{\sqrt{n+1}}$$

Do you see ?

$$\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} = \underbrace{1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}}_{\sum_{i}^{n} \frac{1}{\sqrt{i}}} + \frac{1}{\sqrt{n+1}}$$

$$\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} > \sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n}\sqrt{n+1}+1}{\sqrt{n+1}}$$

$$= \frac{\sqrt{n^2+n} + 1}{\sqrt{n+1}} > \frac{\sqrt{n^2} + 1}{\sqrt{n+1}} = \frac{n+1}{\sqrt{n+1}}$$

$$\sqrt{n^2+n} > \sqrt{n^2}$$

$$= \sqrt{n+1}$$

$\Rightarrow p(n+1)$ TRUE

$$\forall n \geq 1 \quad \sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}$$

EXERCISE 1.12 (a)

$$A = \{ x \in \mathbb{R} : |x - 3| \le 8 \}$$

Property

$$|z| \le a \implies -a \le z \le a$$

Proof

$$z \geq 0 \implies z = |z| \le a$$

$$z \le 0 \implies a \geq |z| = -z$$

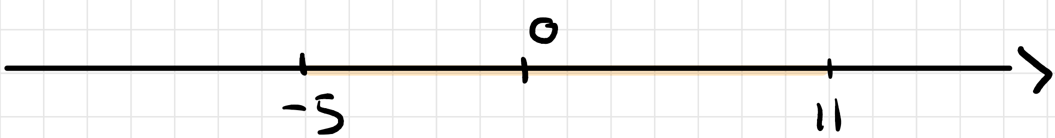
$$\implies z \geq -a$$

$$\implies -a \le z \le a$$

$$|x - 3| \le 8 \implies -8 \le x - 3 \le 8$$

$$\implies -8 + 3 \le x \le 8 + 3 \implies -5 \le x \le 11$$

$$A = [-5, 11]$$



QUESTIONS

1) BOUNDED ? Yes

- Lower Bounded
- Upper BOUNDED

↳ $\underline{\underline{\text{def}}}$ $\exists x \in \mathbb{R} : \forall y \in A \quad x \geq y$

<u>Ex</u> $x = 12$

IS IT UNIQUE ? NO

<u>Ex</u> $x = 11.5$

LEAST Upper Bound

$\sup A = 11$

IS IT a MAXIMUM ? ✓

$\sup A \in A \Rightarrow \sup A = \max A$

ALONG THE SAME LINE

$\inf A = -5$

$\min A = -5$

IS IT CLOSED ?

Exercise 1.12 (ii)

$$B = \{ x \in \mathbb{R} : 0 < |x-2| < \tfrac{1}{2} \}$$

$$= \{ x \in \mathbb{R} : \underbrace{0 < |x-2|}_{(i)} \text{ and } \underbrace{|x-2| < \tfrac{1}{2}}_{(ii)} \}$$

i) $|x-2| > 0 \implies x \neq 2$

ii) $|x-2| < \frac{1}{2} \implies -\frac{1}{2} < x-2 < \frac{1}{2}$

$$\implies -\frac{5}{2} < x < \frac{3}{2}$$

$$B = \left(-\frac{3}{2}, 2\right) \cup \left(2, \frac{5}{2}\right)$$

Same Questions

Bounded? Yes

↳ $\sup B = \frac{5}{2} \neq \max B$

$\quad \inf B = -\frac{3}{2} \neq \min B$

Open!

EXERCISE 1.12 (iii)

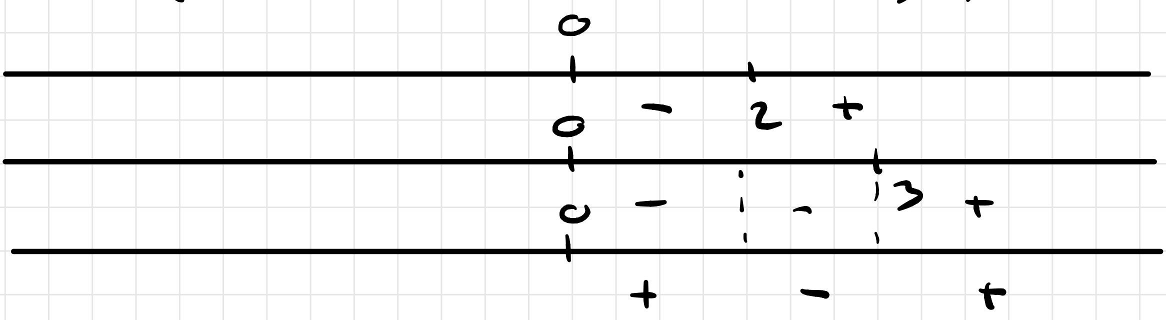
$$C = \{ x \in \mathbb{R} : \underbrace{x^2 - 5x + 6 \geq 0}_{\text{Are you able to solve it?}} \}$$

$$\alpha x^2 + \beta x + \gamma = 0$$

$$x_{\pm} = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

$$x_{\pm} = \frac{+5 \pm \sqrt{25 - 24}}{2} \begin{cases} 3 \\ 2 \end{cases}$$

$$C = \{ x \in \mathbb{R} : (x-2)(x-3) \geq 0 \}$$



$$C = (-\infty, 2) \cup (3, +\infty)$$

IS IT BOUNDED?

EXERCISE 1.14 (iii)

$$C = \left\{ 2 + \frac{1}{n} , \quad n \geq 1 \right\}$$

$\text{Sup } C = 3 = \max C$

$\text{Inf } C = 2 \neq \min C$

BOUNDED? yes

CLOSED? NO

EXERCISE 1.14 (iv)

$$D = \left\{ (n^2+1)/n : n \geq 1 \right\}$$
$$= \left\{ n + \frac{1}{n} : n \geq 1 \right\}$$

$\text{Inf } D = 2 = \min D$

UNBOUNDED