Ex 7.13 (THEORY PART)

$$\lim_{x \to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{\sin(x)}$$

FREE TO USE HÔPITAL

$$D[\sin(x)] = \cos(x)$$

$$D\left[x^2 \sin\left(\frac{1}{x}\right)\right] = +\, 2x \sin\left(\frac{1}{x}\right) - \cancel{x^2} \cos\left(\frac{1}{x}\right)\left(\frac{1}{\cancel{x^2}}\right)$$

WE CALCULATE

$$\lim_{x \to 0} \frac{2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)}{\cos(x)}$$

$$\cos(x) \to 1$$

$\cos\left(\frac{1}{x}\right) \to$ DOES NOT EXISTS

SECOND DERIVATIVE ?

↳ DENOMINATOR

$$\underbrace{2 \sin\left(\frac{1}{x}\right)}_{\nexists} - \underbrace{\frac{2}{x} \cos\left(\frac{1}{x}\right)}_{\to \infty} - \underbrace{\frac{1}{x^2} \sin\left(\frac{1}{x}\right)}_{\to \infty}$$

WHAT CAN WE SAY ?

↳ HÔPITAL USELESS

$$\underbrace{\lim_{x \to x_0} \frac{f'}{g'} = L}_{\text{THE LIMIT EXISTS}} \quad \overset{}{\underset{\text{THEN}}{\Rightarrow}} \quad \lim_{x \to x_0} \frac{f}{g} = L$$

↳ OTHERWISE I CANNOT STATE ANYTHING WITH HÔPITAL

THIS IS ACTUALLY

$$\lim_{x \to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{\sin x} = \lim_{x \to 0} \underbrace{\left(\frac{x}{\sin x}\right)}_{\to 1} \underbrace{\left[x \sin\left(\frac{1}{x}\right)\right]}_{\to 0} = 0$$

SO DO NOT FORGET $\underbrace{\text{HÔPITAL FORMULATION}}$

↳ HYPOTHESIS

$$f, g \begin{cases} \to 0 & \text{(BOTH)} \\ \to \infty & \text{"} \end{cases}$$

$$\lim_{x \to 0} \frac{\cos x}{\underbrace{1 + \log(1+x)}_{\to 1}} \overset{\cos x \to 1}{=} 1$$

HÔPITAL

$$\lim_{x \to 0} \frac{-\sin x}{\frac{1}{1+x}} = 0 \quad \text{FALSE}$$

Ex

$$\lim_{x \to 0} \frac{e^x - 1 - \arctan(x)}{\sin(x^2)}$$

$\longrightarrow$ TAYLOR

$$\boxed{f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \underset{x \to a}{o}\left((x-a)^n\right)}$$

WHERE

$$0! = 1 \quad , \quad f^{(0)}(a) = f(a)$$

FIRST POINT $\quad a = ? \quad a = 0$ (because limit)

SECOND // $\quad n = ? \quad \underbrace{n = 2}$

Because of $\sin(x^2)$

(HÔPITAL)

$$f(x) = \sum_{k=0}^{2} \frac{f^{(k)}(0)}{k!} x^k + o(x^2)$$

$$= f^{(0)}(0) + f^{(1)}(0)\,x + \frac{f^{(2)}(0)}{2} + o(x^2)$$

WE ALREADY KNOW

$$e^x = 1 + x + \frac{x^2}{2} + o(x^2) \quad x \to 0$$

$$\sin(t) = t + o(t) \quad t \to 0$$

$\hookrightarrow$ $t = x^2$, $\quad \sin(x^2) = x^2 + o(x^2)$

WHAT ABOUT ARCTAN? $\quad f(x) = \arctan(x)$

$$f^{(0)}(0) = 0$$

$$f^{(1)}(0) = \frac{d}{dx} \arctan(x) \Big|_0 = \frac{1}{1+x^2}\Big|_0 = 1$$

$$f^{(2)}(0) = \frac{d^2}{dx^2} \arctan(x) \Big|_0 = \frac{d}{dx}\left(\frac{1}{1+x^2}\right)\Big|_{x=0}$$

$$= -\frac{2x}{(1+x^2)^2}\Big|_{x=0} = 0$$

$$\arctan(x) = x + o(x^2)$$

$$\lim_{x\to 0} \frac{\cancel{1}+\cancel{x}+\frac{x^2}{2}+o(x^2)-\cancel{1}-\cancel{x}-o(x^2)}{x^2+o(x^2)}$$

$$\underbrace{o(x^2)-o(x^2)}_{\lambda\, o(f)=o(f)} = o(x^2)+o(x^2) \underset{o(f)+o(f)=o(f)}{=} o(x^2)$$

$$\lim_{x\to 0} \frac{\frac{x^2}{2}+o(x^2)}{x^2+o(x^2)} = \lim_{x\to 0} \frac{\frac{1}{2}+\overbrace{\frac{o(x^2)}{x^2}}^{\to 0}}{1+\underbrace{\frac{o(x^2)}{x^2}}_{0}} = \frac{1}{2}$$

Ex 8.6 x

$$\lim_{x \to +\infty} \left[x - x^2 \log\left(1 + \frac{1}{x}\right)\right]$$

TAYLOR → WHICH POINT ?

$$t = \frac{1}{x}$$

$$\lim_{t \to 0^+} \left[\frac{1}{t} - \frac{1}{t^2} \log(1+t)\right] = \lim_{t \to 0^+} \frac{t - \log(1+t)}{t^2}$$

$$\underbrace{\log(1+t)}_{f(t)}$$ → TAYLOR

↳ WHICH POINT ? $a = 0$

↳ " NUM. ? $n = 2$

$$f(t) = \sum_{k=0}^{2} \frac{f^{(k)}(0)}{k!} t^k + o(t^2)$$

$$f^{(0)}(0) = \log(1) = 0$$

$$f^{(1)}(0) = \frac{d}{dt} \log(1+t)\Big|_0 = \frac{1}{1+t}\Big|_0 = 1$$

$$f^{(2)}(0) = \frac{d^2}{dt^2} \log(1+t)\Big|_0 = -\frac{1}{(1+t)^2} = -1$$

$$\log(1+t) = t - \frac{t^2}{2} + o(t^2)$$

$$\lim_{t \to 0} \frac{1}{t^2}\left[\cancel{t} - \cancel{t} + \frac{t^2}{2} + o(t^2)\right]$$

$$\lim_{t \to 0}\left[\frac{1}{2} + \frac{o(t^2)}{t^2}\right] = \frac{1}{2}$$

PROBLEM 7.25 (i)

$x^7 + 4x = 3$ NUMBER SOLUTIONS ?

7? $x^2 + 1 = 0$ NO SOLUTION (although x^2)

I INTRODUCE

$f(x) = x^7 + 4x - 3$

$\lim_{x \to \pm\infty} f(x) = \pm\infty$ → AT LEAST ONE SOLUTION

(Bolzano EXTENSION)

How many ?

$f'(x) = \underbrace{7x^6 + 4 > 0}_{\text{ALWAYS INCREASING}} \quad \forall x \in \mathbb{R}$

$\Rightarrow \exists!\, c \in \mathbb{R} : f(c) = 0$ (ONE SOLUTION)

PROBLEM 7.25 (ii)

$$x^5 = 5x - 6$$ SAME QUESTION

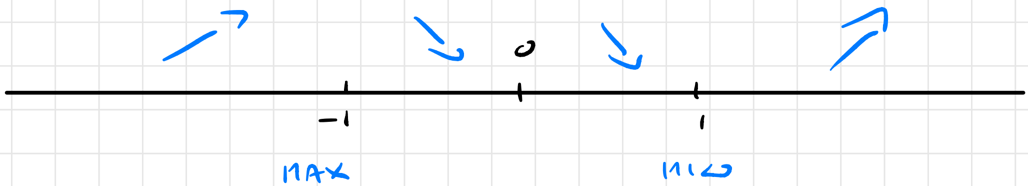
$$f(x) = x^5 - 5x + 6$$

$$\lim_{x \to \pm\infty} f(x) = \pm\infty$$

$$f'(x) = 5x^4 - 5 = 5(x^4 - 1)$$

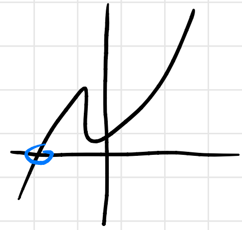
$$f'(x) = 0 \Rightarrow x = \pm 1$$

$$f'(x) \begin{cases} > 0 & x > 1, \quad x < -1 \\ < 0 & -1 < x < +1 \end{cases}$$



$$f(-1) = -1 + 5 + 6 = 10 > 0$$

$$f(1) = 1 - 5 + 6 = 2 > 0$$



$\Rightarrow$ EXISTS ONLY ONE SOLUTION

Problem 7.22 (a)

A FACTORY THAT PRODUCES TOMATO SAUCE WANTS TO STORE IT INTO CYLINDRICAL CANS of VOL. V

r, h such TAT CONSUMES THE LEAST MIN. MATERIAL.

$$\text{MATERIAL} \sim \underbrace{S}_{\text{AREA}} = \overset{\text{up \& Down}}{2}\,\underbrace{\pi r^2}_{\text{BASE}} + \underbrace{2\pi r h}_{\text{SIDES}}$$

Two VARIABLES

$$V = \pi r^2 h \quad \text{FIXED} \Rightarrow h = \frac{V}{\pi r^2}$$

SO

$$S = 2\pi \left(r^2 + \frac{V}{\pi r} \right) = 2\pi f(r)$$

$$f'(r) = 2r - \frac{V}{\pi r^2} = 0 \Rightarrow r^3 = \frac{V}{2\pi}$$

$$h = \left(\frac{4V}{\pi} \right)^{\frac{1}{3}} \quad \square$$