EXERCISE 3.4 (ii)

$$\lim_{n \to +\infty} \frac{n\left(e^{\frac{1}{n}} - e^{\sin\frac{1}{n}}\right)}{1 - n\sin\left(\frac{1}{n}\right)} = L\ ?$$

$$\frac{n\left(e^{1/n} - e^{\sin\frac{1}{n}}\right)}{1 - n\sin\left(\frac{1}{n}\right)} = \cancel{\frac{n}{n}}\ \frac{e^{1/n} - e^{\sin\left(\frac{1}{n}\right)}}{\frac{1}{n} - \sin\left(\frac{1}{n}\right)}$$

$$= e^{\sin\frac{1}{n}}\ \frac{e^{\frac{1}{n} - \sin\left(\frac{1}{n}\right)} - 1}{\frac{1}{n} - \sin\left(\frac{1}{n}\right)}$$

$$\varepsilon_n = \frac{1}{n} - \sin\left(\frac{1}{n}\right) \xrightarrow[n \to +\infty]{} 0$$

$$\lim_n \underbrace{e^{\sin\left(\frac{1}{n}\right)}}_{\substack{\downarrow \\ 1}} \underbrace{\frac{e^{\varepsilon_n} - 1}{\varepsilon_n}}_{\to 1} = 1$$

EXERCISE 3.3 (ii)

$$\lim_n \left(\frac{\sqrt[n]{a} + \sqrt[n]{b}}{2} \right)^n = L \qquad a, b > 0$$

$$\sqrt[n]{a}, \sqrt[n]{b} \xrightarrow[n \to +\infty]{} 1 \quad , \quad 1^\infty \text{ IND.}$$

$$L = e^c$$

$$c = \lim_n \left[\left(\frac{\sqrt[n]{a} + \sqrt[n]{b}}{2} - 1 \right) n \right]$$

$$a_n = (1 + b_n)^{c_n}, \quad b_n \to 0, \quad c_n \to +\infty$$

$$a_n \to e^{\lim_n b_n c_n}$$

$$a_n = b_n^{c_n}, \quad b_n \to 1, \quad c_n \to +\infty$$

$$a_n \to e^{\lim_n (b_n - 1) c_n}$$

$$c = \lim_n \left[\left(\frac{\sqrt[n]{a} + \sqrt[n]{b} - 2}{2} \right) n \right]$$

$$= \frac{1}{2} \lim_n \; n \left(\sqrt[n]{a} - 1 + \sqrt[n]{b} - 1 \right)$$

I INTRODUCE THE NOTATION

$$L_a = \lim_n n \left(\sqrt[n]{a} - 1 \right)$$

$$L_b = \lim_n n \left(\sqrt[n]{b} - 1 \right)$$

THUS

$$c = \frac{1}{2} \left(L_a + L_b \right)$$

I CAN CALCULATE L_a and L_b is the same

$$L_a = \lim_n \frac{a^{\frac{1}{n}} - 1}{\frac{1}{n}} = \lim_n \frac{\left(e^{\log a}\right)^{\frac{1}{n}} - 1}{1/n}$$

$$= \log a \lim_n \underbrace{\frac{e^{\frac{\log a}{n}} - 1}{\frac{\log a}{n}}}_{\to 1} = \log a$$

Overall

$$L_a = \log a$$

$$L_b = \log b$$

$$c = \frac{1}{2}(\log a + \log b)$$

$$= \frac{1}{2}\log(ab) = \log\sqrt{ab}$$

$$L = e^{\log\sqrt{ab}} = \sqrt{ab}$$

EXERCISE 3.5 (i)

$\lim_n a_n = L$?

$$a_n = \left(\underbrace{\cos \frac{b}{n}}_{\to 1} + a \underbrace{\sin \frac{b}{n}}_{\to 0} \right)^{\overset{+\infty}{\nearrow} n}$$

$a_n = b_n^{c_n}$

$b_n \to 1$, $c_n \to \infty$

$a_n \to e^{(b_n - 1)c_n}$

$a_n = (1 + b_n)^{c_n}$

$b_n \to 0$, $c_n \to +\infty$

$a_n \to e^{b_n c_n}$

INDETERMINACY 1^∞

$$L = e^c, \quad c = \lim_n \left[n \left(\cos \frac{b}{n} + a \sin \frac{b}{n} - 1 \right) \right]$$

$$= \underbrace{\lim_n \left[n \left(\cos \frac{b}{n} - 1 \right) \right]}_{B} + \underbrace{a \lim_n n \sin \frac{b}{n}}_{A}$$

A) $a \lim_n n \sin\frac{b}{n} = ab \lim_n \frac{n}{b} \sin\frac{b}{n}$

$$= ab \lim_n \underbrace{\left(\frac{\sin\frac{b}{n}}{\frac{b}{n}}\right)}_{\downarrow\, 1} = ab$$

$\sin\frac{b}{n} \sim \frac{b}{n}$

B) $\lim_n \left[n \left(\cos\frac{b}{n} - 1 \right) \right]$

$$\cos\frac{b}{n} \sim 1 - \frac{1}{2}\left(\frac{b}{n}\right)^2$$

$$\lim_n \left[n \left(\cos\frac{b}{n} - 1 \right) \right] = \lim_n -\frac{n}{2}\left(\frac{b}{n}\right)^2 \underbrace{\frac{\left(\cos\frac{b}{n} - 1\right)}{-\frac{1}{2}\left(\frac{b}{n}\right)^2}}_{\downarrow\, 1}$$

$$= \lim_n -\frac{1}{2n} b^2 = 0$$

$c = 0 + ab$

$L = e^{ab}$

EXERCISE 3.5 (viii)

$$\lim_{n \to +\infty} \frac{1 + 2\sqrt{2} + 3\sqrt[3]{3} + \dots + n\sqrt[n]{n}}{n^2}$$

$$= \lim_{n \to +\infty} \frac{1}{n^2} \sum_{k=1}^{n} k\sqrt[k]{k} \equiv \lim_n C_n = L$$

By SANDWHICH

$$\underbrace{\frac{1}{n^2}}_{\downarrow \, 0} \leq C_n \leq \frac{1}{n^2} \, n\sqrt[n]{n} \underbrace{\sum_{k=1}^{n} 1}_{n} = \underbrace{\sqrt[n]{n}}_{\downarrow \, 1}$$

SANDWHICH $\longrightarrow$ $0 \leq L \leq 1$

Stolz

$$\frac{a_n - a_{n-1}}{b_n - b_{n-1}} \longrightarrow L \implies \frac{a_n}{b_n} \longrightarrow L$$

Recognize a_n, b_n

$$a_n = 1 + 2\sqrt{2} + 3\sqrt[3]{3} + \_\ \_\ \_ + n\sqrt[n]{n}$$

$$b_n = n^2 \quad ($$

$$a_n - a_{n-1} = ?$$

$$a_n = \underbrace{1 + 2\sqrt{2} + 3\sqrt[3]{3} + \_\ \_\ \_ + (n-1)\sqrt[(n-1)]{n-1}}_{a_{n-1}} + n\sqrt[n]{n}$$

$$a_n - a_{n-1} = n\sqrt[n]{n}$$

$$n^2 - (n-1)^2 = \cancel{n^2} - \cancel{n^2} + 2n - 1 = 2n - 1$$

$$\frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \underbrace{\frac{n}{2n-1}}_{\to\ 1/2} \underbrace{\sqrt[n]{n}}_{\to\ 1} \qquad \lim \sqrt[n]{n^p} = 1 \quad \forall p \in \mathbb{R}$$

$$\Rightarrow \lim_n c_n = \lim_n \frac{a_n}{b_n} = \frac{1}{2} \qquad (\text{STOLZ})$$

EXERCISE 3.6 (ii)

$$\lim_{n \to +\infty} \prod_{K=1}^{n} (2K-1)^{\frac{1}{n^2}}$$

↳ do you know this?

$$\prod_{K=1}^{n} a_K = a_1 \cdot a_2 \cdot a_2 \; - - -$$

$$n! = \prod_{K=0}^{n-1} (n-K) = n \cdot (n-1) \cdot (n-2) \cdots 1$$

$\prod$ SHITY → WHAT THE HELL I DO?

↳ Function THAT ALLOWS TO CONVERT X into ·

↳ log

$$\log \left[\prod_{K=1}^{n} (2K-1)^{\frac{1}{n^2}} \right] \to \log L$$

$$\log\left[\prod_{k=1}^{n}(2k-1)^{\frac{1}{n^2}}\right]=\sum_{k=1}^{n}\log\left[(2k-1)^{\frac{1}{n^2}}\right]$$

$$=\sum_{k=1}^{n}\frac{1}{n^2}\log(2k-1)=\frac{1}{n^2}\sum_{k=1}^{n}\log(2k-1)$$

$\longrightarrow$ STOLZ

$$a_n=\sum_{k=1}^{n}\log(2k-1)$$

$$=\underbrace{\log 1+\log(2\cdot 2-1)+\_\_+\log[2\cdot(n-1)-1]}_{\sum_{k=1}^{n-1}\log(2k-1)\,\equiv\,a_{n-1}}+\log[2n-1]$$

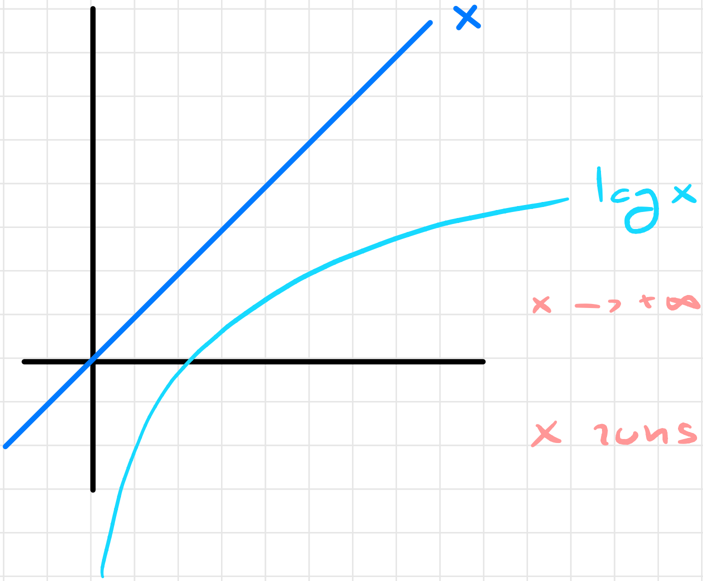
$$a_n-a_{n-1}=\log(2n-1)$$

$$b_n=n^2$$

$$b_n-b_{n-1}=n^2-(n-1)^2=2n-1$$

$$\frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \frac{\log(2n-1)}{2n-1} \longrightarrow L' = 0$$

$$\log(2n-1) \underset{n \to +\infty}{<<} 2n-1$$



$x \to +\infty \quad \log x << x$

x runs FASTER TO ∞

$$\lim_n \frac{a_n}{b_n} = L' = 0 = \log L \Rightarrow L = 1$$

(STOLZ)

EXERCISE (NOT IN THE NOTES)

$$\begin{cases} a_1 = 0 \\ a_{n+1} = a_n^2 + \frac{1}{4} \end{cases}$$

$a_1 = 0$

$a_2 = a_1^2 + \frac{1}{4} = 0 + \frac{1}{4} = \frac{1}{4}$

$a_3 = a_2^2 + \frac{1}{4} = \frac{1}{4}\left(1 + \frac{1}{4}\right) > a_2$

Mon. Inc.? $\rightarrow$ Prove it

$$a_{n+1} - a_n = a_n^2 - a_n + \frac{1}{4}$$

$$= a_n^2 - 2\,\frac{1}{4}\,a_n + \left(\frac{1}{2}\right)^2$$

$$= \left(a_n - \frac{1}{2}\right)^2 > 0 \quad \text{Yes!}$$

Prove THAT $a_n \leq \frac{1}{2} \quad \forall n \geq 1$

$a_1 = 0 \leq \frac{1}{2}$ ✓

Hy) $\exists n \geq 1 : a_n \leq \frac{1}{2}$

$\Rightarrow a_{n+1} \leq \frac{1}{2}$?

$a_{n+1} = a_n^2 + \frac{1}{4} \leq \left(\frac{1}{2}\right)^2 + \frac{1}{4} = 2\,\frac{1}{4} = \frac{1}{2}$ ✓

$\Rightarrow a_n \leq \frac{1}{2} \quad \forall n \geq 1$

$\left.\begin{array}{ll} a_n & \text{UPP. BOUND} \\ b_n & \text{MON. INC.} \end{array}\right\} \Rightarrow L$ EXISTS

L ?

$x_n \rightarrow L$

$x_{n+1} \rightarrow L$

$L = L^2 + \frac{1}{4} \Rightarrow L^2 - L + \frac{1}{4} = 0$

$\Rightarrow \left(L - \frac{1}{2}\right)^2 = 0 \Rightarrow L = \frac{1}{2}$ ✓