DERIVATIVE $\longrightarrow$ OPERATOR

$$f \mapsto \underbrace{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}_{\text{DERIVATIVE FUN.}}$$

INVERSE PROBLEM

$f \longrightarrow F = ?$ s.t. $F' = f$

GIVEN f determine F such that $F' = f$

<u>DEF</u> $\underbrace{F \text{ PRIMITIVE } f}$ $\Leftrightarrow$ $F'(x) = f(x)$

$$F(x) = \int \underbrace{f(x)}_{\text{INTEGRAND}} dx$$

QUESTION? IS IT UNIQUE? NO

↳ ALMOST SO

$$F' = G' = f \Rightarrow F = G + c$$

↳ CONSTANT

AS A CONSEQUENCE OF DERIVATIVE LINEARITY

$$\int [a f(x) + b g(x)] \, dx = a\int f(x)\,dx + b\int g(x)\,dx$$

$(a, b \in \mathbb{R})$

ELEMENTARY PRIMITIVE

$$\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} + c \quad \forall \alpha \in \mathbb{R} / \{-1\}$$

$$\int \frac{1}{x} dx = \log|x| + c$$

$$\int e^{x} dx = e^{x} + c$$

$$\int \sin(x)\, dx = -\cos x + c$$ (SIMILARLY COS)

$$\int \frac{dx}{1+x^2} = \arctan(x)$$

TABLE 9.1 ALL

Today Lecture

↳ Calculating Primitive

- immediate
- by parts
- change variable
- rational fun.

IMMEDIATE

$$\boxed{\int f'(g(x))\, g'(x) = f(g(x)) + c}$$

<u>PROOF</u> $\int f'(g(x))g'(x)\ dx = \int \frac{d}{dx}\Big(f(g(x))\Big)\,dx$

↳ CHAIN RULE

<u>Ex</u> (PARADIGMATIC EXAMPLE)

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{(-\sin(x))}{\cos(x)}\,dx$$

$$f(x) = \log(x)\ ,\quad g(x) = \cos(x)$$

$$\int \tan(x)\,dx = -\log|\cos(x)| + c$$

<u>HOMEWORK</u> $\int \sec x\ dx$

INTEGRAL BY PARTS

$$\boxed{\int f(x)\,g'(x)\,dx = f(x)g(x) - \int f'(x)\,g(x)\,dx}$$

PROOF $\quad f(x)g(x) = \int [f(x)g(x)]'\,dx$

LEIBNIZ $\longrightarrow$ $\quad = \int [f'(x)g(x) + f(x)g'(x)]\,dx$

LINEARITY $\longrightarrow$ $\quad = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$

LEIBNIZ $\rightsquigarrow$ PARTS

Ex $\quad \int \log x\,dx = x\log x - \int dx = x\log x - x + c$

$\hookrightarrow$ $g(x) = x$, $f(x) = \log x$

THIS EXAMPLE PAVES THE WAY . . .

COROLLARY

$$\left.\begin{array}{l} f \text{ INV.} \\ F' = f \end{array}\right\} \Rightarrow \int f^{-1}(x)\,dx = x f^{-1}(x) - F(f^{-1}(x))$$

PROOF $\displaystyle\int f^{-1}(x)\,dx = x f^{-1}(x) - \int x\,(f^{-1})'\,dx$

INT. BY PARTS

$$x = f(f^{-1}(x)) \Rightarrow \int x\,(f^{-1})'(x)\,dx$$

$$= \underbrace{\int f(f^{-1}(x))\,(f^{-1})'(x)\,dx}_{\text{IMMEDIATE}}$$

$$F(f^{-1}(x))$$

Ex $\displaystyle\int \arctan x\,dx = x\arctan x + \log|\cos(\arctan(x))|$

$f = \operatorname{tg} x \;,\; F = \log|\cos x|$

Ex

$$I_n = \int \frac{dx}{(1+x^2)^n} \qquad (n \in \mathbb{N})$$

$$I_1 = \arctan x + c$$

RECURRENCE $I_{n+1} = f(I_n)$

$$I_{n+1} = \int \frac{dx}{(1+x^2)^{n+1}} = \int \frac{1-x^2+x^2}{(1+x^2)^{n+1}}\, dx$$

def

$-x^2+x^2$

$$= I_n - \frac{1}{2} \underbrace{\int x \frac{2x}{(1+x^2)^{n+1}}\, dx}_{\text{INTEGRATION BY PART}}$$

SPLIT

OBSERVATION

$$\frac{d}{dx} \frac{1}{(1+x^2)^n} = \frac{d}{dx} (1+x^2)^{-n}$$

$$= -n (1+x^2)^{-(n+1)} \; 2x$$

$$\int \frac{2x}{(1+x^2)^{n+1}}\, dx = -\frac{1}{n} \frac{1}{(1+x^2)^n}$$

$$\int x \frac{2x}{(1+x^2)^{n+1}}\,dx = -\frac{x}{n}\frac{1}{(1+x^2)^n} + \frac{1}{n}\int \frac{dx}{(1+x^2)^n}$$

$$= \frac{1}{n} I_n(x) - \frac{x}{n(1+x^2)^n}$$

Replacing

$$I_{n+1}(x) = I_n(x)\left(1 - \frac{1}{2n}\right) + \frac{x}{2n(1+x^2)^n}$$

CHANGE VARIABLE

$$\int f(x)\,dx = \int f(g(t))\,g'(t)\,dt$$

(SLOPPY) PROOF

$$\int f(\underbrace{g(t)}_{x})\,\underbrace{g'(t)}_{\frac{dx}{\cancel{dt}}}\,\cancel{dt} = \int f(x)\,dx$$

Ex $\displaystyle\int \frac{e^x}{e^{2x}+1}\,dx \qquad t = e^x,\ dt = e^x\,dx$

$$= \int \frac{t}{t^2+1}\,\frac{dt}{t} = \arctan(e^x) + c$$

Ex

$$\int \frac{dx}{\sqrt[3]{(1-2x)^2} - \sqrt{1-2x}}$$

$$t^m = 1 - 2x$$

$m = ? \longrightarrow$ GET RID of ROOTS

LEAST COMMON MULTIPLE
BETWEEN 3 and 2

$$t^6 = 1 - 2x \Rightarrow x = \frac{1-t^6}{2}$$

$$dx = -3t^5\,dt$$

$$-3\int \frac{t^5}{t^{\frac{6\cdot 2}{3}} - t^3}\,dt = -3\int \frac{t^5}{t^4 - t^3}\,dt$$

$$= -3\int \frac{t^2}{t-1}\,dt = -3\int \frac{t^2-1}{t-1} + \frac{1}{t-1}\,dt$$

$(t-1)(t+1)$

$$= -3\int (t+1)\,dt - 3\int \frac{dt}{t-1} = -3\frac{t^2}{2} - 3t - 3\log|t-1|$$

Replace t and find the solution

$$\int \frac{dx}{\sqrt[3]{(1-2x)^2} - \sqrt{1-2x}} = -\frac{3}{2}\left(1+\sqrt[6]{1-2x}\right)^2 - 3\log\left|1-\sqrt[6]{1-2x}\right| + c$$

Ex

$$\int \frac{dx}{x\sqrt{1-x^2}} =$$

$$x = \sin(t), \quad dx = \cos(t)\,dt$$

$$\int \frac{\cos t}{\sin t \cos t}\,dt = \int \frac{dt}{\sin t} = \underbrace{\int \operatorname{cosec}(t)\,dt}_{\text{Homework}}$$

$$= \log\left|\frac{\sin(x)}{1+\cos(x)}\right| + c$$

Rational Functions

$$R(x) = \frac{P(x)}{\underbrace{Q(x)}_{\text{monic}}} = \underbrace{C(x)}_{\text{easily integrated}} + \frac{M(x)}{Q(x)} \quad \deg M < \deg Q$$

- constant may absorbed

$$Q(x) = (x-a_1)^{n_1} \cdots (x-a_r)^{n_r} (x^2+p_1x+q_1)^{m_1} \cdots \underbrace{(x^2+p_sx+q_s)}_{\text{irreducibles}}{}^{m_s}$$

MULTIPLICITIES

$$p_1^2 < 4q_1$$

$$\frac{M(x)}{Q(x)} = \sum_{i=1}^{r}\left[\frac{A_{11}}{x-a_i} + \cdots + \frac{A_{in_i}}{(x-a_i)^{n_i}}\right] + \sum_{j=1}^{s}\left[\frac{B_{j1}x + C_{j1}}{x^2+p_jx+q_j} + \cdots + \frac{B_{jm_j}x + C_{jm_j}}{(x^2+p_jx+q_j)^{m_j}}\right]$$

Ex

$$\int \frac{2x^2-4x+6}{(x-1)^3}\,dx$$

$$\frac{2x^2-4x+6}{(x-1)^3} = \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{x-1}$$

MULTIPLYING by $(x-1)^3$

$$2x^2-4x+6 = A + B(x-1) + C(x-1)^2$$

$$2x^2-4x+6 = A + Bx - B + Cx^2 + C - 2Cx$$

$$2x^2-4x+6 = (A-B+C) + (B-2C)x + Cx^2$$

$$\begin{cases} A-B+C = 6 \\ B-2C = -4 \\ C = 2 \end{cases} \Rightarrow C=2,\ B=0,\ A=4$$

$$\int dx\left[\frac{4}{(x-1)^3} + \frac{2}{(x-1)}\right] = -\frac{2}{(x-1)^2} + 2\log|x-1| + c$$