TODAY'S LECTURE HAS NOT SO MANY SECRETS...

Limits of Sequences

Exercise 3.3 (v)

$$\lim_{n \to +\infty} \frac{\underbrace{2^{n+1}}_{\to\infty} + \underbrace{3^{n+1}}_{\to\infty}}{\underbrace{2^n}_{\to\infty} + \underbrace{3^n}_{\to\infty}}$$

$\lim_n a^n = \infty \quad (a>1)$

$\infty + \infty = \infty$, $\frac{\infty}{\infty}$ INDETERMINACY

$$\lim_{n \to +\infty} \frac{2^{n+1} + 3^{n+1}}{2^n + 3^n} = \lim_{n \to +\infty} \frac{3^{\cancel{n}+1}}{\cancel{3^n}} \frac{1 + \left(\frac{2}{3}\right)^{n+1}}{1 + \underbrace{\left(\frac{2}{3}\right)^n}_{\to 0}}$$

$= 3$

Remind!

$$\lim_n a^n \begin{cases} 0 & |a|<1 \\ 1 & \text{otherwise} \end{cases}$$

$\lim_n a^n = 0$ if $|a|<1$

Homework $\lim_n \sqrt[n]{a^n+b^n}$ (2.3 (i)) same idea

EXERCISE <u>3.3</u> (vi)

$$\lim_n a_n, \quad a_n = \underbrace{\left(\frac{n^2+1}{n^2-3n}\right)}_{\to 1}^{\overbrace{\frac{n^2-1}{2n}}^{\to +\infty}}$$

$$\lim_n \frac{\alpha_k n^k + \alpha_{k-1} n^{k-1} + \dots + 1}{\beta_i n^i + \dots + 1} \begin{cases} \dfrac{\alpha_k}{\beta_i} & (k=i) \\ 0 & (k<i) \\ \infty & (k>i) \ (\text{SIGN!}) \end{cases}$$

Indeterminacy 1^∞ $\longrightarrow$ sounds e

$$\lim_n \left(1+\frac{1}{n}\right)^n = e$$

I WANNA CAST a_n in this form

$$a_n = b_n^{c_n}$$

$$b_n = \frac{n^2+1}{n^2-3n} = 1 + \frac{n^2+1}{n^2-3n} - 1 \quad (\text{I need a } 1)$$

$$= 1 + \frac{n^2+1-n^2+3n}{n^2-3n} = 1 + \frac{3n+1}{n^2-3n}$$

$$a_n = \left(1 + \frac{3n+1}{n^2-3n}\right)^{\frac{n^2-1}{2n}}$$

$$= \underbrace{\left(1 + \frac{3n+1}{n^2-3n}\right)}_{\to\, e}{}^{\frac{n^2-3n}{3n+1} \frac{3n+1}{n^2-3n} \frac{n^2-1}{2n}}$$

(subsequence of that defines e)

$$\lim_n a_n = e^{\lim_n \left(\frac{3n+1}{n^2-3n} \cdot \frac{n^2-1}{2n}\right)}$$

$$\lim_n \frac{3n+1}{n^2-3n} \frac{n^2-1}{2n} = \lim_n \frac{3n^3+n^2-3n-1}{2n^3-6n}$$

$$= \lim_n \frac{\cancel{n^3}}{\cancel{n^3}} \frac{3 + \overset{\to 0}{\frac{1}{n}} - \overset{\to 0}{\frac{3}{n^2}} - \overset{\to 0}{\frac{1}{n^3}}}{2 - \underset{\to 0}{\frac{6}{n^2}}} = \frac{3}{2}$$

$$\lim_n a_n = e^{\frac{3}{2}}$$

EXERCISE 3.3 (iii)

INDETERMINACY $\infty - \infty$

$$\lim_n a_n = ? \qquad a_n = n\left(\underbrace{\sqrt{n^2+1}}_{x} - \underbrace{n}_{y}\right)$$

$$(x-y)(x+y) = x^2 - y^2$$

$$\Rightarrow x - y = \frac{x^2 - y^2}{x+y}$$

$$a_n = n\left(\frac{\cancel{n^2}+1 - \cancel{n^2}}{\sqrt{n^2+1} + n}\right)$$

$$= \frac{\cancel{n}}{\cancel{n}} \frac{1}{1 + \underbrace{\sqrt{1+\frac{1}{n^2}}}_{\downarrow \atop 1}} = \frac{1}{2}$$

Common WAY TO TREAT ROOT

EXERCISE 3.8

$$\lim_n a_n = ? \qquad a_n = \sum_{k=1}^{3n} \frac{1}{\sqrt{n^2+k^2}}$$

OBSERVE THAT :

$$a_n = \underbrace{\frac{1}{\sqrt{n^2+1}}}_{\text{LARGEST ONE}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+k}} + \cdots + \underbrace{\frac{1}{\sqrt{n^2+k}}}_{\text{Smallest one}}$$

$$\forall k \leq 3n \qquad \sqrt{n^2+k} \leq \sqrt{n^2+3n} \qquad (k \leq 3n)$$

$$\Rightarrow \qquad \frac{1}{\sqrt{n^2+3n}} \leq \frac{1}{\sqrt{n^2+k}}$$

Similarly $\qquad \dfrac{1}{\sqrt{n^2+k}} \leq \dfrac{1}{\sqrt{n^2+1}}$

We stuck into a_n

$$\underbrace{\sum_{k=1}^{3n} \frac{1}{\sqrt{n^2+3n}}}_{\text{does not depend on } k} \leq \underbrace{\sum_{k=1}^{3n} \frac{1}{\sqrt{n^2+k}}}_{a_n} \leq \underbrace{\sum_{k=1}^{3n} \frac{1}{\sqrt{n^2+1}}}_{\text{does not depend on } k}$$

$$\sum_{k=1}^{3n} \frac{1}{\sqrt{n^2+3n}} = \frac{1}{\sqrt{n^2+3n}} \underbrace{\sum_{k=1}^{3n} 1}_{3n} = \frac{3n}{\sqrt{n^2+3n}}$$

Similarly $\displaystyle\sum_{k=1}^{3n} \frac{1}{\sqrt{n^2+1}} = \frac{3n}{\sqrt{n^2+1}}$

So $\displaystyle\underbrace{\frac{3n}{\sqrt{n^2+3n}}}_{\to 3} \leq a_n \leq \underbrace{\frac{3n}{\sqrt{n^2+1}}}_{\to 3}$

$$\lim_n \frac{3n}{\sqrt{n^2+3n}} = \lim_n \frac{3}{\sqrt{1+\frac{3}{n}}} = 3$$

$\Rightarrow a_n \to 3$

EXERCISE 3.14

$$\begin{cases} x_{n+1} = \dfrac{x_n(1+x_n)}{1+2x_n} \\ x_1 = 1 \end{cases}$$

$x_2 = \dfrac{2}{3}$

$x_3 = \dfrac{2}{3}\left(1+\dfrac{2}{3}\right)\dfrac{1}{1+\frac{4}{3}}$

$= 2 \cdot \dfrac{5}{3} \cdot \dfrac{1}{7}$

a) Prove that $x_n > 0 \ \forall n \in \mathbb{N}$

b) // // x_n is monotonically decreasing

c) Calculate its limit

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a) $\longrightarrow$ Induction

$n = 1$, $x_1 = 1 > 0$ ✓

Hy $\exists n \in \mathbb{N} : \underbrace{x_n > 0}_{p(n)}$

$\Rightarrow \underbrace{x_{n+1} > 0}_{p(n+1)}$?

$$x_{n+1} = \frac{x_n(1+x_n)}{1+2x_n}$$

$x_n > 0$ (Hy)

$\Rightarrow x_n + 1 > 0$

$2x_n + 1 > 0$

$x_{n+1} = \dfrac{>0}{>0} > 0$

b) Monotonic decreasing ?

$\longrightarrow$ Definition

$$x_{n+1} - x_n = \frac{x_n(1+x_n)}{1+2x_n} - x_n$$

$$= \frac{x_n(1+x_n) - x_n - 2x_n^2}{1+2x_n}$$

$$= \frac{x_n^2 - 2x_n^2}{1+2x_n} = - \frac{x_n^2 {}^{>0}}{\underbrace{1+2x_n}_{>0}} < 0 \checkmark$$

c) $\lim_n x_n = L$

Why THE LIMIT EXIST?

a) $x_n > 0$ (LOWER BOUNDED)

b) x_n MON. DEC

$x_n \to L$, $x_{n+1} \to L$

$$L = \frac{L(1+L)}{1+2L} \Rightarrow L(1+2L) = L(1+L)$$

$$\Rightarrow L = 0$$

Exercise 1.8

Prove by Induction

$$n! < \left(\frac{n+1}{2}\right)^n \quad \forall n > 1$$

Hint $$2 < \left(1 + \frac{1}{n+1}\right)^{n+1} \sim e \quad \text{Mon. Inc.}$$

n = 2 $$2! = 2 < \left(\frac{2+1}{2}\right)^2 = \frac{3^2}{2^2} = \frac{9}{4} = 2.1$$

Hy $\exists\, n > 1 : \quad n! < \left(\frac{n+1}{2}\right)^n$ $\quad P(n)$

$$\Rightarrow \underbrace{(n+1)! < \left(\frac{n+2}{2}\right)^{n+1}}_{P(n+1)}$$

$$n! < \left(\frac{n+1}{2}\right)^n$$

$$(n+1)\,n! < \left(\frac{n+1}{2}\right)^n (n+1)$$ (MULTIPLY both side $(n+1)$)

QUESTION : why n is preserved? $(n+1) > 0$

$$(n+1)! < \frac{1}{2^n}(n+1)^{n+1}$$ (I need a 2^{n+1})

$$(n+1)! < 2\left(\frac{n+1}{2}\right)^{n+1}$$

I APPLY THE HINT

$$(n+1)! < 2\left(\frac{n+1}{2}\right)^{n+1} < \left(1+\frac{1}{n+1}\right)^{n+1}\left(\frac{n+1}{2}\right)^{n+1}$$

$$= \left(\frac{n+2}{n+1}\right)^{n+1}\left(\frac{n+1}{2}\right)^{n+1} = \left(\frac{n+2}{2}\right)^{n+1} \checkmark$$

$\Rightarrow \forall n \geq 1 \quad p(n)$ TRUE

EXERCISE 3.4 (iv)

$$\lim_{n \to +\infty} n^{-\frac{3}{n}} = \lim_{n \to +\infty} \left(\sqrt[n]{n}\right)^{-3} = 1^{-3} = 1$$

$$\lim_{n} \sqrt[n]{n} = 1$$

EXERCISE 3.4 (v)

$$\lim_{n \to +\infty} \frac{2^n}{n!} = ?$$

$$\frac{2^n}{n!} = \underbrace{\frac{2}{n} \cdot \frac{2}{(n-1)} \cdot - - - \cdot \frac{2}{3}}_{n-2} \cdot \frac{2}{2} \cdot 2$$

NOTE $\frac{2}{n} < \frac{2}{n-1} - - - \frac{2}{3}$

$$0 < \frac{2^n}{n!} < \underbrace{\left(\frac{2}{3}\right)^{n-2}}_{\downarrow \, 0} 2$$

$\lim_n a^n = 0 \ (|a|<1)$