Taylor Expansions

$$\underbrace{f(x)}_{f \text{ diff.}} = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{\text{Polynomial}} + \underbrace{\underset{x \to a}{o(x^n)}}_{\text{Negligible}}$$

$P_{n,a}$

$$\lim_{x \to a} \frac{o(x^n)}{x^n} = 0$$

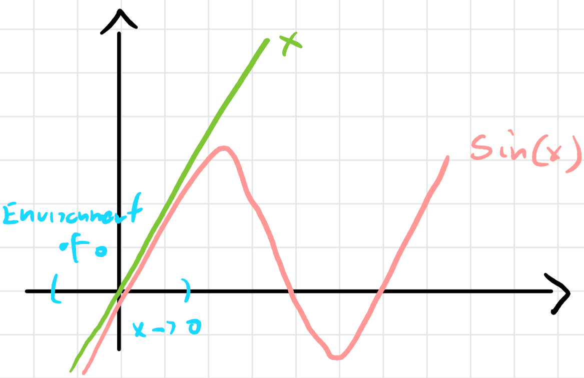
Ex

$$e^x = 1 + x + \frac{x^2}{2} + \underset{x \to 0}{o(x^2)}$$

$$\sin(x) = x + \underset{x \to 0}{o(x)} = x + \underset{x \to 0}{o(x^2)}$$ (Second ord. term is 0)

$$\cos(x) = 1 - \frac{x^2}{2} + \underset{x \to 0}{o(x^2)} = 1 - \frac{x^2}{2} + \underset{x \to 0}{o(x^3)}$$ (Third order term 0)

WHAT DOES IT MEAN?



$O(x^2)$ PRICE TO PAY

↳ COARSE GRAINED INFO.

In GENERAL

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_{n,a}(x)$$

↳ LAGRANGE

↳ CAUCHY

LAGRANGE $$R_{n,a}(x) = \frac{f^{(n+1)}((1-\vartheta)a+\vartheta x)}{(n+1)!}(x-a)^{n+1} \quad \vartheta \in (0,1)$$

Ex EVALUATE ERROR $\underbrace{\text{APPROXIMATION}}$

$\sin(x)$

$a = 0$

$n = 4$

$$\sin(x) = x - \frac{x^3}{6} + R_{4,0}(x)$$

$$R_{4,0} = \frac{f^{(5)}(\vartheta_x)}{5!} x^5 \qquad \vartheta \in (0,1)$$

$$5! = 120 \quad , \quad \frac{d^5}{dx^5} \sin(x)\Big|_{\vartheta_x} = \cos(\vartheta_x)$$

$$R_{4,0}(x) = \frac{\cos(\vartheta_x)}{120} x^5 \qquad \vartheta \in (0,1)$$

$\hookrightarrow$ $x = 0 \Rightarrow R_{4,0} = 0$ (APPROXIMATION IS EXACT)

$\hookrightarrow$ $x = 0.1$?

$x = 0.1 = 10^{-1}$ LEADING ESTIMATE

$x^3 = 10^{-3}$, $\frac{x^3}{6} \approx \frac{10^{-3}}{5} = \frac{2}{10} 10^{-3} = 2\,10^{-4}$

$\sin(0.1) \approx 10^{-1} + 2\,10^{-4} + R_{4,0}$

$|R_{4,0}| = \frac{|\cos \vartheta x|}{120} |x|^5 \leq \frac{|x|^5}{120} \approx \frac{10^{-5}}{100} = 10^{-7}$

$\hookrightarrow \vartheta \in (0,1)$

ERROR NOT SO BAD 10^{-7}

QUESTION $n \to +\infty$

↳ ORDER EXPANSION

$$f(x) = \underbrace{\sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\text{TAYLOR SERIES}} + \underbrace{\lim_{k} \cancel{R_{k,a}}(x)}_{\text{SEQUENCE LIMIT}}$$

$$f(x) = \underbrace{\sum_{n=0}^{\infty} a_n (x-a)^n}_{\text{POWER SERIES}} \qquad \underbrace{a_n = \frac{f^{(n)}(a)}{n!}}_{\text{SERIES TERM}}$$

↳ REPRESENTS FUNCTION

↳ AS FAR IT CONVERGES

$$\underbrace{\sum_n a_n (x-a)^n < \infty}_{\text{ABSOLUTE CONVERGENCE}} \quad ?$$

$$\underbrace{\sum_n |a_n| \, |x-a|^n}_{\text{ROOT TEST}}$$

$$\lim_{n \to +\infty} \sqrt[n]{|a_n|} \, \underbrace{|x-a|}_{\text{depend on } x} < 1$$

$$|x-a| < \rho \quad \text{CONVERGENCE RADIUS}$$

$$\boxed{\frac{1}{\rho} = \lim_n \sqrt[n]{|a_n|}} = \lim_n \frac{a_{n+1}}{a_n}$$

↳ STOLZ

Ex $f(x) = e^x$, $a = 0$

TAYLOR SERIES $\sum_{n=0}^{\infty} \frac{f^{(n)}(n)}{n!} x^n$

$$\forall n \in \mathbb{N} \quad f^{(n)}(x) = e^x$$
$$f^{(n)}(0) = 1$$

$$e^x = \sum_{n=0}^{k} \frac{x^n}{n!} + R_{k,0}(x)$$

$$R_{k,0} = \frac{1}{(k+1)!} f^{(k+1)}(\vartheta x)\, x^{k+1} \quad \vartheta \in (0,1)$$
$$= \frac{e^{\vartheta x}\, x^{k+1}}{(k+1)!}$$

$$\left.\begin{array}{ll} x > 0 & 1 < e^{\vartheta x} < e^x \\ x < 0 & e^x < e^{\vartheta x} < 1 \end{array}\right\} \Rightarrow e^{\vartheta x} < \max\{1, e^x\}$$

$$|R_{k,0}| < \max\{1, e^x\} \frac{|x|^n}{n!} \xrightarrow[n \to +\infty]{} 0$$

$\hookrightarrow e^x$ HOD. INC.

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}} = \sum_n a_n x^n, \quad a_n = \frac{1}{n!}$$

$$\frac{1}{\rho} = \lim_{n \to +\infty} \frac{a_{n+1}}{a_n} = \lim_{n \to +\infty} \frac{n!}{(n+1)!} = \lim_{n \to +\infty} \frac{1}{n} = 0$$

$\boxed{\rho = \infty}$ FORMULA HOLDS $\forall x \in \mathbb{R}$

TAYLOR SERIES IS UNIQUE

Ex

$$\underbrace{f(x) = \sin(x) \quad , \quad a = 0}$$

↳ TAYLOR SERIES

$$f^{(2n)}(0) \propto \sin(0) = 0 \quad \forall n \in \mathbb{N}$$

$$f^{(2n+1)}(0) = (-1)^n \cos(0) = (-1)^n \quad \forall n \in \mathbb{N}$$

$$\boxed{\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}} + \lim_{k} R_{k,0}$$

$$|R_{k,0}| = \frac{|\cos \vartheta x| \, |x|^{2k+1}}{(2k+1)!} \leq \frac{|x|^{2k+1}}{(2k+1)!} \xrightarrow[k \to \infty]{} 0$$

$\int = \infty$ (PREVIOUS EXAMPLE)

OSS f ODD $\Rightarrow$ ODD EXP.

DERIVATIVE

$$\frac{d}{dx} \sin(x) = \boxed{\cos(x)} = \sum_{n=0}^{\infty} (-1)^n (2n+1) \frac{x^{2n}}{(2n+1)!}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}$$

OBS

- $\sin(x)$ ODD $\Rightarrow$ ODD EXP
- f
  - ODD $\longrightarrow$ f' EVEN
  - EVEN $\longrightarrow$ f' ODD

<u>Ex</u> $f(x) = e^{ix}$, $i^2 = -1$, $a = 0$

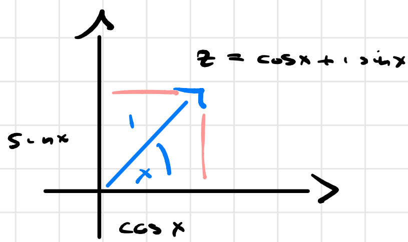
$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \underbrace{\sum_{k=0}^{\infty} \frac{(ix)^{2k}}{(2k)!}}_{\text{EVEN}} + \underbrace{\sum_{k=0}^{\infty} \frac{(ix)^{2k+1}}{(2k+1)!}}_{\text{ODD}}$$

$$(i)^{2k} = (-1)^k, \quad (i)^{2k+1} = (-1)^k i$$

$$e^{ix} = \underbrace{\sum_{k}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}}_{\cos x} + i \underbrace{\sum_{k}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}}_{\sin x}$$

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos x - i \sin x$$



$$\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} \cos x \\ \sin x \end{pmatrix} = \begin{pmatrix} e^{ix} \\ e^{-ix} \end{pmatrix}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

LOOK LIKE?

$$\cosh x = \frac{e^{x} + e^{-x}}{2}, \quad \sinh x = \frac{e^{x} - e^{-x}}{2}$$