Ex. 4.6 (i)

$$\underbrace{\text{CALCULATE SUM}}_{}\quad \sum_{n=0}^{\infty} \underbrace{\frac{3^{n+1}-2^{n-3}}{4^n}}_{a_n}$$

I WANT A NUMBER

→ WE KNOW ONLY for THREE SERIES

- Geometric
- Arithmetic - Geometric
- Telescopic

⟶ Reduce my series to one of THESE

$$\sum_{n=0}^{\infty} a_n = 3\underbrace{\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n}_{\substack{\text{Geometric} \\ x=\frac{3}{4}}} - \frac{1}{2^3}\underbrace{\sum_{n=0}^{\infty}\left(\frac{2}{4}\right)^n}_{\substack{\text{Geometric} \\ x=\frac{1}{2}}}$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x|<1$$

$$\sum_{n=0}^{\infty} a_n = \frac{3}{1-\frac{3}{4}} - \frac{1}{8}\,\frac{1}{1-\frac{1}{2}} = 3\cdot 4 - \frac{2}{8} = 12 - \frac{1}{4} = \frac{47}{4}$$

QUESTION

$$\sum_{n=r}^{\infty} a_n = ?$$

$\hookrightarrow r > 0$

$$\sum_{n=r}^{\infty} x^n = \frac{x^r}{1-x} \qquad |x| < 1$$

Homework

Ex 4.6 (iii)

$$\sum_{n=0}^{\infty} \frac{4n+1}{3^n} = \sum_{n=0}^{\infty} a_n \quad (\text{CALCULATE SUM})$$

$$\sum_{n=0}^{\infty} a_n = 4 \underbrace{\sum_{n=0}^{\infty} n \left(\frac{1}{3}\right)^n} + \underbrace{\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n}$$

WHAT IS THAT?

Geometric Series

$x = \frac{1}{3}$

→ ARITHMETIC - Geometric Series

$$\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2} \qquad |x| < 1$$

$$\sum_{n=0}^{\infty} a_n = 4 \, \frac{1}{3} \, \frac{1}{\left(1-\frac{1}{3}\right)^2} + \frac{1}{1-\frac{1}{3}}$$

$$= \frac{4}{3} \, \frac{3^2}{2^2} + \frac{3}{2} = 3\left(1+\frac{1}{2}\right)$$

$$= 3 \cdot \frac{3}{2} = \frac{9}{2}$$

Ex 4.6 (vi)

Sum $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \log\left[\frac{n(n+2)}{(n+1)^2}\right]$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \log\left[\frac{n}{n+1} \; \frac{n+2}{n+1}\right]$$

$$= \underbrace{\sum_{n=1}^{\infty} \log\left(\frac{n}{n+1}\right)}_{u_n} - \underbrace{\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n+2}\right)}_{u_{n+1}}$$

$$\log\left(\frac{n+2}{n+1}\right) = -\log\left(\frac{n+1}{n+2}\right)$$

Telescopic Series

$$\sum_{n=1}^{\infty} a_n = u_1 - \lim_n u_n$$

$$u_1 = \log\left(\frac{1}{2}\right) = -\log 2$$

$$\lim_n u_n = \lim_n \log\left(\frac{n+1}{n+2}\right) = \log 1 = 0$$

$$\sum_n a_n = -\log 2$$

$\underline{\text{Ex}}$ 4.1 (v)

Determine conv. CHARACTER

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^2+n}$$

$$\underbrace{a_n \geq 0 \quad \forall n \geq 1}_{}$$

Positive $\longrightarrow$ Lot of CRITERIA

$$\frac{|\sin(n)|}{n^2+n} \leq \underbrace{\frac{1}{n^2+n}}_{b_n} \sim \frac{1}{n^2}$$

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty}_{\text{Riemann}} \implies \sum_n b_n < \infty$$ (LIMIT COMP)

$$\sum_n b_n < \infty \implies \sum_n \frac{|\sin(n)|}{n^2+n} < \infty$$ (COMP. TEST)

Ex 4.1 xvi

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}$$

- $a_n > 0$

  ↳ $n \geq 2 \quad \log n > 0$

- MON. DEC.

⟶ CAUCHY

$$\sum_n a_n < \infty \iff \sum_k 2^k a_{2^k} < \infty$$

$$\sum_{k=1}^{\infty} 2^k a_{2^k} = \sum_{k=1}^{\infty} \left[\frac{2^k}{(\log 2^k)^{\log 2^k}} \right] =$$

$\log 2^k = k \log 2$

$$= \sum_{k=1}^{\infty} \frac{2^k}{(k \log 2)^{(k \log 2)}} =$$

$(ab)^c = a^c b^c$

$$= \sum_{k=1}^{\infty} \frac{2^k}{(k^{\log 2})^k (\log 2^{\log 2})^k} =$$

$\frac{a^c}{b^c} = \left(\frac{a}{b}\right)^c$

$$= \sum_{k=1}^{\infty} \left[\left(\frac{1}{k^{\log 2}} \right) \left(\frac{2}{\log 2^{\log 2}} \right) \right]^k = \sum_{k} c_k$$

EXPONENTS ANNOYING $\longrightarrow$ ROOT

$$\lim_{k \to +\infty} \sqrt[k]{c_k} = \lim_{k \to +\infty} \underbrace{\frac{1}{k^{\log 2}}}_{\to 0} \underbrace{\frac{2}{(\log 2)^{\log 2}}}_{\text{CONSTANT}} = 0 < 1$$

$$\sum_{k=1}^{\infty} 2^k a_{2^k} < \infty \qquad \text{(ROOT)}$$

$$\Rightarrow \sum_{n} a_k < \infty \qquad \text{(CAUCHY)}$$

Ex 4.3 (ii)

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^n}{\underbrace{\alpha^n n!}_{}} \qquad (\alpha > 0)$$

$> 0 \to$ QUOTIENT

$\to$ Helps with RATIO

$$\lim_n \frac{a_{n+1}}{a_n} = \lim_n \frac{(n+1)^{\cancel{n+1}}}{\alpha^{\cancel{n+1}} \cancel{(n+1)}\cancel{!}} \frac{\cancel{\alpha^n}\cancel{n!}}{n^n}$$

$$= \lim_n \frac{1}{\alpha}\left(1 + \frac{1}{n}\right)^n = \frac{e}{\alpha} \equiv L$$

$\alpha > e \implies L < 1 \qquad \sum_n a_n < \infty$

$\alpha < e \implies L > 1 \qquad \text{"} \quad = \infty$

$\alpha = e \qquad ?$

$\alpha = e$

$$a_n^{(\alpha=e)} = \frac{n^n}{e^n n!} \sim \left(\frac{n}{e}\right)^n \frac{1}{\sqrt{2\pi n}\, n^n e^{-n}} = \frac{1}{\sqrt{2\pi n}}$$

$\hookrightarrow$ STIRLING

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n}$$

$$\sum_n \frac{1}{\sqrt{2\pi n}} = \infty \implies \sum_n \alpha_n^{(\alpha=e)} = \infty$$

$\hookrightarrow$ LIMIT COMP. TEST

Ex 4.5 (ii)

$$\sum_n a_n = \sum_n \sin\left(n\pi + \frac{1}{n}\right)$$

$$a_n = \underbrace{\sin(n\pi)}_{=\,0} \cos\left(\frac{1}{n}\right) + \underbrace{\cos(n\pi)}_{=\,(-1)^n} \sin\left(\frac{1}{n}\right)$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\sum_n a_n = \sum_n (-1)^n \sin\left(\frac{1}{n}\right)$$ ALTERNATING

ABSOLUTELY CONV? NO

$$|a_n| = \left|\sin\left(\frac{1}{n}\right)\right| \sim \frac{1}{n}$$

$$\sum_n \frac{1}{n} = \infty \Rightarrow \sum_n |a_n| = \infty$$ (LIM. COMP. TEST)

CONDITIONAL CONVERGENT ? Leibniz

- $\sin\left(\frac{1}{n}\right)$ Dec.

- $\lim_n \sin\left(\frac{1}{n}\right) = 0$

$\Rightarrow \sum_n a_n$ C.C.