Ex 10.14

Part (i)

$$\lim_{x\to 0} \frac{1}{x^3}\left(\int_0^{x^2} e^{t^2}\,dt - x\right)$$

OSS $\underbrace{x^3 \to 0, \quad \int_0^{x^2} \cdots \to 0}_{\text{Hôpital}}$

3 TECHNIQ.
- Hô
- Eq. Rel
- Taylor

$$\lim_{x\to 0} \frac{e^{x^2}-1}{3x^2} = \lim_{t\to 0} \frac{e^t - 1}{3t} = \frac{1}{3}$$

↳ $t = x^2$

↳ $\frac{e^t - 1}{t} \to 1$

Similarly $e^t = 1 + t + o(t)$

↳ which order?

→ 1 because of Den.

$$\lim_{t\to 0} \frac{\cancel{1} + t + o(t) - \cancel{1}}{3t} = \lim_{t\to 0} \frac{1}{3}\left(1 + \frac{o(t)}{t}\right) = \frac{1}{3}$$

Ex (EXAM)

$$\lim_{x\to 0} \underbrace{\frac{1}{x^6}}_{\to 0}\underbrace{\left(2\int_0^x \frac{\sin(t^2)}{t}\,dt - x^2\right)}_{\to 0}$$

$\longrightarrow$ HÔPITAL

$$\lim_{x\to 0} \frac{1}{6x^5}\left(2\frac{\sin(x^2)}{x} - 2x\right)$$

$$= \lim_{x\to 0} \frac{1}{6x^5}\left(\frac{2\sin(x^2) - 2x^2}{x}\right)$$

$$= \lim_{x\to 0} \frac{\sin(x^2) - x^2}{3x^6} = \lim_{z\to 0} \frac{\sin(z) - z}{3z^3}$$

$\hookrightarrow z = x^2$

HÔPITAL $\to$ 3rd DERIVATIVE

TAYLOR

$$\sin(z) = z - \frac{z^3}{6} + o(z^3)$$

$$\lim_{z\to 0} \frac{1}{3z^3}\left(\cancel{z} - \frac{z^3}{6} + o(z^3) - \cancel{z}\right) = \lim_{z\to 0} -\frac{1}{18}\left(1 + \frac{o(z^3)}{z^3}\right)$$

$$= -\frac{1}{18}$$

Ex 10.6 (i)

$$F(x) = \int_{x^2}^{x^3} \frac{e^t}{t}\, dt$$

$$F'(x) = ?$$

$$H(x) = \int_{g_1(x)}^{g_2(x)} f(t)\, dt$$

$$H'(x) = f(g_2(x))\, g_2'(x) - f(g_2(x))\, g_1'(x)$$

$$f(t) = \frac{e^t}{t}, \quad g_2(x) = x^3, \quad g_1(x) = x^2$$

$$g_2'(x) = 3x^2, \quad g_1'(x) = 2x$$

$$F'(x) = \frac{e^{x^3}}{x^3}(3x^2) - \frac{e^{x^2}}{x^2}(2x)$$

$$= \frac{1}{x}\left(3e^{x^3} - 2e^{x^2}\right)$$

Ex 10.6 (iii)

$$F(x) = \int_3^{\int_1^x \sin^3(t)\,dt} \frac{dt}{1+\sin^6(t)+t^2}$$

$$F'(x) = \left[1 + \left(\sin \int_1^x \sin^3(t)\,dt\right)^6 + \left(\int_1^x \sin^3(t)\,dt\right)^2\right]^{-1} \cdot \sin^3(x) + \underbrace{0}$$

$$\hookrightarrow \frac{d}{dx}(3) = 0$$

$$F'(x) = \frac{\sin^3(x)}{\left[1 + \left(\sin \int_1^x \sin^3 t\,dt\right)^6 + \left(\int_1^x \sin^3 t\,dt\right)^2\right]}$$

Ex 10.8

$$f(x) = \int_0^{x-1} \left(e^{-t^2} - e^{-2t}\right) dt$$

minimum and maximum $[1, \infty)$

$$f'(x) = e^{-(x-1)^2} - e^{-2(x-1)} = 0$$

$$\Rightarrow (x-1)^2 = 2(x-1) \quad \text{(EXP INJ.)}$$

$$\Rightarrow x^2 + 1 - 2x = 2x - 2$$

$$\Rightarrow x^2 - 4x + 3 = 0$$

$$x_{\pm} = \frac{4 \pm \sqrt{16-12}}{2}$$

$$\frac{4+2}{2} = 3$$

$$\frac{4-2}{2} = 1$$

$$f'(x) > 0 \iff e^{-(x-1)^2} > e^{-2(x-1)}$$

$$-(x-1)^2 > -2(x-1)$$

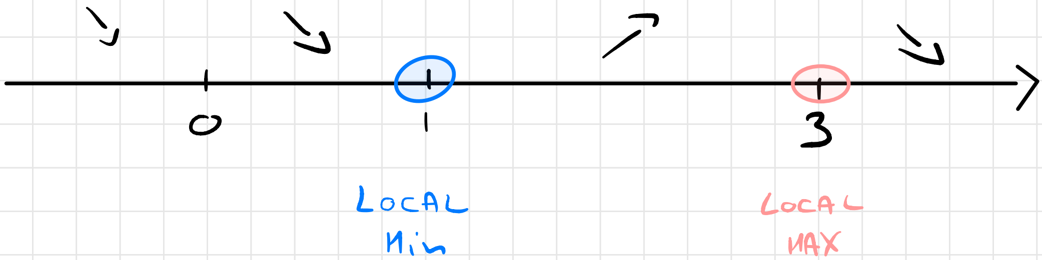
$$(x-1)^2 < 2(x-1)$$

$$x^2+1-2x-2x+2 < 0$$

$$x^2-4x+3 < 0$$

$1 < x < 3$, $f'(x) > 0$

$x < 1$ o $x > 3$, $f'(x) < 0$



Ex 10.9

GIVEN THE EQUATION

$$\int_0^x e^{t^2}\,dt = 1$$

UNIQUE SOLUTION $[0,1]$

$$F(x) = \int_0^x e^{t^2}\,dt - 1$$

F CONTINUOUS

$$F(0) = \int_0^0 e^{t^2}\,dt - 1 = -1 < 0$$

$$F(1) = \int_0^1 e^{t^2}\,dt - 1 \geq \underbrace{\int_0^1 dt}_{1} - 1 = 0$$

$$f \leq g \Rightarrow \int f \leq \int g$$

$$\forall x \in (0,1) \,, \quad 1 \leq e^{t^2}$$

$f(0) = -1 < 0$

$f(1) \geq 0$

$f'(x) = e^{x^2} > 0 \quad \forall x \in (0,1)$

$\Rightarrow \exists! \; c \in (0,1) : f(c) = 0$

Ex 10.10

$f(x) > 0 \quad \forall x \in [0,1] \in I$

$$F(x) = 2\int_0^x f(t)\,dt - \int_x^1 f(t)\,dt$$

How many solution in I ?

$$F(0) = 2\int_0^0 f(t)\,dt - \int_0^1 f(t)\,dt = 0 \underbrace{\int_0^1 f}_{>0} < 0$$

$$F(1) = 2\int_0^1 f(t)\,dt - \int_1^1 f(t)\,dt = 2\int f > 0$$

$$F'(x) = 2f(t) + f(t) = 3f(t) > 0$$

$$-\frac{d}{dx}\int_x^1 f =$$

$$= \frac{d}{dx}\int_1^x f = f$$

$\Rightarrow \exists!\, c \in I : F(c) = 0$

Ex 10.16

$$f(x) = \int_0^{x^2} \frac{\sin(t)}{t}\, dt$$

(1) TAYLOR SERIES of f

$$\boxed{\sin(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!}}$$

$\int = \infty$

$$\frac{\sin(t)}{t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^{x^2} t^{2n}\, dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left. \frac{t^{2n+1}}{(2n+1)} \right|_0^{x^2}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{x^{4n+2}}{2n+1}$$

(ii) $\lim_{x \to 0} \dfrac{f(x)}{1 - \cos(x)}$

$$f(x) = x^2 + o(x^2) \quad \text{(FIRST TERM)}$$

$$1 - \cos(x) = \frac{x^2}{2} + o(x^2)$$

$$\lim_{x \to 0} \frac{x^2 + o(x^2)}{\frac{x^2}{2} + o(x^2)} = \lim_{x \to 0} \frac{1 + \frac{o(x^2)}{x^2}}{\frac{1}{2} + \frac{o(x^2)}{x^2}} = 2$$

(iii) $\displaystyle\sum_{n=1}^{\infty} f\left(\frac{1}{n}\right)$ CONV.

$$f\left(\frac{1}{n}\right) = \frac{1}{n^2} + o\left(\frac{1}{n^2}\right)$$

$$\sum_n f\left(\frac{1}{n}\right) \sim \sum_n \frac{1}{n^2} < \infty$$

(LIMIT COMP. TEST)

Ex 10.3 (i) (RIEMANNS SUM)

$$\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2}$$

THIS IS NOT A SERIES BECAUSE n IS ALSO in the TERM

$$\lim_n \sum_{k=1}^{n} \frac{1}{n} \frac{1}{1+\left(\frac{k}{n}\right)^2} = \lim_n S(f, P_n)$$

$$f(x) = \frac{1}{1+x^2},$$ P_n PARTITIONS of $[0,1]$ in n equally SPACED intervals

$$\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2} = \int_0^1 \frac{dx}{1+x^2} = \arctan(x)\Big|_0^1$$

$$= \frac{\pi}{4}$$

Ex 10.3 (iii)

$$\lim_{n} \sum_{k=1}^{n-1} \frac{1}{\sqrt{n^2-k^2}} = \lim_{n\to+\infty} \sum_{k=1}^{n-1} \frac{1}{n} \frac{1}{\sqrt{1-\left(\frac{k}{n}\right)^2}}$$

$$= \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \arcsin(1) = \frac{\pi}{2}$$

Ex

PROVE THAT $\quad f$ ODD $\Rightarrow \displaystyle\int_{-a}^{+a} f = 0$

$$\underbrace{\int_{-a}^{+a} f(x)\,dx}_{I} = -\int_{+a}^{-a} f(-t)\,dt = \underbrace{\int_{-a}^{+a} f(-t)\,dt}_{-I}$$

$\hookrightarrow \quad t = -x$

$dt = -dx$

$\pm a \mapsto \mp a$

$$\Rightarrow I = -I \Rightarrow 2I = 0 \Rightarrow I = 0$$