Ex SKETCH $f(x) = x^2 e^{-2x}$

$D_f = \mathbb{R}$,

$f(x) \geq 0 \quad \forall x \in D_f$

$\hookrightarrow f(0) = 0$

$$\lim_{x \to +\infty} f(x) = \underbrace{\lim_{x \to +\infty} \frac{x^2}{e^{2x}}}_{\text{HÔPITAL}}$$

$$\lim_{x \to +\infty} \frac{2}{4e^{2x}} = 0$$ (Horizontal As.)

$$\underbrace{\lim_{x \to -\infty} x^2 e^{-2x}}_{} \to \infty \cdot \infty = \infty$$

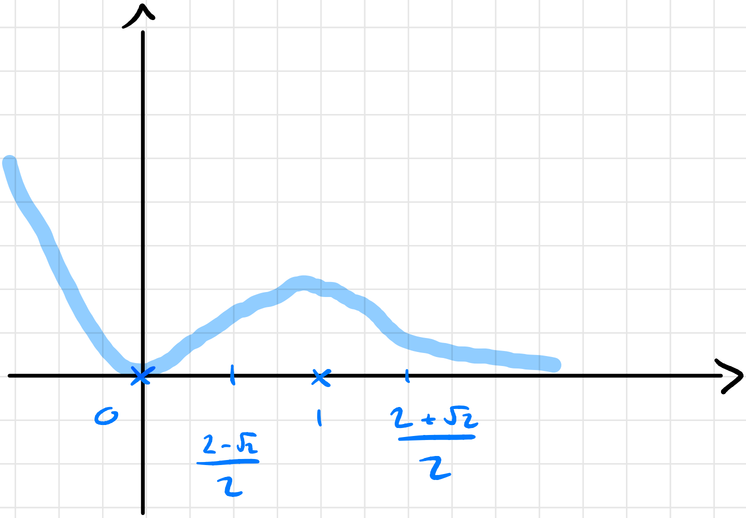
$$f(x) \underset{x \to -\infty}{\sim} x^2$$

$$f'(x) = 2x e^{-2x} - 2x^2 e^{-2x}$$
$$= 2x(1-x)e^{-2x}$$

$$f'(x) = 0 \quad \begin{cases} x = 0 \\ x = 1 \end{cases}$$

$$f''(x) = 2(1-x)e^{-2x} - 2x e^{-2x}$$
$$- 4x(1-x)e^{-2x}$$
$$= 2e^{-2x}[(1-x) - 2(1-x)x - x]$$
$$= 2e^{-2x}[1 - x - 2x + 2x^2 - x]$$
$$= 2e^{-2x}[2x^2 - 4x + 1]$$

$$x_{\pm} = \frac{4 \pm \sqrt{16-8}}{2} = \frac{2 \pm \sqrt{2}}{2}$$


0
$\frac{2-\sqrt{2}}{2}$
1
$\frac{2+\sqrt{2}}{2}$

Ex SKETCH $f(x) = x^2 e^{1/x}$

$$D_f = \mathbb{R} \setminus \{0\}$$

$$\underbrace{f(x) > 0}\quad \forall x \in D_f$$

$\hookrightarrow f(0) = 0$

$$\underbrace{\lim_{x \to \pm\infty} f(x) = +\infty}$$

$$f(x) \underset{x \to \pm\infty}{\sim} x^2$$

$$\lim_{x \to 0^+} f(x) = \lim_{t \to +\infty} \frac{e^t}{t^2} = +\infty$$

$\hookrightarrow$ HÔPITAL TWICE

$$\lim_{x \to 0^-} \quad // \quad = \lim_{t \to -\infty} \frac{\overbrace{e^t}^{\to 0}}{\underbrace{t^2}_{\frac{1}{+\infty}}} = 0$$

$$f'(x) = 2xe^{1/x} - \frac{x^2 e^{1/x}}{x^2} = (2x-1)e^{1/x}$$

$$x < \frac{1}{2}, \quad f'(x) < 0$$

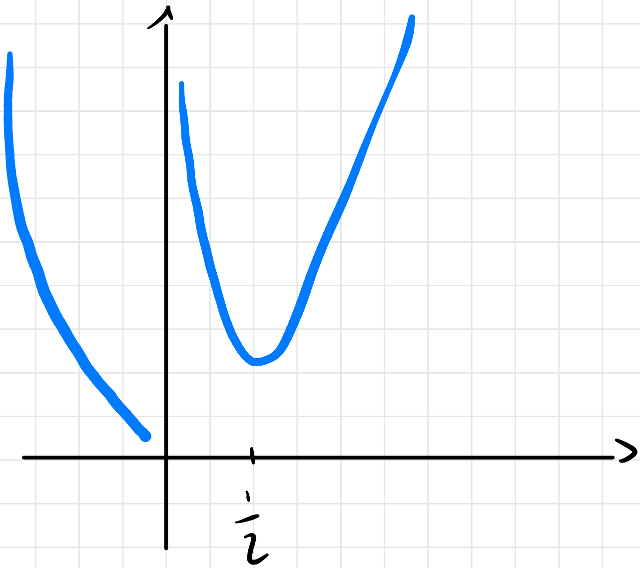
$$x > \frac{1}{2}, \quad f'(x) > 0$$

$$f''(x) = 2e^{\frac{1}{x}} - \frac{1}{x^2}(2x-1)e^{\frac{1}{x}}$$

$$= \frac{2x^2 - 2x + 1}{x^2} e^{\frac{1}{x}}$$

$$x_\pm = \frac{2 \pm \sqrt{4-8}}{2} \notin \mathbb{R}$$

$$f''(x) > 0 \quad \forall x \in D_f$$


$\frac{1}{2}$

Ex 8.31 (vii)

SKETCH $f(x) = \dfrac{x}{\log x}$

$D_f = \{ x \in \mathbb{R} : x > 0 , x \neq 1 \}$
$\equiv (0, +\infty) - \{1\}$

CONT/DIFF ? $\forall x \in D_f$

SYMM? NO

SIGN
$x > 1 \quad f(x) > 0$
$x < 1 \quad // \quad < 0$

$$\lim_{x \to 1^\pm} f(x) = \pm \infty$$

$$\lim_{x \to +\infty} // = \underbrace{+ \infty}_{}$$

↳ How?

$$f'(x) = \frac{\log x - 1}{(\log x)^2}$$

$$x \geq e \Rightarrow f(x) \geq 0$$

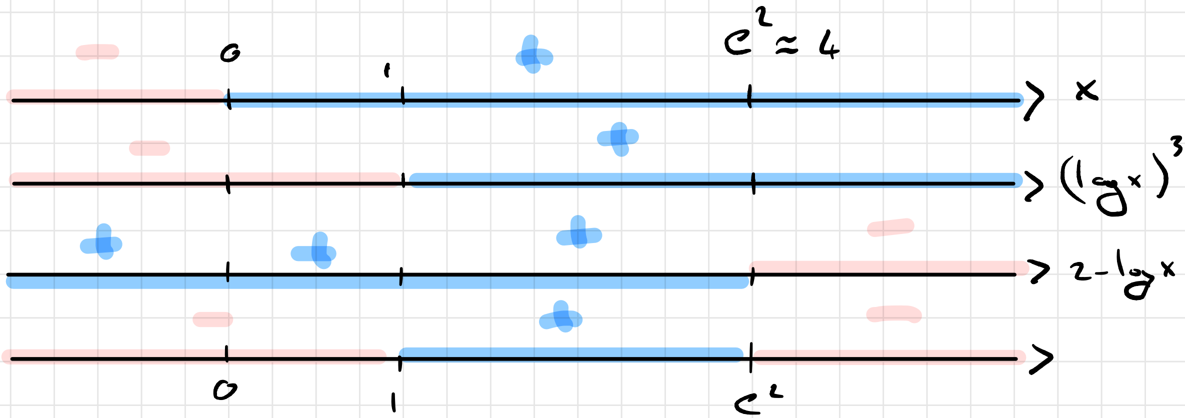
$$\text{〃} < \text{〃} \Rightarrow \text{〃} < 0$$

$$f''(x) = \frac{1}{(\log x)^4}\left[\frac{1}{x}(\log x)^2 - \frac{2}{x}\log x\,(\log x - 1)\right]$$

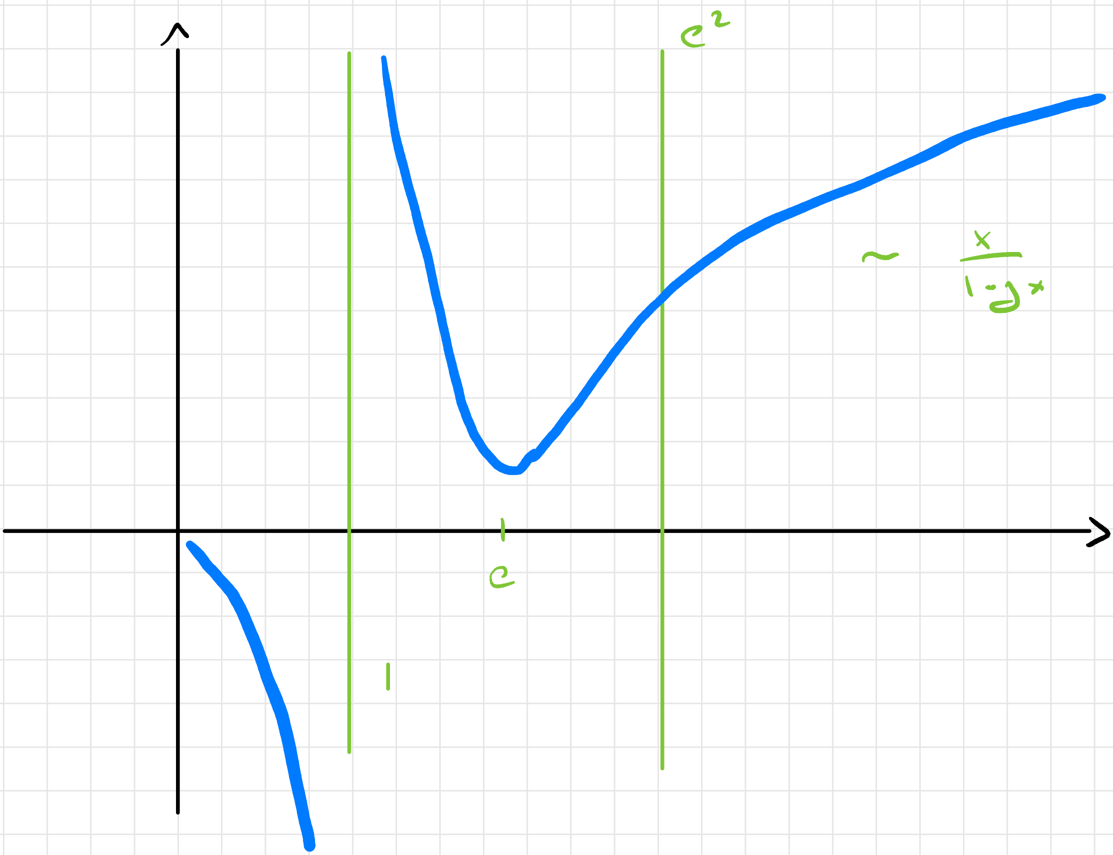
$$= \frac{1}{(\log x)^3}\,\frac{1}{x}\left[\log x - 2\log x + 2\right]$$

$$= \frac{1}{x}\,\frac{2 - \log x}{(\log x)^3}$$

sign $f''(x) = \frac{1}{x} \frac{2-\log x}{(\log x)^3}$



$f''(x) > 0$



Ex 7.25 (v)

$$\underbrace{x^x = 2}_{}$$ How many solutions in $[1, \infty)$

log $\hookrightarrow$ $$\underbrace{x \log x - \log 2}_{f(x)} = 0$$

$f(1) = -\log 2 < 0$

$\lim_{x \to +\infty} f(x) = +\infty$

AT LEAST ONE SOLUTIONS

$f'(x) = \log x + 1 > 0$, $\forall x > \frac{1}{e} < 1$

f MON INC $\forall x > 1$

$\Rightarrow$ 1 SOLUTION

$\underline{\text{Ex}}$ 8.18 (v)

$$\sum_{n=0}^{\infty} (3-2x)^n = \sum_{n=1}^{\infty} (-2)^n \left(x - \frac{2}{3}\right)^n$$

$$\frac{1}{\rho} = \lim_n \sqrt[n]{|2|^n} = 2 \implies \rho = \frac{1}{2}$$

$$\forall x \in \left(\frac{2}{3} - \frac{1}{2}, \frac{2}{3} + \frac{1}{2}\right)$$

$$x : \left|x - \frac{2}{3}\right| < \frac{1}{2}$$

$$x - \frac{2}{3} = \frac{1}{2} = \rho$$

$$\sum_n (-2)^n \frac{1}{2^n} = \sum_n (-1)^n \underbrace{2^{n-n}}_{1} \quad \text{non conv.}$$

Ex

$$f(x) = \frac{e^x}{1-x}$$

$$D_f = \mathbb{R} - \{1\}$$

$$f(x) > 0 \quad \forall x < 1$$

$$< 0 \qquad\qquad >$$

$$\lim_{x \to \pm 1} f(x) = \mp \infty$$

BEHAVIOUR at $\pm\infty$

$$\frac{e^x}{1-x} = \frac{1}{x} \frac{e^x}{\frac{1}{x} - 1} \xrightarrow[x \to -\infty]{} \frac{1}{x}$$

$$\frac{e^x}{1-x} \underset{x \to -\infty}{\sim} \frac{1}{x}$$

$$x \to +\infty \qquad f(x) \sim -\frac{e^x}{x}$$

$$f'(x) = \frac{e^x(1-x) + e^x}{(1-x)^2} =$$

$$= \frac{e^x}{(1-x)^2}(2-x) = \frac{h}{g}$$

$f'(x) > 0 \quad \forall x < 2$

$\quad // \quad < \quad // \quad\quad // \quad > \quad //$

$$h' = e^x(2-x) - e^x = e^x(1-x)$$

$$g' = -2(1-x)$$

$$h'g = e^x(1-x)^3$$

$$hg' = -2(1-x)(2-x)e^x$$

$$h'g - hg' = e^x(1-x)[x^2 + 1 - 2x - 2x - 4]$$

$$f''(x) = \frac{e^x(x^2 - 4x - 5)}{(1-x)^3}$$