W1 - SETS of NUMBERS

$\mathbb{N}$ ELEMENTARY

$\mathbb{R}$ RICHEST

↳ EXCEPT $\mathbb{C}$

W2 - REAL FUNCTIONS

$$f : A \subset \mathbb{R} \to \mathbb{R}$$

W3 - SEQUENCES

$$f : \mathbb{N} \to \mathbb{R}$$

<u>def</u> Sequence / Succession

$$f : \mathbb{N} \longrightarrow \mathbb{R}$$

INPUT $\longrightarrow n \longmapsto a_n$ → real number a which depends on n

$\uparrow$ OUTPUT

very close to $y = f(x)$

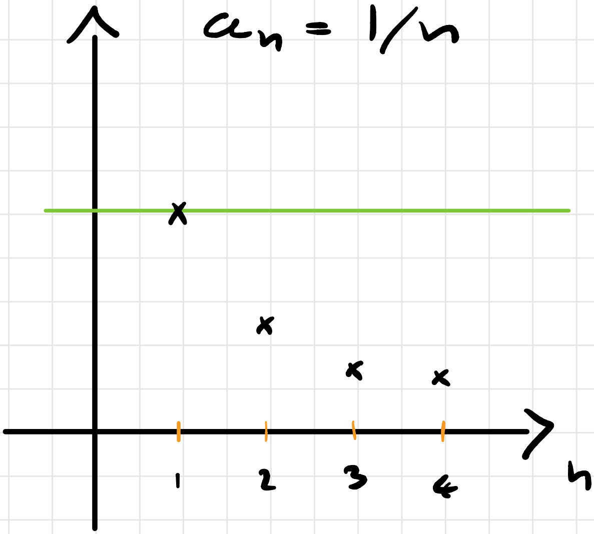
$y \to a$, $x \to n$

ORDERED LIST of REAL NUMBER

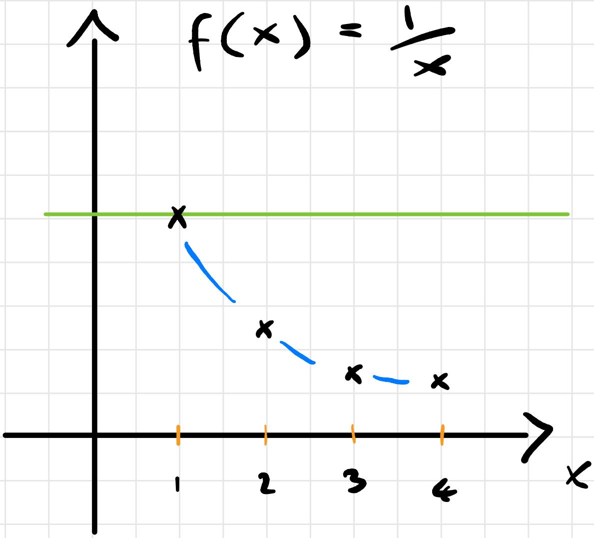
$$\{a_1, a_2, a_3 \underbrace{----}_{\text{until } \infty}\} \equiv \{a_n\}_{n=1}^{\infty} = \underbrace{\{a_n\}}_{\text{MORE COMPACT}}$$

Ex $n \longmapsto a_n = \frac{1}{n}$

$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \; --- \}$



List of Points



Line

SEQUENCE DEFINED by

RECURRENCE

$$\begin{cases} a_1 = 1 \\ a_n = f(a_{n-1}) \end{cases}$$

Ex FACTORIAL

$$\begin{cases} a_1 = 1 \\ a_n = n\,a_{n-1} \end{cases}$$

Ex

$$\begin{cases} a_1 = 1 \\ a_n = \sqrt{a_{n-1}+1} \end{cases}$$

$$\Big\{ \underbrace{1}_{n=1},\ \underbrace{\sqrt{1+1} = \sqrt{2}}_{n=2},\ \underbrace{\sqrt{\sqrt{2}+1}}_{n=3},\ \underbrace{\sqrt{\sqrt{\sqrt{2}+1}+1}}_{n=4} \ldots$$

We ALSO INTRODUCE THE NOTION of

SUBSEQUENCE of $\{a_n\}$

Any CHOICE of ELEMENTS of $\{a_n\}$ according a rule

$$\{a_{n_k}\}_{k=1}^{\infty}$$

EX $a_n = \frac{1}{n} = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$

I introduce THE LAW

$$n_k = 2k-1$$

Keeps only ODD Denominator

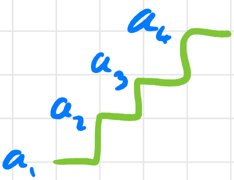
$$b_k = \{1, \frac{1}{3}, \frac{1}{5} \dots\}$$

Properties

def a_n MON. INC $\iff \forall n \in \mathbb{N}$ $a_n \leq a_{n+1}$

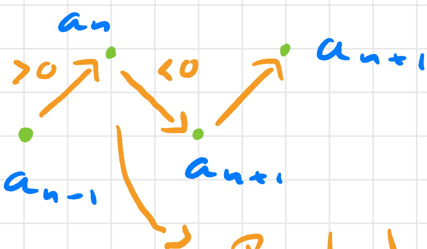
DEC $\geq$

Mon. Inc. Seq $\Rightarrow$ STAIRWAY



$a_2 \leq a_3 \leq a_4$

def a_n ALTERNATING $\iff \forall n \in \mathbb{N}$ $(a_n - a_{n-1})(a_{n+1} - a_n) < 0$

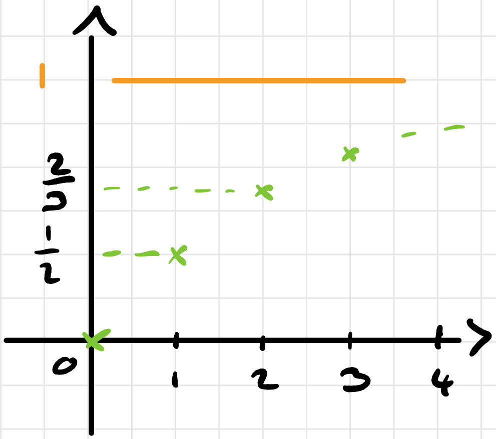


Product < 0

def a_n BOUNDED (ABOVE / BELOW) $\iff \exists c \in \mathbb{R}$

$a_n \leq c$
$\geq$
$\forall n \in \mathbb{N}$

Ex $a_n = \frac{n}{n+1}$



Bounded from above? Yes

$\forall n \in \mathbb{N} \quad a_n = \frac{n}{n+1} < 1$ den. > num.

Monotonic increasing?

$a_n - a_{n+1} = \frac{n}{n+1} - \frac{n+1}{(n+2)} = \frac{n(n+2) - (n+1)^2}{(n+1)(n+2)}$

$= \frac{\cancel{n^2} + \cancel{2n} - \cancel{n^2} - \cancel{2n} - 1}{(n+1)(n+2)} = - \frac{1}{\underbrace{(n+1)}_{\geq 0}\underbrace{(n+2)}_{\geq 0}} < 0$

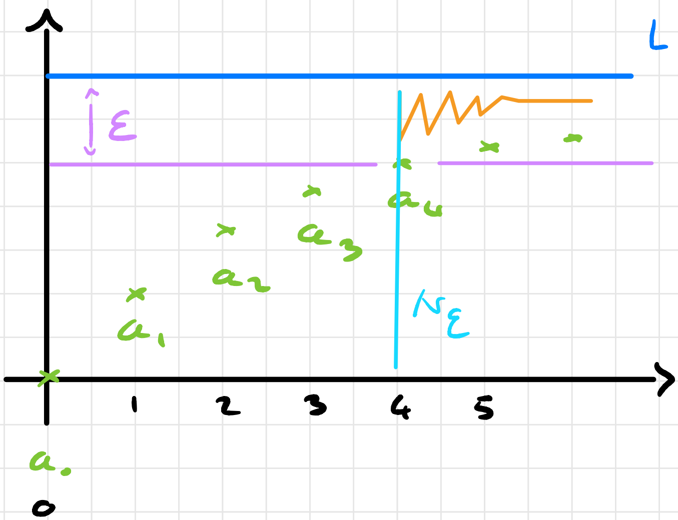
$\{a_n\}$ INCREASING

↳ UNTIL WHAT ? 1

WHAT does it mean

$n \to +\infty \quad a_n \to L = 1$ (LIMIT)

STILL, What does it mean?



$n > N_\varepsilon \Rightarrow |a_n - L|$ SMALL

WHAT does it mean?

$< \varepsilon$

In order to be well defined, A LIMIT requires that for each choice of ε; N_ε everything works

$\underline{\underline{\text{def}}}$ $\lim\limits_{n \to +\infty} a_n = L \iff \forall \varepsilon > 0 \ \exists N_\varepsilon \in \mathbb{N} : \underbrace{n > N_\varepsilon}_{} \Rightarrow \underbrace{|a_n - L| < \varepsilon}_{}$

↳ real number   ↳ depends on ε ↳ n LARGE   small

$\{a_n\}$ CONVERGENT

Similarly when $\{a_n\} \to +\infty$

$\underline{\underline{\text{def}}}$ $\lim\limits_{n \to +\infty} a_n = +\infty \iff \forall c > 0 \ \exists N_c \in \mathbb{N} : n > N_c \Rightarrow a_n > c$

(EQUIVALENT definition when $\{a_n\} \to -\infty$)

↳ IT is in the notes, I don't repeat

$\{a_n\}$ DIVERGENT

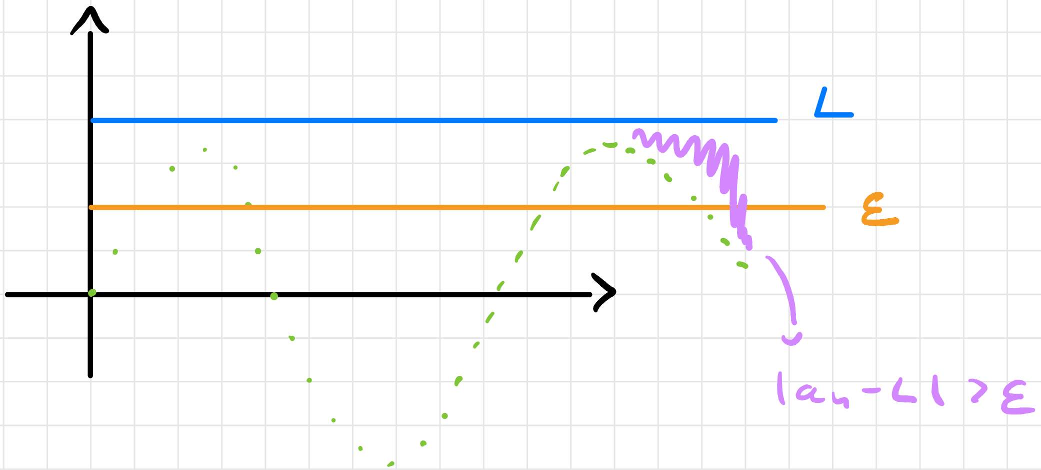
<u>IMPORTANT</u> $\underline{n \to +\infty \text{ (only)}}$

When we TALK ABOUT LIMIT of a REAL FUNCTIONS we relax THIS CONSTRAINT

$\lim\limits_{n \to +\infty} \equiv \lim_n$

ACTUALLY THIS DEFINITION IS MORE USEFUL TO PROVE THAT a limit DOES NOT EXISTS

$a_n = \sin(n)$



LIMIT OF THE SIN DOES NOT EXISTS

$\sin(n)$ NON CONVERGENT

$\cos(n)$ // //

$(-1)^n$ // //

THEY ALWAYS OSCILLATES

FOR THIS EXAM PURPOSE WE NEED

1) ELEMENTARY LIMITS

2) PROPERTIES TO JOIN THEM

Properties (I do not prove)

Prop $\exists!\, L \in \mathbb{R} : a_n \longrightarrow L$

↳ Tends to

Prop $\{a_n\}$

Hy) a_n MON. INC.

Hy) $\sim$ upp. BOUNDED

$\Rightarrow$ (Th) a_n CONVERGENT

SIMILARLY if it is DEC & LOW. BOUNDED

Prop $\{a_n\}$, $\{b_{n_k}\}$ SUBSEQUENCE

Hy) $\lim_n \{a_n\} = L$

$\Rightarrow$ (Th) $\lim_k \{b_{n_k}\} = L$

any SUBSEQUENCE of any CONVERGENT SEQUENCE CONVERGE TO THE SAME LIMIT

THEOREM $\{a_n\}\ \{b_n\}\ \{c_n\}$

Hy) $\lim_n a_n = \lim_n c_n = L$

Hy) $a_n \leq b_n \leq c_n$ (since a certain n)

$\Rightarrow$ (Th) $\lim_n b_n = L$

Ex $a_n = \dfrac{\sin(n)}{n}$

$$\underbrace{-\frac{1}{n}}_{\substack{\downarrow\, n\to+\infty \\ 0}} \leq \frac{\sin(n)}{n} \leq \underbrace{\frac{1}{n}}_{\substack{\downarrow\, n\to+\infty \\ 0}}$$

$\Rightarrow \lim\limits_{n\to+\infty} a_n = 0$

COROLLARY $\lim_n |a_n| = 0 \Rightarrow \lim_n a_n = 0$

Prop $a_n \to a$, $b_n \to b$

- $\lim_n (a_n \pm b_n) = a \pm b$
- $\lim_n a_n b_n = a \cdot b$
- $\lim_n \dfrac{a_n}{b_n} = \dfrac{a}{b}$ $(b \neq 0)$
- $\lim_n a_n^{b_n} = a^b$
- $\lim_n \log a_n = \log a$

Ex $\lim_{n\to+\infty} \dfrac{3n^2+2n-1}{5n^4-2n+7}$

This sounds like Formula 1

you have different cars $3n^2$, $2n$, $5n^4$

the exponents indicates its speed

In general I have $\frac{\infty}{\infty}$ indeterminacy

so you cannot say anything

I proceed like this: I collect the fastest car

$$\frac{n^2}{n^4}\left(\frac{3+\frac{2}{n}-\frac{1}{n^2}}{5-\frac{2}{n^3}+\frac{7}{n^4}}\right)=\frac{1}{n^2}\,\frac{3}{5}\to 0$$

What happens with

$$\frac{3n^4+2n-1}{5n^4-2n+7}\ ? \qquad L=\frac{3}{5}$$

$$\frac{3n^5+2n-1}{5n^4-2n+7}\ ? \qquad L=+\infty$$

Proposition

$$\lim_{n \to +\infty} \frac{\displaystyle\sum_{k=0}^{P} \alpha_k n^k}{\displaystyle\sum_{k=0}^{Q} \beta_k n^k} = L$$

$$P = Q \implies L = \frac{\alpha_P}{\beta_Q}$$

$$Q < P \implies L = 0$$

$$Q > P \begin{cases} \to L = +\infty & \alpha_P \beta_Q > 0 \\ \to L = -\infty & \text{〃} < 0 \end{cases}$$

Propositions

$$\lim_n \sqrt[n]{a} = 1 \qquad \forall a > 0$$

$$\lim_n \sqrt[n]{n^p} = 1 \qquad \forall p \in \mathbb{R}$$

$$\lim_n a^n \begin{cases} \to 0 & (|a| < 1) \\ \to +\infty & (a > 1) \end{cases}$$

$$\lim_n \frac{n^r}{a^n} = 0 \qquad \forall r \in \mathbb{R} \quad |a| > 1$$

Important

$$\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e \approx 2.71\ldots$$

Ex

$$\lim_n \left(1 + \frac{1}{\underbrace{n^2+1}_{k}}\right)^{\overbrace{n^2+1}^{k}} \equiv \lim_n a_n$$

SUBSEQUENCE of $\left(1+\frac{1}{k}\right)^k \longrightarrow e$

$\Rightarrow \quad a_n \to e$

IF a SEQUENCE CONVERGES TO L,
any SUBSEQUENCE CONVERGE TO THE SAME LIMIT

Ex $\lim_n a_n = ?$

$$a_n = \left(1 + \frac{1}{3n^2+1}\right)^{2n^2-3}$$

Idea? Sounds like e but ...

$$a_n = \underbrace{\left[\left(1 + \frac{1}{3n^2+1}\right)^{3n^2+1}\right]}_{e}^{\frac{2n^2-3}{3n^2+1} \to \frac{2}{3}}$$

(Previous exercise)

SUBSEQUENCE of $\left(1+\frac{1}{n}\right)^n$

converges to SAME LIMIT

$$a_n = b_n^{c_n}, \quad b_n \to e, \quad c_n \to \frac{2}{3}$$

$$a_n = e^{\frac{2}{3}}$$