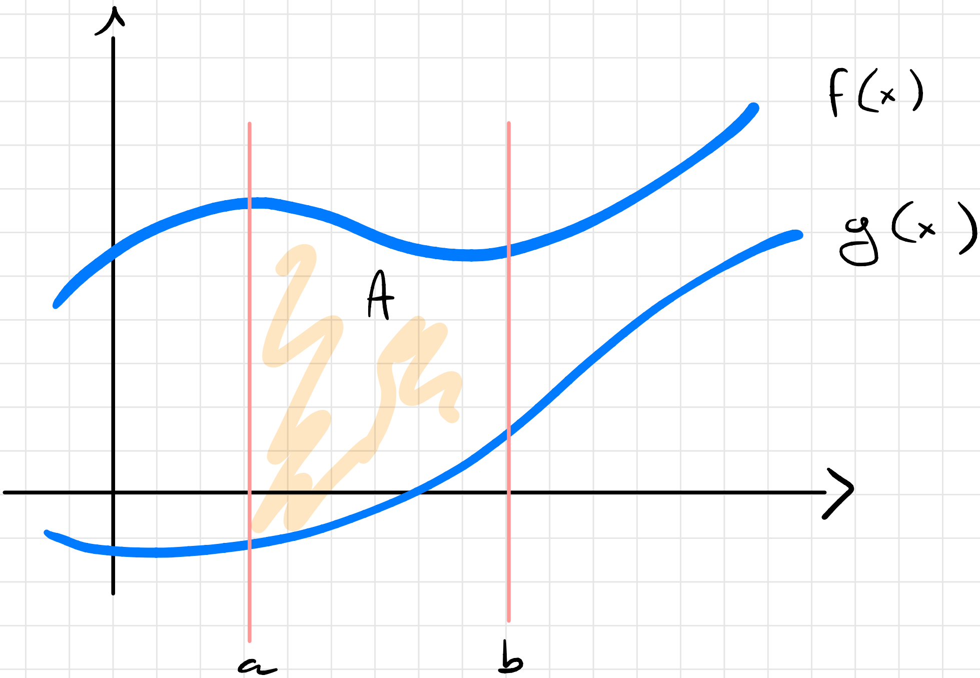
GEOMETRICAL APPLICATIONS of INTEGRALS

- AREAS (1)
  - Between Two Functions (1.1)
  - Polar Coordinates (1.2)
- VOLUMES (Solids of Revolution) (2)
  - x - Axis (2.1)
  - y - Axis (2.2)
- LENGTH of CURVES (3)

CASE (1.1)



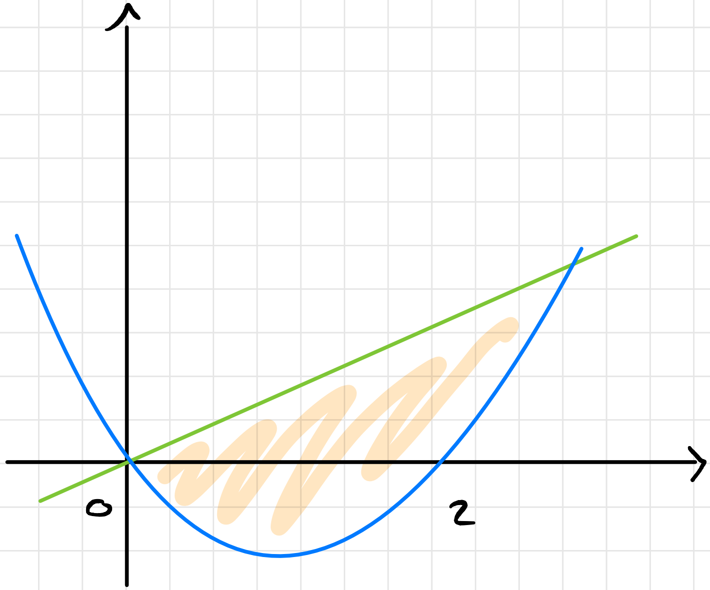
$$A = \int_a^b [f(x) - g(x)]\, dx$$

Ex

$$\boxed{f(x) = x(x-2)}$$

$$\boxed{g(x) = \frac{x}{2}}$$

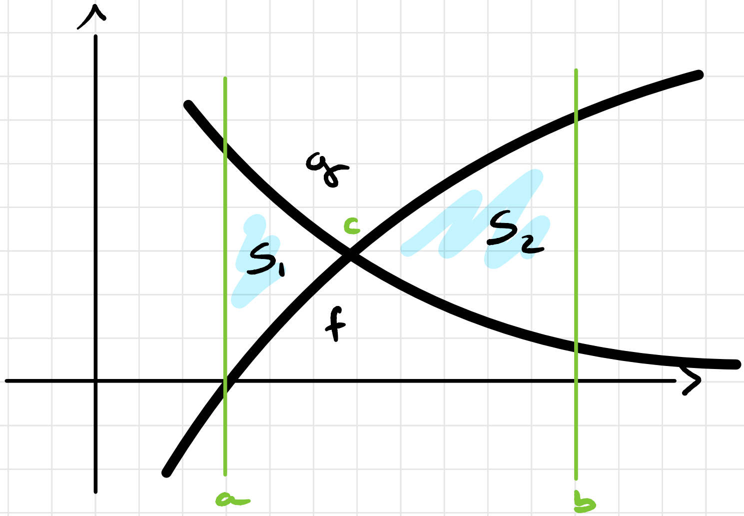
AREA Between 0 and 2



$$A = \int_0^2 \frac{x}{2} - x(x-2)\, dx = \int_0^2 dx \left(\frac{x}{2} - x^2 + 2x\right)$$

$$= \underbrace{\frac{x^2}{4}\Big|_0^2}_{1} - \underbrace{\frac{x^3}{3}\Big|_0^2}_{\frac{8}{3}} + \underbrace{x^2\Big|_0^2}_{4} = \frac{3 - 8 + 12}{3} = \frac{7}{3}$$

Particular Case



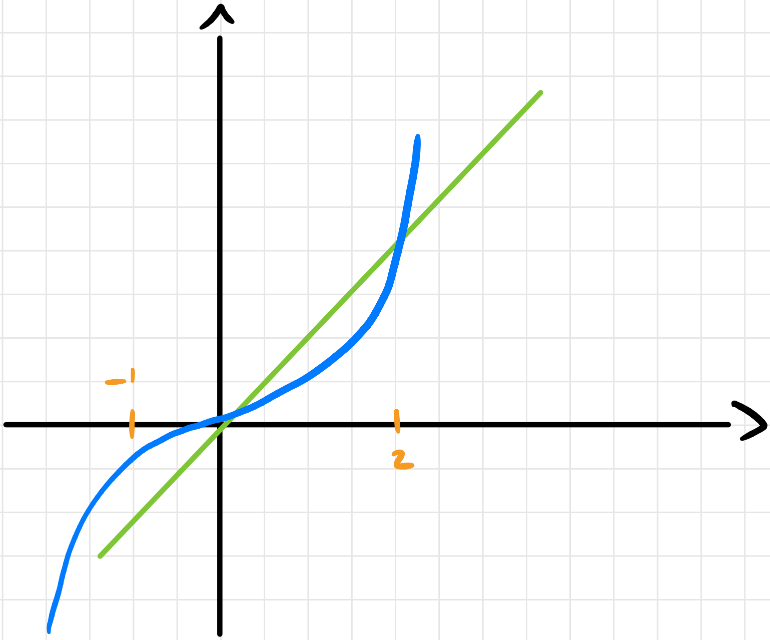
$$S_1 = \int_a^c [g(x) - f(x)]\,dx$$

$$S_2 = \int_c^b [f(x) - g(x)]\,dx$$

$$c \in [a,b], \quad f(c) = g(c)$$

$$A = S_1 + S_2 = \boxed{\int_a^b |f(x) - g(x)|\,dx}$$

Ex $\quad f(x) = x, \quad g(x) = \dfrac{x^3}{4}, \quad a = -1, \ b = 2$



$$A = \int_{-1}^{0} \left(\frac{x^3}{4} - x \right) dx + \int_{0}^{2} \left(x - \frac{x^3}{4} \right) dx$$

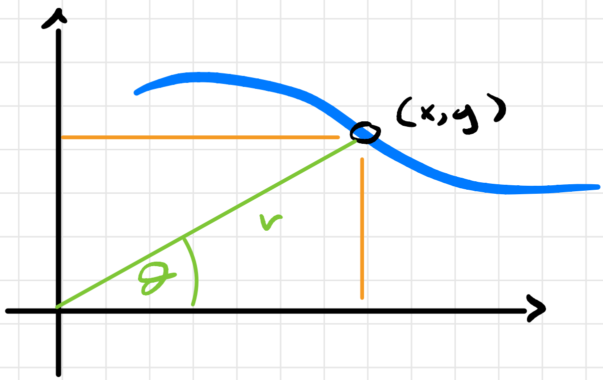
$$= \underbrace{\frac{x^4}{16}\Bigg|_{-1}^{0}}_{-\frac{1}{16}} - \underbrace{\frac{x^2}{2}\Bigg|_{-1}^{0}}_{+\ \frac{1}{2}} + \underbrace{\frac{x^2}{2}\Bigg|_{0}^{2}}_{2} - \underbrace{\frac{x^4}{16}\Bigg|_{0}^{2}}_{-1}$$

$$= \frac{-1+8+32-16}{16} = \frac{23}{16}$$

POLAR COORDINATE (1.2)

GRAPH of a FUNCTION

$$G = \{ (x,y) \in \mathbb{R}^2 : x \in D_f , \ y = f(x) \}$$



$(x,y) \longrightarrow (r, \theta)$ (CHANGE of BASIS)

$x = r \cos \theta$

$y = r \sin \theta$

$y = f(x) \rightarrow r = \tilde{f}(\theta)$

Ex 1 $r = 1, \theta = 0$

$x = r\cos\theta = 1$

$y = r\sin\theta = 0$

$(r = 1, \theta = 0) \to (x = 1, y = 0)$

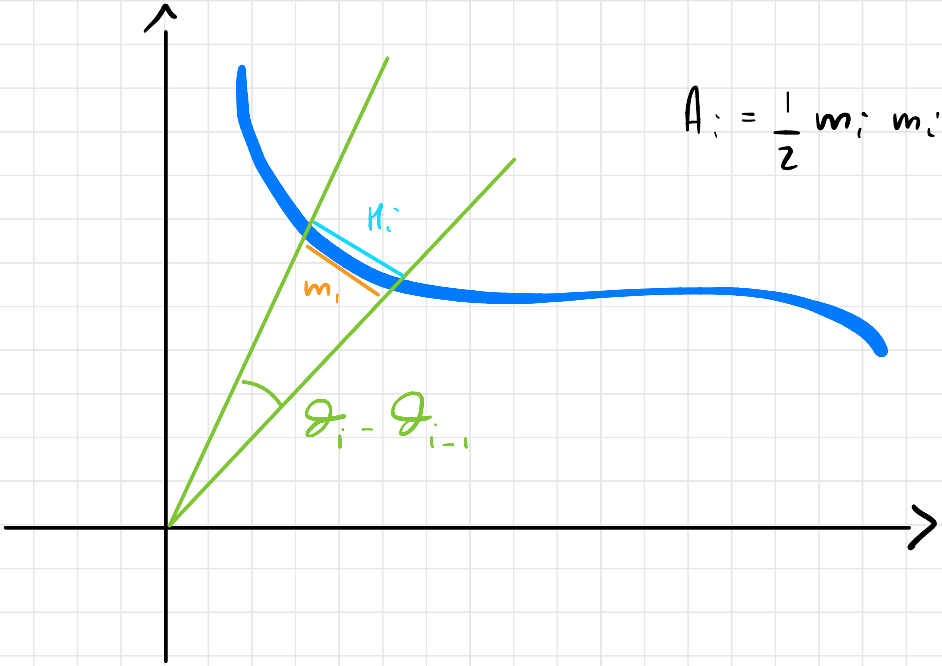
Ex 2 $r = 3, \quad \theta = \frac{\pi}{4}$

$x = 3\cos\frac{\pi}{4} = \frac{3}{2}\sqrt{2}$

$y = 3\sin\frac{\pi}{4} = \frac{3}{2}\sqrt{2}$

$(r = 3, \theta = \frac{\pi}{4}) = (\frac{3}{2}\sqrt{2}, \frac{3}{2}\sqrt{2})$

$$A = \{(r,\vartheta) : a \leq \vartheta \leq b,\ 0 < r < f(\vartheta)\}$$



$$A_i = \frac{1}{2} m_i\, m_i\, (\vartheta_i - \vartheta_{i-1})$$

$$P = \{a = \vartheta_0, \vartheta_1, \ldots, \vartheta_N = b\}$$

$$\sum_{i=1}^{n} \frac{1}{2} m_i^2 (\vartheta_i - \vartheta_{i-1}) \leq A(s) \leq \sum_{i=1}^{n} \frac{1}{2} M_i^2 (\vartheta_i - \vartheta_{i-1})$$

$r = f(\vartheta)$ INTEGRABLE

$$\boxed{A = \frac{1}{2} \int_a^b f(\vartheta)^2 \, d\vartheta}$$

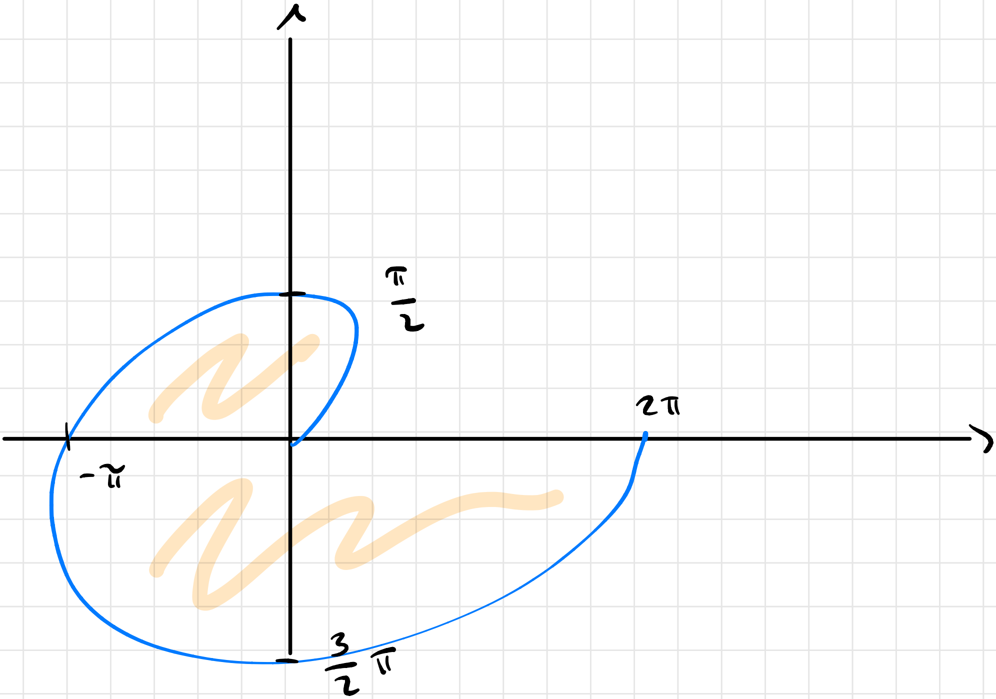
CIRCLE AREA

$r = R$ (CONSTANT)

$$A = \frac{1}{2}\int_0^{2\pi} \underbrace{f(\theta)^2}_{R^2}\, d\theta = \frac{1}{2}\, 2\pi R^2 = \pi R^2$$

Ex 11.3

$r = \vartheta$ (ARQUIMEDE SPIRAL)

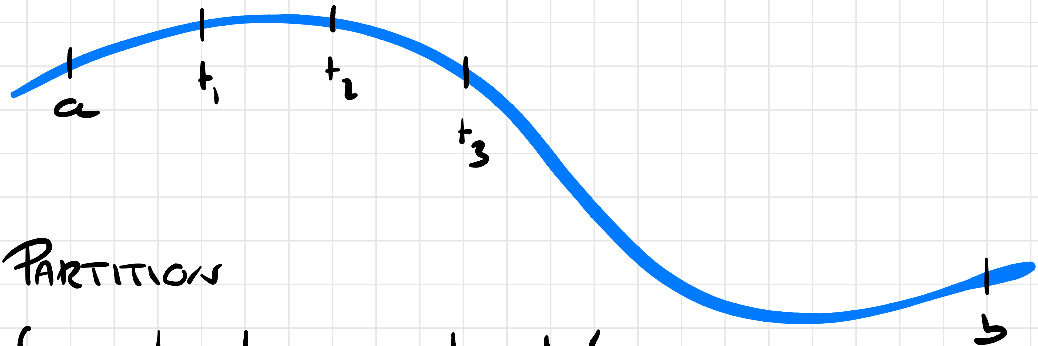


$$A = \frac{1}{2}\int_0^{2\pi} \vartheta^2 \, d\vartheta = \frac{1}{2}\,\frac{1}{3}\,\vartheta^3 \Big|_0^{2\pi}$$

$$= \frac{1}{2}\,\frac{1}{3}\,8\pi^3 = \frac{4}{3}\pi^3$$

LENGTH of CURVES (3)

$$C = \{ r(t) \in \mathbb{R}^n , \; a \leq t \leq b \}$$



PARTITION

$$\{ a = t_0 , t_1 \, - - - , t_N = b \}$$

$$\mathcal{L}_N^{(P)} = \sum_{i=1}^{N} \| r(t_i) - r(t_i - 1) \|$$

$$\| \underline{v} \| = \sum_i v_i^2 \quad \text{(EUCLIDEAN NORM)}$$

$$\mathcal{L}_N^{(P)} = \sum_{i=1}^{N} \left\| \frac{r(t_i) - r(t_{i-1})}{t_i - t_{i-1}} \right\| (t_i - t_{i-1})$$

$$\mathcal{L}_C = \lim_{N \to +\infty} \mathcal{L}_N^{(P)} = \int_a^b \| r'(t) \| \, dt$$

__Ex__ CIRCUMFERENCE RADIUS a

$$r(t) = (a\cos t,\ a\sin t)$$

$$\mathcal{L} = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

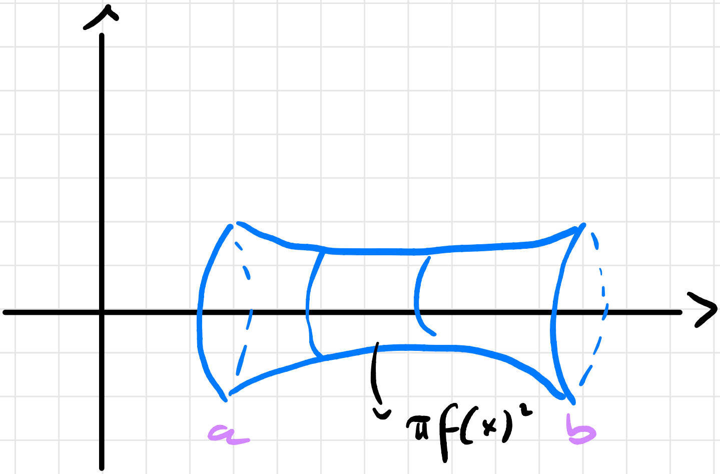
$$= \int_0^{2\pi} a \underbrace{\sqrt{\sin^2 t + \cos^2 t}}_{1}\, dt = 2\pi a$$

Solid of Revolution

↳ x - Axis 2.1

↳ y - Axis 2.2

Case (i)

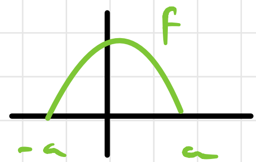


$$V = \int \pi f(x)^2 \, dx$$

Ex Volume of a SPHERE of RADIUS a

$$V = \int_{-a}^{a} \pi f^2(x)\, dx$$

$$f(x) = \sqrt{a^2 - x^2}$$



$$V = \int_{-a}^{a} \pi (a^2 - x^2)\, dx$$

$$= \pi a^2 x \Big|_{-a}^{a} + \frac{\pi}{3} x^3 \Big|_{a}^{-a}$$

$$= \pi a^3 - (-\pi a^3) - \frac{\pi}{3} a^3 - \frac{\pi}{3} a^3$$

$$= 2\pi \left(a^3 - \frac{a^3}{3}\right) = \frac{4}{3} \pi a^3$$

Case 2.2

$$V = 2\pi \int_a^b x f(x)\,dx$$

Still, it comes from a Riemann sum

Ex Donuts

$$V = 2 \cdot 2\pi \int_{R-a}^{R+a} x f(x)\,dx$$

$$f(x) = \sqrt{a^2 - (x-R)^2}$$

$$V = 4\pi \int_{R-a}^{R+a} x \sqrt{a^2 - (x-R)^2}\,dx$$

$$x = R + a\sin\theta$$

$$dx = a\cos\theta\, d\theta$$

$$V = 4\pi \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} (R + a\sin\theta)\, a^2 \cos^2\theta\, d\theta$$

$$\sin\theta \cos^2\theta \quad \text{ODD} \Rightarrow \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \sin\theta \cos^2\theta\, d\theta = 0$$

$$V = 4\pi R a^2 \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$\hookrightarrow 1 + \cos 2\theta$$

$$= 2\pi R a^2 \left(\pi + \frac{1}{2} \sin(2\theta) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \right)$$

$$= (2\pi R)(\pi a^2)$$