# Ex. 5.3 (vi)

$$\lim_{x \to \pm\infty} \frac{x-2}{\sqrt{4x^2+1}}$$

IND. $\frac{\infty}{\infty}$

↳ WHO RUNS FASTER

$$\lim_{x \to \pm\infty} \frac{x\left(1-\frac{2}{x}\right)}{\sqrt{x^2}\sqrt{4+\frac{1}{x^2}}}$$

$$\sqrt{x^2} = |x|$$

$$= \lim_{x \to \pm\infty} \frac{x}{|x|} \frac{1-\overbrace{\frac{2}{x}}^{\to 0}}{\sqrt{4+\underbrace{\frac{1}{x^2}}_{\to 0}}} = \lim_{x \to \pm\infty} \frac{1}{2} \frac{x}{|x|}$$

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$\lim_{x \to +\infty} \frac{1}{2} \frac{x}{x} = \frac{1}{2} , \quad \lim_{x \to -\infty} \frac{1}{2} \frac{x}{-x} = -\frac{1}{2}$$

Ex. 5.2 (v)

$$\lim_{x \to 0} \frac{\log(1-2x)}{\sin x} = L$$

INDETERMINACY $\frac{0}{0}$

$\hookrightarrow$ WHO RUNS FASTER? log or sin?

$$\left.\begin{array}{l} \log(1+t) \sim t \\ \sin t \sim t \end{array}\right\} t \to 0$$

$$L = \lim_{x \to 0} -2 \cdot \underbrace{\frac{\log(1-2x)}{-2x}}_{+1} \underbrace{\frac{x}{\sin x}}_{1} = -2$$

# Ex. 5.2 (xii)

$$\lim_{x \to 0} \frac{a^x - b^x}{x} \qquad a, b > 0$$

WHICH EQUIVALENCE RELATIONSHIP?

$\hookrightarrow \quad a^x \to e^x$

$$e^t - 1 \sim t$$

$\hookrightarrow$ I NEED A ONE

$$\lim_{x \to 0} \frac{1}{x}\left(a^x - 1\right) - \frac{1}{x}\left(b^x - 1\right)$$

I NEED A $e$

$$a^x = \left(e^{\log a}\right)^x = e^{x \log a}$$

$$b^x = \left(e^{\log b}\right)^x = e^{x \log b}$$

$$\lim_{x \to 0} \frac{e^{x \log a} - 1}{x} - \frac{e^{x \log b} - 1}{x}$$

I NEED THE SAME UPSTAIRS AND DOWNSTAIRS

$$\lim_{x \to 0} \log a \underbrace{\left( \frac{e^{x \log a} - 1}{x \log a} \right)}_{\downarrow \atop 1} - \log b \underbrace{\left( \frac{e^{x \log b} - 1}{x \log b} \right)}_{\downarrow \atop 1}$$

$$= \log a - \log b = \log \left( \frac{a}{b} \right)$$

Ex. 5.2 (IX)

$$\lim_{x \to 0} \left( \frac{x}{\sin x} \right)^{\frac{\sin x}{\sin x - x}} = L \qquad \text{IND } 1^{\infty}$$

$$f(x) = a(x)^{b(x)}$$

$$a(x) \to 1 \;, \quad b(x) \to +\infty$$

$$f(x) \longrightarrow \mathrm{Exp}\left( \lim_{x \to 0} (a(x) - 1)\, b(x) \right)$$

$$L = e^{c} \;, \quad c = \lim_{x \to 0} \left( \frac{x}{\sin x} - 1 \right) \frac{\sin x}{\sin x - x}$$

$$c = \lim_{x \to 0} \left( \frac{x - \sin x}{\sin x} \right) \frac{\sin x}{\sin x - x} = -1$$

$$L = e^{-1} = 1/e$$

Ex. 5.2 (x)

$$\lim_{x \to 0} (\cos x)^{1/x^2} = L \qquad \text{IND. } 1^{\infty}$$

$$L = e^{c}, \quad c = \lim_{x \to 0} (\cos x - 1)\frac{1}{x^2}$$

$$= -\frac{1}{2} \underbrace{\lim_{x \to 0} \frac{2}{x^2}(1 - \cos x)}_{1}$$

$$= -\frac{1}{2}$$

$$L = e^{-\frac{1}{2}} = \sqrt{\frac{1}{e}}$$

# Ex. 5.3 (iv)

$$\lim_{x \to +\infty} \sqrt{x^2+4x} - x$$

IND. $\infty - \infty$

$$\lim_{x \to +\infty} \frac{\left(\sqrt{x^2+4x} - x\right)\left(\sqrt{x^2+4x} + x\right)}{\sqrt{x^2+4x} + x}$$

$$= \lim_{x \to +\infty} \frac{\cancel{x^2} + 4x - \cancel{x^2}}{\sqrt{x^2+4x} + x} = \lim_{x \to +\infty} \frac{4x}{\underbrace{\sqrt{x^2}}_{|x| = x \atop x \to +\infty}\sqrt{1+\frac{4}{x}} + x}$$

$$= \lim_{x \to +\infty} \cancel{\frac{x}{x}} \frac{4}{\sqrt{1+\underbrace{\frac{4}{x}}_{\downarrow \atop 0}} + 1} = \frac{4}{2} = 2$$

# Ex. 6.3

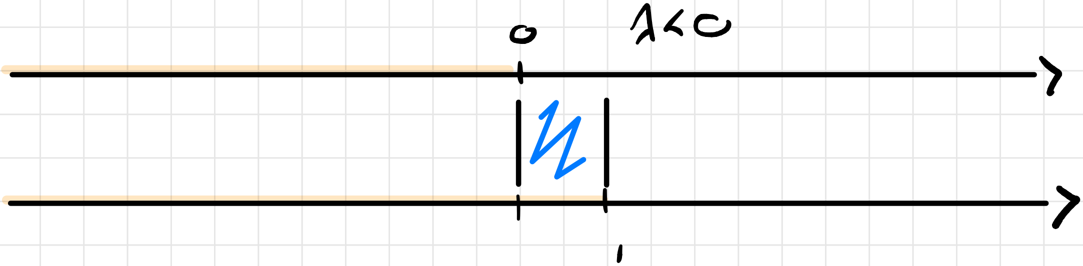
$$f(x) = \frac{1}{\lambda x^2 - 2\lambda x + 1} \quad \text{cont } \mathbb{R}$$

$$\lambda x^2 - 2\lambda x + 1 \neq 0 \quad \forall x \in \mathbb{R}$$

$$x_\pm = \frac{2\lambda \pm \sqrt{4\lambda^2 - 4\lambda}}{2\lambda}$$

$$x_\pm \notin \mathbb{R} \iff 4\lambda^2 - 4\lambda < 0$$

$$\lambda(\lambda - 1) < 0 \quad , \quad \lambda \in [0, 1)$$



## Ex. 6.4 (ix)

$$f(x) = \begin{cases} e^{\frac{1}{x}} & x < 0 \\ 0 & x = 0 \\ \dfrac{\tan x}{\sqrt{x}} & x > 0 \end{cases} \qquad \text{cont?}$$

$x > 0$ $\quad \sqrt{x}$ cont

$\tan x$ cont. in $\{x \in \mathbb{R} : \cos x \neq 0\}$

$\{x \in \mathbb{R} : x \neq (2n+1)\frac{\pi}{2}, \underbrace{n \in \mathbb{N}}_{x>0}\}$

$x < 0$ $\quad e^{1/x}$ cont

$x = 0$ ?

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} e^{\frac{1}{x}} = e^{-\infty} = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\tan x}{\sqrt{x}}$$

$$= \lim_{x \to 0^+} \frac{\sin x}{\cos x} \frac{1}{\sqrt{x}}$$

$$= \lim_{x \to 0^+} \underbrace{\frac{\sin x}{x}}_{1} \underbrace{\frac{\sqrt{x}}{\cos x}}_{1}^{0} = 0$$

f cont 0

# Ex. 6.5 (vi)

$$\frac{1}{4}x^3 - \sin(\pi x) + 3 = \frac{7}{3}$$

SOLUTION $[-2, 2]$ ?

$\longrightarrow$ BOLZANO

$$f(x) = \frac{x^3}{4} - \sin(\pi x) + \frac{9-7}{3} = 0$$

$$= \frac{x^3}{4} - \sin(\pi x) + \frac{2}{3} = 0$$

SUM OF CONT FCN

$$f(2) = \frac{8}{4} + \frac{2}{3} - \underbrace{\sin(2\pi)}_{0} = 2 + \frac{2}{3} = \frac{8}{3} > 0$$

$$f(-2) = -2 + \frac{2}{3} + \underbrace{\sin(2\pi)}_{0} = \frac{2-6}{3} = -\frac{4}{3} < 0$$

$$f(2)\,f(-2) < 0$$

$$\Rightarrow \exists\, c \in [-2, 2] : f(c) = 0$$