INTEGRALS

~ PRIMITIVES

~ DERIVATIVES

$\underbrace{\text{PRIMITIVES} \quad \text{DERIVATIVES}}_{\text{FDT}}$

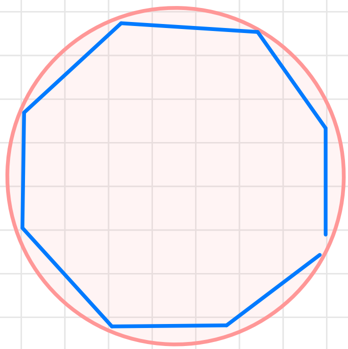
INTEGRALS $\xrightarrow{\text{AIM}}$ AREAS

AREAS $\downarrow$ A LONG TIME AGO

ARCHIMEDE

$\downarrow$

CIRCLE



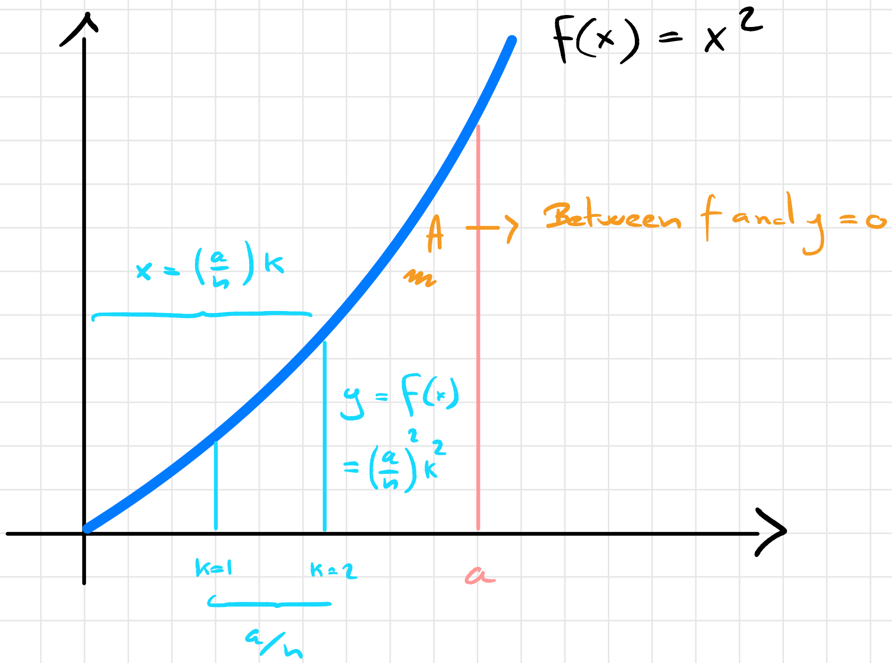
n SIDES

POLYGON SEQUENCE

$$\underbrace{\text{POLYGON SEQUENCE} \quad n \to +\infty}$$

- SEQUENCE (OF POLYGONS)
- LIMIT

LET'S SWITCH TO MORE COMPLICATED CASES



n rectangles ⟶ WIDTH $\frac{a}{n}$

(divide x axis into...)

HEIGHT $f\left(\frac{a}{n}\right) = \left(\frac{a}{n} k\right)^2$

AREA = WIDTH · HEIGHT = $\left(\frac{a}{n}\right)^3 k^2$

$$A_n = \sum_{k=1}^{n} \underbrace{\left(\frac{a}{n}\right)^3 k^2}_{\text{AREA RECT.}} = \left(\frac{a}{n}\right)^3 \underbrace{\sum_{k=1}^{n} k^2}$$

SUM over rectangle

$$\underbrace{\sum_{k=1}^{n} k^2 = \frac{n}{6}(n+1)(2n+1)}_{\text{Prove By IND.}}$$

$$A_n = \frac{n}{6}\left(\frac{a}{n}\right)^3 (n+1)(2n+1) \underset{n \to +\infty}{\sim} \frac{2n^3}{6n^3} a^3$$

$$A = \lim_{n \to +\infty} A_n = \frac{a^3}{3}$$

LET'S FORMALIZE - -

RIEMANN'S INTEGRAL

$$\underbrace{f : [a, b]}_{} \longrightarrow \mathbb{R} \qquad \text{AREA ?}$$

↳ PARTITION $\underbrace{\{x_0, x_1, x_2, \dots, x_n\}}_{}$

DOES NOT DEPEND ON RECTANGLE CHOICE

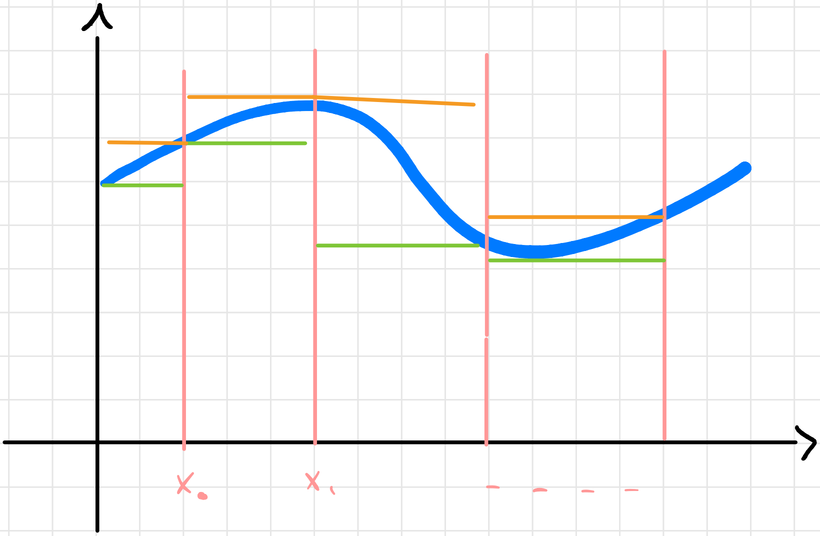
SEQUENCE of ORDERED num.

$$a = x_0 < x_1 < x_2 \dots < x_n = b$$

Hy : f BOUNDED

$$m_i = \inf_{x_{i-1} < x < x_i} f(x)$$

$$M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x)$$



AREA

- LOWER $L(f,P) = \sum_i m_i (x_i - x_{i-1})$

- UPPER $U(f,P) = \sum_i M_i (x_i - x_{i-1})$

OF COURSE $\underbrace{L(f,P) \leq U(f,P)}$

FIRST ESTIMATION OF AREA

$\rightarrow$ $\underbrace{\text{WE CAN DO SOMETHING BETTER}}$

REFINEMENT

Q REFINEMENT $P \iff P \subset Q$

OF COURSE

$$L(f,P) \leq L(f,Q)$$

$$U(f,Q) \leq U(f,P)$$

FOR TWO PARTITIONS P_1, P_2, $Q = P_1 \cup P_2$

$$\underbrace{L(f,P_1) \leq L(f,Q)} \leq \underbrace{U(f,Q) \leq U(f,P_2)}$$

$\sup_P L(f,P)$ $\qquad\qquad$ $\inf_P S(f,P)$

$\hookrightarrow$ ALL POSSIBLE PARTITIONS

DEF f BOUNDED $I = [a, b]$

$$f \text{ INTEGRABLE } I \iff \underbrace{\sup_P L(f, P) = \inf_P U(f, P)}_{\int_a^b f(x)\,dx}$$

ALTERNATIVE NOTATIONS

$$\int_a^b f(x)\,dx = \int_I f(x)\,dx = \int_a^b f$$

OBS.

PRIMITIVE $\int f\,dx \to$ FUNCTION

INTEGRAND $\int_a^b f \to$ NUMBER $\to$ AREA

$$\int_b^a f \gtrless 0 \qquad \text{SIGNED AREA}$$

$$\int_a^a f = 0$$

Prop f int. $\Rightarrow$ f bounded

$\cancel{\Leftarrow}$

Any bounded function is integrable?

No! Dirichlet function

$$\begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

$$\forall n \in \mathbb{N},\ \forall k \leq n\ \exists q \in \mathbb{Q} : q \in [x_k, x_{k+1}]$$

$$\Rightarrow \sup_{P} L(f, P) = 0 \neq x_{k+1} - x_k = \inf U(f, P)$$

WHEN A FUNCTION IS INTEGRABLE ?

THEOREM

f INT $[a,b] \iff \exists \{P_n\}_{n=1}^{\infty}$ PARTITION SEQ.

$$\lim_{n \to +\infty} L(f, P_n) = \lim_{n \to +\infty} U(f, P_n)$$

BENEFIT : $\iff$ EQUIVALENCE

HANDICAP : $\underbrace{\text{NOT SO PRACTICE}}$

CRITERIA

f CONTINUOUS $[a,b] \implies f$ INT. $[a,b]$

// MONOT. // // //

INTEGRAL PROPERTIES

- $$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g \qquad (\text{LIN})$$

- $$f \leq g \;\; \forall x \in [a, b] \Rightarrow \int_a^b f \leq \int_a^b g \qquad (\text{BOUND})$$

  - $\forall x \in [a,b]$, $f(x) \geq 0 \Rightarrow \int_a^b f(x) \geq 0$

  - $$\underset{\text{inf}}{m}(b-a) \leq \int_a^b f \leq \underset{\text{sup}}{M}(b-a)$$

- $$\left| \int_a^b f \right| \leq \int_a^b |f| \qquad (\text{ABS. INE})$$

- $$a < b < c \Rightarrow \int_a^c f = \int_a^b f + \int_a^b f$$

$$\Rightarrow \int_a^b f = -\int_b^a f$$

We consider

$$F(x) = \int_a^x \underbrace{f(t)}_{\text{INTEG.}}\,dt \qquad (\underbrace{a \in \mathbb{R}}_{\text{CONSTANT}})$$

Properties of F

- F CONT. $c \in [a, b] = I$

Proof f INT $\Rightarrow$ f BOUNDED $\Rightarrow$ $\exists M = \sup_{x \in I} |f|$

$$\forall x \in I, \quad |F(x) - F(c)| = \left| \int_a^x f - \int_a^c f \right|$$

$$- \int_a^c = \int_c^a \qquad = \left| \int_a^x f - \int_a^c f \right| = \left| \int_c^x f \right|$$

$$\left|\int\right| < \int |f| \qquad \leq \int_c^x |f| \leq \int_c^x M = M\,|x - c|$$

$$\lim_{x \to c} |x - c| = 0 \overset{\downarrow}{\underset{\text{SANDWICH}}{\Rightarrow}} \underbrace{\lim_{x \to c} |F(x) - F(c)| = 0}_{\text{CONTINUITY}}$$

Ex

$$f(x) = \begin{cases} 0 & x \le 1/2 \\ 1 & x > 1/2 \end{cases}$$

$x \le \frac{1}{2}$, $F(x) = \int_0^x f(t)\,dt = \int_0^x 0\,dt = 0$

$x > \frac{1}{2}$ $\quad F(x) = \underbrace{\int_0^{1/2} f}_{=0} + \int_{1/2}^x f = \int_{1/2}^x 1 = x - \frac{1}{2}$

↳ you used Barrow Rule

$$F(x) = \begin{cases} 0 & x \le 1/2 \\ x - \frac{1}{2} & x > 1/2 \end{cases}$$

WHAT ABOUT DERIVATIVES ?

THEOREM (FDT)

$f : [a,b] \to \mathbb{R}$

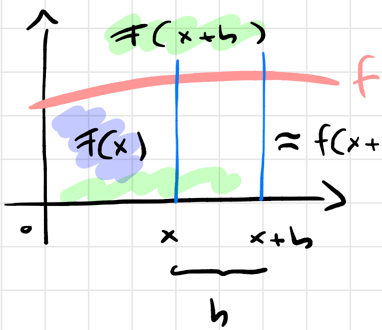
Hy) f cont. $[a,b]$

$\Rightarrow$ (Th) $F(x) = \int_0^x f(t)\,dt$ DIFF.

$$F'(x) = f(x)$$

(SLOPPY PROOF)

$$\underbrace{F(x) = \int_0^x f(t)\,dt}_{\text{AREA } f \text{ and } y=0}$$



$\approx f(x+h) \approx f(x)$ $\quad h \to 0$

CONTINUITY

$$F(x+h) \approx F(x) + f(x+h)\,h$$

$$\Rightarrow f(x+h) = \frac{F(x+h) - F(x)}{h} \xrightarrow[h \to 0]{} \underbrace{\lim_{h \to 0} \frac{F(x+h) - F(x)}{h}}_{F'(x)} = f(x)$$

AREA $\longrightarrow$ PRIMITIVE

$\downarrow$

WHICH ONE? $\longrightarrow$ ANY ONE?

Theorem (Barrow Rule)

G Primitive f cont $[a,b] \Rightarrow \int_a^b f = G(b) - G(a)$

Proof F, G Prim. f

$$F(x) = \int_a^x f(t)\,dt$$

$$G(x) = F(x) + c$$

$$G(a) = F(a) + c = \overbrace{\int_a^a f}^{=0} + c = c$$

$$F(x) = G(x) - G(a)$$

$$F(b) = \boxed{\int_a^b f = G(b) - G(a)}$$

Corollary

$$H = \int_{g_1(x)}^{g_2(x)} f(t)\,dt$$ (composition of fun.)

$$\frac{d}{dx} H(x) = f(g_2(x))\, g_2'(x) - f(g_1(x))\, g_1'(x)$$

Ex $H(x) = \int_0^{x^3} \cos(t)\,dt$

$f(t) = \cos(t)$

$g_2(x) = x^3$

$g_1(x) = 0$

$g_2'(x) = 3x^2$

$$H' = 3\cos(x^3)\,x^2 - \underbrace{\cos(0)\frac{d}{dx}0}_{= 0}$$

INTEGRALS

↳ BARROW'S RULE

- INT by PARTS

$$\int_a^b f g' = fg\Big|_a^b - \int_a^b f' g \, dt$$

- CHANGE OF VARIABLE

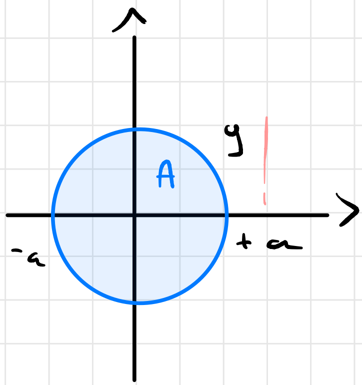
$$\int_{g(a)}^{g(b)} f(u)\, du = \int_a^b f(g(t))\, g'(t)\, dt$$

EXTREME CHANGE

Ex AREA of a CIRCLE of RADIUS a

CIRCUMFERENCE $\underbrace{x^2 + y^2 = a^2}$

$\hookrightarrow y = \pm\sqrt{a^2 - x^2}$



$$A = 2\int_{-a}^{+a} \sqrt{a^2 - x^2}\, dx$$

$$= 2a \int_{-a}^{a} \sqrt{1 - \left(\frac{x}{a}\right)^2}\, dx$$

$$= 2a^2 \int_{-1}^{+1} \sqrt{1 - t^2}\, dt$$

$t = \frac{x}{a}, \quad dx = a\, dt$

$\pm a \longmapsto \pm 1$

NEXT CHANGE OF VARIABLE

$$t = \sin(\theta) \quad , \quad dt = \cos(\theta)\, d\theta \quad , \quad \pm 1 \mapsto \pm \frac{\pi}{2}$$

$$A = 2a^2 \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos^2(\theta)\, d\theta = \quad \cos^2(\theta) = \frac{1}{2}\left(1 + \cos(2\theta)\right)$$

$$= a^2 \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \left(1 + \cos(2\theta)\right) d\theta$$

$$= a^2 \left[\pi + \underbrace{\frac{1}{2} \sin(2\theta) \Big|_{-\pi/2}^{\pi/2}}_{=0} \right] = a^2 \pi$$

Ex

$$\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{1}{n + k}$$

It's not a Series

$$\sum_{k=1}^{n} \frac{1}{n+k} = \sum_{k=1}^{n} \frac{1}{n} \frac{1}{1 + \left(\frac{k}{n}\right)}$$

Partition of $[0,1]$ in n rectangle

$$\lim_{n} \sum_{k=1}^{n} \frac{1}{n+k} = \int_0^1 \frac{dx}{1+x}$$