ENERGY

EXTRA CLASS

↳ THURSDAY 5TH

13 - 15 h

2.3.C03

SERIES

$$\sum_{n=0}^{\infty} a_n \qquad \text{Sum of } \infty \ \overbrace{\text{NUMBERS}}^{\text{Terms of Sequence } a_n}$$

Main Issue: SUM CONVERGE / DIVERGE ?
↳ WHICH VALUE ?

Ex

✓ $\sum_{n=0}^{\infty} x^n$ (GEOMETRIC)

✓ $\sum_{n=0}^{\infty} n x^n$ (AR. - GEO.)

✓ $\sum_{n=0}^{\infty} u_n - u_{n+1}$ (TELE)

lim KNOWN (ONLY SITUATION)

GENERAL GOAL

- RATHER THAN LIMIT ...
- ASSESS CONVERGENCE
  - → CRITERIA

Prop $\underbrace{\sum_{n=0}^{\infty} a_n < \infty}_{Hy} \Rightarrow \underbrace{\lim_n a_n = 0}_{Th}$

CONV. SERIES $\Rightarrow$ LIMIT TERMS ZERO

$\lim_n a_n \neq 0 \Rightarrow \sum a_n$ non conv.

Ex $x > 1$ $\lim_n x^n = \infty$ and $\sum_n x^n = \infty$

Ex $\lim_{n \to \infty} \frac{1}{\sqrt[n]{n}} = 1 \Rightarrow \underbrace{\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}} = \infty}_{\text{does not converge, not necessarily diverge}}$

$\sqrt[n]{n} \to 1$

Ex $\lim_n (-1)^n \neq 0 \Rightarrow \sum_{n=0}^{\infty} (-1)^n$ non conv.

UNFORTUNATELY, THE OPPOSITE IMPLICATION IS NOT TRUE

$$\lim_n a_n = 0 \;\cancel{\Rightarrow}\; \sum_n a_n < \infty$$

Ex HARMONIC SERIES

$$\lim_n \frac{1}{n} = 0 \Rightarrow \underbrace{\sum_n \frac{1}{n} = \infty}_{\text{WE'LL PROVE in a WHILE}}$$

PROPOSITION → DIVERGENCE rather than CONV.

List of Criteria for

Series of non-negative terms

<u>def</u> $\sum_{n}^{\infty} a_n$ non-neg $\iff a_n \geq 0 \ \forall n \in \mathbb{N}$

OBSERVATION

<u>Prop</u> $\underbrace{a_n \geq 0}_{\text{non-negative}} \Rightarrow S_k = \sum_{n=0}^{k} a_n$ Mon. Inc.

<u>Proof</u> $S_k - S_{k-1} = a_k \geq 0$

$S_k \nearrow$ UPPER BOUNDED $\Rightarrow S_k$ CONV

$S_k \searrow$ UNBOUNDED $\Rightarrow$ " NON CONV

THEOREM (COMPARISON TEST)

Hy) $\exists N \in \mathbb{N} : \forall n \geq N \quad \underline{0 \leq a_n \leq b_n}$

$$\Rightarrow (\text{Th}) \begin{cases} \sum_{n=1}^{\infty} b_n < \infty \Rightarrow \sum_{n=1}^{\infty} a_n < \infty \\ \text{〃}\ a_n = \infty \Rightarrow \text{〃}\ b_n = \infty \end{cases}$$

Ex $\log n < n \Rightarrow \frac{1}{n} < \frac{1}{\log n}$

$$\Rightarrow \underbrace{\sum_{n=2}^{\infty} \frac{1}{n}}_{} < \sum_{n=2}^{\infty} \frac{1}{\log n}$$

$= \infty$ (Before)

$$\Rightarrow \sum_{n} \frac{1}{\log n} = \infty$$

Ex $\sum_{n=0}^{\infty} \frac{1}{n!}$

FIRST WEEK

$n! > 2^n \quad \forall n \geq 4$

$\Rightarrow \frac{1}{n!} < \frac{1}{2^n}$ //

$$\sum_{n=0}^{\infty} \frac{1}{n!} < \underbrace{\sum_{n=0}^{\infty} \frac{1}{2^n}}_{\text{WHAT IS THIS}} = \underbrace{\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n}_{\text{Geometric Series}} < \infty$$

$|x| = \frac{1}{2} < 1$

Theorem (Cauchy's Condensation Test)

Hy 1) $a_n \geq 0$ (non-negative)

" 2) a_n Mon. Dec.

$\Rightarrow$ (Th) $\sum_n a_n < \infty \iff \underbrace{\sum_k 2^k a_{2^k} < \infty}$

LOOKS LIKE TRICKY but in some circumstances may HELP

Ex Riemanns (or Generalized Harmonic)

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \quad (\alpha \in \mathbb{R})$$

$$\sum_{n=1}^{\infty} \underbrace{\frac{1}{n^{\alpha}}}_{a_n} < \infty \iff \sum_{k=1}^{\infty} 2^{k} \underbrace{\frac{1}{2^{\alpha k}}}_{a_{2^k}} < \infty$$

$$\underbrace{\sum_{k=1}^{\infty} \left(\frac{1}{2^{(\alpha-1)}}\right)^{k}}_{\text{Geometric Series}} < \infty \iff \frac{1}{2^{\alpha-1}} < 1$$

$$x = \frac{1}{2^{\alpha-1}}$$

$$2^{\alpha-1} > 1$$

$$\alpha - 1 > 0$$

$$\boxed{\alpha > 1}$$

Cauchy → Log

Homework $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{\alpha}}$

THEOREM (LIMIT COMPARISON TEST)

$a_n, b_n \geq 0$ (non-negative)

$$a_n \sim b_n \quad \begin{cases} \sum_n a_n < \infty \Rightarrow \sum_n b_n < \infty \\ \sum_n a_n = \infty \Rightarrow \sum_n b_n = \infty \end{cases}$$

$$a_n \ll b_n \quad \begin{cases} \sum_n b_n < \infty \Rightarrow \sum_n a_n < \infty \\ \sum_n a_n = \infty \Rightarrow \sum_n b_n = \infty \end{cases}$$

Ex $a_n = \dfrac{1}{\sqrt{3n^2+2n+7}} \sim \dfrac{1}{\sqrt{3n^2}} = b_n$

$$\sum_n b_n = \frac{1}{\sqrt{3}} \sum_n \frac{1}{n} = \infty \Rightarrow \sum_n a_n = \infty$$

<u>THEOREM</u> (ROOT TEST)

<u>Hy i</u> $a_n \geq 0$

<u>Hy ii</u> $\lim_n \sqrt[n]{a_n} = L \geq 0$

$\Rightarrow$ (Th)

$L > 1 \Rightarrow \sum_n a_n = \infty$

$L < 1 \Rightarrow \sum_n a_n < \infty$

$L = 1 \Rightarrow$ CANNOT SAY ANYTHING

$\hookrightarrow$ WITH THIS CRITERION

PAST YEAR TEST

$$\sum_{n=1}^{\infty} \underbrace{\frac{1}{5^n}\left(1+\frac{1}{n}\right)^{n^2}}_{\text{non-negative}} \equiv \sum_n a_n$$

↳ ROOT TEST

$$\lim_n \sqrt[n]{a_n} = \frac{1}{5} \lim_n \left(1+\frac{1}{n}\right)^n = \frac{e}{5} < 1$$

$$\Rightarrow \sum_n a_n < \infty$$

WHAT HAPPENS IF $a_n = \frac{1}{2^n}\left(1+\frac{1}{n}\right)^{n^2}$

$$\lim_n \sqrt[n]{a_n} = \frac{e}{2} > 1 \Rightarrow \sum_n a_n = \infty$$

Recalling the Stolz Theorem

Corollary (Quotient Test)

Hy i $a_n \geq 0$

Hy ii $\lim_n \frac{a_n}{a_{n-1}} = \lim_n \frac{a_{n+1}}{a_n} = L > 0$

$\Rightarrow (T_n)$

- $L > 1 \Rightarrow \sum_n a_n = \infty$
- $L < 1 \Rightarrow \sum_n a_n < \infty$

$L = 1 \Rightarrow$ we cannot say anything

$\hookrightarrow$ with this criterion

Ex 4.1 (ix)

$$\sum_n a_n, \quad a_n = \frac{n^n}{3^n n!}$$

QUOTIENT TEST IS USEFUL FOR RATIO

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{(n+\cancel{1})}}{3^{n+\cancel{1}}} \frac{1}{\cancel{(n+1)!}} \frac{\cancel{3^n}\,\cancel{n!}}{n^n}$$

$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^n = \frac{1}{3}\left(1+\frac{1}{n}\right)^n \to \frac{e}{3} < 1$$

Same QUESTION $\quad a_n = \frac{1}{2^n}\frac{n^n}{n!}$

WHAT ABOUT NEGATIVE SERIES?

$\underline{\underline{\text{def}}}$ $\sum_n a_n$ ABS. CONV.

$$\iff \underbrace{\sum_n |a_n| < \infty}_{\text{CRITERIA FOR POSITIVE SUMS}}$$

$$\Rightarrow \sum_n a_n \text{ INC. CONV}$$

- ANY Perturbation of THE TERMS does NOT ALTER THE FINAL SUM

THIS IS IN GENERAL NOT TRUE

ALTERNATING SERIES

$$\sum_n (-1)^n a_n \qquad (a_n \geq 0)$$

THEOREM (Leibnit)

Hy 1 a_n dec. MON.

Hy 2 $a_n \to 0$

$$\Rightarrow \sum_n (-1)^n a_n < \infty$$

Ex $\sum_n \frac{(-1)^n}{n}$

$$\left.\begin{array}{l} a_n = \frac{1}{n} \quad \text{MON. DEC.} \\ \\ a_n \to 0 \end{array}\right| \Rightarrow \sum_n a_n < \infty$$