INTEGRALS

AREAS

$$\left.\begin{matrix}\text{INTEGRALS} \\ \text{AREAS}\end{matrix}\right\} \quad \underbrace{\int_I f \longrightarrow \sum_n \underbrace{S(f, P_n)}_{\text{RECTANGLES}}}_{\text{RELATIONSHIP BETWEEN TWO OBJECTS}}$$

FINITE SUM

$$\underbrace{\sum_{n=0}^{3} \left(\frac{1}{2}\right)^n}_{\text{FINITE SUM}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$

$$\underbrace{\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n}_{\text{INFINITE SUM}} = \underbrace{\frac{1}{1 - \frac{1}{2}}}_{\text{IN SOME CASE IT CONVERGES}} = 2$$

$$\int_a^b f \quad \text{AREA OVER FINITE RANGE}$$

$$\underbrace{\int_a^{\infty} f \quad \text{" INFINITE "}}_{\text{IMPROPER INT.}} \longrightarrow \text{STILL, MAY CONVERGE OR NOT}$$

DEF

$$\int_a^{\infty} f(x)\,dx = \lim_{b \to +\infty} \int_a^b f(x)\,dx$$

TRIVIALLY $\displaystyle\int_{-\infty}^{b} f = \lim_{a \to -\infty} \int_a^b f$

$$\boxed{\text{Ex} \quad \int_1^{\infty} \frac{dx}{x^{\alpha}}}$$

$\alpha = 1$ $\displaystyle\int_1^{\infty} \frac{dx}{x} = \lim_{b \to +\infty} \int_1^b \frac{dx}{x} = \lim_{b \to +\infty} \log b = \infty$

$\alpha \neq 1$ $\displaystyle\int_1^{\infty} \frac{dx}{x^{\alpha}} = \lim_{b \to +\infty} \frac{x^{1-\alpha}}{1-\alpha}\Bigg|_1^b = \boxed{\left(\frac{1}{1-\alpha}\right) \lim_{b \to \infty} \left(b^{1-\alpha} - 1\right)}$

$\alpha > 1$, $b^{-\alpha+1} \to 0$, $\displaystyle\int_1^{\infty} \frac{1}{x^{\alpha}} = \frac{1}{\alpha - 1}$

$\alpha < 1$ $\quad$ // $\quad \infty \quad \quad \infty$

Like Rieman Series

Convergence/Divergence IS BACK

Strong connection with series

Theorem (Mc Laurin)

Hy) $\forall x \in [1, \infty) \quad \underbrace{f(x) \geq 0}_{\text{non-negative}}$

Hy) f mon. dec.

$\Rightarrow$ (Th) $\displaystyle\int_1^\infty f(x)\,dx < \infty \iff \sum_{n=1}^{\infty} f(n) < \infty$

Integral test for non-negative series

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^\alpha} < \infty \ ? \qquad (\alpha > 1)$$

N.B. $\quad f(x) = \dfrac{1}{x} \dfrac{1}{(\log x)^\alpha}$ mon. dec.

$$\int_2^\infty \frac{dx}{x(\log x)^\alpha} \qquad t = \log x, \quad dt = \frac{dx}{x} = \frac{dx}{e^t}$$

$$\int_{\log 2}^{\infty} \frac{e^t}{e^t\, t^\alpha}\,dt = \int_{\log 2}^{\infty} \frac{dt}{t^\alpha} < \infty \iff \alpha > 1$$

Comparison Test f, g integrable

Hy) $0 \le f \le g$

$\Rightarrow$ (TH) $\begin{cases} \int_a^\infty g < \infty \Rightarrow \int_a^\infty f < \infty \\ \text{or} \\ \int_a^\infty f = \infty \Rightarrow \int_a^\infty g = \infty \end{cases}$

TRIVIAL EXTENSION OF WHAT YOU ALREADY KNOW

Ex

$$\int_0^\infty \frac{dx}{1+x^\alpha} \quad (\alpha \in \mathbb{R})$$

COMPARE WITH SOMETHING THAT I ALREADY KNOWS

FOR INSTANCE $\displaystyle\int_1^\infty \frac{dx}{x^\alpha}$

$$\int_0^\infty \frac{dx}{1+x^\alpha} = \underbrace{\int_0^1 \frac{dx}{1+x^\alpha}}_{\text{ORDINARY INTEGRAL}} + \int_1^\infty \frac{dx}{1+x^\alpha}$$

ORDINARY INTEGRAL $\longrightarrow$ NO CONV. PROBLEM

$$\forall x \geq 1 \qquad 2x^\alpha = x^\alpha + x^\alpha \geq 1 + x^\alpha \geq x^\alpha$$

$$\text{"} \qquad \frac{1}{2x^\alpha} \leq \frac{1}{1+x^\alpha} \leq \frac{1}{x^\alpha}$$

$\alpha > 1$ $$\int_1^\infty \frac{1}{1+x^\alpha} \leq \int_1^\infty \frac{1}{x^\alpha} < \infty$$

$\alpha < 1$ $$\int_1^\infty \text{"} \geq \int_1^\infty \frac{1}{2x^\alpha} = \infty$$

LIMIT COMPARISON TEST f, g INT.

Hy) $f \geq 0$, $g > 0$

Th 1 $f = g + o(g) \iff \frac{f}{g} = 1 + \frac{o(g)}{g} \to 1$ $(f \sim g)$

$$\Rightarrow \int_0^\infty f < \infty \iff \int_0^\infty g < \infty$$

Th 2 $f = o(g) \iff \frac{f}{g} = \frac{o(g)}{g} \to 0$ $(f \ll g)$

$$\int_0^\infty g < \infty \Rightarrow \int_0^\infty f < \infty \quad \text{OR} \quad \int_0^\infty f = \infty \Rightarrow \int_0^\infty g = \infty$$

Ex

$$\int_1^{\infty} e^{-x} \frac{dx}{x^{\alpha}} \qquad (\alpha \in \mathbb{R})$$

WE KNOW THAT

$$\int_1^{\infty} \frac{dx}{x^2} = 1 \ , \qquad \underbrace{\frac{e^{-x}}{x^{\alpha}} = o\left(\frac{1}{x^2}\right)}_{\displaystyle \lim_{x \to +\infty} \frac{e^{-x} x^{-\alpha}}{x^{-2}} = \lim_{x \to +\infty} e^{-x} x^{2-\alpha} = 0} \quad x \to +\infty$$

EXP KILLS EVERYTHING

$$\Rightarrow \int_1^{\infty} e^{-x} \frac{dx}{x^{\alpha}} < \infty$$

Difference with Respect to Series

$$\sum_n a_n < \infty \Rightarrow \lim_n a_n = 0$$

$$\int_a^\infty f < \infty \not\Rightarrow \lim_{x \to +\infty} f = 0$$

Ex $f(x) = \begin{cases} 1 & n \leq x \leq n + \frac{1}{n^2} \\ 0 & \text{otherwise} \end{cases}$

BEFORE WE STUDIED

$$\int_1^{\infty} \frac{dx}{x^{\alpha}} \quad - -$$

NOW

$$\int_0^1 \frac{dx}{x^{\alpha}}$$ WHAT HAPPENS in 0?

$\underbrace{\phantom{x^{\alpha}}}$ ↳ α VALUES on Finite RANGE

DEF IMPROPER INTEGRALS of SECOND KIND

$$f : (a, b] \to \mathbb{R}$$

↳ NOT DEFINED in a

$$f \text{ INT } [a, b] \iff \exists \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^{b} f$$

Ex

$$\int_0^1 \frac{1}{x^\alpha} \qquad (\alpha \in \mathbb{R})$$

$$\alpha = 1 \qquad \int_0^1 \frac{1}{x} = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{1}{x} = \lim_{\varepsilon \to 0^+} \left(\frac{1}{\varepsilon} \right) = -\infty$$

$$\alpha \neq 1 \qquad \int_0^1 \frac{1}{x^\alpha} = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{1}{x^\alpha} = \lim_{\varepsilon \to 0} \left. \frac{x^{1-\alpha}}{1-\alpha} \right|_\varepsilon^1$$

$$= \lim_{\varepsilon \to 0^+} \frac{1 - \varepsilon^{1-\alpha}}{1-\alpha} < \infty \iff \alpha < 1$$

Ex

$$\int_0^\infty \frac{1}{x^\alpha} = \underbrace{\int_0^1 \frac{1}{x^\alpha}}_{\alpha < 1} + \underbrace{\int_1^\infty \frac{1}{x^\alpha}}_{\alpha > 1}$$

never conv.

Ex

$$\int_0^1 \frac{dx}{\sqrt{x}}, \quad t = \frac{1}{x}, \quad dt = -\frac{dx}{x^2} = -\frac{dt}{t^2}$$

$1 \to 1, \quad 0 \to +\infty$

$$\int_1^\infty \sqrt{t}\,\frac{dt}{t^2} = \underbrace{\int_1^\infty \frac{1}{t^{3/2}}\,dt}$$

SWITCH FROM FIRST TO SECOND KIND

Biomedic Sciences

↳ Signals

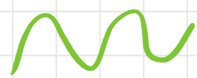
↳ Improper Integrals

frequency

Fourier

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t)\, \underbrace{e^{-i\omega t}}\, dt$$

↳ Plane wave



Basis

Projection of f signals on basis of plane wave

Laplace

$$\mathcal{L}(f) = \int_{0}^{+\infty} f(t)\, e^{-st}\, dt \qquad (s \in \mathbb{C})$$

Partic. case

Gamma

$$\Gamma(z) = \int_{0}^{\infty} t^{z-1} e^{-t}\, dt$$

Generalization factorial

Ex 12.5 (c)

$\Gamma(n+1) = n! \quad \forall n \in \mathbb{N}$

Prove By INDUCTION

$$\int_0^{+\infty} t^n e^{-t}\, dt = n! \quad \forall n \in \mathbb{N}$$

<u>$n=0$</u> $\displaystyle\int_0^{+\infty} e^{-t}\, dt = -e^{-t}\Big|_0^{+\infty}$

$$= 1 - \underbrace{\lim_{b \to +\infty} e^{-t}}_{=0} = 1 = 0!$$

Hy $\exists n \in \mathbb{N} : \underbrace{\int_0^{+\infty} t^n e^{-t} dt = n!}_{p(n)}$

$p(n+1)$ TRUE ?

$$\underbrace{\int_0^{+\infty} t^{n+1} e^{-t}}_{} = n+1 \ ?$$

INTEGRATION by PART

$$\int_a^b f g' = fg \Big|_a^b - \int_a^b f' g$$

$$f = t^{n+1}, \quad g' = e^{-t}$$

$$\int_0^{+\infty} t^{n+1} e^{-t}\, dt = -t^{n+1} e^{-t} \Big|_0^{+\infty} + (n+1)\int_0^{+\infty} t^n e^{-t}\, dt$$

OBS. $-t^{n+1} e^{-t} \Big|_0^{+\infty} = -\lim_{b \to +\infty} b^{n+1} e^{-b}$

$$= \lim_{b \to +\infty} \underbrace{\frac{b^{n+1}}{e^b}}_{} = 0$$

↳ Hôpital $n+1$ TIMES

OBS $(n+1) \underbrace{\int_0^{+\infty} t^n e^{-t}\, dt}_{n! \ (H_y)} = (n+1)n! = (n+1)!$

$p(n)$ TRUE $\Rightarrow$ $p(n+1)$ TRUE

$\forall n \in \mathbb{N} \quad \Gamma(n+1) = n!$