ENERGY	3/11	13-15h	2.3.C06
BIOMEDIC	31/10	"	7.1.J05

Revision → Past Year Problems

$$\begin{cases} a_1 = 1 \\ a_{n+1} = \dfrac{2a_n}{3a_n + 2} \end{cases}$$

First terms

$a_1 = 1$

$$a_2 = \frac{2a_1}{3a_1 + 2} = \frac{2}{5}$$

$$a_3 = \frac{2a_2}{3a_2 + 2} = 2\left(\frac{2}{5}\right)\frac{1}{3\left(\frac{2}{5}\right) + 2}$$

a) Prove by INDUCTION THAT $a_n > 0 \quad \forall n \geq 1$

FIRST STEP $\quad a_1 = 1 > 0 \quad \checkmark$

$\underline{Hy}$ $\exists n \geq 1 : \quad a_n > 0$

$$\underbrace{\Rightarrow \quad a_{n+1} > 0}_{\text{USE Hypothesis}} \quad ?$$

$a_n > 0$ $\begin{cases} 2a_n > 0 & \text{(Product positive number)} \\ 3a_n > 0 \Rightarrow \underline{\text{sum positive num.}} \; 3a_n + 2 > 0 \end{cases}$

$$\Rightarrow \quad a_{n+1} = \underbrace{\frac{2a_n}{3a_n + 2} > 0}_{\text{RATIO Positive num.}}$$

$a_n > 0 \quad \forall n \geq 1$

b) Prove a_n MON. DEC.

DEFINITION

$$a_{n+1} - a_n = \frac{2a_n}{3a_n+2} - a_n$$

$$= \frac{\cancel{2a_n} - \cancel{2a_n} - 3a_n^2}{3a_n+2}$$

$$= - \frac{3a_n^2 \; {>0}}{3a_n+2 \; {>0}} < 0$$

$\hookrightarrow$ MON. DEC.

c) LIMIT ?

$\hookrightarrow$ $\exists$ because a_n LOWER BOUND and MON. DEC.

$\hookrightarrow$ CALCULATION

$n \to +\infty$, $a_n \approx a_{n+1} \approx L$

$$L = \frac{2L}{3L+2} \Rightarrow 3L^2 + \cancel{2L} - \cancel{2L} = 0$$

$$\Rightarrow 3L^2 = 0$$

$$\Rightarrow L = 0$$

PROBLEM 4.1 (.x)

$$\sum_{n=1}^{\infty} \frac{n^n}{3^n n!} \equiv \sum_{n=1}^{\infty} a_n$$

CONVERGENT CHARACTER?

$$\underbrace{a_n \geq 0 \quad \forall n \geq 1}$$

NON-NEGATIVE $\longrightarrow$ QUOTIENT

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = L \quad \begin{cases} L > 1 & \sum_n a_n = \infty \\ L < 1 & \quad \text{"} \quad < \infty \end{cases}$$

$\hookrightarrow L = 1?$

$$a_{n+1} = \frac{1}{3^{n+1}} \frac{1}{(n+1)!} (n+1)^{n+1}$$

$$\frac{1}{a_n} = \frac{3^n n!}{n^n}$$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{3^n}}{3^{\cancel{n+1}}} \frac{\cancel{n!}}{(n+1)\cancel{!}} \frac{(n+1)^{n+1}}{n^n} = \frac{1}{3} \frac{1}{\cancel{(n+1)}} \frac{(n+1)^{\cancel{n+1}}}{n^n}$$

$$= \frac{1}{3}\left(1+\frac{1}{n}\right)^n \to \frac{e}{3} < 1 \Rightarrow \sum_n a_n < \infty$$

Ex (Past Year Exam

$$\lim_{x\to 0} \frac{\sin 3x}{5^x - 3^x} = 3\underbrace{\left(\frac{\sin 3x}{3x}\right)}_{\to 1}\underbrace{\left(\frac{5^x-3^x}{x}\right)^{-1}}_{\text{I wanna you understand THIS}} = L$$

$$\left(\frac{5^x-3^x}{x}\right) = \frac{1}{x}\left(e^{x\log 5} - e^{x\log 3}\right)$$

$$= \frac{e^{x\log 3}}{x}\left(e^{x(\log 5-\log 3)} - 1\right)$$

$$= \frac{e^{x\log 3}}{x}\left(e^{x\log\frac{5}{3}} - 1\right)$$

$$= \underbrace{e^{x\log 3}}_{\to 1}\log\frac{5}{3}\underbrace{\left(\frac{e^{x\log\frac{5}{3}}-1}{x\log\frac{5}{3}}\right)}_{\to 1} = \log\frac{5}{3}$$

$$L = \frac{3}{\log 5 - \log 3}$$

Ex

$$I = \{ x \in \mathbb{R} : \underbrace{|x-1| \leq 3}_{} \}$$

$$-3 \leq x-1 \leq 3$$

$$-3+1 \leq x \leq 3+1$$

$$-2 \leq x \leq 4$$

$$x \in [-2, 4]$$

$$\sup I = 4 = \max I$$

$$\inf I = -2 = \min I$$

Same question for $|x-1| < 3$

PROBLEM 7.25 (i)

$x^7 + 4x = 3$ NUMBER SOLUTIONS?

?? $x^2 + 1 = 0$ NO SOLUTION (although x^2)

I INTRODUCE

$$f(x) = x^7 + 4x - 3$$

$$\lim_{x \to \pm\infty} f(x) = \pm\infty$$ → AT LEAST ONE SOLUTION (BOLZANO EXTENSION)

HOW MANY?

$$f'(x) = \underbrace{7x^6 + 4}_{\text{ALWAYS INCREASING}} > 0 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow \exists! \, c \in \mathbb{R} : f(c) = 0$$ (ONE SOLUTION)

PROBLEM 7.25 (ii)

$$x^5 = 5x - 6$$

SAME QUESTION

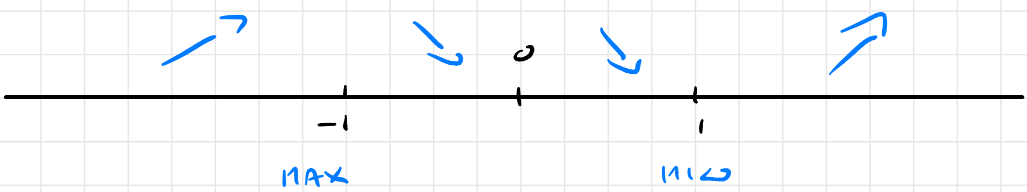
$$f(x) = x^5 - 5x + 6$$

$$\lim_{x \to \pm\infty} f(x) = \pm\infty$$

$$f'(x) = 5x^4 - 5 = 5(x^4 - 1)$$

$$f'(x) = 0 \Rightarrow x = \pm 1$$

$$f'(x) \begin{cases} > 0 & x > 1, \quad x < -1 \\ < 0 & -1 < x < +1 \end{cases}$$



$$f(-1) = -1 + 5 + 6 = 10 > 0$$

$$f(1) = 1 - 5 + 6 = 2 > 0$$

$\Rightarrow$ EXISTS ONLY ONE SOLUTION

Problem 7.22 (a)

A FACTORY THAT PRODUCES TOMATO SAUCE WANTS TO STORE IT INTO CYLINDRICAL CANS of VOL. V

r, h such THAT CONSUMES THE LEAST MIN. MATERIAL.

up & DOWN

$$\text{MATERIAL} \sim \underset{\text{AREA}}{S} = 2\,\underbrace{\pi r^2}_{\text{BASE}} + \underbrace{2\pi r h}_{\text{SIDES}}$$

Two VARIABLES

$$V = \pi r^2 h \quad \text{FIXED} \Rightarrow h = \frac{V}{\pi r^2}$$

SO

$$S = 2\pi\left(r^2 + \frac{V}{\pi r}\right) = 2\pi f(r)$$

$$f'(r) = 2r - \frac{V}{\pi r^2} = 0 \Rightarrow r^3 = \frac{V}{2\pi}$$

$$h = \left(\frac{4V}{\pi}\right)^{\frac{1}{3}} \quad \square$$