TAYLOR SERIES

$$\underbrace{f(x) = \sum_{n}^{\infty} a_n (x-a)^n}_{\underbrace{< \infty \,?}_{\forall x \in (-\rho, \rho)}}$$

Ex 8.18 (i)

$$\sum_{n=1}^{\infty} \frac{x^n}{2^n n^2} \qquad \rho = ?$$

CONVERGENCE RADIUS

$\hookrightarrow x : \sum < \infty$ ABS.

$$\sum_n a_n x^n \ , \quad \frac{1}{\rho} = \lim_{n \to +\infty} \sqrt[n]{|a_n|}$$

STOLZ

$$= \lim_{n \to +\infty} \frac{a_{n+1}}{a_n}$$

$$a_n = \frac{1}{2^n n^2} \ ,$$

$$\frac{1}{\rho} = \lim_{n \to \infty} \sqrt[n]{\frac{1}{2^n n^2}} = \lim_{n \to \infty} \frac{1}{2} \underbrace{\frac{1}{\sqrt[n]{n^2}}} = \frac{1}{2}$$

$\hookrightarrow \lim_n \sqrt[n]{n^p} = 1$

$$\rho = 2 \qquad \forall x \in (-\rho, \rho) \quad \sum_n |a_n| |x^n| < \infty$$

ABS. CONV.

$$x = 2 \ ? \qquad \sum_{n=1}^{\infty} \frac{\cancel{2^n}}{\cancel{2^n} n^2} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{Riemann}} < \infty$$

Ex 8.18 (ii)

$$\sum_{n=1}^{\infty} \frac{n!\, x^n}{n^n} = \sum_{n=1}^{\infty} a_n x^n, \quad a_n = \frac{n!}{n^n}$$

$$\frac{1}{\rho} = \lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \frac{(n+1)\cancel{!}}{(n+1)^{(n+1)}} \frac{n^n}{n\cancel{!}}$$

$$= \lim_{n\to+\infty} \frac{\cancel{(n+1)}}{(n+1)^{n+\cancel{1}}} n^n = \lim_{n\to+\infty} \left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to+\infty} \left[\left(1+\frac{1}{n}\right)^n\right]^{-1} = \frac{1}{e}$$

$$\rho = e, \quad \forall x \in (-e, e), \quad \sum_n \frac{n!}{n^n} x^n < \infty$$

$$x = e\,? \qquad \frac{n!\, e^n}{n^n} \sim \sqrt{2\pi n}\, \cancel{e^{-n}}\, \cancel{n^n}\, \frac{\cancel{e^n}}{\cancel{n^n}}$$

$$\sum_n \sqrt{2\pi n} = \infty \implies \sum_n \frac{n!\, e^n}{n^n} = \infty$$

Ex 8.22

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \quad , \quad \int = ? \quad , \quad \text{sum} ?$$

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \Rightarrow f'(x) = \sum_{n=1}^{\infty} \frac{\cancel{n} x^{n-1}}{\cancel{n}}$$

$$f'(x) = \sum_{n=1}^{\infty} x^{n-1} = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad |x|<1$$

$k = n-1$

$1 \mapsto 0$

$$f'(x) = \frac{1}{1-x} \Rightarrow f(x) = -\log(1-x) + c$$

$$f(x) = \int \frac{1}{1-x} dx$$

$$c ? \quad , \quad f(0) = c = \sum_n \frac{0^n}{n} = 0$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x) \quad , \quad |x|<1$$

Ex 8.24 (i)

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \quad \text{sum?}$$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

$$\sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Bigg|_{x=-1/2} = e^{-\frac{1}{2}}$$

Ex

$$\sum_{n=2}^{\infty} \frac{2^{n-1}}{3^{n+2}} \frac{1}{n!} = \sum_{n=0}^{\infty} \frac{2^{n-1}}{3^{n+2}} \frac{1}{n!} - \underbrace{\frac{1}{2 \cdot 3^2}}_{n=0} - \underbrace{\frac{1}{3^2}}_{n=1}$$

$$= \frac{1}{2 \cdot 3^2} \sum_{n=0}^{\infty} \frac{\left(\frac{2}{3}\right)^n}{n!} - \frac{1}{2 \cdot 3^2} - \frac{2}{3 \cdot 2 \cdot 3^2}$$

$$= \frac{1}{2 \cdot 3^2} \left(e^{\frac{2}{3}} - 1 - \frac{2}{3} \right)$$

Ex

$$\lim_{x \to 0} \frac{(\sin x)^2 - (e^x - 1)^2}{\log(1+x^3)} =$$

$$\sin x = x + o(x^2)$$

$$e^x = 1 + x + \frac{x^2}{2} + o(x^2)$$

$$\underbrace{\log(1+x^3) = x^3 + o(x^3)}_{x \to 0} \qquad \log(1+t) = t + o(t)$$

$$(\sin x)^2 = [x + o(x^2)]^2 = x^2 + \underbrace{\underbrace{2x\,o(x^2)}_{2\,o(x^3)}}_{o(x^3)} + \underbrace{o(x^2)\,o(x^2)}_{o(x^4)}$$

$$= x^2 + o(x^3) + o(x^4)$$

$$(e^x - 1)^2 = \left(\cancel{1} + x + \frac{x^2}{2} + o(x^2) \cancel{-1}\right)^2$$

$$= x^2 + \frac{x^4}{4} + o(x^2)\,o(x^2)$$

$$+ \; x^3 + x\,o(x^2) + \frac{1}{2}x^2 o(x^2)$$

$$= x^2 + x^3 + \underbrace{\frac{x^4}{4}}_{o(x^3)} + \underbrace{x\,o(x^2)}_{o(x^3)} + \underbrace{o(x^2)\,o(x^2)}_{o(x^4)} + \underbrace{\frac{x^2}{2} x^2 o(x^2)}_{o(x^4)}$$

$$= x^2 + x^3 + o(x^3) + \underbrace{o(x^4)}_{o(o(x^3))}$$

$$= x^2 + x^3 + o(x^3)$$

REPLACING INTO THE LIMIT

$$\lim_{x \to 0} \frac{[\cancel{x^2} + o(x^3)] - [\cancel{x^2} + x^3 + o(x^3)]}{x^3 + o(x^3)}$$

$$= \lim_{x \to 0} \frac{-x^3 + o(x^3)}{x^3 + o(x^3)} = -1$$

Ex 8.13

$$\lim_{n\to+\infty} \sin\left(\pi\sqrt{1+n^2}\right)$$

COLLECT n^2

$$= \lim_{n\to+\infty} \sin\left(\pi n\sqrt{1+\frac{1}{n^2}}\right)$$

$n \to \infty$, $\frac{1}{n} \approx x \to 0$

$\hookrightarrow$ n LARGE $\frac{1}{n}$ gets CONTINUOUS

$$\sqrt{1+x} = 1 + \frac{x}{2} + o(x)$$

$$\hookrightarrow \left.\frac{d}{dx}\sqrt{1+x}\,\right|_0 = \left.\frac{1}{2\sqrt{1+x}}\right|_0 = \frac{1}{2}$$

$$\sqrt{1+\frac{1}{n^2}} = 1 + \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)$$

$$\lim_n \sin\left(\pi n + \frac{\pi}{2n} + \underbrace{\pi n\, o\left(\frac{1}{n^2}\right)}_{o\left(\frac{1}{n}\right)}\right)$$

$$\varepsilon_n = \frac{\pi}{2n} + o\left(\frac{1}{n}\right) \longrightarrow 0$$

$$\sin\left(\pi n + \varepsilon_n\right) = (-1)^n \sin(\varepsilon_n)$$

$$= (-1)^n \left[\, \varepsilon_n + o(\varepsilon_n) \,\right]$$

$$= (-1)^n \varepsilon_n \left(1 + o(1) \right)$$

$$= (-1)^n \underbrace{\frac{\pi}{2n}}_{\to 0} \underbrace{\left(1 + \frac{o\left(\frac{1}{n}\right)}{\frac{\pi}{2n}} \right)}_{\to 1} \underbrace{\left(1 + o(1) \right)}_{\to 1}$$

$$\lim_n \sin \pi \sqrt{1 + n^2} = 0$$

$$\sum_n \sin^2\left(\pi\sqrt{1+n^2}\right) \equiv \sum_n a_n$$

$a_n \sim \frac{1}{n^2}$ Geometric Series Term

$\sum_n a_n <$ COMP TEST.