W6 LIMITS

$$\lim_{x \to x_0} f(x) = L$$

APPROXIMATES f WHEN $\underbrace{x \to x_0}_{\text{x gets close to } x_0}$

$\underbrace{\qquad\qquad}$

f CONST $\Rightarrow f = L$

IN GENERAL, APPROXIMATION

TODAY WE DELVE INTO THIS IDEA

W8, 9 : TAYLOR EXPANSION

APPROXIMATIONS

→ APPROXIMATELY 50 MILLIONS OF PEOPLE LIVE IN SPAIN

$$P_{SP} = 50 \text{ mill.} \pm \underbrace{\varepsilon}_{\substack{\text{SOMETHING} \\ \text{WHICH I} \\ \text{DON'T CARE}}}$$

$$\underbrace{}_{\text{NEGLIGIBLE}}$$

$$\frac{\varepsilon}{50_{\text{mill}}} \approx 0$$

$$= 0$$

↓

IN THE POPULATION LIMIT GROWS

WHAT DOES IT MEAN NEGLIGIBILITY?

NEGLIGIBILITY

DEF

$$\underbrace{f \ll g \quad x \to a}_{} \iff \lim_{x \to a} \frac{f(x)}{g(x)} = 0$$

$\downarrow$ NEW NOTATION

$$\underbrace{f = o(g) \quad x \to a}$$ SMALL o

LANDAU NOTATION

Ex

$$\log x = o(x), \quad x \to +\infty$$

$$\lim_{x \to +\infty} \frac{\log x}{x} = 0$$ (HÔPITAL)

Ex

$$\sin(x) = o(\sqrt{x}), \quad x \to 0^{+} \to \sqrt{\ }$$

$$\lim_{x \to 0^+} \frac{\sin(x)}{\sqrt{x}} = \lim_{x \to 0^+} \underbrace{\sqrt{x}}_{\to 0} \underbrace{\frac{\sin(x)}{x}}_{\to 1} = 1$$

Ex $x^\alpha = o(x^\beta) \quad x \to 0 \quad (\alpha > \beta \geq 0)$

$$\lim_{x \to 0} \frac{x^\alpha}{x^\beta} = \lim_{x \to 0} \frac{\cancel{x^\beta}\, x^{\overbrace{\alpha-\beta}^{>0}}}{\cancel{x^\beta}} = 0$$

Ex $x^3 = o(x^2) \quad x \to 0$

$x^4 = o(x^2) \quad x \to 0$

Ex $x^\alpha = o(x^\beta) \quad x \to \pm\infty \quad (\beta > \alpha \geq 0)$

$$\lim_{x \to \pm\infty} \frac{x^\alpha}{x^\beta} = \lim_{x \to \pm} \frac{\cancel{x^\alpha}}{\cancel{x^\alpha}\, x^{\underbrace{\beta-\alpha}_{>0}}} = 0$$

Combine Small o

↳ Algebra Small o

$\underbrace{\hspace{6cm}}$

Follows from that of limits

- $f = o(g)$ $\lambda \in \mathbb{R} \Rightarrow \lambda f = o(g)$
  $x \to x_0$ $\underline{x \to x_0}$
  $-\infty \leq x_0 \leq \infty$
  I do not repeat any time

- $f_1, f_2 = o(g) \Rightarrow f_1 + f_2 = o(g)$

- $\left.\begin{array}{l} f_1 = o(g_1) \\ f_2 = o(g_2) \end{array}\right\} \Rightarrow f_1 f_2 = o(g_1 g_2)$

- $\left.\begin{array}{l} f = o(g) \\ g = o(h) \end{array}\right\} \Rightarrow f = o(o(h)) = o(h)$

- $f = o(g) \Rightarrow hf = o(hg)$

<u>Ex</u> $[2 + x + o(x^2)][x^2 + o(x^3)]$ $x \to 0$

$$= 2x^2 + 2\,o(x^3) + x^3 + x\,o(x^3)$$

$$+ x^2 o(x^2) + o(x^2)\,o(x^3)$$

$$= 2x^2 + x^3 + \underbrace{2\,o(x^3)}_{o(x^3)} + \underbrace{x\,o(x^3)}_{o(x^4)}$$

$\lambda\, o(f) = o(f)$ $\qquad$ $h\, o(f) = o(hf)$

$$+ \underbrace{x^2 o(x^2)}_{o(x^4)} + \underbrace{o(x^2)\,o(x^3)}_{o(x^5)}$$

$h\, o(f) = o(hf)$ $\qquad$ $o(g_1)\,o(g_2) = o(g_1 g_2)$

$$= 2x^2 + x^3 + o(x^3) + \underbrace{o(x^4) + o(x^4) + o(x^5)}_{o(x^4)}$$

$$= 2x^2 + x^3 + o(x^3) + o(\underbrace{o(x^3)}_{x^4}) + o(\underbrace{o(x^3)}_{x^5})$$

$$= 2x^2 + x^3 + \underbrace{o(x^3)}_{\text{NEGLIGIBLE QUANTITY}}$$

APPROXIMATIONS of a FUNCTION

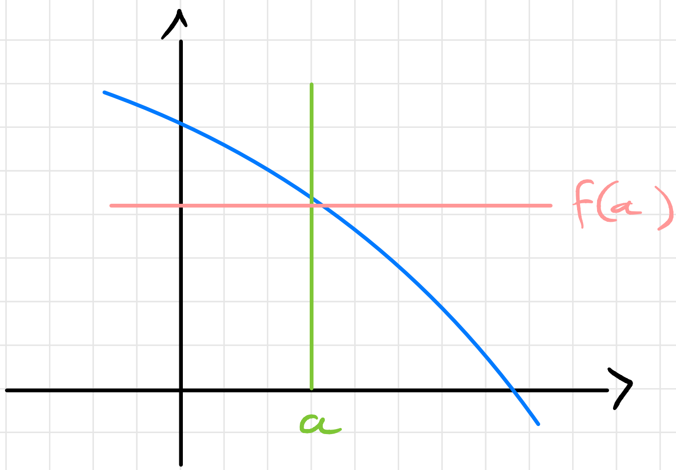
f , $\underbrace{x \to a}_{\text{ENVIRONMENT } a}$ | WHERE WE APPROXIMATE

f CONT a $\Rightarrow$ $\lim_{x \to 0} f(x) = f(a)$

(WE ASSUME)

$$\lim_{x \to 0} [f(x) - f(a)] = 0$$

$$f(x) = f(a) + \underbrace{o(1)}_{\text{something which goes to 0 when } x \to 0}$$



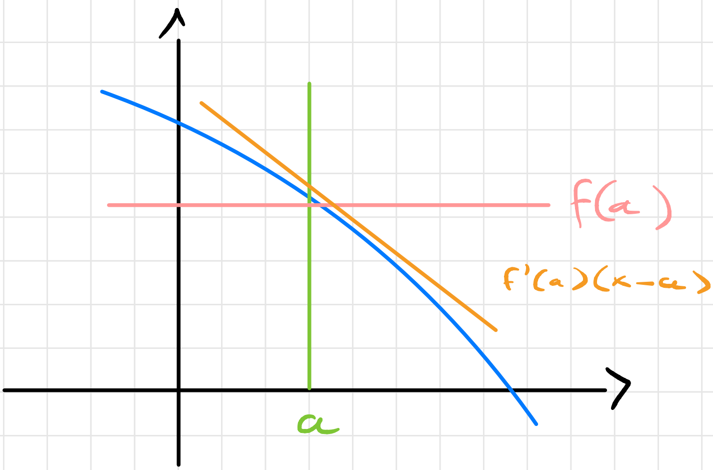
$f(a)$ APPROXIMATE $f(x)$ AROUND a

SUCH A BAD APPROXIMATION!

$$f \text{ DIFF } a \Rightarrow \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} - f'(a) \right] = 0$$

$$\Rightarrow \lim_{x \to a} \left[\frac{f(x) - f(a) - f'(a)(x-a)}{x - a} \right] = 0$$

$$\Rightarrow f(x) = \underline{f(a)} + \underline{f'(a)(x-a)} + o(x-a)$$



$f(x) \approx$ TANGENT

↳ WE GAIN INFO

↳ PARABOLA

WE PUSH THE IDEA A LITTLE BIT FURTHER

$$f(x) = f(a) + f'(a)(x-a) + c_2 + o(x-a)^2$$

$$c_2 = \lim_{x \to a} \frac{f(x) - f(a) - f'(a)(x-a)}{(x-a)^2}$$

HÔPITAL $$\lim_{x \to a} \frac{1}{2} \frac{f'(x) - f'(a)}{x-a} = \underbrace{\frac{1}{2} f''(a)}_{c_2}$$

ALONG THE SAME LINE

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + o(x-a)^3$$

n TIMES ...

THEOREM (TAYLOR)

f diff n times

$$\Rightarrow f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{P_{n,a}(x)} + o(x-a)^n \quad x \to a$$

k^{th} derivative

Provided is diff.

VERY BEAUTIFUL THEOREM

↳ EACH FUNCTION MAY BE APPROXIMATED BY MEANS OF A POLYNOMIALS

o(- - -) IS THE ERROR

↳ NEGLIGIBLE IN A CERTAIN LIMIT

Ex

$\cos(x) = \_\ \_\ \_ \qquad x \to 0$

$f^{(0)}(0) = \cos(0) = 1$ (THE FUNCTION ITSELF)

$f^{(1)}(0) = -\sin(0) = 0$

$f^{(2)}(0) = -\cos(0) = -1$

$$\underbrace{\cos(x) = 1 - \frac{x^2}{2}}_{} + o(x^2)$$

WE RECOVER EQUIVALENCE RELATIONSHIP

Ex

$e^x = \_\ \_\ \_ \qquad x \to 0$

$f^{(0)}(0) = e^0 = 1$

$f^{(1)}(0) = e^0 = 1$

$f^{(2)}(0) = e^0 = 1$

$$\underbrace{e^x = 1 + x}_{} + \frac{x^2}{2} + o(x^2)$$

EQUIVALENCE RELATIONSHIP

Ex

$\sin(x) = x + o(x^2)$

APPLICATIONS → LIMITS

$$\lim_{x \to 0} \frac{\cos x - e^x + x}{x^2}$$

$$= \lim_{x \to 0} \frac{\cancel{1} - \frac{x^2}{2} + o(x^2) - \cancel{1} - \cancel{x} - \frac{x^2}{2} - o(x^2) + \cancel{x}}{x^2}$$

$$= \lim_{x \to 0} \frac{1}{\cancel{x^2}} \left(- \frac{2 \cancel{x^2}}{2} + \underbrace{o(x^2) - o(x^2)}_{} \right)$$

$o(x^2) + o(x^2) = o(x^2)$

$\hookrightarrow \lambda o(f) = o(f)$

$$= -1 + \underbrace{\lim_{x \to 0} \frac{o(x^2)}{x^2}}_{0 \text{ by definition}}$$

COMMENT

↳ EXPANSIONS ORDER 2

↳ WHY ?

↳ LIKE HOPITAL

↳ HOW MANY TIME do you DERIVATE

→ 2 (Because of DENOMINATORE)

HÔPITAL
TAYLOR } → DERIVATIVES

↳ SAME THINGS

Ex

$$\lim_{x \to +\infty} x^2 \left(1 - \sec \frac{1}{x} \right)$$

$$\sec \alpha = \frac{1}{\cos \alpha}$$

NEW VARIABLE $t = \frac{1}{x}$, $t \to 0^{+)} \to \underbrace{+\infty}_{\text{we don't care}}$

$$\lim_{t \to 0} \left(\frac{1}{t^2}\right) \left(1 - \frac{1}{\cos t} \right)$$

$$\lim_{t \to 0} \left(\frac{1}{t^2}\right) \left(1 - \frac{1}{1 - \frac{t^2}{2} + o(t^2)} \right)$$

$$\lim_{t \to 0} \frac{1}{t^2} \frac{\cancel{1} - \cancel{1} - \frac{t^2}{2} + o(t^2)}{1 - \frac{t^2}{2} + o(t^2)}$$

$7 o(f) = o(f)$

$$\lim_{t \to 0} \left[-\frac{1}{2} + \underbrace{\frac{o(t^2)}{t^2}}_{\to 0} \right] \left[1 - \underbrace{\frac{t^2}{2}}_{\to 0} \left(1 + \underbrace{\frac{o(t^2)}{t^2}}_{\to 0} \right) \right]^{-1} = -\frac{1}{2}$$

TAYLOR

↳ FUNCTION = Polynomials + $o(x)$

↳ ERROR

QUANTIFY

$$f(x) = P_{n,a}(x) + \underbrace{R_{n,a}(x)}_{\text{REST}}$$

$$R_{n,a}(x) \begin{cases} \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} & \text{LAGRANGE} \\[2ex] \dfrac{f^{(n+1)}(c)}{(n+1)!}(1-\vartheta)^{n}(x-a) & \text{CAUCHY} \end{cases}$$

$$c = (1-\vartheta)a + \vartheta x \,, \quad \vartheta \in [0,1]$$