LAST WEEK

↳ LIMIT

↳ INDETERMINACIES

$$\frac{0}{0}, \ \frac{\infty}{\infty}, \ 0\cdot\infty, \ 0^0, \ \infty^0, \ \infty-\infty, \ 1^\infty$$

OPERATIONS THAT ARE NOT DETERMINED

WARNING:

↳ THESE ARE NOT INDETERMINACIES

$$\infty^\infty=\infty, \ 0^\infty=0, \ \infty+\infty, \ \frac{0}{\infty}=0, \ \frac{\infty}{0}=\pm\infty$$

THERE IS NOT A GENERAL RULE

↳ THEY NEED TO BE EVALUATED CASE BY CASE

$$a_n = (1+\underset{0}{\underset{\downarrow}{b_n}})^{\underset{+\infty}{\underset{\downarrow}{c_n}}} \to 1^\infty$$

↳ $L = e^{b_n c_n}$

Ratio of Polynomials

AN USEFUL TOOL IS

STOLZ THEOREM $\{a_n\}\ \{b_n\}$

Hy) b_n STRICTLY MONOTONIC (INCREASING OR DECREASING)

either

Hy) $b_n \to \pm\infty$

or

$a_n, b_n \to 0$

$\Rightarrow$ (Th)

$$\underbrace{\lim_n \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = L}_{} \Rightarrow \underbrace{\lim_n \frac{a_n}{b_n} = L}_{}$$

IF I KNOW THIS . . .

$\longrightarrow$ THE LIMIT IS THE SAME

I WANT TO CALCULATE THIS

LOOKS LIKE INVOLVED

EX

$$\lim_n \frac{1+\frac{1}{2}+\frac{1}{3}\cdots+\frac{1}{n}}{\log n} = \lim \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k}$$

Idea? → STOLZ → RECOGNIZE a_n, b_n

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$$b_n = \log n$$

STOLZ SUGGES TO LOOK INTO

$$\lim_n \frac{a_n - a_{n-1}}{b_n - b_{n-1}}$$

IF THIS EXISTS, IT MATCHES THE WISHED LIMITS

$$a_n - a_{n-1} = \underbrace{\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}+\frac{1}{n}\right)}_{a_n} - \underbrace{\left(1+\cdots+\frac{1}{n-1}\right)}_{b_n} = \frac{1}{n}$$

$$b_n - b_{n-1} = \log n - \log(n-1) = \log\left(\frac{n}{n-1}\right)$$

$$= \log\left(\frac{n-1+1}{n-1}\right) = \log\left(1+\frac{1}{n-1}\right)^n$$

$$\lim_n \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \lim_n \frac{1}{n \log\left(1+\frac{1}{n-1}\right)}$$

$$= \lim_n \frac{1}{\log\left(1+\frac{1}{n-1}\right)^n} = \frac{1}{\log\Big[\underbrace{\lim_n \left(1+\frac{1}{n-1}\right)^n}_{\text{EQUIVALENT TO } 1+\frac{1}{n-1}}\Big]}$$

$$= \frac{1}{\log e} = 1$$

$$\Rightarrow \lim_n \frac{a_n}{b_n} = \lim_n \frac{1+\frac{1}{2}+\dots+\frac{1}{n}}{\log n} = 1$$

BECAUSE OF STOLZ

STOLZ THEOREM HAS THE FOLLOWING

$$\lim_n \frac{a_n}{a_{n-1}} = L \Rightarrow \lim_n \sqrt[n]{a_n} = L$$

$$\lim_n a_n = L \Rightarrow \lim_n \sqrt[n]{a_1 \cdot a_2 \cdots a_n} = L$$

$$\lim_n a_n = L \Rightarrow \lim_n \frac{1}{n}(a_1 + \dots + a_n) = L$$

ANOTHER POINT OF VIEW TO DEAL WITH INDET.

ASYMPTOTICAL COMPARISONS

$\{a_n\}, \{b_n\}$ SEQUENCES (which both converge/div.)

$a_n \sim b_n$ (EQUIVALENT) $\Leftrightarrow \lim_n \frac{a_n}{b_n} = 1$

$a_n << b_n$ (NEGLIGIBLE) $\Leftrightarrow \lim_n \frac{a_n}{b_n} = 0$

a_n NEG b_n

Ex $\varepsilon_n \xrightarrow[n\to+\infty]{} 0$ a generic sequence going to zero

WE KNOW

$$(1+\varepsilon_n)^{\frac{1}{\varepsilon_n}} \longrightarrow e$$

and so

$$\lim_n \log\left[(1+\varepsilon_n)^{\frac{1}{\varepsilon_n}}\right] = 1$$

namely

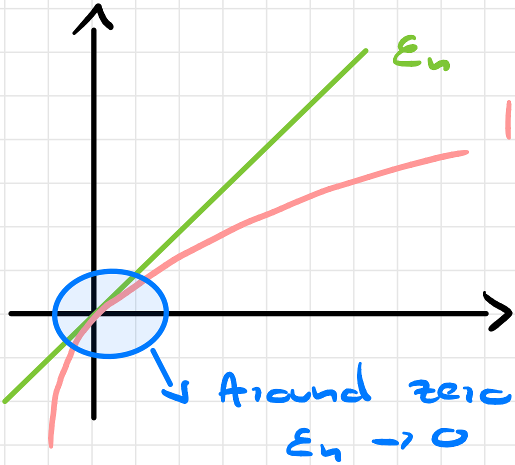
$$\lim_n \frac{1}{\varepsilon_n}\log(1+\varepsilon_n) = 1 \iff \boxed{\varepsilon_n \sim \log(1+\varepsilon_n)}$$

Ex $\underbrace{\log(1+\varepsilon_n)}_{\delta_n} \sim \varepsilon_n$

$\hookrightarrow e^{\delta_n} - 1$

I get $\boxed{\delta_n \sim e^{\delta_n} - 1}$

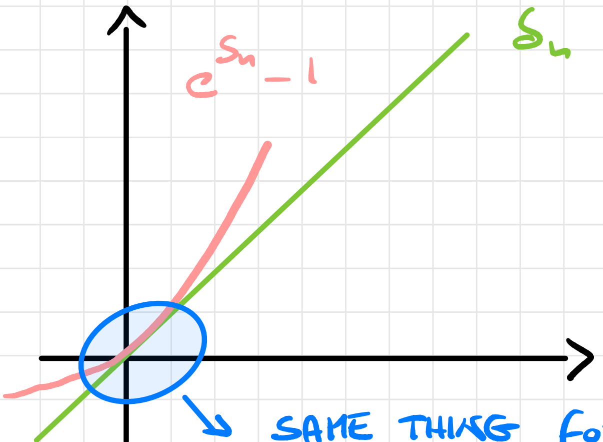
WHAT DOES IT MEAN GRAPHICALLY?



of course

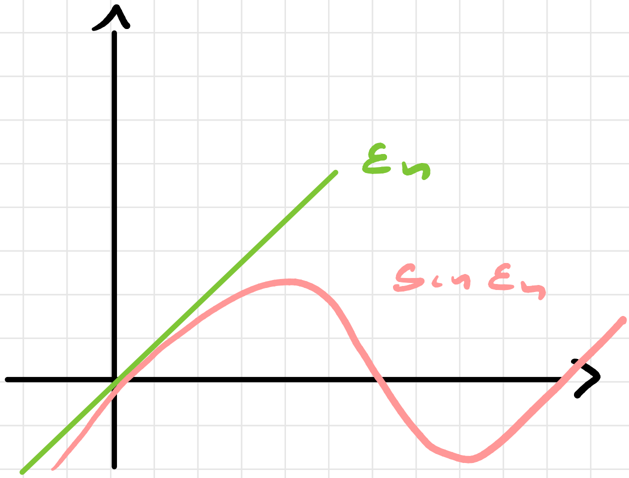
Around zero THEY ARE THE SAME THING

$\varepsilon_n \to 0$



SAME THING for $\varepsilon_n \to 0$

Similarly



$$\boxed{\sin \varepsilon_n \sim \varepsilon_n}$$

ALONG THE SAME LINE

$$\boxed{\tan \varepsilon_n \sim \varepsilon_n}$$

Overall

$$\varepsilon_n \sim \sin \varepsilon_n \sim \tan \varepsilon_n \sim (e^{\varepsilon_n} - 1) \sim \log(1+\varepsilon_n)$$

WE ALSO HAVE

$$\boxed{\cos \varepsilon_n = 1 - \frac{\varepsilon_n^2}{2}}$$

and

$$\boxed{n! \sim \sqrt{2\pi n}\, n^n e^{-n}}$$ (STIRLING)

Ex

$$\lim_n \frac{\sin\left(\frac{1}{n}\right)}{\log\left(\frac{n+1}{n}\right)} = \lim_n \frac{\sin\left(\frac{1}{n}\right)}{\log\left(1+\frac{1}{n}\right)}$$

$$= \lim_n \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \frac{\frac{1}{n}}{\log\left(1+\frac{1}{n}\right)} = 1 \cdot 1$$

WHAT ABOUT NEGLIGIBILITY?

$$(\log n)^a \ll n^b \ll c^n \ll \underbrace{n! \ll n^n}$$

$(a, b > 0, \quad c > 1)$

I prove THIS

$$\lim_n \frac{n!}{n^n} = \lim_n \underbrace{\frac{n}{n}}_{\downarrow \atop 1} \underbrace{\frac{n-1}{n}}_{\downarrow \atop 1} \cdots \underbrace{\frac{1}{n}}_{\downarrow \atop 0} = 0$$

A NATURAL EXTENSIONS of THE NOTION of SEQUENCE IS :

SERIES

SEQUENCE made up by SUM of TERMS of another seq.

Sequence $\{a_n\}$

$$S_K = \sum_{n=1}^{K} a_n = a_1 + \dots + a_K \quad \text{(PARTIAL SUM)}$$

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$S_4 = a_1 + a_2 + a_3 + a_4$$

$$\lim_{K \to +\infty} S_K = \sum_{n=1}^{\infty} a_n \quad \text{(SERIES)}$$

Sum of ∞ NUMBERS

Ex Geometric Series

$$\boxed{\sum_{n=0}^{\infty} x^n} = \underbrace{1 + x + x^2 + \_ \_ + x^n + \_ \_ \_}$$

How does this ∞ sum behave?

$x = 1$, $S_K = \sum_{n=0}^{K} x^n = 1 + 1 + \_ \_ = K + 1 \to +\infty$ (TRIVIAL)

CASE $x \neq 1$

MULTIPLY BOTH SIDE by x

$$x S_K = x \sum_{n=0}^{K} x^n = \sum_{n=0}^{K} x^{n+1}$$

$$= \underbrace{x + x^2 + \_ \_ \_ + x^K}_{S_K - 1} + x^{K+1}$$

$$= S_K - 1 + x^{K+1}$$

$S_K = 1 + x + \_ \_ + x^K$

$$\Rightarrow (x-1) S_K = x^{K+1} - 1$$

$$S_K = \frac{x^{K+1} - 1}{x - 1}$$

$k \to +\infty$

$|x| < 1 \qquad \lim_k \frac{x^{k+1} - 1}{x - 1} = \frac{1}{1-x}$ CONVERGES

$x \geq 1 \qquad \lim_k \quad '' \quad = +\infty$ DIVERGES

$x < -1 \qquad '' \quad '' \quad = \times$ ALTERNATING

I WILL USE THIS NOTATION TO SAY

$\sum_{n=0}^{\infty} a_n = \infty$ DIVERGENT

$= \quad < \infty$ CONVERGENT

WHAT HAPPENS IF I START BY A DIFFERENT NUM?

$$\sum_{n=r}^{\infty} x^n = \frac{x^r}{1-x} \qquad |x|<1$$

- CONVERGENT PROPERTIES does NOT CHANGE

  $|x|<1 \qquad \sum < \infty$

  $x > 1 \qquad = \infty$

  $x < -1 \qquad$ ALT

- LIMIT CONVERGENT CASE CHANGES

Homework

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2} \qquad |x|<1$$

(ARITHMETIC - GEOMETRIC SERIES)

Ex TELESCOPIC SERIES

$$S_K = \sum_{n=1}^{K} a_n = \sum_{n=1}^{K} (U_n - U_{n+1})$$

$$= U_1 - U_{K+1}$$

So - -

$$\boxed{\sum_{n=1}^{\infty} a_n = U_1 - \lim_K U_K}$$

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \underbrace{\frac{1}{n}}_{U_n} - \underbrace{\frac{1}{n+1}}_{U_{n+1}}$

$$= U_1 - \lim_n U_n = 1 - \lim_n \frac{1}{n} = 1$$