W5 LIMITS

↳ OPERATIONS on FUNCTIONS

↳ TREND

TO WHICH VALUE TENDS THE FUNCTION WHEN ITS ARGUMENT APPROACHES A CERTAIN POINT

↳ HOW FAST?

↳ NEW OPERATION.

W6 DERIVATIVE

def $f : D_f \subset \mathbb{R} \to \mathbb{R}$, $a \in D_f$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

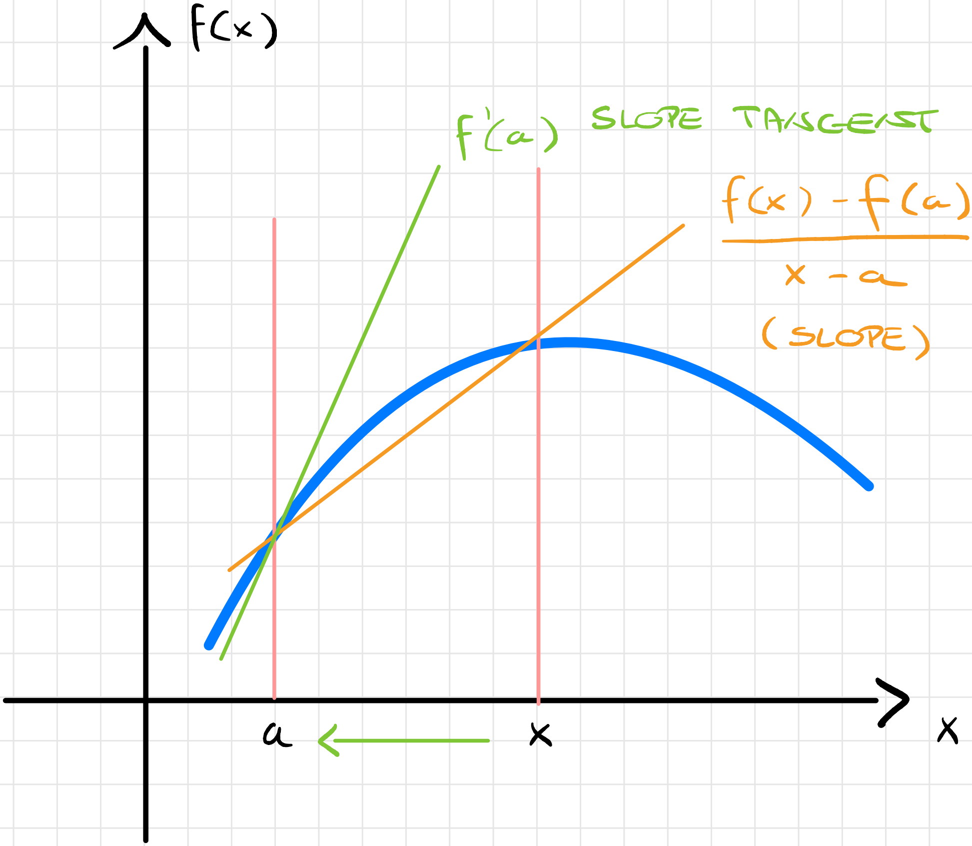
$x = a + h$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

NOTATION

$$f'(a) = \underbrace{\frac{d}{dx} f(x)}_{\text{LEIBNIZ}} \Big|_{x=a}$$

def f DIFF. $a \iff \exists\, f'(a)$



MATHEMATICS

$f'(a)$

GEOMETRY

TANGENT SLOPE

PHYSICS

VELOCITY

Ex $f(x) = x^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + h^2 + 2xh - \cancel{x^2}}{h} = \lim_{h \to 0} h + 2x = 2x$$

HOMEWORK $f(x) = x^n$

$$f'(x) = n x^{n-1} \quad \forall n \in \mathbb{N}$$

HINT NEWTON'S BINOMIAL

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$ BINOMIAL COEFFICIENT

Ex $f(x) = e^{ax} \quad (a \in \mathbb{R})$

$$f'(x) = \lim_{h \to 0} \frac{e^{a(x+h)} - e^{ax}}{h}$$

$$= e^{ax} \lim_{x \to 0} \frac{e^{ah} - 1}{h}$$

$$= a e^{ax} \underbrace{\lim_{x \to 0} \frac{e^{ah} - 1}{ah}}_{1} = a e^{ax}$$

$\underline{\text{Ex}}$ $f(x) = \sin x$

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cosh + \sinh \cos x - \sin x}{h}$$

$$= \sin x \lim_{h \to 0} \underbrace{\left(\frac{\cosh - 1}{h} \right)}_{\sim -\frac{h^2}{2h}} + \cos x \underbrace{\lim_{h \to 0} \frac{\sinh}{h}}_{1}$$

$$= \cos x - \sin x \underbrace{\lim_{x \to 0} \frac{h^{\cancel{2}}}{2\cancel{h}}}_{0} = \cos x$$

$\underline{\text{HOMEWORK}}$ $f(x) = \cos x$

$f'(x) = -\sin x$

PREVIOUS DEFINITIONS → DERIVATIVE IN A POINT

<u>def</u> f DIFF. $(a,b) \iff \forall x \in (a,b)$ f DIFF x

WE CAN INTRODUCE

<u>def</u> $D : (a,b) \to \mathbb{R}$, $x \mapsto f'(x)$ DERIVATIVE FUNCTION

NOTE

- $f'(x)$ NUMBER
- $D(f)$ FUNCTION
  ↳ ALONG THE SAME LINE

<u>def</u> $D^{(2)} : (a,b) \to \mathbb{R}$

$$x \mapsto f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

↓ SECOND DERIVATIVE

↳ and so on .. $D^{(n)}$

THEOREM $f : D_f \subseteq \mathbb{R} \to \mathbb{R}$

Hy) f diff. $x_0 \in D_f$

$\Rightarrow$ (Th) f cont x_0

Proof $f(x) = f(x) - f(x_0) + f(x_0)$

$$= \frac{f(x) - f(x_0)}{x - x_0} (x - x_0) + f(x_0)$$

$$\lim_{x \to x_0} f(x) = \underbrace{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}_{\substack{f'(x_0) \\ \text{(EXISTS BY Hy)}}} \underbrace{\lim_{x \to x_0} (x - x_0)}_{= 0} + f(x_0)$$

$$\lim_{x \to x_0} f(x) = f(x_0) \quad (f \text{ cont})$$

MESSAGE USE Hy TO GET Th

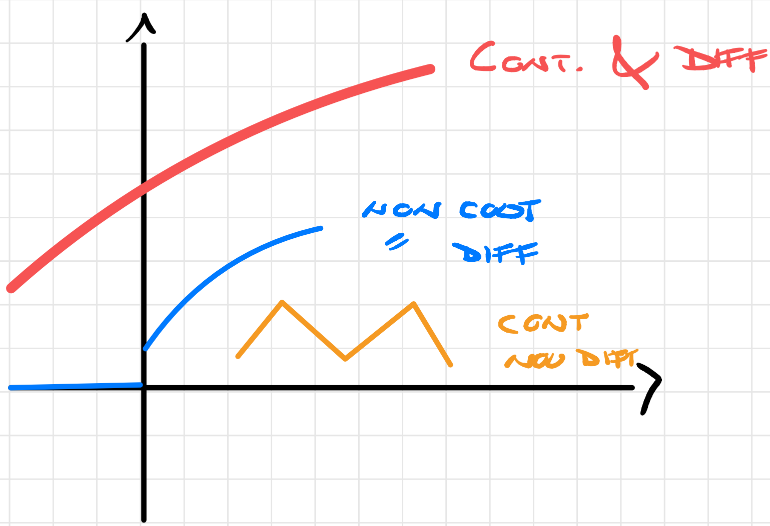
OPPOSITE IS TRUE ?

f CONT x_0 $\cancel{\Longrightarrow}$ f DIFF x_0

<u>Ex</u> $\overbrace{|x|}^{f}$ CONT in 0

But . . .

$$\left.\begin{aligned} &\underbrace{\lim_{x \to 0^+} \frac{|x| - 0}{x - 0}}_{f'_+(0)} = +1 \\ &\underbrace{\lim_{x \to 0^-} \frac{|x| - 0}{x - 0}}_{f'_-(0)} = -1 \end{aligned}\right\} \quad \nexists f'(0)$$



ALGEBRAIC PROPERTIES f, g DIFF.

(i) $\mu, \lambda \in \mathbb{R}$ $\quad (\mu f + \lambda g)' = \lambda f' + \mu g'$ LINEARITY

(ii) $(fg)' = f'g + g'f$ LEIBNIZ

(iii) $(f \circ g)' = (f' \circ g)\, g'$ CHAIN / UNION

(iv) $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$ QUOTIENT

(v) $(f^{-1})' = \dfrac{1}{f' \circ f^{-1}}$

Ex $f(x) = \sinh(x)$

$$\frac{d}{dx} f(x) = \frac{1}{2}\left(\underbrace{\frac{d}{dx} e^{x}}_{e^{x}} - \underbrace{\frac{d}{dx} e^{-x}}_{-e^{-x}}\right)$$

$$= \cosh x$$

$$\frac{d}{dx} \cosh x \quad ?$$ HOMEWORK

Ex $\frac{d}{dx} \tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \equiv \frac{d}{dx}\left(\frac{f}{g}\right)$

$f(x) = \sin x \quad , \quad f'(x) = \cos x$

$g(x) = \cos x \quad , \quad g'(x) = -\sin x$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$$

$$\frac{d}{dx} \tan x = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\frac{d}{dx} \cot x \quad ?$$

$\underline{\text{Ex}}$ $\frac{d}{dx} \log(x)$ → INVERSÉ

$$f^{-1}(x) = \log x \quad , \quad f(x) = e^x$$

$$(f^{-1})' = \frac{1}{f' \circ f^{-1}} = \frac{1}{e^{\log x}} = \frac{1}{x}$$

$\underline{\text{Ex}}$ $\frac{d}{dx} \operatorname{arctg}(x)$

$$f^{-1}(x) = \operatorname{arctg} x \quad , \quad f(x) = \operatorname{tg} x$$

$$(f^{-1})' = \frac{1}{f' \circ f^{-1}}$$

$$f' = \frac{1}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1 + \operatorname{tg}^2 x$$

$$f' \circ f^{-1} = 1 + (\operatorname{tg}(\operatorname{actg}))^2 = 1 + x^2$$

$$(f^{-1})' = \frac{1}{1 + x^2}$$

Ex $f(x) = x^{\alpha} \quad (\alpha \in \mathbb{R})$

$$f'(x) = \frac{d}{dx} e^{\alpha \log x} = e^{\alpha \log x} \frac{\alpha}{x} = \alpha x^{\alpha - 1}$$

$\hookrightarrow$ CHAIN

GRAPHICAL APPLICATIONS

def $f : D_f \subset \mathbb{R} \to \mathbb{R}$

$x_0 \in D_f$ LOCAL MIN. $f \iff \exists \delta > 0 : f(x) \geq f(x_0)$

$$\forall x \in \underbrace{(x_0 - \delta, x_0 + \delta)}_{\text{ENVIRONMENT}}$$

x_0 ABSOLUTE MIN. $\iff \forall x \in D_f \quad f(x) \geq f(x_0)$

(SIMILARLY FOR MAX.)

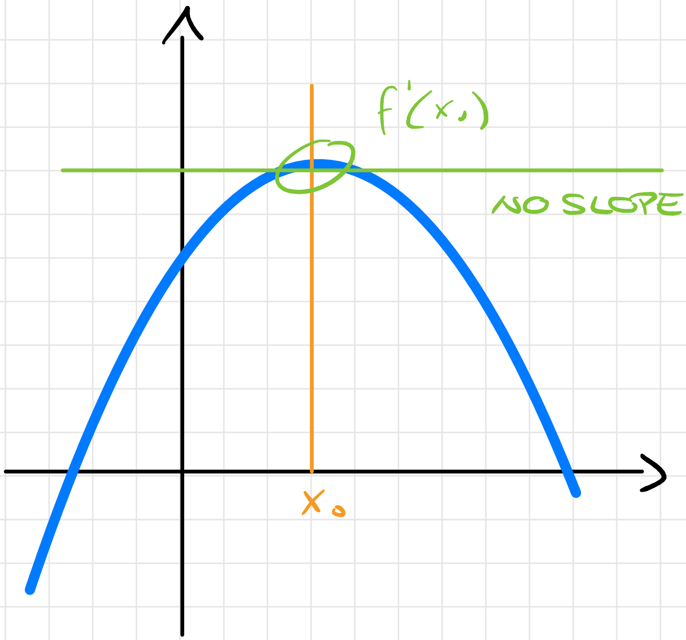
THEOREM $f: D_f \subset \mathbb{R} \to \mathbb{R}$

Hy) $\exists\, x_0 \in D_f$ LOCAL MIN. (o MAX)

Hy) f DIFF x_0

$\Rightarrow$ (Th) $f'(x_0) = 0$

Proof SEE NOTES



SAME for

- MAX.
- ABS.

QUESTIONS

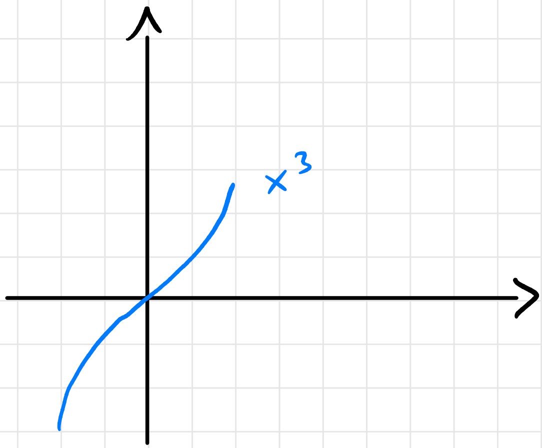
↳ OPPOSITE TRUE? NO!

Ex $f(x) = x^3$

$x_0 = 0 \quad f'(x_0) = 3x_0^2 = 0$

But ...

x_0 NON MIN.



THEOREM (ROLLE)

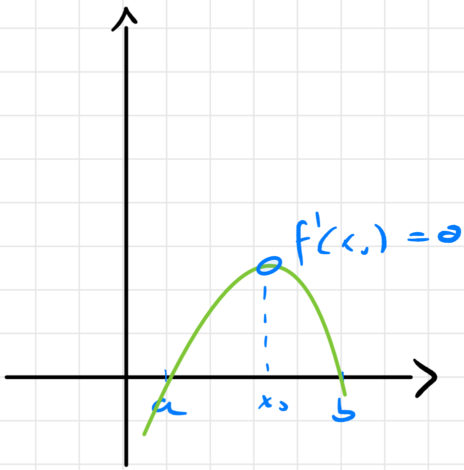
$f : D_f \subset \mathbb{R} \to \mathbb{R}$

Hy) f CONT. $[a,b]$

Hy) " DIFF. "

Hy) $f(a) = f(b)$

$\Rightarrow$ (Th) $\exists\, c \in (a,b)$ $f'(c)$



THEOREM (MEANS VALUES THEOREM)

$f : D_f \subset \mathbb{R} \to \mathbb{R}$

Hy) f CONT. $[a,b]$

Hy) " DIFF. "

$\Rightarrow$ (Th) $\exists c \in (a,b)$ $f(b)-f(a) = f'(c)(b-a)$

COROLLARY

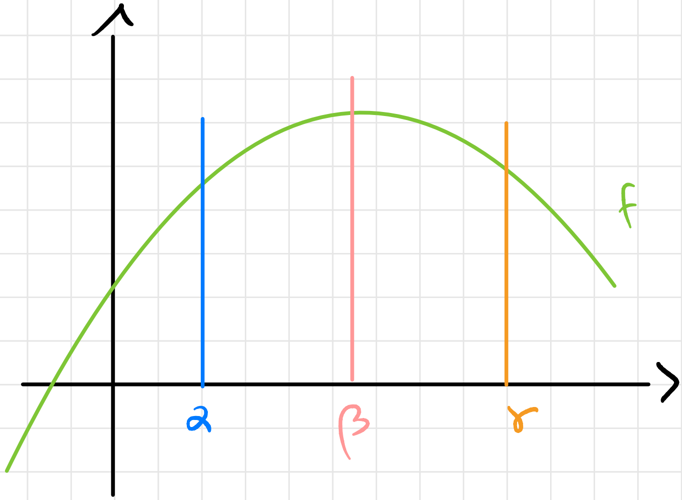
$f'(x) > 0 \quad \forall x \in (a,b)$

$\Rightarrow$ f STRICTLY INCREASING

$\Leftarrow$ OK TOO!

$$f'(c) > 0 \quad \Rightarrow \quad \frac{f(b)-f(a)}{b-a} > 0$$

$$\Rightarrow \quad \underset{<}{b > a} \quad \Rightarrow \quad f(b) \underset{<}{>} f(a)$$



$f'(\alpha) > 0$

$f'(\beta) = 0$

$f'(\gamma) < 0$

Ex $f(x) = x^2$

$x > 0$ $\quad x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ (INC)

$x < 0$ $\quad$ " > " $\quad$ " $\quad$ " > " (DEC)

$x = 0$ $\quad$ MIN.

$f'(x) = 2x$

↳ $f'(0) = 2 \cdot 0 = 0$

$x < 0 \Rightarrow 2 \cdot x < 0$ (DEC)

$x > 0 \Rightarrow 2 \cdot x > 0$

THEOREM (HÔPITAL) f, g DIFF.

Hy) $\lim_{x \to x_0} f = \lim_{x \to x_0} g = 0$

or

$\quad \text{〃} \qquad \text{〃} \qquad \pm\infty$

$\Rightarrow$ (Th) $\lim_{x \to x_0} \frac{f'}{g'} = L \Rightarrow \lim_{x \to x_0} \frac{f}{g} = L$