# CÁLCULO INTEGRAL con más variables

→ REGIONES ELEMENTALES

$$Q = [a,b] \times [c,d]$$

$$\iint_Q dA\, f(\ldots) \rightarrow \text{Teorema Fubini}$$

$$\int_a^b dx \int_c^d dy\, f(x,y)$$

→ Funciones $x(y)$ sencillas

$$Q = \{(x,y) \in \mathbb{R}^2 : x \in [a,b],\ \varphi_1(x) \leq y \leq \varphi_2(x)\}$$

$$\iint_Q dA\, f(\ldots) \rightarrow \text{FÓRMULA ITERADA}$$

$$\int_a^b dx \underbrace{\int_{\varphi_1(x)}^{\varphi_2(x)} dy\, f(x,y)}_{\text{Función } x}$$

¿Qué ocurre si consideramos una región que x e y simple a la vez

→ CÍRCULO

$$D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq a^2\}$$
$$= \{ \text{'' ''} : 0 \leq y \leq \sqrt{a^2 - x^2} \,,\; x \in [0,a] \}$$
$$= \{ \text{'' ''} : 0 \leq x \leq \sqrt{a^2 - y^2} \,,\; y \in [0,a] \}$$

→ Se puede cambiar el ORDEN INTEGRACIÓN

$$\int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} dy \; f(x,y) = \int_c^d dy \int_{\tilde{\varphi}_1(y)}^{\tilde{\varphi}_2(y)} dx \; f(x,y)$$

→ ¿POR QUÉ es ÚTIL?

→ Puedo cambiar el ORDEN y pasar a integral mas sencilla

E)

Tengo que integrar $\sqrt{1+e^{2y}}$

$$\int_1^2 dx \int_0^{\log x} dy \,(x-1)\sqrt{1+e^{2y}} = \iint_D dA \, f(x,y)$$

$$D = \{(x,y)\in \mathbb{R}^2 : 1 \leq x \leq 2 ,\ 0 \leq y \leq \log x\}$$

$$= \{\ \text{''}\ \ \text{''} : 0 \leq y \leq \log 2 ,\ e^y \leq x \leq 2\}$$

OBTENGO

$$\int_0^{\log 2}\int_{e^y}^{2} (x-1)\sqrt{1+e^{2y}}\, dx\, dy$$

$$= \int_0^{\log 2} dy \sqrt{1+e^{2y}} \left( \int_{e^y}^{2} (x-1)\, dx \right)$$

$$= \int_0^{\log 2} dy \sqrt{1+e^{2y}} \left[ \frac{x^2}{2} - x \right]\Bigg|_{e^y}^{2}$$

$$= \int_0^{\log 2} dy \sqrt{1+e^{2y}} \left( -\frac{e^{2y}}{2} + e^y - \cancel{2} + \cancel{2} \right)$$

OBTENEMOS

$$\underbrace{-\int_0^{\log 2} dy\, \frac{1}{2} e^{2y}\sqrt{1+e^{2y}}}_{A} + \underbrace{\int_0^{\log 2} e^{y}\sqrt{1+e^{2y}}\, dy}_{B}$$

A) $u = e^{2y}$, $du = 2u\,dy$, $dy = \frac{du}{2u}$

$0 \mapsto 1$, $\log 2 \mapsto 4$

$$-\frac{1}{4}\int_1^4 u^2\sqrt{1+u^2}\, du$$

B $z = e^{y}$, $dz = u\,dy$, $dy = \frac{dz}{u}$

$0 \mapsto 1$, $\log 2 \mapsto 2$

$$\int_1^2 \frac{\cancel{z}}{\cancel{z}}\sqrt{1+z^2}\, dz = \int_1^2 \sqrt{1+z^2}\, dz$$

A)

$$-\frac{1}{4}\int_1^4 \sqrt{1+u}\,du = -\frac{1}{4}\int_2^5 \tilde{u}^{\frac{1}{2}}\,d\tilde{u} = -\frac{1}{4}\,\frac{2}{3}\,\tilde{u}^{\frac{3}{2}}\Big|_2^5$$

$$= -\frac{1}{6}\left(5^{\frac{3}{2}} + 2^{\frac{3}{2}}\right)$$

B) $\displaystyle\int_1^2 \sqrt{1+z^2}\,dz$ → SOSTITUCIÓN TRIG.

$z = \tan\theta$ , $dz = \dfrac{d\theta}{\cos^2\theta} = \sec^2\theta\,d\theta$

$1 + \tan^2\theta = \dfrac{1}{\cos^2\theta} = \sec^2\theta$

$1 \mapsto \frac{\pi}{4}$ , $2 \mapsto \arctan(2)$

$\displaystyle\int_{\frac{\pi}{4}}^{\arctan(2)} d\theta\,\sec^3\theta$ → ¿cómo lo soluciono?

→ x PARTES

$$\int \sec^3\vartheta \, d\vartheta = \int \sec\vartheta \, (\sec\vartheta)^2 \, d\vartheta$$

$$= \sec\vartheta \tan\vartheta$$

$$- \int \tan\vartheta \underbrace{\frac{d}{d\vartheta} \sec\vartheta}$$

$$\frac{d}{d\vartheta}\left(\frac{1}{\cos\vartheta}\right) = -\frac{\sin\vartheta}{(\cos\vartheta)^2}$$

$$= -\tan\vartheta \sec\vartheta$$

$$= \sec\vartheta \tan\vartheta + \int (\tan\vartheta)^2 \sec\vartheta \, d\vartheta$$

$$= \sec\vartheta \tan\vartheta - \int \underbrace{(\sec^2\vartheta - 1)} \sec\vartheta \, d\vartheta$$

$$\tan^2\vartheta = \frac{\sin^2\vartheta}{\cos^2\vartheta} = \frac{\sin^2 - \cos^2\vartheta + \cos^2\vartheta}{\cos^2\vartheta}$$

$$= \frac{\sin^2 + \cos^2}{\cos^2} - 1 = \sec^2\vartheta - 1$$

$$= \sec\vartheta \tan\vartheta - \int \sec^3\vartheta \, d\vartheta + \int \sec\vartheta \, d\vartheta$$

$$\int \sec\theta \, d\theta = \int \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta} \, d\theta$$

MULTIPLICO Y DIVIDO X $\sec\theta + \tan\theta$

$$u = \sec\theta + \tan\theta$$

$$du = \left(\frac{d}{d\theta}\sec\theta + \frac{d}{d\theta}\tan\theta\right) d\theta$$

$$= \underbrace{\frac{d}{d\theta}\left(\frac{1}{\cos\theta}\right)} + \frac{d}{d\theta}\frac{\sin\theta}{\cos\theta}$$

$$\tan\theta\sec\theta + \sec^2\theta$$

$$= \sec\theta\,(\tan\theta + \sec\theta)$$

$$\int \sec\theta \, d\theta = \int \frac{du}{u} = \ln|\sec\theta + \tan\theta|$$

OBTENÉIS

$$\int \sec^3\theta \, d\theta = \sec\theta \tan\theta + \log|\sec\theta + \tan\theta| - \int \sec^3\theta \, d\theta$$

ENTONCES

$$\int \sec^3\theta = \frac{1}{2}\left[\sec\theta\tan\theta + \log|\sec\theta + \tan\theta|\right]$$

Vuelvo a las variables del principio y termino

¿Cómo se EXTIENDE TODO esto a 3D

→ BASTANTE TRIVIAL

$$\begin{cases} 2D & \text{Rectángulo} \\ 3D & \text{CAJAS} \end{cases}$$

$$\iiint_B (x+2y+3z)^2 \, dx\, dy\, dz$$

$$B = [0,1] \times \left[-\tfrac{1}{2}, 0\right] \times \left[0, \tfrac{1}{3}\right]$$

$$\int_0^1 dx \int_{-\frac{1}{2}}^0 dy \int_0^{\frac{1}{3}} dz \, (x+2y+3z)^2$$

$$u = x+2y+3z, \quad du = 3dz, \quad dz = \frac{du}{3}$$

$$\frac{1}{3} \mapsto x+2y+1, \quad 0 \mapsto x+2y$$

$$\frac{1}{3}\int_{x+2y}^{x+2y+1} u^2\, du = \frac{1}{3\cdot 3} u^3 \Big|_{x+2y}^{x+2y+1} = \frac{1}{9}\left[(x+2y+1)^3 - (x+2y)^3\right]$$

Se reduce a

$$\frac{1}{9}\int_0^1 dx \int_{-\frac{1}{2}}^{0} dy \left[ (x+2y+1)^3 - (x+2y)^3 \right]$$

$$= \underbrace{\frac{1}{9}\int_0^1 dx \int_{-\frac{1}{2}}^{0} dy (x+2y+1)^3}_{a} - \underbrace{\frac{1}{9}\int_0^1 dx \int_{-\frac{1}{2}}^{0} (x+2y)^3 dy}_{b}$$

a) $u = x+2y+1$

$du = 2\,dy \,,\quad dy = \frac{du}{2}$

$0 \mapsto x+1 \,,\quad -\frac{1}{2} \mapsto x$

$$\frac{1}{2}\frac{1}{9}\int_0^1 dx \int_x^{x+1} du \; u^3 = \frac{1}{2\cdot 4\cdot 3^2}\int_0^1 dx \left[ (x+1)^4 - x^4 \right]$$

b) $u = x+2y \,,\quad du = 2y \,,\quad dy = \frac{du}{2}$

$0 \mapsto x \,,\quad -\frac{1}{2} \mapsto x-1$

$$\frac{1}{2}\frac{1}{9}\int_0^1 dx \int_{x-1}^{x} du \; u^3 = \frac{1}{2^3 3^2}\int_0^1 dx \left[ x^4 - (x-1)^4 \right]$$

Finalmente obtenemos

$$\frac{1}{2^3 3^2}\int_0^1 dx\ (x+1)^4 - 2x^4 + (x-1)^4 ]$$

$$= \frac{1}{2^3 3^2}\int_1^2 du\ u^4 - \frac{2}{2^2 3^2}\frac{x^5}{5}\Big|_0^1 + \frac{1}{2^2 3^2}\int_{-1}^0 u^4\ du$$

$$= \frac{1}{2^3 3^2}\frac{1}{5}\left[2^5 \cancel{-1} - 2 \cancel{+1}\right]$$

$$= \frac{1}{2^3\cdot 5\cdot 9}[2^5 - 2] = \frac{1}{360}[2^5 - 2]$$

Vamos a calcular

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz\; e^{x+y+z}$$

$$= \int_0^1 dx \int_0^1 dy \int_{x+y}^{x+y+1} du\; e^{u} = \int_0^1 dx \int_0^1 dy\; e^{x+y}(e-1)$$

$$= (e-1) \int_0^1 dx \int_x^{x+1} du\; e^{u} = (e-1)\int_0^1 dx\; e^{x+1} - e^{x}$$

$$= (e-1)^2 \int_0^1 dx\; e^{x} = (e-1)^3$$

Este TB se podía tratar así

$$\int_0^1 dx\; e^{x} \int_0^1 dy\; e^{y} \int_0^1 dz\, e^{z} = \left(\int_0^1 e^{x}\, dx\right)^3$$

$$= (e-1)^3$$

Ej

$$\iiint_{W} dx\, dy\, dz$$

$$W = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 1\}$$

→ ESFERA RADIO 1

→ ¿QUÉ RESULTADO os esperáis?

→ $V = \frac{4}{3}\pi$

Vamos a demostrar la fórmula para el volumen de la esfera.

→ HAY QUE ENTENDER CÓMO ESCRIBIR la integral.

→ ESTA es la dificultad

$$\int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz = 2\int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \sqrt{1-x^2-y^2}$$

OBSERVAMOS QUE

$$\int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, dy = \underbrace{\int_{-a}^{a} \sqrt{a^2-y^2}\, dy}_{\text{¿cómo lo soluciono?}}$$

$$\to \quad y = a \sin t$$

$$\sqrt{a^2-y^2} = a\sqrt{1-\sin^2 t} = a \cos t$$

$$dy = a \cos t \, dt \, ,$$

$$\pm a \mapsto \pm \frac{\pi}{2}$$

cómo se soluciona

OBTENEMOS $\displaystyle \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2 \cos^2 t \, dt$

$$= \frac{a^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Big(1 + \underbrace{\cos(2t)}_{\text{NO CONTRIBUYE}}\Big)\, dt = \frac{\pi}{2} a^2$$


$\to 0$

Finalmente obtenemos

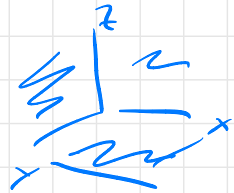
$$2\pi\int_{-1}^{1}\left(\frac{1-x^2}{2}\right)dx = 2\pi\int_{0}^{1}(1-x^2)\,dx$$

$$= 2\pi\left(x-\frac{x^3}{3}\right)\Bigg|_{0}^{1} = 2\pi\left(1-\frac{1}{3}\right) = \frac{4}{3}\pi$$

E)

$W \to$ yace entre planos

$x = 0, \; y = 0, \; z = 2$




$x^2 + y^2 = z$

CUADRANTES $x \geq 0, \; y \geq 0$

Integrar $f(x,y,z) = x$

$\to$ ¿cómo escribimos la integral?

$$x^2 + y^2 \leq z \leq 2$$

$$0 \leq y \leq \sqrt{2 - x^2}$$

$$\int_0^{\sqrt{2}} dx \int_0^{\sqrt{2-x^2}} dy \int_{x^2+y^2}^{2} dz \; x$$

$$= \int_0^{\sqrt{2}} dx \int_0^{\sqrt{2-x^2}} dy \; x \, (2 - x^2 + y^2)$$

$$\int_0^{\sqrt{2-x^2}} x\,(2-x^2-y^2)\,dy$$

$$= x\,(2-x^2)\sqrt{2-x^2} - \frac{x}{3}(2-x^2)^{\frac{3}{2}}$$

$$= x\,(2-x^2)^{\frac{3}{2}} - \frac{x}{3}(2-x^2)^{\frac{3}{2}}$$

$$= \frac{2}{3}x\,(2-x^2)^{\frac{3}{2}}$$

$$\int_0^{\sqrt{2}} dx\,\frac{2}{3}\,x\,(2-x^2)^{\frac{3}{2}} \to$$ LO HACEIS VOSOTROS