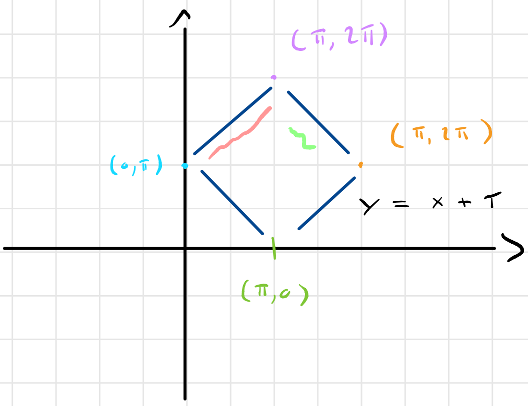
Ej 3.2.1

$$\int_S (x-y)^2 \sin^2(x+y)\, dx\,dy$$

S AREA PARALELOGRAMA con vértices

$(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$, $(0, \pi)$

Vamos a dibujarlo

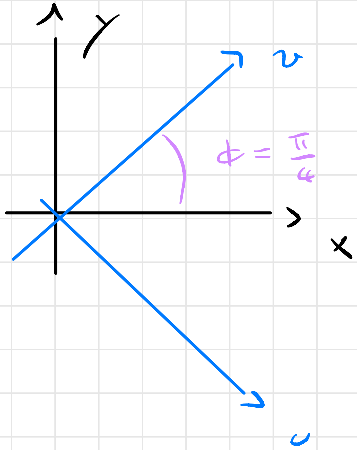


en la práctica es UN RECTÁNGULO

→ pero se integra con respecto a x e y es la suma de dos dominio x/y sencillo

$$\left[\int_0^{\pi} dx \int_{\pi}^{x-\pi} dy\, f + \int_{\pi}^{2\pi} dx \int_{\pi}^{3\pi - x} dy\, f \right] + \ldots$$ OTROS C.

Voy a rotar los ejes



$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \cos\phi & \sin\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$u = x - y \quad , \quad x = \frac{1}{2}(u + v)$

$v = x + y \quad , \quad y = \frac{1}{2}(v - u) \; *$

$$J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & +1 \end{pmatrix}$$

$\det J = \frac{1}{2}$

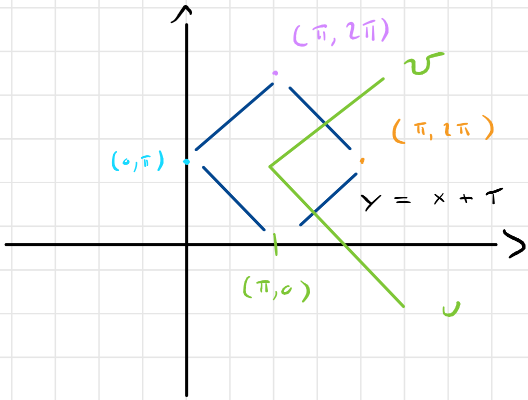
$* \; u - v = x - x - y - y$

$u - v = -2y \Rightarrow y = \frac{1}{2}(v - u)$

NUESTRA INTEGRAL SE ESCRIBE COMO

$$\frac{1}{2} \int u^2 \sin^2 v \, du \, dv$$

¿Y LOS EXTREMOS?



v varia entre $(0,\pi)$ y $(\pi, 2\pi)$

$(0, \pi) \longrightarrow v_1 = \overbrace{0 + \pi}^{\text{dot } v} = \pi$

$(\pi, 2\pi) \longrightarrow v_2 = \pi + 2\pi = 3\pi$

u varia entre $(0,\pi)$, $(\pi, 0)$

$(0, \pi) \longrightarrow u_1 = 0 - \pi = -\pi$

$(\pi, 0) \longrightarrow u_2 = \pi - 0 = \pi$

Finalmente Tenemos

$$\frac{1}{2}\int_{-\pi}^{\pi} du \int_{\pi}^{3\pi} dv \quad u^2 \sin^2 v$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} u^2\, du \int_{\pi}^{3\pi} \sin^2 v\, dv$$

$$\int_{-\pi}^{\pi} u^2\, du = \frac{u^3}{3}\Bigg|_{-\pi}^{\pi} = \frac{2}{3}\pi^3$$

$$\int_{\pi}^{3\pi} \sin^2 v\, dv = \frac{1}{2}\int_{\pi}^{3\pi} 1-\cos(2v)\, dv$$

$$= \frac{1}{2} v\Bigg|_{\pi}^{3\pi} - \frac{1}{2}\int_{\pi}^{3\pi}\cos(2v)\, dv$$

$$= \pi - \frac{1}{4}\int_{2\pi}^{6\pi}\cos(z)\, dz$$

$z = 2v$, $dz = 2dv$
$dv = \frac{dz}{2}$, $3\pi \to 6\pi$
$\pi \to 2\pi$

$$= \pi - \frac{\sin(z)}{4}\Bigg|_{2\pi}^{6\pi} \to 0 \qquad = \pi$$

$$\frac{1}{2}\int_{-\pi}^{\pi} u^2\, du \int_{\pi}^{3\pi} \sin^2 v\, dv = \frac{1}{2}\left(\frac{2}{3}\pi^3\right)\left(\pi\right) = \frac{\pi^4}{3}$$

Vamos a rematar un poco estas nociones

Ej 3.2.3

$$\begin{cases} x = u + v \\ y = v - u^2 \end{cases}$$

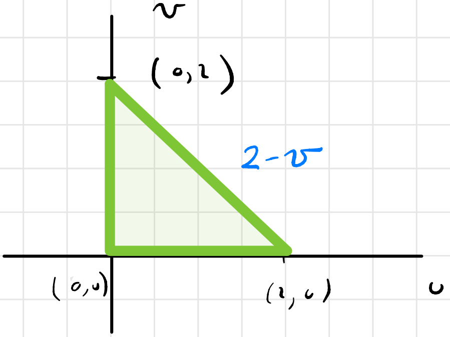
I) JACOBIANO

$$J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2u & 1 \end{pmatrix}$$

$$\det J = 1 + 2u$$

II) $S_{xy} = \operatorname{Im} T_{uv}$, T_{uv} Tr en u-v con vértices

$(0,0)$, $(2,0)$, $(0,2)$



$$(0,0) \longrightarrow (0+0,\ 0-0^2) = (0,0)$$

$u \quad v \qquad x = u+v \qquad y = v - u^2$

$$(2,0) \longrightarrow (2+0,\ 0-2^2) = (2,-4)$$

$$(0,2) \longrightarrow (2,\ 2)$$

(iii) Area S_{xy}

$$\int_{S_{xy}} dx\,dy = \int_{T_{uv}} (1+2u)\,du\,dv$$

$$= \int_0^2 dv \int_0^{\boxed{2-v}} du\,(1+2u)$$

$$= \int_0^2 dv \quad u\Big|_0^{2-v} + u^2\Big|_0^{2-v}$$

$$= \int_0^2 dv \quad \overbrace{(2-v)}^{\tilde{v}} + (2-v)^2$$

$$2 + \frac{8}{3} = \frac{14}{3}$$

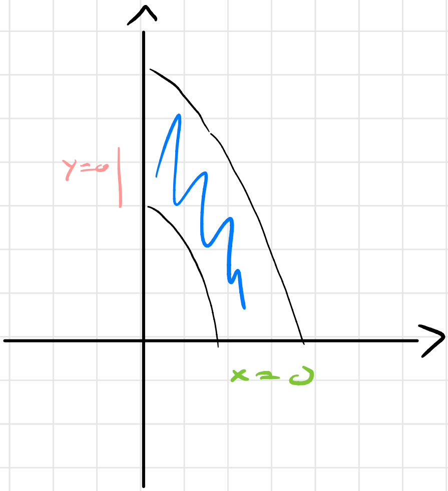
$$= \int_0^2 d\tilde{v}\ \tilde{v} + \int_0^2 \tilde{v}^2\,d\tilde{v} = \frac{2^2}{2} + \frac{2^3}{3}$$

E) 3.2.8

$$\iint_S \frac{x\,dx\,dy}{4x^2+y^2}$$

$S \to$ región entre $x=0$, $y=0$

$1 \leq 4x^2+y^2 \leq 16$



INTRODUZCO

$$\begin{cases} x = \dfrac{r}{2}\cos t \\ y = r\sin t \end{cases}, \quad \begin{cases} t \in [0, \frac{\pi}{2}] \\ r \in [1, 4] \end{cases}$$

$$\int_0^{\frac{\pi}{2}} dt \int_1^4 dr \; |J| \; \overbrace{\frac{r}{2} \cos t}^{x} \; \frac{1}{\underbrace{r^2}_{4x^2+y^2}}$$

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial t} \end{pmatrix}$$

$$\frac{\partial x}{\partial r} = \frac{\cos t}{2} \quad , \quad \frac{\partial x}{\partial t} = -\frac{r}{2} \sin t$$

$$\frac{\partial y}{\partial r} = \sin t \quad , \quad \frac{\partial y}{\partial r} = r \cos t$$

$$\det J = \frac{r}{2} \cos^2 t + \frac{r}{2} \sin^2 t = \frac{r}{2}$$

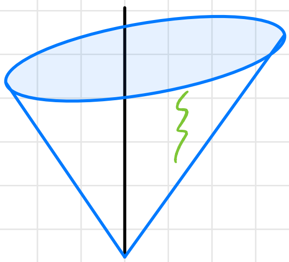
$$\int_0^{\frac{\pi}{2}} dt \int_1^4 dr \; \left(\frac{r}{2}\right)^2 \frac{1}{r^2} \; \cos t$$

$$= \frac{1}{4} \int_0^{\frac{\pi}{2}} dt \cos t \int_1^4 dr = \frac{3}{4} \sin(t) \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{3}{4}$$

E) 3.2.9

$$R = \{(x,y,z) \in \mathbb{R}^3 : z = \sqrt{x^2+y^2},\ z \in [0,3]\}$$



PARTE (ii)

$$\int_R \sqrt{9-x^2-y^2}\,dx\,dy\,dz$$

$$= \int_0^{2\pi} d\theta \int_0^3 dz \int_0^z dr\, \sqrt{9-r^2}\; r$$

→ viene jacobiano

$$= 2\pi \int_0^3 dz \int_0^z dr\, \sqrt{9-r^2}\; r$$

Voy a considerar

$$\int_0^z dr \quad r \sqrt{9 - r^2}$$

$$r = 3 \sin(t), \qquad dr = 3\cos(t)\, dt$$

$$9 \int_0^z dt \quad \sin(t) \cos(t) \sqrt{9 - 9\sin^2(t)}$$

$$= 27 \int_0^z dt \ \sin(t) \cos^2(t)$$

$$u = \cos t \quad , \qquad du = - \sin(t)\, dt$$

OBTENGO

$$- 27 \int_0^z du \ u^2 = - 9 \ u^3 = - 9\cos^3 t$$

Lo tengo que evaluar entre

$$r = 0 \longrightarrow t = 0$$

$$r = z \longrightarrow t = \arcsin\left(\frac{z}{3}\right)$$

$$- 9 \cos^3(t) \Big|_{t=0} = -9$$

$$- 9 \cos^3(t) \Big|_{t = \arcsin\left(\frac{z}{3}\right)}$$

$$= -9 \left[1 - \left(\frac{z}{3}\right)^2 \right]^{\frac{3}{2}}$$

$$= -9 \left(\frac{9 - z^2}{9} \right)^{\frac{3}{2}} = -\frac{1}{3} (9 - z^2)^{\frac{3}{2}}$$

Finalmente obtengo

$$\int_0^z dr \, r \sqrt{9 - r^2} = 9 - \frac{1}{3} (9 - z^2)^{\frac{3}{2}}$$

Tengo que integrar con respecto a z

$$\int_0^3 \Big(9 - \underbrace{\frac{1}{3} (9 - z^2)^{\frac{3}{2}}} \Big) \, dz$$

CASA $\rightarrow$ SOSTITUCIÓN TR.

E) 3.2.9 (iii)

$$\int_R z\, e^{x^2+y^2+z^2}$$

$$= \int_0^{2\pi} d\theta \int_0^3 dz\; z\, e^{z^2} \int_0^1 r\, e^{r^2}\, dr$$

viene de J

$$= 2\pi \int_0^3 dz\; z\, e^{z^2} \int_0^z \frac{\partial}{\partial r}\left(\frac{1}{2} e^{r^2}\right) dr$$

$$= \pi \int_0^3 dz\; z\, e^{z^2}\left(e^{z^2} - 1\right) dz$$

$$= \underbrace{\pi \int_0^3 dz\; z\, e^{2z^2} - \pi \int_0^3 z\, e^{z^2}\, dz}_{\text{MISMA Técnica}}$$

$$= \pi \int_0^3 dz\, \frac{\partial}{\partial z}\left(\frac{e^{2z^2}}{4}\right) - \pi \int_0^3 dz\, \frac{\partial}{\partial z}\left(\frac{e^{z^2}}{2}\right)$$

$$= \frac{\pi}{4}\left(e^{18} - 1 + 2 - e^{9}\right) = \frac{\pi}{4}\left(e^{18} - 2e^{9} + 1\right)$$

Ej 3.2.10

$$\int_W f(x,y,z)\,dx\,dy\,dz$$

$$f(x,y,z) = e^{-(x^2+y^2+z^2)^{\frac{3}{2}}}$$

W región entre

- cono $z = \sqrt{x^2+y^2}$
- esfera $x^2+y^2+z^2 = 9$

→ COORDENADAS ESFÉRICAS

→ radio · 2

$$\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} d\varphi\, \underbrace{\sin\varphi \int_0^3 e^{-\rho^3}\rho^2}_{J}\, d\rho$$

$$= 2\pi\left(1-\frac{\sqrt{2}}{2}\right)\int_0^3 e^{-\rho^3}\rho^2$$

$$= 2\pi\left(1-\frac{\sqrt{2}}{2}\right)\int_0^3 \frac{\partial}{\partial\rho}\left(\frac{e^{-\rho^3}}{3}\right)$$

$$= \frac{\pi}{3}(2-\sqrt{2})\,e^{-\rho^3}\Big|_3^0 = \frac{\pi}{3}(2-\sqrt{2})(1-e^{-27})$$