Derivadas PARCIALES

$\hookrightarrow \underbrace{f_x, f_y \ ---}$ $--$

DERIVADAS ORDEN SUPERIOR

$\frac{\partial^2}{\partial x^2} f \quad --$ } por qué es interesante?

$\rightarrow$ DERIVADAS MIXTAS $\underbrace{\frac{\partial}{\partial y \partial x} f}$

Quiero comentar cosas sobre ese objeto

Andiamo con ordine

def $f: \mathbb{R}^n \rightarrow \mathbb{R}$

$\forall i \le n \quad \exists \frac{\partial f}{\partial x_i}$ CONTINUAS $\Rightarrow f \in C^1$

" $\frac{\partial^2}{\partial}$ " $\Rightarrow f \in C^2$

y así seguimos hasta... C^∞

Ej) $\frac{\partial^2}{\partial x^2} f = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f\right)$, $\underbrace{\frac{\partial}{\partial_{xy}} f = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f}$

Derivada MIXTA

Ej $f(x,y) = xy + (x+2y)^2$

$$\frac{\partial}{\partial x} f(x,y) = y + 2(x+2y)$$

$$\frac{\partial}{\partial y} f(x,y) = x + 4(x+2y)$$

Vamos a calcular la derivada MIXTA

$$\frac{\partial}{\partial x y} f(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f\right)$$

$$= \frac{\partial}{\partial x}\left(x + 4(x+2y)\right)$$

$$= 1 + 4 = 5$$

¿Qué pasa con la otra?

→ ¿MISMO RESULTADO? ¿QUÉ OS ESPERAIS?

$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f\right)$$

$$= \frac{\partial}{\partial y}\left(y + 2(x+2y)\right)$$

$$= 1 + 4 = 5$$

¡LA MISMA!

LA PREGUNTA QUE NOS HACEMOS ES --

¿BAJO QUÉ CONDICIÓN ESTO OCURRE?

Teorema $f : \mathbb{R}^n \to \mathbb{R}$

$$f \in C^2 \Rightarrow \frac{\partial f}{\partial xy} = \frac{\partial f}{\partial yx}$$

↳ ESTA es la condición

Ej $f(x,y) = xe^y + yx^2$ combinación de funciones siempre continuas

$$\frac{\partial f}{\partial x} = e^y + 2xy$$

$$\frac{\partial f}{\partial y} = xe^y + x^2$$

$$\frac{\partial f}{\partial yx} = e^y + 2x$$

Son iguales

$$\frac{\partial f}{\partial xy} = e^y + 2x$$

CAMPOS VECTORIALES

def CAMPO VECTORIAL en $\mathbb{R}^n$

$$F : A \subset \mathbb{R}^n \to \mathbb{R}^n$$
$$\bar{x} \mapsto F(\bar{x})$$

EJ ∇f es un CAMPO VECTORIAL

def $F = \nabla \varphi$ (CAMPO EXPRESADO COMO GRADIENTE FUNCIÓN ESCALAR)

$\to F$ CAMPO CONSERVATIVO

EJ $V(x,y) = -y\hat{i} + x\hat{j}$

Vectores Base Canónica: con $\hat{i} = (1,0)$, $\hat{j} = (0,1)$

CAMPO VECTORIAL $\mathbb{R}^2$

$\longrightarrow$ MOVIMIENTO ROTACIONAL

(Figura 5.2 en las notas)

$\to$ COORDENADAS POLARES

$$V(x,y) = r\cos\theta\,\hat{j} - r\sin\theta\,\hat{i}$$

Introducimos un operador en el CAMPO VECTORIAL

def $F : A \subset \mathbb{R}^n \to \mathbb{R}^n$ (CAMPO VECTORIAL $\mathbb{R}^n$)

$$\operatorname{div} F = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}$$

deriva la componente i-esima con respecto a la VARIABLE i-esima

→ Lo hago con respecto a TODAS LAS VARIABLES

→ Y sumo

Ej $F = x^2 y \,\hat{i} + z\,\hat{j} + xyz\,\hat{k}$

$$\operatorname{div} F = \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(xyz)$$

$$= 2xy + 0 + xy$$

$$= 3xy$$

OBSERVACIÓN : Se puede indicar TB

$$\operatorname{div} F = \underbrace{\nabla \cdot F}$$

Producto escalar entre ∇ y F

DIVERGENCIA

↳ SIGNIFICADO FÍSICO

$\underbrace{\vec{F} \to \text{Velocidad FLUIDO}}_{\text{A CADA Punto espacio }(x,y,z)\text{ ASOCIO vector velocidad}}$

A CADA Punto espacio (x,y,z)
ASOCIO vector velocidad)

→ ¿Qué me dice la divergencia?

$\underbrace{\operatorname{div} \vec{F}}$ → TASA EXPANSIÓN FLUIDO

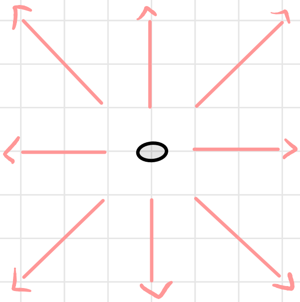
$\operatorname{div} \vec{F} < 0$ → FLUIDO se COMPRIME

$\operatorname{div} \vec{F} > 0$ → FLUIDO se EXPANDE

$\operatorname{div} \vec{F} = 0$ → ni se comprime,
ni se expande

Ej $\vec{F} = x\hat{i} + y\hat{j}$

→ Líneas rectas que se alejan del origen



$\operatorname{div} \vec{F} = ?$

¿Qué os esperáis?

Positiva? NEGATIVA?

¿Por qué?

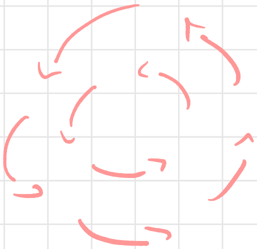
$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y$$

$$= 1 + 1 = 2 > 0$$

→ EL FLUIDO SALE

Ej $\vec{F} = -y\,\hat{i} + x\,\hat{j}$

$\longrightarrow$ MOVIMIENTO ROTACIONAL



$\operatorname{div} \vec{F} = \ ?$

¿Qué os esperáis?

$$\operatorname{div} \vec{F} = +\frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x)$$

$$= \quad 0 \quad + \quad 0 \quad = 0$$

EL FLUIDO da VUELTAS, NO SALE

Vamos a ver otra operación

<u>def</u> $F = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ campo vectorial $\mathbb{R}^3$

Rotor / Rotacional F (curl, inglés)

$$\nabla \times F = \nabla \wedge F = \det\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j}$$

$$+ \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{y}$$

<u>Ej</u> $F(x,y,z) = \underline{x\hat{i} + xy\hat{j} + \hat{k}}$

$$\underline{\nabla \wedge F} = \det\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & xy & 1 \end{pmatrix}$$

$$= \left(\frac{\partial}{\partial y} 1 - \frac{\partial}{\partial z} xy\right)\hat{i} + \left(\frac{\partial}{\partial x} 1 - \frac{\partial}{\partial z} x\right)\hat{j}$$

$$+ \left(\frac{\partial}{\partial x} xy - \frac{\partial}{\partial t} x\right)\hat{k} = 0\hat{i} + 0\hat{j} + \underline{y\hat{k}}$$

SIGNIFICADO FÍSICO

$\rightarrow$ CAPACIDAD DESARROLLAR ROTACIÓN alrededor de un punto

$\rightarrow$ Será más claro cuando se estudie el teorema de STOKES

Teorema $\forall f \in C^1(\mathbb{R}^3)$

$$\nabla \times \nabla f = 0$$

dim $\nabla f = \left(\frac{\partial}{\partial x} f, \frac{\partial}{\partial y} f, \frac{\partial}{\partial z} f\right)$

$$\nabla \times \nabla f = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial_{yz}} f - \frac{\partial}{\partial_{zy}} f\right) + \hat{j}\left(\frac{\partial}{\partial_{xz}} f - \frac{\partial}{\partial_{zx}} f\right)$$

$$+ \hat{k}\left(\frac{\partial f}{\partial_{xy}} - \frac{\partial f}{\partial_{yx}}\right) = 0$$

$= \hookrightarrow C^1$

COR $\mathcal{F}$ CAMPO CONSERVATIVO

$\Rightarrow \nabla \times \mathcal{F} = 0$

Ej $V(x,y,z) = y\hat{i} - x\hat{j}$

CONSERVATIVO ?

$\rightarrow \nabla \times \mathcal{F} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{pmatrix} = -2\hat{k} \neq 0$

$\rightarrow$ NO es CONSERVATIVO

Teorema $\nabla \cdot (\nabla \times \mathcal{F}) = 0 \quad (\mathcal{F} \in C^2(\mathbb{R}^3))$

dem CALCULAR EXPLICITAMENTE y usar hecho de que las derivadas MIXTAS son iguales

Ej $V(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$

$\exists \tilde{V} : V = \nabla \times \tilde{V}$?

$\operatorname{div} V = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z = 3 \neq 0$

NO, o si no $\operatorname{div} V = 0$

ULTIMO OPERADOR

LAPLACIANO

$f : \mathbb{R}^n \to \mathbb{R}$

$$\Delta f = \underbrace{\nabla^2 f = \nabla \cdot \nabla f}$$

SABRÍAIS decirme la formula sabiendo que se puede indicar así?

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

COORDENADAS POLARES

$$f : A \subset \mathbb{R}^2 \to \mathbb{R}$$

$$x = r\cos\vartheta \quad , \quad y = r\sin\vartheta$$

$$r = \sqrt{x^2+y^2} \quad , \quad \vartheta = \arctan\left(\frac{y}{x}\right)$$

<u>def</u> LAPLACIANO $\Delta f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}$

Transformo la derivada en coordenadas polares

$$\frac{\partial}{\partial x} f = \frac{\partial r}{\partial x}\frac{\partial f}{\partial r} + \frac{\partial \vartheta}{\partial x}\frac{\partial f}{\partial \vartheta}$$

$$\frac{\partial}{\partial y} f = \frac{\partial r}{\partial y}\frac{\partial f}{\partial r} + \frac{\partial \vartheta}{\partial y}\frac{\partial f}{\partial \vartheta}$$

$$\left(\frac{\partial r}{\partial x}\right) = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{r\cos\vartheta}{r} = \cos\vartheta$$

$$\left(\frac{\partial r}{\partial y}\right) = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{r\sin\vartheta}{r} = \sin\vartheta$$

$$\left(\frac{\partial \vartheta}{\partial x}\right) = -\frac{x^2}{x^2+y^2}\,\frac{y}{x^2} = -\frac{\sin\vartheta}{r}$$

$$\left(\frac{\partial \vartheta}{\partial y}\right) = +\frac{x}{x^2+y^2} = \frac{\cos\vartheta}{r}$$

Se obtiene

$$\frac{\partial f}{\partial x} = \cos\theta \frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}$$

$$\frac{\partial f}{\partial y} = \sin\theta \frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}$$

Ahora tenemos que calcular las derivadas segundas

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \cos\theta \frac{\partial}{\partial x}\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial x}\frac{\partial}{\partial \theta} f$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial r} = \frac{\partial r}{\partial x}\frac{\partial^2 f}{\partial r^2} + \frac{\partial \theta}{\partial x}\frac{\partial^2 f}{\partial \theta \partial r}$$

$$= \cos\theta \frac{\partial^2 f}{\partial r^2} - \frac{\sin\theta}{r}\frac{\partial^2 f}{\partial r \partial \theta}$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial \theta} = \frac{\partial r}{\partial x}\frac{\partial^2 f}{\partial r \partial \theta} + \frac{\partial \theta}{\partial x}\frac{\partial^2 f}{\partial \theta^2}$$

$$= \cos\theta \frac{\partial^2 f}{\partial r \partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 f}{\partial \theta^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \cos^2\theta \frac{\partial^2 f}{\partial r^2} - \frac{\sin 2\theta}{\cancel{2} r}\frac{\partial^2 f}{\partial r \partial \theta}$$

$$- \frac{\sin 2\theta}{2r}\cancel{\frac{\partial^2 f}{\partial r \partial \theta}} + \left(\frac{\sin\theta}{r}\right)^2 \frac{\partial^2}{\partial \theta^2} f$$

(incompleto)