Single Qubit Interactions

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1 What are qubits

Qubits are computational bits used to hold information in quantum computers. Similar to regular bits they can be 0 or 1. However they can also be in superposition where they are neither 0 or 1 but instead have a certain probability of being 1 or 0. This superposition state can be influenced by gate operations. This enables qubits to hold a lot more information than regular bits. This project will explore this physically examining electrons in a vacuum which can act as qubits.

Electrons can be in a spin up or a spin down state. Applying a magnetic field can create this split. Different states affect the energy of the electron when measured. Spin cannot be directly measured, however the energy of the electron can be measured. This is why Hamiltonian are used to simulate quantum interactions. The Hamiltonian of a system is just a expression equating the total energy of the system. The energy Hamiltonian can be related to the wavefunction of the electron by the Schrodinger's equation. Solving Schrodinger's equation is the first step to simulating a quantum system using it's Hamiltonian.

2 Solving Schrodinger's equation

Schrodinger's equation.

$i\hbar \frac{d}{dt}\psi(t) = H(t)\psi(t)$

H(matrix) is the Hamiltonian operator of the system. It is the energy of the system, so its an observable; its eigenvalues are real and represent the energies of the system.

$HE_n = E_nE_n$

E_n is an eigenvector, and E_n is an eigenvalue.
we can diagonalize the matrix to find the eigenvalues and eigenvectors.
Diagonalization involves solving the following equation for E_n

$det(H - E_n I)$

the output is the energy eiganstates and this can be used to calculate the value of $\psi(t)$. Here $c_n(t)$ are the expansion coefficients used to represent the state in terms of the eigenvectors

$\psi(t) = \sum_n c_n(t) E_n$

The basis of eigenvectors is orthonormal, so

$E_k | E_n = \delta_{kn}$

δ_{kn} is the Kronecker delta, and it is 1 if orthogonal and 0 if not orthogonal. This can be substituted into Schrodinger's equation

$i\hbar \frac{d}{dt} \sum_n c_n(t) E_n = H \sum_n c_n(t) E_n$

Using energy eigenvalue equation Since the 2 eigenvectors are orthonormal, the following relation can be made

$E_k i\hbar \sum_n \frac{dc_n(t)}{dt} E_n = E_k H \sum_n c_n(t) E_n E_n$

This can be simplified into a differential equation for each of the possible energy states k = 1...n of the system

$\frac{dc_k(t)}{dt} = -i \frac{E_k}{\hbar} c_k(t)$

This solution of this differential equation is a complex exponential

$c_k(t) = c_k e^{-iE_k t/\hbar}$ c_k is the initial condition

Each coefficient in the energy basis expansion of the state obeys this time dependence. Time evolution of this state using a time independent Hamiltonian is:

$\psi(t) = \sum_n c_n e^{-iE_k t/\hbar} E_n$

The probability of measuring a observable A, the probability of measuring an eigenvalue a_j at t = 1 is given by

$P_{a_j} = |a_j | \psi(t)|^2$

$= |a_j | e^{-iE_1 t/\hbar} | E_1|^2$

$= |a_j | E_1|^2$

a measurement of the system energy at time t would yield the value E1 with a probability

$\psi(t) = c_1 e^{-iE_1t/\hbar} E_1 + c_2 e^{-iE_1t/\hbar} E_2$

$P_{E_1} = |E_1|\psi(t)^2|$

$= —E_1[c_1 e^{-iE_1t/\hbar} E_1 + c_2 e^{-iE_1t/\hbar} E_2]|^2$

$= |c_1|^2$

The probability of measuring eigenvalue a_1 is

$P_{a_1} = |a_1|\psi(t)|^2$

$= |[a_1^* E_1 + a_2^* E_2][c_1 e^{-iE_1t/\hbar} E_1 + c_2 e^{-1E_2t/\hbar} E_2]|^2$

this time evolution oscillates according to the below angular frequency

$\omega_{21} = \frac{E_2 - E_1}{\hbar}$

this is the bohr frequency

3 Spin precession

Simulating the spin of a free electron in a vacuum, the expectation value or average spin value of all free electrons can be calculated for the X, Y, and Z directions. When a static magnetic field is applied, the spin of the qubit can precess along orthogonal directions. in a spin 1/2 system, the Hamiltonian operator represents the total energy of the system $\mu = g\frac{q}{2m_e}$S
here μ is the magnetic moment, g is a constant that defines how strongly the particle's spin couples with the magnetic field, q is the charge of the free electron. m_e is the mass of the electron The Hamiltonian is
$H = -\mu \cdot B = g\frac{q}{2m_e} \cdot B$
B is the magnetic field This Hamiltonian can be solved using Schrödinger's equation. But I used the mesolve solver in the qutip library to generate this visualization of the expectation value.

If a magnetic field is applied along the Z axis,

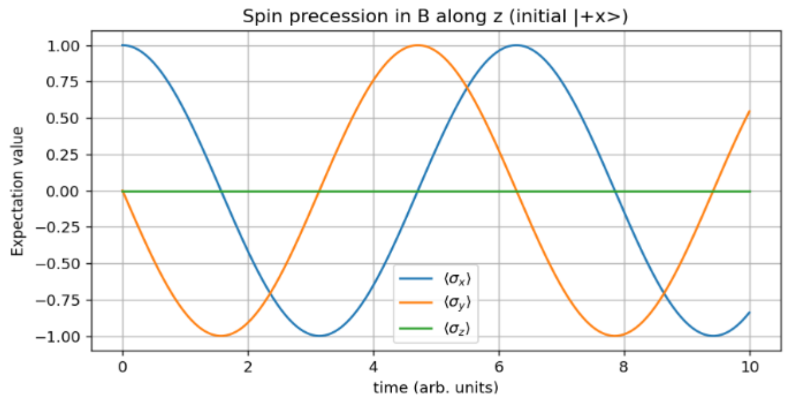

Figure 1: Spin Precession when Magnetic Field is in the Z axis

Here we can see that X and Y do oscillate and differ by a phase $\pi/2$. Z does not oscillate because the magnetic field is in the same direction of the spin.

4 Rabi Frequency

Just like in the previous experiment, there is a magnetic field splitting the spin into 2 possible states and causes the spin to process. Rabi oscillations are caused when another magnetic field is applied perpendicular to the original magnetic field, and this magnetic field is rotating at a frequency of ω.

The Rabi frequency(Ω) can be written as: Ω=γB_1 where B_1 is the perpendicular magnetic field, and γ is the gyromagnetic ratio (how strongly the spin couples to a magnetic field). From this, the following Hamiltonian can be written:

$$H = \begin{bmatrix} -\Delta & \Omega \\ \Omega & \Delta \end{bmatrix}$$

There Δ is the difference between the spin frequency of the electron (ω) and the rotation frequency of the magnetic field (ω_0) of $\Delta = \omega - \omega_0$.

Solving this Hamiltonian using Schrodinger's equation results in the following probability equation $\frac{\Omega^2}{\sqrt{\Omega^2+\Delta^2}} \sin^2(\frac{\sqrt{\Omega^2+\Delta^2}t}{2})$

This can be graphed using a qutip simulation to show the probability and expectation value at different values of Δ keeping Ω constant at 1.

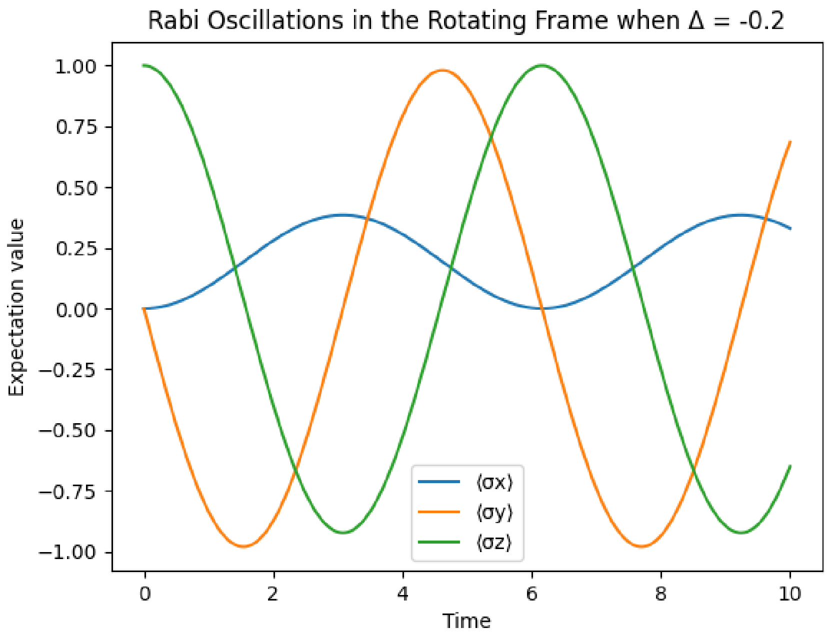

Figure 2: Expectation value of System when Δ = -0.2

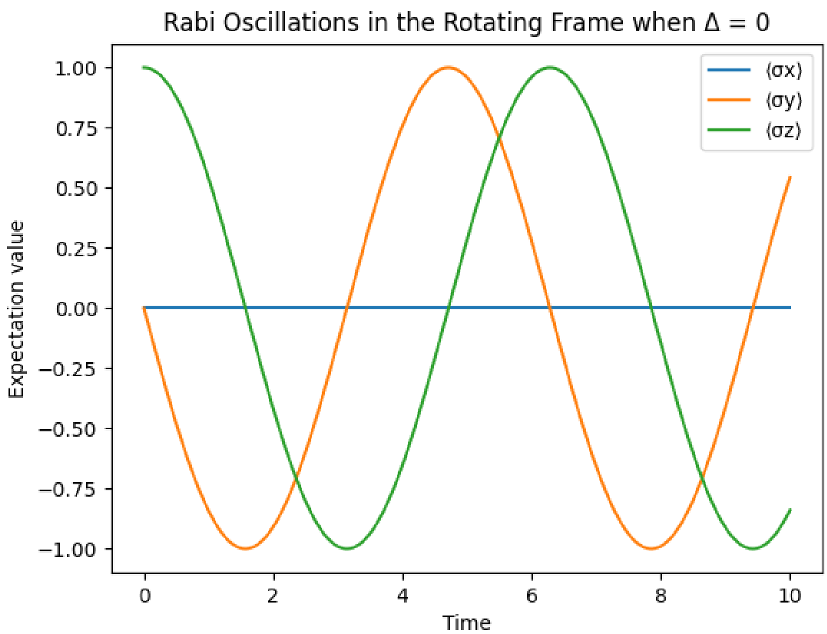

Figure 3: Expectation value of System when $\Delta = 0$

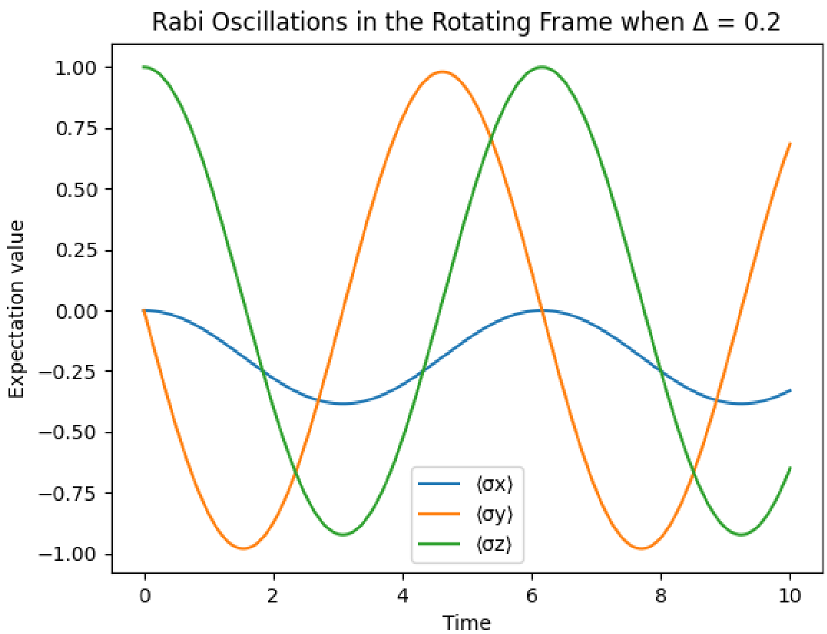

Figure 4: Expectation value of System when $\Delta = 0.2$

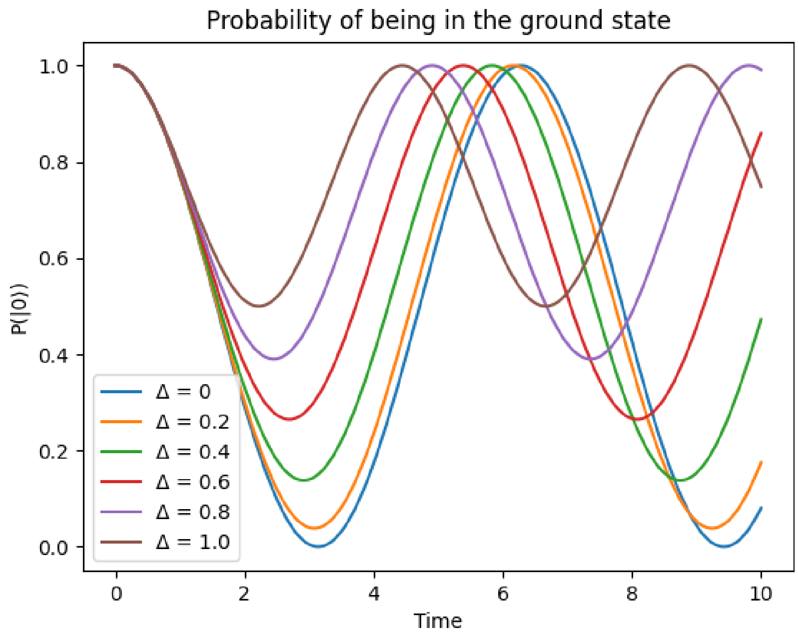

Figure 5: Probability of being in the ground of 0 state

As shown by the figures changing the detuning value or the difference between the two affects the expectation value in the x direction. This also affects the probability of the electron appearing in the 0 or 1 state. The probability also oscillates but the phase and probabilities change with the value of Δ. This can be seen in the graph of probabilities.
As you can see when the probability density function can change depending on the value of Δ this can be used to change the phase of the system and control the superposition state of the qubits.

5 Conclusion

Through this exploration, the process and math behind simulating and solving quantum mechanical systems was explored. We explored single qubits interactions such as superpositions and spin precession. The next step would be to explore entanglement and multi qubits systems.

6 Sources

6.1 Advisors

ISR advisor: Chhavi Goenka
Independent Advisor: Nishant V Sule

6.2 Sources

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Townsend, John S. A Modern Approach to Quantum Mechanics. United States ,University Science Books, 2000.

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